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Preface

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its twentieth annual Conference on Research in Undergraduate Mathematics Education in San Diego, California from February 23 - 25, 2017.

The conference is organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education.

The proceedings include several types of papers that represent current work in the field of undergraduate mathematics education, each of which underwent a rigorous review by two or more reviewers:

– Conference Papers are elaborations of selected RUME Conference Reports
– Contributed Research Reports describe completed research studies
– Preliminary Research Reports describe ongoing research projects in early stages of analysis
– Theoretical Research Reports describe new theoretical perspectives for research
– Posters are 1-page summaries of work that was presented in poster format

The proceedings begin with the winner of the best paper award, the paper receiving honorable mention, and the paper receiving meritorious citation; these awards are bestowed upon papers that make a substantial contribution to the field in terms of raising new questions or providing significant or unique insights into existing research programs. These papers are followed by the pre-journal award winner, which was selected based on its potential to make a substantial contribution to the field; this award is limited to authorship teams that only includes graduate students, recent PhDs (within 2 years of graduation), and/or mathematicians who are transitioning to mathematics education research.

The conference was hosted by San Diego State Univerisity and the University of California San Diego. Their faculty and students provided many hours of volunteer work that made the conference possible and pleasurable, and we greatly thank them for their support.

Many members of the RUME community volunteered to review submissions before the conference and during the review of the conference papers. We sincerely appreciate all of their hard work.

We wish to acknowledge the conference program committee for their substantial contributions to RUME and our institutions. Without their support, the conference would not exist.

Last but not least, we would like to thank Tim Fukawa-Connelly, the previous conference organizer, for his work setting up the 2017 conference and providing guidance throughout the process. His efforts over the past four years have significantly contributed to the growth of the conference and the strengthening of our community.

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Leveraging Real Analysis to Foster Pedagogical Practices

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Real analysis is frequently a required course for prospective secondary mathematics teachers. However, most teachers view real analysis as unnecessary and unrelated to the work of teaching secondary mathematics. The purposes of this paper are to (i) explore why real analysis, as it is conventionally taught, is not helpful to many teachers, (ii) present a new instructional model for how the course can be taught to increase its relevance, and (iii) present a case study in which our instructional model was implemented in a real analysis course and led to productive changes in teachers' actual pedagogical practice.

Key words: Real Analysis, Secondary Teacher Preparation, Advanced Mathematics, Pedagogical Practice

1. Introduction

In the United States and elsewhere, prospective teachers of secondary mathematics are required to complete extensive coursework in undergraduate mathematics. Such coursework usually includes advanced upper-level courses for mathematics majors (e.g. CBMS, 2012), with many institutions now requiring future teachers to complete the equivalent of an undergraduate degree in mathematics (Ferrini-Mundy & Findell, 2010). Consequently, prospective secondary mathematics teachers often complete more courses in advanced mathematics from a mathematics department than mathematics education courses that focus on teaching methods and secondary content. The key point is that prospective mathematics teachers' experiences in their advanced mathematics courses are a significant part of their preparation for teaching mathematics.

The requirement that prospective mathematics teachers complete advanced mathematics courses raises three important questions. First, if prospective teachers complete a course in advanced mathematics such as real analysis, are they better able to teach secondary mathematics? Second, given that experiences in advanced mathematics courses do not appear to be productive for secondary teacher preparation, is there a way to restructure advanced mathematics courses such as real analysis so that they better meet the needs of prospective teachers? Third, what evidence is there that an innovative instructional approach in an advanced mathematics course can influence teachers' actual pedagogical practice in secondary classrooms?

Our paper is centered around investigating these three questions. We answer the first research question in sections 2 and 3. In section 2, we synthesize the extant literature, which demonstrates that prospective teachers do not appear to benefit from taking advanced mathematics courses. We use existing theoretical frameworks regarding transfer (Lobato, 2012) to explain why this is the case in section 3. We address the second research question in section 4, where we present a transformative instructional model for teaching advanced mathematics to prospective mathematics teachers and describe theoretical reasons for why instruction based on this model can be beneficial to prospective teachers. We illustrate this instructional model with one example module in the beginning of Section 5. We address the third question in Section 5.3, showing that teachers who participated in this module in a real analysis course changed their pedagogical practice. At a minimum, this provides a theoretically-motivated existence proof that advanced mathematics courses can potentially benefit teachers and influence their classroom practice if taught in a productive manner.

2. Literature Review

2.1. The Influence of Advanced Mathematics Courses on Subsequent Teaching

Prospective secondary mathematics teachers are typically required to complete many courses in advanced mathematics. However, as several scholars have noted, there is little research on whether or how these courses influence prospective teachers' future pedagogical practice (e.g., Deng, 2008; Moriera & David, 2007; Ticknor, 2012), which is fundamentally important for practice-based approaches to teacher knowledge and teacher education (e.g., Ball, Thames, & Phelps, 2008). Here, we discuss two findings that suggest that completing such courses have only a modest effect on prospective teachers' pedagogical behavior. First, large-scale studies have found a weak relationship between the number of advanced mathematics courses that a teacher has completed and the achievement of that teacher's students (Darling-Hammond, 2000; Monk, 1994). For instance, Monk (1994) wrote:

> [T]he use of alternative 'cut-points' revealed that the model performed best when the distinction was drawn between having five or fewer versus more than five undergraduate mathematics courses. The addition of courses beyond the fifth course has a smaller effect. In contrast to 1.2% increase in pupil performance reported earlier, the addition of a mathematics course beginning with the 6th course is associated with a 0.2% increase (p. 130).

Because most mathematics majors do not take advanced mathematics courses until after they have completed five courses, including a four-semester calculus sequence and a course in linear algebra, Monk's analysis suggests that prospective teachers will reap only a small benefit from completing a subsequent advanced mathematics course.

Second, when practicing secondary mathematics teachers have been asked how their experiences in advanced mathematics courses have influenced their teaching, many teachers claimed that their advanced coursework did not contribute to their development as teachers (e.g. Goulding, Hatch, & Rodd, 2000; Ticknor, 2012; Zazkis & Leikin, 2010). Few could cite specific instances of how their knowledge of advanced mathematics influenced their secondary teaching (Rhoads, 2014; Wasserman & Ham, 2013; Zazkis & Leikin, 2010). Wasserman et al. (2015) found that this occurred even when the teachers demonstrated an understanding of the advanced mathematics that they were taught.

2.2. Postulated Reasons for why Advanced Mathematics does not benefit Prospective Teachers

Researchers have proposed two reasons for why advanced mathematics courses might not benefit prospective mathematics teachers, even if the prospective teachers understood the content that they were studying. The first reason is that the representation systems used in advanced mathematics differ substantially from those used in secondary mathematics (Deng, 2008; Moreira & David, 2007). For instance, Moreira and David (2007) presented a theoretical analysis of how advanced mathematics courses framed concepts from the secondary curriculum. They noted that in advanced mathematics courses, concepts usually were introduced using a single canonical formal representation. For example, the familiar concept of fractions was defined as an equivalence class of ordered pairs in $\boldsymbol{Z} \times \boldsymbol{Z}\backslash\{0\}$ where (a, b) and (c, d) were equivalent if ad =

bc. However, Moreira and David argued that effective teaching of secondary mathematics often required the use of multiple representations, many of which were visual but not necessarily formal. For example, fractions might be represented both numerically and pictorially as pie charts.

A second reason is that the goals of teachers of advanced mathematics and mathematics educators may not align with those of prospective teachers. Ticknor (2012) argued that many prospective teachers' primary goal in their advanced mathematics courses is merely to pass the course, which is in part due to students being intimidated by the material and having a fear that they may fail (e.g., Ticknor, 2012; Pinto & Tall, 1999; Weber, 2008). Ticknor used this theoretical frame to account for her case studies of prospective teachers in an abstract algebra course learning the skills to earn passing grades on examinations but not reflecting on how the material might relate to solving equations in secondary algebra. The coping strategies that prospective teachers adapt to survive their advanced mathematics courses may bear little relationship to what they do in their classrooms.

3. Theoretical Perspective

3.1. A Trickle-Down Model

From our point of view, the anticipated benefits of having prospective teachers complete a course in advanced mathematics can be exhibited by the "trickle-down" model presented in Figure 1 (Wasserman, et al., in press), which considers the relationships between i) advanced mathematics, ii) secondary mathematics, and iii) *teaching* secondary mathematics. This model highlights that most of the material covered in an advanced mathematics course consists of advanced mathematics, where little or no attention is paid to secondary mathematics or issues of teaching. However, the hope is that the advanced mathematics provides an opportunity for the prospective teacher to better understand certain aspects of the *content* of secondary mathematics. For instance, by learning the zero divisor property about rings in abstract algebra, the prospective teacher may develop a deeper understanding for why one can solve polynomial equations by factoring polynomials. Some instructors of advanced mathematics may be explicit about such connections between advanced mathematics and the content of secondary mathematics, but in many other cases, prospective teachers are asked to make these connections themselves. Next, the expectation is that prospective teacher's better understanding of the secondary mathematics content will inform their future *teaching* of mathematics. In our experience, exactly how prospective teachers should teach differently is rarely discussed in advanced mathematics courses. Prospective teachers are expected to use their understanding of advanced and secondary mathematics to improve their teaching more or less on their own.

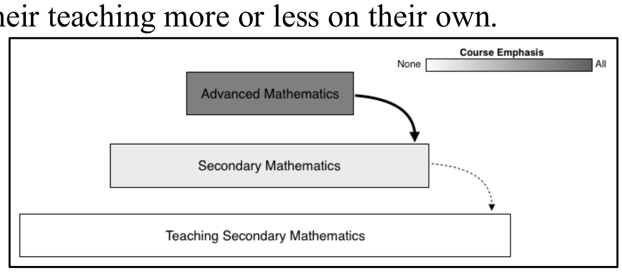

Figure 1. Trickle-down model for teaching advanced mathematics to secondary teachers.

3.2. An Issue of Transfer

As Figure 1 illustrates, the justification for having prospective teachers complete advanced mathematics courses is based on the belief that a transfer of knowledge will occur. The expectation is that prospective teachers' experience in a source domain (as students in an advanced mathematics course) will lead to improved performance in a target domain (as teachers in secondary mathematics), even though the exact nature and mechanisms of this transfer are often unspecified (Wasserman et al., in press).

Lobato (2012) has proposed two distinct lenses by which the issue of transfer can be studied: the mainstream cognitive perspective (MCP) on transfer and actor-oriented transfer (AOT). We believe both can offer insight into why the model in Figure 1 may not help prospective teachers with regard to their teaching.

According to Lobato (2012), using the MCP, transfer occurs when an individual generalizes a desired abstract piece of knowledge from a source domain, recognizes this piece of knowledge as relevant in a future target domain, and then successfully adapts this piece of knowledge to respond to a situation in the target domain. The extent to which this will occur depends on (i) the type of behavioral change that is desired and (ii) similarities between the source domain and the target domain. Wagner (2010) noted that the disposition of many researchers in MCP is to treat judgments about (i) and (ii) as independent of the participant who is doing the transferring, "as if situational structure can be directly perceived in the world" (p. 364).

For (i), transfer is more likely to occur between situations if the change in behavioral performance involves executing a previously learned procedure more quickly. Transfer is less likely to occur if the behavioral change relies on an abstract principle to change how one would approach the problem and there is no prompting that the abstract principle is relevant (Barnett & Ceci, 2002). The latter behavioral change is an instance of *far transfer*, which is widely accepted as difficult to achieve. Of course, it is exactly the type of change we are hoping to see in prospective teachers when they take an advanced mathematics course.

For (ii), Barnett and Ceci (2002) described six ways in which the target situation may differ from the source situation: (a) knowledge domain, (b) physical context, (c) temporal context, (d) functional context (academic vs. play), (e) social context (individual vs. large group), and (f) modality (lecture vs. writing). For each aspect that Barnett and Ceci highlighted, there are substantial differences between the context in which prospective teachers learn advanced mathematics and the contexts in which they will be teaching. We elaborate each point: (a) prospective teachers learn about advanced mathematics as it is expressed with formal symbolic notation but when they teach secondary mathematics, they often use multiple informal representations (Moreira & David, 2007); (b) prospective teachers learn in university classrooms but teach in high schools; (c) the time gap between completing a real analysis course and teaching can be several semesters; (d) the goal of advanced mathematics involves establishing claims via proof while the goal of a secondary mathematics teacher involves enhancing student understanding; (e) the prospective teacher goes from being a student usually working individually (directed by the course instructor) to being a teacher (and leader) of many students; and (f) the teacher switches from writing homework solutions to lecturing, preparing lessons, and leading other instructional activities. Again, these descriptions indicate the goal of improving prospective teachers' pedagogy by teaching them advanced mathematics is an instance of far transfer. Given the distance between the source and target domains, we would expect the influence of advanced mathematics on the teaching of secondary mathematics to be minimal.

Lobato (2012) offered an alternative approach to the MCP that she labeled Actor Oriented Transfer (AOT). In AOT, the researcher seeks to account not by the similarities and differences that the researcher observes between source and target domains, but rather the similarities and differences that the individual doing the transferring finds relevant and salient. In this perspective, the researcher is not so much interested in predicting or measuring whether normative transfer occurred, but rather aims to document any transfer that occurred (i.e., specify all the ways that an individual's experience in a target domain affected subsequent performance in a source domain, including ways that the researcher might find unproductive) and account for this by the individual's perceptions of the source and target domains. AOT offers an additional insight into why prospective teachers might not benefit from completing a course in advanced mathematics. Wasserman et al. (2015) interviewed 14 prospective and in-service teachers about their experiences in real analysis. Many participants found real analysis irrelevant for their teaching because the participants expected to use the proofs that were presented as explanations that they could provide to students in their secondary mathematics courses. Because the real analysis proofs were lengthy and used technical terminology (the concept of limit, epsilons and deltas), the participants found them useless for their teaching. Hence, when studying real analysis, these participants were not focusing on the relationships between concepts (as researchers often do) but what was salient to them was the literal proofs and explanations that were presented and whether or not they could be transported "as is" to their own classrooms.

4. An Alternative Model for Teaching Advanced Mathematics to Secondary Teachers

We present an alternative instructional model for how advanced mathematics can be taught to prospective teachers in Figure 2. In essence, we are "book-ending" the study of advanced mathematics by beginning and ending our lessons with a discussion of teaching secondary mathematics.

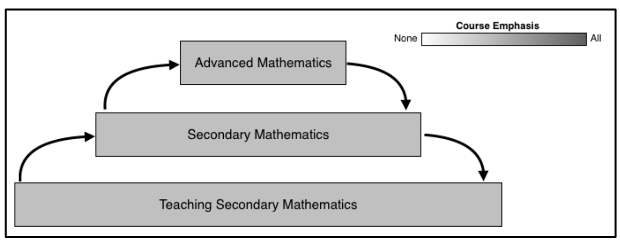

Figure 2. Alternative model for teaching advanced mathematics to secondary teachers.

The left side of the figure consists of *building up from (teaching) practice.* In this phase, we begin an instructional unit with a secondary teaching situation – oftentimes in the form of a comic strip using LessonSketch Software[1] (Herbst et al., 2011). Teachers are asked to evaluate a teachers' pedagogical action or to provide a pedagogical response to that situation (or both). These situations were chosen so that prospective teachers had to engage in the high-leverage teaching practices (HLPs) that Ball and her colleagues documented as central to the work of teaching (TeachingWorks, 2013). Such HLPs include explaining and modeling content, establishing productive classroom norms, eliciting and interpreting student thinking, and providing feedback to students. The right side of the figure is *stepping down to (teaching) practice*. After engaging with the advanced mathematics, such as studying definitions, theorems, and proofs, prospective teachers are asked to reconsider the secondary mathematics and pedagogical situations that they had previously discussed in light of the advanced content. Their homework consists of them doing traditional advanced mathematics exercises (i.e., writing proofs), answering questions about secondary mathematics, and providing responses to other pedagogical situations.

[1] This software is available at Patricio Herbt's Lesson Sketch webpage: https://www.lessonsketch.org/

Our model (Wasserman, et al., in press) in Figure 2 has several theoretical benefits over the "trickle-down" model presented in Figure 1 that undergirds the way advanced mathematics is typically taught to prospective teachers. First, our learning goals include specific pedagogical behaviors that we would like prospective teachers to exhibit as a result of completing our module, goals that we are able to convey to our students and that we assess. As Barnett and Ceci (2002) argued, it is easier for individuals to transfer specific behavioral actions to a new domain than it is to transfer abstract principles. Second, in accord with our practice-based view of secondary mathematics teacher education, our lessons are situated within the context of teaching secondary mathematics. Hence, our modules are closer to the context of secondary teaching than is typical of a real analysis course. Prospective teachers are asked to act as the teacher when responding to pedagogical situations, including providing oral explanations to a hypothetical class of students, interpreting hypothetical students' mathematical justifications, and providing oral and written feedback to student work, all of which should be accessible to a secondary student. Of course, there are differences between responding to our pedagogical situations and the actual craft of teaching (e.g., in our situations, teachers do not need to worry about time management or classroom management) and there are contextual differences between the two that cannot be addressed (e.g., there still is a time lapse and a change in physical location). Nonetheless, compared to the traditional model, the gap between learning real analysis and teaching secondary mathematics has been closed considerably, which Barnett and Ceci (2002) contended will increase the likelihood of successful transfer.

Perhaps most importantly, our model will help prospective teachers notice the aspects of real analysis that we think are important for teaching. Previous research has demonstrated that teachers do not find advanced mathematics helpful for their teaching because they used unproductive coping strategies (e.g., rote memorization) to survive their courses with a passing grade (Ticknor, 2012) or that they take advanced mathematics courses with an eye toward learning proofs that they could share with their future students (Wasserman et al., 2015). Under our alternate instructional model, prospective teachers' grades depend in part on their responses to pedagogical situations, which means they cannot simply ignore the pedagogical implications of the real analysis course. As successfully responding to pedagogical situations goes beyond reciting a proof that was learned in real analysis, prospective teachers are asked to broaden their views on how real analysis can be beneficial to them in their future professional work.

5. An Illustration of our Model

The goal of this section is threefold. First, we illustrate how we used our instructional model to design real analysis modules for prospective teachers. Second, we portray how this was implemented in a real analysis classroom and how prospective teachers interacted with one module. Third, we describe the results of a study in which we followed six teachers into their secondary classrooms and illustrate how their pedagogy changed as a result of completing our modules. This provides an existence proof that real analysis can be relevant for the teaching of secondary mathematics as well as an illustration that instruction based on our module has the potential to foster these changes.

5.1. Overall Module Design

To generate our modules, we engaged in the following process. Our research team examined the first six chapters of Fitzpatrick's (2006) real analysis textbook, identifying any overlap between the concepts covered in Fitzpatrick and the Common Core State Standards in

Mathematics (CCSS-M, 2010). Once these topics were identified, we tried to generate pedagogical situations with the following attributes: (i) the pedagogical situations were authentic, (ii) the prospective teachers were asked to engage in a High Leverage Practice (TeachingWorks, 2013) that is central for teaching, and (iii) successfully engaging in these High Leverage Practices required mathematical knowledge that could be informed by real analysis. We developed 12 modules based on this approach.

5.2. One Module: Considering Derivative Proofs as "Attending to Scope"

5.2.1. Research context and data collection. The data presented in this section is part of a larger corpus of data. We are currently in the second year of a three-year project using design research. We are engaged an iterative process in which we teach a real analysis course using the 12 modules that we generated. The data presented here are from the first iteration of the course that was taught by the first author. The course had 31 prospective and in-service teachers who agreed to participate. In this paper we focus on the "Attending to Scope" module, which took place across two 100-minute sessions. All classes were videotaped and all teacher conversations were audio-taped, and all homework assignments and reflective journal entries were collected.

5.2.2. Building up from (teaching) practice. In their professional work, teachers must explain content, practices, and strategies (e.g., TeachingWorks, 2013); for this particular module, we focus on a mathematical practice that mathematics teachers should engage in – attending to the scope to which an explanation, idea, or justification applies. For example, in trying to help elementary students understand subtraction, some teachers might state: "you cannot subtract a larger number from a smaller one." This explanation has a limited scope – it is only accurate when one is considering positive numbers. Acknowledging the limitation of scope in explanations is an important component of teaching. As Leinhardt, Zaslavky, and Stein (1990) noted, "a primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases" (p. 6). Real analysis, with its careful attention to stating explicitly the conditions for when a statement applies, is a domain that is well-suited for helping prospective teachers recognize the importance of being careful with their language, particularly as it relates to considering the mathematical scope to which statements or theorems apply.

We began the module by presenting teachers with the cartoon in Figure 3, asking the teachers to evaluate the pedagogical quality of two of Mr. Ryan's explanations – the *exponent statement* and the *power rule statement*. (Mr. Ryan was a fictitious comic teacher who appeared in many of our modules.) In addition to a variety of other pedagogical issues that might be of concern, both explanations that Mr. Ryan gave were limited in scope. During the module, all groups of teachers highlighted limitations of the explanations, but only two groups highlighted the limitation in scope of both explanations (e.g., the power rule statement doesn't when you need the "chain rule"). Rather than the "*mathematics* of the explanation," most teachers' comments instead dealt with the "*explanation* of the mathematics" – for example, that students might misinterpret what Mr. Ryan had said (e.g., 'repeated multiplication' might mean "2x3x5x7") or that Mr. Ryan's explanations were too procedural. The next part of the module focused on the secondary mathematics, asking teachers to state the number sets (or objects) for which Mr. Ryan's explanations would be valid. Therefore, this task prompted the groups that did not attend to scope previously to consider this facet of his explanations; all groups identified the limited scope to which these statements would apply during this portion of the module.

Classroom Scenario: Mr. Ryan teaches everything from Pre-Algebra to Calculus. The following scenes are snapshots from his classes at different times during the year.

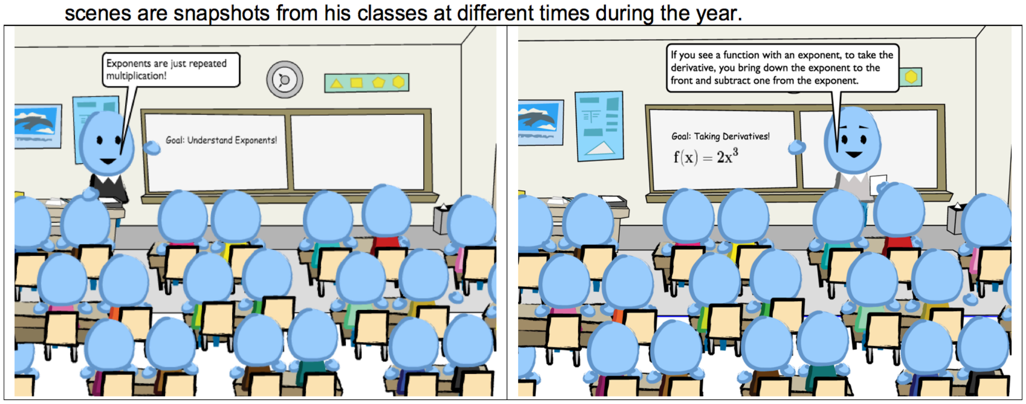

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Figure 3. A classroom scenario.

5.2.3. Real analysis. During the real analysis portion, we first presented a standard proof of the power rule for derivatives using the binomial theorem that appears in many secondary calculus textbooks. Teachers were asked to which number set each step in the proof applies (***N***, ***Z***, ***Q***, or ***R***). All groups of teachers eventually recognized that the proof using the binomial theorem was only correct for the natural numbers, with many expressing surprise because they had always assumed that the proof that they read applied to all real numbers (as they knew that the power rule is valid for all real numbers). Afterwards, the instructor presented a sequence of proofs, each of which expanded the scope (***Z***, ***Q***, ***R***) to which the power rule for derivatives applied, which, along the way, involved proving other derivative rules like the product, quotient, chain, and inverse function rules. These derivative proofs are nearly ubiquitous in real analysis courses. What is critical is that the product, quotient, chain and inverse function rules were presented as lemmas for the goal of proving the general power rule.

5.2.4. Stepping down to (teaching) practice. After the proofs were presented, we stepped back down to practice by asking the teachers to discuss, "When, if ever, would the statements made by the teacher be appropriate? Describe the specific context." For their consideration, they were also given specific classroom contexts for each statement (graphing exponential functions for the *exponent statement*, and an end-of-year review for the *power rule statement*). During this time, all groups of teachers further examined the mathematical limitations of each statement, identifying more exact constraints around which each was true. In addition, the groups discussed various contextual factors. For example, one group mentioned "this only makes sense for little kids" (with regard to the *exponent statement*), and another group noted "they're reviewing" as a reason that someone might make somewhat informal statements (with regard to the *power rule statement*). Some groups also discussed that the *power rule statement* would not ever be pedagogically appropriate unless the teacher was only intending to teach a procedure; other groups suggested "fixes" to the statements to make them more mathematically precise. What is important is that the teachers had meaningful discussions around the use of statements with limited scope in mathematics education; indeed, one student claimed, "I feel like I just discovered gold, math gold," which we took to indicate that she found the discussion of the limitations of statements to be very valuable.

5.2.5. Teacher performance on homework tasks. In addition to asking students to prove several real analysis theorems, teachers were presented with two statements: "The perimeter is just the sum of all side lengths" (*perimeter statement*) and "Remember, to multiply a number by ten, just add a 0 to the end" (*add zero statement*). The teachers were asked to, "Determine for what set(s) of objects the statement is true. If there are any, provide an example of a set(s) of objects for which the statement is not true. Discuss in what mathematical contexts might the statement by the teacher be appropriate, if ever? When might it be in appropriate, if ever?" For both tasks, all 31 teachers identified at least one valid counterexample that revealed the limitations in scope. For instance, 29 of the 31 teachers noted that the statement about the perimeter would not apply to a curved figure and 29 of the 31 participants cited a rational number as an instance in which you cannot multiply by 10 by adding a zero at the end.

The teachers were also asked to submit a reflective journal entry in which they described what, if anything, they learned from this week's class. Of the 27 teachers who completed this assignment, 26 mentioned the importance of language and 17 cited the importance of attending to the scope of an explanation and letting students know if there were sets of objects to which the statement would not apply. For instance, the following response was representative:

> *We tend to make general statements for the sake of time without even realizing we are doing it. This past week in class I learned that as teachers, we must think about the statements we make and the limitations we place on students learning moving forward… This statement made by the teacher does not hold for all cases, therefore this is not a mathematically precise statement.*

In summary, the teachers' homework assignment provided some suggestive evidence that our module went well. The prospective teachers' journal entries indicated that the majority of the prospective teachers indicated an increased appreciation for attending to the scope of the explanations that they provide, and, the prospective teachers could recognize the limitations of scope in mathematical explanations if they were prompted to do so. However, our primary concern was not teachers' performance on these written assignments but whether completing this module would influence their actual pedagogical practice.

5.3. Influence on Teachers' Practice

5.3.1. Data collection. The first iteration of our experimental real analysis course was in Spring 2016. For the 2016-2017 school year, six teachers volunteered to participate in a follow-up study. Of the volunteers, five were in-service teachers with under seven years experience, and one was a pre-service teacher who obtained her first job (Ms. J, discussed below). Five participants taught in public schools and one taught in a private school; all schools were around a large urban metroplex. For the courses we observed of these teachers, two teachers taught geometry, two taught calculus, one taught Algebra II, and one taught pre-calculus.

For each teacher, the first and third authors of the paper visited their classroom up to six days during the academic year. For each visit, we observed a one-period lesson that was 45 minutes or 90 minutes in length. Lessons were audiotaped (from a microphone on the teacher) and the researcher kept field notes for each lesson, including transcribing all inscriptions on the whiteboard or projector. We videotaped the lessons as well when it was possible to do so (but school district regulations sometimes did not allow this). After each classroom visit, the teachers participated in a post-interview in which the interviewer recalled common occurrences or key events in their class and asked them to describe why they engaged in those specific behaviors during the lesson. We then asked them whether any of their classroom actions were influenced by their experience in our experimental real analysis course.

We say a teaching unit displayed "attention to scope" when the following three conditions occurred: (i) during the lesson, the teacher paid explicit attention to the scope of a teacher or student-generated claim in their lesson; (ii) in the post-interview, they highlighted this excerpt as an instance where their experience in real analysis influenced their teaching; and (iii) in the post-interview, they explicitly mentioned how the real analysis course led them to be more mindful of the language that they used and/or the limitations in the claims that they made.

5.3.2. Results. Across the six teachers, we made 31 observations. (Due to scheduling constraints by the school districts, we still have to make 4 observations for 3 of the teachers). In these 31 observations, the observed teaching unit displayed “attention to scope” on 17 occasions. That is, in the majority of our observations, we observed the teachers attending to scope in their explanations *and* citing their experiences in real analysis as a reason that they did so. We use two lessons to illustrate.

In our first example, Ms. J, a first-year teacher, was teaching a pre-calculus course about polynomials. During her lesson, she covered the Remainder Theorem, the Factor Theorem (i.e., for a polynomial *f(x)*, *f(c)* = 0 if and only if the factor *(x-c)* divides *f(x)*), synthetic division, and the Rational Zero Theorem. Throughout the lesson, Ms. J emphasized the limitations to the scope of statements that she gave. We illustrate two excerpts from the lesson in Figures 4 and 5.

Mathematics	Explanations and class dialogue (emphasis added)
Remainder Theorem: If a polynomial $f(x)$ is divided by $x-k$, then the remainder is $r=f(k)$. (Example. Find the remainder of $f(x)=2x^2-x+12$ when divided by $x+3$.) Proof. What’s the “r” of $f(x)$ when $f(x)$ is divided by $(x-k)$? $f(x)=(x-k)q(x)+r$ $f(k)=(k-k)q(k)+r$ $f(k)=0\cdot q(k)+r$ $f(k)=r$	Ms. J: So let’s think about that. I have some polynomial and I’m dividing by $x-k$. Instead, I can take this k and evaluate my function at that value and that will be my reminder when I divide by it .. Do you want to see why this works? It’s pretty quick. And I think it's worth seeing so we have some understanding of how this ends up working…It cancels out because I have $(k-k)$, so that’s zero times some polynomial. So then I get $f(k)=r$. Kind of cool to actually see why that works. And then it feels a little less arbitrary. It’s just I plug it in and I know why because Ms. J told me to. (pause) **Again, this only works when it’s $(\mathbf{x}-\mathbf{k})$ of degree one divisor, so keep that in mind. If it’s $(\mathbf{x}^2-\mathbf{5})$, we can’t do this.** Kind of like synthetic division, remember synthetic division only worked when we had terms, divisors of “*x* plus or minus a number.”

Figure 4. Ms. J’s classroom instruction around the Remainder Theorem.

Mathematics	Explanations and class dialogue (emphasis added)
Example. $f(x)=3x^3-2x^2+x-5$ divided by $x-1$. $\begin{array}{c\|cccc} 1 & 3 & -2 & 1 & -5 \\ & \downarrow & 3 & 1 & 2 \\ \hline & 3 & 1 & 2 & -3 \end{array}$ So, $3x^2+1x+2+\frac{-3}{x-1}$.	Ms. J: We’re saying $f(x)=3x^3-2x^2+x-5$ divided by $x-1$. **Synthetic division is a tool we can use to divide polynomials when our divisor is a degree one polynomial, right? “*x* plus or minus a number.” That’s the only time we can use synthetic division. I can’t say that enough.** … Student A: For both divisions, it’s only if it’s to the degree one? Ms. J: For both divisions, meaning what? Student A: (Long) Division and synthetic division Ms. J: Long division… Synthetic division is an algorithm that mathematicians figured out, when **if it’s a specific case, it saves you time from long division**. So that’s why it looks different because it’s an algorithm for a specific type of situation, type of division problem. **The long division works for everything.**

Figure 5. Ms. J's classroom instruction about synthetic division.

In Figures 4 and 5, Ms. J explicitly noted a limitation in the scope of both the Remainder Theorem and synthetic division – namely, that the divisor must be a linear factor of degree one, of the form $x - k$. At times, these were her explanations, and in other cases they were prompted by student questions – but in both situations, she was very explicit about the limitations in scope. Further, at a different point, one student asked her about other linear factors, such as $2x - 2$. The guiding notion in her response was that synthetic division works for factors of the form, $x - k$, and this explicit attention to scope helped her resolve how to respond to the student. In the post-interview, Ms. J discussed why she insisted on clarifying the scope of many of her statements.

Ms. J: I think that there are common mistakes made as a student if you hear a rule. Like great, a rule, I can do synthetic division, especially if it looks like *x* plus or minus something. But they might think x^2 plus or minus something, or just like it's a trick, I'll plug the last number in for the example of synthetic division. So *knowing the common mistakes and trying to prevent them is one reason I think it's important to say that* […] *I think that this reminds me of like the exponents lesson, where I make these vast like generalizations, like exponents, just like repeated multiplication. And it's like, well, that's true if, you know, it's a positive integer and things like that.* But the thing is if you don't say them, it could get confusing later, I think for them. So like that one would have caused them to make mistakes, but it's just explaining I'm simplifying this and here's the restriction that I need to say to make this so.

We wish to highlight two things in this excerpt. In the first italicized section, Ms. J shows how she was able to apply the theme of attending to scope flexibly in coordination with her other pedagogical knowledge – in this case, her knowledge of common student mistakes. She hoped that by highlighting limitations in scope, it would reduce the chance of student mistakes in the future. In the second italicized section, Ms. J references the exponents lesson (i.e., the module described previously) as part of the rationale for her justification of why teachers should avoid vast generalizations that could confuse students at a later time. Later in the post-interview, Ms. J again emphasized the importance of her experiences in the real analysis course.

Ms. J: *I attribute that probably one of the biggest things, like one of the biggest take-aways I had from the analysis course because the exponents one really threw me off.* Like I remember we had to do a problem—I think I even talked about this before, but I just really remember it. We had to do a problem for homework or something where it was, you know, what's two squared? What's two to the one-half? And then it was like, the last one was what's like pi to pi? And I was like I have no idea how to be thinking about this right now. *So it just made—it really stuck out to me how it's important to (pause) really limit your – it's trying to present a smaller view of the topic, but just set the situation so that when you're presenting a smaller view, it's an accurate view still. So very much so to analysis.*

The highlighted excerpts again illustrate how the exponents lesson (i.e., the module described previously) emphasized to Ms. J that she should be careful in her language and even when presenting a "smaller" view on a topic, she should still be mathematically accurate.

In our second illustration, we highlight Ms. T, a teacher with six years experience, teaching an AP Calculus BC course. In this lesson, she covered the Extreme Value Theorem, Fermat's Theorem, Rolle's Theorem, and the Mean Value Theorem. When presenting the theorems, Ms. T attended to the scope of the theorems in an interesting way. She first asked individual students to draw a graph of a function that satisfied the conditions of the theorem. When commenting on the graph, she explicitly checked and then emphasized that each condition was satisfied. We illustrate this in Figure 6.

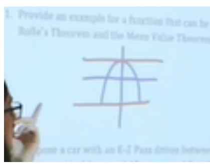
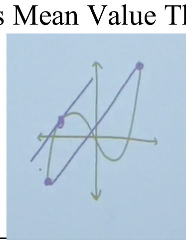
Mathematics	Explanations and class dialogue (emphasis added)
Rolle's Theorem: Let $f: [a, b] \to R$ be a function that is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and satisfies $f(a) = f(b)$. Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$. [Student-drawn example of graph that satisfies Rolle's Theorem] 	Ms. T: We want an example. So, we want a sketch of a function that can be used to help interpret Rolle's Theorem and the Mean Value Theorem. So, it might be easier to sketch something that you can easily see Rolle's Theorem with. So, can I have a volunteer? Your what goes up must come down. … Ms. T: So, let's think about it. *Is she continuous on a closed interval? Is she differentiable on the open? Does $f(a) = f(b)$ somewhere? Is the derivative zero somewhere? So, did she satisfy Rolle's? Okay.*
… Mean Value Theorem: Let $f: [a, b] \to R$ be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then, there exists at least one $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. [Student-drawn example of graph that satisfies Mean Value Theorem] 	… Ms. T: I need a volunteer. And the volunteer just has to sketch a function where the Mean Value Theorem will work. You're thinking of our two conditions. Continuous on the closed. Differentiable on the open. … So, the Mean Value Theorem has to work. You need continuity and differentiability. … Ms. T: *Good. Is that continuous? It's continuous. She gave us a closed interval. I'm just gonna highlight it. Is it differentiable? Yeah.*

Figure 6. Ms. T's classroom instruction about Rolle's Theorem and the Mean Value Theorem.

In each italicized excerpt, Ms. T explicitly highlighted how each condition of the relevant theorems was satisfied. In other instances, Ms. T would use student-generated examples not only to illustrate the necessity of the conditions of a theorem, but also to explore the ways in which examples that did not fulfill the conditions resulted in the implication being invalid. Ms. T was able to apply the themes of the real analysis class on attending to scope in a flexible manner, in this case one that involved making students active in the construction of examples in her course. Later in the post-interview, Ms. T affirmed that she attended to scope because of her experience in the real analysis course.

Interviewer: Every time you brought up a theorem, you like specifically referenced the conditions for the theorem.

Ms. T: Yeah.

Interviewer: So, what was your rationale for that?

Ms. T: Yeah. *So, that's still from the course, the Real Analysis course, because the specificity of language* … so it was really being clear like when these things actually will occur, and *that also inspired like, you know, draw a graph where it failed to meet a condition so you can see that it's actually not always going to work out*.

Interviewer: Okay. And so, would you say that—is that something that you normally do? Like would you normally mention it every single time you reference it on your own?

Ms. T: No. I think, now it's just – my own better understanding is why I keep repeating it so that it's kind of more ingrained to them under what conditions things can happen.

In this section, we have documented that in the majority of the instances that we observed, the teachers attended to the scope of mathematical statements and verified in their post-interview that they did so because of their experience in the real analysis course. The excerpts that we presented showed how teachers were able to adapt this theme to their own style of teaching.

6. Discussion and Conclusion

The aim of our paper was to address three research questions. We synthesize our answers to these questions. First, we argue that advanced mathematics courses usually do not benefit prospective mathematics teachers due to the difficulty of the far transfer that we expect to occur and also because what is salient to the students in these classes may differ from what is important to the course instructor or mathematics education researchers. Second, we describe an alternative model for how students can be taught real analysis – one in which the lessons are situated in classroom practice. We first build up from classroom practice in order to motivate studying the real analysis content. We then step back down to classroom practice so students can practice applying the mathematics they have learned to the pedagogical situations. Third, we document that teachers who completed a module on attending to the scope of explanations in real analysis attended to scope in their subsequent instruction and attributed this pedagogical practice to what they learned in real analysis.

To our knowledge, this paper is innovative in several respects. First, although several researchers have documented the limited impact that advanced mathematics courses have on teachers' pedagogical practice (e.g., Goulding, Hatch, & Rodd, 2000; Moreira & David, 2007; Ticknor, 2012; Wasserman et al., 2015; Zazkis & Leikin, 2010), there have been few research-based attempts to ameliorate the situation. In general, attempts to make the study of advanced mathematics courses relevant to secondary teachers have involved making connections to the *content* of school mathematics; in this study, we have attempted to be more explicit about connections to the *teaching* of school mathematics. Second, when attempts have been made to highlight the connections between advanced mathematical content and secondary teaching, they have occurred primarily in "capstone" or "connections" courses (e.g., Murray & Star, 2013) or in professional development programs. Our innovative instructional model, instead, alters the structure of an advanced mathematical course such as real analysis to specifically meet the needs of secondary teachers. Third, the standard by which we evaluated the efficacy of our course did not only rely on how prospective teachers did on our in-course assessments but also on the changes that were reported in their actual teaching practice. This is, of course, the ultimate measure of teacher development, but one that is not often used in evaluating the efficacy of advanced mathematics courses in secondary teacher education.

For the reasons above, we believe our model offers a potentially valuable alternative to the typical model in which advanced mathematics is taught with limited regard for changing prospective teachers' pedagogical practice. By design, the real analysis content in our modules was both tightly connected to and framed by this pedagogical practice. To substantiate the value of our model, however, merits further research. In particular, what we have offered here is a proof of concept. Our model *can* be used to develop real analysis instruction that can improve teachers' secondary teaching. However, we certainly do not have the data to generalize beyond that. For instance, we only explored one specific module in this paper. In the module discussed in this paper, for example, although one does not have to learn real analysis to be able to attend to the scope of secondary mathematics explanations, since a real analysis course already inherently models this idea in both the precision of statements and progression of proofs, it seems sensible to exploit this connection for teachers. We would want to know, at a minimum, the effectiveness of the other 11 modules – some of which paid more attention to specific theorems or proofs in real analysis, others of which had different kinds of connections to teaching. We are currently in the process of evaluating and refining these modules. Further work studying how best to mathematically prepare secondary teachers is needed, including the degree to which this particular model is productive and/or needs refinement, and could help guide improved design and implementation of advanced mathematics courses for secondary teachers.

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Correction, July 2019: Figure 3 has been corrected to indicate proper copyright attribution.

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Students' Epistemological Frames and Their Interpretation of Lectures in Advanced Mathematics

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In this paper, we present a comparative case study of two students with different epistemological frames watching the same real analysis lectures. We show that students with different epistemological frames can interpret the same lecture in different ways. These results illustrate how a student's interpretation of a lecture is not inherently tied to the lecture, but rather depend on the student and her perspective on mathematics. Thus, improving student learning may depend on more than improving the quality of the lectures, but also changing student's beliefs and orientations about mathematics and mathematics learning.

Key words: epistemological frames, real analysis, student understanding of lecture

1. Introduction

In recent years, several researchers have explored the relationship between students' epistemological beliefs and their learning of advanced mathematics. In particular, some scholars have claimed that some students struggle to learn mathematics because they lack the epistemological beliefs to support this learning (e.g., Alcock & Simpson, 2004; Bressoud, 2016; Dawkins & Weber, in press; Lew et al., 2016; Solomon, 2006). The primary aim of this paper is to extend this research. In particular, we adapt the notion of epistemological frames (e-frames), a construct from physics education (e.g., Redish, 2004), and illustrate how students who hold different e-frames can interpret the same advanced mathematical lecture in different ways. In particular, we first give an account of two students' e-frames in an advanced mathematical setting; we then use these e-frames to give a fine-grained account for these students' different interpretations of the same utterances by a lecturer.

2. Theoretical perspective and related literature

2. 1. Epistemic frames

Goffman (1997) introduced the notion of *frame* to describe how individuals develop expectations to help them make sense of the complex social spaces that they inhabit. For instance, most adults in the Western world have a "restaurant frame" consisting of expectations that are activated when they enter a restaurant. When frequenting a restaurant, an individual likely would expect that the restaurant employees will prepare food for the individual, the individual will be obligated to pay for this food, and so on (Schank, 1990). Such restaurant frames are usually helpful; these frames allow individuals to act sensibly in restaurants that they have never visited before. However, frames can occasionally be counterproductive if two individuals frame the same situation in different ways. For instance, a European diner may offend a waiter in the United States if she was not aware of the United States custom to leave at least a 15% gratuity.

Physics educators have introduced the notion of an individual's *epistemic frame*, or e-frame, as consisting of their epistemological expectations about a pedagogical situation. These consist of an individual's responses to questions such as "what do I expect to learn?" and "what counts as knowledge or an intellectual contribution in this environment?" (Redish, 2004). If a teacher and her students approach the same pedagogical activity with different e-frames, the students likely will not learn what their teacher intends. For instance, Redish (2004) described a physics tutorial in which students were asked to form a hypothesis. The teacher's aim of this activity was for students make qualitative predictions using their conceptual understanding of physics principles. Redish found that a student who viewed intellectual contributions in physics as consisting of numeric answers derived from textbook formulas responded to such tasks by engaging in computations, thereby avoiding the conceptual considerations the activity was designed to elicit.

We are not aware of any mathematics education research that has specifically used the notions of e-frames to account for students' behaviors. However scholars have explored the relationship between students' epistemological beliefs and their concomitant mathematical cognition. In many of these cases, the claims of these scholars can be expressed using the construct of e-frames. For instance, Thompson (2013) presented a situation in which a teacher provided many conceptual explanations to her high school algebra class but these explanations were ignored by some students in the class who had a procedural orientation. We might interpret Thompson's claim with e-frames as follows. In the students' e-frames, an intellectual contribution in an algebra class consisted of using a procedure to solve a problem symbolically. The teacher viewed part of the intellectual contribution of her presentation as explaining the meaning of the procedure that she was implemented. Since students did not recognize this as a legitimate intellectual contribution in a mathematics, they simply ignored the conceptual explanations.

We can use similar reasoning to characterize other mathematical constructs. For instance, the didactical contract (Brousseau et al., 2014) includes expectations about what mathematical contributions the teacher is required to make, establishing sociomathematical norms (Yackel & Cobb, 1996) involves the negotiation of what an acceptable mathematical contribution is, and institutional meanings of proof (Recio & Godino, 2001) are expectations about what constraints a justification must satisfy in different contexts. In summary, while we are introducing the notion of e-frames to mathematics community, this work builds upon a rich tradition of scholarship examining the links between students' epistemology and cognition. Our contribution is offering a more fine-grained account of how specific e-frames influence students' interpretations of specific mathematical utterances in advanced mathematics.

2. 2. Logical versus psychological understandings in advanced mathematics

In this paper, we distinguish between two ways of knowing a mathematical concept. An individual knows a concept *psychologically* if she believes the statement is true and feels that they understand why the statement is true. An individual knows a concept *logically* if she can provide a deductive justification demonstrating the statement is true from previous statements (usually definitions and axioms) that are assumed to be true.

We make three observations about this distinction. First, in many mathematical settings, psychological and logical knowing are inextricably intertwined. Mathematicians often believe a statement is true exactly when they see how it can be logically deduced from other things that are known or assumed to be true (e.g., Harel & Sowder, 2007). Second, psychological knowing and logical knowing are nonetheless distinct. Some mathematicians hold rational certainty in the veracity of unproven conjectures (e.g., Goldbach's conjecture) and others retain some doubt in claims that have been proven (on the grounds that they cannot be certain that their proofs are correct) (c.f., Weber, Inglis, & Mejia-Ramos, 2014). This reflects the view that the acceptability of a proof is dependent upon a reference theory specifying what facts are allowed to be assumed (Mariotti, 2006). Third, in some cases, the purpose of proof is not to enhance one's psychological knowing, but to provide logical justification (c.f., Dawkins & Weber, in press). Mamona-Downs (2010) expressed this clearly when she wrote that "the point [of proof] is not so much about conviction, but how we can clarify the bases of the reasoning employed" (p. 2338). Arzarello (2007) expressed a similar sentiment, arguing that the purpose of proof is to give meaning to a statement by placing the statement into a network of mathematical knowledge in the form of logical consequence.

2. 3. Systematization

In this paper, we focus on a particular type of activity that deVilliers (1990) has coined *systematization*. In this activity, mathematicians transform an existing theory—i.e., a constellation of concepts and related statements that are accepted as true—into a unified whole. Mathematicians do so by creating a system of axioms and definitions and then demonstrating that commonly accepted statements within the existing theory are deductive consequences from this system of axioms and definitions. As deVilliers (1990) noted, with systematization, "the main objective is clearly not *to check whether statements are really true*" (p. 21, emphasis was the author's). In our interpretation, the purpose of systematization is not to enhance one's *psychological knowledge*; the statements being justified are already accepted as true. Rather, the purpose is to create a system of axioms and definitions that lets us provide logical justifications for things that are accepted as true.

In this paper, we explore two students interpretation of the same real analysis lectures. In these lectures, the instructor is using the integers to define the systems of rational numbers and real numbers. He then derives some well-known consequences of these number systems. For instance, the professor defined the rational numbers as equivalence classes of pairs of integers, and addition on rational numbers as an operation on these equivalence class of ordered pairs. The professor then sketched a proof that addition on the rational numbers is a well-defined operation. We can thus say that the rational numbers are being *re-presented* to the students as we can assume that mathematics majors in a real analysis course have had extensive prior experience with rational numbers. In our view, the mathematical contribution being made is many facts that students already knew *psychologically* about rational numbers can now be seen as *logical* consequences of how the rational numbers were defined. In this paper, we analyze two different student views on what the purpose of this re-presentation is.

2. 4. Students' understanding of lectures in proof-oriented math courses

Most proof-oriented courses in advanced mathematics are taught via lectures (e.g., Fukawa-Connelly, Johnson, & Keller, 2016). Although there have been several studies on how these lectures are given (e.g., Gabel & Dreyfus, in press; Lew et al., 2016), there have been few studies on how students perceive these lectures. We describe one such study, and how e-frames can account for the results of this study below.

Lew et al. (2016) described a real analysis professor, Dr. A, who presented a proof to illustrate a heuristic that would be useful in proving other theorems. However, when Lew et al showed a videotape of this lecture to six students in the class, none of them recognized this heuristic as Dr. A's reason for presenting this proof. In his commentary on Lew et al., Bressoud (2016) conjectured that students "understood the instructor's intention as one of communicating that this is a valid result worthy of being noted and remembered" (paragraph 6) whereas the professor was trying to showcase a new approach that could be used to prove a new class of propositions. We can interpret Bressoud's commentary in terms of e-frames as follows: The professor in Lew et al's study believed the intellectual contribution of his proof presentation was describing an important proving heuristic. In the students' e-frame, the intellectual contribution of a lecture proof was communicating a valid result that was worthy of being remembered. Hence, the main points that the professor was trying to convey were thought of as tangential by the students.

3. Methods

3. 1. Rationale

The goal of this study was to understand how a student's e-frames can shape how that student interprets a real analysis lecture. To accomplish this, we interviewed two students as they watched a publicly available video-recording of a pair of real analysis lectures. This methodology had the advantage that students could act as if they were attending an actual lecture, yet the interviewer or student could pause the video to discuss their in-the-moment impressions of what was being discussed. In this study, we viewed a student's answer to one of the following two questions as one of her e-frames:

- What counts as a legitimate mathematical contribution?
- By what standards should the mathematical contribution be evaluated?

3. 2. Participants

Two participants, Alice and Brittany (pseudonyms), agreed to participate in this study. Both participants were mathematics majors at a large state university in the northeast United States. At the time of the study, both students had completed a transition-to-proof course in the previous semester, which the university requires as a prerequisite for a real analysis course. However, neither student was taking real analysis in the semester when this study occurred. Alice and Brittany were instead taking other proof-oriented courses and planned to take real analysis in a future semester. Thus, both students could potentially have enrolled in a real analysis course, but had not done so.

Alice and Brittany both claimed that they enjoyed their transition-to-proof course. Both students evinced a deductive proof scheme (Harel & Sowder, 2007) and exhibited competence at writing and understanding proofs. While watching the lectures, both students actively tried to make sense of the material.

Alice was an honors sophomore mathematics major who was intending to pursue certification to teach secondary mathematics, who earned an A in her transition-to-proof class. Brittany was a junior mathematics major who earned a B in her transition-to-proof class. To avoid misinterpretation, our aim is not to compare the productivity of Alice's and Brittany's frame or to evaluate the quality of Alice or Brittany's interpretation of the lectures that they observed. We do not make claims about who is the better mathematics student. Instead our aim in this paper is to show that Alice and Brittany's different interpretation of the same lecture can be attributed to the different e-frames that they hold.

3. 3. The lecture

The lecture studied in this paper consisted of the first two class videos from a real analysis course.[1] Each class video lasted between 60 and 70 minutes. The instructor of the course was Professor Francis Su, who won two major national teaching awards from the Mathematical Association of America. The lectures consisted of Professor Su beginning the real analysis course by constructing the rational numbers and then the real numbers from the integers.

3. 4. Procedure

Prior to conducting the study, the research team studied the lecture and parsed the lecture into five to ten minute segments in which coherent mathematical content was being presented. In

[1] The lectures are available at analysisyawp.blogspot.com

each clip we identified any important ideas that we felt were being conveyed. We used this parsing to create an interview protocol. After each segment of the lecture was played, participants were asked, "what, if anything, was valuable to you?" and "what did you take away from this?" as well as specific questions pertaining to the mathematical content that Su presented in that clip.

Each participant met individually with the first author of the paper once a week for four video-recorded clinical interviews (Hunting, 1997). Interview 1 was a one-hour interview in which the participant discussed their experience in their transition-to-proof course to provide the interviewer with a sense of the participants' understanding of the content of the course (particularly with number theory, functions, and proof) as well as their learning strategies and dispositions.

Interviews 2, 3, and 4 were two-hour interviews in which the research team attempted to explore the e-frames, ways of knowing, and any associated mental schemes that each participant used to interpret the mathematical lectures. During each interview, the participant watched Professor Su's lecture and was instructed to stop the video whenever they observed something that was important, interesting, confusing, or otherwise noteworthy. The interviewer would also stop the tape to probe the participant's thinking when the professor had stated something that the research team had identified as important or at the end of a segment, as we described above.

Between the interviews, the members of the research team engaged in concurrent data analysis. After each interview, each member of the research team listened to recordings of the interview and formed initial hypotheses about the e-frames that the participants were using to interpret the lectures. The research team would meet to synthesize these initial hypotheses and develop questions that would allow us to test the viability of these hypotheses. The next interview began with the interviewer asking the participants these questions, which was then followed by them resuming watching the lecture videos. After all four interviewers were conducted, we transcribed each interview.

3. 5. Retrospective analysis

Our retrospective analysis had two main purposes:

(i) we first aimed to analyze broad characteristics of Alice and Brittany's behavior in our interview to give *an account of* the e-frames that they are using;

(ii) we then analyzed specific interpretations that they gave to Professor Su's lectures and used their e-frames to give *an account for* these interpretations (c.f., Mason, 2002).

We first gave an account of Alice and Brittany's e-frames as follows: for each segment of the lecture, we summarized Alice and Brittany's comments. We focused on any instance in which Alice or Brittany described or evaluated what they felt was the mathematical contribution that Professor Su made or was intending to make. Because we regularly asked Alice and Brittany what they thought Professor Su was trying to convey and what, if anything, was valuable, our data contained many comments from Alice and Brittany about Professor Su's mathematical contributions. By analyzing commonalities in each participant's responses across the interviews, we revised the hypotheses developed during concurrent analysis and, as needed, developed new hypotheses about their e-frames.

We then engaged in the following iterative process to refine our hypotheses: For each hypothesis about a participant's e-frame, each member of the research team read the transcripts in their entirety, identifying all instances that either supported or contradicted the hypothesis. We evaluated a specific hypothesis as not viable if one of the three conditions occurred: (i) we found few instances supporting the hypothesis; (ii) we found many instances that were inconsistent with the hypothesis; or (iii) we found a single significant instance that strongly contradicted our hypothesis. If a hypothesis was evaluated as unviable, we would often refine our hypothesis and repeat this process. Other times, we would judge the hypothesis as fundamentally inaccurate and discard it. After this iterative process, we had a set of e-frames for each participant that were highly grounded in the data for each participant.

We then used these e-frames to give an account for students' interpretations of lectures as follows: We identified excerpts in which Alice and Brittany had different interpretations of what Professor Su said in a segment of the lecture, which was a common occurrence. We chose differences that we felt were representative of the data set. With those differences, we described the difference between Alice and Brittany's interpretations in terms of how they framed and evaluated the mathematical contribution that they felt that Professor Su was making. Through an interpretive analysis, we accounted for Alice and Brittany's different interpretations using the e-frames that we posited.

4. Results

4. 1. Alice's e-frames

4. 1. 1. One needed to define a concept to be able to reason about it.

E-frame 1: Making claims and providing justifications about a concept requires a precise definition of the concept.

E-frame 2: A precise definition of a concept is a mathematical contribution.

E-frame 1 provided Alice with a criterion on which she judged whether a mathematical justification was a legitimate contribution. E-frame 1 warranted e-frame 2, that providing a precise definition of a concept was a legitimate contribution.

There were multiple instances that indicated Alice held these e-frames throughout the interviews. For example, in the first interview, the interviewer asked Alice what the real numbers were. Alice's response was revealing: "That's an excellent question. [long pause] I don't know. I don't know the formal definition of a real number". This was representative of Alice's tendency to express an epistemic need to see concepts defined, which she displayed throughout the interviews. For instance, in Interview 1, Alice was asked if the fractions 9/15 and 12/20 were "the same thing". She responded:

> "You need to assign a definition. 'Same thing' does not tell me anything[…]So if I have two of the same shirt, are they the same shirt? No, if I'm wearing one, then one is being worn and one's not. But in terms of just shirts yeah, they're the same shirt. So based on how we want to define 'the same thing', they may or may not be".

In Interview 4, Alice reiterated this point, stressing, "we need very well-defined definitions so that we can get very clear implications". At four other points in our interviews, Alice objected to questions about concepts that were not explicitly defined, saying she found them ambiguous and unanswerable because terms in the concepts were not defined. The importance that Alice assigned to concepts being defined led her to continually seek out definitions when she watched the lectures.

4. 1. 2. When constructing a system, you need to distinguish between what you know through experience and what you are allowed to know within the system.

E-frame 3: Justifications contributing to logical knowing are legitimate mathematical contributions.

E-frame 4: When providing a logical justification, you can only employ claims that are *a priori* or have been previously logically justified. You cannot employ claims that you know psychologically.

E-frame 3 specifies that logical justifications are contributions while e-frame 4 delineates criteria on which they should be evaluated.

As Professor Su constructed the rational numbers, Alice continually distinguished between what she knew psychologically (i.e., what she knew based on her previous experience with the rationals) and what had been justified logically (i.e., axioms, definitions, and statements proven in the classroom context). At 14 different points, Alice stressed the need to differentiate between the two, reminding herself and the interviewer that "we only assumed that we have knowledge about the integers"[2] and "we don't know anything about what [rational numbers] do or look like if they are not an integer". Alice was careful to only use definitions and propositions that were proven in class when justifying claims about the rationals.

4. 1. 3. The mathematical contribution of these lectures.

E-frame 5: The intellectual contribution of these lectures was providing a framework so we can know statements about the rational numbers logically.

E-frame 6: Systematization is valuable when you characterize a more complicated system in terms of a simpler system.

We believe that E-Frame 5 was evoked by Alice in these lectures as a consequence of the prior e-frames that we discussed. Alice desired logical justifications for claims, which is only possible if these claims are defined precisely. E-frame 6 is a criterion that Alice used to judge the significance of the particular systematization that she observed, which we illustrate below. When Alice was asked why Professor Su was constructing the rational numbers, she responded as follows:

Alice: One thing that I have always seen in both physics and math is that, if we can, we always want to go with something more elementary. [In math] I know first we start with our natural numbers. Then we say, what the natural numbers are, a representation of the empty set, and recursive sets of the empty set, ... so 0 was the empty set, 1 was the set of the empty set, and 2 was the set of the set of the set of the empty set, something like that. I feel like there is a search to get even more elementary.

At several other points, Alice became reflective on the nuanced relationship between what she knew psychologically and what she knew logically about the rationals. For instance, in the conclusion of the last lecture, the interviewer asked Alice "how do you understand the rationals?". Her response was as follows:

Alice: [I understand the rationals] on a very simplified level. [The rationals] are just fractions of an integer, numerator and denominator, and I've been working with those types of fractions all my life. So I know exactly what they make, what they look like, how to treat them on a very simplistic level. But on a construction level, we are trying to build them. It's like I want to already know this but the attitude is that it is newly explored material which is a little ironic. It's the attitude that you kind of have to have.

In this excerpt, Alice distinguishes between what she knows on a "simplified level" (what we call knowing in a psychological sense) and at the "construction level" (knowing in a logical sense), noting that you are trying to construct what you already know simplistically (justifying logically what you know psychologically).

[2] Professor Su began his lectures by quoting Kronecker, who said, "God made the integers. All else is the work of man," and did not define the set of integers or operations upon them.

4. 2. Brittany

4. 2. 1. Definitions were used to enhance understanding.

E-Frame 1: The mathematical contribution of a definition is to help Brittany psychologically know a concept.

E-frame 1 describes a criterion by which the significance of the contribution of a definition should be judged. E-frame 1 was evidenced by Brittany in multiple ways. First, at six points Brittany recognized that Professor Su's characterization of the rational numbers (as equivalence classes of ordered pairs) differed from her understanding of fractions, stating at the end of the lectures that she "wouldn't do it [constructing the rationals as equivalence classes] that way" and that rationals were "basically just a fraction of one integer on top of another one". She assumed that the purpose of these alternative characterizations was to provide her with an alternative way to think about the rational numbers. For instance, when asked what she thought the purpose of this presentation was, she responded that she learned "a new way of thinking about the rationals".

Brittany also did not see merit in Professor Su defining aspects of the rational numbers, saying "it's too obvious", because presumably the entire class already understood fractions. Our interpretation is that from Brittany's perspective, such definitions could not enhance her understanding (e.g. "but it's [the definition of positive rationals] not valuable for clarification of anything because we already know"), because she already psychologically understood the concepts being defined.

4. 2. 2. Good definitions are comprehensible.

E-frame 2: Good mathematical definitions are comprehensible

E-frame 2 also provides a criterion on which the quality of a definition should be judged. While Brittany never explicitly described what she thought was a "good definition", at five points she complimented Professor Su's definitions because of their clarity and simplicity. For instance, Professor Su's definition of order on the rationals (i.e., how you define a < b when a and b are rational numbers) was "pretty good" because "it's simple and understandable".

4. 2. 2. You could use what you knew about the rationals to answer the questions that Professor Su discussed.

E-frame 3: When justifying a mathematical statement, it is permissible and desirable to use one's psychological understanding.

E-frame 3 provides a criterion by which mathematical justifications can be evaluated. Brittany rarely expressed a distinction between what she knew logically and what she knew psychologically. Only twice during our four interviews did Brittany question what she was allowed to assume. At 18 other points, she invoked facts about the rational numbers that had not been stated in the lectures to answer questions about the rational numbers. To elaborate on this further, we consider how Brittany reacts to Professor Su's justifications of statements about the rational numbers, which proceeded logically from the definitions that he produced (definitions that Brittany thought were "a new way of thinking about the rationals"). For instance, Professor Su illustrated how each integer z could be represented as the equivalence class of ordered pairs $[(z, 1)]$, Brittany was able to explain why this made sense, but then remarked, "it's just simple to think of it as Z is a subset of Q. I think that would be a simpler way of saying it"[3]. Brittany often expressed frustration at these justifications, which she felt were needlessly complicated.

[3] In our interviews, Brittany was able to explain how the equivalence classes of integers that Professor Su introduced in his lectures represented the integers, which we do not include in this paper for brevity. We only note here that

4. 2. 4. The mathematical contribution of these lectures

E-frame 4: The goal of the lectures was to enhance her psychological knowledge of the rational numbers.

We believe that Brittany evoked this e-frame as a consequence of the earlier e-frames. Each of the earlier e-frames indicated that mathematical contributions of this lecture—that is, Professor Su's definitions and proofs about rational numbers—were presented to enhance Brittany's understanding of the rational numbers and therefore should be easy to comprehend. One way that Brittany evinced e-frame 4 was when she viewed the purpose of the entirety of the lectures as to provide a review about the rational numbers. This failed to satisfy the criterion in e-frame 4 because she felt that she already psychologically knew the relevant statements about the rational numbers introduced by Professor Su.

When asked about the main purpose of the lectures, Brittany cited the construction of the rational numbers and remarked that this "was important to take away." When asked what it meant to construct the rationals, Brittany responded, "I think he was just going over properties of it—order, addition, multiplication, what it means putting them all on the number line". In general, Brittany was frustrated because she wanted to learn new material and did not find value in what she perceived as an extended review, as illustrated in the following exchange:

Interviewer: So what I'm hearing you say is it's more interesting to talk about things you don't know than things you do know, to answer some questions that you might not really know that are interesting?

Brittany: Yeah, I think that's like true for everything.

4. 3. Different interpretations of the same lecture

In this sub-section, we present clips from Professor Su's lectures and describe Alice's and Brittany's reactions. Our aim is to illustrate how Alice's and Brittany's different e-frames led them to interpret these clips in different ways and then to make the broader point that a professor effectively conveying the material depends on the e-frame of the student observing the lecture.

4. 3. 1. Motivating questions. Early in lecture 1, Professor Su presented motivating questions on a PowerPoint slide. These questions included, "What does it mean for a series of numbers to converge? What is a limit? Are there 'enough' numbers to capture all limits?[...]What does it even mean for a sequence of numbers to converge when you're not referencing a limit? There's a question. Some really tough questions." Alice interpreted these questions as follows:

Alice: I started realizing that these were questions that were probably going to be addressed throughout the course so then I understood why he was asking them. It was kind of a mindset I need to be in. I wasn't trying to answer these questions whenever he asked them, but really trying to get myself into the mindset of questioning certain definitions. It prompted me to get in what I think would be a good mindset in this class.

Interviewer: I was going to ask you if you tried answering any of these questions, and you said that you didn't.

Alice: I briefly thought about them but then I realized that I really didn't have the grounds. I didn't know the definitions. I thought about each question. I understood that really I haven't questioned these at all and that I accepted many of them. That allowed me to get into a mindset of, 'I can't just accept these facts anymore. I have to bring these questions back and this is the attitude you have to face in this class'.

Brittany's understanding was beside the point. She did not see the need for a complicated explanation for why Z was represented in Q because clearly the integers were a subset of the rationals.

In this excerpt, Alice discussed how the questions prompted her to consider the definitions of the concepts involved in the questions and motivated the lectures in the early part of the course, which consisted of defining the terms in the questions in terms of the integers. We account for this with Alice's e-frame 1: that making and justifying claims about concepts requires a definition of that concept. Alice's desire for these definitions can be accounted for by her e-frame 2: that providing precise definitions of concepts is a mathematical contribution. Finally, Alice described Professor Su's concepts as invoking an "attitude" and "mindset" that one cannot simply rely on their prior understanding (what we refer to as Alice's psychological knowing) which aligns with Alice's e-frame 3 and e-frame 4 calling for Alice to distinguish between her psychological and logical knowledge of the rationals.

When Brittany was asked what she was thinking about when she saw the questions, she said she found the questions "interesting" and "I was visualizing a number line in my head". Hence, Professor Su's questions were also motivating for Brittany, but they prompted her to use her intuition to think about what the answers to these questions might be and how they might relate to her mental models for understanding the real numbers. Hence, Professor Su's questions also evoked an e-frame for Brittany, but it was a different one than Alice. It was Brittany's e-frame 4, that the goal of these lectures was to enhance her psychological knowledge of rational and real numbers.

4. 3. 2. Well-defined operations. In lecture 2, Professor Su defined addition on the rational numbers (<(a, b)> + <(c, d)> = <(ad+bc, bd)>). He then wanted to show that this binary operation was well-defined. To illustrate what he meant by well-defined, Professor Su presented two other candidates for addition, one of which was well-defined but useless (a binary operation whose output is always <(0, 1)>) and another that was not well-defined (<(a, b)> + <(c, d)> = <(a+c, b+d)>). Alice claimed she understood what Professor Su meant by the term well-defined, saying "we can put in different elements of the same equivalence class, and we should still expect the same result". Nonetheless she objected, "when he says this definition is well-defined, the specific definition requirements for something being well-defined was not gone over. The term well-defined was actually not well-defined". Finally, Alice noted that if she were actually taking this class, she would look up the definition of well-defined outside of class. Our interpretation was that Alice understood the concept of a well-defined operation psychologically, but without a formal definition, she could not understand the concept logically. This is because of e-frame 1 that logical reasoning about the concept requires precise definitions. Therefore, she found Professor Su's presentation inadequate.

Brittany viewed Professor Su's definition of well-defined favorably:

Brittany: I like the definition of well-defined. It was really clear and understandable because well-defined is a word we use a lot.

Our interpretation is that Brittany found Professor Su's examples as adequate to get a good sense (or psychological understanding) of what the concept of well-defined meant. We account for Brittany's comments with Brittany's e-frame 2: that good definitions are comprehensible. Because Brittany psychologically understood Professor Su's definition, she valued it.

4. 3. 3. The definition of addition. In lecture 2, Professor Su defined addition by <(a,b)> + <(c,d)> = <(ad+bc, bd)>. When Alice was asked what Professor Su was trying to convey, Alice responded that providing this definition was necessary.

Alice: [Without the definition], we wouldn't know what addition is. We want to keep that mentality that the whole thing that we are doing is we are defining that construction, so we need to make these definitions.

We account for Alice's comment with her e-frame 1 and e-frame 2. One could not reason about addition without defining it so it was a necessary contribution.

> Alice: Half of me wants how we get to that exact definition[...]As we have seen, there are well-defined functions that [we don't want...]Assume we don't have any knowledge of the rationals, why this and not the others?

Alice proposed one such criterion for evaluating a definition for addition was verifying that the definition implied that <(a, 1)> + <(b, 1)> = <(a+b, 1)>.

We account for Alice's comments with e-frame 4, that one could not use their previous knowledge about the rationals in justifying claims about the rationals. Alice desired a justification for the adequacy and choice of that particular well-defined definition *independently* of what Alice knew psychologically. As the following excerpt reveals, Alice knew Professor Su's definition of addition would "work", but for now, we must "assume we don't have any knowledge of the rationals":

> Alice: The other half of me, well I know how to get to this. Do I really want to see him lay it out or do we just accept this definition? I know why cause it works and that's just what I'm told[…]I feel like a lot of this would be considered valuable but I wouldn't say its significant and new. It's hard because it's like we are discovering something we already know.

Our interpretation of this excerpt is that although Alice appreciates the need to justify that Professor Su's definition of addition is an adequate one (it "would be considered valuable"), a part of her does not want to see this justified because, based on her experience, she knows it is going to work, even though e-frame 4 requires logical knowledge to justify Professor Su's choice.

When Brittany was asked about the definition, she thought the definition that Professor Su provided was adequate, saying, "I liked the definition because it's true. I can see how he got it. I thought it was going to be that. It proves I know what's going on". However, later in the interview, Brittany also complained that she saw little value in the lecture in its entirety, saying, "it's not that useful because I already know what addition is, know what rational numbers are, and what fractions are".

We account for Brittany's comments as follows. Brittany thought the definition provided by Professor Su was comprehensible, satisfying the criteria specified in e-frame 2. However, Brittany did not think that Professor Su's description of addition enhanced her understanding since she already knew how to add rational numbers. Therefore, the definition did not satisfy the criterion for a mathematical contribution specified in Brittany's e-frame 1.

4. 3. 4. An alternative proof. Professor Su would often state well-known facts about the rational numbers as theorems and then prove these theorems. In Interviews 2 and 3, Alice said these proofs were useful because they reminded the class that they needed to have the attitude that they knew nothing about the rational numbers. Thus, due to her e-frame 4, Alice believed that even obvious claims needed to be justified from basic principles about the integers. In contrast, Brittany was unsure why Professor Su was justifying things that were obviously true and sometimes remarked that the notation that he was using was unnecessarily cumbersome. We conjectured that Brittany's frustration was due to the fact that she felt that she already (psychologically) knew how to justify claims about fractions based on her experiences as an elementary and secondary student. Thus Brittany's e-frame 3 suggests that using psychological knowledge in reasoning is desirable. To explore this hypothesis, we started Interview 4 by presenting Alice and Brittany with the following proof of the addition of any two rational numbers and asking the participant if it would be useful for Professor Su to use:

Claim. Assume a, b, c, and d are integers and assume that $b \neq 0$ and $d \neq 0$. For any rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

Proof.
$\frac{a}{b} + \frac{c}{d}$
$= \frac{ad}{bd} + \frac{c}{d}$ (since $\frac{a}{b} = \frac{ad}{bd}$)
$= \frac{ad}{bd} + \frac{bc}{bd}$ (since $\frac{c}{d} = \frac{bc}{bd}$)
$= \frac{ad+bc}{bd}$ (since the two terms have like denominators).

Alice immediately rejected the proof as inadequate.

> Alice: So here [the second line of the proof] what we did is we multiplied by 1 essentially, by multiplying by d over d but *we have to know how to multiply rationals to do that*. We have to know that 1 times a certain rational does not change the amount[...]But then here [the last step] we have to assume that we also know how to add rational numbers with like denominator. *But that was also not really part of how we constructed or defined the rationals.* Or at least not in the last class. (italics were our emphases).

In the two italicized excerpts, we see Alice's recognition that one cannot use common (psychological) knowledge about the rationals to justify properties when you are constructing the rationals. We account for this with Alice's e-frame 4, that justifications cannot import facts that are known psychologically if they have not been established logically.

In contrast, Brittany thought the proof was adequate, saying several times, "I guess it would be a good proof". Her only concern was that the proof would be too simple, saying, "it just seems so obvious, like everyone knows, so I don't think it would be necessary". Hence, this proof would be a permissible mathematical contribution, although perhaps not a useful one. We account for this with Brittany's e-frames as follows. Brittany's e-frame 3 specifies that it is acceptable and desirable to use one's psychological knowledge when writing a proof. The proof that she evaluated did so and Brittany saw no problems with this. However, Brittany's e-frame 4 specified that the purpose of these lectures were to enhance her psychological understanding. Since the reasoning in the proof was obvious to her, the proof failed to satisfy that criterion.

5. Discussion

5. 1. Summary of main results

The purpose of this paper was to illustrate the phenomena that students with different e-frames may interpret the same mathematics lecture in different ways. Alice distinguished between logical and psychological ways of knowing mathematics and she viewed the intellectual contribution of Professor Su's lectures as providing a logical basis for her psychological understanding of the rational numbers. Providing this logical basis involved supplying definitions and justifying facts using only what had been explicitly defined by Professor Su until that point. Although Alice sometimes did not find this to be interesting, she understood why it was necessary (section 4.3.3). Brittany perceived the contribution of the lecture to be to enhance her psychological understanding of the rational numbers. Since she already felt that she had a robust psychological understanding of the rationals, she did not see much value in the lectures.

In this paper, we focus on a lecture of the systematization of the rational numbers. However, we use the comparative case study to illustrate what we believe is a more general phenomenon: students' e-frames about what is a mathematical contribution act as an interpretive filter to the mathematics that they observe.

5. 2. Limitations

Having Alice and Brittany view videotaped lectures allowed us to understand their perceptions of lectures as they occurred, but was inauthentic in several respects. For instance, Alice and Brittany were not enrolled in the course, their task during the study was not to master the material, and they did not work on real analysis outside of our interviews. The students might have behaved differently if they were attending actual lectures and had a strong motivation to learn the material. Further, we deliberately chose to take a non-evaluative stance in evaluating Alice's and Brittany's e-frames or the quality of their interpretation of Professor Su's lectures. We recommend future research in more authentic contexts and in other content domains that explicitly explore the link between students' e-frames and their understanding of the mathematics that they study.

5. 3. Implications for lecturing

A key takeaway from this paper is that even if a mathematics professor clearly explains the ideas that she wishes to convey, students may not grasp the point of the lecture if they do not hold e-frames that allow them to perceive the mathematical contribution that the lecturer is making. This implies that a lecturer not only needs to provide students with the opportunities to internalize the mathematical contributions that she is making but she must also help students develop the e-frames that enable them recognize the mathematical contributions and capitalize on these opportunities. We concur with Solomon (2006) that if advanced mathematics courses are to be effective, epistemology cannot be ignored.

We observe an interesting pedagogical challenge in using a lecture as an impetus for students to adjust their e-frames. In section 4.3.1, Professor Su's motivating questions prior to defining the rational numbers evoked e-frames in both Alice and Brittany, but the e-frames these questions evoked were different. To Alice, these questions evoked a mindset to seek out the definitions of the concepts of these questions. To Brittany, this led her to seek out comprehensible answers to the questions that Professor Su posed. In our opinion, both interpretations of Professor Su's comments were sensible. This illustrates an important point. The e-frames that a professor's comments invokes in a student are likely related to the e-frames that the student already holds.

The preceding analysis suggests one reason that collaborative inquiry-based learning may be an attractive alternative to lecturing. What counts as a mathematical contribution in a classroom is a sociomathematical norm. The analysis of Yackel and Cobb (1996) illustrates how sociomathematical norms can be established with negotiation between the teacher and students as the students are engaged in authentic mathematical activity. Having students engaged in activities such as systematization with feedback from their teacher and classmates may provide students with a better sense of what the mathematical contributions of these activities are than simply having a professor explain this to the students in lecture.

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A Case Study in Constructing Set-based Meanings for Conditional Truth

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We present a case study of Hugo's construction of Euler diagrams to develop set-based meanings for mathematical conditionals. This episode arose in a teaching experiment guiding students to reinvent mathematical logic from their reasoning about meaningful mathematical statements. We intended for Hugo to develop a subset meaning for conditional truth. Hugo successfully identified and used this condition, but he also introduced another formally equivalent meaning for conditional truth. We discuss the shifts in his thinking necessary for developing set-based reasoning and how this case influenced our goals for logic learning.

Keywords: logic, Euler diagrams, conditionals, teaching experiment

Though the modern formalizations of mathematical logic and language are relatively young (Frege, 1879; Russell, 1903), these accounts have become integral to the normative understanding of the language of proof-oriented mathematics. Azzouni (2006) argues that the essential novelty was developing a formal language in which meaning and truth are defined in purely syntactic ways. Because students must abide by these formal conventions of language to some degree, many transition to proof courses teach mathematical logic (Selden, 2012). However, the existing literature provides relatively little insight about logic instruction and the meanings that students must develop from that instruction (exceptions include Antonini, 2001; Bardelle, 2013; Barnard, 1995; Durand-Guerrier, 2003; Hawthorne & Rasmussen, 2014).

Regarding instruction, logic can be taught using everyday statements (e.g. Epp, 2003), formal syntax (e.g. Hawthorne & Rasmussen, 2014), or mathematical statements (e.g. Dubinsky & Yiparaki, 2000). Regarding student learning, there are various ways in which a student may affirm a conditional such as "If an integer *x* is a multiple of 6, then *x* is a multiple of 3":

1. as an empirical generalization inducted from a series of examples,
2. based on properties such as the spacing of these multiples on the number line,
3. as the result of a proof (maybe implicitly) using the theorem "if a|b and b|c, then a|c,"
4. as a subset relation between the set of multiples of 6 and the set of multiples of 3, or
5. as not false because there is no multiple of 6 that is not a multiple of 3.

We know little about students' *meanings* (Thompson, Carlson, Byerley, & Hatfield, 2013) for conditional truth, how they develop, and which should be privileged by logic instruction. We find it useful to distinguish between those meanings for conditional truth that intrinsically rely on the mathematical content of the sentence (examples 1-3) and those that rely on generalizable criteria (examples 3-5). We place 3 in both categories because the theorem used to prove is mathematically specific, but one can generalize the criterion that a conditional is true if there is a proof of the conclusions from the hypotheses (Weber & Alcock's, 2005, warranted conditional). How and when is it important for students to develop generalizable meanings for conditional truth and even to become reflectively aware of such meanings?

The Current Investigation

In this report, we lay some foundation for investigating these questions through a detailed case study of one students' learning about conditional truth and contrapositive equivalence. By documenting particular students' pathways of learning, we can discern important challenges and opportunities for instruction. This case comes from a series of teaching experiments attempting to guide undergraduate students to reinvent mathematical logic (Dawkins & Cook, 2015, in press). We taught logic by presenting students with meaningful mathematical statements all of the same logical form (disjunctions, conditionals, then multiply quantified). By letting students assign truth-values, reflect on their strategies, and construct negations, we intended for students to reinvent truth-functional logic as a formalization of their own reasoning and languaging.

We frame our findings in terms of student *meanings*, which Thompson et al. (2013) define as the set of inferences available to a student as a result of understanding something in a particular way. To illustrate this in our context, consider the tools available to a student for interpreting a mathematical statement and deciding its truth-value. While our participants exhibited various strategies for interpreting general statements, many times their reasoning focused either on examples, properties, or sets (which correspond respectively to examples 1, 2-3, and 4 above). By interpreting the sentence in terms of examples or properties, varying truth conditions and insights became available to students, which are parts of the *meaning* of the sentence for that student. We will thus discuss example-based, property-based, or set-based meanings. While students assigned statements their normative truth-values using all three, we found that reasoning with sets often afforded students the most fruitful strategies (Dawkins & Cook, 2016). Thus, in the experiment featured in this paper we attempted to guide students toward set-based truth conditions, which for conditional statements can be stated: *The conditional "If for $x \in S, P(x)$, then $Q(x)$" is true if and only if* $\{x \in S|P(x)\} \subset \{x \in S|Q(x)\}$*."* We call this the *subset meaning for conditional truth* (or *subset meaning* for brevity). The primary contributions of this study are 1) documenting this student's resources and challenges in developing the subset meaning and 2) documenting the novel meaning he created to affirm the contrapositive of a true conditional.

Conceptual analysis of conditional truth

We value Euler diagrams as a means of representing set relations relative to compound statements. The student featured in this report was familiar with similar diagrams from previous instruction (likely Venn diagrams), but he did not have fully normative meanings for how the diagram referred to mathematical objects. Dawkins and Cook (2015) point out at least one important meaning for understanding such diagrams as mathematicians do: the negation of a property corresponds to the complement of the set with the original property. Dawkins and Cook (2015) demonstrate that not all students associate the negation of a property with the complement set of examples, but such an understanding seems necessary for understanding why contrapositive conditionals have the same truth value (Figure 1).

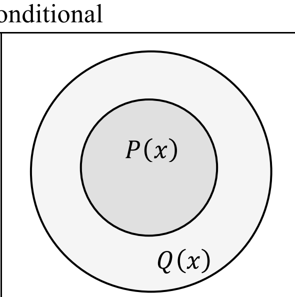
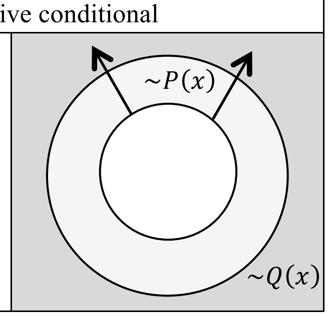
Original conditional		Contrapositive conditional	
"If for $x \in S$ $P(x)$, then $Q(x)$." is true whenever $\{x \in S\|P(x)\}$ $\subset \{x \in S\|Q(x)\}$		"If for $x \in S$ not $Q(x)$, then not $P(x)$." is true whenever $\{x \in S\|\sim Q(x)\}$ $\subset \{x \in S\|\sim P(x)\}$	

Figure 1: Euler diagrams demonstrating why contrapositives have the same truth-value.

Rather, students may associate the negation of a property with a proper subset of the complement – e.g. "not acute" means "obtuse," or "not even" means "odd" – or they may associate the negation with an overlapping property – e.g. "not a rectangle" means "is a parallelogram." Such *semantic substitutions* (Dawkins & Cook, 2016) do not afford the set structure that we intend for students to develop. Students' choices of semantic substitutes demonstrate a strong preference for familiar categories precluding the Euler diagram's novel partitions.

Background and Study Design

This study investigates Hugo's learning during the time he and his partner, Elya, worked on the set of conditional statements provided by the teacher/researcher (the second author). These students were recruited from a Calculus 3 class at a mid-sized university in the United States, but the experiment took place in one-hour sessions outside of class. The task progression for each type of statement – disjunction, conditional, and multiply-quantified – included: 1) assigning truth-values to all of the provided statements, 2) look for patterns in how they determined whether the statements were true or false, 3) consider the sets of examples that made each statement true and the sets that made them false, and 4) constructing negations. Hugo and Elya spent 2.5 sessions studying mathematical disjunctions prior to working on conditionals. This paper focuses on the subsequent 2.5 sessions they spent studying conditionals, especially the second such session from which Elya happened to be absent. Table 1 presents some conditionals they studied. After the students assigned truth-values and looked for patterns, the interviewer asked Hugo and Elya to "Think about the set of all things that satisfy the *if* part and the set of all things that satisfy the *then* part. And tell me about the relationship between those two." This was particularly intended to prompt students to formulate the subset meaning for conditional truth.

1. If a number is a multiple of 3, then it is a multiple of 4.
2. If a number is a multiple of 3, then it is a multiple of 6.
3. If a number is a multiple of 6, then it is a multiple of 3.
4. If a number is not a multiple of 6, then it is not a multiple of 3.
5. If a number is not a multiple of 3, then it is not a multiple of 6.
6. If a triangle is not acute, then it is obtuse.
7. If a triangle is obtuse, then it is not acute.
8. If a triangle is not acute, then it is not equilateral.
9. If a quadrilateral is a rectangle, then it is a square.
10. If a quadrilateral is a rectangle, then it is a parallelogram.
11. If a quadrilateral is not a rhombus, then it is not a rectangle.

Table 1: Sample conditionals that Hugo and Elya studied.

Consistent with teaching experiment methodology (Steffe & Thompson, 2000), the teacher/researcher consistently formed second-order models of student understanding and tested those models through subsequent questioning. These models were informed by findings from previous studies (Dawkins & Cook, 2015, 2016), as were the learning goals articulated earlier in the paper. The data analysis consisted of iterative analysis throughout the experiment and retrospective analysis of the video recordings and artifacts of student work afterward. The analysis presented in this paper focused particularly on the pair's construction of set-based meanings for conditionals, especially as facilitated by Euler diagrams. Retrospective analysis similarly consisted of developing second-order models of Hugo's understanding by forming and testing hypotheses using the corpus of his mathematical activity. Dawkins and Cook (2015, 2016) present more thorough accounts of the teaching and data analysis methodologies.

Results

On Hugo and Elya's first pass through the set of conditionals, they assigned to each the normative truth-value. They quickly recognized that an example that satisfied the *if* part and not the *then* part was a counterexample (their *counterexample criterion*). Based on their work with disjunctions, they understood that a single counterexample made a general statement false and they could articulate general conditions for declaring statements false. For instance, after the first five statements Hugo explained, “If we can come up with a case that fits the first [antecedent] but does not fit the second [consequent], then it [conditional] has to be false. We only need one case.” Even in the case of positive statements, Hugo focused on examples by populating Elya’s set-based explanations with particular examples. When she explained, “[Statement] Five would be true, because every multiple of 3—every multiple of 6 is a multiple of 3. So if it’s not a multiple of 3 it can’t be a multiple of 6,” Hugo elaborated, “Like 17, is not multiple of 3 but it is also not a multiple of 6.” Elya inferred statement 5 from statement 3, which we call a *contrapositive inference*, but her later work suggests that she did not understand contrapositive equivalence as a general property of conditionals. We hypothesize that her understanding of multiples supported her inference rather than reasoning about abstract logic or set relations. In addition to focusing on examples, Hugo also affirmed some conditionals using property relations. Regarding statement eight, he said, “Not acute would mean either a right triangle or an obtuse triangle. Neither of those can be equilateral so that would be true.”

After Elya and Hugo assessed all the provided conditionals, the interviewer asked them to look for patterns in the statements and why they were true or false. The pair noticed that many statements contained the same properties and identified the relationships commonly referred to as inverse (e.g. #3,4) and converse (e.g. #6,7) conditionals. After some discussion and conjectures about the truth-values of these related statements, the interviewer asked the pair to consider the sets related to each part of the statements. The conversation proceeded:

H(1): I’d say, if the statement is true then the set for the first part—I’m sorry the set of the second part will be included in the set of the first part.

I(2): Okay. Why do you say that?

H(3): Um, because if we said that it’s true then when we pick—something that’s true for the first part, then it has to be included in the second part for the whole statement to be true.

I(4): So its sounds to me like you’re saying there’s two possibilities. One is to say that the set—the *if* set can be sort of inside of—or contained in the *then* set. Or you can say the *then* set is contained in the *if* set. Which one do you—are claiming? So you’re saying, if the statement is true then what was the relationship here?

H(5): Then the—*then* will be inside *if*.

I(6): Okay. What do you think [Elya]?

E(7): I think the *if* has to be in the *then* but *then* doesn’t have to be in the *if*. ‘Cause there—when we looked at 3 there’s all the multiples of 6 are contained in multiples of 3 but all multiples of 3 are not contained in multiples of 6.

I(8): Do you agree with that, [Hugo]?

H(9): Yeah I think I’m good with that […]

I(10): What are thinking about in a particular example?

H(11): Uh, you wanna talk about [statement] 3. Um in like a circle, and multiples of 3—3,6,9,12. Um, multiples of 6 will be included in that circle. Like 6 and 12 are multiples of 6. So there’s an additional circle inside that includes some numbers but does not include others.

I(12): Okay, but sort of which are you calling the *if* part and which circle are you calling the *then*?

H(13): The *then* part would be the bigger one. The inside would be *then*—sorry other way around. *Then* is on the outside. *If* is in.

This appeared to be the first time Hugo reasoned about conditionals using sets, and, from the researcher perspective, his general explanation (turn 5) was inconsistent with his earlier semantic reasoning about the given statements. Once the interviewer pushed Hugo to explain his reasoning in a particular semantic context (turn 10), he recognized the subset meaning for conditional truth (turn 13) but even after this episode he sometimes returned to his intuition that the *then* set is contained in the *if* set. We conjecture that this intuition may be supported by the relationship between the properties in true conditionals. For instance, statement three is true because being a multiple of 6 means being a multiple of 2 and a multiple of 3, meaning the *if* property entails the *then* property. This explanation seems consistent with Hugo's use of the word *included* in turns 1 and 3. Hugo's population of the Euler diagram with numbers (Figure 2) appeared to serve as a bridge between his example-based meanings and the set reasoning the interviewer invited.

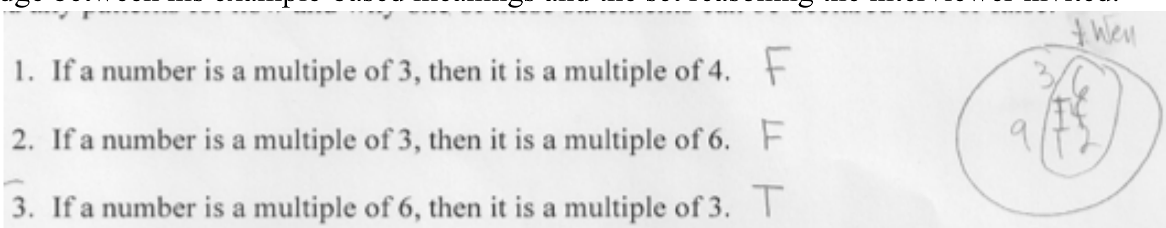


Figure 2: Hugo's Euler diagram for statement 6.

Hugo and Elya began to use these Euler diagrams to explore the truth of the various conditionals provided. By the end of the first session on conditionals, Elya was able to produce an explanation for why two contrapositive statements were both true as portrayed in Figure 1, but Hugo showed little evidence of understanding her reasoning. The next session afforded Hugo opportunity to independently develop his understanding of conditional truth because Elya happened to be absent. To encourage rediscovery of contrapositive equivalence, the interviewer invited Hugo to explore multiple related statements – original, inverse, converse, and contrapositive – with the same Euler diagram. The dialogue proceeded:

I(14): So look at [statements] 2 through 5 then. 'Cause 2 through 5 are gonna be—again—its a bunch of the same options but the same kinds of parts.

H(15): Two—the number is a multiple of 3. We got a circle—they're all multiples of 3—3, 6, 9, 12. Then it is a multiple of 6. [draws a circle around the 6 and 12].

I(16): Okay so is this true or false? So if it's a multiple of 3, then it's a multiple of 6.

H(17): We said it was false because our set was this circle of multiples of 3, and then if you only want multiples of 6—you have this but you still have the 3 and the 9 which aren't included in the little circle.

In turn 15, Hugo produced the diagram shown in Figure 3, labeling the outer circle "if 3x" and the inner circle "then 6x." In turn 17, Hugo did not rely on his counterexample meaning, but rather declared statement 2 false because the sets failed to satisfy the subset condition ("aren't included in the little circle"). When the interviewer asked him to consider statement 3 (the converse), he expected Hugo to refer to the same diagram. Instead Hugo began modifying the position of the circles around the example numbers. This suggested that Hugo's meaning for the diagram did not merely reflect the invariant relationship between the multiples of 3 and 6. Instead, Hugo tied the circles' meaning to the statement structure – *if* and *then* – such that shifting to the converse altered the diagram. The interviewer recreated the original diagram and encouraged Hugo to interpret statements 3-5 using the same diagram:

I(18): Okay, so we can sort of stick with the circles. But now what's—so in the second statement, or now actually it's number 3, what is it we're saying?

H(19): If is a multiple of 6, we put the *if* here [switches *if/then* labels]. Then it's a multiple of 3. So if we only talk about this, this set of numbers, does that included inside of 3x? Yes.
I(20): Yea, all the multiples of 6 here are multiples of 3. Okay, right. I agree, so this one is true.
H(21): So again, if the *if* is encased, enclosed, inside the bigger then circle, then it's true. […]
I(22): I want you to try to use the same picture to talk about 4 and 5.
H(23): If it is not a multiple of 6, so—if its not in this circle, so we're talking about outside the little circle. […] Does that make sense?—If not multiple of 6 then it is not a multiple of 3. So we're talking about anything outside of that. Well 9 is still a multiple of 3, 15 is still multiple of 3. So that would be false.
I(24): Okay, where are the non-multiples of 3?
H(25): Non-multiples of 3 would be outside the little circle infinitely, so that would be—oh non multiples of 3. That would be—[marks diagram]—So if it is not a multiple of 6. So we said it's outside the little circle. Then it is not multiple of 3, not multiple of 3 would be outside the bigger circle. So our set of numbers is inside the big circle and outside of this, but also outside the big circle.
I(26): Right, so what numbers are outside the big circle? Just give me a few examples.
H(27): 7, 11 […]
I(28): If it's a non multiple of 6 then it is a non-multiple of 3.
H(29): It's not a multiple of 6, like 3, 3 is multiple of 3 though.
I(30): Yea, so your counter-examples, like you said are these. 'Cause they're non-multiples of 6 that in fact are multiples of 3. Which, so I agree. [Statement] 4 is false for that reason. 3 is a counter-example. […]
H(31): So we have a ton of numbers that prove it yes but we have all the counterexamples would be inside the big circle and outside the little circle. [Reads statement 5] So if we're only talking about number that are not multiples of 3. So if anything outside the big circle—well our multiples of 6 are inside the circle, so if we talk about anything outside of the circle, obviously we're not going to contain multiples of 6. So if our set of numbers are outside the big circle, then yeah, none of those are multiples of 6. So that would be true.

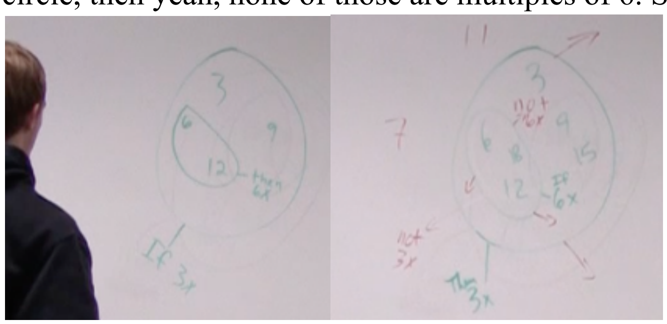


Figure 3: Hugo's diagram at turns 17 and 25

Once Hugo recognized that he could use the same diagram to assess the converse (turn 19), he explicitly cited the subset meaning to affirm statement 3 (turn 21). The interviewer's prompt to use the same diagram for statements 4 and 5 required Hugo to attend to the negation/complement relationship. In turn 23 he associated the complement of the inner circle with the non-multiples of 6, but he avoided coordinating the complement of both sets by identifying a counterexample (the criterion he used the previous day). In turns 25 and 31, Hugo recognized that his counterexamples exemplified a class of numbers represented by the space between the inner and outer circles. Thus while his identification of a counterexample built upon his previous day's work, he now displayed a set-based meaning for counterexamples afforded by the Euler diagram.

When Hugo considered statement 5, he associated the first condition with the complement of the outer circle. However, instead of affirming this statement by the subset meaning as portrayed in Figure 1, Hugo argued that non-multiples of 3 (numbers outside the outer circle) could not be multiples of 6 (inside the inner circle). Stated another way, the intersection between the non-multiples of 3 and the multiples of 6 was empty. This *empty intersection meaning for conditional truth* again allowed Hugo to avoid coordinating two negations by parsing statement 5 as "if a numbers is a *non-multiple of 3*, then it is not a *multiple of 6*."

The interviewer invited Hugo to consider the relationships among statements 2-5 in terms of converses and inverses. With interviewer prompting, Hugo acknowledged that since #4 was the inverse of #3 and #5 was the converse of #4, one could go from #3 to #5 by taking both the converse and inverse. Hugo said, "But if we have an if-then statement that's true, we take the inverse and the switch [their term for converse]—we switch it and take the inverse, then—so far we've proved that it would be true. […] I'm observing [this pattern] but I'm trying to articulate why that is." Thus, he observed that these statements shared the same truth-value and conjectured that this pattern might hold generally, but he was unable to justify why this occurred.

Discussion

Our initial goals in this section of the teaching experiment were for Hugo and Elya to develop the subset meaning for conditional truth and to use that meaning to justify contrapositive equivalence. Elya used set-based meanings spontaneously while Hugo needed to develop tools to move beyond his example and property-based meanings. We claim that Hugo's Euler diagram, initially produced to record his example-based reasoning, served as a transformational record (Rasmussen & Marrongelle, 2006). It both allowed him to generalize his example-based meaning for counterexample and later allowed him to relate inverses, converses, and contrapositives.

Unlike prior study participants who struggled to associate negative properties with complement sets (Dawkins & Cook, 2015), Hugo developed this relationship in his interpretation of the diagram. However, he avoided coordinating two complements simultaneously for statements 4 and 5, as would be required by the subset meaning. He instead used counterexample and empty intersection meanings. While we could view this negatively as it falls short of adopting a purely syntactic and content general understanding of conditional truth – as one might desire in teaching formal logic – Hugo's empty intersection meaning is logically equivalent to the subset meaning. Furthermore, he showed flexibility in parsing and interpreting the given statements, which could be fruitful in proof-oriented mathematical activity. We appreciate how the Euler diagram afforded Hugo valid mathematical inferences even if he could not justify those inferences, much as Elya's semantic understanding supported contrapositive inferences. Hugo alternatively drew upon semantic information (about multiples or geometric shapes), linguistic competencies (parsing negative statements), and logical criteria (the subset meaning) to interpret the various statements provided, which we anticipate to be more consistent with proof-oriented reasoning than purely syntactic operations in a formal language.

This paper contributes to our understanding of students' construction of logical structure in semantically rich settings and reveals hurdles and opportunities in students' development of set-based meanings for compound statements. As was portrayed in the introduction, we prioritize students like Hugo and Elya developing generalizable meanings for mathematical truth. However, based on cases like Hugo's we also value the flexibility to affirm the same statement in multiple, formally equivalent ways. Further study should continue to shed light on viable pathways for students to abstract their semantic reasoning into generalizable, syntactic tools and how these tools can be harnessed in students' subsequent proof-oriented activity.

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A New Methodological Approach for Examining Mathematical Knowledge for Teaching at the Undergraduate Level: Utilizing Task Unfolding and Cognitive Demand

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In 2010, Charalambous published an article that examined the relationship between mathematical knowledge for teaching and task unfolding at the elementary level. As a result of this study, Charalambous evidence to support the claim that there is a positive relationship between a teacher's MKT and the cognitive level of enacted task. Drawing upon this finding, the purpose of this study is to propose a new methodological approach examining mathematical knowledge for teaching at the undergraduate level. While this approach draws upon results concerning MKT at the K-12 level, it primarily focuses on examining undergraduate instruction through the lens of task unfolding and cognitive demand. To illustrate how this methodology can be used, the paper concludes by presenting two case studies that demonstrate how the methodology can be used to examine mathematical knowledge for teaching undergraduate Precalculus courses.

Keywords: Mathematical Knowledge for Teaching, Cognitive Demand, Task Unfolding, Undergraduate Instruction

The combination of low pass rates in first-year undergraduate mathematics courses (Saxe & Braddy, 2015) with low percentages of students who persist on to complete STEM majors (Ellis, Kelton, & Rasmussen, 2014) has brought attention to the need to improve mathematics instruction at the undergraduate level. Commonly, undergraduate instructors are viewed as qualified teachers because they are considered experts in the content area they are expected to teach. However, Bass (1997) points out that "knowing something for oneself or for communication to an expert colleague is not the same as knowing it for explanation to a student" (p. 19). Even though they may spend half of their professional live teaching, professors often receive little to no professional preparation or development as teachers. In order to better understand how we can prepare teachers at all levels, it is important to understand the knowledge that is entailed in teaching. Just as accountants, engineers, and economists rely on specific mathematical knowledge, mathematics teachers rely on their own special domain of mathematical knowledge.

Clearly a teacher must know the math that they expect their students to learn, but what other mathematical knowledge is part of this domain? To answer this question and better define this domain, educational researchers have begun to focus on mathematical knowledge for teaching (MKT). Mathematical knowledge for teaching has been defined as "the mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball, Thames, & Phelps, 2008, p. 399). At the K-12 level, MKT has been studied extensively, but few studies exist at the undergraduate level. While researchers have studied MKT in various ways, this study focuses on examining MKT at the undergraduate level by observing and interviewing experienced instructors. Examining teaching by examining the actual practice of teaching will not only bring to light the knowledge that instructors use in the classroom, but also provides a more accurate description than one could take from hypothetical reasoning, personal reflection, or third-party insight.

Mathematical Knowledge for Teaching

Studies have shown that content knowledge is not a predictor of teaching quality and student outcomes (Begle, 1972; Greenwald, Hedges, & Laine, 1996; Hanushek, 1981, 1996). However, Lee Shulman proposed in 1986 that there is content knowledge that matters for teaching. The key difference that Shulman identified was that teacher content knowledge must be connected to domain-specific pedagogical knowledge. Shulman formalized the idea of *pedagogical knowledge for teaching*, "which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p. 9). In 1987, Shulman called for researchers and practitioners to pay more attention to the *professional* knowledge of teaching, including pedagogical content knowledge.

Existing Research on MKT

Several frameworks have been developed to describe the professional mathematical knowledge that teachers use (e.g., Rowland, Huckstep, & Thwaites, 2005; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012; Baumert & Kunter, 2013) including Ball et al.'s (2008) framework for mathematical knowledge for teaching. Ball and her colleagues built off of Shulman's (1987) idea of pedagogical content knowledge and found that there was content knowledge that mattered for teaching and that a focus on this content benefited teaching and learning. Hill, Rowan, and Ball (2005) showed that at the elementary level that "teacher's *content knowledge for teaching mathematics* [emphasis added] was a significant predictor of student gains" (p. 396). While content knowledge for teaching does require general content knowledge, it also includes content knowledge that is *usually* not taught in undergraduate or graduate mathematics courses (i.e., common unproductive ways of thinking and developmentally appropriate definitions). Therefore, the content knowledge that undergraduate instructors have gained through their formal education may not be the same as content knowledge they need to know for teaching.

More recently, research on MKT has been conducted at all levels of K-12 (McCrory et al., 2012; Krauss, Baumert, & Blum, 2008); however, there still are relatively few studies that study MKT at the undergraduate level. Speer, Smith, and Horvath (2010) conducted a literature review to search for empirical research on the practices of collegiate teachers of mathematics. As a result, the authors found that they only were able to identify five articles in their search. While some may argue that this gap exists because MKT frameworks developed at the K-12 level can be extended to the collegiate level, the authors point out that "there are important differences between college and pre-college teachers and teaching" (p. 100), such as level and depth of content and pedagogy knowledge. In another article, Speer, King, and Howell (2014) focus on the problems that result from assuming that research on MKT at the K-12 level can be extended to MKT at the collegiate level. The authors claim that "relatively little attention has been paid to the ways in which MKT theory is or is not applicable to teachers at secondary and post-secondary levels" (p. 106) and challenge researchers to explore "the types of knowledge entailed in the work of [collegiate] teaching...through the same kinds of careful study of the mathematical demands of teaching that sparked the early work on mathematical knowledge for teaching (Ball and Bass 2000)" (p. 119).

Studying MKT at the Undergraduate Level

To address this gap in the research, the purpose of this study is to draw upon previous research in order to propose a methodological approach to examining MKT at the undergraduate level from the perspective of practice. In particular, the methodological approach used in this study was inspired by Charalambous' (2010) study that found a positive relationship between teachers' MKT and the cognitive demand of tasks enacted in the classroom. Studies have shown that the cognitive demand of a task is related to student learning (Boaler & Staples, 2008; Hiebert & Wearne, 1993; Stein & Lane, 1996), but enacting tasks at a high level of cognitive demand is difficult for teachers to do (Stein, Grover, & Henningsen, 1996; Hiebert & Stigler, 2004). Building upon these findings, Charalambous (2010) hypothesized that there was a connection between teachers' MKT and their ability to enact tasks at a high level of cognitive demand. To test this hypothesis, he utilized the Learning Mathematics for Teaching (LMT) assessment to measure teachers' MKT and analyze the cognitive demand of enacted tasks. What he found was that teachers' scores on the LMT were positively associated with their ability to enact tasks at a high level of cognitive demand. While it would be desirable to replicate this study to look for similar results at the undergraduate level, no comparable measure of MKT exists. However, I propose that it is possible to use the combination of task unfolding and cognitive demand as a lens to examine MKT at the undergraduate level.

Task Unfolding and Cognitive Demand

Before I explain the details of the new methodological approach I propose for studying MKT at the undergraduate level, I first want to further develop the underlying theoretical frameworks that this methodological approach builds upon: task unfolding and the Task Analysis Guide (Smith & Stein, 1998). In the third subsection, I connect these frameworks by presenting Charalambous' (2010) characterization of task unfolding by the cognitive demand.

Task Unfolding

Stein et al. (1996) defined a *mathematical task* as "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (p. 460). They also describe the phases involved in the unfolding of a mathematical task and the factors that influence this unfolding. In 2007, Stein, Remillard, and Smith generalized task unfolding to apply to curriculum unfolding more generally, but the underlying process remained the same. In Figure 1, the rectangle boxes represent the three phases of task unfolding. The *written task* describes how the mathematical task is represented in the written curriculum or instructional materials. The *intended task* describes the teacher's plan for implementing the task during instruction. Finally, the *enacted* task captures how the mathematical task is actually implemented during instruction. While each phase has an impact on student learning (represented by the triangle in Figure 1), studies have shown that the enacted task has the greatest impact (Carpenter & Fenemma, 1988). The bottom oval identifies some factors that influence how teachers plan out a task for implementation in the classroom and how the task is actually implemented in the classroom. Finally, it is important to note that the return arrows from the enacted task and student learning represent the impact that these will have on future teaching actions.

Cognitive Demand of Tasks

In order to differentiate between tasks of different types, Smith and Stein (1998) also considered the cognitive demand of a task. They defined *lower-level demand tasks* as "tasks that ask students to perform a memorized procedure in a routine manner" and *higher-level demand tasks* as "tasks that require students to think conceptually and that stimulate students to make connections" (p. 269). Each of these categories was then broken down into two subcategories:

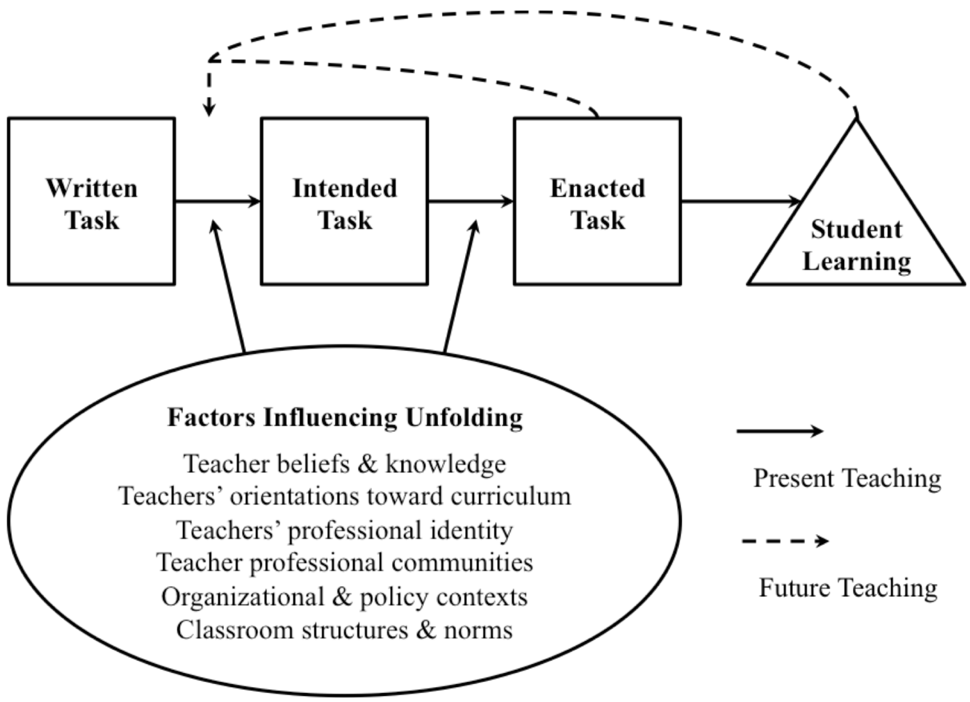


Figure 1. The phases and factors influencing task unfolding.

memorization, *procedures without connections*, *procedures with connections*, and *doing mathematics*. Smith and Stein (1998) differentiated procedures *with* and *without* connections as representing differing levels of cognitive demand. They separated these two types of tasks in order to categorize mathematical tasks that "use procedures, but in a way that *builds connections to the mathematical meaning*" of the underlying concept as a higher-level demand task. Doing mathematics tasks are categorized as higher-level demand tasks that require "students to explore and understand the nature of relationships" (p. 347). To aid in differentiating between the different types of tasks, Smith and Stein developed the Task Analysis Guide, which lists characteristics of the four types of mathematical tasks. Later, when utilizing the Task Analysis Guide to code the third phase of task unfolding, Stein et al. (1996) added a third type of lower-level demand task called *unsystematic exploration*. This type of task, which applies to only the third phase of task unfolding, describes declines in cognitive demand that are characterized by "motivated student engagement, well-intentioned teacher goals for complex work, and well-managed work" but "the cognitive activity…was not at a high enough level to be characterized as engagement in complex mathematical thinking and reasoning" (p. 478).

Categorizing Task Unfolding Using Cognitive Demand

In their 1996 study, Stein et al. utilized the Task Analysis Guide to analyze a sample of 144 tasks that were implemented in reform-oriented classrooms. They focused on the transition from the second to the third phase of task unfolding and found that the majority of the tasks were coded as maintaining or declining in cognitive demand. They also found that "the higher the cognitive demands of tasks at the set-up phase, the lower the percentage of tasks that actually remained that way during implementation" (p. 476). This finding provides confirming evidence for the claim that tasks with high cognitive demand are difficult to enact (NCTM, 2014, p. 17). In 2010, Charalambous conducted a similar case study, but explicitly categorized task unfolding

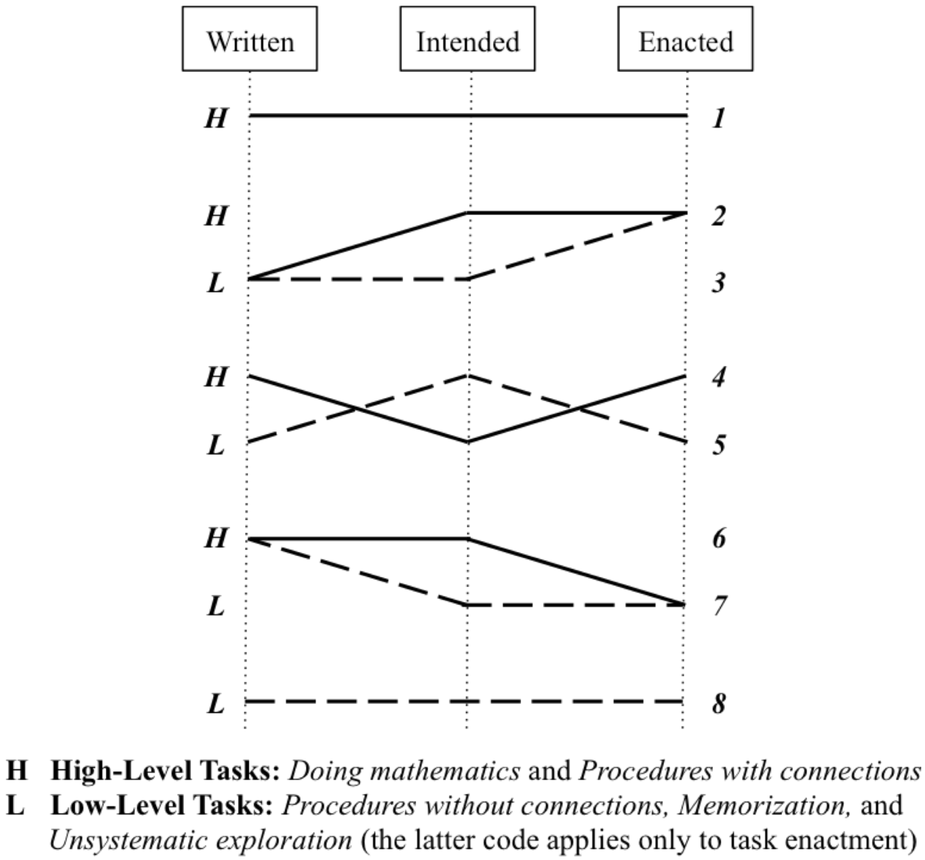


Figure 2. Categorization of possible types of task unfolding.

by the type of path they follow (Figure 2). In his categorization, Charalambous utilized the Task Analysis Guide to code cognitive demand as high or low at each phase in task unfolding, which resulted in eight possible types that a task unfolding could follow. It is worth noting that Charalambous only observed five of the eight possible task unfolding (Types 1, 5, 6, 7, & 8 in Figure 2) in the cases he studied and I added in the type numberings for ease of reference.

Studying MKT at the Undergraduate Level

Using these frameworks, I propose that it is possible to use the combination of task unfolding and cognitive demand as a lens to examine MKT at the undergraduate level. This methodological approach to studying MKT at the undergraduate level theoretically makes sense for several reasons. First, tasks are central elements of teaching due to the fact that they focus on the work that students do in the classroom. In particular, Doyle (1983) argues that tasks are central to students' learning because their enactment focuses students' attention on mathematical ideas and defines students' ideas of what it means to do math. Second, there is a growing body of research that supports the view that instruction should focus on engaging students in mathematical rich tasks. "Principles to Actions" (NCTM, 2014) synthesizes the research on mathematical tasks as resulting in three major findings: (1) not all tasks are equal in terms of the opportunities they provide for student thinking and learning, (2) student learning is greatest in classrooms where tasks are consistently enacted at a high level of cognitive demand, and (3) tasks with high cognitive demand are the most difficult to enact (p. 17). Third, it is important to note that cognitive demand is not an invariable feature of a task. Factors that can influence the cognitive demand of enacted tasks include students' understanding of the task objectives, teachers' interpretation and setup of the task, and teachers' content knowledge (Stein et al., 1996; Charalambous, 2010). The final reason why I believe that this methodological approach is defensible is due to the fact that studies have found a positive relationship between teachers' MKT and enacting cognitively demanding tasks (Charalambous, 2010; Baumert et al., 2010).

In the new methodological approach that I describe below, there are two stages of analysis: one that focuses on categorizing the unfolding of the task by considering the cognitive demand and another that examines the mathematical knowledge for teaching that influenced the unfolding of the task. After explaining each stage of analysis, I provide the reader with two case studies of what it looks like to apply this methodological approach in practice.

Stage 1: Categorizing Task Unfolding by Cognitive Demand

Data. In order to characterize task unfolding by cognitive demand, data should be collected for each phase in the task unfolding process. To capture the written task, any formal curriculum materials, such as textbooks and teacher guides, or informal instructional materials, such as activities developed by the teacher, should be collected. To capture the intended task, data corresponding to how the teacher intends to implement the task during instruction, such as the teacher's lesson plan, should be collected. Finally, to capture the enacted class, data corresponding to the actual implementation of the task in the classroom, such as a video observation, should be collected. To aid in capturing the enacted task, the researcher may find it helpful to refer to the observation protocol used by Rogers & Steele (2016).

Process. Once the data has been collected, the cognitive demand of the written, intended, and enacted tasks should be analyzed using the Task Analysis Guide (Smith & Stein, 1998). In each phase of unfolding, the task should be coded as memorization, procedures without connections, procedures with connections, doing mathematics, or unsystematic exploration. Recall that unsystematic exploration should only be used to categorize the final enactment phase of task unfolding. Also, the Task Analysis Guide contains detailed descriptors of these tasks that should aid the researcher in assigning codes. While Smith and Stein have not published any formal training on how to use the Task Analysis Guide, Charalambous (2010) describes how he trained his coders. One thing that is important to note is that the rest of the analysis only depends on the level of cognitive demand and not the finer-grained analysis of whether or not a high-level task is coded as procedures with connections or doing mathematics. However, utilizing the Task Analysis Guide will aid in making the distinction between high- and low-level tasks more clear.

Product. Once the cognitive demand at each phase in the task unfolding has been analyzed using the Task Analysis Framework, the researcher can categorize the task unfolding using the path types (Figure 2). One could examine MKT without first categorizing the task unfolding, Charalambous' finding that MKT and the cognitive demand of task enactment are positively related suggests that examining tasks that are enacted at a high-level of cognitive demand will provide more opportunities to examine MKT. For this reason, the researcher should ideally identify paths of Types 1, 2, 3, or 4 and utilize these in the second stage of analysis.

Stage 2: Examining MKT

Data. To examine MKT, the researcher needs to look at the factors that influence the transformation of task between phases. To capture how the written task is interpreted by the teacher and transformed into the intended task, the researcher should conduct a semi-structured interview with the teacher before the observation. To capture how the interpreted task is transformed into the enacted task during instruction, the researcher should conduct another semi-structured interview with the teacher after the observation. To aid in the development of the pre- and post-observation interview protocol, refer to Appendix B from Rogers and Steele (2016) and Appendix A of this paper. To allow for an in-depth analysis, it may be beneficial to choose only one or two of the tasks that were enacted at a high-level of cognitive demand to probe into during the post-observation interview. Also, since the intent of this methodology is to study MKT at the undergraduate level from the perspective of practice, stimulated-recall (Bloom, 1953) should be used during the post-observation interview in an attempt to get at the mathematical knowledge that the teacher used in the moment during instruction, as opposed to purely reflective thoughts.

Process. Once the data has been collected, the researcher can begin to analyze the transition between phases of task unfolding in order to examine MKT. Since the purpose of this analysis is to examine MKT at the undergraduate level, the researcher is essentially building a theory for undergraduate MKT. For that reason, the pre- and post-observation interview responses should be analyzed using grounded theory (Strauss & Corbin, 1994). To do this type of analysis, the researcher begins with open coding of the data. As the researcher begins to identify categories that emerge during open coding, they should utilize the constant comparison method (Glaser & Strauss, 1999) to organize the codes into categories and subcategories. To fully develop a theory, the researcher must reach saturation, which is the point "when no new information seems to emerge during coding" (Strauss & Corbin, 1998, p. 136). It is important to note that if the researcher's purpose is to only better understand how some aspect of the world works, as opposed to create a formal theory that explains this aspect, they may conclude their study before reaching theoretical saturation.

Product. Depending upon whether or not the grounded theory analysis was taken all the way through theoretical saturation, the results of the second stage of analysis will differ. If the researcher's purpose is to better understand MKT at the undergraduate level looks like, then completing this analysis will give us a partial description. Although this partial description may fall short of a formal theory, it would still contribute much to the field, since there are currently so few studies that examine MKT at the undergraduate level. However, eventually, I believe that our goal should be to develop a formal theory for MKT at the undergraduate level. By developing a complete picture of MKT at the undergraduate level, we will have a more complete picture of what knowledge is required when teaching at the undergraduate level. Also, developing a theory from the ground up for MKT at the undergraduate level will allow us to see how it is similar to and different from MKT at the K-12 level.

Case Studies

To demonstrate how to use this methodological approach to study MKT at the undergraduate level, I have included two case studies below. The data used in these case studies were collected during the Fall 2015 semester at a mid-size research university in the Midwest. The instructors were selected primarily due to their experience teaching these courses and because they had been identified within the department as strong teachers. At the time of data collection, Greg was a fifth-year graduate student who was teaching Trigonometry for the third time. He had also earned his M.S. in pure mathematics and was nearing the end of his doctoral work. Kelly was a third-year graduate student who was teaching College Algebra + Trigonometry for the third time. After earning her Masters in Engineering, Kelly earned her M.S. in pure math and was just beginning her doctoral work. The data I collected for analysis included the written lesson guides (which are provided by the department), the instructor's intended lesson plans, pre-observation interviews, video observations of enacted lesson, and post-observation interviews. For the pre-and post-observation interviews, I utilized the protocols in Appendix A. For this case study, I will focus on one task that each instructor enacted at a high-level of cognitive demand.

Greg. The task that I analyzed for Greg focused on demonstrating how to use the law of sines to find a side length in a non-right triangle (Figure 3). Since this task called for the specific procedure that was to be used, required limited cognitive demand for completion, was not connected to the concept underlying the procedure, focused on producing a correct mathematical answer, and required no explanation, it was categorized as a procedures without connections task. In his lesson plan, Greg decided to remove the goal statement from the task so that students would have to recognize which procedure to use. Since this task was situated in the lesson right after the law of sines and cosines had been discussed, I still categorized it as procedures without connections because the procedure was evident from prior instruction and the placement of the task. However, when this task was enacted, it took on another form.

Goal: To study how to use the law of sines to find the length of an unknown side.
Exercise: Solve for x in the triangle below.

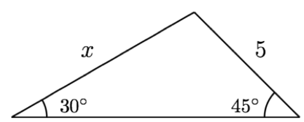


Figure 3. Greg's written task.

When Greg introduced this task during class, a student immediately suggested that they use the law of sines to solve for *x*. However, when Greg asked, "What does the law of sines tell us in this case?", a student responded by saying, "Break this into two triangles." Greg recognized that the student was attempting to follow the procedure that they had used previously when deriving the law of sines. Greg affirmed that the student's idea would work, but then focused on explaining why we don't have to follow the steps of the derivation and instead can just use the final result. During this enactment of the task, the focus shifted from producing a correct answer to developing a deeper understanding of the mathematical process of derivation. For this reason, I coded the enacted task as procedures with connections. By doing this first stage of analysis, I was able to identify Greg's task as a Type 3 task unfolding.

During the second stage of analysis, several aspects of Greg's MKT were brought to the surface. In analyzing how Greg transformed the written task in his lesson plan, Greg depended on his previous experience teaching this course to identify areas where students might struggle with this problem. During the pre-observation interview, Greg said that he knew that students often struggle to figure out what procedure will help them solve a problem. For that reason, Greg removed the goal statement in order to give students an opportunity to learn how to identify what procedure is appropriate. When Greg enacted the intended task during class, he shifted his attention from identifying the procedure to unpacking the mathematical process of derivation. To do this, Greg had to interpret the mathematical statements made by his students. This involved both assessing whether or not the student's idea was mathematically sound and what they did and did not understand about the process of derivation. Finally, Greg had to determine an appropriate way to explain the process of derivation and connect it back to the original task.

Kelly. The task that I analyzed for Kelly focused on introducing the short-run behavior of polynomials and reviewing the concept of long-run behavior (Figure 4). Instead of introducing and reviewing these ideas generally, the task provided a concrete example to work with.

However, since the task required limited cognitive demand for completion, was not connected to the concept underlying the procedure, focused on producing a correct mathematical answer, and required no explanation, I coded it as a procedures without connections task. In her lesson plan, Kelly modified the task by graphing the polynomial at the beginning, asking students to identify relationship between the graph's behavior around zeros and the multiplicities of the zeros (as opposed to just telling them), connecting short-run behavior to the simple power functions x, x^2, and x^3, relating the multiplicities and number of zeros to the degree, and considering how long-term behavior would change if the leading coefficient was negative. Because her intended task now required students to explore and understand the mathematical concept of short-run behavior, access relevant knowledge and make appropriate use of it when working through the task, analyze the task and actively examine constraints, and put forth considerable cognitive effort, I coded it as doing mathematics. When Kelly enacted this task in the classroom, the only thing that she modified was when she graphed the function. Instead of graphing the polynomial first, they utilized the function to identify the zeros, talked about the multiplicities, and briefly explored the general idea of short-run behavior and how it is connected to the multiplicities before graphing $p(x)$. The enacted task added the complexity of requiring students to think generally about short-run behavior instead of relying on a graphical representation and did not remove any complexity of the problem. Therefore, I still coded it as doing mathematics. By doing this first stage of analysis, I was able to identify Kelly's task as a Type 2 task unfolding.

During the second stage of analysis, several aspects of Kelly's MKT were brought to the surface. In analyzing how Kelly transformed the written task in her lesson plan, Kelly intentionally highlighted mathematical connections and patterns instead of focusing on providing definitions. To help students make connect the idea of multiplicities with short run behavior, Kelli utilized "anchor" examples (x, x^2, and x^3) that captured the complexity of the relationship, but through simplified and easily accessible representations. One way that Kelly attended to complexity was by ensuring that the task provided enough variety (such as multiple zeros with both even and odd multiplicities) that students could recognize patterns and waiting to consider additional complexity (such as a negative leading coefficient) at the end. When Kelly enacted the intended task during class, her decision to wait and graph and polynomial after talking generally about short-run behavior required her to critically analyze the mathematical ability of her students. Forming mental representations of graphs and using these to identify graph features is a complex task for students. However, Kelly determined that her students were able to handle this challenge and successfully implemented this modification during the task enactment.

Discussion

As these cases illustrate, utilizing the frameworks of task unfolding and cognitive demand can help reveal MKT by focusing the analysis on the mathematical knowledge entailed in transforming written task into a classroom activity. The cases presented provide a glimpse into what MKT at the undergraduate level looks like, but are far from providing a complete picture. In particular, the analysis done here is really only representative of the first phase of grounded theory of open coding. To develop categories and subcategories or potentially even a general theory for MKT at the undergraduate level, further analysis must be done. As I mentioned previously, these two cases were selected from a larger data set that contains many more cases that can be analyzed. In the Fall 2016, I observed 39 instances of task unfolding in Precalculus classrooms. This semester (Spring 2016), I'm observing six Precalculus instructors three times each, so I anticipate that my complete data set will include around 90 instances of task unfolding.

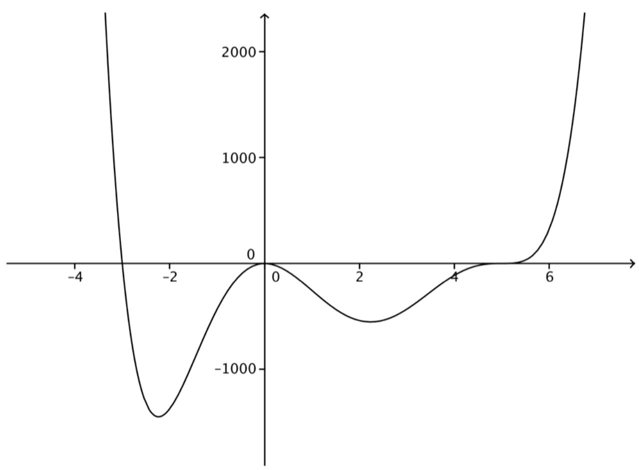
Use the example $p(x) = x^2(x+3)(x-5)^3$ to introduce vocabulary ("zeros" and "multiplicity") and prepare students for the upcoming activity. *On the Board: (When drawing the graph of* $p(x)$*, it may make sense to demonstrate inputting the formula into your calculator and finding the graph that way. Display the graph first in the standard window. Then point out that it's hard to see behavior because the lines are too steep. Adjust the window to have ymin=-500 and ymax=500. Point out that we don't need to spend more time figuring out the zeros on the calculator because we already know these values from our previous work. Be sure to label the zeros as ordered pairs on the graph to emphasize their meaning.)*

Students may have trouble understanding that having a factor of x implies having 0 as a zero. To make this clearer, write $x^2(x+3)(x-5)^3 = (x-0)^2(x+3)(x-5)^3$.

Ex. 1 Let $p(x) = x^2(x+3)(x-5)^3$.
Find zeros and their multiplicities:
To find zeros, find when $p(x) = 0$.
Solve: $0 = x^2(x+3)(x-5)^3$
Happens when $x^2 = 0$, $x+3 = 0$, or $(x-5)^3 = 0$
Happens when: $x = 0$, $x = -3$, or $x = 5$
Hence,

zero	multiplicity
0	2
-3	1
5	3

Expand: $p(x) = x^6 - 12x^5 + 30x^4 + 100x^3 - 375x^2$
Degree of $p(x)$: 6



Long term behavior: Resembles $y = x^6$
Short-run behavior:
Bounces off x-axis: at $(0,0)$
Crosses x-axis: at $(-3,0)$ and $(5,0)$

Figure 4. Kelly's written task.

There are several limitations to both the case studies I presented and the general methodological approach I suggested. For sake of continuity, I will address the limitations of the case studies first and then consider the methodology. One limitation of the cases presented here is that the instructors were graduate students and not faculty members. This distinction may impact the MKT I uncover, since graduate students are generally less-experienced teachers than faculty members and are still learning how to teach. However, studies have shown that experience is actually not a predictor of MKT (Ball, Lubienski, & Mewborn, 2001). It is also important to note that at many doctoral-granting universities, graduate students primarily teach first-year courses, so my sample is representative of the population. Also, my case study is limited in that it considers instructors who have taught a course 3 or more times as experienced. If utilizing a more-traditional K-12 definition of experienced teachers, these instructors would still be considered novices. However, relative to the total population of graduate instructors teaching Precalculus courses, my instructors would be considered experienced.

There are two main limitations that I have identified in the methodological approach I suggest for studying MKT at the undergraduate level. First, one must consider how to handle intended tasks that are not from a written curriculum, intended tasks that are not enacted, and enacted tasks that are not included in the intended lesson plan. Of the 39 instances of task unfolding that I observed last fall, 19 fit into one of these categories. However, I believe that this limitation does not trivialize the analysis. In these cases, the Stage 1 and 2 analyses can still be done if data was collected for at least two of the phases. A second limitation of the methodology is that it is designed under the assumption that MKT only influences the transition between phases. While research does support the fact that it is involved in these transitions (Brophy, 1991, 2001; Stein, Baxter, & Leinhardt, 1990), it is worth considering whether or not any opportunities to examine MKT may be lost by limiting the analysis in this way. Finally, by restricting the unit of analysis to mathematical tasks, the methodology does not consider how MKT might be related to other aspects of teaching, such as designing and providing feedback on assessments. However, if one considers the 19 high-leverage practices (TeachingWorks, 2017) that Ball and colleagues identified as the basic-fundamentals of teaching, task unfolding captures the majority of these practices.

Even with its limitations, the methodological approach I proposed for examining MKT has many strengths. First, the methodology itself is independent of the content and level of instruction. Therefore, it can be used to study MKT K-12 courses, other undergraduate courses besides Precalculus, or even graduate courses. Second, the methodology is flexible in that it can be used to generate formalized theories of MKT or less-formal descriptions and characterizations. Third, utilizing the frameworks for task unfolding and cognitive demand focuses the analysis on specific teaching actions. This is beneficial examining teaching holistically can easily overwhelm a researcher and make it difficult to focus on MKT. However, by utilizing the frameworks of task unfolding and cognitive demand, this methodology provides a structure that begins to reveal the mathematical knowledge involved in the complex work of teaching. Finally, the primary strength of this methodology is that it provides a way to study MKT at the undergraduate level through careful study of the practice of teaching. Instead of utilizing existing frameworks for MKT that were developed at the K-12, it examines MKT at the undergraduate level independently. However, it still draws upon the research and findings at the K-12 level, but in a careful and systematic way that still attends to the unique characteristics of undergraduate instruction.

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Appendix A

Pre-Observation Interview Protocol

1. Have you previously used this task in a class before?
2. Where did this task come from?
3. Did you make any changes to this task?
4. What is the mathematics that you intend students to learn through this task?
    a. Why did you want your students to learn this mathematics?

   b. What about this task made you believe it is an appropriate way to learn this mathematics?

Post-Observation Interview Protocol

1. Did you get to all of the tasks that were in your lesson plan?
   a. Why or why not?
2. Did you change any of the tasks that were in your lesson plan?
   a. Why or why not?
3. During the pre-observation interview, you said that the intended learning outcome for this task was __________________. Was that learning outcome the same during class?
   a. Why or why not?
   b. Do you believe that the intended learning outcome was achieved?
4. During this part of the task, it seemed like you were unpacking the mathematics to make it comprehensible for your students.
   a. What exactly were you trying to unpack?
   b. Why did you decide to unpack this?
   c. How did you determine a way to unpack this?
5. During this part of the task, it seemed like you were making mathematical connections across topics, assignments, representation, or domain.
   a. What exactly were you trying to connect?
   b. Why did you want to connect these things?
   c. How did you make these connections?
6. During this part of the task, it seemed like you removed some mathematical complexity to make it more comprehensible for your students?
   a. What exactly did you remove?
   b. Why did you decide to remove that?
   c. How did you maintain the mathematical integrity of the task?
7. During this part of the task, it seemed like you added some mathematical complexity to make it more challenging for your students?
   a. What exactly did you add?
   b. Why did you decide to add that?
   c. How did you maintain the mathematical integrity of the task?
8. During this part of the task, you elicited and interpreted student thinking.
   a. What response(s) did you anticipate?
   b. Why did you elicit student thinking here?
   c. How did you interpret what the student said mathematically?
9. During this part of the task, you used the following mathematical representation(s).
   a. What exactly were you trying to represent?
   b. Why did you use this (these) representation(s)?
   c. How did you use this (these) representation(s)?

Mathematical Actions, Mathematical Objects, and Mathematical Induction

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Proof by mathematical induction is arguably the most difficult proof technique for students to master. We explain this difficulty within an action-object framework. Specifically, we report on results from clinical interviews with two mathematics majors in which the first author administered tasks designed to elucidate each student's understanding of logical implications as mental objects. We found that the framework explains much of the difficulty inherent in proof by induction, even the students' struggles with hidden quantifiers.

Keywords: Action-object theory, Logical implication, Mathematical induction, Proof

Beyond the difficulties students experience with proof in general (Weber & Alcock, 2004; Mariotti, 2006; Stylianideas, 2007), mathematical induction poses particular challenges (Baker, 1996; Harel, 2002; Michaelson, 2008; Movshovitz-Hadar, 1993; Stylianides, Stylianides, & Philippou, 2007). Proof by induction involves the implication that if a proposition holds for some integer k, then the proposition holds for the integer k+1. Students often conflate this inductive assumption with the assumption that the proof holds for any k (Avital & Libeskind, 1978; Ron & Dreyfus, 2004). Using the notation P(k) to represent the proposition applied to k, there needs to be a distinction between the implication $P(k) \rightarrow P(k+1)$ and P(k) itself.

For example, consider the proposition, P(n): The sum of the first n odd natural numbers is n^2. The proposition holds for n=1, and assuming it holds for some natural number k, we can show that it also holds for k+1. Suppose that $1 + 3 + \cdots + (2k-1) = k^2$; then adding 2k+1 (the next odd number) to both sides of the equation, we get the sum of the first k+1 natural numbers on the left side of the equation, and on the right side of the equation, we get $k^2 + 2k+1 = (k+1)^2$. Thus, we have proven that the proposition holds for n=1 and that, if P(k) is true, then P(k+1) is also true. Therefore, P(n) is true for all natural numbers.

The purpose of this paper is to investigate the cognitive origins of students' difficulties in mastering proofs by induction. More specifically, we apply an action-object theory to the logical implication $P(k) \rightarrow P(k+1)$ in order to study how a complete understanding of induction might develop. Most mathematics majors can prove logical implications (Harel & Sowder, 2007), but proof by induction imposes an additional requirement: the inductive implication, $P(k) \rightarrow P(k+1)$, must be taken as a single object rather than a relation between two objects, P(k) and P(k+1) (Dubinsky, 1986). Our study contributes empirical results to support this claim within a revised action-object framework.

We begin our report with a review of literature on students' difficulties in understanding proof in general and proof by induction. Then we introduce our action-object framework for investigating such difficulties. Next, we describe the tasks we used to investigate students' understandings within that framework. Finally, we report on results that answer the following four questions.

1. How do college mathematics majors understand logical implications?
2. Are action-object distinctions useful in modeling these understandings?
3. How do these understandings contribute to their mastery of proof by induction?

4. What other factors contribute to, or detract from, mastery of proof by induction?

Research on Students' Difficulties with Proofs and Proving

Harel and Sowder (2007) defined a conjecture as "an assertion made by an individual who is uncertain of its truth" (p. 808). Correspondingly, they defined proving as the process of removing doubts about such assertions. Recognizing the reciprocal role that conjecturing plays in proving, Boero, Lemut, and Mariotti (1996) referred to a cognitive unity between these two activities, and several researchers have described ways in which students switch back and forth between conjecturing and proving as they attempt to construct proofs (Arzarello, Andriano, Olivero, & Robutti, 1998; Herbst, 2006; Saenz-Ludlow, 1997; Weber & Alcock, 2004). For example, Cifarelli (1997) found that, "[students'] self-generated hypotheses went hand-in-hand with their conception of carrying out of purposive activity designed to test the viability of their hypotheses" (p. 20).

Harel and Sowder (2007) referred to a second kind of switching, between ascertaining and persuading. The role of persuasion in proof emphasizes its social dimension and subjective nature. In learning to make convincing arguments, students need to do more than ascertain truth for themselves; they must also find ways to convince others. Mathematical communities (such as mathematics classrooms) can specify criteria for convincing arguments. de Villiers (1999) specified six purposes these arguments might serve: verification, explanation, systemization, discovery, communication, and intellectual challenge. Additionally, mathematicians often place value on the aesthetic qualities of a proof (Sinclair, 2006).

To classify ways that students might attempt to ascertain and persuade, Harel and Sowder (2007) identified three broad proof schemes: external, empirical, and deductive. Generally, mathematics educators aspire for their students to progress toward deductive proofs because: (1) unlike external proof schemes, they include personal meaning that relates to ascertaining; and (2) unlike empirical proof schemes, they provide persuasive power via logical explanation (NCTM, 1989). Table 1 summarizes the three broad proof schemes, along with their subcategories.

In contrast to our aspirations, students generally rely on empirical or external proof schemes (Harel & Sowder, 2007), and poor performance in proving persists in college, even among mathematics majors (Selden & Selden, 2003; Weber, 2001). Proofs by mathematical induction pose particular challenges for mathematics students (and teachers), from high school through college (Avital & Libeskind, 1978; Baker, 1996; Ron & Dreyfus, 2004; Stylianides, Sylianides, & Philippou, 2007). In a conceptual analysis, Ernest (1984) speculated several possible reasons for students' difficulty, including their understandings of logical implication in general. Several empirical studies have followed, elucidating the role of such factors.

In a study of elementary and secondary school preservice teachers, Stylianides, Stylianides, and Philippou (2007) identified three specific difficulties underlying students' poor performance with proofs by mathematical induction: (1) understanding the necessity of establishing a base case (usually $n=1$); (2) interpreting the meaning of the inductive step, $P(k)$ implies $P(k+1)$; and (3) accepting that the proposition might hold beyond the cases covered by induction. In discussing the first two difficulties, the researchers cited prior work by Dubinsky (1986) suggesting that, in order to develop a mature understanding of proof by mathematical induction, students need to understand logical implication as an object: "Similar to what we found, many sophomores in Dubinsky's study tried to prove $P(k+1)$ rather than $P(k) \rightarrow P(k+1)$" (p. 162).

Table 1. *Harel and Sowder's (2007) Proof Schemes*

Class	Sub-category	Description
External Proof Schemes		
	Authoritarian	Relies on external authority, such as a text or teacher
	Ritual	Focuses on format (such as two-column proof) over substance
	Non-referential symbolic	Focuses on symbol manipulation rather than underlying concepts
Empirical Proof Schemes		
	Inductive	Relies on measurements from specific examples
	Perceptual	Relies on perceptions of specific examples
Deductive Proof Schemes		
	Transformational	Based on operational thinking and logical inference that generalizes across an entire class
	Axiomatic	A transformational proof scheme that begins from axioms.

We return to Dubinsky's work in the next section as part of a more general discussion of action-object theory. Here, we note that the conflation of the proposition P(k) with the implication P(k) → P(k+1) can lead students to conflate proofs by induction with the fallacy of assuming what is to be proved (Movshovitz-Hadar, 1993). Further, even successful mathematics students have difficulty accepting the truth of the implication without knowing the truth of the proposition itself: "How can you establish the truth of P(k+1) if you don't even know if P(k) is true?" (Avital & Libeskind, 1978, p. 430).

Harel (2002) explicitly connected students' poor performance in mathematical induction to their proof schemes, as characterized by Harel and Sowder (2007; see Table 1). In a study of preservice secondary school teachers enrolled in a college number theory course, Harel (2002) found that students' proof schemes largely fell into the empirical and external categories, particularly the authoritative and non-referential symbolic subcategories of the external proof scheme. Furthermore, he found that students' proof schemes strongly influenced the ways they understood the method of mathematical induction. He argued that students rely on authoritative schemes because they are introduced to the method before they have an intellectual need for it. He suggested a need-driven instructional approach that could build from students' empirical proof schemes toward transformational proof schemes that would support a complete understanding of the method.

The instructional approach utilizes pattern generalization, which is in the purview of the empirical proof scheme. However, the approach emphasizes patterns in the process rather than patterns in the results of that process, supporting a form of reasoning that Harel (2002) calls *quasi-induction*. In the previously shared example of summing odd integers, instruction that supports quasi-induction might involve drawing students' attention to the way one perfect square follows from the previous one, rather than the pattern of perfect squares itself. For instance, $(4+1)^2 - 4^2 = (2 \cdot 4+1)$, and this pattern holds across any pair of consecutive perfect squares so that

$(k+1)^2 - k^2 = (2k+1)$. Because quasi-induction and its process pattern generalizations focus on students' own mental actions rather than empirical observations, it is transformational in the Piagetian sense (Piaget, 1970), and Harel (2002) refers to it as a manifestation of the transformational proof scheme.

Action-Object Framework

Action-object theories of mathematical development derive from Piaget's (1970) genetic epistemology, in which mathematics is understood as a product of psychology: Mathematical objects arise as coordinations of mental actions through a process called reflective abstraction. Within that framework, the enterprise of mathematics education is to specify the mental actions that underlie mathematical objects and how they might be coordinated--composed and reversed--with one another to construct those objects. For example, mathematics education researchers have described how whole numbers, like 5, arise as objects for children through coordinated activities of unitizing, iterating, partitioning, and disembedding (Piaget, 1942; Steffe & Cobb, 1988; Ulrich, 2015).

Dubinsky (1986) adopted a Piagetian perspective to extend action-object theories to advanced mathematics. He developed the APOS framework to explain students' mathematical development through processes of interiorizing *actions* as *processes*, then encapsulating those processes as *objects* that can be acted upon; *schema* organize processes and objects so that students can make sense of mathematical situations. Similarly, Sfard (1991) described the reification of actions as objects, thus distinguishing objects from pseudo-objects. Unlike mathematical objects, pseudo-objects are merely figures or symbols, with no basis in action, so they cannot be de-encapsulated. For example, high school students learn rules for manipulating expressions within algebraic equations, but for many students, the expressions themselves have no reference to underlying actions (Sfard & Linchevski, 1994).

Action-object theories point to two essential features of logico-mathematical development—that students begin to construct new mathematical objects by coordinating their available mental actions and that new mental actions become available for acting on those objects. For example, students can construct the cube as a mathematical object by coordinating mental rotations, and once they have constructed the cube, they can consider new actions, like reflecting the cube about a plane through its center. The double arrow in Figure 1 represents these two essential features.

$$\text{Actions} \rightrightarrows \text{Objects}$$

Figure 1. Actions and objects.

In the domain of proof and proving, we might consider logical implication as a mental action that transforms one assertion into another. In formal logic, this transformation is referred to as *modus ponens* (P implies Q, and P is true, therefore Q is true). It has three kinds of reverse actions: negation (P is true and Q is false); inversion, which relies on the converse of the implication (Q implies P); and *modus tollens* (P implies Q, and Q is false, therefore P is false), which relies on the contrapositive of the implication (not Q implies not P). Whereas the contrapositive of the implication is logically equivalent to the original implication, its converse is not.

“Performance on logical inferences involving modus ponens is usually reasonably good, but performance on those tasks involving modus tollens is weak, as is a full understanding of inferences involving if-then statements” (Harel & Sowder, 2007, p. 826). From a Piagetian perspective, these latter two findings go hand-in-hand. A full understanding of any mathematical object relies on the ability to reason reversibly (Piaget, 1970), so students will not have a full understanding of inferences involving if-then statements until they can reason with modus tollens as well as modus ponens. In other words, a logical implication would arise as a mathematical object for students only after they begin to coordinate modus ponens and modus tollens as reverse actions.

In referring to quasi-induction, Harel (2002) was making an action-object distinction in the development of mathematical induction. The logical implication $P(k) \rightarrow P(k+1)$ begins as an action wherein students have to carry out the transformation from the k^{th} case to then $(k+1)^{st}$ case. True induction arises from the objectification of this action (see Figure 2). “In quasi-induction one views the inference, $P(n-1) \rightarrow P(n)$, just as one of the inference steps—the last step—in a sequence of inferences that leads to P(n). In mathematical induction on the other hand, one views the inference, $P(n-1) \rightarrow P(n)$, as a variable inference form, a placeholder for the entire sequence of inferences” (Harel, 2002, p. 26). Based on this action-object framework, our study focuses on the actions and objects of mathematical induction, including the two sides of the implication, the implication itself, and three ways of reversing it: converse, contrapositive, and negation.

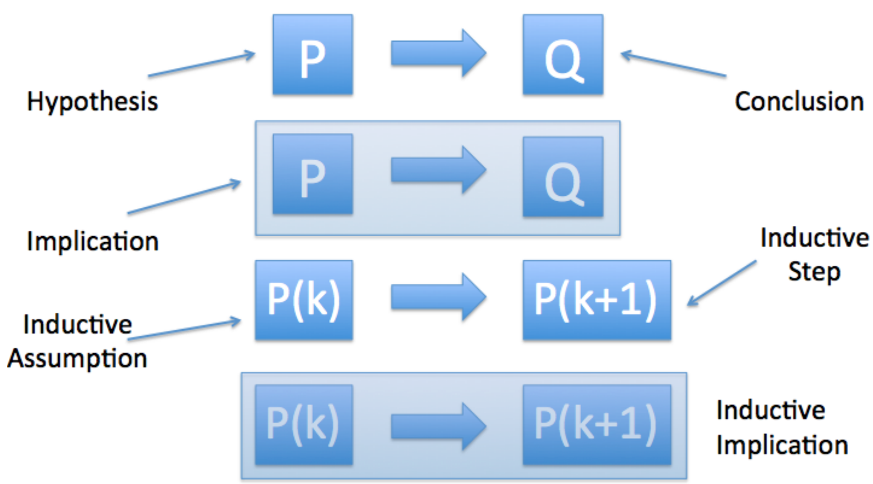

Figure 2. The actions and objects of logical implication.

Methods

To investigate the actions and objects of mathematical induction, the first author conducted clinical interviews with each of two college students, Trevor and Laura, who had completed an *Introduction to Proofs* course, which included instruction on mathematical induction. One student, Trevor, earned an A in the course, and the other student, Laura struggled in the course, earning a grade of C. In this paper, we share results from our analysis of the interview with the higher performing student, Trevor.

The one-hour interview was video-recorded and consisted of tasks designed to elicit the actions and objects those students had available for reasoning with proofs by mathematical induction. In the remainder of this section, we describe those tasks and our video analysis of the interview with Trevor.

Tasks

Interview tasks included three types (see Table 2). Type A tasks were designed to assess student understanding of logical implication. We included questions in both a familiar context (number theory) and an unfamiliar context (homology). In both contexts, students were given a statement and asked to provide truth values for its converse, its contrapositive, and its negation.

Type B tasks assessed student understanding of the components of mathematical induction (e.g., P(1) and P(k)) and how they might support an inductive proof. Type C tasks assessed student ability to construct a formal proof, both in general and by induction. Sample tasks are listed in Table 2.

Table 2. Sample interview tasks.

<table>
<tr><th>Task Type</th><th>Sample Task</th></tr>
<tr><td>A: Logical implication</td><td>Suppose the statement S is true. Evaluate whether the statements (a)-(c) are true, false, or uncertain.
1. S: If two topological spaces are homeomorphic, their homology groups are isomorphic.
a. If two topological spaces have isomorphic homology groups, the spaces are homeomorphic.
b. If the homology groups of two topological spaces are not all isomorphic, the spaces are not homeomorphic.
c. There is a pair of homeomorphic topological spaces whose homology groups are not all isomorphic.
2. S: Every even natural number can be written as the sum of two prime numbers.
a. If a number is not the sum of two primes, it is odd.
b. There is an even number that is not the sum of two primes.
c. If a number is odd, it is not the sum of two primes.</td></tr>
<tr><td>B: Induction components</td><td>Each of the following scenarios relates to a proposition P(n), where n is a positive integer. Decide whether: (a) the given information is enough to prove P(n) without induction (i.e., induction is not necessary); (b) the given information is enough to prove P(n) with induction; or (c) or the given information is not enough to prove the proposition.
1. P(1) is true; there is an integer $k \geq 1$ such that P(k) is true.
2. P(1) is true; there is an integer $k \geq 1$ such that $P(k) \rightarrow P(k+1)$.
3. P(1) is true; for all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.</td></tr>
<tr><td>C: Non-inductive formal proof</td><td>Let k be an integer. Prove that if 36 divides k, then 81 divides k^2.</td></tr>
<tr><td>D: Inductive formal proof</td><td>Prove that beginning with zero, every third even number is divisible by 6.</td></tr>
</table>

During each interview, the first author posed tasks to the student one at a time by handing the student a slip of paper. The student was given paper to write notes and was provided opportunities to ask clarifying questions. After the student's response to each task, the interviewer would ask follow-up questions, probing the student's reasoning. For example, for Type B tasks (see Table 2), the interviewer might asked the student what additional information (s)he would need in order to show that the proposition would hold for all positive integers.

Video Analysis

Each researcher independently engaged in video analysis through an action-object lens. In this initial analysis, we focused on collecting facts about student understanding from each task type, without consideration of how performance on tasks of one type predicted performance on tasks of another type. We analyzed the students' spoken explanations for the Type A tasks to assess if they understood logical implication as an object, or if they instead viewed implication as an action between two objects, the hypothesis and the conclusion. For Type B tasks, we inferred which components of mathematical induction the students objectified and what actions they could perform on those objects. Finally, for Type C tasks, we evaluated the students' success in proving statements with and without mathematical induction.

In a second iteration of analysis, the researchers jointly considered how well the action-object framework explained student responses and how the students' performance on tasks of one type predicted their performance on later tasks. In particular, we looked for connections between the students' conceptualization of logical implication and their success in proof by mathematical induction. More specifically, we examined how the students' performance on Type A tasks predicted their objectification of the components of induction and the subsequent actions the student could perform on these components in Type B tasks. We considered whether students' objectification of logical implication and the components of mathematical induction (Type A and Type B tasks) explained additional challenges they experienced in proof by mathematical induction when compared to proof without induction (Type C tasks). In light of previous research, we considered explanations alternative to action-object theory for these differences. For example, what challenges did students experience in the components of mathematical induction because of hidden quantifiers (Shipman, 2016)? Selected transcription was used to support our analyses.

Results

We focus on the results of our video analysis of the interview with the higher-performing student, Trevor. Trevor seemed to treat logical implication as an action on two objects (P and Q), rather than a single object itself ($P \rightarrow Q$). This was evidenced throughout the interview, but particularly when he explained his reasoning for his responses to Type A tasks.

As we outline below, we infer from Trevor's spoken and written reasoning that he conceptualizes the negation, converse, and contrapositive of the implications in Task A via transformations on the objects of the implication, the hypothesis and conclusion. The more complicated the transformation process became, the more Trevor struggled with assessing the validity of the new statement.

The easiest transformation for him was by far the negation. In Task A1c, Trevor was asked to state the negation of the implication $P \rightarrow Q$ as a follow up to his response that the statement was false. Trevor replied, “the negation is P implies not Q … if you’ve already violated your assumption by saying that P is false then it doesn’t really matter what happens to Q because you don’t really care because P isn’t true.” He also wrote “P → ~Q” on his interview paper. Trevor’s explanation of why he believes the negation is P → ~Q seems to indicate the following thought process. To negate the implication, he first considers negating P and Q individually. We infer that Trevor is considering whether ~P → ~Q could be the negation. He concludes that P should not be negated, misusing the fact that an implication is vacuously true when P is false. Thus, he arrives at his conclusion that the negation should be P → ~Q. In considering the negation of the hypothesis and conclusion separately, Trevor treated implication as an action on objects, rather than an object itself. And, because Trevor believed the negation was P → ~Q, he needed only to construct ~Q as an object to assess the truth value of the entire statement.

Determining the truth value of the converse statements in Task A was slightly trickier for Trevor because it involved a reverse transformation of the implication. When considering the converse in Task A1a, Trevor began by writing out and separating the two statements in the implication (see Figure 3). He then claimed that the statement was uncertain, using the following justification: “Just because you know that the forward direction is true, there’s nothing implying that the reverse direction is true, in this case.” Prompted for a term to describe the statement in question, Trevor correctly labeled it as the converse statement. These responses indicate that Trevor treated the original statement in two parts, with implication as an transformation between them--a transformation that he could reverse. His representation of the original statement seemed to support his reasoning in comparing the reverse (converse) statement to the original.

Figure 3. Trevor’s representation of the original implication in Task 1a.

In response to Task A2c, Trevor paused for about 12 seconds, looking at the statement in question. Finally, he responded as follows:

> That one I’m going to say ‘uncertain’ because this one [pointing to the original statement] just says that if you have a naturally even number then you can express it as the sum of two primes. But it doesn’t [flips right hand over] flip the… Inverse statement isn’t true necessarily, saying that, if a number is odd, it’s not the sum of two primes. But… I feel like I’m drawing outside information into saying this next part, but being a prime number, you can’t be an even number…

Trevor went on to explain (based on his assumption that all primes are odd) that the sum of two primes will always be even. Thus, he justified the truth of the converse statement based on the context of the task. He knew that, in general, the converse would not necessarily follow from the original implication. However, once again, determining the relationship between the original statement and its converse required Trevor to perform a "flip" wherein he treated the two sides of the implication as separate objects.

While Trevor seemed to recognize the statements in tasks A1a and A2c as the converse of each implication relatively quickly, Trevor did not recognize the contrapositive statements in both tasks of Type A without significant prompting. In fact, the most challenging Type A tasks for Trevor were the contrapositive statements because they appeared to require him to combine two transformations, reversal and negation, on the objects of the implication. We note that this additional transformation, as compared to his handling of the converse, created substantially more struggle for Trevor. This is evidenced by Trevor's responses to the Task A1b, where the mathematical context was unfamiliar. After several minutes of little progress and after the first author prompting him to see the relationship between the statement *S* and its contrapositive, Trevor stated, "It's the negation of the reverse order, which is true in general. I do remember that… I don't remember the word but, 'If P then Q is true', then 'not Q implies not P' is generally true." Trevor's explanation indicates the two transformations, reversal and negation, he performs to obtain the contrapositive from *S*. Further, like in Task 1Ac, Trevor uses the term "negation" to mean the negation of each individual statement P and Q. If he was indeed referring to the "negation of the reverse order," he would have $\sim(Q \rightarrow P)$ and hence Q and $\sim P$, which is not the contrapositive of $P \rightarrow Q$.

In Task A2a, Trevor had to again determine the truth value of the contrapositive of statement *S* in the familiar setting. Despite having just solved Task A1b, he still did not recognize the statement as the contrapositive and struggled to determine its validity. This supports our previously mentioned inference that the additional transformations necessary for Trevor to mentally construct the contrapositive are enough to blur his connection of the truth value of *S* to its contrapositive. Further, Trevor relied on his knowledge of primes to reason through his answer, ignoring the logical equivalence connection altogether. Rather than viewing the entire implication as an object that could be manipulated, he focused on the meaning of the hypothesis and conclusion separately. Consequently, we infer that Trevor's reliance on mathematical context demonstrated that he was not viewing the logical implication as a single object, rather an action on objects.

Contrasting Trevor's performance on Type A tasks in the unfamiliar versus familiar mathematical setting reveals the mental actions and objects that Trevor seemed to have available. First, because the hypothesis and conclusion of statement *S* in the unfamiliar setting (Task A1) did not carry mathematical meaning for Trevor, he seemed to conceptualize them rather quickly as pseudo-objects (Sfard, 1991). He did not devote time to framing the statements as objects with mathematical meaning. Therefore, he was more able to perform his transformations on *S* that were necessary for him to construct each new statement in Tasks A1a-c. However, in the familiar setting, Trevor got stuck trying to first construct the pieces of the implication as mental objects because they carried mathematical meaning that he believed he could unpack. As a result, he was delayed in the process of carrying out his transformations. This was particularly evidenced in Trevor's performance on Task A2a when he struggled to execute his two step action sequence of transforming the implication *S* into its contrapositive.

Next, Trevor's responses to the Type B tasks indicated that he understood how to combine components of an inductive proof, but he seemed to rely heavily on a procedure learned in class. In this way, he seemed to rely on an authoritarian proof scheme (Harel & Sowder, 2007, see Table 1). In particular, he treated induction as a sequence of objects: a base case, an inductive assumption, and an inductive step. Like his treatment of logical implication in the Type A tasks, Trevor separated the inductive implication into two objects—the assumption and the step. This reasoning was evidenced in his response to Task B2 in Table 2 where he was given P(1) and the existence of an integer k for which P(k) implies P(k+1).

> We have everything we need to do induction on this case because we have a base case that P(1) is true. And then can lump P(k) into our inductive assumption and then use P(k) implying P(k+1) to form our inductive step and follow that all the way through to all of the natural numbers.

These criteria helped him successfully distinguish which scenarios could generate a proof by induction, but a conceptual limitation became apparent. Specifically, it was clear that Trevor viewed induction as a connected sequence of three objects as follows. Trevor first checked for a base case, object one. He then linked the base case to his second object, the inductive assumption, by checking that k began at 1 for the remaining information. Once he was satisfied that this connection existed, he was able to consider the final object, the inductive step. Because of this ordered thought process, Trevor did not seem to notice that the inductive implication was missing from Task B1. In what follows, Trevor was concerned with making sure that k began at 1. He argued that if k were 3, he would still have a kind of inductive step but that the step size would be 2 instead of 1.

> We can say that P(k) is true for this given integer, but we don't really know where k is, so I don't think we can construct an inductive argument because we don't know where k is relative to 1. But if k is 3, for example, we don't know what happens at 2, and so we haven't proved it for all of the natural numbers. So even if we were to say like skip two steps, then we leave out all of the evens, for example.

We contrast this to Trevor's performance on Task B2 where he was given P(1) and that there exists an integer k≥1 such that P(k) implies P(k+1). Here, overlooking the existential quantification of k (which we address below), Trevor was satisfied with his initial action of joining the base case to the inductive assumption and completed his object sequence as follows.

> We have everything we need to do induction on this case because we have a base case that P(1) is true. And then can lump P(k) into our inductive assumption and then use P(k) implying P(k+1) to form our inductive step and follow that all the way through to all of the natural numbers.

Trevor's above response to Task B2 also surfaced a new issue in successful proof by induction: a student's ability to recognize the role of quantification in the inductive implication. Initially, Trevor overlooked the quantification of k in Tasks B1 and B2. His sequencing of inductive objects led to a correct conclusion (but for the wrong reason) that Task B1 did not have enough information, and an incorrect conclusion that Task B2 had enough information for proof by induction. However, upon seeing Task B3—which gave P(1) and that for *all* integers k≥1, P(k) implies P(k+1)—and after much prompting from the interviewer, Trevor exposed the role of quantification. He then went back to Task B2 and said there was not enough information to use induction.

While Trevor's object sequence may not lead to a mastery understanding of induction, it did allow for him to systematically outline the necessary components in a proof by induction. For the Type C task shown in Table 2, Trevor had no trouble establishing base cases, and he seemed to understand the purpose of induction.

> We can establish as many base cases as we want… 0 can be represented as something times 6; 6 can be represented as something times 6; so can 12; 18; all the way up. So you're going to have to use induction because you can't really prove every number, without sitting here and writing them all out. So you know that the inductive assumption is going to say that, assume for all numbers, k, greater than or equal to 0, but…

He struggled in establishing an inductive step and consequently completing the proof of the inductive implication: "I know what argument I want to make, but I'm not sure how to make it." His difficulty stemmed from an inability to formulate a useful representation of every third even integer. However, when prompted about what P(k+1) meant in this case, Trevor clearly understood, and also confirmed, that "k+1" did not literally mean add 1 to k. He proposed that "k+1" really meant k+6 in this case, and when asked why, he replied "because [k] is just an arbitrary number and you want to prove that the *next one* [is true]." So, Trevor did seem to understand how the inductive step should work, and arguably he could have been successful in his proof without this formulation issue.

Conclusions and Implications

Prior research identified several potential hurdles in students' mastery of proof by induction. Among these, Stylianides, Stylianides, and Philippou (2007) highlighted understanding the necessity of establishing a base case and interpreting the meaning of the inductive step. Neither student in our study demonstrated any difficulty in understanding the necessity of the base case. In fact, Trevor consistently included the base case as a critical component for inductive proofs. However, both students misconstrued the meaning of the inductive step, and the action-object framework was especially helpful in explaining why. Additionally, we uncovered a new issue not captured by prior research: students' struggles with the role of quantification in proofs by induction. We believe that our action-object framework can also be used to explain this struggle.

In line with research on proof in general (Harel & Sowder, 2007; Selden & Selden, 2003; Weber, 2001), the two students in our study relied on external proof schemes to make inductive arguments. Still, a student's ability to follow a procedural sequence of objects (base case, then inductive assumption, followed by inductive step) without a mastery of induction can allow for successful proof by induction. Separating the inductive implication into two objects, P(k) and P(k + 1), makes the process more accessible to the typical student because the typical student already handles implications in pieces (Avital & Libeskind, 1978; Dubinsky, 1986; Movshovitz-Hadar, 1993). Thus, students who have not constructed logical implications as objects can write successful proofs. However, they do not have a complete understanding for how the process works. When the inductive proof calls for some modification to the standard format (as arose in Trevor's struggles to formulate P(k + 1)) students can become confused about how to proceed.

Our results support findings from prior studies indicating that treatment of logical implications is a major mediator in students' understandings of proof by induction (Ernest, 1984; Dubinsky, 1986). Our study also affirms limitations on students' treatment of logical implications–even among high performing students like Trevor (Harel & Sowder, 2007). We consider some ways to address these limitations using our action-object framework, but first we consider the unanticipated limitation regarding students' treatment of hidden quantifiers.

Both of the students we interviewed struggled to recognize quantifiers in the statements of Type B tasks. When Trevor was asked whether there was a difference between Tasks B2 and B3 (see Table 1), he replied that they were the same. When asked if he was sure, he noticed that one sentence had the words "such that" and again overlooked the quantifiers. Other researchers have noticed students' difficulty in accounting for hidden quantifiers in proving mathematical statements (Seldon & Seldon, 1995; Shipman, 2016). In particular, Barbara Shipman discussed just how prevalent this issue of hidden quantifiers is among students and how it leads to errors in logic in proof by contradiction. The inductive implication $P(k) \rightarrow P(k + 1)$ is an implication with hidden quantifiers. $P(k)$ and $P(k + 1)$ are open statements that have no truth value until k is quantified. What we really mean when we write $P(k) \rightarrow P(k + 1)$ is "for all $k \geq 1$, $P(k) \rightarrow P(k + 1)$." Students often suppress the significance of the hidden quantification of k in the inductive implication, and consequently in their conceptualization of proof by induction.

In proof by contradiction, Shipman (2016) noted that failure to recognize hidden quantifiers can lead to correct conclusions for the wrong reason, or incorrect conclusions about the validity of a statement. She also noted that oftentimes students' mistreatment of quantifiers leads to the erroneous proof of a "for all" statement by example. In our study, we found that Shipman's observations also appear to hold true in the context of proof by induction. In Task B1, Trevor came to the correct conclusion that more information was needed but for the wrong reason. He bypassed the hidden quantification of k and focused on whether $P(k + 1)$ was true. In Task B2, Trevor's oversight of the quantification of k led to an erroneous induction proof by example. Trevor conflated showing the inductive implication was true for one k with showing the implication was true for all k. We conclude that students might be able to complete the proof of the unquantified inductive implication $P(k) \rightarrow P(k + 1)$ by breaking it down into procedural steps. However, in the absence of a memorized, quantified inductive assumption, their proofs by induction are not quite logically complete.

Students' struggles with the role of the quantification of the inductive implication in creating a logically complete proof by induction may be related to their construction of implication as a mental object. In particular, dissecting quantified statements in the context of mathematical induction places an increased cognitive demand on students that is less easily navigated if a student conceptualizes an implication as an action on objects. Because a student must work with additional objects when they are unable to mentally construct the implication as a single object, the student's cognitive resources available for addressing quantification are reduced. Our claim is supported by Trevor's performance on problems in his introduction to proofs course where the sole focus was quantification. For example, on the first exam, Trevor was given the following two statements and asked to label them as true or false and justify his answer.

1. There exists a real number x such that for all real numbers y, $2x - 3y + 7 = 14 - 6y$.
2. There exists a real number x such that for all real numbers y, $xy + 3x = 2y + 6$.

Trevor earned a perfect score on this problem, showing no problems with understanding quantifiers, even when mixed. Thus, his struggles with quantification during his clinical interview were unexpected. We speculate that Trevor's treatment of logical implication as an action on objects was an inhibiting factor.

Our study indicates that constructing logical implications as objects and identifying hidden quantifiers are prerequisite knowledge for developing transformational proof schemes for mathematical induction. Thus, similar to Harel (2002), we consider instructional activities that should be included in Proofs courses, leading into formal instruction on mathematical induction. Harel had suggested introducing quasi-induction as a means of focusing students' attention on the logical implication that related the inductive assumption to the inductive step, by explicitly relating P(k) to P(k + 1) for specific values of k. Results from our study attest to the value of that approach, assuming it supports the objectification of the implication, in general.

Tasks of Type A (see Table 1), in addition to their value in assessing whether students have constructed logical implications as objects, might also support that construction as students are challenged to transform a given logical implication into other forms (negation, converse, and contrapositive). Furthermore, our study suggests that instructors should give attention to how students handle hidden quantifiers, and tasks of Type B might reinforce the role of hidden quantifiers in proofs by induction. We recommend further study to test the efficacy of these different task types in supporting students' development of transformational schemes for proof by induction.

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Examining Students' Procedural and Conceptual Understanding of Eigenvectors and Eigenvalues in the Context of Inquiry-Oriented Instruction

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This study examines students' reasoning about eigenvalues and eigenvectors as evidenced by their written responses to two open-ended response questions. This analysis draws on data taken from 126 students whose instructors received a set of supports to implement a particular inquiry-oriented instructional approach and 129 comparable students whose instructors did not use this instructional approach. In this chapter, we offer examples of student responses that provide insight into students' reasoning and summarize broad trends observed in our quantitative analysis. In general, students in both groups performed better on the procedurally oriented question than on the conceptually oriented question. The group of students whose instructors received support to implement the inquiry-oriented approach outperformed the other group of students on the conceptually oriented question and performed equally well on the procedurally oriented question.

Key words: eigenvalues, eigenvectors, linear algebra, inquiry-oriented, student thinking

Linear algebra is a mandatory course for many science, technology, engineering, and mathematics (STEM) students. The theoretical nature of linear algebra makes it a difficult course for many students because it may be their first time to deal with abstract and conceptual content (Carlson, 1993). Carlson (1993) also posited that this difficulty arises from the prevalence of procedural and computational emphases in students' coursework prior to linear algebra, and that it might therefore be difficult for students to connect new linear algebra topics and their previous knowledge. To address this issue, researchers have developed instructional materials for Inquiry-Oriented Linear Algebra (IOLA; http://iola.math.vt.edu/) and strategies to help students develop more robust, conceptual ways of reasoning about core topics in introductory linear algebra (e.g. Wawro, Rasmussen, Zandieh, & Larson, 2013; Zandieh, Wawro, & Rasmusen, 2016; Andrews-Larson, Wawro, & Zandieh, 2017).

Instructors who were not involved in the development of these kinds of research-based, inquiry-oriented instructional materials have been shown to encounter challenges when implementing such materials (Johnson, Caughman, Fredericks, & Gibson, 2013). Under an NSF-supported project Teaching Inquiry-Oriented Mathematics: Establishing Supports (TIMES), Johnson, Keene, & Andrews-Larson (2015) designed and implemented a system of instructional supports based on research in instructional change in undergraduate mathematics education and teacher learning and professional development in settings ranging from K-20 (e.g. Henderson, Beach, & Finkelstein, 2011). These supports included sequences of student activities with implementation notes, a three-day summer workshop, and weekly online workgroups during the semester instructors implemented the materials in their teaching. This chapter examines differences in performance and reasoning of students whose instructors received these supports through the TIMES project (TIMES students) as compared to students whose instructors did not receive these supports (Non-TIMES students). In particular, we examine assessment data to identify differences in student performance and reasoning about eigenvectors and eigenvalues.

In this work, we draw on data from an assessment that was developed to align with four core introductory linear algebra concepts addressed in the IOLA instructional materials: linear independence and span; systems of linear equations; linear transformations; and eigenvalues and eigenvectors (Haider et al., 2016). In the assessment, there were two questions that addressed eigenvalues and eigenvectors: question 8 and 9. Question 8 was a procedurally oriented question related to the eigenvalue of a given matrix and question 9 focused on conceptual understanding of the eigenvectors. The research questions for this analysis are:

- How does the performance of students whose instructors received TIMES instructional supports for teaching linear algebra compare to the performance of other students?
- How did students reason about eigenvectors and eigenvalues in the context of questions designed to assess aspects of student's procedural and conceptual understanding? How did reasoning differ for students of TIMES versus Non-TIMES instructors?

Literature

Linear algebra is a course in which students struggle to develop conceptual understanding (Carlson, 1993; Dorier & Sierpenska, 2001; Dorier, Robers, Robinet & Rogalski, 2000). Across the literature on the teaching and learning of eigenvalues and eigenvectors, procedural thought processes feature prominently. For example, Stewart and Thomas (2006) pointed to ways in which students often learn about eigenvalues and eigenvectors, where a formal definition is often linked to a symbolic presentation and its manipulation. For the purpose of this paper, we will draw on the following formal definition for eigenvectors and eigenvalues:

> Suppose A is an nxn real-valued matrix and x is a non-zero vector in $\mathbb{R}^n$. We say the vector x is an eigenvector of the matrix A if there is some scalar λ such that $Ax = \lambda x$. Further, in this case, we say that λ is the eigenvalue associated with the eigenvector x.

Thomas & Stewart (2011) highlighted a difficulty students find when faced with formal definitions for eigenvalues and eigenvectors: these definitions contain an embedded symbolic form ($Ax = \lambda x$), and instructors often move quickly into symbolic manipulations of algebraic and matrix representations such as transforming $Ax = \lambda x$ to $(A - \lambda I)x = 0$. Their findings that students struggle to make sense of formal definitions, struggle to make use of geometric representations of eigenvectors, and exhibit procedural orientations toward eigenvectors suggest that such treatments might not provide sufficient opportunities for students to make sense of the reasons behind these symbolic shifts (Thomas & Stewart, 2011).

Schoenfeld (1995) used eigenpictures in the 2x2 case ("stroboscopic" pictures) to show x and Ax at the same time by using multiple line segments on the x-y-axis. He observed that graphical representations of eigenvalues and eigenvectors got little attention in the literature and that a picture may benefit more than algebraic presentations. It is also documented more generally in linear algebra that students struggle to coordinate algebraic with geometric interpretations (e.g. Stewart & Thomas, 2010; Larson & Zandieh, 2013) and the students' understanding of eigenvectors is not always well connected to concepts of other topics of linear algebra (Lapp, Nyman, & Berry, 2010).

To support students in developing a better understanding of the formal definition and associated interpretations of the eigenvalues and eigenvectors, researchers have developed a variety of instructional interventions (e.g. Tabaghi & Sinclair, 2013; Zandieh, Wawro, & Rasmusen, 2016). This paper examines student learning outcomes associated with the geometrically motived instructional approach detailed in Zandieh, Wawro, & Rasmussen (2016) when paired with TIMES instructional supports; the approach will be described in Data Sources & Study Context.

Theoretical Framing

Researchers often make reference to conceptual understanding and procedural understanding when discussing students' reasoning about mathematical concepts (Hiebert, 1986). Hiebert and Lefevre (1986) defined conceptual knowledge as a "knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relatonships are as prominent as the discrete pieces of information" (pp. 3-4). According to Hiebert and Lefevre (1986) students have procedural knowledge if they can combine the formal language and symbol representation systems with algorithms or rules in order to complete mathematical tasks.

In this paper, we also draw on Larson and Zandieh's (2013) framework for students' mathematical thinking about matrix equations of the form $Ax = b$. This framework details three important interpretations, relationships between geometric and symbolic representations within each interpretation, and the complexity entailed in shifting among interpretations. The interpretations are: (1) a linear combination interpretation, in which b is viewed as a linear combination of the column vectors of A with x functioning as the set of weights on the column vectors of A, (2) a system of equations interpretation in which x is viewed as a solution and A is seen as a set of coefficients, and (3) a linear transformation interpretation in which x is viewed as an input vector, b as an output vector, and A as the matrix that transforms x into b.

We argue these interpretations are helpful for making sense of students' reasoning, but that the framework may need to be modified or expanded to more fully account for student reasoning in the context of eigenvalues and eigenvectors. In the context of eigenvectors and eigenvalues, students need to coordinate a transformation interpretation with the equation $Ax = \lambda x$, where the matrix A transforms the vector x by stretching, shrinking, and/or reversing the direction of vector x. Additionally, students need to shift to a systems interpretation and consider when the equivalent system $(A - \lambda I)x = 0$ has a non-trivial solution in order to make sense of standard procedures for computing eigenvalues and eigenvectors.

Data Sources & Study Context

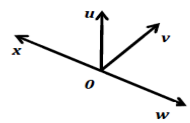
In previous work, we have developed an assessment aligned with the inquiry-oriented linear algebra (IOLA) instructional materials used in the TIMES project (Haider et al. 2016). This paper-and-pencil assessment consists of 9 items, most of which include an open-ended response component. The assessment was administered at the end of the semester, and students were allocated one hour to complete the assessment.

In this analysis, we examine assessment data from 126 students across six TIMES instructors and 129 students across three Non-TIMES instructors from different institutions in the US. Non-TIMES linear algebra instructors were selected from either the same institutions as TIMES instructors or a similar institution (e.g. preferably one from a similar geographic area, with similar size of student population, with similar acceptance rate) to collect assessment data for comparison of TIMES and Non-TIMES students. In this study, we focused on an in-depth analysis of students' reasoning on the assessment questions related to eigenvalues and eigenvectors. Both items are shown in Figure 1.

8. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$? Why or why not?

9. Suppose the vector x is a real-valued eigenvector of the matrix M and that the entries of M are also real-valued.

a. What could be the result of the product Mx? (Check all that apply.)



- ☐ i: Mx could be u
- ☐ ii: Mx could be v
- ☐ iii: Mx could be w
- ☐ iv: Mx could be $\mathbf{0}$
- ☐ v: Mx could be x
- ☐ vi: None of the above

b. Explain your reasoning for your choice(s) in part a.

Figure 1. Assessment items related to eigenvectors and eigenvalues[1]

The inquiry-oriented approach to learn eigenvalues and eigenvectors associated with this study is characterized in detail elsewhere (Zandieh, Wawro & Rasmussen, 2016).

Methods of Analysis

To answer our research questions, our analysis has two main components. The first component of our analysis is quantitative in nature, as we aim to compare learning outcomes of students whose instructors received TIMES instructional supports to those who did not. The second component of our analysis is qualitative in nature, as we work to identify students' ways of reasoning on both the more procedurally oriented assessment item (Q8) and the more conceptually oriented item (Q9). We follow Kwon, Rasmussen & Keene's (2005) approach for distinguishing assessment items that are conceptually oriented from those that are procedurally oriented. In particular we consider Q8 to be more procedurally oriented in that there is a commonly taught procedure that students can directly invoke (with some interpretations) to produce a correct answer to the question. There is no such standard procedure for Q9, so we consider it to be more conceptually oriented. In our qualitative analysis, we also look for similarities and differences that emerge from considering the two groups.

To facilitate our quantitative analysis, we needed to score students' responses to the two assessment items. Specifically, we needed to develop a uniform system for assigning a number of points to students' responses that provide an overall assessment of the quality of their response and the understanding reflected in that response. Question 9a required students to select which subset of 6 possible options were appropriate responses, so 1 point was awarded to each of the possible options for correctly selecting or not selecting that option. Both Question 8 and Question 9b were open ended response questions, and both of these were scored on a scale of 0 to 3 points. The condensed version of the grading scheme for assigning points to open ended response questions can be found in Appendix A. Additionally, Appendix A includes some explanation of

[1] Question 9 was retrieved from http://mathquest.carroll.edu and developed as part of an NSF-supported project entitled Project MathVote: Teaching Mathematics with Classroom Voting. For related research, see Cline, Zullo, Duncan, Stewart, & Snipes (2013).

how this grading scheme aligns with our coding categories for *how* students reasoned, which are described in greater detail below. Student work exemplifying common ways of reasoning with explanation of points awarded are provided in the findings section.

To ensure agreement regarding points assigned to each response, two researchers looked at every student's attempt and assigned a score independently before comparing with each other. If the two researchers assigned a different score to a particular student, they then discussed according to the codebook and agreed on a common score for that student. If both researchers disagreed about a particular score, then a third researcher was consulted to reach a consensus.

Once scores had been assigned to all student responses, descriptive statistics were generated to examine the overall performance of students on the eigenvalue and eigenvector questions and to compare TIMES students with Non-TIMES students for both questions. We were unable to control for factors such as students' mathematical background, major, and instructor's teaching experience, so this is an unavoidable limitation for our statistical analysis. However, we tried our best to choose TIMES and Non-TIMES students either from same school or from similar schools. This helps us establish similarity of students in TIMES and non-TIMES classrooms. Hence, we compared the mean scores of TIMES and Non-TIMES students using two-tailed t-tests to identify when differences of means were statistically significant.

In order to facilitate our qualitative analysis of students' reasoning, we examined student responses to the open-ended portions of question 8 and question 9. After examining the data several times and refining the categories of the students' reasoning about item 8, we sorted students' responses into 5 broad categories: (1) reasoning about the determinant, (2) reasoning about $A - \lambda I$ without computing a determinant, (3) other, (4) students who explicitly indicated they did not know, and (5) who left the item blank. Details about the categories are in Table 2.

In examining students' responses to question 9, we found it helpful to distinguish responses that were conceptually aligned with the formal definition for eigenvectors and eigenvalues from those that were not. We were specifically interested in student reasoning that appropriately coordinated interpretations of $A, x,$ and λ in the context of the matrix equation $Ax = \lambda x$. In particular, we say a student response "uses the eigen-concept" when there is evidence a student is coordinating $M, x,$ and λ in at least one of the following ways:

- Algebraically: The matrix M is a fixed matrix that transforms the (nonzero) eigenvector x in a particular way, namely such that the resulting vector Ax is a scalar multiple (λ) of x.
- Geometrically: this can be interpreted to mean that multiplying x by A has the effect of
    - stretching x in the same direction or opposite direction, or
    - causing the resultant vector to lie along the same line as the vector x.

If a student drew on a transformation interpretation to make sense of Ax but did not coordinate this appropriately with λx in one of the ways mentioned above, we did not say that the student's response used the eigen-concept.

We grouped students' responses to question 9 into five categories: (1) responses that used the eigen-concept, (2) responses that focused on the role of the matrix M in a way that did not use the eigen-concept, (3) other, (4) responses in which the student explicitly indicated he or she did not know, and (5) responses that were left blank. Details about the categories are in Table 3.

After coding students' responses to Q8 and Q9, we aggregated these responses into tables, organized by the category assigned to each response and number of points awarded. We also separated TIMES from Non-TIMES students in counting the number of responses in these discrete categories. This allowed us to look for patterns in which approaches were conceptually oriented, which approaches lent themselves to arriving at correct answers, and differences in approaches taken by TIMES and Non-TIMES students.

Findings

In order to answer our research question about how TIMES students compared to Non-TIMES students, we first present our quantitative analysis of students' performance on questions (Q8) and (Q9), separating students of TIMES instructors from students of Non-TIMES instructors. We then summarize findings from our coding of students' approaches to these same questions, providing examples of responses that highlight important trends in student reasoning.

Overview of differences in student performance

We highlight three central trends from our quantitative analysis. First, TIMES students outperformed Non-TIMES on both items, with a strongly significant difference of means on the conceptual item. Second, both TIMES and Non-TIMES students did better on the procedurally oriented item than on the conceptually oriented item. Third, correlations between students' performance on both the conceptual and procedural items were weak for students in both groups, suggesting that the two items assessed relatively different aspects of student understanding. Note that the last trend is not part of answering our research questions, it is more of a side observation that emerged from our quantitative analysis and what we take it to mean.

To compare the performance of TIMES students with Non-TIMES students, we first computed the mean and standard deviation for question 8, question 9a and 9b. To make a 'cleaner' comparison, we have separately included the mean and standard deviation of part a and part b of question 9. We also compared question 8 with question 9b as they are naturally comparable items.

The data presented in Table 1 showed that on the procedurally oriented question (Q8) the mean score of TIMES students was greater than that of Non-TIMES students, but this difference of means was not statistically significant with the available sample size. Similarly, there was not a statistically significant difference in means on question 9a. However, in comparing the performance of students in both groups on question 9b, we noticed that TIMES students performed significantly better than the Non-TIMES students. The results of t-test indicated that this difference of means was statistically meaningful. In this way, TIMES students outperformed Non-TIMES students on the conceptually oriented question.

Question	All Students	TIMES Students	Non-TIMES Students	p-value (two-tailed)
Q 8 3 Points	Mean: 1.85 SD: 1.31	Mean: 1.98 SD: 1.24	Mean: 1.71 SD: 1.37	$t(125) = 1.73$ $p = .08 > .05$
Q 9 (part a only) 6 Points	Mean: 3.73 SD: 1.68	Mean: 3.74 SD: 1.76	Mean: 3.71 SD: 1.61	$t(249) = 0.56$ $p = .88 > .05$
Q 9 (part b only) 3 Points	Mean: 0.79 SD: 1.03	Mean: 1.05 SD: 1.12	Mean: 0.54 SD: 0.86	$t(125) = 4.29$ $p < 0.001$

Table 1: Summary of results of quantitative analysis

Overall, students performed better on the procedurally oriented question (Q8) than the conceptually oriented question (Q9). We compared Q8 to Q9b and found that the difference of means for all students between Q8 and Q9b was also statistically meaningful with p-value (two-tailed) less than 0.001.

Since both problems we investigated in this study were related to eigenvectors and eigenvalues, one might think that students' performance on the two items should be correlated. However, quantitative analysis revealed a positive but weak correlation between students' performance on the two questions, the Pearson correlation coefficient r = 0.30 for all students. Recall that a correlation coefficient measures the degree of relationship between two variables and ranges from -1 to 1, where the sign indicates the direction of the relationship and the distance from zero indicates the strength of the relationship (e.g. 1 means the two variables are highly correlated and 0 means there is very little or no correlation between the two variables). For TIMES students, the correlation between the two items was r = 0.36 as compared to the correlation for Non-TIMES which was r = 0.22. This suggests two things: first, that the two items measure *different* aspects of student understanding of eigenvalues and eigenvectors. Second, it shows that performance on the procedurally and conceptually oriented questions was more highly correlated for TIMES students.

Trends in student reasoning on procedurally oriented question (Q8)

In this section, we provide our qualitative analysis of question 8. In particular, we highlight two common approaches to this problem: approaches that involve reasoning about the determinant, and approaches that involve reasoning about $A - \lambda I$ without computing a determinant. The majority of students who reasoned about the determinant responded correctly. Reasoning about $A - \lambda I$ was a less common approach but more frequently observed among TIMES students. Further, we argue that students who reasoned about $A - \lambda I$ showed more evidence of conceptual understanding. A summary of our coding and scoring of student responses is shown in Table 2.

Reasoning about the determinant was the most common approach observed in students' responses to question 8, and students who used this kind of approach tended to do so without making conceptual errors;[2] 146 out of 255 students (57% of all students). We note two interesting trends within those who used this approach distinguishing TIMES from Non-TIMES students. First, more TIMES students using this approach made computational errors (usually when factoring the characteristic polynomial) than did Non-TIMES students – such errors are evidenced by 2 point responses in our coding. On the other hand, fewer TIMES students using this approach made *conceptual* errors than did Non-TIMES students – such errors are evidenced by 1 point responses in our coding. In the TIMES instructional approach (previously described under study context), the standard algorithm for finding eigenvalues and eigenvectors is intended to emerge in relation to student-invented strategies on the third of fourth day of instruction in the unit, so we conjecture Non-TIMES students may have spent more time practicing this procedure in comparison to TIMES students.

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[2] We align our conceptions of conceptual and procedural errors with our definitions for conceptual and procedural understanding. We refer to an error as conceptual when there is evidence that a student does not understand an important underlying idea or relationship. We refer to an error as procedural when a student incorrectly performs a step in a mathematical process that is not central to the idea being assessed (e.g. an error in computation or algebraic manipulation). Examples of conceptual errors include incorrectly interpreting the value of the determinant to decide if something is an eigenvalue, or computing the determinant of A rather than the determinant of $A - \lambda I$. Examples of procedural errors include incorrectly factoring the characteristic polynomial or making an error when row reducing $A - \lambda I$.

A less common approach to solve problem 8 was by reasoning about $A - \lambda I$ without computing a determinant. Overall, 48 out of 255 students (19%) used such determinant-free approach to solve the problem. This approach was more common among TIMES than Non-TIMES students, and far more TIMES students successfully responded to the problem in this way with no or few conceptual errors. Indeed, 70% (19 out of 27) of TIMES students and only 38% (8 out of 21) Non-TIMES students who used this approach did so with minimal conceptual errors.

		TIMES, n = 126				Non-TIMES, n = 129			
Category		3 pts	2 pts	1 pts	0 pts	3 pts	2 pts	1 pts	0 pts
Reasoning about the determinant Total: 146 (57%) TIMES: 75 (60%) Non-TIMES: 71 (55%)	Solving the Characteristic Equation	43	12	4	0	49	4	4	4
	Plugging 2 into the Characteristic Equation	3	1	0	0	3	0	0	0
	Finding $\det(A - 2I) \neq 0$	10	2	0	0	5	2	0	0
Reasoning about $A - \lambda I$ Total: 48 (19%) TIMES: 27 (21%) Non-TIMES: 21 (16%)	Solving the system $(A - 2I)\begin{bmatrix} x \\ y \end{bmatrix} = 0$ or $Av = 2v$	9	2	2	1	5	1	2	0
	Considering linear dependence of the set of column vectors of $(A - 2I) \cong \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$	2	0	1	0	0	2	1	1
	Noting $rref(A - 2I)$ does not have a free variable	3	3	4	0	0	0	6	3
Other		0	0	1	7	0	0	0	16
Student indicated he/she did not know		0	0	0	5	0	0	0	9
Item left blank		0	0	0	11	0	0	0	10

Table 2: Summary of Students' Approaches and Scores on Q8

Students whose responses were categorized as "other" showed little or no evidence of understanding related to the definition or computation of eigenvectors and eigenvalues. We noticed that twice as many Non-TIMES students as TIMES students gave a response categorized as 'other.' However, TIMES and Non-TIMES students left the item blank at similar rates, but a larger number of Non-TIMES students explicitly mentioned that they "don't know".

Examples of student reasoning about Q8

In this section, we examine examples of common approaches identified in our analysis of students' responses to question 8. We provide one example response coded as 'reasoning about the determinant.' and one example responses coded as 'reasoning about $A - \lambda I$ without using the determinant.' We highlight the use of multiple representations in these responses, and connections between these representations and the formal definition of eigenvectors and eigenvalues.

Response 2.a. was awarded full points because the student correctly found the roots of the characteristic polynomial, noted that 2 was not one of those roots, and concluded that 2 is not an eigenvalue.

We note that in response 2.b., the student began with the equation $Ax = \lambda x$, rewrote this as $Ax - \lambda x = 0$, and then factored this to write $(A - \lambda I)x = 0$. The student then computed the

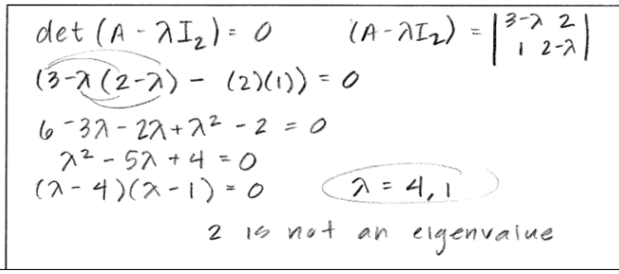
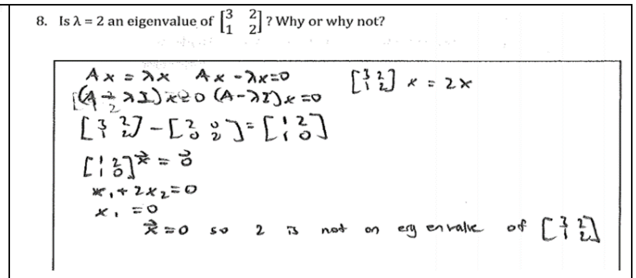
	
2.a. Response awarded three points	2.b. Response awarded three points

Figure 2: Responses to Q8 coded as "reasoning about the determinant"

entries of the matrix $A - 2I$, rewrote this as a homogeneous matrix equation which he or she translated into a system of equations, correctly solved, and correctly concluded that because the solution is the zero vector that 2 **is not** an eigenvalue of the given matrix.

Trends in student reasoning on conceptually oriented question (Q9b)

We now focus on responses to question 9. Overall, students' responses to this item were split relatively evenly among responses that used the eigen-concept, responses that focused on the role of the matrix M without using the eigen-concept, and students who wrote that they did not know or left the answer blank. However, TIMES students' responses used the eigen-concept at much higher rates than Non-TIMES students, and with greater success. See Table 3.

The most commonly observed response to Q9 involved using the eigen-concept, with 99 out of 255 (39%) total responses coded in this way. This approach was more common among TIMES students than Non-TIMES students (61/126 vs 38/129). Further, TIMES students who used this approach gave correct responses to the question at higher rate than Non-TIMES students; the ratio of TIMES students who used the eigen-concept in fully or mostly correct ways to those who used the eigen-concept in mostly incorrect ways was 44:17 whereas that ratio for Non-TIMES students is 18:20.

The second most commonly observed trend on Q9 involved responses that focused on the role of the matrix M without using the eigen-concept. We noted that students using this approach tended to be mostly or completely incorrect, and that more Non-TIMES students than TIMES students used this approach (29/126 TIMES as compared to 40/129 Non-TIMES students). We noticed that 14/29 (48%) of the TIMES students and 12/40 (30%) Non-TIMES students used this approach did so with some conceptual understanding but not using the eigen-concept. We argue these responses indicated some conceptual understanding because they drew on appropriate transformation interpretation of a matrix times a vector. However, the understanding reflected in these responses was incomplete in that the interpretation did not explicitly use the eigen-concept by coordinating that interpretation with the result of that multiplication also corresponding to a scalar times that same vector.

		TIMES, n = 126				Non-TIMES, n = 129			
		3 pts	2 pts	1 pts	0 pts	3 pts	2 pts	1 pts	0 pts
Responses using the eigen-concept Times: 61 Non-Times: 38	$Mx = \lambda x$ with explanations	9	14	4	0	5	3	2	0
	• Stretching • Same direction • Scalar multiple • Same line as x	10	11	12	1	2	8	16	2

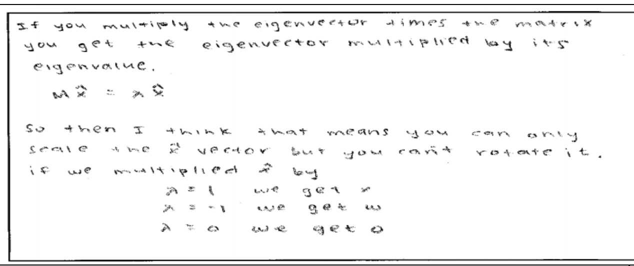
Responses focusing on the role of the matrix M without using the eigen-concept Times: 29 Non-Times: 40	Specific entries are suggested for matrix M that would yield different outputs	0	0	12	8	0	0	11	15
	Descriptions are given about how the matrix M transforms the input vector to yield different outputs (without suggesting specific entries of M)	0	0	2	7	0	0	1	13
Others		0	0	0	11	0	0	0	15
I don't know		0	0	0	10	0	0	0	11
Blank		0	0	0	17	0	0	0	25

Table 3: Summary of Students' Approaches and Scores on Item 9

There was little difference between TIMES and Non-TIMES Students who used approaches classified as 'other.' In this category, we saw no evidence of using the eigen-concept. TIMES and Non-TIMES students said they "Don't know" at similar rates, but more Non-TIMES students left the item blank than TIMES students.

Examples of student reasoning about Q9

We provide examples of common approaches identified in our analysis of students' responses to question 9. Specifically, we provide two example of responses coded as "using the eigen-concept" and two example responses coded as "focus on the role of the matrix M without using the eigen-oncept." Response 4.a. used the eigen-concept by writing the equation $Mx = \lambda x$ and suggesting values of λ (e.g. 1,-1,0) that corresponded appropriately to possible outputs. It was awarded full credit because the student linked this reasoning to all three possible outputs. Responses 4.b. used the eigen-concept in a slightly different way than the previous example. Rather than writing $Mx = \lambda x$ and suggesting appropriate values of λ, this student justified selections of correct output vectors by describing the role of M as stretching the vector x by a factor or in its direction. It was awarded just 2 points due to the omission of the 0 vector. Many students in our study who used the eigen-concept omitted the 0 vector as a possible eigenvector.


If you multiply the eigenvector times the matrix you get the eigenvector multiplied by its eigenvalue.

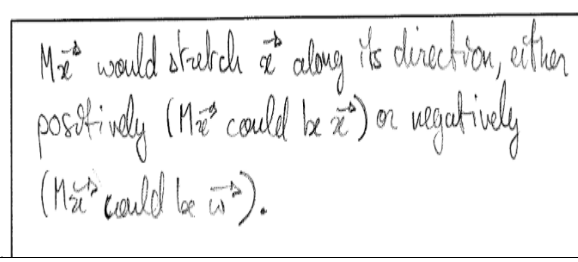
$M\hat{x} = \lambda\hat{x}$

So then I think that means you can only scale the $\hat{x}$ vector but you can't rotate it. if we multiplied $\hat{x}$ by

$\lambda = 1$ we get x

$\lambda = -1$ we get w

$\lambda = 0$ we get 0


$M\vec{x}$ would stretch $\vec{x}$ along its direction, either positively ($M\vec{x}$ could be $\vec{x}$) or negatively ($M\vec{x}$ could be $\vec{w}$).

4.a. Response awarded three points	4.b. Responses awarded two points

Figure 4: Responses to Q9 coded as using the eigen-concept

The next two examples presented in Figure 5 responses focus on the role of the matrix M as a transformation that can transform the vector x in many ways (not limiting to outputs that must lie along the same line as x). Student 5.a.'s response suggests that the student sees the matrix M not as a fixed matrix that transforms the eigenvector in a particular way; the student suggested different matrices that correctly produced the output vectors he/she selected – he/she considered M as the identity matrix I to produce x, $-I$ to produce w and the zero matrix to produce the zero vector. In addition, a matrix M with generic entries also was suggested as a transformation that can transform x into the other two incorrect vectors u and v.

$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\vec{x}$ $\quad\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\vec{u},\vec{v}$ $\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$ $\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\vec{w}$ x is a real valued eigenvector & M is a matrix and matrix m could transform $\vec{x}$ into anything $0, \vec{w}, \vec{u}, \vec{v}$ given the right 'm'.	The matrix M could rotate X to either u or v. It could also stretch reflect it to be w. And it could also be the identity matrix to give back X.
5.a. Response awarded one point	5.b. Response awarded one point

Figure 5: Responses to Q9 focused on the role of M without using the eigen-concept

Response 5.b. similarly focuses on the role of the matrix M, arguing it could rotate x to produce u or v, "stretch reflect" to produce w, and that it could be the identity matrix to "give back" x. This combination of what the student believes the matrix could be indicates that the student did not use the eigen-concept. Responses 5.a. and 5.b. were both awarded 1 point because both were interpreting the matrix M as a transformation and making some true statements, though in ways that did not use the eigen-concept.

Discussion

We see this paper contributing to the literature in three primary ways. First, we document the effectiveness of a particular instructional approach that is detailed in the literature (see Zandieh, Wawro, and Rasmussen, 2016; Plaxco, Wawro, & Zandieh, this volume). Second, we document aspects of students' reasoning about eigenvectors and eigenvalues (including how students draw on a transformation interpretation in ways that do and do not use the eigen-concept). Finally, we consider and discuss links between conceptual and procedural understandings of eigenvectors and eigenvalues documented in our study.

Our findings presented above are consistent with findings of previous studies examining student learning outcomes in inquiry-oriented instructional settings at the undergraduate level (e.g., Kwon, Rasmussen, & Allen, 2005), though we are excited that this study was conducted on a larger scale involving instructors not involved in the development of the curricular materials. These findings also are consistent with a broader body of literature documenting the benefits of student-centered approaches to learning in undergraduate mathematics (Freeman et al., 2014). We conclude our paper with a discussion of the kinds of conceptual understandings observed in our analysis, and the insights these offer into what is entailed in a conceptual understanding of eigenvectors and eigenvalues.

As mentioned in our theoretical framework, conceptual understanding has been broadly defined by some in terms of the richness of connections among ideas (Vinner, 1997; Hiebert & Lafevre, 1986). More recently, Star (2005) has argued that conceptions of conceptual and procedural knowledge in mathematics education are under-articulated in a way that promotes ideological rather than empirical examination, and relationships between conceptual and procedural understandings merit greater examination. With this in mind, we now reflect on the kinds of conceptual understandings observed in our analysis, and discuss three different kinds of connections we consider to be important aspects of students' conceptual understanding of eigenvectors and eigenvalues.

First, we consider the use of appropriate interpretations of a matrix times a vector to be an important aspect of students' understanding of eigenvalues and eigenvectors. On the conceptually oriented question (Q9), this involved drawing on a transformation interpretation as characterized by Larson & Zandieh (2013). In our data, many students showed evidence of interpreting Mx, the product of a matrix M and its eigenvector $x,$ in ways that use the eigen-concept. A smaller number of students interpreted Mx with a transformation lens, but in a way that did not use the eigen-concept.

This leads to our second aspect of students' understanding of eigenvalues and eigenvectors: using the eigen-concept in the context of finding eigenvalues. Relatively few showed evidence of using the eigen-concept on the procedurally oriented question by reasoning about $A - \lambda I$ without taking the determinant. We argue this approach provided more evidence of conceptual understanding: providing and converting between multiple representations (e.g. $Ax = \lambda x$ and $(A - \lambda I)x = 0$, written as matrix equations and systems of equations), linking those representations to the eigen-concept, and offering reasons for their conclusion in terms of a matrix equation or system of equations in their response. It is possible that a student who used the standard procedure to determine if 2 is an eigenvalue on this problem also had a deep conceptual understanding of how and why that procedure works; it is also possible that a student who used the standard procedure knew this procedure only as a sequence of steps to be executed without knowing how or why the procedure worked. Further work is needed to tease out this distinction.

This leads to the final aspect of conceptual understanding of eigenvectors and eigenvalues relevant to our analysis, which includes coordinating with the Invertible Matrix Theorem (IMT). A standard procedure for finding eigenvalues and eigenvectors draws on the argument that $Ax = \lambda x$ has a non-trivial solution x for some scalar λ if and only if $(A - \lambda I)x = 0$; one can argue through the IMT that this happens when $\det(A - \lambda I) = 0$. Among students who did not use the determinant in their response to the procedurally oriented question, there was a need to draw on equivalent ideas from the invertible matrix theorem. In these responses, we observed students noting and leveraging the following relationships:

- $(A - \lambda I)$ is invertible if and only if $(A - \lambda I)x = 0$ has a trivial solution. If $(A - \lambda I)x = 0$ has only the trivial solution, then λ is <u>not</u> an eigenvalue of the matrix A.
- If the columns of $A - \lambda I$ are linearly dependent or one column is a scalar multiple of the other (in the case of a 2x2 matrix), then $(A - \lambda I)x = 0$ has nontrivial solution so λ is an eigenvalue of the matrix A.
- If $rref(A - \lambda I)$ has no free variable then $(A - \lambda I)x = 0$ has only the trivial solution, which means λ is not an eigenvalue of the matrix A.

We argue that these kinds of responses from students who did not use the previously mentioned standard procedure offer insight into conceptual connections that are both important and potentially natural for students to make as they come to make sense of standard algorithms.

Overall, students in our study correctly solved a procedural question related to eigenvalues (as in Q8) at about twice the rate they offered an appropriate conceptual understanding of $Ax = \lambda x$ (as in Q9). This suggests there is a disconnect between students' understanding of standard procedures for finding eigenvalues and the formal definition of an eigenvector and eigenvalue, and that students are more able to execute the standard procedure than draw a conceptual understanding aligned with the formal definition. This points to a need to push students to think more about core understandings as they connect to procedures rather than just assess students' ability to execute standard procedures. Indeed, many connections are needed to explain why a standard procedure for finding eigenvalues and eigenvectors works and how it connects to the formal definition of eigenvalues and eigenvectors. However, we argue that there is little value in being able to compute eigenvectors and eigenvalues without being able to appropriately interpret the meaning of the result of such computations. The inquiry-oriented approach of the IOLA instructional materials taken up by instructors who received TIMES instructional supports appears to be a promising way of beginning to address this issue, but more work is needed to better understand the ways in which students come to develop and coordinate the interpretations needed for a robust understanding of eigenvectors and eigenvalues.

Acknowledgments

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Appendix A: Grading scheme for assigning points to open-ended response questions 8 and 9b

Q #	Points Awarded and Criteria
8	3 points: Method 1: Full points were awarded to students who reasoned about the determinant to arrive at the correct conclusion without making computational or conceptual errors. Examples of this kind of reasoning are shown below. i) $\det(A-\lambda I)=0$ implies $(\lambda-1)(\lambda-4)=0$ implies $\lambda=1$ or $\lambda=4$ implies $\lambda=2$ is not an eigenvalue for the matrix A. ii) $\det(A-2I)=-2\neq 0$ implies $\lambda=2$ is not an eigenvalue for the matrix A. iii) $\det(A-\lambda I)=\begin{vmatrix}3-\lambda & 2\\ 1 & 2-\lambda\end{vmatrix}=(3-\lambda)(2-\lambda)-2=\lambda^2-5\lambda+4$. Substituting 2 in the characteristic equation gives $4-10+4=-2$ implies $\lambda=2$ is not an eigenvalue for the matrix A. Method 2: Full points were awarded to students who reasoned about $A-\lambda I$ without using the determinant to arrive at the correct conclusion without making any computational or conceptual errors. Examples are shown below. i) $(A-2I)\begin{bmatrix}x\\ y\end{bmatrix}=0$ implies $x=0$ and $y=0$ which is the trivial solution, so $\lambda=2$ is not an eigenvector for the matrix A. ii) $(A-2I)\cong\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$, and the column vectors of this matrix are not linearly dependent, so $\lambda=2$ is not an eigenvalue. iii) $rref(A-2I)$ does not have a free variable, so $\lambda=2$ is not an eigenvalue iv) The first column of $(A-2I)$ is not a scalar multiple of the second column
	2 points: Two points were awarded to students to students who take a conceptually correct approach (either by reasoning about the determinant or by reasoning about $A-\lambda I$ without using the determinant) but either • made a computational error (e.g. factoring the characteristic polynomial incorrectly) or • did not offer a clear conclusion about whether 2 is an eigenvalue or not, or • arrived at the correct conclusion without a full explanation of why
	1 point: One point was awarded to students whose response included some evidence of conceptual understanding, but who made a conceptual error (which might be accompanied by a computational error).
	0 points: No points were awarded to students who left the page blank, or whose response: (i) gave no evidence of conceptual understanding, or (ii) said something like "I don't know." Example of responses we considered to include no evidence of conceptual understanding are "Yes, because $A=PDP^{-1}$" and "I say it is… because… there are 2's in the problem."
9b	3 points: Three points were awarded to students whose response appropriately coordinated with the eigen-concept, referenced (either by directly naming or by explicitly referring to their work shown in 9a) all three correct vectors and provided a correct rationale for this selection.
	2 points: Two points were awarded to students whose response provided at least two correct explanations (e.g. $Mx=\lambda x$ is written and student writes that "an eigenvector tells you the direction of stretching") but did not identify and explicitly describe what happens to all three correct vectors.
	1 point: One point was awarded to students who either i) Provided one correct explanation (e.g. by either writing "$Mx=\lambda x$" or "an eigenvector tells you the direction of stretching") and explicitly connected this explanation to at most one correctly selected vector ii) Suggested components of M that would transform x into one of the given choices, such as $M=I, -I, or\ 0$.
	0 point: No points were awarded to responses that do not coordinate with the eigen-concept, do not suggest components of M that would transform x into one of the given choices, says I don't know, or leaves the page blank. An example of student response to question 9 which was awarded 0 point was "all are the same size."

Students' Understanding of Test Statistics in Hypothesis Testing

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Hypothesis testing is a key concept included in many introductory statistics courses. Yet, due to common misunderstandings of both scientists and students, the use of hypothesis testing to interpret experimental data has received criticism. With statistics education on the rise as well as an increasing number of students enrolling in introductory statistics courses each year, there is a need for research that investigates students' understanding and curriculum effectiveness of hypothesis testing. This paper describes results obtained from a larger study designed to explore introductory statistics students' understanding of one sample hypothesis testing. In particular, this paper explores students' understanding of test statistic as a component of hypothesis testing. APOS Theory is used as a guiding theoretical framework. This paper focuses on three students' understandings of test statistic when performing hypothesis tests on real world data.

Key Words: Hypothesis Testing, Statistics, Test Statistic

Introduction

The use of statistics is crucial for numerous fields, such as business, medicine, education, and psychology. Due to its importance, according to the *Guidelines for Assessment and Instruction in Statistics Education College Report*, more students are studying statistics, and at an increasingly younger age (GAISE College Report ASA Revision Committee, 2016). In the United States today, the *Common Core State Standards for Mathematics* calls for students to "understand statistics as a process for making inferences about population parameters based on a random sample from that population" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 81). More recently, the *GAISE College Report* calls for nine goals for students in introductory statistics courses, including "Students should demonstrate an understanding of, and ability to use, basic ideas of statistical inference, both hypothesis tests and interval estimation, in a variety of settings" (GAISE College Report ASA Revision Committee, 2016, p. 8). In other words, in an introductory statistics course, students should understand and be able to apply hypothesis testing in various situations.

Hypothesis testing is an important tool of statistical inference (Krishnan & Idris, 2015). However, the use of hypothesis testing to interpret experimental data has received criticism (Nickerson, 2000; Nuzzo, 2014) due to the common misunderstandings of both scientists and students when using this method (Batanero, 2000; Dolor & Noll, 2015; Vallecillos, 2000). Rather than abandon the inference method entirely, researchers have called for improving the education and understanding of hypothesis testing. For example, LeMire (2010) developed a framework to revise and improve instructional content on hypothesis testing. Even still, there are few studies on student understanding of hypothesis testing as a whole (Smith, 2008). Our larger research goal is to supplement these efforts by first analyzing how students come to understand hypothesis testing and then develop instructional materials to cultivate this understanding. What follows is a brief summary of the literature on student understanding of hypothesis testing.

Literature Review

Research suggests that although students are able to perform the procedures surrounding hypothesis testing, students lack a strong understanding of the concepts and their use (Smith, 2008). It is suggested that hypothesis testing and the area of inferences "is probably the most misunderstood, confused and abused of all statistical topics" (Batanero et al., 1994, p. 541). Textbooks and instructors often give a specific step-by-step script to follow when performing hypothesis testing, not connecting or allowing the students to see the idea as a whole. Link (2002) suggested this practice as a six-part procedure, which leads many students to look for key words and phrases as guides when solving hypothesis testing problems. He also found evidence that supports the belief that students can correctly substitute values into a formula selected from a formula sheet, yet they do not have a full understanding of hypothesis testing in its entirety.

In an attempt to move away from a procedural approach, and due to the rise of statistical education, calls for reform have led to a shift from an emphasis on procedural understanding to conceptual understanding (GAISE College Report ASA Revision Committee, 2016; Krishnan & Idris, 2015). Ways to teach for conceptual understanding have been varied. For example, Hong & O'Neil (1992) suggested that to foster this conceptual understanding in hypothesis testing, conceptual instruction should be presented prior to the procedural instruction with emphasis on the use of diagrammatic problem representations. In contrast, Chandrakantha (2014) suggested that utilizing technology that allows students to visualize and work hands-on with data will enhance student understanding of concepts such as hypothesis testing.

Other research focuses on identifying students' misconceptions with various parts of hypothesis testing in order to improve conceptual understanding. Specifically in introductory statistics courses, students appear to experience a "symbol shock" (Schuyten, 1990), which provides an obstacle for students interpreting particular questions (Dolor & Noll, 2015; Lui & Thompson, 2005; Vallecillos, 2000). Vallecillos (2000) found that students have trouble with not only the symbols, but also with the formal language and meaning behind the concepts involved in hypothesis testing, including words such as 'null' and 'alternative' hypothesis. Students interviewed were not able to accurately describe what these terms mean and how they impact the decision to either accept or reject within the test (Vallecillos, 2000). Furthermore, Williams (1997) during two different interviews with 18 students, found that the term 'significance' was not well understood. Students gave vague and inadequate descriptions of what 'significance' means in the context of hypothesis testing. Even after the final exam, students continued to confuse terminology and still had a poor understanding of the concept. Mathematical symbolism presents challenges across all levels (Rubenstein, 2008), especially as most students' school mathematics experiences give little or no attention to the idea of reading mathematics as a language (Adams, 2003).

With statistics education reform on the rise, as well as an increasing number of students enrolling in introductory statistics courses each year, there is a need for research that investigates students' understanding and curriculum effectiveness of hypothesis testing, a concept taught in almost every introductory statistics course (GAISE College Report ASA Revision Committee, 2016; Krishnan & Idris, 2015). Literature in this area focuses on students' lack of understanding of hypothesis testing, ways to teach for conceptual understanding of hypothesis testing, and misconceptions with various parts of hypothesis testing. Particular parts of hypothesis testing that have received the most attention in literature are hypotheses and p-value. Test statistics contribute to understanding a p-value, but there is a lack of literature on this concept (Smith, 2008). Thus, we focus our attention on the following research question:

> What are students' understandings of test statistic, as a component of hypothesis testing as a whole, in two distinguished real world situations?

The next section will introduce test statistics within the scope of one-proportion hypothesis testing.

Test Statistic

A test statistic, according to the textbook *Elementary Statistics Using Excel*, is "a value used in making decisions about the null hypothesis," (Triola, 2014, p. 415). While the definition provided is simplistic, the actual concept of test statistic in hypothesis testing is complex. Assuming the null hypothesis is true, a test statistic is found by converting a sample statistic, such as a sample proportion or a sample mean, to a standardized score. Although we discuss test statistic as one mathematical term or value, there is a distinction between test statistics calculated from sample proportions versus sample means. Specifically, in our study, the distinction is between the normal distribution and the Student's t distribution.

When calculating a test statistic for a sample representing a population mean or proportion, we are referring to a standardized value that represents the extremeness of a sample in regards to what is expected. For proportions, students use the normal distribution as an approximation to the binomial distribution, and thus calculate test statistics, which are z-scores. For means, students learn about test statistics in hypothesis testing using the normal distribution (z-scores) and the Student's t distribution (t-scores). Although these distributions appear similar, the distinction occurs depending on what we know about our sample. In particular, if we know the population standard deviation, then we know the center, shape, and spread of the data, and can use the normal distribution (z-scores). However, if the population standard deviation is unknown and we only have the sample standard deviation as an estimate, then we can use the Student's t distribution to represent the center, shape and spread of the data. Because of the greater variability associated with smaller sample sizes, t-scores are greater than or equal to z-scores for the same value of n, but as n approaches infinity, the limiting value of the Student's t distribution is the normal distribution (see Figure 1).

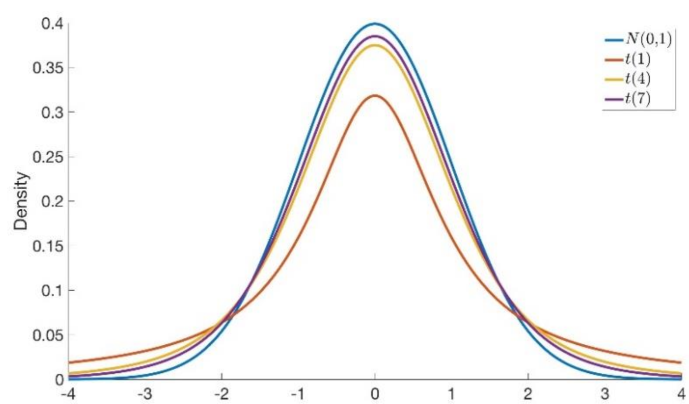

Figure 1. Graphical Representation of Normal Distribution and Student's t Distribution
(Note: key represents $t(n)$, where n is the sample size)

Theoretical Framework

A theoretical framework is necessary in order to analyze and describe how students understand a particular concept. The guiding theoretical framework for our larger study is APOS Theory (Asiala et al., 1996). APOS Theory is a framework which models an individual's mathematical conception using Actions, Processes, Objects, and Schema. An Action is an externally driven transformation of a mathematical object (or objects). An Action can be described as an individual needing an external cue to complete a task, such as a step-by-step example to follow. For our study, in the context of test statistic, an example of this would be a student calculates the test statistic by the correct formula (even with wrong distribution label), but cannot interpret the test statistic verbally or graphically. Another example, in the context of test statistic, would be a student who uses key words or phrases from the problem as external cues to decide which formula to use to calculate the test statistic.

Once Actions are repeated and reflected on, an individual can start to interiorize them to become a Process. A Process no longer requires step-by-step external cues. An individual is now able to internally imagine the steps in a transformation, without having to actually perform them for specific examples. In the context of test statistic, an example would be a student calculates test statistic by using the correct formula (even with wrong distribution label) and describes their calculations in general terms. Although the student can calculate the test statistic, they do not have to perform the steps in order to describe the calculations. Another example would be a student who interprets the test statistic verbally or graphically by describing how it corresponds to $\hat{p}$ or $\bar{x}$ or the distance from the sample value to the expected value ($\hat{p}$ to p or $\bar{x}$ to μ).

When an individual is then able to see the Process as a totality, is aware that transformations can be applied to it, then the Process has been encapsulated into an Object. In the context of test statistic, several examples of this are as follows: a student describes the test statistic as an input of a function that determines the p-value; student interprets test statistic graphically as determining the edge of the region whose area is the p-value; or student describes the test statistic as large or small in comparison to a "usual" value of a test statistic (not just describing the number as large or small in general). As defined by the textbook, a "usual" value of a test statistic refers to z-scores between -2 and 2 (Triola, 2014).

Schemas are "structures that contain the descriptions, organization, and exemplifications of the mental structures that an individual has constructed regarding a mathematical concept" (Arnon et al., 2014, p. 25). Schemas may also be included within another Schema. For example, the distribution schema plays a role in the development of the schema of test statistic. This report is devoted to providing examples of Action and Process conceptions of test statistic from a sample of student interviews conducted for a larger study.

Methodology

The focus of our study is on university students who are enrolled in an introductory statistics course based on the emporium model. The emporium model, originated at Virginia Tech, includes key components of "interactive computer software, personalized on-demand assistance, and mandatory student participation" (Twig, 2011, p. 26). For this particular institution, each week students were required to spend three academic hours in a computerized mathematics lab, as well as attend one academic hour class each week with an instructor. The time in the mathematics lab was spent actively learning using the mathematical software *MyStatLab* by Pearson. Students also read, viewed videos, and discussed material with peers, lab assistants, and instructors.

Relevant data was collected during Fall 2014 and Spring 2015. All students enrolled in six sections of an introductory statistics course (approximately 240 students) were invited to participate in a problem solving session and semi-structured interview pertaining to hypothesis testing. Twelve students volunteered to participate. During the problem solving sessions, each participant worked alone on two hypothesis test questions. They were allowed to use Excel when needed since the use of it was required as part of the class. The first question asked the student to conduct and interpret a hypothesis test for a single population proportion. The second question asked the student to conduct and interpret a hypothesis test for a single population mean. The questions were as follows:

1. In a recent poll of 750 randomly selected adults, 588 said that it is morally wrong to not report all income on tax returns. Use a 0.05 significance level to test the claim that 70% of adults say that it is morally wrong to not report all income on tax returns. Use the *P*-value method. Use the normal distribution as an approximation of the binomial distribution.
2. Assume that a simple random sample has been selected from a normally distributed population and test the given claim. In a manual on how to have a number one song, it is stated that a song must be no longer than 210 seconds. A simple random sample of 40 current hit songs results in a mean length of 231.8 seconds and a standard deviation of 53.5 seconds. Use a 0.05 significance level to test the claim that the sample is from a population of songs with a mean greater than 210 seconds.

Students had seen these exact questions with variable parameters on their homework and quizzes when using the *MyStatLab* software. Students also engaged in active learning associated with these concepts both in the lab and in class. Thus, students were expected to know how to conduct and interpret hypothesis tests for both questions, and in particular, they were expected to know to use the normal distribution to find the test statistic for Question 1 and the Student's t distribution to find the test statistic for Question 2. In other words, they were expected to know the procedure.

Immediately following the problem solving sessions, the students participated in semi-structured interviews. There were ten interviews, eight with one participant each and two with two participants each (12 students in total). During the semi-structured interviews, participants were asked to elaborate on their answers and thought processes. The interviews were conducted and divided among multiple members of the research team. To standardize the interviews, an interview protocol was developed beforehand. The relevant data for this paper consists of participants' written work, Microsoft Excel files, and transcribed discussion from the follow up interviews.

Data analysis took place after all interviews were conducted. The recordings of the interviews were distributed and transcribed by each of the six members of the research team. After transcriptions were completed, analysis was organized in a way so that each transcript was reviewed by two different pairs of researchers. The data was coded and analyzed according to APOS Theory. After codes were developed and agreed upon by each pair of researchers, the team came together for discussion as a whole. Six concepts of hypothesis testing emerged in participants' reasoning. This paper focuses on one of these six concepts, test statistic. The data and codes were then used to develop individual learning trajectories for each participant that merely served as a method to explore and organize the ways of understanding of each concept for this group of individuals. The focus of this paper will be on the individual learning trajectories for test statistic.

Results

In this section, we illustrate examples from the data analysis that are most relevant and indicative of the Action and Process conceptions of test statistic according to APOS Theory. Our results will focus on three students: Haley, Lana, and Steve. Each student was representative of different subgroups of the twelve learning trajectories, and exhibited different conceptions of test statistic.

Haley

Haley demonstrated evidence of an Action conception of test statistic. For both questions, Haley looked for indicator phrases to identify which test statistic formula to use. She identified that a problem about proportions implies that the test statistic will be a *z*-score. Haley also mentioned that if in a problem she is not given the population standard deviation, this indicates when she would use a *t*-score.

I: Okay, and um is this a t-score or a z-score?
H: This is a z… yeah this is a z-score.
I: And how did you know that?
H: Because we're using proportions so you just use z.
I: Okay, good. Um, and then when would you use a t-score?
H: When you don't know the.. like when you don't know the um… what is it? You don't know… when you don't know this [draws sigma on the paper].
I: Okay!
H: But like you know it for a sample! [laughs]
I: So this is.. so.. I know that this is referring to the standard deviation right?
H: Yeah.
I: So and you said if you know the sample… so I'm assuming you're saying this is not the sample standard deviation?
H: No, that's the population. So if you don't know the population standard deviation I should use t.

Haley appeared to base her ideas of which test statistic to use off of key words found in the problems. When calculating the test statistic, Haley again looked for indicator phrases in the problem to identify the values to plug in. She also goes on later in the interview to state that she has memorized the formulas for test statistics.

I: Okay, then for the test statistic… how did you figure out all those numbers and stuff?
H: Well, I took the [yawn] I took x bar minus mu and then I divided it by the standard deviation divided by the square root of n.
I: Okay and did you just find that formula on the sheet or did you have that one memorized?
H: I had it memorized.

As more evidence of a lack of full understanding of what a test statistic represents, for the second problem, she identified the given sample mean in addition to her test statistic in different places on her graph when in actuality the test statistic is representative of the sample mean.

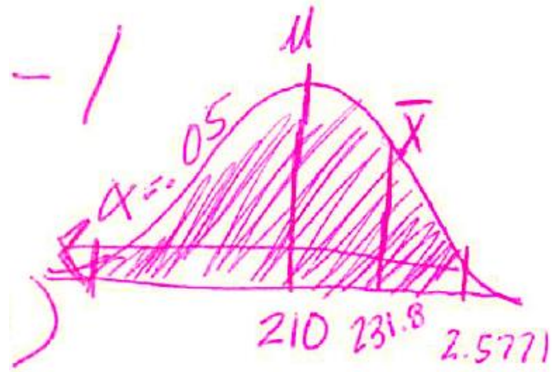

Figure 2. Haley's Graphical Representation of Question 2

Her memorization of formulas and what appears to be dependency on key words in the problem to identify which test statistic to use and what values to plug in, with no verbal explanation of the reasoning behind this, suggests Haley illustrates an Action conception of test statistic.

Lana

Lana exhibited evidence of a Process conception of test statistic. Lana, for Question 1, used the normal distribution to find the test statistic based on the fact that the problem was about proportions. After recognizing which test statistic to use, she described in general how to calculate the test statistic using the formula and she double checked all her work in Excel.

> L: And then I double checked by using the formula, the phat minus p over the square root of pq over n. I put that in Excel to double check and make sure that was right.

Lana was then prompted to explain the test statistic. She described, in words, what she imagined.

> L: I think that, I'm picturing the big curve, the bell curve, and I'm picturing the test statistic is where the point that falls on there … Okay, so this is the mean right in the middle, and the test statistic is one side of it, saying this is how far away from what they are saying is the mean, this is what the mean of this.

By reflecting on her previous steps and calculations, Lana appeared to interiorize the action of calculating test statistics. She internally imagined calculating or finding test statistics in relation to the mean, without having to actually perform any calculations. This suggests Lana exhibited a Process conception of test statistic. After her explanation, she drew a picture to illustrate her thinking (Figure 2) of a graphical representation of the normal distribution.

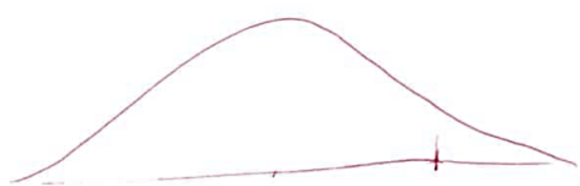

Figure 3. Lana's Graphical Representation of the Normal Distribution

In her written work, Lana initially used the normal distribution to find the test statistic for Question 2. During the interview she became worried when asked if the question was a *z*-test. She mentioned that she remembered from high school to use a *t*-score if the sample size is 30 or less. After prompting, she realized she should have used a *t*-score, however, she also immediately recognized that her *z*-score would "probably not" be very different from the *t*-score. She made the observation that t-scores and z-scores are "really close together". This is evidence of the role that distribution Schema plays in the development of the Schema of test statistic.

It appears that Lana illustrates a Process conception of test statistic because she pictured in her head a graphical interpretation of test statistic without relying on specific calculations, and also interpreted the test statistic as describing how far away the sample statistic is from the expected (population) value. It also appears that she possesses a distribution Schema that is in an early stage, as noted in her confusion between *z*-scores and *t*-scores. However, her understanding that a *z*-score and a *t*-score are "really close together" suggests that her distribution Schema is progressing and emerging.

Steve

Steve is included in the results as illustrative of a student whose analysis suggests discrepancies in his conceptual understanding of test statistic. For Question 1, Steve identified key words in the problem to recognize which test statistic to use. He explained, "I immediately thought of these two formulas, and at first I wasn't sure which one to use, and then I was like, oh wait, there's no x-bar or mu or standard deviation. So that makes it pretty easy." He used a system of elimination to decide which formula not to use. Even though he correctly identified the test statistic, he went on to say that this is "just a formula that I've learned like any other" and that he "doesn't understand why we use that formula, other than we just use it". He concluded his explanation stating that Question 1 used a *z*-value because the problem was of proportions, what appears is likely a memorized rule applied to the problem.

For Question 2, Steve mentioned that "it's basically the same problem", other than now being a question of means. Steve identified the distribution as normal based on the language in the problem, "a simple random sample has been selected from a 'normally distributed' population". He further explained that he would not know how to deal with a non-normally distributed population, that he could not recall learning anything other than a normal distribution, and that he did not know much about a Student's *t* distribution. Ironically, later in the interview, when asked if his answer was a *z*-value or *t*-value, he responded that it is "a t-value because you're testing means".

The excerpts above are suggestive of Action conception of test statistic. Steve appeared to rely on external cues based on key words identified in the problem and what appeared to be memorized rules of when to use a particular formula. However, when prompted by the interviewer to explain what a test statistic is, Steve elaborated as follows, "And also your test statistic is very large. I'm not totally sure what a test stat is, but it reminds me of z-scores, and I remember when you have a z-score that gets above 3. It starts to get pretty, pretty crazy. So 5 is huge, which is also the reason that you're getting a bunch of zeros or very close to 1 [for the p-value]." Although Steve's interpretation of z-scores seems fairly advanced in comparison to other students, it is not clear from his description whether he considers a *z*-score to be an example of the concept of test statistic or a similar concept that is distinct from the test statistic. The former would be suggestive of a Process conception of test statistic, while the latter would be suggestive of an Action conception of test statistic. Since these discrepancies exist, we suggest that Steve is emerging from the Action conception to the Process conception of understanding of test statistic.

Discussion and Concluding Remarks

Our results provide evidence of three students' understanding of test statistic, one who appears to exhibit an Action conception of test statistic, one who appears to exhibit a Process conception of test statistic, and one who appears to be emerging from an Action conception to a Process conception of test statistic. Consistent with Link (2002), many students used a procedural approach to hypothesis testing which included students plugging in correct values for a formula. In reference to test statistic, Haley looked for indicator phrases and key words in the problem to decide which test statistic formula to use and to decide which values to plug in, yet she did not have a full conceptual understanding of hypothesis testing. According to APOS Theory, she is illustrative of an Action conception of test statistic.

Another student, Lana, reflected on her calculations and appeared to interiorize the action of calculating test statistics by describing the test statistic in relation to the mean. However, for Question 2, Lana initially used the normal distribution to find the test statistic, rather than the Student's *t* distribution. In our study, we did not initially consider the role that the stage of distribution Schema would have in the development of the concept of test statistic. However, our results suggest that an individual can possess a Process conception of test statistic, while possessing only an early stage of Schema for distribution. Further research is suggested to explore the extent to which an individual's distribution Schema has in the development of the concept of test statistic.

Lastly, discrepancies in data analysis suggest Steve is emerging from an Action conception of test statistic to a Process conception of test statistic. Excerpts are suggestive that he was relying on key words and memorized formulas to decide which test statistic to use, evidence of Action conception of test statistic. However, Steve did verbally elaborate about *z*-scores, indicative of a possible Process conception. Nevertheless, inconsistency exists in the fact that Steve did not equate z-scores to the test statistic and instead suggested they remind him of each other. If Steve had a Process conception of test statistic, we would have expected him to note that test statistics are standardized values, which in this case *z*-scores are an example representative of test statistic as a whole.

As hypothesis testing is a key concept included in the majority of introductory statistics courses (Krishnan & Idris, 2015), and is arguably one of the most misunderstood statistical topics (Bantanero, 1994), it is important to continue research that investigates students' understanding and curriculum effectiveness of hypothesis testing. This paper, specifically focused on test statistic, a component of hypothesis testing, revealed students' understanding in two distinguished real world situations. By analyzing data according to APOS Theory, we have illustrated examples of Action conception, Process conception, and an example of a student who is emerging from Action conception to Process conception of test statistic. This knowledge can then be used to describe how students may develop an understanding of test statistic – an important concept within hypothesis testing and one that has not received much attention within the research community.

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The Role of Visual Reasoning in Evaluating Statements about Real-Valued Functions: A Comparison of Two Advanced Calculus Students

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The purpose of this study is to examine the characteristics of students' visual reasoning in the context of evaluating statements about real-valued functions. We conducted clinical interviews with nine undergraduate students in which we asked them to evaluate several mathematical statements using graphs to explain their reasoning. In this paper, we focus on two Advanced Calculus students and the differences in their visual reasoning in these tasks. Our findings indicate that students' visual reasoning accounts for key differences in their understandings of mathematical statements. In this paper, we introduce a visual reasoning framework which emerged from our data. We also provide examples from the two students to highlight the use of the framework to characterize students' visual reasoning as value-thinking or location-thinking.

Key words: Visual Reasoning, Empirical Study, Graphical Interpretations, Undergraduate Students, Intermediate Value Theorem

Undergraduate Calculus courses, from elementary through advanced Calculus, are comprised of many definitions and theorems about real-valued functions. Often, these statements are accompanied by visual representations in the form of graphs of relevant functions. For example, the Intermediate Value Theorem (IVT) is one such statement commonly associated with a visual representation. The IVT can be stated as follows: "Suppose that f is a continuous function on $[a, b]$ with $f(a)\neq f(b)$. Then, for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b) such that $f(c)=N$." This theorem is often shown in Calculus textbooks with graphs to accompany the statement (e.g., Briggs, Cochran, & Gillett, 2011). For instance, a graph of a continuous function such as the one in Figure 1 (left) may be used to illustrate the IVT in the case of a monotone function. Additionally, a graph like Figure 1 (right) may be shown to demonstrate that for a given N value, the corresponding value of c need not be unique.

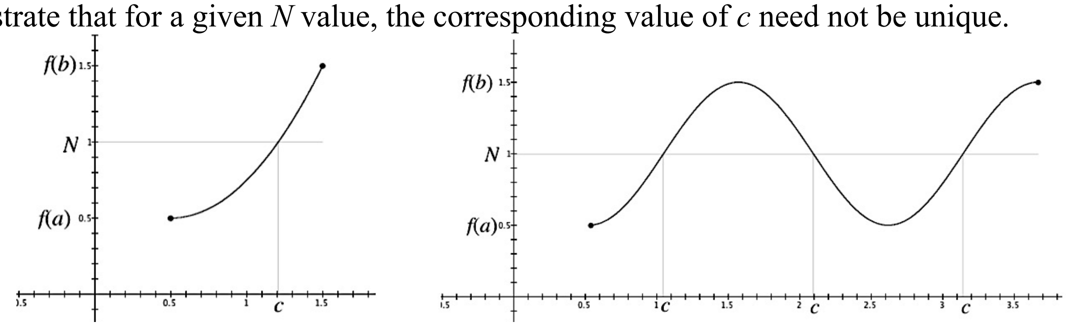

Figure 1. Possible graphical illustrations of the Intermediate Value Theorem (IVT)

Visual representations, such as the ones provided in Figure 1, are often included in textbooks to support students in understanding the given statement or theorem (e.g., Stewart, 2012). While research has called for the inclusion of such visual representations (Arcavi, 2003; Davis, 1993), few empirical studies have been conducted to look at students' visual reasoning about graphs of real-valued functions. Although several studies have looked at students' understanding of graphs as a whole (Monk, 1992; Moore & Thompson, 2015; Moore, 2016), it is not widely known what meanings students have for various aspects of graphs, such as the input, output, and points on graphs. While it is hoped that students focus on the details of graphs that highlight the intended concept, some students may construe other properties of a given graph rather than the intended ones. For instance, a student looking at Figure 1 may presume that the IVT refers to a single value of *N*, or that *N* can be selected from the entire range of the function. Thus, students' interpretations of the IVT, based on the information they perceived from the provided graphs, might hinder their subsequent mathematical activities, such as rigorous proofs.

The purpose of this study is to investigate the role of students' visual reasoning in understanding and evaluating mathematical statements from Calculus. In particular, this study focuses on the IVT and statements similar to the IVT. Through analyzing two advanced Calculus students' evaluations and interpretations of these statements, we address the following research questions: *What are characteristics of undergraduate students' visual reasoning about graphs related to statements from Calculus contexts?* Specifically,

a. *How do students interpret outputs of a function on a graph, points on a graph, and a graph as a whole?*
b. *How do students' visual reasoning affect their understanding and evaluation of the Intermediate Value Theorem and similar statements?*

Literature Review

Mathematics education research has included numerous studies and discussions of the role of visualization throughout mathematics teaching and learning (e.g., Arcavi, 2003; Bishop, 1980; Davis, 1993). Studies have both suggested and reported the success of various instructional interventions that utilize visualization in undergraduate mathematics contexts in various levels of Calculus (e.g., Tall, 2010; Thompson, Byerly, & Hatfield, 2013). Researchers have also examined students' understanding of mathematical concepts using visualization tools, including their understanding of graphs of functions (Kidron & Tall, 2015; Monk, 1992; Pinto & Tall, 2002). These studies provide evidence that visualization can be a powerful tool for students learning concepts in advanced mathematics.

Instructional Interventions Using Visualization

There have been calls throughout the literature to encourage and promote visual reasoning in mathematics instruction, as an essential part of mathematical activity (Arcavi, 2003; Davis, 1993). Several researchers have developed visual reasoning tasks that instructors can use to support students in understanding formal definitions of concepts in undergraduate mathematics (Kidron & Tall, 2015; Roh, 2010; Tall, 2010). Proposed instructional interventions, such as the ones noted above, show promise in supporting students' understanding (e.g., Cory & Garofalo, 2011; Kidron & Tall, 2015; Roh & Lee, 2017). However, visual representations and interventions that rely on them are not without limitations. Students may over-generalize from a single visual representation they view as "prototypical" (Harel & Sowder, 1998), focus on irrelevant details of them (Presmeg, 1986), or rely too heavily on self-generated visual representations (Alcock & Simpson, 2004). Arcavi (2003) accounts for some differences in what students gather from images, claiming that an individual's perceptions are "conceptually driven" (p. 234). Because visual representations may be interpreted differently by different onlookers or in different contexts, it is imperative to investigate what sense *students* make of visual representations of concepts that are commonly provided.

Students' Visual Reasoning about Graphs and Points

Several studies in mathematics and science education have documented students' creation, interpretation, and use of graphs (e.g., Padillo, McKenzie, & Shaw, 1986; Pinto & Tall, 2002). Some studies have classified students' conceptions of graphs as a whole, while others suggested models of students' understanding of points on a graph (Lakoff & Nunez, 2000; Moore & Thompson, 2015; Moore, 2016; Thompson & Carlson, 2017).

Moore and Thompson (2015) highlight a distinction in students' visual reasoning about graphs as a whole. In their study, some students whom they classified as engaged in *static shape-thinking* reasoned about graphs as though the graph itself was an object, such as a wire. In contrast, they describe students who engage in *emergent shape-thinking* as conceiving of a graph as a trace which emerges from the coordination of two varying quantities. Extending this work, Moore (2016) claims that students may attend to both visual properties of graphs and the relationships of the quantities represented, but one of the two will guide students' reasoning.

While most literature on students' visual reasoning focused on students' conception of graphs in their entirety, other researchers have attended to students' conception of points on a graph. Thompson and Carlson (2017) use the term *multiplicative object* to characterize student thinking about points as representing two values or quantities simultaneously in the form of a single object. Lakoff and Nunez (2000) offer a cognitive model for how individuals come to understand numbers as points on a number line, claiming this requires the use of what they call a *conceptual metaphor*, "an inference preserving cross-domain mapping" (p. 6). The idea of mapping numerical values to physical locations on a line preserves properties of values such as ordering. In two-dimensions, then, the conception of plotting a point relies on using this conceptual metaphor for each value of the pair represented, which may explain students' difficulties with interpreting points on graphs.

While these theoretical discussions can help distinguish certain aspects of student visual reasoning, we seek to investigate the ways students understand outputs and points on a graph at a more fine-grained level. In our effort to delve deeper into students' meanings for relevant aspects of graphs, we define the constructs *value-thinking* and *location-thinking*, as explained in the following theoretical framework section.

Theoretical Perspective and Visual Reasoning Framework

This study is grounded in a constructivist perspective. We adopt von Glasersfeld's (1995) view that students' knowledge consists of a set of action schemes that are increasingly viable given their experience. This perspective implies that we, as researchers, do not have direct access to students' knowledge and can only model their visual reasoning based upon what we can observe. Thus, our analysis reflects our best attempt at creating a hypothetical model of student visual reasoning grounded in evidence found in their words, gestures, and markings on graphs.

We adopt components of Arcavi's (2003) definition of visualization for this study:

> The ability, the process, and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper, or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (p. 217).

Arcavi (ibid) argues that visualization has a natural role in mathematics, and gives examples of visual representations such as graphs, or diagrams. While Arcavi's description of visualization is broad, in this study, we use the term *visual reasoning* to refer specifically to how students interpret, use, and reason about graphs of real-valued functions. We also use the terms "thinking" and "reasoning" interchangeably to describe students' mathematical activity in the context of graphs. In particular, we focus on how students perceive the elements that constitute graphs.

Our Visual Reasoning Framework

In this section, we describe the framework that we developed and employed to categorize students' visual reasoning. In particular, this framework is rooted in distinguishing students' visual reasoning about aspects of points on graphs.

Points in a two-dimensional Cartesian system are intended to represent the concurrence of the values of two quantities, conventionally notated as an ordered pair (x, y). This coordination of values x and y also has a spatial location in the Cartesian plane due to the geometric nature of a point in a plane as a coordination of two signed distances. To graph a point (x, y) in the Cartesian plane, one marks the spatial location that is a distance of x units from the origin along the x-axis (conventionally left or right) and a distance of y units from the origin along the y-axis (conventionally up or down). Thus, we see a point on a graph as dual-natured, simultaneously representing a pair of values, as well as a specified location in the space of the Cartesian plane.

These two properties of points in a Cartesian coordinate system, values and locations, are possible foci of a student's attention while reasoning about graphs. Accordingly, we developed our visual reasoning framework in terms of *value-thinking* and *location-thinking*. Students who primarily focus on the *values* represented by the coordinates of a point are said to engage in *value-thinking.* In contrast, *location-thinking* refers to reasoning that primarily attends to the spatial *location* of the point. We detail characteristics of both categories of visual reasoning in Table 1 by listing the meanings for three aspects of the graph for each category and observable behaviors indicative of these meanings. The three aspects of graphs in this framework are named from our perspective as researchers, whereas the meanings explicated under each type of visual reasoning are from the perspective of the student. To be clear, the distinction between these two ways of reasoning lies in the *place of the student's attention* when reasoning about graphs.

Table 1
Comparison of Characteristics of Value-Thinking and Location-Thinking

<table>
<tr><th colspan="2"></th><th colspan="2">Value-Thinking</th><th colspan="2">Location-Thinking</th></tr>
<tr><th colspan="2"></th><th>Visual Reasoning</th><th>Evidence</th><th>Visual Reasoning</th><th>Evidence</th></tr>
<tr><td rowspan="3">Aspects of a Graph</td><td>Output of Function</td><td>The resulting value from inputting a value in the function</td><td>▪ Labels output values on output axis
▪ Speaks about output values</td><td>The resulting location in the Cartesian plane from inputting a value in the function</td><td rowspan="3">▪ Labels output on the graph
▪ Labels point as output
▪ Speaks about points as a result of an input into the function (e.g., "an input maps to a point on the graph")</td></tr>
<tr><td>Point on Graph</td><td>The coordinated values of the input and output represented together</td><td rowspan="2">▪ Labels points as ordered pairs
▪ Speaks about points as the result of coordinating an input and output value</td><td>A specified spatial location in the Cartesian plane</td></tr>
<tr><td>Graph as a Whole</td><td>A collection of coordinated values of the input and output</td><td>A collection of spatial locations in the Cartesian plane associated with input values</td></tr>
</table>

Value-thinking. By *value-thinking*, we mean visual reasoning that relies on the values represented by the coordinates of a given point on a graph. *Value-thinkers*, then, refer to those who engage in value-thinking. One of the key characteristics of value-thinking is distinguishing the output of a function from the corresponding point on a graph. Students who engage in value-thinking label points as ordered pairs (e.g., $(a, f(a))$) and speak about points as representing both input and output values simultaneously. When considering the output of a function, value-thinkers attend to the resulting value of the function from a given input value. Students focusing on values tend to label relevant output values on the output axis of graphs, and specifically speak about *values*. These actions are due to the student's attention to the corresponding values represented by a given point on a graph, which they consider to be a coordination of both an input and output value. Value-thinkers, then, treat graphs as a collection of ordered pairs that relate corresponding input and output values.

Location-thinking. By *location-thinking,* we mean visual reasoning that relies on the spatial locations of the points in the Cartesian plane. *Location-thinkers*, then, refer to those who engage in location-thinking. Location-thinkers focus on the location of the point, while the values of the coordinates are either in the background of their reasoning or absent from it. In contrast with value-thinking, one of the key characteristics of location-thinking is treating the output of the function as the location of the point on the graph of the function. Accordingly, location-thinkers often label points on the graph as outputs (e.g., $f(a)$) rather than ordered pairs and speak about points in terms of their location in the coordinate plane. While value-thinkers label outputs on the output axis, location-thinkers place the output label (e.g., $f(a)$) at the location of the point on the graph. Instead of speaking about output *values*, location-thinkers speak about *points on the graph* as the result of a value inputted to the function. Location-thinkers, then, treat graphs as a collection of locations in space associated with input values.

Use of Framework

The intent of our framework is to characterize students' visual reasoning with graphs of functions. In Figure 2, the two sets of labels on the same graph illustrate distinctive characteristics of value-thinking (left) and location-thinking (right), respectively.

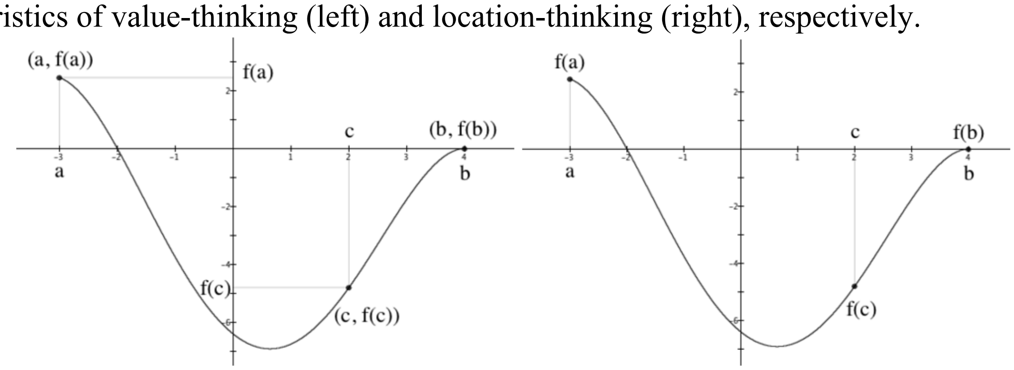

Figure 2. Sample labeling activity of a value thinker (left) vs. a location thinker (right)

In Figure 2 (left), input values are labeled on the input axis, and output values are labeled on the output axis. Furthermore, points on this graph are labeled as ordered pairs. In Figure 2 (right), while the input values are labeled on the input axis, the output values are not labeled on the output axis. In contrast to Figure 2 (left), points on the graph in Figure 2 (right) are labeled with only output notation, rather than as ordered pairs. Additionally, the placement of output labels differs between the two figures. For example, in Figure 2 (left), $f(c)$ is a label of a value on the output axis, whereas in Figure 2 (right), $f(c)$ is a label of a point at a particular spatial location.

This framework is intended to model students' visual reasoning through evidence found in the student's (repeated) *actions*, rather than their *abilities* to think in various ways. Thus, the classification of a student as a location-thinker does not imply that the student is unable to consider values of the output. On the contrary, it is likely that students may be classified as location-thinkers and may still at times refer to or find specific values of the output. Rather, location-thinkers focus on the location of the relevant points in their reasoning, as revealed through their words and actions. Accordingly, no single piece of evidence (e.g., one labeled point, one gesture) is viewed as enough to categorize a student's visual reasoning. We also recognize that the same student presented with different contexts may engage in value-thinking or location-thinking, depending on what he or she focuses on. We thus emphasize that the distinction between value-thinking and location-thinking is one of the placement of students' attention, rather than an exclusive way of thinking.

Visual Reasoning Framework in the IVT Context

In the context of the IVT and related statements, location-thinkers and value-thinkers likely interpret the phrase "N between $f(a)$ and $f(b)$" differently. Due to the differences in their meanings for outputs and points on a graph, value-thinkers interpret "N between $f(a)$ and $f(b)$" as referring to output *values* between the *values* of $f(a)$ and $f(b)$. This meaning stems from a conception of a point as a coordination of input and output values. In contrast, *location-thinkers* interpret "N between $f(a)$ and $f(b)$" as referring to *locations* between the *locations* of $f(a)$ (at $(a, f(a))$) and $f(b)$ (at $(b, f(b))$), an interpretation rooted in a conception of the location of points *themselves* as outputs of a function. Whether a student chooses to focus on the values or the locations of points in a graph influences how that student will interpret this phrase.

Methods

As part of a larger study, we conducted two-hour clinical interviews (Clement, 2000) with nine undergraduate students from a public southwestern university. We selected three undergraduates who had just completed one of the following three mathematical courses that may use the theorem: Calculus I, Introduction to Proof (also known as Transition-to-Proof), and Advanced Calculus. During each interview, one of the four members of our research team served as the interviewer with the other three acting as witnesses. All four researchers utilized laptops, whose screens the participant could not see, to communicate their current models of the participant's thinking and offer clarifying questions in real-time via group chat.

Interview Tasks

During the interview, the interviewer asked students to evaluate each of the four mathematical statements in Table 2 and to provide their justification. The second statement, the Intermediate Value Theorem (IVT), was the only true statement we presented. The remaining three statements (1, 3, and 4), all of which are false, were variations on the IVT with the quantifiers (for all, there exists) reordered or the variables (N, c) reversed. Table 2 contains the statements in the order presented to the students.

Table 2
Statements Presented to Participants, in Order

Statement 1	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$ such that $f(c)=N$.
Statement 2 (IVT)	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b) such that $f(c)=N$.
Statement 3	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, there exists a real number N between $f(a)$ and $f(b)$, such that for all real numbers c in (a, b), $f(c)=N$.
Statement 4	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, there exists a real number c in (a, b), such that for all real numbers N between $f(a)$ and $f(b)$, $f(c)=N$.

The interviewer later asked students to re-examine each statement along with six graphs we created, with the chance of changing their evaluation. The graphs, shown in Figure 3, were intended to represent a spectrum of possible functions and relevant counterexamples and included: a polynomial with extrema beyond the endpoints of the displayed function (Graph 1), a vertical line segment (Graph 2), a continuous sinusoidal function (Graph 3), a monotone increasing function (Graph 4), a constant function (Graph 5), and a function that is discontinuous on $[a, b]$ (Graph 6). Graphs 1, 3, and 4 were chosen to represent three cases of continuous functions where $f(a)\neq f(b)$. Graphs 2, 5, and 6 were chosen to represent graphs each of which does not meet one of the conditions of the four statements.

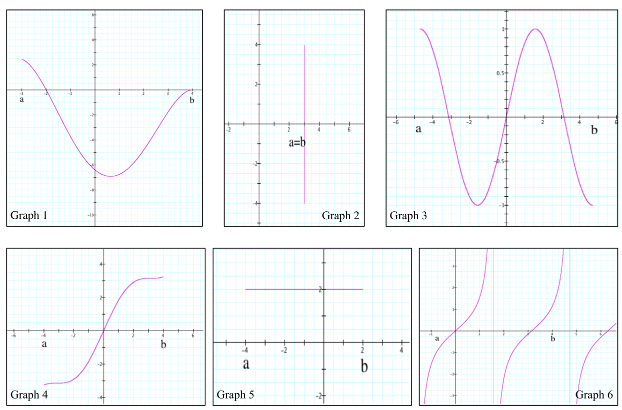

Figure 3. The six graphs we presented to the participants

The participants were also asked to explain how they interpreted various aspects of these graphs and to label relevant points and values on each graph where appropriate. As each participant explained his or her interpretations of the graphs, all four researchers developed models of the student's thinking in the moment. Where necessary, researchers proposed additional questions to confirm the current model of the student's thinking. If a student's responses to these questions did not confirm the model, the model was changed or refined to accommodate the student's response.

Data Analysis

Our data analysis was consistent with Corbin and Strauss' (2014) description of grounded theory, in which categories of student visual reasoning emerged from the data analysis. We began preliminary analysis during and immediately following each interview to note relevant findings. After all the interviews were conducted, we employed open coding (Corbin & Strauss, 2014), through which students' interpretation of the phrase "N between $f(a)$ and $f(b)$" emerged as highly relevant to their reasoning about graphs and subsequent evaluations of the given statements. We developed two codes, *value-thinking* and *location-thinking*, to broadly characterize our participants' visual reasoning about graphs. Finally, we refined these categories and re-coded the video interview data using axial coding (Corbin & Strauss, ibid). Through this process, we developed the theoretical framework described in the previous section (see Table 1) which emerged in our analysis. In the next section, we share our findings from this data analysis.

Results

The purpose of this study is to characterize students' visual reasoning with graphs related to statements from Calculus contexts. In particular, we examined students' interpretations of various aspects of graphs, as well as the effects of these interpretations on their understanding and evaluation of the IVT and three similar statements (see Table 2). We categorized the visual reasoning of each of the nine students we interviewed according to the theoretical framework (see Table 1) which emerged in the process of data analysis as previously described. In this paper, we report the cases of Nate and Jay, Advanced Calculus students from each category, in order to highlight the defining characteristics of each category of visual reasoning. We provide gestures, labeling on graphs, and verbal explanations of their interpretation of the graphs to describe details of each student's visual reasoning in terms of our theoretical framework (see Table 1).

Jay: A Value-Thinker

In this section, we describe characteristics of value-thinking in terms of Jay's meanings for outputs, points, and the phrase "N between $f(a)$ and $f(b)$" through examples from his visual reasoning. We present his verbal explanations along with his labeled graphs from two episodes of his interview to illustrate value-thinking.

Even before the interviewer provided him with graphs, Jay engaged in visual reasoning by creating his own graphical representations to explain his understanding of the statements. For instance, when the interviewer asked Jay to evaluate Statement 2 (IVT), he responded by saying that the statement was true, and drew a graph to explain his reasoning. Figure 4 shows Jay's hand-drawn graph.

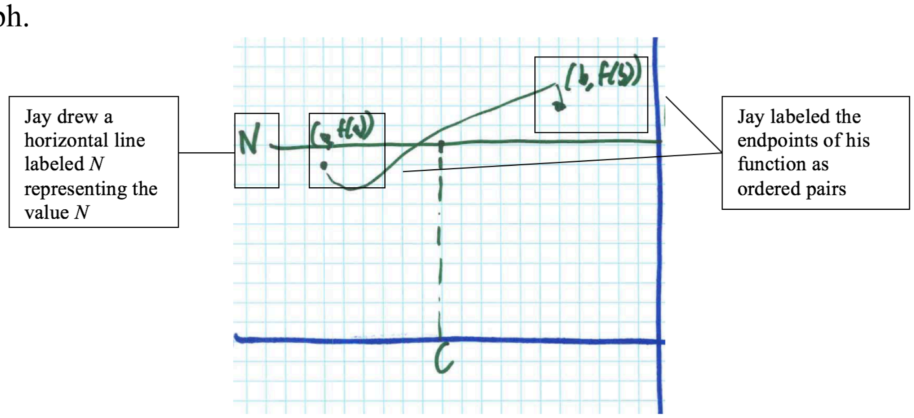

Figure 4. Jay's hand-drawn graph explaining why the IVT is true[1]

[1] Jay intended that c be placed at the intersection of the graph of the function and the horizontal line $y=N$.

In creating his graph in Figure 4, Jay first plotted two points to represent endpoints of a graph of a function, which he labeled as the ordered pairs, $(a, f(a))$ and $(b, f(b))$, respectively. Then, Jay drew a horizontal line, which he labeled 'N,' to represent a value of N between $f(a)$ and $f(b)$. Jay explained that, from his perspective, Statement 2 must be true because any graph of a continuous function that he could draw from $(a, f(a))$ to $(b, f(b))$ must cross the horizontal line (y=N) that he labeled N. He then labeled c on the x-axis and explained that if the function crossed the line y=N at the value c, then $f(c)$ equals N. Jay then drew a graph from $(a, f(a))$ to $(b, f(b))$ as an example of a continuous function that he was speaking of.

Outputs as values. Jay consistently treated outputs of the function as values. For instance, Jay drew a graph of a horizontal line y=N and claimed that where the function "cross[es] this line" is "where $f(c)$ is N." Jay's purpose in drawing this horizontal line was to identify a particular output value of the function, namely, N. We take both Jay's graph of the horizontal line y=N, and the way he used this line in his explanation as evidence that he was attending the value of outputs (similar to Figure 2 left). Confirming that he attended to the value of outputs of the function, Jay later explained that Statement 2 meant to him that continuous functions do not "skip over any *values* between $f(a)$ and $f(b)$." We thus take Jay's language and labeling on the graph as evidence of his consideration of *outputs of the function as values*, a characteristic of value-thinking as described in our visual reasoning framework (see Table 1).

Points as ordered pairs. Jay also labeled the endpoints of the graph as ordered pairs: $(a, f(a))$ and $(b, f(b))$. Additionally, he referred to the intersection point of a continuous function, f, and a horizontal line, y=N, as an ordered pair, $(c, f(c))$. Jay's labeling and description of points as ordered pairs indicate that he attended to both the values of the input and output represented at each point. We thus take Jay's consistent treatment of *points as ordered pairs* as evidence of value-thinking, as noted in Table 1.

In addition to considering outputs as values and points as ordered pairs, Jay's interpretation of the phrase "N between $f(a)$ and $f(b)$" in the later portion of the interview provides further evidence of Jay's value-thinking. We present a second episode from Jay's interview to highlight his meaning for this phrase.

Earlier in the interview, Jay had already evaluated all four statements correctly. Later, when the interviewer presented with graphs in Figure 3, Jay again explained his evaluation of the statements while looking at Graph 1. In particular, Jay used a curly brace along the y-axis to mark off what he took to be the relevant interval for possible N values between $f(a)$ and $f(b)$ (see Figure 5 for his work with Graph 1).

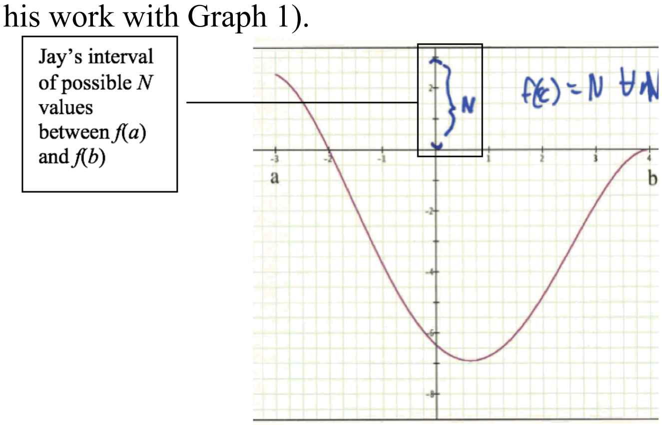

Figure 5. Jay's possible values of N on Graph 1

When explaining why he concluded that Statement 1was false, Jay referenced the interval for possible N values that he drew in Figure 5. In his explanation, he approximated this interval to be [0, 4] (Jay did not carefully attend to the scale of the y-axis), and explained that although 0 was an input value between a and b, i.e., in the interval (–3, 4) in Figure 5, its corresponding output $f(0)$, or –7, was not between $f(a)$ and $f(b)$, i.e., in the interval [0, 4]. The transcript below provides Jay's verbal explanation.

Jay	*Statement 1*	*Graph 1*
Interviewer: Can you use this graph to explain why the statement is false?		
Jay: This [Statement 1] is false in this case because it would say that in the open interval (a, b) (*points to a and b on x-axis*), okay, so for all numbers between these two numbers, –3 and 4, (*gestures along x-axis between –3 and 4*) there exists a real number between, and this is the important part, between 0 and 4 or 0 and 3, so between these two values, (*gestures on graph parallel to y-axis similar to interval he marked off above*) such that when I evaluate the function here (*gestures along x-axis between –3 and 4*), I get the value in here (*gestures to indicate interval on y-axis from f(a) to f(b)*). Okay…so if I am looking at $f(0)$, right? $f(0)$ is –7 which is outside of the interval of [0, 4], which means that there is no N between 0 and 4 such that… –7 which is... $f(c)$ is equal to N, that value between 0 and 4.		

N as a value "between f(a) and f(b)." As previously noted, Jay marked off what he took to be the interval of possible N values between $f(a)$ and $f(b)$ using a curly brace along the y-axis (see Figure 5). Using $f(0) = –7$ as an example in which the input value is between a and b, but the output value is not between $f(a)$ and $f(b)$, he determined Statement 1 to be false. We thus take Jay's interval of possible N values between $f(a)$ and $f(b)$ in Figure 5 as further evidence that he engaged in value-thinking, as explained in our Theoretical Framework.

Nate: A Location-Thinker

Unlike Jay and the other students whom we classified as value-thinkers, some students did not consider outputs as values, points as ordered pairs, or N as values "between $f(a)$ and $f(b)$." Instead, these students labeled *outputs at locations* on the graph rather than the y-axis, labeled *points as outputs* rather than ordered pairs, and considered N as a *spatial location* rather than a value. In this section, we present an illustrative episode from one such student, Nate, whose visual reasoning we categorized as *location-thinking*. To highlight the characteristics of *location-thinking,* we present his verbal explanations along with his gestures and graph labels to illustrate defining characteristics of location-thinking in terms of his meanings for outputs, points, and the phrase "N between $f(a)$ and $f(b)$."

In one portion of the interview, Nate explained why he evaluated Statement 1 as true using Graph 1. In contrast with value-thinkers who considered outputs of the function to be *values*, Nate considered both outputs of functions and points on graphs to be *locations* on graphs. Specifically, Nate first labeled the endpoints of the graph as $f(a)$ and $f(b)$, respectively. Then, Nate highlighted the x-axis with his pen, and explained that for every c on this axis, he could find an N on the curve that it maps to. He labeled several c's on the x-axis as he explained this. He also motioned from the x-axis vertically down to the graph when describing that c's were mapped to N's on the graph. Similarly, when describing N's, he swept along the entire graph of the function from what he marked as $f(a)$ to $f(b)$. His transcript in this portion of the interview is provided below and his labels on Graph 1 are shown in Figure 6.

Nate	*Statement 1*	*Graph 1*
Nate: So for all these c's (*sweeps pen along x-axis between a and b)* you can see that it mapped to a point on the curve. For every single c there is a point on the curve that it maps to…So after that *c, N* would be here (*marks c around 1 on x-axis, N at ordered pair location of (c, f(c))*. So it maps to that. And this c would be in here (*marks c around 2 on x-axis*). And this c would be like, N right here (*marks corresponding N's for each c on Graph 1*).		

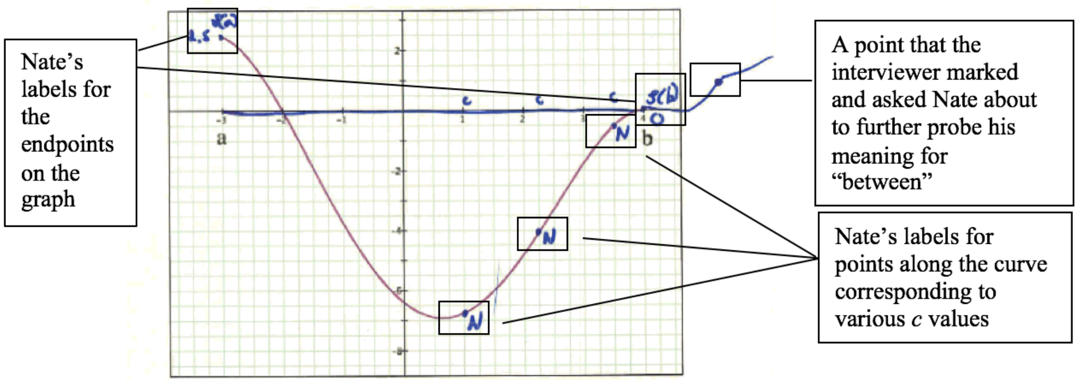

Figure 6. Nate's labeling on Graph 1 when he explained why he evaluated Statement 1 as true

Outputs as locations. Unlike the value-thinkers, who placed output labels on the y-axis, Nate, in Graph 1, used outputs, $f(a)$, $f(b)$, and N, to label points on the graph as shown in Figure 6. In fact, Nate consistently placed output labels at points on the graphs of the functions he worked with throughout the interview. His work on the graphs indicates that he considered these outputs to be locations on the graph, rather than values on the y-axis. Nate also spoke about inputs mapping to N's which he labeled on the graph. Thus, for Nate, outputs of the function were locations on the graph. We thus take Nate's gestures and graphical activity as evidence of his consideration of outputs of the function as locations, indicative of location-thinking as described in our visual reasoning framework (see Table 1).

Points as locations. We also take the placement of Nate's labels of $f(a)$, $f(b)$, and N at points as evidence that he attended to the different spatial locations of these points. Rather than label the points as an ordered pair of input and output values, Nate chose to label the endpoints as the output alone. For Nate, there was no difference between outputs of the function and points on the graph, as both referred to spatial locations on the graph. We thus conclude that Nate conceived of points as locations, an indication of location-thinking, as described in our framework.

When Nate explained his meaning for the phrase "N between $f(a)$ and $f(b)$,"his visual reasoning in terms of location-thinking was most prominent. The following exchange between Nate and the interviewer more clearly reveals his location-thinking. Nate labeled possible N's on the graph that were, from our perspective, not between $f(a)$ and $f(b)$, as the output values at these points were less than the value of $f(b)$. Noticing Nate's placement of N labels, the interviewer extended the graph to the right to further examine Nate's meaning for the phrase "N between $f(a)$ and $f(b)$" (see Figure 6). The interviewer picked a point on the extended graph, with an output value between the values of $f(a)$ and $f(b)$ and asked Nate if this output was between $f(a)$ and $f(b)$. In response, Nate hesitated and explained that there were two possible interpretations of the phrase "N between $f(a)$ and $f(b)$." The transcript below contains his explanation and Figure 6 contains his corresponding labels on Graph 1.

Nate	*Statement 1*	*Graph 1*
Interviewer: …Okay. Let's say we picked a point over here (*points to a point beyond endpoints of the graph, on portion of extended graph, the rightmost marked point on Figure 6*). Would we say that that output would be between $f(a)$ and $f(b)$?		
Nate: I would not say it's between $f(a)$ and $f(b)$. Even though the, yeah. This is the confusing part, where the actual numbers 2.5 and 0 (*marks the points 2.5, 0, boxed on Figure 6*). This would be, if you are looking at numbers 2.5, 0, this would be in between that interval. But it's in between that number interval. But it's not in between the functional interval in this case, so $f(a)$. The interval refers to all these points between $f(a)$ and $f(b)$ (*sweeps pen along the graph)*. All points of the function. That's what I am interpreting.		

N as a location "between f(a) and f(b)." The interviewer's prompt in the exchange above allowed Nate to consider more carefully his meaning for "N between $f(a)$ and $f(b)$." While Nate said the point that the interviewer plotted was not between $f(a)$ and $f(b)$, he acknowledged that the point was "in the number interval" between 2.5 and 0 (the values of $f(a)$ and $f(b)$). We take Nate's number interval to be referencing the interval of output values between 0 and 2.5, which he labeled on the endpoints of the graph. He clarified that the point was not "in the function interval," which we take as located on the graph of the function, since Nate swept along the graph as he described the function interval. Although Nate acknowledged the numerical interval, he considered his notion of the "function interval" as more relevant for interpreting the phrase "N between $f(a)$ and $f(b)$." We thus take Nate's interpretation of "between" in terms of the *location* of the points, rather than the values of the outputs, as indicative of *location-thinking*.

Conclusion

This study reveals that students' interpretations of outputs, points, and graphs differ based on their visual reasoning. Like Jay, the students whom we categorized as *value-thinkers* considered outputs as values, points as coordinates of input and output values, and N as a value. They consistently labeled outputs on the y-axis and points as ordered pairs. Additionally, they referred to the y-axis when speaking about possible values of N in the various statements. These characteristics of their visual reasoning are consistent with the characteristics of value-thinking described in our visual reasoning framework in Table 1. On the other hand, students like Nate whom we categorized as *location-thinkers* considered both outputs and points as locations on the graph of the function, and, consequently, N as a location between the endpoints of the graph. They consistently labeled outputs at points on the graph rather than label outputs on the y-axis or label points as ordered pairs. Additionally, they referred to the whole graph when speaking about possible values of N in the various statements. These characteristics of their visual reasoning are consistent with the characteristics of location-thinking described in our visual reasoning framework in Table 1.

Our analysis of the data in this study also reveals that students' visual reasoning impacts their understanding and evaluation of the statements. Five students including Jay were identified as value-thinkers and four students including Nate were identified as location-thinkers. Among the five value-thinkers, three students evaluated all four statements correctly as FTFF. In contrast, no location-thinker evaluated all four statements correctly.

The two students whose visual reasoning we reported in this paper share many similarities. Both Jay and Nate had recently completed Advanced Calculus and both showed evidence of having a proper understanding of multiple quantifiers and conditional structure. However, they evaluated Statement 1 differently. While Jay correctly evaluated this statement as false, Nate evaluated it as true. The role of visual reasoning, then, becomes central in explaining Nate's and Jay's opposite evaluations of Statement 1, since both students understood the quantifiers and conditional structure of the statement in the same way while their visual reasoning differed. In short, the sole reason Nate evaluated Statement 1 incorrectly was due to his location-thinking. Unlike value-thinking, which seemed to support students, location-thinking did not support students' understanding and evaluation of the statements.

Discussion

In summary, through probing students' understanding and evaluation of the IVT and related statements, along with several graphs, we discovered a significant distinction in how students interpreted these graphs. Location-thinkers confounded the output of a function with the location of the coordinate point. They thus interpreted the IVT and related statements differently from value-thinkers, who distinguished between output values and points that coordinated both an input and output value. We hypothesize that the differences in value-thinkers' and location-thinkers' visual reasoning may lead to differences in how these students understand other mathematical statements about real-valued functions.

Significance of findings & relation to literature

Through our analysis of student visual reasoning, we conclude that a subtle difference in place of a student's focus when reasoning about a graph has significant implications on mathematical ideas. As illustrated through the cases of Jay and Nate, different methods of visual reasoning contributed to different evaluations and understandings of a complex mathematical statement about real-valued functions, of which there are many in undergraduate mathematics courses.

Our results highlight and explain some important aspects of students' graphical activity not previously accounted for by current theories and studies on visual reasoning (Moore & Thompson, 2015; Moore, 2016). We view our findings as complimentary to Moore's (2016) work in classifying students' graphical reasoning through the source of their attention. In terms of Moore's (2016) constructs, students who were labeled as value-thinkers were seen as engaging in operative thought and emergent shape-thinking, as their visual perception of the graph was subordinate to their meanings for output value and they coordinated the values of the varying quantities. Location-thinkers, on the other hand, did not clearly fall into the alternate category of figurative thought, which Moore (2016) aligns with static shape-thinking. The visual cues from the graph did dominate their thinking in some aspects of the graph, such as the spatial location of the points, which informed their interpretation of "N between $f(a)$ and $f(b)$." Although location-thinkers engaged in figurative thought in this way, they did not conceive of the graph statically. Instead, these students conceived of graphs as emerging from the coordination of input values with the corresponding spatial locations of the points on the graph. This conception of a graph is neither captured by the construct of static shape thinking (not conceiving of the graph as a wire) nor emergent shape thinking, (not imagining the values of both the input and output represented simultaneously) as described by Moore and Thompson (2015). In this way, we consider our constructs of value-thinking and location-thinking as contributing to the existing body of literature in describing students' graphical activity. Thus, the use of our constructs of value-thinking and location-thinking could progress the depth of analysis in the field of student visual reasoning, especially with regard to ideas from Calculus.

Implications for curriculum and instruction

Our findings in this study provide insight into distinctions in students' visual reasoning, which instructors at both the high school and undergraduate level may find informative for understanding students' mathematical activity. In teaching concepts involving graphical representations, instructors' attention to the distinction between value-thinking and location-thinking may in turn provide students with opportunities to reflect on their conception of points and graphs. In the classroom, it may also be beneficial for instructors to be aware of the various ways in which students may interpret information from a visual representation.

While in our study, value-thinking helped students to understand the IVT and varied versions of the statement, in other contexts, such as Geometry, location-thinking may be preferable. Ideally, students should possess the ability to think in both ways, as well as the ability to discern when to use each. We also recognize that our findings indicate overcoming various perceptual cues found in graphs, beyond conceiving of the graph as a static shape, is a nontrivial achievement, even for advanced students. Teachers utilizing such representations may seek to support students in overcoming adherence to visual cues.

Our findings also have implications for curriculum and textbook design. Curriculum developers may consider including tasks designed to highlight the distinction in the two modes of visual reasoning. In the context of the IVT specifically, we note that the examples provided in Figure 1 do not readily provide students with the opportunity to reflect on the meaning of the phrase "N between $f(a)$ and $f(b)$" in the Calculus context as referring to values. Because the endpoints of the interval in Figure 1 coincide with the absolute minimum and maximum values of the function, students would arrive at the same conclusion, regardless of their mode of visual reasoning. Both curriculum developers and instructors may consider the use of graphs like our Graph 1, in which the function includes output values beyond the range of values between $f(a)$ and $f(b)$ to provide opportunities to distinguish students' visual reasoning. We hope that our findings raise awareness of the subtle yet significant details of students' visual reasoning and may thus inform decisions both in curriculum design and instruction.

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Conceptual Understanding of Differential Calculus: A Comparative Study

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The research community shares a concern for students' conceptual understanding of calculus and commonly advocates for student-centered approaches as a way to promote it. In this study, we investigated the effect of different instructional approaches on 151 undergraduate students' conceptual understanding of differential calculus in context-specific, natural settings. We collected data on the pre- and posttest of the Calculus Concept Inventory in three classes. In one class, most of the time was dedicated to conceptually oriented problem solving, another class implemented practice problems for students, and the third class was a traditional lecture class. The results showed that there was no difference in students' conceptual understanding of differential calculus controlling for their initial understanding. Thus, our findings do not support the research that advocates for student-centered instruction suggesting that the approaches' implementation and contextual differences may be sources of variation in their effectiveness.

Key words: Calculus, Conceptual Understanding, Active Learning, Instruction, Concept Inventory

Mastery of calculus, a desired and necessary student learning outcome (Sofronas et al., 2011), needs to include not only mastery of procedures but mastery of concepts, as well (Zerr, 2010). Multiple attempts have been made to identify instructional approaches that lead to greater conceptual understanding of STEM disciplines (Freeman et al., 2014; Prince, 2004) and specifically of calculus (Laursen, Hassi, Kogan, & Weston, 2014; Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006), typically advocating for student-centered instruction. However, those calculus studies either used measures with limited evidence of validity and reliability or aggregated data across classrooms, potentially different in instruction implementation or contextual factors. With our ex post facto study, we aimed to overcome these limitations and investigate students' conceptual understanding of differential calculus (measured by a validated instrument) in three calculus classes with distinct instructional approaches taking contextual factors into account.

Literature Review

The education research community has been working on identifying instructional approaches effective for students' learning and specifically for their conceptual understanding of content for a long time (Prince, 2004). Many researchers have advocated for student-centered instruction as an effective one, typically contrasting it with the teacher-centered instruction. For example, in physics, one of the largest studies was conducted by Hake (1998) where he compared student conceptual understanding in interactive engagement classes and traditional classes. The results of that study suggested that students in the former classes had higher conceptual understanding than students in the latter.

In undergraduate mathematics, several studies exist that explored the influence of student-centered instruction - specifically inquiry-based learning (IBL) - on student conceptual understanding. One of such studies is a study of Laursen et al. (2014) where learning gains of students in IBL mathematics classes were compared to those of students in non-IBL mathematics classes. The results showed that students' cognitive gains in understanding and thinking, among others, were greater in IBL classes than in non-IBL. However, the measurement of learning gains in this study is important to note. The learning gains were self-reported by students, i.e., the gains were students' subjective perceptions of their learning. Perceived learning, though it has its advantages, might not always be an accurate estimation of actual learning.

Another relevant study was conducted by Rasmussen et al. (2006) where students conceptual understanding of differential equations was explored in IBL and traditional classes. The results also supported the effectiveness of IBL. However, several notes need to be made about the measurement of conceptual understanding in this study, as well. First, the validity evidence for the instrument used was limited (Kwon, Allen, & Rasmussen, 2005). Second, the measure was administered only as a posttest assessment (without a pretest) and only to volunteers after the final exam.

Both studies also used data that were aggregated across classrooms. While data aggregation has its pros in terms of increasing sample sizes and, therefore, increasing the power of statistical comparisons, it may have cons, as well. Our main concern is that by considering students from different classes as one sample, important class-level differences may be overlooked. These differences, may contribute to differences in learning outcomes between classes. Examples of such class-level differences may include different quality of teaching of different instructors or different implementation of the same teaching approach.

Due to the limitations of the studies of Laursen et al. (2014) and Rasmussen et al. (2006) discussed above, we decided to explore the effects of student-centered instruction on student conceptual understanding using a validated content instrument, administered during class time at the beginning and end of the semester. We specifically focused on differential calculus as (1) it is one of the fundamental college mathematics courses, and (2) the content measure for this material was already developed and validated. We also decided to consider each class individually to explore the effects of instructional approaches holistically. In this study, we examined two different student-centered instruction types and one traditional instruction type. The decision to study two different student-centered instruction types instead of one is consistent with the suggestions drawn from the meta-analysis of Freeman et al. (2014). This meta-analysis encouraged further research to focus on "second-generation research," which compares courses that differ in active learning implementation, rather than on "first-generation research," which compares active learning courses with traditional ones. The three studied classes are described in the next section.

Methods

Context

Course. The study was conducted in the three course sections of Calculus I, the first course in the calculus sequence. This mainstream course has the traditional material on limits, derivatives, the integral, and culminates in the fundamental theorem of calculus. All three sections met twice a week for a lecture with a professor (1 hour and 50 minutes each) and once a week for a recitation with a graduate teaching assistant (50 minutes). The study was conducted during the same academic year with the data collection in the first two classes done in the fall semester and in the third class in the spring semester.

Lectures. The lecture portion of the first class was taught in an active learning classroom with most of the class time dedicated to conceptually oriented problem solving (COPS) and whole class discussion (the COPS class). The ALT classroom has 8 round tables with 9 seats at each table (a total room capacity is 72). The room also contains flat screen displays (one per table), and whiteboards that cover the walls. In the COPS class, class periods typically started with resolving any questions or problems that students encountered doing homework or that remained from the last class period. The professor would ask students to write their concerns on a whiteboard, and then have a whole class discussion to address the concerns. Then, the professor would lecture for a short period of time (10-20 minutes), followed by student active work that would take the majority of the class time. The active work typically included student group work on worksheets that consisted of conceptually oriented problem sets. The groups were self-selected and included 4-5 students each. The students were also encouraged to work on whiteboards to show their solutions. During this part of the class, the professor and undergraduate learning assistants walked around the classroom and talked to students to monitor their progress and answer or pose questions. If a common question or misconception arose, the professor would often address it via a whole-class discussion. To wrap up the active work, the professor would ask students to do a gallery walk and/or would hold a whole class discussion. At the end of the class, students typically turned in their completed worksheets.

The lecture portion of the second class was taught in a traditional lecture hall and implemented practice problems (PP) during lectures (the PP class). Similar to the COPS class, this class also started with the professor answering student questions. Then, the professor would present new material and work through an example problem. Next, students were asked to solve a similar problem in groups (i.e., their neighbors) or individually, as they preferred. During this part of the class, the professor and learning assistants circulated around the classroom to monitor student progress and answer questions. After most students finished, the professor would write down the solution suggested by the students and then discuss it with the whole class.

The lecture portion of the third class was also taught in a lecture hall but utilized primarily direct instruction (DI), the DI class. This professor prepared handwritten notes of the material (typically, proofs) and projected them on the screen in class. He/she would talk through the projected notes and then show an example problem on the board. This professor also incorporated graded quizzes in class which usually consisted of true-false questions (typically conceptual) to check student understanding of the material. Answers to the quizzes were discussed during the following lecture. In this class, no group work was utilized.

Recitations. Recitations for the COPS and DI classes were taught by the same teaching assistant in a primarily lecture style. This teaching assistant would typically answer student questions, if any, conduct quizzes if required by the professor, and then explain the material and show solutions for example problems. Recitations for the PP class were taught in an active style with most of the class time dedicated to answering students' questions, addressing their concerns, and clarifying misconceptions.

Participants

The three professors who participated in the study were experienced mathematics faculty members with similar goals for Calculus I classes. All of them aimed for students to have a mastery of both concepts and procedures by the end of the course. In addition, all of them wanted students to be actively involved in class and ask questions. The teaching assistants – recitation instructors – were both graduate students studying mathematics. The recitation instructor for the PP class was an experienced teaching assistant; the recitation instructor for the

Table 1
Sample Demographic Information

Characteristic	COPS (N=49)		PP (N=64)		DI (N=38)		Overall (N=151)	
	Frequency	%	Frequency	%	Frequency	%	Frequency	%
Student classification	N=47		N=63		N=37		N=147	
- Freshman	14	29.8%	43	68.3%	16	43.2%	73	49.7%
- Sophomore	21	44.7%	11	17.5%	14	37.8%	46	31.3%
- Junior	5	10.6%	5	7.9%	6	16.2%	16	10.9%
- Senior	7	14.9%	4	6.3%	1	2.7%	12	8.2%
GPA	N=46		N=56		N=35		N=137	
- 3.5 or better	15	32.6%	22	39.3%	6	17.1%	43	31.4%
- 3.0 up to 3.5	17	37.0%	23	41.1%	10	28.6%	50	36.5%
- 2.5 up to 3.0	11	23.9%	9	16.1%	11	31.4%	31	22.6%
- 2.0 up to 2.5	3	6.5%	2	3.6%	8	22.9%	13	9.5%
Gender	N=45		N=63		N=37		N=145	
- Male	23	51.1%	40	63.5%	25	67.6%	88	60.7%
- Female	22	48.9%	23	36.5%	12	32.4%	57	39.3%
Race/Ethnicity	N=44		N=62		N=37		N=143	
- White	23	52.3%	33	53.2%	18	48.6%	74	51.7%
- African-American	2	4.5%	4	6.5%	7	18.9%	13	9.1%
- Hispanic	4	9.1%	4	6.5%	2	5.4%	10	7.0%
- Asian	13	29.5%	16	25.8%	7	18.9%	36	25.2%
- American Indian or Pacific Islander	0	0%	0	0%	1	2.7%	1	0.7%
- Other/Mixed	2	4.5%	5	8.1%	2	5.4%	9	6.3%
Age	M=20.33 (SD=3.46); N=45		M=19.02 (SD=2.06); N=61		M=20.11 (SD=3.04); N=37		M=19.71 (SD=2.87); N=143	

COPS and DI classes was a new teaching assistant at the university where the study was conducted, though he/she had teaching experience at a different institution.

A total of 151 undergraduate students participated in the study (49 in the COPS class, 64 in the PP class, and 38 in the DI class). The students were enrolled in the Calculus I course at a large, suburban public university located on the east coast of the U.S. Student demographic information is presented in Table 1. In the COPS class, most participants were sophomores; in the PP class, the majority were freshman; in the DI class, freshman and sophomore students were enrolled in about the same proportion. Students' GPAs (self-reported) in all classes varied greatly. In terms of gender, in the COPS class, about a half of students were male, while in the PP and DI classes, the majority of students were male. Students also varied in race and ethnicity. In all classes, about a half of students were White and about a quarter were Asian. Notably, the DI class had more African-American students than the other two classes. Lastly, in the PP class, students were, on average, 19 years old; in the COPS and DI classes, they were, on average, 20 years old.

Procedure

The Calculus Concept Inventory (CCI; Epstein, 2007), a measure of conceptual understanding of differential calculus, was administered in all three classes at the beginning and end of the semester during recitations. Additionally, at the end of the semester, students were also asked to complete a demographic form. Students received a small amount of extra credit for participating in the study. They also received their individual scores on the inventory. The instructors received only class average scores. After the semester was over, the instructors also participated in interviews, during which they were asked mainly about their teaching practices in the classes in question and their teaching philosophies.

Results

Data Exploration

We computed descriptive statistics of CCI scores for each class measured at each point of time (the beginning and end of the semester). The averages and standard deviations are presented in Table 2. First, we were interested in whether students in each class showed growth over time. To answer this research question, we conducted three dependent samples t-tests with a Bonferroni correction ($\alpha=0.017$). The results revealed a significant effect of Time for the PP class, $t(63)=3.303$, $p=0.002$, but not for the COPS class, $t(48)=2.165$, $p=0.035$, or for the DI class, $t(37)=2.371$, $p=0.023$.

Next, we wanted to know if students in the three classes differed in their conceptual understanding on the pre- and posttest. To answer this research question, we conducted two ANOVA tests with a Bonferroni correction ($\alpha=0.025$). The results showed a significant effect of Class for both the pretest ($F(2,148)=5.466$, $p=0.005$) and the posttest ($F(2,148)=5.700$, $p=0.004$). Multiple comparisons – Tukey HSD tests – revealed a significant difference between the PP and COPS classes ($p=0.013$ for the pretest; $p=0.012$ for the posttest) and between the PP and DI classes on the pretest only ($p=0.017$).

Table 2
Descriptive Statistics for CCI

	Mean (SD)			
	COPS (N=49)	**PP (N=64)**	**DI (N=38)**	**Total (N=151)**
Pretest	6.43 (3.03)	8.06 (3.18)	6.37 (2.56)	7.11 (3.08)
Posttest	7.33 (3.60)	9.28 (3.77)	7.37 (3.11)	8.17 (3.66)

Differences in Conceptual Understanding between Classes over Time

To determine whether there was a difference in student conceptual understanding of differential calculus between classes over time, we conducted a mixed design ANOVA with Time as a within subjects factor and Class as a between subjects factor (see Figure 1). The results indicated a main effect of Time ($F(1,148)=19.160$, $p=0.000$), i.e., students' conceptual understanding, averaged across classes, was higher at the end of the semester ($M=7.11$; $SD=3.08$) than at the beginning ($M=8.17$; $SD=3.66$). The results also showed a main effect of Class ($F(2,148)=6.811$, $p=0.001$). Multiple comparisons – Tukey HSD tests – revealed that students in the PP class had significantly higher conceptual understanding, averaged across time, than students in the COPS ($p=0.005$) or DI ($p=0.009$) classes. No interaction effect between Time and Class was found ($F(2,148)=0.187$, $p=0.830$). Thus, the changes in students' conceptual understanding in all three classes were not significantly different from each other.

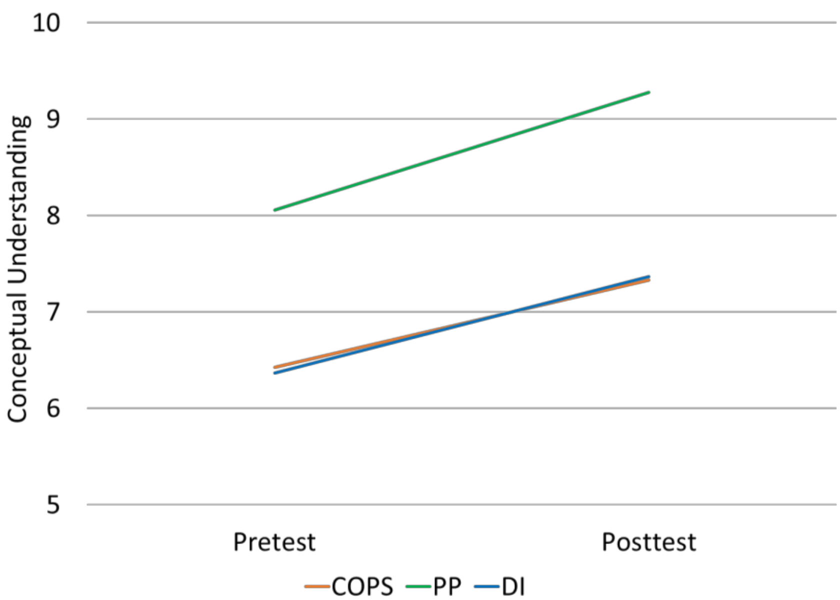

Figure 1. Pretest and posttest means for each class

Differences in Conceptual Understanding at the End of the Semester Controlling for Initial Understanding

To determine whether students differed in their conceptual understanding of differential calculus at the end of the semester controlling for their initial understanding, we conducted an ANCOVA test (see Figure 2). The results showed no difference between the classes in students' conceptual understanding at the end of the semester controlling for their initial understanding, $F(2,147)=1.065$, $p=0.347$. The adjusted means were as follows: 7.837 ($SE=0.398$) for the COPS class, 8.561 ($SE=0.353$) for the PP class, and 7.924 ($SE=0.452$) for the DI class.

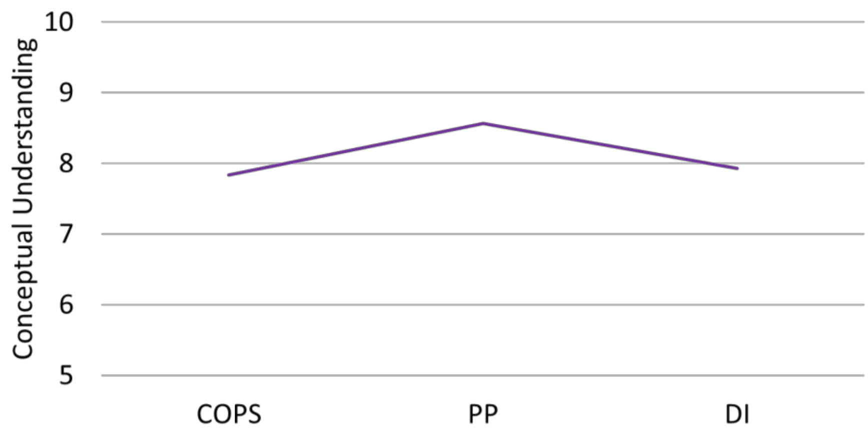

Figure 2. Adjusted posttest means for each class

Discussion

Our findings suggest that students' conceptual understanding of differential calculus is independent from the type of instruction when (1) conceptual understanding is measured by a validated, content instrument, and (2) the study is context-specific, i.e., when each class is considered individually instead of averaging across multiple classes. These nuances may explain why our results appeared to be different from the results of Laursen et al. (2014) and Rasmussen et al. (2006). In other words, implementation characteristics of a particular approach by a particular instructor in a particular course offering may lead to different levels of conceptual understanding and, therefore, need to be taken into account.

Among advantages of the study, we consider its ex post facto design, as no intervention was made. We aimed to explore the effects of teaching approaches in the most natural environment possible, and, therefore, chose to investigate the effects of the approaches typical to the instructors. Thus, this design provides a comprehensive picture of instruction implementation, where all elements of the instructional approaches are considered together. At the same time, a comprehensive picture of instruction has the disadvantage of making those elements with the most influence on the outcome challenging to identify. Therefore, future research should explore more context specific variations in approaches' implementation to determine potential commonalities between the effective ones. Another disadvantage of our ex post facto, context-specific design is a possibility for confounding variables to occur, as no control over the approaches is used. For example, the direct instruction in the recitations of the COPS class may have cancelled out the effect of conceptually oriented problem solving in the lecture periods. Finally, our study design substantially limits the generalizability of the findings.

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Student's Ways of Thinking In a Traditional and Inquiry-Based Linear Algebra Course

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ABSTRACT: *In the evaluation study presented in this paper, the authors compared the mathematical thinking of undergraduate students (as they responded to class work and interview prompts) who participated in an inquiry-based linear algebra course to a comparison group of students who participated in a traditional course.*

Key words: linear algebra, realistic mathematics education, inquiry based learning

The Problem

The literature on undergraduate mathematics shows that in traditional linear algebra classrooms students often memorize algorithmic methods that "work" even when not properly understood. Students develop understandings of matrix algebra and solving systems of linear equations using the Gaussian elimination, yet have problems with the more abstract notions of spanning set, linear independence and linear transformation (Stewart & Thomas, 2010). In the language of Sierpinska's modes of thinking (2000), the student dependent upon reducing a matrix to echelon form to determine whether vectors are linearly independent are thinking in *arithmetic mode*, whereas a student who is able to think more generally about objects as parts of a system, by applying a definition or theorem when appropriate, is thinking in *structural mode.* According to Bogomolny (2007), a notion of linear dependence as an object requires an understanding beyond the actual or intended procedures of row reduction toward an understanding of the structure of linear dependence relations as a set of vectors. While one mode of thinking is not given precedence over the other, and row-reduction and similar techniques are an important component of linear algebra, it is important for students to be able flexibly use an appropriate mode in a given context. According to Sierpinska (2000), despite a decade of innovations in the teaching of linear algebra "the students in our experiments could not understand the theory because they appeared to want to grasp it with a 'practical' rather than a 'theoretical' mind." (p. 211). Dubinsky's APOS theory (1994) has been used extensively since to characterize the ways in which students struggle with linear algebra concepts (eg. Stewart & Thomas 2010, 2009; Wawro, Rasmussen, Zandieh, Sweeney & Larson, 2012;). An inquiry based learning (IBL) course in linear algebra was designed based on the notion of mathematics as "mathematizing" (Freudenthal, 1991) in order to circumvent a purely arithmetic understanding of linear algebra concepts. This study compared the thinking of linear algebra students that participated in the IBL course to a comparison group of students who participated in a traditional course.

Theoretical Framework

Our framework is adapted from Stewart & Thomas (2009, 2010). Tall's three mathematical worlds (embodied, symbolic and formal) depict a progression in the development of mathematical thinking (Tall, 2004)) and coincide with the modes of reasoning purported in Sierpinska (2000). The *embodied world* uses physical attributes to build conceptions. Similarly the synthetic-geometric mode of thinking is one that is intuitive, practical and is used as a heuristic tool to aid thinking in the analytic modes. Tall's symbolic world is one in which symbolic representations of concepts are acted upon and manipulated. This corresponds to Sierpinska's *arithmetic mode* in which the appropriate use of formulas and techniques that allow one to solve problem concerning vectors and matrices are developed. The *symbolic world* is where Dubinsky's (1994) actions, processes and objects are constructed and symbolized.

According to APOS theory an action occurs as an instance of problem solution by the student—it is dependent upon a particular problem (Dubinsky & McDonald 2001). When this action is reflected upon, in such a way that the action can be imagined without being carried out, then the action has become a process. A process becomes an object when the student becomes aware of the totality of the process and is able to encapsulate it as a single entity or object. In Tall's formal world the properties of objects become formalized into axioms. This corresponds to Sierpinska's *structural mode* emphasizing connections between concepts and is concerned with vector spaces and linear transformations. The formal world with its emphasis on structural thinking is where objects become part of the learner's schemata. To illustrate consider Figure 1 below.

	Embodied World-Visual-Geometric Thinking	**Symbolic World - Arithmetic Thinking**	**Formal World – Structural Thinking**
Action	Can add multiples of two given vectors in R^2 or R^3 to visually determine whether a third is a linear combination of the given vectors.	Can test if a set of vectors is linear independent by constructing a matrix with the vectors as columns and row reducing it.	
Process	Can generalize this visualization process to two arbitrary vectors v_1, v_2, v_3 in R^2 or R^3.	Can think about the action above without actually carrying it out.	
Object	Can operate on this visualization of linear independent vectors (eg. transforming them via reflection, rotation, etc)	Understands process as above and can operate on the resulting matrix (eg. knows that if matrix has a pivot in every column then the original set of vectors is linear independent.	Shows set of given vectors is linear independent by definition by considering the vectors space that the vectors are in (eg. gives a dimensional argument)

Figure 1. Framework Using Three Worlds and APOS With Illustrations of Linear Independence.

Inquiry Based Learning Course

The IBL courses offered at this university are based on the premise that all of mathematics teaching and learning should consist of students mathematizing, as proposed in Realistic Mathematics Education (RME) (Freudenthal 1991, Gravemeijer, 1994, 1997). Within RME a learner comes to "know" by re-inventing mathematics, with the teacher as guide, as the learner responds to authentic problem contexts that are chosen to bring forth big ideas in mathematics (Schifter & Fosnot, 1993). Big ideas are the central, organizing ideas of mathematics—principles that define mathematical order such as linear independence, basis and linear transformation. The emergence of big ideas in learners occurs as a certain shift in mathematical thinking often evidenced by changes in strategy and model use. When learners participate in this kind of mathematical inquiry they are said to be mathematizing. According to Treffers (1997) *horizontal mathematizing* involves students in developing mathematical tools that help them to organize and solve a problem located in an embodied context that they can then generalize to use in other embodied contexts. *Vertical mathematizing* involves the forming of conjectures while working on problems that encourage finding shortcuts and discovering connections between concepts and strategies and then applying these discoveries (Treffers, 1987). Both types of mathematizing were incorporated into this course.

Horizontal mathematizing influenced the choice of course materials (Wawro, Rasmussen, Zandieh, Sweeney & Larson 2012), whereas the vertical counterpart influenced the structure of the centerpiece assignment in the course – the course wiki. In lieu of a textbook, students worked from a scaffolded sequence of tasks (the course notes) and recorded their findings in the form of a whole-class wiki, which ultimately served as their reference text. All of the Inquiry Oriented Linear Algebra task sequences were incorporated into the course notes (Inquiry-Oriented Instructional Materials website, n.d.). However, instead of presenting students with theorems, the students generated conjectures, which they recorded, revised and supported in the course wiki, which was intended to support student ownership of the material. Students could support conjectures with examples or proofs and this support was revised throughout the semester. These activities were intended to support students as they began to transition from embodied and symbolic thinking to structural thinking and participation in the formal world (Tall, 2004). Additionally, students build ``concept pages'' in the wiki, which included i) the definition of the concept, ii) information on and hyperlinks to key examples that illustrate applications of the concept and iii) additional information that provides intuition for the concept, information on the importance of the concept or explains the origins of the concept, including hyperlinks to relevant section and concept pages. All intended as opportunities for mathematizing.

Methods

During Fall 2014 data were collected from the linear algebra course as part of a larger research study to trace the efficacy of inquiry-based courses. Data were collected via examination questions and task-based interviews in order to compare the thinking of students in the inquiry and traditional classes. Two common final exam problems were given. Responses were collected and compared between student in the inquiry-based course (n=20) and a comparison group of student in the traditional course (n=20). Students in the comparison group were top course performers purposively sampled from those students who had expressed an interest in taking the inquiry-based course but who had not fit that course into their schedule. These students were enrolled in a traditional course with an experienced instructor.

Participants

In addition to the student responses described above, four students in the inquiry group and four students in the traditional group agreed to participate in an interview. The interviews were conducted during week 10 of the quarter and took between 30 and 40 minutes each.

Data

Final examination questions. Two common problems (see Figure 2 below) were placed on the final in both classes. In order to ensure that the items were accessible to the traditional students, Professor 1 who was teaching the traditional course shared his exam with Professor 2, the instructor for inquiry-based course, 3 days before the final was to take place. Professor 2 selected the two items from Professor 1's exam. Responses to these items were obtained from the course instructors. The data were used to compare students' ways of thinking about linear algebra concepts.

Problem 1a and b

3. Suppose the reduced row echelon form of a matrix A is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. How many solutions are there to $A\vec{x} = \vec{0}$? Does $A\vec{x} = \vec{b}$ always have a solution, or is it inconsistent for some vectors $\vec{b}$?

Problem 2

. Is $\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizable?

Figure 2. Common Final Exam Problems.

Task-based questions from interviews. Participants were asked to (1) describe their learning experiences in their linear algebra course, (2) express the big ideas in the linear algebra class, (3) talk about how they would solve two linear algebra problems that involved an understanding of span, linear independence, basis and linear transformation (figure 7 below).

Analysis and Results

Final exam question analysis. Student graded responses to final exam items and verbal and written/verbal responses to the task-based interview questions were coded in three ways: (1) Responses were categorized as correct/ incorrect; (2) Responses were open coded number based on the type of solution strategy used; (3) Strategies codes where grouped into themes using the framework in figure 1 as a guide.

Table 1. *Comparison of Errors Between Groups*

	Inquiry- Based Group (n=20)		Traditional Group (n=20)	
	# Computational errors	# Conceptual errors	# Computational errors	# Conceptual errors
Problem 1a	1	2	0	3
Problem 1b	0	1	1	4
Problem 2	3	3	6	3

While the number of errors in the groups were quite similar (slightly favoring the inquiry group) very different solution methods between the groups for problem 1a and 1b were observed. In problem 1a, the inquiry group most often found the kernel or the solution set to the homogeneous system to show that there are infinitely many solutions (9/17 with correct answers), whereas the majority of the traditional groups stated, "there was a free variable" (13/17 with correct answers).

Arithmetic thinking with connection to embodied world.	1. (15 pts) Suppose A is a 3×3 matrix with reduced row echelon form $\begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\vec{0}$ $x+2y=0$ $z=0$ $x=-2y$ $z=0$ $s\begin{bmatrix}-2\\1\\0\end{bmatrix}$ (a) How many solutions are there to $A\vec{x}=\vec{0}$? There are an infinite number of solutions to $A\vec{x}=\vec{0}$ such that $\vec{x}$ lies on the line defined by: $s\begin{bmatrix}-2\\1\\0\end{bmatrix}, s\in\mathbb{R}$
Arithmetic Mode of Thinking	(b) Does $A\vec{x}=\vec{b}$ always have a solution, or is it inconsistent for some vectors $\vec{b}\in\mathbb{R}^3$? If, after row reductions to RREF, the 3rd entry is no zero. (i.e 3rd row looks like: $0\,0\,0 \mid b \Rightarrow b \neq 0$) b/c there is no way for $0x+0y+0z=b$ to have a solutio
Structural Thinking/ connection to embodied world.	(b) Does $A\vec{x}=\vec{b}$ always have a solution, or is it inconsistent for some vectors $\vec{b}\in\mathbb{R}^3$? $A\vec{x}=\vec{b}$ does not always have a solution because the column space of A does not include all of $\mathbb{R}^3$ (the first and se columns are linearly dependent). As a result, any vector $\vec{b}$ does not lie on the x, y plane will not satisfy this equation.

Figure 3. One of the Common Final Exam Problems

The expectation in the inquiry-based class for the student to justify their answers may be responsible for this difference. For problem1b, most students across groups gave an example of a vector that made the non-homogeneous system inconsistent (comparison group 9/15, inquiry group 7/ 19), however 10 of the 19 inquiry based group for whom the problem was scored as correct and only 3 of the 14 in the comparison group gave an argument that took into consideration the fact that they were working with the augmented matrix rather than the matrix itself.

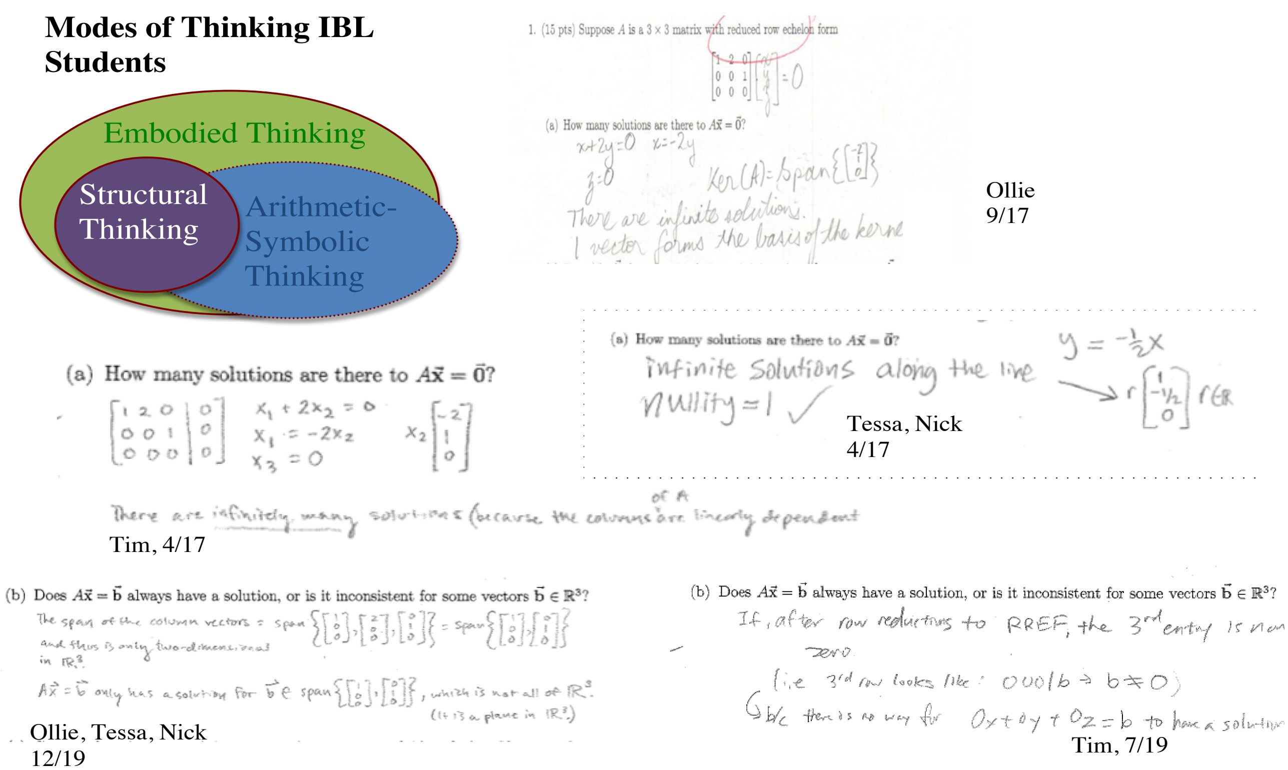

Figure 5. IBL Student Thinking

Using the framework to qualitatively classify student responses results indicate that in some instances 2 times as many students in the inquiry-based group were operating at both the *arithmetic* and *structural modes* (Exam question 1 part a: 7, 5) (Exam question 1 part b 10, 5 respectively). The solution methods were quite different between the groups with the inquiry-based group flexibly using two modes of reasoning--solving the system arithmetically followed by structural thinking to create the solution space while the traditional group used purely arithmetic modes of reasoning.

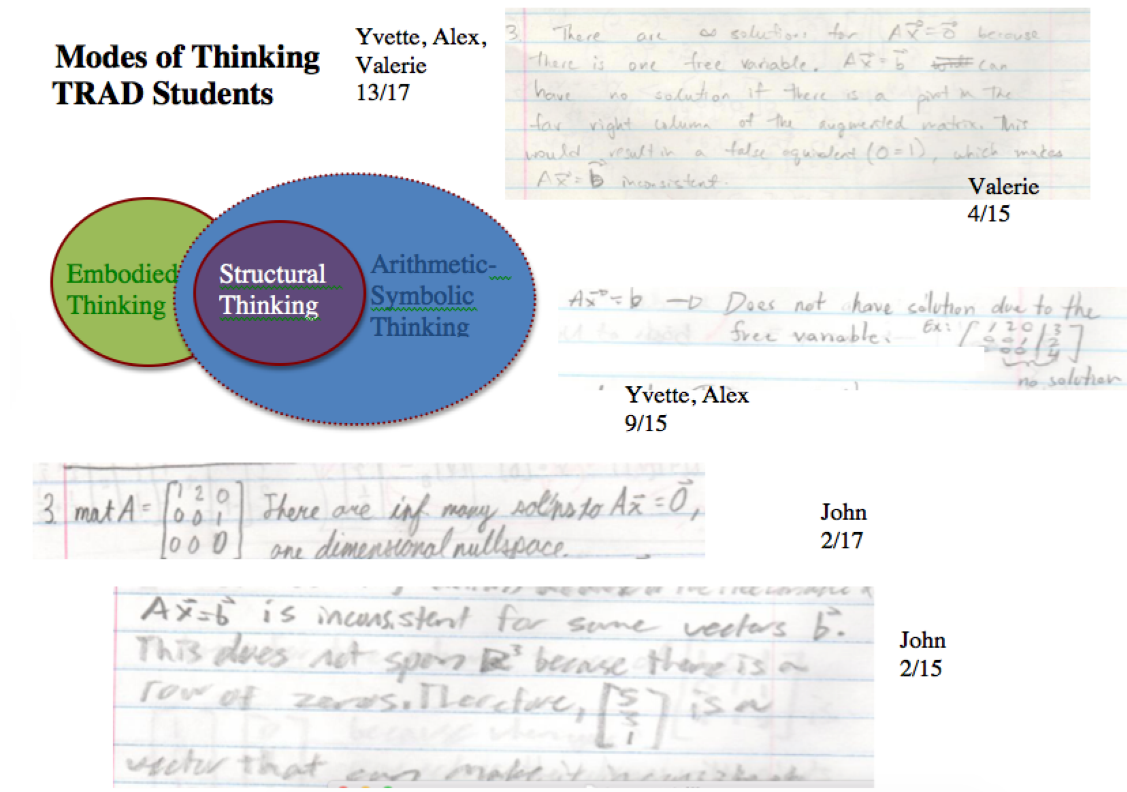

Figure 6. TRAD Student Thinking

Interview Analysis

In order to investigate students' modes of thinking in more depth, we interviewed eight students (four traditional and four students in the inquiry-based course). Table 2 below shows these interview participant demographics.

Table 2. *Interview Participant Demographics*

	Physics		Chemistry		Computer Science/Engineering	
	Male	Female	Male	Female	Male	Female
4AI (n=4)	2	1	1			
4A (n=4)	1				2	1

All participants were freshman, had comparable mathematics backgrounds and this course their first college math course. The purpose of the interviews was to further explore students' thinking about the linear algebra topics they were learning—in particular to explore their concept image (Tall & Vinner, 1991) of spanning set, linear independence, basis and transformation and their perspectives on the big ideas in linear algebra. Analysis of students' ideas about the big ideas in linear algebra is currently being analyzed to triangulate the findings from the common final items and the interview tasks. Table 3 below shows how these eight students' thinking on the final exam its were categorized.

Table 3. *Interview Participants' Thinking on Final Exam Items.*

Name	Part A		Part B	
	Strategy	Type of Thinking	Strategy	Type of Thinking
Alex (TRAD)	States free variable	Arithmetic Thinking/Process conception	Uses free variable	Cannot determine
John (TRAD)	Uses dimension of the null space	Structural reasoning	Columns don't span R^3	Structural reasoning
Yvette (TRAD)	States free variable	Arithmetic Thinking/Process conception	Uses free variable	Cannot determine
Valerie (TRAD)	States free variable	Arithmetic Thinking/Process conception	Gives a vector that doesn't have a solution	Arithmetic thinking
Nick (IBL)	Solves and presents solution set as a line	Geometric/Arithmetic	Span of LI column vectors does not span all of R^3	Structural reasoning
Ollie (IBL)	Uses dimension of the kernel	Structural reasoning	Span of LI column vectors does not span all of R^3	Structural reasoning
Tessa (IBL)	Solves and presents solution set as a line	Geometric/Arithmetic	Span of LI column vectors does not span all of R^3	Structural reasoning
Tim (IBL)	Uses columns are linearly independent	Structural reasoning	Gives a vector that doesn't have a solution	Arithmetic thinking

Figure 7 below depicts the strategies used by students on interview tasks and illustrates how the interview responses were coded similarly to the common final items and according to the three modes of thinking (Sierpinska, 2000). The first interview item was given to see if students had made connections between linear independence and spanning set. It was also used to see whether students were able to instantiate formal definitions and use them to form logical arguments. All four of the inquiry-based and two of the traditional group were able to instantiate and act upon their understanding of the connection between span and linear dependence to solve the problem. These students had an object conception of span and linear dependence and were operating in the formal word.

Interview Problem 1 *Suppose that Span {$v_1, v_2, v_3, \ldots\ldots, v_n$} = V and w is a vector in V.* *Is the set {$v_1, v_2, v_3, \ldots, v_n$, w} linearly independent (Justify your answer)?*	IBL Group (n=4)	TRAD Group (n=4)
<u>Structural Thinking.</u> Acts upon the definition of span to argue structurally that since w is in the spanning set then by definition it is a linear combination of a subset of the v_i and hence by definition of LI the set is not LI.	3 (Tim, Tessa, Ollie)	1 (Valerie)
<u>Geometric Thinking</u> Acts upon the definition of span and LI and thinks geometrically. When I think of span I think of where we can go if these v_1, v_2, ...v_n are means of transport. Span is all of the possible regions you can reach with these linear combinations. So now w is a vector in V and so is already reached by this set of vs – it doesn't really take me anywhere new so therefore the set is linearly dependent.	1 (Nick)	
<u>Erroneous Structural Thinking</u> Incorrectly acts upon the definition in the problem. Erroneously assumes that the space is n-dimensional and uses dimensional argument.		2 (Alex, John)
<u>Arithmetic Thinking</u> Thinks arithmetically, bypasses notion of span and thinks of LI as a process. "Row reduce" homogenous matrix or set up an augmented matrix.		1 (Yvette)

Interview problem 2:

Let $T : R^2 \to \mathbb{R}^3$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 3x-y \\ 2x+4y \end{pmatrix}$$

Is it possible to find v_1, v_2, $v_3 \in \mathbb{R}^2$ such that $T(v_1)$, $T(v_2)$, $T(v_3)$ is a basis for $\mathbb{R}^3$?

<u>Geometric-Embodied/Structural Thinking.</u> Recognizes that the image of T is a two-dimensional subspace of R^3 hence not possible to find a basis for R^3.	Ollie, Tim	John
<u>Arithmetic-Structural Thinking</u> Starts to row reduce the matrix that defines the transformation to find the basis and concludes that no such basis can exist based on a dimensional argument.	Nick	
<u>Arithmetic Thinking</u> Say that it is possible by mapping the standard basis in R^2 via the transformation but does not complete the calculations.	Tessa	
<u>Arithmetic Thinking</u> Says that the basis exists and the way to find it is by row reducing the matrix that defines the transformation.		Valerie
Says it is possible but does not give reasoning.		Alex
Doesn't attempt the problem		Yvette

Figure 7. Coding Examples of Interview Questions

Even though this item was constructed to circumvent an approach that used matrix multiplication two of the traditional group demonstrated that they were dependent upon an action concept of linear dependence by saying that they would set up the vectors as columns in an *m x n* matrix and row reduce to see if the RREF was the identity matrix. This type of solution is dependent upon being given actual numbers and hence these students were not able to solve the problem (see Yvette example in Figure 8 below).

Suppose that Span $\{v_1, v_2, v_3, \ldots\ldots, v_n\} = V$ and w is a vector in V. Is the set $\{v_1, v_2, v_3, \ldots, v_n, w\}$ linearly independent (Justify your answer)?	
Category	**Example of student thinking**
Arithmetic thinking with process conception of linear independence.	"OK so for this one I would, wait ... [writes the vectors v_1, $v_2, \ldots, v_n$ as the columns of a m x n matrix]. I would just row reduce this and see if there are any free variables in the set. If there were then that would mean that the set were linearly dependent. If it were linearly independent there would be no free variables and there would be a pivot in every column." [Yvette]
Geometric thinking, using the "model for" in MCR (Wawro, Rasmussen, Zandieh, Sweeney & Larson, M. 2012)	When I think of span I think of where we can go if these v_1, v_2, ….v_ns are means of transport. Span is all of the possible regions you can reach with these linear combinations. So now w is a vector in V and so is already reached by this set of vs – it doesn't really take me anywhere new so therefore the set is linearly dependent.
Structural thinking but unable to "operationalize" the notation in the given definition.	The span of these vectors is this vector space and w is a vector in v, so is this set linearly independent? Well let's see, I am going to try to do a counter example if we have…so what I am going to do I know I have n vectors so I know that I can have at most n vectors in R^n, so if I added w to the set then that would exceed the number of vectors that I can have therefore this is not linearly independent." [John]
Structural thinking with object conception of linear independence.	The set containing w would be linearly dependent because if you have a span of all of these vectors and w is in that span then by definition you can add up some number of these, a linear combination of the vs, to get w [Tim, Tessa, Ollie, Valerie]

Figure 8. Examples of Student Thinking on Interview Item 1.

The remaining two students from the traditional group erroneously assumed that $V = R^n$ and incorrectly used the proposition proved in class that states there are at most n linearly independent vectors in a basis for R^n. These students appeared to be trying to operate in the formal world but possibly have not developed an object conception of linear dependence (see John example in Figure 8 above).

The second item, depicted in Figure 9 below, was used to understand whether students had formed an object conception of basis. Two of the inquiry students and one of the traditional students combined geometric and structural thinking (see John example). One of the inquiry students showed an object conception of basis by operating between the arithmetic-structural modes by reducing the matrix that defines the transformation to RREF and concluding that a basis does not exist as the matrix has 2 linearly independent columns (see Nicks example). Two traditional students showed erroneous arithmetic thinking by stating that such a basis did exist by reducing the transformation matrix to echelon form.

<table>
<tr><td colspan="2">Let $T : R^2 \to \mathbb{R}^3$
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 3x-y \\ 2x+4y \end{pmatrix}$$
Is it possible to find $v_1, v_2, v_3 \in \mathbb{R}^2$ such that $T(v_1), T(v_2), T(v_3)$ is a basis for $\mathbb{R}^3$?</td></tr>
<tr><td>Geometric Thinking</td><td>“I would have to say that you cannot because you are starting with a two dimensional space and this just transforms it to a plane. A plane cannot represent the whole space.” [John]</td></tr>
<tr><td>Thinking structurally across worlds</td><td>“It gives you a transformation and asks if it possible to find 3 vectors in R^2 such that the transformation in R^3 forms a basis for R^3. So if these three are a basis in R^3 then that means that they have to be independent. Now the transformations come from R^2 over here so one thing we can do is just try to find a matrix for the transformation. So just using these components work, this has to be 3,1 this has to be 3, -1 and this has to be 2, 4. Now the transformation acts on these vectors in R^2 we might have v_1, v_2 and v_3 and we want to check that $T(v_1)$, $T(v_2)$, $T(v_3)$ are independent, right?

If we get this to RREF, if we get the matrix associated with the transformation into RREF then that is going to end up being this matrix right here, because we only have 2 pivots because this…if you want to multiply a matrix and you are given three vectors here, v_1, v_2 and v_3, let’s make them into a matrix V. Actually lets call it [x_1, y_1; x_2, y_2; x_3, y_3] that gives us a 2 by 3 matrix and if we want the columns of this matrix to be independent then we are going to have 3 pivots but this matrix has only 2 pivots and this matrix at the best can only have 2 pivots because the pivots can never exceed the number of rows and columns. In matrix multiplication all you are doing is looking at linear combinations of rows and columns you can never have more than 2 pivots in the answer so this is impossible. “ [Nick]</td></tr>
</table>

Figure 9. Example Student Responses From Interview Item 2.

One student from the inquiry-based class started by applying the transformation to the standard basis in R^2 to get a basis for R^3. She convinced herself that this was a viable approach and that a basis did exist. One traditional student stated that such a basis could be found but did not give any rationale. The final traditional student did not attempt the problem.

Significance

Few studies have compared the understanding of students in inquiry-based versus traditional courses. We have presented data that suggests ways in which students in an inquiry-based linear algebra course have different understandings of concepts than students who have taken a traditional course. One reason that comparison studies are rare is due to the self-selection of students into non-traditional courses. This study circumvented the issue by forming the comparison group with students who self-selected into the non-traditional course but who could not take it due to schedule conflicts.

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Data Cleaning in Mathematics Education Research: The Overlooked Methodological Step

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The results of educational research studies are only as accurate as the data used to produce them. Drawing on experiences conducting large-scale efficacy studies of classroom-based algebra interventions for community college and middle school students, I am developing practice-based data cleaning procedures to support scholars in conducting rigorous research. The poster identifies common sources of data errors in mathematics education research and offers a framework and related data cleaning process designed to address these errors.

Key words: Research methodology, Efficacy studies, Algebra

The results of educational research studies are only as accurate as the data used to produce them. Screening data for potential errors and ensuring anomalies do not influence analyses is an essential step of the research cycle (Wilkinson, 1999). Odom and Henson (2002) demonstrated how regression models of high school seniors' mathematics achievement varied depending on the level of screening applied to the publicly available High School and Beyond National Survey data set. As another example, Whalley (2011) analyzed the Panel Study of Income Dynamics data set to show how the choice of trimming procedures (i.e., methods for removing outliers) effected estimates of the relationship between education and labor income volatility. Educators and policy makers rely on study results to make decisions that influence the lives of many. It is important that scholars understand and apply appropriate methods for ensuring high quality data.

The process of identifying, resolving, and documenting data inconsistencies is called *data cleaning* (Rahm & Do, 2000). Despite the importance of data cleaning in rigorous research practice, most methodology courses only give cursory attention to the topic (Osborne, 2013). Therefore, scholars often acquire cleaning strategies heuristically, making it difficult for others to accurately judge or replicate studies (Leahey, Entwisle, & Einaudi, 2003). Furthermore, my conversations with scholars new to large-scale research suggest many underestimate the amount of time and resources required to properly clean data. Data collection and data preparation each can take about 20% of project time (Munson, 2012). Well-established standards for data cleaning could facilitate the integration of this topic into researcher training and support educational researchers in accurately planning their studies.

Drawing on my experiences conducting statewide and nationwide efficacy studies of mathematics interventions, I am developing practice-based data cleaning processes to support research in classroom settings. While data cleaning can be applied to many forms of data (e.g., interviews, observations, documents), I focus on quantitative data gathered from surveys, questionnaires, assessments, and demographic records. Specifically, I ask: (1) What are the sources of data errors and challenges in educational research studies conducted in authentic mathematics learning environments? (2) How can a data cleaning process be designed to consistently produce accurate, reliable, confidential, and timely datasets?

Methods

Two large-scale efficacy studies inform the framework presented here. *Study A* was a three-year, nationwide study of revisions to a popular mathematics curriculum involving over 10,000 middle school students and 180 mathematics teachers. Students completed between five and eight end-of-unit assessments on paper, two attitude surveys, and two mathematics assessments either on computers or on paper. Teachers completed weekly logs and two teaching knowledge assessments, all electronically. School districts provided demographic data and state test scores for participating students. *Study B* is a two-year statewide study of the use of a computerized interactive learning platform in community college elementary algebra courses. The first year of the college-level study involved approximately 400 students and 89 instructors across the state; the second year of the study is underway. In this study, all data is collected electronically. Students complete two mathematics assessments, a background questionnaire, an academic motivation questionnaire, and an end-of-semester survey. Log data of student interactions in a web-based activity and testing system are also collected. Instructors complete assessments of knowledge of technology in teaching and content knowledge for teaching, background questionnaires, and weekly logs.

An initial data cleaning process was created for Study A based on prior experiences with small-scale research and evaluation projects. Across the three years of Study A, the data team documented their data related challenges and associated resolutions, revising the Study A data cleaning process as they went along. The revised process was also compared against rigorous research standards in the What Works Clearinghouse Procedures and Standards Handbook (2016), ethnical research guidelines around privacy and confidentiality (OHRP, 1993), general data modeling rules from the field of computer science, and data management practices used in educational survey research (e.g., Schleicher & Saito, 2005) and in statistics (e.g., de Jonge & van der Loo, 2013). The modified data cleaning process is being implemented in Study B.

Researcher's Role and Background

Describing one's background and one's role in research allows readers to understand the perspective researchers bring to their work (Creswell, 2012). Drawing on my undergraduate training in computer science, I applied my knowledge of data modeling and databases to the framework described in this paper. My research training in learning sciences provided me with the domain knowledge needed to understand the contexts of Study A and Study B, to understand how data could be organized for useful analysis, and to identify anomalies that might signal problems at other stages of the research cycle.

I was involved in multiple aspects of Study A including participant management, data collection, data cleaning, data analysis, instrument creation, classroom observations, meetings with the research team, and professional development workshops for participating teachers. This level of involvement gave me a chance to confront data issues directly at many points across the study. For example, answering participant phone calls about the study provided insights into the ways teachers implemented study data collection tasks, which sometimes conflicted with researcher expectations (e.g., administering student assessments across multiple days). Working with researchers to conduct item-response theory (IRT) analyses and teams to score constructed response items highlighted the importance of distinguishing the reasons for which data were sometimes missing. My involvement in Study B was constrained to data management, working with a team to clean data files, and interacting with the project staff to stay abreast of study developments. This narrower role allowed me to focus more on the refinement of the data cleaning process.

Results

Data Errors and Challenges

Despite standardized procedures for administering and gathering data, data collection in large scale educational studies often result in a host of data cleaning errors that are, to some extent, unavoidable. Errors can include duplicated records, illegal values, missing values, or misspellings (Rahm & Do, 2000). In the studies described here, errors in the gathered data created the need to make decisions about issues such as handling duplicate records, the validity of an assessment completed on an incorrect form, and how to link records in a hierarchical research design when participant identifiers changed. Challenges in study implementation hindered the timely collection and cleaning of study data. Common sources of error and challenges in data cleaning for both studies are described below.

Variations in assessment administration. Schools and colleges differed in their schedules and access to computers. For example, many middle school class periods lasted 45 minutes while other schools operated on block schedules where class periods lasted 90 minutes. Some teachers with shorter class periods would administer paper-based assessments across two days, having students complete selected response items on the first day and constructed response items on the second day. The educational institutions within which Study A and Study B were conducted varied widely in their computer availability. Teachers with computers inside their classrooms easily administered online assessments for the study. However, teachers who had to reserve a computer lab often lost time in transitioning to the lab room or they held multiple administration sessions due to an insufficient number of computers for their students. A handful of teachers had no computer access and administered study assessments on paper. Lastly, some teachers requested Spanish versions of study assessments, which were only available on paper.

Understandably, teachers were responding to the realities of their school environments. However, some administration choices led to students completing part of an assessment on the wrong form or completing the assessment more than once. Differences in administration also complicated data cleaning processes. An assessment completed on paper, where students could write what they wanted, required different cleaning checks than the same assessment completed online, where computer-based forms restricted the possible answers permitted. Also, it became more difficult to account for test completion, both at the student level and the class level, when some items were received on paper and others online.

Participant mobility and late joiners. Some participants joined the studies after baseline data were collected and others changed institutions during the studies. This mobility was explained by various factors. First, families moved and in doing so placed their children into new school zones. Student mobility is common at the K-12 level in the U.S., particularly amongst students from urban areas, lower income families, or migrant, military, or immigrant families (Welsh, 2016). Student mobility occurred in Study A, particularly in schools near the U.S. border with Mexico that served large numbers of migrant families. Second, many community college instructors work across multiple institutions. In California, the location of Study B, 36% of associate faculty (e.g., part-time or adjunct instructors) teach at more than one institution in order to make a livable wage (Smith, 2013). While instructor mobility was not a significant issue to data collection for Study B, one participant unexpectedly taught the study-target course at different colleges each semester of the study. Third, research teams in both studies were confronted with low enrollment numbers and participant dropouts. Recruitment and retention of study participants in large-scale educational studies is challenging because it requires a long-term commitment from teachers who already have busy schedules (Gallagher, Roschelle, & Feng, 2014). This issue required (a) additional rounds of recruitment to obtain participant counts that allowed for sufficiently powered statistical analyses and (b) an extension of administration windows to allow for greater data collection. Lastly, a teacher strike that occurred during Study A delayed data collection for one district containing several consented participants.

Participant mobility resulted in some student participants in Study A moving between treatment and control groups and completing pre- and post-intervention assessments under different experimental conditions. In other instances, records at a given level of the study design appeared to be missing a link to records at the other levels of the study design. For example, when a teacher in Study B changed institutions, she was assigned a new participant identifier. During the cleaning process, data from students in her first semester course appeared unconnected to any teacher. Issues resulting from participant mobility introduced the need to (a) create new versions of data files that included late joiners and (b) make decisions about how to resolve participants linked to multiple classes, teachers, or schools.

Multiple participant names. Some participants became associated with multiple names and e-mail addresses and some had names that changed. In Study A, we often saw the name a student wrote on assessments differed from the name on the teacher's roster, which differed from the name provided by districts when collecting demographic information. In Study B, students used both school-provided and personal email addresses when completing study tasks. At the teacher level, names occasionally changed when participants changed marital status during the study.

As a consequence, participants sometimes appeared to have missing records because their data could not be matched to the legal name or school email address provided to the research team. Connecting individuals to the correct names and e-mails, a process called identity resolution in the field of computing, was time consuming and usually required direct communication with participants. Identity resolution was further complicated by the fact that some study participants had the same or similar names, sometimes within the same classroom.

External Vendor Systems. Assessment vendors (e.g., a company that hosts a website through which participants complete a test) and school districts were *external vendors* who provided participant data and hosted instruments in both studies. Over 65 school districts across 22 states provided demographic information and state standardized test scores for middle school student participants in Study A. Assessment providers host copyrighted instruments on their own websites and had specific rules regarding how paper versions of their assessments could be administered. Study A and Study B both used assessments offered through the University of California at San Diego's Mathematics Diagnostic Testing Project (MDTP).

External vendors typically used varied and conflicting conventions for data values. For example, in Study A, the ethnicity definitions across elementary school districts were wide-ranging. The category of Black or African American had values such as: 1, 4, Black/African American, Bl, and African Am. We decided to use standard categories provided by the National Center for Education Statistics[1] and transformed data values into these conventions. Confidentiality was also an issue because external vendors needed to identify participants but could not be provided with the identifiers used by our research team. If external vendors saw our research identifiers, they could easily identify specific participants in our publicly released datasets. This necessitated an additional set of interim identifiers to allow our data cleaning team to map data received from external vendors with our own participant records.

[1] Common Education Data Standards (CEDS) also provides a data dictionary for information related to pre-school through post-secondary educational environments that can be used to establish conventions for an educational research study.

Data Cleaning Process

My experiences with Study A highlighted the need to attend to data cleaning at all phases of the research cycle. I developed a list of tasks to accomplish during data planning, data collection, and data cleaning to minimize the issues likely to occur in educational data sets (see Table 1). My goal was not to eliminate issues from occurring, but rather, to create a reproducible process to improve the identification and handling of data errors in the cleaning process. Below I describe these tasks and their rationales.

Table 1

Data Cleaning Tasks

Task
Data Planning Phase
Create visually distinct instrument forms
Clarify data requirements and timeline with external vendors
Set administration windows and a last enrollment date
Determine decision rules for handling duplicate records
Develop codebooks for each instrument to describe variables and their possible values
Data Collection Phase
De-identify study data as early as possible in the data collection process
Collect details on how assessments were administered and any anomalies that occurred
Create three sets of identifiers for participants: one used by data collectors, one used by researchers, and one used by external vendors
Make sure identifiers do not depend on malleable participant characteristics
Verify administration dates on completed study instruments are valid
Data Cleaning Phase
Use tools that allow you to log and retrace your data cleaning steps
Establish a review process so data cleaning work can be checked by another person
Make a copy of your raw data file and only work with the copy
Check data files for missing and extra data columns
Apply codes to indicate types of missing data (e.g., not completed, not administered, optional)
Transform categorical values into pre-determined standard values
Check identifier columns for duplicate values
Flag records with errors
Indicate administration format in final data sets (e.g., completed on paper or online)

Data Planning Phase. In preparing to launch a study, research teams can facilitate future data cleaning by carefully planning the design of instrument forms, data collection timelines, decision rules for rejecting data, and instrument codebooks. Although these tasks are presented sequentially, in practice they can occur in any order and even concurrently. First, instrument forms should be visually distinct to help participants, administrators, and data collectors notice when an instrument was completed on the wrong form. This can be accomplished, for example, by using different colors, specifying instrument names and dates, and customizing templates to display the number of items and answer choices corresponding to the instrument form (see Figure 1). Second, research teams need to establish timelines for data collection and participant enrollment. This will require working with external vendors to understand the time and information they need to set up instruments or gather data. During this phase, research teams should familiarize themselves with the school calendars of study participants. For example, it would be extremely difficult for a middle school teacher to administer a study assessment during the week of state testing. Lastly, research teams need to consider the structure of their data and rules for rejecting data when issues occur. This can be accomplished by creating codebooks for each study instrument. Decisions for rejecting data may need to be made on a case-by-case base, but during the data planning phase research teams can consider the following question:

- When is it too late to accept data?
- How much of an instrument needs to be completed to be included in the dataset?
- How do we handle duplicate responses?

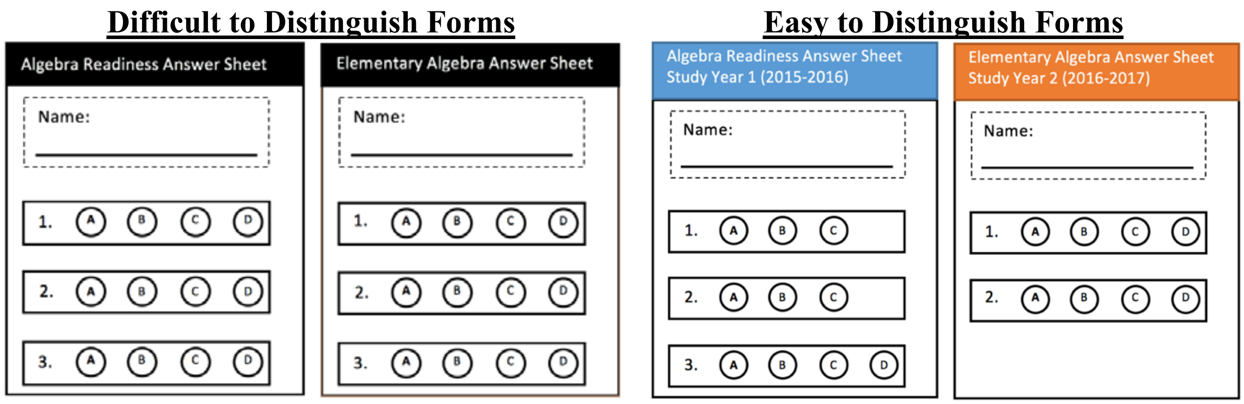

Figure 1. Sample Instrument Forms.

Data Collection Phase. Once a study has started, research teams can implement processes that make it easier to track data and to share real-time information with external vendors. First, de-identify study data as early as possible. The rationale behind this task is that people closer to participants will find it easier to identify who completed an instrument. When using paper instruments, we send teachers a list of barcode stickers that they distribute to specific students who then place the barcodes on their own assessments. These barcodes contain the student's study identifier and the instrument name which can be quickly entered into a tracking system using a barcode scanner. Second, research teams should collect information on instrument administration and any anomalies that occurred during administration. This can take the form of a feedback sheet administrators complete and send back with study data. When using online systems, this might require communicating with external vendors about any issues that appeared during administration (e.g., servers going down). Third, create unique identifiers for each study participant. Identifiers should not depend on characteristics that might change during a study. For example, a student identifier should not include the identifier of the student's teacher, because the student may change teachers or schools during the study. Also, if working with external vendors or external analysts, then an additional set of identifiers should be created to share with these audiences. Figure 2 provides an example of a data file shared with external vendors and analysts. Lastly, data collectors should also check administration dates to make sure they are within acceptable ranges. This can help identify anomalous data (e.g., students entering birthdates).

Original Data File

Internal_ID	Vendor_ID	Analyst_ID	Item_1	Item_2	Item_3
STU023	VSTU010	ASTU771	A	A	B
STU055	VSTU222	ASTU823	A	B	B
STU023	VSTU987	ASTU802	C	E	B

External Vendor Data File

Vendor_ID	Item_1	Item_2	Item_3
VSTU010	A	A	B
VSTU222	A	B	B
VSTU987	C	E	B

Analyst Data File

Analyst_ID	Item_1	Item_2	Item_3
ASTU771	A	A	B
ASTU823	A	B	B
ASTU802	C	E	B

Figure 2. Sample Data Files.

Data Cleaning Phase. After data has been collected, it should undergo a series of validation checks to identify and repair anomalies. Prior to beginning this process, data cleaners should identify a tool that will log steps used to transform data files into their final format. This will make it easier to reproduce files and to retrace steps in case errors are identified later. Our teams have worked with R (https://www.r-project.org/) and OpenRefine (http://openrefine.org). Second, data cleaners should establish a review process so that their work can be verified by another person. When working with many data files, it is easy to make simple mistakes (e.g., assigning a string value like 'treatment' the wrong numeric value). We implement this review process in two ways. First, we have each data cleaning script reviewed by someone who did not author the script. Second, we ask our participant managers and researchers to compare our file counts against their records to identify missing or extra participant records. Next, data cleaners should create copies of their raw data files and only work with the copies. This allows you to return to the original version if needed. To distinguish these files, we add the suffix *_raw* to our original files. Once data cleaners have setup these processes, they can begin working on the data cleaning checks listed in Table 1.

Figures 3 to 6 provide an example of data cleaning checks applied to a student questionnaire. First, the data file is compared against a codebook to identify any missing or extra data columns (Figure 3). Extra columns arise frequently in data files provided by external vendors or captured through online survey tools. Columns can go missing due to software failures or because they were simply overlooked. Second, missing data values are replaced with codes indicating the reason for their absence (Figure 4). I distinguish three types of missing data: data I expected to receive that was not provided, data I did not expect to receive, and data that were optional. Next, categorical values from an open-ended response item are transformed into a limited number of standard values (Figure 5). Demographic data often require mapping to standard values. Where possible, I use values common in educational work (e.g., NCES conventions) to support external researchers in comparing their own data against the data files I produce. Lastly, the unique identifier column is reviewed for duplicate values (Figure 6).

Check for missing and extra columns
By comparing the data file against the codebook, we see *Math Club* is missing and *School ID* was added. We need to work with data collectors to retrieve the missing Math Club information. The School ID column can be deleted.

School ID	Study ID	Grade	Grade 9 GPA	Grade 10 GPA	Elective
111-11-1111	STU01	9	2.0		Engineer
222-22-2222	STU01	9	3.2		Choir
333-33-3333	STU02		3.5	3.0	French

Codebook

Study ID	Unique ID
Grade	9-12
Grade 9 GPA	0.0 – 4.0
Grade 10 GPA	0.0 – 4.0
Elective	STEM Non-STEM
Math Club	Yes, No

Figure 3. Checking questionnaire data for missing and extra columns.

Apply codes to indicate types of missing data
The students in the first two rows are missing a Grade 10 GPA. Since they are in 9th grade, we expect their Grade 10 GPA values to be missing. We indicate missing values that are expected with 888888. The student in the last row is missing a value in the grade column, but we expect all students to have a grade level. We indicate missing values that are unexpected with 999999. Missing codes should stand out from other values in the same column.

Study ID	Grade	Grade 9 GPA	Grade 10 GPA	Elective	Math Club
STU01	9	2.0	888888	Engineer	Yes
STU01	9	3.2	888888	Choir	Yes
STU02	999999	3.5	3.0	French	No

Figure 4. Applying missing codes to questionnaire data.

Transform categorical values into standard values
According to the codebook, the *Elective* column should only contain the values of STEM or Non-STEM. We map fields such as engineering to STEM and other fields to Non-STEM.

Study ID	Grade	Grade 9 GPA	Grade 10 GPA	Elective	Math Club
STU01	9	2.0	888888	STEM	Yes
STU01	9	3.2	888888	Non-STEM	Yes
STU02	999999	3.5	3.0	Non-STEM	No

Figure 5. Transforming categorical values in questionnaire data.

Check identifier columns for duplicate values
The first two rows contain the same value in the *Study ID* column, but our codebook indicates this column should be unique. Looking back to our original data file, we see these students had different values in *School ID*. We would need to work with data collectors to identify if these records represent two different students. In the meantime, we add a column to flag that there is an error with these records.

Study ID	Grade	Grade 9 GPA	Grade 10 GPA	Elective	Math Club	Has Error
STU01	9	2.0	888888	STEM	Yes	1
STU01	9	3.2	888888	Non-STEM	Yes	1
STU02	999999	3.5	3.0	Non-STEM	No	0

Figure 6. Checking questionnaire data for duplicate values.

Communication Processes

Given the large scopes of Study A and Study B, data cleaning was completed by several individuals and required working with staff involved in other aspects of the projects. In Study B, for example, staff were divided into (a) a participant team responsible for recruitment and participant management, (b) a data management team responsible for data collection and cleaning, (c) a research team responsible for study design and analysis, and (d) a management team responsible for project planning and coordination. Working within and across teams to accomplish data cleaning work was not straightforward nor free from error.

As an example, several participating teachers were erroneously excluded from Study A data files because of differences between the participant team and the research team's understanding of *participant*. For the participant team, a 'participant' was someone who started study tasks and had communicated with the project staff at some point. However, for the research team, a 'participant' was any person who enrolled in the study and was randomized into a study condition, regardless of the number of study tasks completed. The error was uncovered when a research team member compared a data file record count against the randomization record count. As a bridge between participant teams and research teams, our data teams needed to navigate across different group norms and discourses to accomplish our work. Next, I briefly summarize the communication and collaboration processes I now use with other project staff to facilitate data cleaning tasks.

Prior to the data cleaning phase, I meet with both the research team and the participant team to discuss their study plans. For the research team, I review their list of data sources and prepare a low-tech sketch of each data file that includes a file name, data columns, and sample values (see Figure 7). Reviewing this sketch with the research team provides confirmation on the data files to be produced, helps to identify if additional files are needed (e.g., a master data file combining information from multiple files), and establishes a shared terminology. With the participant team, I gather information on recruitment deadlines, administration windows they have shared with participants, and school calendars, which inform data validity checks and the data cleaning timeline. During these conversations, I also attend to the ways in which the research team and the participant team discuss their work, being vigilant for possibly confounding terminology. After I meet with both teams, I produce an accounting spreadsheet listing each data file to be created and an estimated due date (see Figure 8).

Once data cleaning begins, I meet regularly with members of the research and participant teams to stay abreast of study progress that might impact data cleaning. For example, the participant team may decide to extend the administration window for a background questionnaire because a new class of students joined the study later than expected. Or, the research team may decide to include additional items on an attitude survey before the second administration of that instrument. These frequent meetings help to identify study plan changes that other project staff may not realize impact data cleaning procedures. When such changes occur, I record them in an appropriate documentation source such as the data accounting spreadsheet.

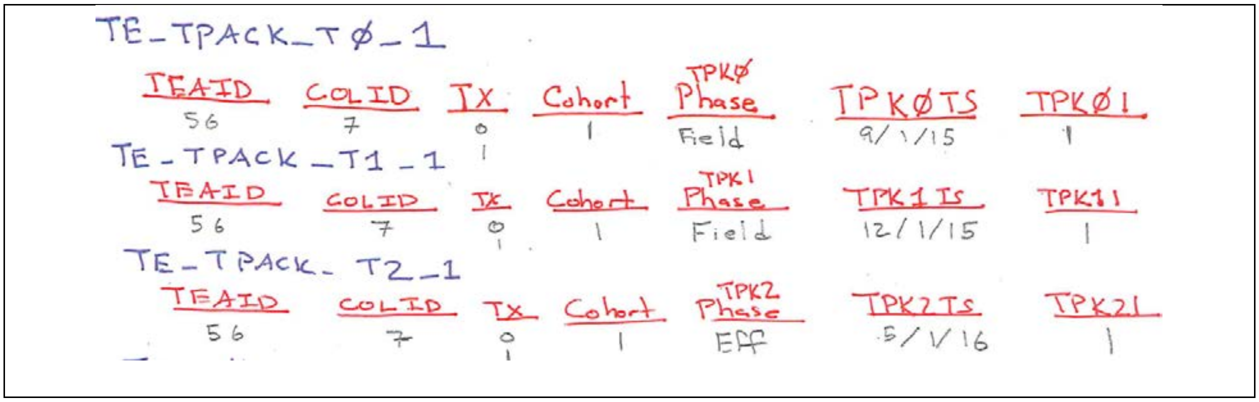

Figure 7. Low-tech sketch of a teacher (TE) assessment of teaching knowledge (TPACK). Teachers completed the assessment at the beginning (T0), middle (T1), and end (T2) of the school year.

Instrument	Pre/Post	Assigned to	Base File Name	Date Due	Status
TE_TPACK	Pre	Tammi	F.1.Tea.Tpk.0.V0	11/25/15	Finished
TE_TPACK	Mid	Tammi	F.1.Tea.Tpk.1.V0	01/15/16	Started
TE_TPACK	Post	Jean	F.1.Tea.Tpk.2.V0	06/30/16	Not started

Figure 8. Data accounting spreadsheet.

Discussion

While the topic of data cleaning may seem tangential to research in undergraduate mathematics education, I hope I have highlighted the critical role data cleaning plays in our research practices. The messiness of environments within which educational research studies are conducted necessitate attention to how we collect and prepare study data. The work presented here demonstrates that processes can be put in place to facilitate the efficient production of quality data sets. The data cleaning process, list of common data error sources, and communication processes offered here provide a framework for other researchers to evaluate their current data management strategies and to provide more comprehensive methods training for researchers.

For readers implementing the framework or embarking on their own data cleaning projects, I offer two recommendations. First, it is important to acknowledge that a comprehensive data cleaning process cannot be created a priori. Studies evolve, participants change, and unexpected events occur. It is impossible to predict all of these variations before a study begins. Implementing an initial data cleaning process that is flexible can help you plan for common errors while giving you the space to adopt procedures as needed in the future. Second, documentation of decisions, processes, and data files is essential. Recording such information helps with accountability and communicating across teams during the data cleaning phase. Documentation also helps researchers recall their procedures and reproduce their work months (even years) after studies have finished and data cleaning decisions are distant memories.

Lastly, some readers may wonder if the procedures presented here apply beyond large-scale, quantitative studies. I argue that all research data needs to undergo some level of cleaning before analysis. While the specific checks of the framework may not apply to all studies (e.g., duplicate records may not be an issue in a case study with one participant), its underlying ideas are relevant to other types of research. In the qualitative research I conduct, my data undergo condensation or "the process of selecting, focusing, simplifying, abstracting, and/or transforming the data that appear in the full corpus (body) of written-up field notes, interview transcripts, documents, and other empirical materials" (Miles & Huberman, 2013, p. 12). I still review my transformed qualitative data to ensure all records are accounted for and no erroneous values exist (e.g., a participant being mistakenly labeled as a teacher instead of a student). At the heart of data cleaning is the acknowledgement that data errors can occur in our studies and that rigorous research practices involve correcting them before analysis.

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Mathematicians' Evaluations of the Language of Mathematical Proof Writing at the Undergraduate Level in Three Different Pedagogical Contexts

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This paper presents the findings from a survey used to investigate how mathematicians perceive the genre of mathematical proof writing at the undergraduate level. Mathematicians were asked whether proof excerpts were unconventional in three contexts: undergraduate textbooks, what instructors write on the blackboard in undergraduate courses, and how students write in these courses. There are four main findings. First, participants found some potential breaches unconventional regardless of the context in which they occur. Second, mathematicians perceived the linguistic conventions in blackboard proofs and student-produced proofs differently in some cases. Third, textbook authors are expected to adhere to stricter norms than instructors and students when writing proofs. Fourth, there were potential breaches that the literature suggests were unconventional, which were not evaluated as unconventional by the mathematicians.

Key words: Mathematical language, Proof, Mathematics textbooks, Mathematics lectures

Introduction

Research has shown that undergraduate mathematics students have difficulties when constructing (Weber, 2001), reading (Conradie & Frith, 2000), and validating (Selden & Selden, 2003) mathematical proofs. Among several reasons for why undergraduate students struggle with constructing mathematical proofs, Moore (1994) included unfamiliarity with the language of mathematical proof writing. However, there is a dearth of empirical research in the field of mathematics education on the language of mathematical proof writing at the undergraduate level. In particular, how undergraduate mathematics students and mathematicians understand and use the technical language of mathematical proof writing is largely unknown to the field.

In Lew & Mejía-Ramos (2016), we conducted semi-structured clinical interviews with mathematicians and undergraduate students, asking them to identify and discuss potential breaches of mathematical language conventions in student-produced proofs. The findings of this study showed that mathematicians and undergraduate students disagreed on the extent to which one should attend to English grammar, the introduction of new mathematical objects, and the context in which the proof was constructed. While these interviews provided a clearer picture of how some mathematicians and undergraduate students perceived the language of proof writing at the undergraduate level, the present study investigated how a larger sample of mathematicians evaluated parts of the same proofs used in Lew & Mejía-Ramos (2016) via an online survey.

This quantitative approach lends a different perspective on how mathematicians understand this technical language and further informs researchers' and mathematics instructors' understanding of mathematicians' expectations regarding the presentation of proofs at the undergraduate level. In turn, such understandings could enable the design of interventions and curriculum to help undergraduate students in their transition to advanced mathematics courses.

One of the main findings from Lew & Mejía-Ramos (2016) was that the participating mathematicians focused on the context in which the proof was written when evaluating the exposition of a mathematical proof. As such, this consideration of context frames the present study, which investigates the extent to which the findings from Lew & Mejía-Ramos (2016) generalize to a larger sample. In particular, we investigate the following two questions:

1. To what extent do mathematicians differentiate between the contexts of textbook proofs, blackboard proofs, and student-produced proofs when evaluating the exposition of a mathematical proof?
2. To what extent do mathematicians agree among themselves on what the linguistic conventions of mathematical proof writing are in each one of these three contexts?

Related Literature and Theoretical Perspective

There is little systematic, empirical work on the language of mathematical proof writing. In one of the few systematic studies of published mathematical proofs, Konior (1993) studied over 700 mathematical proofs written in academic textbooks and mathematical monographs. He identified a common structure that authors used to frame the arguments of a proof, which included highlighting the "plan of procedure" of the proof and using cues to direct the reader through the parts of a proof. In another study, Burton and Morgan (2000) analyzed the writing of mathematicians in 53 published research papers, and found that the norms discussed in professional writing guides (e.g. Gillman, 1987; Krantz, 1998) were often broken by mathematicians (e.g. using of the word "we" to include the reader and avoid the use of passive voice), especially by those who were highly regarded in the field. Selden and Selden (2014) described seven conjectured features of the style in which mathematicians write proofs (e.g., statements of entire definitions are not included within written proofs) and reported having interviewed an undisclosed number of mathematicians who identified these features in their own published papers. While these studies begin to further our understanding of proof writing at the professional level, research on the language of proof writing at the undergraduate level is scarce.

As referred above, a number of mathematicians (e.g. Gillman, 1987; Krantz, 1997) have written texts describing how to properly and effectively use the language of mathematics for professional purposes such as published journal articles, dissertations, and books. Common suggestions for proof writing included: (1) making the logical structure evident in the exposition of the proof, (2) avoiding statements using too many symbols and mathematical notation, (3) being consistent with respect to both notation and word choice, (4) aiming to be concise while still using full and correct sentences, 5) using correct grammar when considering symbols as words and mathematical expressions as phrases, and 6) avoiding using a passive voice. A number of mathematicians and a mathematics educator have written similar guides for undergraduate students, three of which (Vivaldi, 2014; Alcock, 2013; Houston, 2009) have sections addressing the language of mathematical proof writing specifically. While there is overlap in the suggestions given in writing guides for mathematicians and students, the suggestions for students focused on issues of mathematical grammar and clarity, including: 1) using correct words to describe mathematical objects, 2) using connecting words and phrases, 3) avoiding using unclear referents, 4) defining symbols and notation used in a proof, 5) using symbols and notation commonly used in practice, and 6) revising their mathematical writing.

Since the suggestions provided in guides for mathematicians and students were written based on the authors' own assumptions and personal experiences with mathematical writing, further work is necessary to investigate the extent to which these expectations of advanced mathematical proof writing are shared by mathematicians. This study serves as a first step toward this goal.

Linguistic Conventions of Proof Writing in Different Contexts

As a particular type of mathematical text, we see mathematical proof as a genre of mathematical writing. Mathematician Armand Borel (1983) equated mathematical proofs to the genre of poetry in natural language, saying, “our poems are written in a highly specialized language, the mathematical language […], unfortunately, these poems can only be understood in the original” (p .15). Borel emphasized not only that the language of mathematics is distinct from the vernacular, but also that one *must* be knowledgeable in the language of mathematics in order to understand mathematical proofs. In this work, we assume that the genre of proof is defined by both the formal properties and linguistic structures of this type of mathematical text, as well as the communicative purposes of using these texts in particular contexts. This view of genre is consistent with the genre theory literature (Hyland, 2002). Our consideration of proofs in this light is in the pursuit of helping students to understand the linguistic conventions and other characteristics of the genre, as others have done in other genres and discourses (Hyon, 1996).

In our study of mathematical proof writing, we sought to identify the different linguistic conventions of this genre. Understanding conventions as rationally justifiable customs of practice to which members of that practice are expected to conform to (Jackman, 1998), we take linguistic conventions to be rationally justifiable customs of linguistic communication. Existing literature (e.g. Gillman, 1987; Krantz, 1998) has suggested possible conventions of writing proofs for professional contexts, such as correctly situating notation within a sentence according to proper grammar and structuring the proof to guide a reader through the argument.

Meanwhile, it is important to consider how the context of the proof might affect how these conventions are followed as suggested by the mathematicians in Lew & Mejía-Ramos’s (2016) study. In particular, we investigate how mathematicians believe conventions of proof writing vary in the contexts of undergraduate textbooks, and in two classroom contexts: the way proofs are written on the board in class and the ways in which proofs are written by students. The regard of this variation of context within the genre of proof allows this work to highlight similarities and differences in the contexts created by mathematical discourse, as Bondi (1999) identified in her study of research papers, textbooks, and newspaper articles in economic discourse.

Researchers in higher education (Becher, 1987), linguistics (Hyland, 2004), and composition (Bizzell, 1982; Batholomae, 1985) have highlighted that different disciplines have characteristic discourse practices. Berkenkotter, Huckin, and Ackerman (1988) explained “students entering academic disciplines must learn the genres and conventions that members of the disciplinary community employ. Without this knowledge, [Bizzell and Batholomae] contend, students remain locked outside of the community’s discourse” (p. 10). We extend this necessity to acquire specialized literacy to undergraduate students of advanced mathematics, who—we argue—must understand the conventions of mathematical discourse, including the genre of proof in contexts that pervade their study. Given the fundamental role of proof in mathematical practice (e.g. Thurston, 1994) and Borel’s (1983) sentiment above, understanding the language in which proofs are written is of utmost importance for advanced undergraduate mathematics students.

In the present study, we investigate the conventions of mathematical proof writing from the perspective of mathematicians—the most prevalent instructors and examiners of undergraduate students’ proof writing. This investigation was motivated by the mathematicians’ focus on the context in which the proof was constructed in Lew & Mejía-Ramos (2016). The mathematicians in that study discussed the importance of knowing in what course the student was enrolled, where the proof was written (in a textbook, on the board, in a graded assignment), and what type of assessment the proof was a part of (in a homework assignment, or a timed exam setting). As a result of these discussions with mathematicians in Lew & Mejía-Ramos (2016), the current study examines how three different proof contexts (textbook, blackboard, and student-produced) affect mathematicians’ evaluations of potential breaches of mathematical language.

Methods

Following the methods of data collection employed by Inglis and Mejía-Ramos (2009), this study uses an online survey in order to maximize the sample size of mathematicians. Using an online survey to conduct research presents some practical difficulties including the possibility of individual participants submitting multiple responses. Meanwhile, using the steps described by Reips (2000), Gosling et al. (2004), and Krantz and Dalal (2000) showed that online studies can produce results consistent with more traditional methods of research. Moreover, mathematics education publications have used these methods of web-based research (see Inglis, Mejía-Ramos, Weber, & Alcock, 2013; Lai, Weber, & Mejía-Ramos, 2012; Mejía-Ramos & Weber, 2014).

Participants

Participants were recruited from 25 of the top mathematics departments in the United States through email solicitation through their department secretaries. In total, 128 mathematicians (75 PhD students, 16 Postdoctoral fellows, and 37 faculty members) participated in the survey.

Design of the Study

The survey website included fourteen pages asking participants to make evaluations regarding the language used in partial proofs (that were truncated to discourage participants from focusing on the logical validity of the purported proof being evaluated, and to instead focus the evaluation on the use of mathematical language). For an example survey page, see the Appendix. The tasks of the survey were designed based on the design and analysis of Lew & Mejía-Ramos (2016). We chose four of the seven partial proofs from Lew & Mejía-Ramos (2016) to include in the survey as shown in Table 1. Partial proofs were chosen to maximize the number of types of potential breaches of mathematical language included and to include proof excerpts for which mathematicians and students disagreed in Lew & Mejía-Ramos (2016). Each of the proofs in the survey included three or four types of potential breaches of mathematical language.

Potential breaches of the language of mathematical proof writing

Each of the potential breaches of mathematical language included in the survey is briefly described in Table 1. The table includes the highlighted portions of the partial proofs for each of the potential breaches along with the verbatim explanations provided. The explanations are based on the mathematicians' discussions of the same potential breaches and proofs in Lew & Mejía-Ramos (2016). These potential breaches of mathematical proof writing are at the core of both this study and the study presented in Lew & Mejía-Ramos (2016). These breaches were identified as common, potentially unconventional uses of mathematical language found in student-produced proofs from 149 exams at the introduction to proof level (Lew & Mejía-Ramos, 2015). The breaches were categorized based on suggestions from the mathematical writing guides discussed above, our personal experiences with undergraduate proof writing, and existing literature discussing the genre of mathematical proof writing (Selden & Selden, 2003).

The fourteen potential breaches of the language of mathematical proof writing considered in this study are provided in Table 1. More specifically, Table 1 shows each of the four partial proofs marked for all of the potential breaches of the language of mathematical proof writing included in the survey. Below each of the partial proofs, proof excerpts are given with the marked potential breach and the explanation provided in the survey for why one might believe the potential breach is unconventional. In each page of the survey presenting a potential breach of the language of mathematical proof writing, the entire partial proof appeared marked with only one of the potential breaches (for an example, see the Appendix).

<table>
<tr><th rowspan="2"></th><th colspan="2">Marked Partial Proof</th></tr>
<tr><th>Potential Breach and Corresponding Proof Excerpt</th><th>Explanation for Potential Breach
(A mathematician suggested this is unconventional mathematical writing because…)</th></tr>
<tr><th rowspan="5">Partial Proof 1</th><td colspan="2">Suppose $f : A \to B$, $g : B \to C$, $h : B \to C$, for sets A, B, and C.
Prove: If f is onto B and $g \circ f = h \circ f$, then $g = h$.
Suppose f is onto B and $g \circ f$ NTS: $g=h$
If f is onto B, $Rng(f)=B$
$\boxed{\forall y \in B,\ \exists x \in A \text{ such that } f(x)=y}$
let $(x,z) \in g \circ f$ such that $\exists y \in B$ $g(y)=z$</td></tr>
<tr><td>Uses non-statement
Suppose f is onto B and $g \circ f$ NTS: $g=h$</td><td>… the statement "suppose g∘f" is incomplete/meaningless.</td></tr>
<tr><td>Uses an unspecified variable
let $(x,z) \in g \circ f$ such that $\exists y \in B$ $g(y)=z$</td><td>… the variable z should be introduced prior to its use in the proof.</td></tr>
<tr><td>Includes statements of definitions
$\boxed{\forall y \in B,\ \exists x \in A \text{ such that } f(x)=y}$</td><td>… complete statements of definitions should not be stated in proofs, rather they should be applied.</td></tr>
<tr><td>Lacks punctuation and capitalization
let $(x,z) \in g \circ f$ such that $\exists y \in B$ $g(y)=z$.</td><td>… proofs should be written in full sentences, which includes correct capitalization and punctuation.</td></tr>
<tr><th rowspan="5">Partial Proof 2</th><td colspan="2">Prove that $\sqrt{3}$ is irrational.
Assume for the sake of contradiction that $\sqrt{3} \in \mathbb{Q}$.
Then, $[(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})](\sqrt{3} = \frac{a}{b}) \wedge (b \neq 0)$ with no $k \in \mathbb{Z}$ such that $k \mid a$ and $k \mid b$.
So $a^2 = 3b^2$ and $3 \mid a^2$. Since 3 is prime, so $3 \mid a$.
Given $3 \mid a$, we're able to conclude that $(\exists b \in \mathbb{Z})(a = 3b)$.
Note that their greatest common divisor $= 1$.</td></tr>
<tr><td>Uses formal propositional language
Then, $[(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})](\sqrt{3} = \frac{a}{b}) \wedge (b \neq 0)$ with no $k \in \mathbb{Z}$</td><td>… entire statements of formal propositional logic are difficult to read.</td></tr>
<tr><td>Uses unclear referent
Note that their greatest common divisor $= 1$.</td><td>… this use of pronouns is imprecise and makes what the writer is discussing unclear.</td></tr>
<tr><td>Overuses variable names
Then $[(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})](\sqrt{3} = \frac{a}{b}) \wedge (b \neq 0)$ with no $k \in \mathbb{Z}$ such that $k \mid a$ and $k \mid b$.
So $a^2 = 3b^2$ and $3 \mid a^2$. Since 3 is prime, so $3 \mid a$.
Given $3 \mid a$, we're able to conclude that $(\exists b \in \mathbb{Z})(a = 3b)$.</td><td>… the variable b is used in two different ways, representing different values.</td></tr>
<tr><td>Mixes mathematical notation and text
Note that their greatest common divisor $= 1$.</td><td>… the equal symbol should not be used to connect a verbal statement and a number.</td></tr>
</table>

<table>
<tr><td rowspan="4">Partial Proof 3</td><td colspan="2">Prove that $\mathcal{P}$ is a partition of $\mathbb{Z}$.
None of the sets are $\emptyset$.
All are pairwise disjoint, since the positive sets share nothing with the negative sets and the evens share nothing with the odds, and $\{0\}$ shares nothing with the rest.</td></tr>
<tr><td>Fails to make the proof structure explicit
Prove that $\mathcal{P}$ is a partition of $\mathbb{Z}$.
None of the sets are $\emptyset$.</td><td>… the writer should include a statement about the structure of the proof to indicate the flow of the argument to the reader.</td></tr>
<tr><td>Uses mathematical symbols or notation as an incorrect part of speech
None of the sets are $\emptyset$.</td><td>… the symbol ø represents the phrase "empty set". So the first line reads "none of the sets are empty set", which is not a grammatically correct sentence.</td></tr>
<tr><td>Uses informal language
All are pairwise disjoint, since the positive sets share nothing with the negative sets and the evens share nothing with the odds, and $\{0\}$ shares nothing with the rest.</td><td>… the second sentence of the proof is insufficiently rigorous and formal.</td></tr>
<tr><td rowspan="4">Partial Proof 4</td><td colspan="2">Let A be a set. Prove: If S is a relation on A, then the relation $R = S \circ S^{-1}$ is symmetric.
let $x,z \in A$ s.t.
$(x,z) \in R$
$R = S \circ S^{-1}$
$(x,z) \in S \circ S^{-1}$
$\exists y$ s.t. $(x,y) \in S^{-1}$ and $(y,z) \in S$
$(y,z) \in S$ $\therefore$ $(z,y) \in S^{-1}$
$(x,y) \in S^{-1}$ $\therefore$ $(y,x) \in S$</td></tr>
<tr><td>Fails to state assumptions of hypotheses
Let A be a set. Prove: If S is a relation on A, then the relation $R = S \circ S^{-1}$ is symmetric.
let $x,z \in A$ s.t.
$(x,z) \in R$</td><td>… the writer should include a statement about what is being assumed in the proof (in this case that S is a relation on A).</td></tr>
<tr><td>Uses an unspecified variable with an existential quantifier
$\exists y$ s.t. $(x,y) \in S^{-1}$ and $(y,z) \in S$
$(y,z) \in S$ $\therefore$ $(z,y) \in S^{-1}$</td><td>… the introduction of the variable y should specify the set to which y belongs.</td></tr>
<tr><td>Lacks verbal connectives
let $x,z \in A$ s.t.
$(x,z) \in R$
$R = S \circ S^{-1}$
$(x,z) \in S \circ S^{-1}$
$\exists y$ s.t. $(x,y) \in S^{-1}$ and $(y,z) \in S$
$(y,z) \in S$ $\therefore$ $(z,y) \in S^{-1}$
$(x,y) \in S^{-1}$ $\therefore$ $(y,x) \in S$</td><td>… the writer should use verbal connectives (e.g. therefore, then, since) to indicate the flow of the argument to the reader.</td></tr>
</table>

Table 1. Example potential breach and explanation presented in the survey by partial proof.

Survey Tasks

For each of the potential breaches presented, participants were provided an explanation of why a colleague might believe the corresponding proof excerpt was written in an unconventional manner. Participants were asked if they agreed this proof excerpt was indeed unconventional for the stated reason, and to what extent it affected the quality of the proof. These questions were asked for the context of a textbook proof, and for both classroom contexts: a blackboard proof and a student-produced proof. Finally, mathematicians were asked if they would make a note or deduct points for this use of language in a homework assignment and an exam. An example survey page is provided in the Appendix. The four proofs were presented in a randomized order.

Analysis

The analysis for this study investigated how the mathematicians answered the various aspects of the survey–in particular, whether they agreed the potential breaches were unconventional in each of the three contexts (textbook proofs, blackboard profs, and student-produced proofs), whether they agreed on the extent to which these potential breaches affected the quality of the proof in each of the three contexts, whether they believed points would be deducted or a note would be made in a student-produced proof, and the extent to which participants viewed these contexts differently. The findings from the study are summarized in Tables 2, 3, and 4.

Table 2 focuses on the mathematicians responses indicating if they agreed that the proof excerpt was unconventional for the reason provided in each context. In order to evaluate if the proportions of agreement that a potential breach was unconventional indicated a high level of agreement within the samples, we considered 75% of the sample to be the threshold. Similarly, we considered 25% of the sample to be the threshold of a high level of agreement that a potential breach was not unconventional within the samples. As such, we conducted Chi-squared tests for equality of proportions checking for proportions $p=0.25$ and 0.75 with a level of significance of $\alpha=0.05/42$. The results of these Chi-squared tests are indicated with ++ and - - as described below Table 2. The proportions of agreement were categorized in the following ways: high agreement that the use is unconventional (significantly different from and greater than 75%), high agreement that the use is not unconventional (significantly different from and less than 25%), or inconclusive (not shown to have high agreement within the sample).

Table 3 provides pairwise comparisons of the responses across the three contexts. To investigate if the mathematicians evaluated the three different contexts differently when evaluating if the potential breaches were unconventional of mathematical language, we conducted pairwise Cochran Q tests comparing how the participants responded in the following contexts: textbook proofs vs. blackboard proofs, textbook proofs vs. student-produced proofs, and blackboard proofs vs. student-produced proofs. These tests were also evaluated with a level of significance of $\alpha=0.05/42$ and are indicated in Table 3 with * as described below Table 3.

Table 4 presents the mathematicians' expectations of how a student-constructed proof would be assessed in classroom assignments. We conducted Stuart-Maxwell Tests with a level of significance $\alpha=0.05/28$ to evaluate if the mathematicians evaluated the contexts of proofs written in homework assignments and proofs written in an in-class exam differently. Categories in which mathematicians graded the contexts differently are indicted in bold as described below Table 4.

Results

To what extent does context affect mathematicians' evaluations of proof writing?

As shown in Table 3, the pairwise Cochran Q tests indicate that context did play a role in participants' evaluations of whether or not these potential breaches were unconventional for the reasons provided. Indeed, for twelve of the fourteen potential breaches, mathematicians' level of agreement differed significantly depending on the context of the evaluation. Overall, the comparison between participants' level of agreement in the textbook context and the classroom contexts yielded a different outcome to the corresponding comparison between the two classroom contexts.

	Potential Breach of Mathematical Language	Do you agree that this is an unconventional use of mathematical language for the reason provided? *(% Agree)*		
		Textbook Context	Blackboard Context	Student Context
Partial Proof 1	Uses non-statement	100% ++	100% ++	98% ++
	Uses an unspecified variable	59%	34%	37%
	Includes statements of definitions	41%	18%	12% --
	Lacks punctuation and capitalization	95% ++	34%	50%
Partial Proof 2	Uses formal propositional language	88% ++	74%	66%
	Uses unclear referent	93% ++	67%	70%
	Overuses variable names	98% ++	95% ++	93% ++
	Mixes mathematical notation and text	88% ++	28%	45%
Partial Proof 3	Fails to make the proof structure explicit	70%	29%	28%
	Uses mathematical symbols or notation as an incorrect part of speech	72%	19%	24%
	Uses informal language	77%	45%	48%
Partial Proof 4	Fails to state assumptions of hypotheses	64%	34%	40%
	Uses an unspecified variable with an existential quantifier	85% ++	55%	54%
	Lacks verbal connectives	97% ++	52%	72%

++ Significantly different from and greater than 75% of the sample, -- Significantly different from and less than 25% of the sample (These tests were all evaluated with a level of significance α=0.05/42.)

Table 2. Mathematicians' responses indicating if they agree that the proof excerpt was unconventional for the reason provided in each context.

	Potential Breach of Mathematical Language	Are there differences in how mathematicians view different contexts of mathematical proof writing at the undergraduate level? *(% Agree in Context 1 / % Agree in Context 2)*		
		Textbook Context vs Blackboard Context	Textbook Context vs Student Context	Blackboard Context vs Student Context
Partial Proof 1	Uses non-statement	100% / 100%	100% / 98%	100% / 98%
	Uses an unspecified variable	59% / 34% *	59% / 37% *	34% / 37%
	Includes statements of definitions	41% / 18% *	41% / 12% *	18% / 12%
	Lacks punctuation and capitalization	95% / 34% *	95% / 50% *	34% / 50% *
Partial Proof 2	Uses formal propositional language	88% / 74% *	88% / 66% *	74% / 66%
	Uses unclear referent	93% / 67% *	93% / 70% *	67% / 70%
	Overuses variable names	98% / 95%	98% / 93%	95% / 93%
	Mixes mathematical notation and text	88% / 28% *	88% / 45% *	28% / 45% *
Partial Proof 3	Fails to make the proof structure explicit	70% / 29% *	70% / 28% *	29% / 28%
	Uses mathematical symbols or notation as an incorrect part of speech	72% / 19% *	72% / 24% *	19% / 24%
	Uses informal language	77% / 45% *	77% / 48% *	45% / 48%
Partial Proof 4	Fails to state assumptions of hypotheses	64% / 34% *	64% / 40% *	34% / 40%
	Uses an unspecified variable with an existential quantifier	85% / 55% *	85% / 54% *	55% / 54%
	Lacks verbal connectives	97% / 52% *	97% / 72% *	52% / 72% *

* The sample's responses for the two contexts were significantly different. (Evaluated with a level of significance α=0.05/42.)

Table 3. Pairwise comparisons across contexts of the mathematicians' responses indicating if they agree that the proof excerpt was unconventional for the reason provided.

	Potential Breach of Mathematical Language and Context of the Assignment	Points deducted for this unconventional use of mathematical language in a student-produced proof? *(% Points off / % Note made / % No mark)*	
		Homework Context	**Exam Context**
Partial Proof 1	**Uses non-statement**	58% / 41% / 2%	55% / 45% / 1%
	Uses an unspecified variable	20% / 27% / 54%	20% / 27% / 54%
	Includes statements of definitions	3% / 9% / 88%	2% / 13% / 86%
	Lacks punctuation and capitalization	**6% / 38% / 56%**	**2% / 32% / 66%**
Partial Proof 2	**Uses formal propositional language**	**14% / 63% / 23%**	**8% / 60% / 32%**
	Uses unclear referent	**25% / 50% / 25%**	**17% / 48% / 34%**
	Overuses variable names	57% / 37% / 6%	49% / 43% / 8%
	Mixes mathematical notation and text	**3% / 39% / 58%**	**1% / 31% / 68%**
Partial Proof 3	**Fails to make the proof structure explicit**	9% / 34% / 57%	11% / 21% / 68%
	Uses mathematical symbols or notation as an incorrect part of speech	2% / 27% / 71%	2% / 21% / 77%
	Uses informal language	34% / 27% / 39%	30% / 29% / 41%
Partial Proof 4	**Fails to state assumptions of hypotheses**	16% / 30% / 54%	14% / 25% / 61%
	Uses an unspecified variable with an existential quantifier	19% / 42% / 39%	20% / 38% / 42%
	Lacks verbal connectives	**27% / 48% / 24%**	**21% / 45% / 34%**

* Bold rows show that mathematicians expected the proof to be graded differently in the two contexts by a level of significance α=0.05/14.

Table 4. Reponses by mathematicians of their expectations of how a student-constructed proof would be assessed in classroom assignments.

The context of textbook proofs

For twelve of the fourteen breaches presented in the survey, a significantly higher percentage of mathematicians agreed that the potential breach was unconventional in the context of textbook proofs than in either the context of blackboard proofs or student-produced proofs. In other words, a large majority of the potential breaches were more commonly judged to be unconventional in the context of textbooks than the classroom contexts. For instance, 85% of participants agreed that using an unspecified variable with an existential quantifier in the way shown in Partial Proof 4 (which contains the line "$\exists y$ s.t. $(x,y) \in S^{-1}$ and $(y,x) \in S$") was unconventional in the textbook context, given that the introduction of the variable y should specify the set to which it belongs. However, only 55% and 54% of mathematicians agreed that this constituted an unconventional use of mathematical language (for the reason provided) in the contexts of blackboard proofs and student-produced proofs, respectively.

Contexts of blackboard proofs and student-produced proofs

In contrast to the prevalent differences in agreement levels between the textbook and classroom contexts, when comparing the contexts of blackboard proofs and student-produced proofs, the corresponding agreement levels were significantly different in only three of the fourteen potential breaches. In particular, the Cochran-Q tests indicated that the mathematicians judged the proof excerpts that lacked punctuation and capitalization, mixed mathematical notation and text, and lacked verbal connectives differently in the contexts of blackboard proofs and student-produced proofs. Moreover, for each of these three potential breaches, a significantly larger percentage of mathematicians found the potential breach to be unconventional in the context of student-produced proofs than in the context of blackboard proofs. For instance, 28% of participants agreed that mixing mathematical notation and text in the way done in the last line of Partial Proof 2 (i.e. "Note that their greatest common divisor = 1.") was unconventional in the blackboard context given that the equal symbol should not be used to connect a verbal statement and a number. However, 45% of mathematicians agreed this constituted an unconventional use of mathematical language (for the reason provided) in the context of student-produced proofs.

When the context of the proof may not matter

The responses from mathematicians were not significantly different across contexts for two of the potential breaches: using non-statements and overusing variable names. Furthermore, mathematician widely agreed that these potential breaches were unconventional, with agreement levels between 93% and 100% across contexts. In other words, the only two potential breaches for which context did not make a difference were potential breaches that were widely considered to be unconventional. For instance, 100% of participants agreed that using non-statements in the way shown in the first line of Partial Proof 1 (i.e. "Suppose f is onto B and $g \circ f$.") was unconventional in the textbook and the blackboard contexts, given that "suppose $g \circ f$" is an incomplete/meaningless statement. In the student-proof context, 98% of mathematicians evaluated this use of language as unconventional for the reason provided.

Summary

This analysis revealed that mathematicians evaluated a large majority of the potential breaches differently depending on the context, failing to differentiate between the three given contexts in only two of the fourteen potential breaches. These results are consistent with the findings from Lew & Mejía-Ramos's (2016) interview study, in which mathematicians focused on the context in which the proof was written when evaluating the exposition of a mathematical proof. Furthermore, we found most of these differences were accounted by the fact that potential breaches were more commonly judged to be unconventional in the context of textbooks than in the classroom contexts, which makes sense given the extensive editorial work devoted to improving the presentation of mathematics (including proofs) in this context. Finally, we found that the only two potential breaches for which context did not make a difference, were potential breaches that were widely considered to be unconventional. Participants' evaluations of these two types of potential breaches provide evidence of *the existence* of uses of mathematical language that are widely perceived to be unconventional, regardless of context.

In contrast to comparisons to the textbook context, comparisons between the evaluations in the contexts of blackboard proofs and student-produced proofs were *not* found to be significantly different in eleven of the fourteen potential breaches. Furthermore, for the three remaining breaches, a significantly larger percentage of mathematicians found the potential breach to be unconventional in the context of student-produced proofs than in the blackboard proofs. Taken together, these findings may suggest mathematicians generally do not see a difference between the contexts of blackboard proofs and student-produced proofs when it comes to the language of proof writing, but when they do differentiate between these contexts, mathematicians may hold students' proofs to higher linguistic standards than the proofs they themselves write on the blackboard in class. This suggests, in certain respects, that the formality of a blackboard proof is less important to mathematicians than that of a student-produced proof. This is particularly interesting as mathematicians most often present proofs to their students on the blackboard. As such, these results suggest the most common way in which mathematicians present proofs to their students is less formal than the way they might expect their students to produce proofs.

To what extent do mathematicians agree among themselves in these contexts?

Figure 5 shows the percentage of mathematicians who agreed the potential breaches were unconventional for the reason provided in each context. Lines connect the agreement percentages for evaluation in the same context and the shaded sections indicate percentages significantly different and greater than 75% or significantly different and less than 25%. The un-shaded, center section of the graph presents the results of the participants' surveys, which did not show high levels of agreement according to the Chi-squared tests for equality of proportions. In this section, we first discuss the types of potential breaches for which the mathematicians' responses showed high level of agreement. Then, we provide a post hoc analysis of the types of potential breaches for which responses did not show this high agreement.

Types of potential breaches for which the mathematicians' responses showed high agreement

As shown on Figure 5, (significantly) more than 75% of mathematicians found eight of the fourteen types of potential breaches (using non-statements, overusing variable names, lacking punctuation and capitalization, mixing mathematical notation and text, lacking verbal connectives, using formal propositional language, using unclear referents, and using an unspecified variable with an existential quantifier) to be unconventional in the context of textbook proofs for the reasons presented in the survey. Moreover, there was a high level of agreement among mathematicians that the proof excerpts exhibiting the use of non-statements or overuse of variable names were also unconventional in each of the other two contexts.

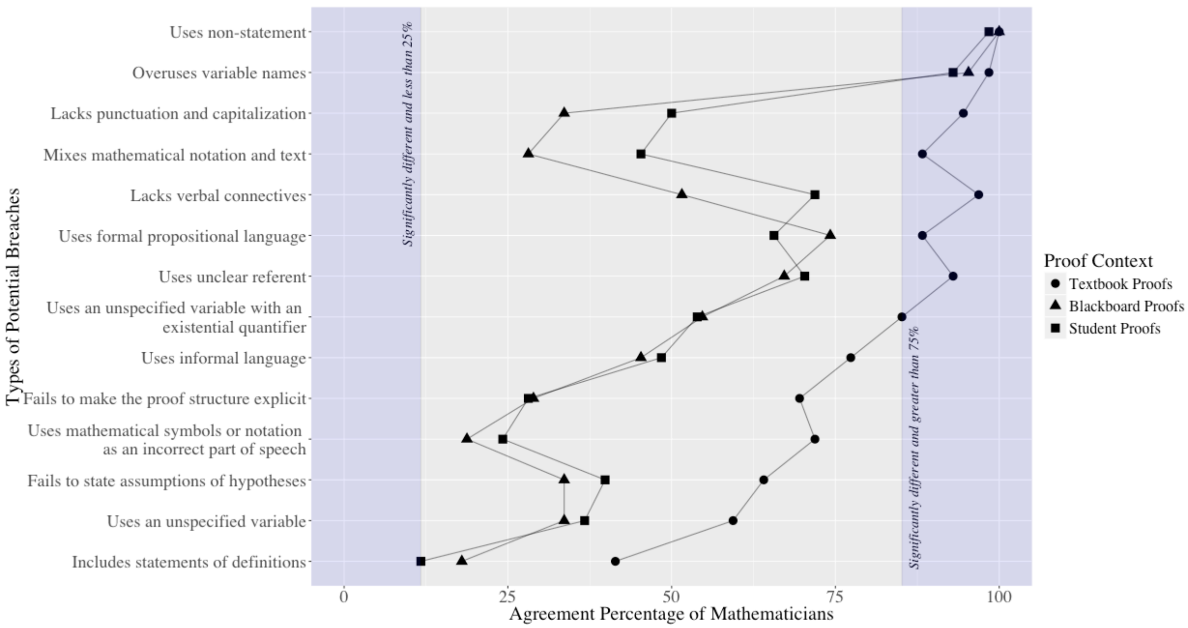

Figure 5. The mathematicians' agreement percentage of agreement for each potential breach in each of the three contexts.

These findings indicate that these eight potential breaches of the conventions of mathematical language are indeed unconventional in the context of textbook proofs for the reasons provided in the survey and that overusing variable names and the use of non-statements are also unconventional in classroom contexts. Figure 5 also shows there was high agreement amongst the mathematicians that the inclusion of statements of definition was not unconventional in the context of a student-produced proof. Moreover, fewer than 42% of mathematicians agreed that the inclusion of statements of definitions was unconventional in any of the contexts considered. We note that this is in contrast to claims made by Selden and Selden (2014) that mathematicians do not include the statements of entire definitions within their written proofs. Thus, it may not be the case that the features of proof writing described by Selden and Selden (2014) extend to different contexts of proofs written by mathematicians (including textbook and blackboard proofs), or to the contexts of student-produced proofs.

When the mathematicians' responses did not show high agreement

For 29 of the 42 judgments made by mathematicians (fourteen potential breaches in three contexts) the agreement percentages were inconclusive, i.e. percentages were not significantly different from and higher that 75%, or significantly different from and lower than 25%. Figure 5 further shows that for five of the potential breaches, there was no high agreement among mathematicians in *any of the three contexts*, and that when we restrict the analysis to only the classroom contexts, the results did not show high agreement for up to eleven of the fourteen types of potential breaches. Finally, Figure 5 further highlights that a number of these agreement percentages are close to 50%. In particular, eight of the 42 judgments had percentage agreements between 40% and 60%, including two judgments in the context of textbook proofs.

Summary

Mathematicians displayed a high level of agreement (as defined above) in 13 out of the 42 evaluations they were asked to make in this survey (fourteen potential breaches in each of three contexts), with 8 of them about potential breaches in the textbook context. This provides evidence that the language used in some of the partial proofs they evaluated indeed breached the conventions of proof writing in those contexts. However, beyond failing to provide confirmation that the remaining 29 proof passages were indeed breaches of linguistic conventions in proof writing, these findings suggest that the disagreement among mathematicians may be higher in the classroom contexts, and that for some specific types of potential breaches the disagreement amongst mathematicians may be particularly extreme, even in the context of textbook proofs.

These findings suggest a possible lack of clarity from the mathematicians of what a student-produced proof should look like. If it were the case that mathematicians do not have a shared understanding of how mathematical language should be used in student-produced proofs, it would then be unsurprising if undergraduate students similarly lacked clear understandings of undergraduate proof writing. A lack of a shared understanding between mathematicians and students would indicate a need for a discussion among mathematicians and students on the expositional expectations of students' proofs in introduction to proof courses.

Conclusion

This paper reported on the results of a survey asking mathematicians to agree or disagree that potential breaches of the language of mathematical proof writing in different contexts were unconventional for reasons provided in the survey. As such, this study identified breaches of linguistic conventions, in the sense of Jackman (1998), based on the participants' assessments of rational justifications of potential linguistic customs in mathematics. There are four main findings from this survey: First, there are some potential breaches of mathematical language that are unconventional regardless of the context in which they occur. Second, while we only found a few instances in which mathematicians differentiated between the contexts of blackboard proofs and student-produced proofs, when they did mathematicians held students' proofs to higher linguistic standards than the proofs they themselves write on the blackboard in class. Third, textbook authors are expected to adhere to stricter writing norms than mathematics instructors and undergraduate students when writing proofs. Fourth, the participants' responses indicated there are some potential breaches of mathematical language that the literature (Selden & Selden, 2003) might suggest are unconventional, which the participants agreed were not unconventional.

The findings of this report highlight some potential breaches of mathematical language that mathematicians agree are unconventional in the context of published proofs and provide insight on how mathematicians consider the language of mathematical proof writing in the classroom context at the introduction to proof level. In particular, the results regarding the linguistic conventions of mathematical proof writing in classroom contexts suggest that it is unclear what a student-produced proof is expected to look like. The mathematicians' responses did not indicate significantly high levels of agreement for eleven of the fourteen types of potential breaches in the student context, which may indicate the possibility that there is no standard universal understanding or expectation among mathematicians of how students should write proofs.

Moreover, despite findings indicating that mathematicians may not have a shared understanding of linguistic conventions of student-produced proofs, results do suggest the way mathematicians present proofs in their introduction to proof courses is less formal than the manner in which they may expect their students to produce proofs. However, research studies on note taking indicate that students are less likely to recall details of a lecture that were not included in their notes (Kiewra, 2002) and are more likely to record what is written on the blackboard (Johnstone & Su, 1994). Thus, one would expect students to model their proof writing based on how their professors write proofs in class. As a result, this paper's finding that mathematicians may not present proofs in class in a manner that represents their linguistic expectations of their students' proofs is certainly troubling.

Discussions amongst mathematicians, especially those who teach introduction to proof courses, concerning their expectations for the writing of mathematical proofs by their students would be a useful step towards a shared understanding of linguistic conventions of proof writing in the context of student-produced proofs. Meanwhile, if there is not currently a consensus among mathematicians of how their students in introduction to proof courses should be writing their proofs, then a natural question is, how are the instructors of these courses presenting mathematical proof writing to their students?

One possible avenue for future research entails considering mathematical proofs, which are written by mathematicians for their students to read outside of class time such as homework solutions, and how students perceive the differences between the exposition of the proofs they produced in their own homework and the exposition of their professor's solution.

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Appendix: Survey Page Example

Please keep in mind this study is particularly focused on the mathematical writing of the proof. While it is natural to focus on the validity of the logical argument of the proof, this is not the goal of this study.

Also, keep in mind the three different contexts of undergraduate proof writing:

1. A ***textbook proof*** is a proof that is written in the way that proofs in undergraduate textbooks are normally written.
2. A ***blackboard proof*** is a proof that is written in the way that proofs normally appear on the blackboard in an undergraduate proof-based course taught by a mathematician.
3. A ***student produced proof*** is a proof that is written in the way that you expect that a mathematician would prefer an ideal undergraduate student to write mathematical proofs.

Consider the following task and subsequent partial proof:

Let A be a set. Prove: If S is a relation on A, then the relation $R = S \circ S^{-1}$ is symmetric.

let $x, z \in A$ s.t.

$(x,z) \in R$

$R = S \circ S^{-1}$

$(x,z) \in S \circ S^{-1}$

$\exists y$ s.t. $(x,y) \in S^{-1}$ and $(y,z) \in S$

$(y,z) \in S \therefore (z,y) \in S^{-1}$

$(x,y) \in S^{-1} \therefore (y,x) \in S$

	A mathematician suggested that this is unconventional mathematical writing because ***the writer should use verbal connectives (e.g. therefore, then, since) to indicate the flow of the argument to the reader.*** Would you agree or disagree in the following contexts?		To what extent does this use of mathematical language lower the quality of the exposition of the partial proof in the following contexts? *(Please select "N/A" if you do not find this writing to be unconventional.)*			
	Agree	Disagree	Significantly	Moderately	Not at all	N/A
Textbook Proofs	○	○	○	○	○	○
Blackboard Proofs	○	○	○	○	○	○
Student Produced Proofs	○	○	○	○	○	○

In an introduction to proof course, would you take points off for this type of unconventional use of mathematical language in the following contexts?

	I would take points off (and perhaps make a note).	I would make a note, but not take points off.	I would not take points off or make any note to the student.
Homework	○	○	○
Exam	○	○	○

Spatial Training and Calculus Ability: Investigating Impacts on Student Performance and Cognitive Style

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Despite concerted efforts on the part of educational policy makers, women are still underrepresented in the STEM fields. Researchers have shown that calculus plays a major role in this gender disparity since it requires spatial skills to success -- skills that women tend to utilize differently compared to men. However, previous studies have shown that spatial ability is malleable and spatial skills can be improved with training. This pilot study employed a form of spatial training in a third-term calculus course and measured the effects of this training on students' calculus ability, spatial rotation ability, and cognitive style. Associations between cognitive style and task performance were also measured. Preliminary results indicate that spatial training did not significantly impact student performance on a calculus skills assessment or a test of mental rotations, but effects on students' cognitive style were present.

Key words: spatial training, calculus skills, cognitive learning style

Introduction & Literature Review

A concentrated effort is being made to achieve greater diversity in students graduating with bachelor's degrees in Science, Technology, Engineering and Mathematics (STEM) (Stieff & Uttal, 2015). Unfortunately, the demographics of STEM graduates have been stagnant in recent years (Stieff & Uttal, 2015) and women are often underrepresented in STEM fields in North America and Europe (Nimmesgern, 2016; Schlenker, 2015). Indeed, those women who do undertake studies in the STEM fields seem to have a higher probability of *not* working in STEM occupations after graduation compared to men (OECD, 2012).

While there are a number of psychosocial reasons for the underrepresentation of women in STEM fields (e.g., Nimmesgern, 2016; Saucerman & Vasquez, 2014; Skolnik, 2015), one often-cited reason is that females may have less developed spatial abilities than males (as per Ferrini-Mundy, 1987), and perform spatial tasks differently than males starting as young as four years of age (Levine et al., 1999; Voyer, Voyer, & Bryden, 1995). Gender has also been found to associate with the accuracy and organization of spatial learning (Gãrling et al., 1981; Kirasic, Allen, & Siegel, 1984) often (but not always) in favor of males (Acredolo, 1988; Brown et al., 1998; Cutmore et al., 2000; Gifford, 2007; Lehnung et al., 2003; Ward et al., 1986; Webley & Whalley, 1987). Although a number of studies have found insignificant differences between the performances of men and women on various spatial tasks, these results seem to depend on the type of task and the age of the study (see Casey, 2013). However, in general, men have been found to consistently perform better than women on spatial perception and mental rotation tasks (Linn & Petersen, 1985).

Although the body of literature on spatial ability indicates that this skill is an important facet of general intelligence (Johnson & Bouchard, 2005), and that spatial ability can be used as a key performance indicator for success in the STEM fields (National Science Board, 2010), more must be done to understand these relationships with respect to gender. Recently, the association between spatial ability and success in the STEM fields have been outlined in a landmark longitudinal study (titled 'Project Talent') following the lives of high school students from the

1950s to the present day (Wai, Lubinski, & Benbow, 2009). The study indicates that those who score highly on spatial tests are more likely to enjoy the STEM fields and gravitate toward STEM careers over and above the effects of mathematical and verbal ability (Wai, Lubinski, & Benbow, 2009). A previous study done by the same authors found that mathematically-talented participants who chose careers in math and science also excelled in object-based skills earlier in life (Lubinsky & Benbow, 2006).

One way to approach closing the gender gap in the STEM fields is to utilize the encouraging evidence that one can train to improve spatial ability exists (Newcombe, 2010; Stieff & Uttal, 2015; Uttal, 2009). Spatial training refers to the explicit teaching of spatial skills, often through the use of mental rotation tasks. At this time, the body of literature concerning the benefits of spatial training is not conclusive on whether an increase in spatial ability has a direct effect on performance in the STEM fields. Although studies done with students enrolled in engineering courses have found an association between spatial training and academic performance, as well as a closing of the gender gap (Sorby et al., 2013), few studies have been done to correlate spatial training and mathematics performance. One thirty-year old study by Ferrini-Mundy (1987) required undergraduate students to complete spatial training exercises during a calculus course but did not find significant increases in calculus performance (although female students were better able to visualize solids of revolution than male students post-training). Other research has found that while spatial training can reduce the gender gap in performance on spatial tasks, it fails to eliminate it (Uttal, 2009). Thus, a call has been made in the spatial cognition literature to extend this line of enquiry and test the potential for spatial training to close the gender gap in the STEM fields by investigating mediating variables and extending periods of spatial training (Casey, 2013).

One correlate to spatial and mathematical ability may be the psychological construct of cognitive style. Cognitive style represents consistency in an individual's manner of cognitive functioning (i.e., information acquisition and processing) (Harvard Mental Imagery and Human-Computer Interaction Lab, 2013). Kozhevnikov, Kosslyn, and Shephard (2005) have investigated cognitive styles that describe individuals' preferences to, or self-assessments of, the use of object, spatial, or verbal modes of information processing. Because the human visual system distinctly processes properties about objects (color, shape) and space (location and spatial relations), they used this neuropsychological evidence to propose the Object-Spatial-Verbal theoretical model of cognitive style. The model outlines three independent dimensions (object imagery, spatial imagery, and verbalization) to propose that object visualizers prefer to construct vivid, concrete, and detailed images of individual objects, while spatial imagers schematically represent spatial relations of objects and spatial transformations. Verbalizers prefer to process and represent information verbally and rely on non-visual strategies (Kozhevnikov, Kosslyn, & Shephard, 2005). In their examination of the three types of cognitive styles, Kozhevnikov, Kosslyn, and Shephard (2005) found that verbalizers performed at an intermediate level on imagery tasks (not at the low level one might expect), and that object visualizers "encode and process images holistically as a single perceptual unit, while spatial visualizers generate and process images analytically, part-by-part" (p. 710).

It seems that individuals often prefer to use one style over another (Kozhevnikov et al., 2005) and that one's preference for one of the three styles directly relates to performance on either mathematical, object imagery ability, or spatial ability tests (MM Virtual Design, 2016). Therefore, cognitive styles may assist math educators to tailor content, assignments, and visualization tools to students' individual differences in cognitive style. Further, information presented in ways that satisfy all cognitive styles could augment student engagement with information presented in the classroom and, perhaps, encourage students' willingness to contribute and collaborate with others, regardless of gender. But, presumably, learning and performance based on visual information presented in a manner congruent to one's cognitive style would be more consistent and effective and, perhaps, help close the gender gap in STEM.

Women tend to report higher object imagery ratings (Blajenkova et al. 2006) and have a negligible advantage in verbal ability (Hyde & Linn, 1988). Casey (2013) pointed out that one reason why large gender differences are often found for mental rotation tasks is because verbal strategies are less effective in solving them than holistic mental rotation approaches used more often by men. Thus, measuring cognitive style in association with spatial ability and an understanding of calculus may afford information about whether those with predominant verbal cognitive styles are women, as well as whether their performance in calculus improves with spatial training. Since success in calculus often predicts success in STEM fields (and calculus involves spatial reasoning), it may be that congruence between cognitive style and material presented or assigned in calculus courses can increase learning and retention (as suggested by Blajenkova & Kozhevnikov, 2008) and, perhaps, increase confidence – a psychological factor that may also help to close the gender gap in STEM fields

Study aims

This pilot study aimed to augment the body of literature regarding the usefulness of spatial training in undergraduate mathematics and answer the following three research questions: (1) What are the impacts of spatial training on undergraduate students' performance in a calculus course? (2) Are differences present in the effects of spatial training between male and female students? (3) What are the impacts of spatial training on students' cognitive style and does cognitive style predict success in the course?

Because academic success in at least one term of calculus is often required in undergraduate STEM programs, calculus courses can serve as "gatekeepers" for STEM fields (Bressoud et al., 2009). If our study reveals that spatial training is beneficial to performance in a calculus course, math educators may have reason to recommend a spatial training module be added to courses that require students to think abstractly. Indeed, if results reveal this training to be particularly effective for female students, an argument may exist that spatial training be studied further with respect to gender differences. Finally, if cognitive styles change over the course of a term to be more spatial as students receive spatial training, or if one style predicts success in the course over another style, it may be reasonable to include an assessment of cognitive style at the start of STEM courses in order for instructors to tailor the delivery of information to students depending on the predominant style of the group (or to assist individual students with styles that are more or less likely to respond to spatial requirements of a course). Overall, connecting research on cognitive styles with spatial training methodology may help educators to diversify the pool of students studying STEM at the undergraduate level while bolstering the success of those who enter into the STEM fields with a non-spatial cognitive style.

Methods

Context

The study took place in the summer quarter of 2016 (June to August) with student participants enrolled in TMATH 126 (Calculus with Analytic Geometry III) at a mid-sized state-funded university in the United States. TMATH 126 covers calculus of sequences and series, vectors and parametric equations and properties of three-dimensional surfaces, as well as integration techniques and approximation, applications of integration and differential equations. Because we assessed introductory-level calculus knowledge, students in TMATH 126 could be expected to understand items in the knowledge assessment we used in both studies.

Participants

Participants were undergraduate students who had already completed two quarters of calculus and, thus, had some prior knowledge of the subject. Seventeen students (8 males, 9 females) attended class for both rounds of data collection but all but one took part in the study (n = 16; 8 males, 8 females, mean age = 21 years). Five students (31% of the sample) reported to be concurrently taking a differential equations course during the summer term, while 11 participants (69% of the sample) reported that they were not receiving other forms of math training at the time of the study). These students also reported that they expected to work an average of 12 hours a week (SD = 5.00) on studying TMATH 126 course material.

Materials

The Calculus Concept Inventory. The Calculus Concept Inventory (CCI) was designed by Epstein (2013) to evaluate how students think about the fundamental concepts in calculus and was used in the present study to gather a baseline of students' knowledge of calculus as well as to determine the effects of spatial training on conceptual knowledge of calculus at the end of the term. The CCI contains 22 questions about limits and differential calculus only, many of which are visual and require an interpretation of a graph. The use of the CCI was deliberate: we did not want to measure mastery of concepts learned in the current course but to measure instead whether spatial training could impact understanding of previously-learned visual topics.

The Purdue Spatial Visualization Test: Rotations. To test students' spatial ability, a shortened version (15 items) of the Purdue Spatial Visualization Test: Rotations (PSVT:R) (Guay, 1977) was also administered. The PSVT:R is a multiple choice test that, per shape, asks students to choose from 5 possibilities a shape that is equivalently rotated as a given shape. This test established each student's baseline spatial ability to assess improvement throughout the quarter and to determine whether spatial training had any effect on students' cognitive style.

Object-Spatial Imagery and Verbal Questionnaire. The Object-Spatial Imagery and Verbal Questionnaire (OSIVQ) developed by Blazhenkova and Kozhevnikov (2009) (but copyrighted jointly by MM Virtual Design, LLC and Rutgers University) was administered to determine students' predominant cognitive style. The OSIVQ is a "self-report questionnaire designed to distinguish between three different types of people: 1) object imagers who prefer to construct vivid, concrete and detailed images of individual objects (e.g., visual artists), 2) spatial imagers who prefer to use imagery to schematically represent spatial relations among objects and to perform complex spatial transformations (e.g., scientists), and 3) verbalizers who prefer to use verbal-analytical tools to solve cognitive tasks (e.g., philosophers and linguists)" (MM Virtual Design, 2016, paragraph 1). The OSIVQ consists of 45 questions (an equal number concerning object imagery, spatial imagery ability, and verbal ability) to assess object imagers, spatial imagers, and verbalizers and takes approximately 10 minutes. Each item asked on a 5-point Likert scale ranging from "totally disagree" (1) to "totally agree" (5). Four items (one spatial, three verbal) from the OSIVQ required reverse coding.

Spatial training. All participants received spatial training during the term consisting of the administration and discussion of several tests of spatial ability, such as those described in Wai et al. (2009). In addition, exercises from a spatial training workbook developed by Sorby et al. (2013) was used with permission. Students completed an average of 15 minutes of spatial training during each day of class. Exercises in the workbook ranged from assessments of what a given shape would look like when rotated around a given axis, to asking students to draw an object from different angles using different cross-sections, to showing a 2-D expression of an object when asking students to draw an analogous 3-D object. During spatial training, students were asked to discuss the exercises in small groups and come to a consensus on the correct answer before answers were discussed among the class at large.

Procedure

Participants completed all tasks in the assigned TMATH 126 classroom for both rounds of data collection. The four tasks were offered to participants in the following order: consent form; OSIVQ; Visualization and Rotation Purdue Spatial Visualization Test; Calculus Concept Inventory. Thus, the tasks were not completed simultaneously: only when a task was completed was the next task offered to a participant by one of the researchers. Each participant was asked to create a unique codename for him or herself to include on the front page of each task in order afford direct measurement of change in task performance over time. No calculators or other electronic devices were used during task completion. Each participant was dismissed from the classroom after he or she had completed all tasks.

Results

Data from the CCI was analyzed to determine students normalized gain on the assessment using the formula $g = \frac{posttest - pretest}{100 - pretest}$. Calculating normalized gain was appropriate because it takes into account the amount of improvement possible for each student. This is important because students may have widely variable spatial abilities upon entry into the course. Each student's score was tallied on both the pre-test and post-test and a paired-samples *t*-test was performed to determine significant improvements in calculus knowledge. No significant improvement was revealed over the term ($p > .05$).

Although male participants' average scores were higher on the CCI than women at the start of term ($M = 9.63$, $SD = 4.47$ and $M = 7.75$, $SD = 7.75$, respectively), they were not significantly higher ($p > .05$). This result was also borne out at the end of term whereby male students' average scores on the CCI were insignificantly higher than female students' scores ($M = 10.25$, $SD = 2.09$ and $M = 8.25$, $SD = 1.40$, respectively).

To determine an association between students' spatial ability and mathematical ability, a correlation coefficient for a student's final grade in Calculus III and their final PSVT:R score was calculated. This correlation ($r = 0.4283$) was not significant ($p > .05$). Additionally, a paired-samples *t*-test did not determine a significant improvement in students' rotational ability during the term ($p > .05$). Consistent with previous studies (Levine et al., 1999; Voyer, Voyer, & Bryden, 1995), male participants' average scores were higher on the PSVT:R than women at the start of term ($M = 9.87$, $SD = 4.29$ and $M = 9.75$, $SD = 2.05$, respectively), and at the end of the term ($M = 10.5$, $SD = 1.45$ and $M = 9.25$, $SD = 1.24$, respectively). However, these differences were not significant ($p > .05$). And, similar to results pertaining to performance on the CCI, a paired-samples *t*-test did not determine a significant improvement in students' rotation ability during the term ($p > .05$).

Table 1:
Descriptive Statistics for Test Variables Per Gender Type

Variable	Gender	Mean	Standard Deviation
		Round 1	*Round 1*
CCI (scored out of 22)	Male	9.63	4.47
	Female	7.75	3.45
PSVT:R (scored out of 15)	Male	9.88	4.29
	Female	9.75	2.05
OSIVQ: Spatial (scored out of 75)	Male	49.75	2.74
	Female	46.13	3.44
OSIVQ: Object (scored out of 75)	Male	47.00	2.67
	Female	50.13	2.26
OSIVQ: Verbal (scored out of 75)	Male	42.75	1.40
	Female	39.50	1.58
		Round 2	*Round 2*
CCI (scored out of 22)	Male	10.25	2.09
	Female	8.25	1.40
PSVT:R (scored out of 15)	Male	10.5	1.45
	Female	9.25	1.24
OSIVQ: Spatial (scored out of 75)	Male	54.00	2.19
	Female	47.50	3.70
OSIVQ: Object (scored out of 75)	Male	52.38	3.42
	Female	49.25	2.04
OSIVQ: Verbal (scored out of 75)	Male	45.38	0.80
	Female	40.13	1.42

Similar to scores on the CCI, male participants' average scores were higher on the PSVT:R than women at the start of term ($M = 9.87$, $SD = 4.29$ and $M = 9.75$, $SD = 2.05$, respectively), and at the end of the term ($M = 10.5$, $SD = 1.45$ and $M = 9.25$, $SD = 1.24$, respectively). However, these differences were not significant ($p > .05$).

On average, more students at the start of the term self-identified as object learners ($M = 48.56$, $SD = 7.00$) than they did as spatial learners ($M = 47.94$, $SD = 8.69$) or verbal learners ($M = 41.13$, $SD = 4.41$). Indeed, scores on the object sub-scale were significantly higher than on the verbal sub-scale, $t(15) = 3.97$, $p = .001$. In addition, scores on the spatial sub-scale were significantly higher than those on the verbal sub-scale, $t(15) = -2.98$, $p < .01$. No significant differences were found between scores on the spatial and object subscales ($p > .05$).

After 10 weeks, the object style remained the predominant style for the class as a whole ($M = 50.81$, $SD = 7.87$) and, similar to the start of term, the second-most common style among the class was spatial ($M = 50.75$, $SD = 8.96$), followed by verbal ($M = 42.75$, $SD = 4.16$). Similar to the start of the term, these differences were significant: students self-scored significantly higher on the object style sub-scale compared to the verbal style subscale, $t(15) = 3.57$, $p < .01$. They also self-scored significantly higher on the spatial sub-scale compared to the verbal subscale, $t(15) = 3.55$, $p < .01$. Again, no significant differences were found between scores on the spatial and object subscales ($p > .05$).

Although mean scores on each of the three sub-scales increased over the term, paired-samples *t*-tests revealed only one significant difference between subscale scores over time. Students did not self-score significantly better or worse on the object or verbal subscales over time (all *p*s > .05). However, they did self-score significantly higher on the spatial sub-scale at the end of the term after receiving spatial training, $t(15) = -2.59$, $p < .05$.

At the start of the term, the men in the sample identified most, on average, as spatial learners (*M* = 49.75, *SD* = 2.74) and least as verbal learners (*M* = 42.75, *SD* = 1.40). This was also the case at the end of the term (see Table 1). In contrast, the highest average score among the three cognitive styles for women was on the object sub-scale (*M* = 50.13, *SD* = 2.26) while the lowest was on the verbal subscale (*M* = 39.50, *SD* = 1.58) and remained so at the end of term.

Although participants' general perceptions of dominant cognitive styles remained stable over 10 weeks, each gender's scores increased on each sub-scale except that women's scores on the object style sub-scale decreased slightly (but insignificantly, $p > .05$) over time.

After the first round of data collection, independent samples t-tests revealed no significant differences between male and female participants' scores on the three OSIVQ sub-scales (all *p*s $> .05$). However, at the end of the term, a significant difference between men and women's scores on the verbal sub-scale of the OSIVQ was revealed, $t(14) = 3.22$, $p < .01$.

Finally, scores on the spatial sub-scale of the OSIVQ did not correlate significantly with high scores on the PSVT:R at the start of the term ($p > .05$) but did so at the end of term ($r = .62$, $p = .01$). No other significant correlations were revealed between scores on the PSVT:R and other sub-scales of the OSIVQ, nor were there any significant associations between scores on the OSIVQ sub-scales and the CCI at the start or end of the term.

Discussion & Future Work

This pilot study revealed that students who self-identified more as object or spatial learners than verbal learners did so significantly more so after spatial training. In particular, a strong identification as a spatial learner linearly associated with mental rotation ability after spatial training. However, spatial training alone did not significantly impact the calculus or mental rotation abilities of undergraduate students. The results of this research informed an adjustment to the methods when a second study was undertaken with a second-term calculus course in 2016. Since the present study was done during a summer term, only a total of 18 classes occurred, with approximately 15 minutes of spatial training done per class -- this may not have been sufficient for students to be influenced by the training because less than half of the items in the workbook were completed during the term.

In general, the second study further explores our research questions in a Calculus II course ($n = 14$), as well as in a section of Calculus II that was not engaged in spatial training to serve as a control group ($n = 9$). The type of data collected was the same as the pilot study except that students were required in the second study to complete some spatial training modules independently and then bring their responses to the classroom to discuss with their peers in order to allow more time for an effect. Another difference between the present study and second iteration concerns the PSVT:R assessment. In the pilot study, 15 items were chosen ranging in difficulty from easy to medium. This range did not determine a reliable baseline in the pilot study because several students obtained a perfect score on this instrument pre- and post-spatial training. Therefore, in the second study, 15 items were chosen where 5 were categorized within the PSVT:R as "easy," 5 as "medium," and 5 as "difficult." No participant in the second study scored perfectly on the pre-test of the PSVT:R. Other results from the second study are currently being analyzed.

We believe that continuing this interdisciplinary line of enquiry is compelling for math educators and cognitive psychologists in better understanding how best to present calculus to students and assess their knowledge over time. Some evidence exists that object visualizers experience difficulties in interpreting science graphs as abstract schematic representations because they interpret them more literally, as though it were a picture (e.g., Heagarty & Kozhevnikov, 1999; Kozhevnikov et al., 2002). Perhaps students with particular cognitive style preferences can be offered opportunities in the classroom to pair physical events (or words) with graphical representations moving in real time to explore connections between how they perceive the world, as Kozhevnikov & Thornton, 2006, suggest. We hope that understanding more about how cognitive styles associate with spatial training and success in mathematics courses will advance pedagogical principles and practices and, ultimately, help to close the gender gap in STEM fields.

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Exploring Student Conceptions of Binary Operation

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Students' conceptions about binary operations often reflect a lifetime of situated learning about concepts such as arithmetic operations and functions in their K-12 years. This prior knowledge base bleeds into their experience of binary operation in abstract algebra, leading students to have differing and incomplete notions of what constitutes a binary operation. We present a qualitative study in which we classify students' conceptions of binary operation in terms of variation theory and concept image and definition. These frameworks helped us to focus on critical and noncritical aspects of binary operation found in student reasoning. Our results indicate that students' enacted objects about binary operation fall into one of three domains (i.e. function meaning, arithmetic meaning, and structural meaning) and often depend on task context. Additionally, students' reasoning reflected a set of critical aspects that diverge from the conventional critical aspects of binary operation.

Key words: Abstract Algebra, Binary Operation, Conceptual Understanding

Binary operations are woven into mathematics from early elementary school and throughout the K-20 spectrum. However, binary operation is not defined abstractly until advanced mathematics courses such as abstract algebra. Despite the role that operations play in both familiar structures such as our base ten number system, and non-routine and abstract settings of group, ring and field, little attention has been paid to binary operation directly. This is reflected both in its treatment in abstract algebra textbooks (e.g. Fraleigh, 2003, Gallian 2012), and the body of mathematics education literature (Melhuish, 2015). In fact, during a recent survey with abstract algebra instructors, binary operation was deemed one of the most important topics in introductory group theory, yet identified as one of the least difficult topics in the course (Melhuish & Fasteen, 2016). This divergent sentiment may account for the lack of specific attention to binary operations as an abstract concept.

However, as part of a large study exploring student conceptions in introductory group theory, we found that student conceptions of binary operations had the potential to interfere with their performance on tasks related to a number of essential topics in group theory (Melhuish & Fasteen, 2016). We developed a survey centered on binary operation to empirically explore whether abstract algebra students have alternate or incomplete views on binary operation. In this paper, we report on student conceptions of binary operation through three lenses: function, arithmetic, and structure. We also leverage ideas from variation theory to dissect enacted objects of learning, specifically focusing our analysis on which critical aspects students appeared to attend to during their engagements in the tasks. We conclude with some discussion of how variations related to binary operation in typical classrooms may align with students' understanding (or lack thereof) of binary operation.

Theoretical Framing

There are a number of ways to make sense of concept understanding in advanced mathematics. In this study, we merge the construct of concept image/concept definition (Tall & Vinner, 1981) with elements of the variation theory of learning (Marton & Booth, 1997). Our underlying assumption is that a given theory of learning can meaningfully inform the analysis of student understanding at a point in time. The concept image and definition framework provides a parsing of concept understanding into definitions and surrounding cognitive structures associated with a given concept. A concept has a conventional definition in the mathematics community, but also personal definition for each student. Concept images capture both this personal definition, and all associations such as metaphors, examples, applications, and representations of a concept. Tall and Vinner's work is especially powerful in its ability to capture a lack of coherence that often exist between and within student concept images and definitions.

Variation Theory (Marton & Booth, 1997; Watson & Mason, 2006) can provide one lens for making sense of concept images and definitions by both focusing on contrasting elements across a particular construct as well as bringing to light analogous features that may also exist (Watson, 2017). This is done by leveraging the underlying assumption in variation theory that, "nothing is one thing only, and each thing has many features" (Lueng, 2017, p. 68). Variation frames aspects of an object as either critical aspects, features an object possesses that are shared across all instantiations (e.g. the sum the interior angles of an n-gon is $(n-2) * 180$ for all n-gons), while others features, permissible variations, may be unique to a specific instantiation or subclass of object when compared to others (e.g. the sum of the interior angles of a triangle is 180). For instance, variation theory could be used to explore how a child is taught about oranges in the context of apples. Through contrasting oranges and apples, students may attend to what aspects of orange are critical to "orange-ness" versus what aspects are shared amongst the two fruits. In this way, variation draws attention to critical aspects through *contrast*, that is by comparing examples and non-examples, students can attend to the features that are critical to the concept by seeing them vary in the non-examples.

Beyond contrast, variation also allows generalizing via attending to sameness, or features that are shared with elements contained in a student's prior knowledge. Gu (1994) identifies the use of examples, standard and non-standard, and non-examples as *conceptual variation.* To return to our orange example, by experiencing varying types of oranges such as common oranges and bitter oranges, students may apprehend commonalities such as texture and color to determine critical aspects of "orange-ness" and remove noncritical features such as a requirement for sweetness. Other metaphors have been aptly used to explain variation theory: seeing the forest through the trees (Leung, 2017, p. 68) in the sense of being able to separate features and have a holistic treatment of a topic, and putting a puzzle together (Marton & Booth, 1997) where essential features, or *critical aspects* are pieced together to build an accurate conception of the whole concept.

In order to meaningfully leverage a theory of learning (variation), and theory of understanding (concept image and definition), we use variation theory's construct of *enacted object of learning* (Marton & Pang, 2006). The enacted object refers to what is at the forefront of a student's awareness when engaged in a task, including the critical aspects they attend to. Additionally, the *lived object of learning* refers to the critical aspects students are aware of after leaving a learning environment. In a survey situation, students enact an object of learning that reflects their lived object of learning. In this way, to make sense of snapshot understanding, we can identify what *critical aspects* appear to be part of the student's concept image or object of learning. See Figure 1. A students' concept image can be parsed in terms of critical and noncritical aspects brought to the forefront by students when engaged in survey tasks. We also adapt Bussey, Orgill, & Crippen's (2013) view that a students' lived object of learning is not just a result of exposure to variation in class, but also combines with *prior knowledge* in order to develop their larger conception.

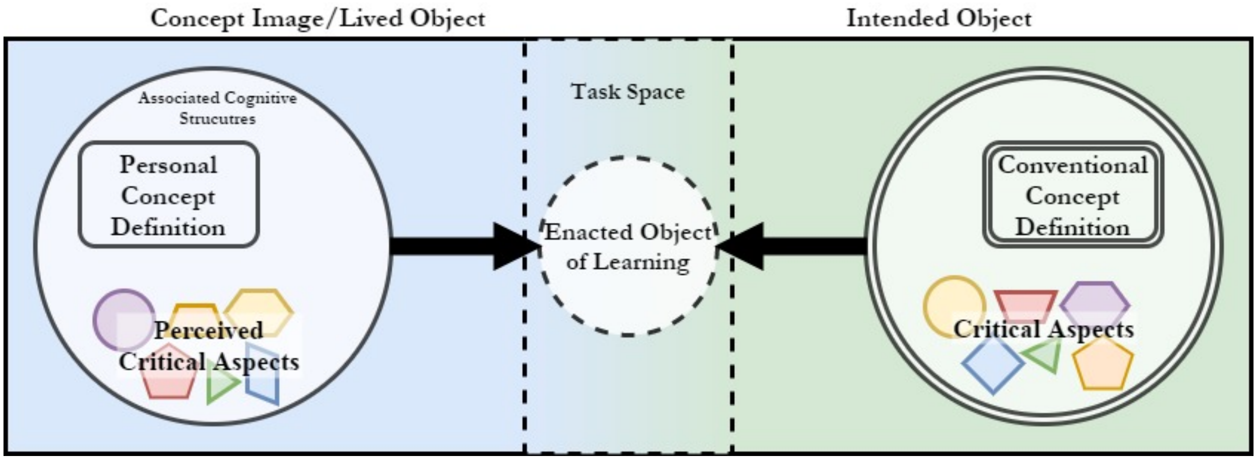

Figure 1. The relationship between a students' concept image, the intended object and the enacted object of learning within the task space.

Prior Research on Student Conceptions of Binary Operation

Binary operation is often a key underlying component of educational studies in abstract algebra, although its role is often left implicit. Group, ring and field are defined via the way binary operation(s) provide structure on a relevant set. Binary operation played an essential role in Larsen's (2009) student-driven reinvention of the group structure (Larsen, 2009) and Simpson and Stehlíková's (2006) work having a student apprehend the structure of a commutative ring. Informally, a binary operation can be thought of as a rule for combining two elements of a set to produce a single element (from the same set). Addition is a binary operation on the set of integers because the sum of any two integers is another integer. Formally, binary operations are typically defined as:

A *binary operation* $*$ on a set S is a function mapping $S \times S$ into S.

In this way, binary operation brings together two concepts that may have been seen as disjoint in prior situations: operations and functions. In the operation literature, functions may be treated as special cases of operation (e.g. Slavit, 1998), or alternately, binary operations may be treated as special cases of functions (e.g. Dubinsky, Dautermann, Leron, & Zazkis, 1994).

A majority of the literature on operation falls into the former category with focus on arithmetic operations (with the occasional use of function as an operation example.) Slavit (1998) posits one such framework: *operation sense*. He discussed operation sense in a series of stages built around familiarity with standard arithmetic operations in terms of underlying models, symbolic notation, their relationships to other operations, properties they may possess, and their meaning independent of concrete inputs. However, this framework is built in terms of operations that are not arbitrarily defined but rather represent a standard process. For example, a process of combining groups underlies the operation of addition. One of the driving components behind operation sense is the operation's duality as both a process and object. Gray and Tall (1994) use the construct of *procept* to explain the process and concept associated with a given operation symbol. For instance, an expression such as "$3 + 2$" represents both the process of adding 3 and 2, as well as the resulting sum. This literature is emblematic of the larger field of research on operations in that it remains contextualized with specific examples of operations.

When attending to binary operations as a special case of functions, student conceptions around functions may play an additional role in their understanding. A large body of literature exists highlighting the many complexities of the function concept and students have been documented with many alternate and incomplete understandings through the K-20 spectrum (Oehrtman, Carlson, & Thompson, 2008). For example, students may prefer a certain representation such as a written, symbolic rule (Breidenbach, Dubsinky, Hawks, & Nicols, 1992; Vinner & Dreyfus, 1989). From a variation standpoint, this may reflect students overgeneralizing standard examples where a general rule is often an aspect. Additionally, students have frequently proceduralized functions such as evaluating $f(x+a)$ as being equal to $f(x)+a$ (Carlson, 1998). This highlights the complexity involved in making sense of what constitutes the inputs and outputs of a function. The input and output language (and to an extent input/output machine metaphor) often provides the basis for conceptualizing function (Tall, McGowen, & DeMarois, 2000). Frequently, much of the difficulty with functions is attributed to the process/object duality of functions (e.g. Dubinsky & Wilson, 2013). Functions can be conceptualized as the *action* of mapping individual inputs to outputs, the more general *process* of mapping from domain to range, or as an *object* itself to be operated and compared with other functions. As with operations research, function research generally does not breach binary operation specifically, but tends to remain in the more familiar K-12 context of inputs that are singular elements rather than two inputs (or equivalently ordered pairs of inputs).

In terms of the abstract concept of binary operation, the literature does provide some framework options. Brown, DeVries, Dubinsky, and Thomas (1997) presented a genetic decomposition of binary operation where students may have an action conception (explicitly combining two inputs to arrive at an output), a process conception (a general process for combining inputs to arrive at outputs), or an object conception (seeing binary operations as things that can be acted on as objects themselves). Novotná, Stehliková, and Hoch (2006) provided a structure sense framework capturing the transition from familiar to unfamiliar operations. They divided understanding of binary operations into four levels: *Recognise a binary operation in familiar structures; Recognise a binary operation in non-familiar structures; See elements of the set as objects to be manipulated, and understand the closure property; and See similarities and differences of the forms of defining the operations (formula, table, other).* Rather than considering stages of mental constructions in terms of process/object reification, the focus was on apprehending structure. Ehmke, Peasonen, and Haapasalo (2001) presented a framework leveraging different representations to distinguish students with procedural and conceptual understandings. They identified students as having procedure-based understanding of binary operation if they could match binary operations if presented in different representations. The next level is procedure-oriented where students could also create different representations when prompted. The highest level is conceptual where students could not only move between representations, but also determine if a given relation was a binary operation. The variation theory-informed approach in this paper aims to complement these process frameworks with a more nuanced view of exactly what critical aspects/noncritical aspects of binary operation may be influencing student conceptions.

Some of these attributes have been broached in the misconception literature in other subjects. Mevarech (1983) found statistics students overgeneralized properties (noncritical aspects) such as associativity onto binary operations. Zaslavsky and Peled (1996) had pre-service and in-service teachers generate examples of binary operations resulting in a number of issues that can be attributed to critical aspects. For example, some of the participants defined a unary operation which may indicate that an operation defined on two elements may not be a critical aspect of these students' conceptions of binary operation.

The study presented below serves a follow-up to initial results from the Group Theory Concept Assessment (GTCA), formerly, Group Concept Inventory (Melhuish & Fasteen, 2016). We conjectured that student conceptions of binary operation accounted for performance on questions targeting subgroups, the associative property, and groups themselves. Of 486 students asked if a binary operation could be defined on the set $\{1,2,4\}$ in a way to form a group, only 23% responded correctly that "an operation can be defined on any three element set to form a group." We conjectured that the students were limiting their view of acceptable binary operation to those with a familiar symbolic name. In a question targeting subgroups, students were asked whether the set $\{\overline{1},\overline{2},\overline{3}\}$ forms a subgroup in $\mathbb{Z}_6$ (adapted from Dubinsky, et al., 1994; Hazzan & Leron, 1996). A little over a third of the 429 students who responded, indicated that the set forms a subgroup by claiming that $\mathbb{Z}_3$ is a group itself. Although $\mathbb{Z}_3$ has a different operation, the students appeared unaware of this fact possibly owing to students possessing a view of binary operations that only allowed them to attend to the similarities in $(+ \bmod 3)$ and $(+ \bmod 6)$ (i.e. the additive structure) and unaware of the contrasting restrictions placed by the moduli. In a third question, students were asked if the binary operation defined as $a \diamond b = 1/2\,(a+b)$ was associative. A sizable number of students (17% of 432) treated $1/2$, a, and b as the inputs claiming the operation was not associative because: $1/2\,(a+b) \neq (1/2\,a) + b$. These students appeared to have issues identifying what exactly was the binary operation as they attempted to attend to some sort of artificial structural feature that they felt was shared amongst all binary operators.. In each of these cases, conceptual issues with binary operations appeared to be a logical cause for the students' reasoning. In this paper, we directly unpack conceptual understandings associated with binary operation in terms of *meanings* and *critical aspects*.

Methods

The surveys analyzed below were given to two introductory, undergraduate-level, modern algebra classes (n=12, n=8 respectively) at a large, public university. The surveys were created to cover a variety of binary operations in terms of standard examples, non-standard examples, and non-examples. We selected four activity domains that research has shown are tightly linked to understanding of mathematical concepts:

1. *Is or is not.* Determining if a given instantiation is an example of a concept (e.g. Ehmke, Peasonen, and Happasalo, 2011)
2. *Same or different.* Determining if two instantiations are mathematical the same (e.g. Novotná, Stehliková, and Hoch, 2006)
3. *Properties.* Determining what properties an example may or may not have. (e.g. Dubinsky, et al., 1994)
4. *Generating.* Creating an example meeting some criteria (e.g. Zazkis, & Leiken, 2007)

The survey consisted of questions corresponding to each of the above categories. In each case, the student was also prompted to explain their reasoning. A set of sample tasks can be seen in Table 1. Additionally, four think aloud interviews were conducted in order to test the robustness of survey interpretations. The interviews were semi-structured to allow for the interviewer to follow the student reasoning in an organic manner.

Table 1

Sample Survey Tasks

Determine if the following define a binary operation on the given set. Explain why or why not. • Addition mod 3 on the set $\{0, 1, 2\}$ • $\diamond(a) = \sqrt[3]{a}$ on $\mathbb{R}$ (the set of real numbers) • $\diamond(a) = a^2$ on $\mathbb{R}$
Determine if the following binary operations are the same. Explain why or why not. • **Op 1:** Division on $\mathbb{R}$, **Op 2:** Multiplication on $\mathbb{R}$
Consider the operation $\diamond$ on the set of real numbers defined as follows $a \diamond b = \frac{1}{2}(a + b)$ Is this operation associative? Why or Why not?
Is it possible to define a binary operation on the set $\{1,2,4\}$ such that the set and operation form a group? Why or why not?

Analysis

The surveys were analyzed through a content analysis heavily informed by phenomenographic methods. Phenomenographic studies attempt to study *meaning* through the interaction between an individual and a task (Trigwell, 2006). The assumption is that students experience different phenomena in a variety of ways. One use of this type of study is to "investigate the range of different ways that students experience technical concepts" (Reed, 2006, p. 1). The focus is often on the range of ways students ascribe meaning to ideas. The analysis is not about classifying students into categories, but rather classifying their reasoning based on ways of experiencing binary operation in the given tasks. Similarly, the classification of student reasoning is not meant to ascribe hierarchical values to different forms of thinking, but to report the qualitatively different ways in which students thought about binary operations themselves.

From this vantage point, the data was analyzed in two manners to attend to our focal aspects of concept image: (1) Global: overarching understanding of binary operation and (2) Local: properties treated as critical aspects of binary operations. In phenomographic tradition student responses were stripped from individual surveys and interviews to put attention on the reported reasoning in tasks rather than analyze an individuals' responses as a whole. In the next section, we outline three ways of thinking about binary operation and various critical and noncritical aspects were found across surveys.

Results and Discussion

At the global level, we identified three meanings associated with binary operation: *function meaning, arithmetic meaning,* and *structural meaning.* We begin by explaining each type of meaning. We then explore the enacted objects of learning via looking at aspects of binary operation treated as critical throughout the range of tasks. At each point, we also reflect on the connections between these aspects and relevant meanings for binary operation.

Function Meaning for Binary Operation

We define a *function meaning* as one that leverages notions of function when attributing meaning and engaging in tasks related to binary operation. From the parent study (Melhuish, 2015), this type of definition has been articulated by a number of students both formally and informally. One undergraduate student described binary operation (on $\mathbb{Q}$) as, “you have some operation that goes from $\mathbb{Q}\times\mathbb{Q}$ to $\mathbb{Q}$” explicitly focusing on the ordered pair domain. Other students provided more informal definitions such as, "[A binary operation] takes two elements as input and outputs a single element but it has to be defined on the entire... on all possible pairs in $\mathbb{Q}$ or in your group.”

This type of definition aligns with what definition is found in standard undergraduate textbooks for introductory abstract algebra (e.g. Fraleigh, 2003, Gallian 2012). With a function treatment of binary operation, various properties associated with prior knowledge of function may be paired with the specific binary operation case such as those of being well-defined, range-domains, or one-to-one. Similarly, function language such as “input” and “outputs” often become part of the vernacular around binary operation.

Arithmetic Meaning for Binary Operation

We define an *arithmetic meaning* for binary operation as a focus on a binary operation as a way to combine two elements to produce a third. For example, one student stated that an operation was, “a binary operation does something to two elements from a set and that what it does to those two at the very end is still in that set.” This meaning more prominently connects to prior knowledge of operations such as arithmetic operations. An operation such as addition is thought of as a way to sum two numbers to produce an third. Rather than conceptualizing adding the numbers 2 and 3 as inputs with an output of 5, students are likely to think of combining 2 and 3 to arrive at 5. This language aligns with the literature where operation is often framed as often abstracting and formalizing some process to arrive at a result such as combining groups as the process behind addition.

Structural Meaning for Binary Operation

We define *structural meaning* for binary operation as meaning ascribed at a more global level where binary operation can be thought of a general pattern or way that elements interact without necessarily attending to the role of either input/output or combining two elements to arrive at a third. Structural meaning manifested in reasoning that looked holistically at operations often identifying familiar traits. This type of treatment tended to emerge in tasks prompting students to determine if two binary operations are the same. In fact, students applying a structural meaning may not even look at the inputs at all, but rather focus on the set of outputs or even the general utility of an operation. The meaning language surrounding binary operation often contained “pattern” or that the tables “behave the same.” The term structural was selected because (1) students leveraging this type of meaning where generally not attending to individual element interactions, and (2) they may be leaning on prior class knowledge on “structural properties” and their intimate relationship to binary operations.

At this point, we want to caution that these meanings a) are not always mutually exclusive in a given enacted object of learning, b) may internally have varying levels of sophistication, and c) are not necessarily consistent and coherent amongst individuals as they engage in tasks. In fact, of the interviewed students, most used all three meanings at different points when engaging in these tasks. In terms of sophistication, a number of frameworks exist to capture sophistication of function understanding. For example, if a student has a schema understanding (in the sense of Dubinsky & Wilson, 2013) of the binary operation function, they can likely approach majority of the tasks with high sophistication via treating a binary operation as a function object, process, or action at appropriate times. However, a robust understanding of binary operation using an arithmetic approach can also allow for successful treatment of binary operation tasks. As we know from the Concept Image / Definition (Tall & Vinner, 1981) literature, students' degree of sophistication with one meaning or definition may or may not align with other aspects. In this way, the purpose of the below analysis is to attend explicitly to critical aspects of binary operation appearing in students' enacted object of learning throughout the survey tasks.

In the next section we briefly outline the most prevalent aspects of binary operation found in the students' reasoning amongst surveys. The aspects can be found in the Table 2.

Table 2

Critical Aspects in Students' Concept Images

Aspect	Description	Critical/Noncritical
Closure	A binary operation is closed on the set	Critical Aspect
Two Elements	A binary operation is defined on two elements	Critical Aspect
Defined Element-wise	A binary operation is determined by where each pair of elements map	Critical Aspect
General Rule	A binary operation has a familiar, general name	Noncritical Aspect
Element-Operator-Element	A binary operation is of the form $a * b$	Noncritical Aspect

Critical Aspects of Binary Operation

We begin by exploring three critical aspects of binary operation reflected in students' enacted objects of learning. These are aspects of binary operation that are generally part of the intended object of learning and reflect the conventional approach to binary operation. We share the degree to which they were found in students' reasoning.

Closure. The most commonly addressed critical aspect of binary operation enacted was closure. In fact, reasoning about closure emerged during the *Is or Is Not* questions almost universally. Only one survey was absent of reasoning about closure. For example, when asked if x^2 was a binary operation on the set of real numbers, a typical student response was: "yes, because any $a \in \mathbb{R}$ squared will still be in $\mathbb{R}$." A number of tasks included functions that were not binary operations, but rather unary operations. These tasks may lead to students leveraging a function meaning focused on a single input and output to explore closure. For example, one student shared the following sentiment while exploring this prompt, "One input could only give you one unique output." This language reflects a function meaning for binary operation.

Two elements. The critical aspect of binary operation that was often not seen in the enacted objects of learning was that of an operation being defined on two elements, or on an ordered pair. This aspect was addressed only twice amongst the responses to $\sqrt[3]{x}$ and x^2 being binary operations respectively. As seen in the closure section above, the focus was exclusively on returning to the same set without attention to the type of input (or equivalently the domain of the function). The students who did attend to this critical aspect used language reflecting an arithmetic meaning using "two things" rather than "ordered pair" or input/output language. During a follow-up interview, one student reflected, "Do we need two things? Probably, because the word binary…Initially I was thinking closure, then I was like, wait should there be two things?" This student originally attended only to closure, but revised his thinking in light of the intentional variation between unary and binary operations in the tasks. His enacted object of learning was influenced by the particular tasks illustrating an example of a critical aspect moving from the background to forefront of his concept image.

Defined element-wise. When students were asked to determine if two binary operations were the same, their reasoning often varied depending on the task. For example, when students were provided with a table of elements and a list of elements defined element-by-element (see Figure 2), they were able to check each individual pair of inputs to determine if the outputs match. This approach likely reflects an arithmetic meaning for operation.

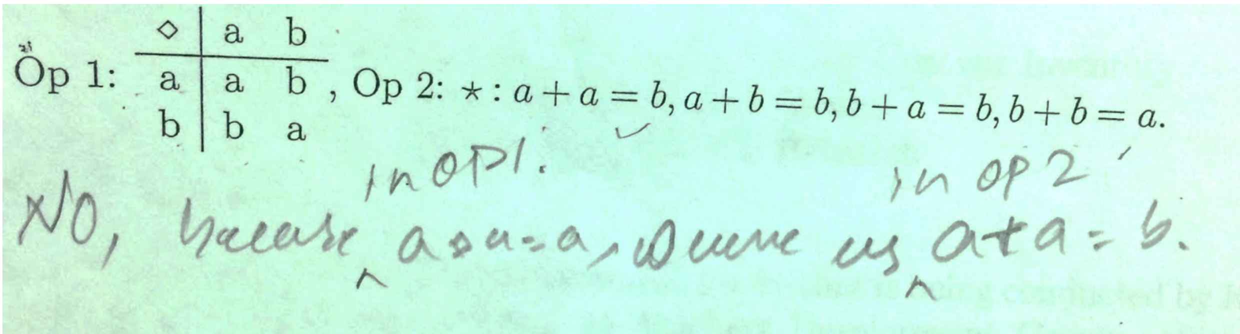

Figure 2. Student identifying two operations as different based upon an element-wise conception of binary operation.

However, when asked about the operations in the Cayley tables in Figure 3, different reasoning emerged reflecting a different enacted object. Student reasoning often included language like the "same pattern" or alternately they were the same type of operation, "addition." The differing number of elements and differing outputs for the same inputs were not treated as problematic in much of the student reasoning. In this task, the students did not seem to attend to where the elements map as a critical aspect of binary operation. This patterning approach aligns with a more structural meaning for binary operation. Rather than looking at combining individual elements, or thinking about the function mapping, the students looked holistically at the binary operations.

Op 1:

$\diamond$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

, Op 2:

$\star$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Figure 3. $\mathbb{Z}_3$ and $\mathbb{Z}_6$ presented as Cayley Tables

This also occurred when asked if multiplication and division on the real numbers were the same operation. This occurred validly (such as stating one is associative and the other is not) and invalidly such as explaining that the operations were the same because, "basically to undo division, you apply multiplication" or that division could be rewritten as multiplication: "Yes because a/b is the same as $a * 1/b$." This structural approach likely encapsulates a number of ways of making sense of sameness in mathematics from identifying families (such as the operations in Figure 3 being types of "addition") or related operations (such as multiplication and division which are inverses). We do not want to imply that addressing patterns holistically may limit approaching this family of tasks. A high level of sophistication around this idea may be reflected in identifying structural or essential sameness in the sense of isomorphism (e.g. Weber & Alcock, 2004, Leron, Hazzan, & Zazkis, 1995). In fact, treating operation as a structure on sets is a fundamental goal in abstract algebra. To adopt APOS-language, seeing operations as structures rather than just processes or actions is necessary for an object conception. However, differing tasks illustrated a variety of student reasoning related to a structural meaning with both accurate and inaccurate approaches.

Noncritical Aspects of Binary Operation

While students treated intended critical aspects of binary operation to varying degrees, students' reasoning reflected a number of noncritical aspects as well.

Element-operator-element. Student responses reflected a struggle with identifying the binary operation in a situation where the binary operation did not appear as "element operator element" (E-O-E). The treatment of E-O-E format emerged in a number of tasks. When asked if $a \diamond b = 1/2\,(a+b)$ is associative, many student responses contained: $1/2\,(a+b) \neq (1/2\,a) + b$. In follow-up interviews, one student explained, "I don't know what is the binary rule." This particular student explained she lacked confidence in her response because she was not sure what the binary operation was. Her enacted binary object did not align with the object found in the task. Another student providing a similar response was asked to explain what this binary operation was and identified the "$a + b$" portion of the expression. In this sense, the student was focusing on the portion of the operation that appeared in E-O-E format. She had some cognitive dissonance when attempting to apply her definition of the associative property of: $a * (b * c) = (a * b) * c$ to explore whether the midpoint binary operation in fact met this requirement. When the interviewer asked what operation the $*$ represented between the b and c, the student responded, "the plus sign." The student responses to this prompt tended to link to the arithmetic meaning of binary operation, however, with perhaps a lower level of sophistication where the idea of combining two inputs is necessarily linked to a rule in the E-O-E format. This noncritical aspect was also seen in other tasks including the previously discussed task prompting students to determine if x^2 is a binary operation. Several student responses explained that x^2 could be written as "$x * x$" and therefore was a binary operation. They converted from a non-E-O-E form to an E-O-E form.

A general rule. In addition to binary operations typically represented in the E-O-E format, most binary operations have general names. When asked to generate a binary operation that forms a group on an atypical set $\{1,2,4\}$, a majority of students provided examples where they checked known operations such as modular arithmetic operations. The responses in this study were consistent with our previous finding (Melhuish & Fasteen, 2016), where students reasoned by checking a list of known operations. This desire for a general rule was further found in other tasks such as determining if the operation defined element-wise was a binary operation. A task of this sort was found in Figure 4.

$$1+1=1, 2+2=1, 1+2=2, 2+1=2 \text{ on the set } \{1,2\}$$

Figure 4. This task prompted students to determine if the operation defined element-wise was a binary operation.

This task was frequently left blank. When interviewing a student who did so, she explained, "I was just- I don't know. What are they trying to get at?" She explained she had not seen this binary operation and was attempting to define a general rule without success. She noted, "I guess because it didn't specifically say mod something- it blew me away because of that." A general rule appeared to be a critical aspect of her enacted object. This attention to a general rule reflects the similar desire for functions to have a general rule in the literature on student conceptions of function (Breidenbach, Dubsinky, Hawks, & Nicols, 1992). The interview subject primarily used language reflecting a structural meaning such as searching for a "pattern." However, her enacted object may be linked to her function knowledge (that of a general rule being a critical aspect), her arithmetic knowledge (a set of named, general rules), or more generally a desire to reduce abstraction level (Hazzan, 1999) to familiar operations.

Other properties. The list of critical aspects and permissible variations of binary operation does not present an exhaustive list of features attended to in students' enacted objects. Rather, they provide an outline of some of the most prevalent aspects found in responses to the survey tasks. In Figure 5, we provide a list of additional aspects of binary operation that were found in student reasoning. In the blue are critical aspects explicitly found in definitions. In red are critical aspects implicit in the binary operation definition. In purple are noncritical aspects of binary operation found in student reasoning. Additional noncritical aspects are related to functions (one-to-one), and groups (associative property, and group structure). Binary operations that impose a group structure are the typical examples in abstract algebra, and therefore their critical aspects may be adopted to the students' lived binary operation object.

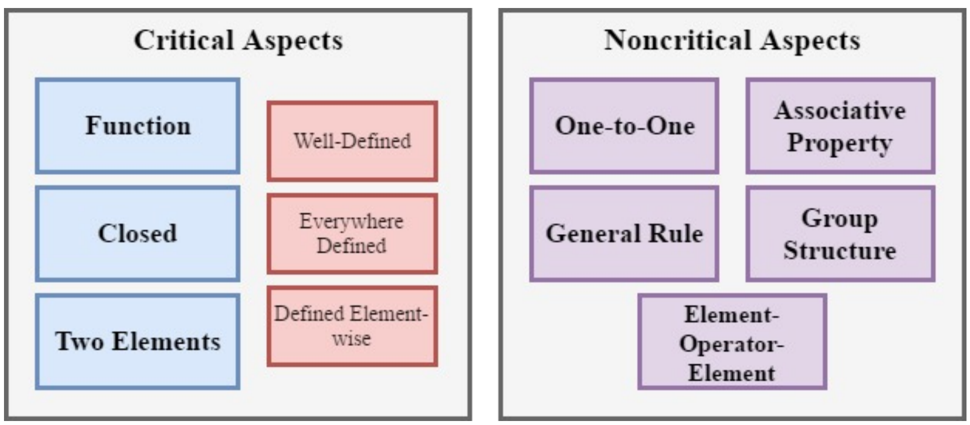

Figure 5. Critical and noncritical aspects of binary operation found in students' survey responses.

A Reflection on Variation as Attended to By Students

As a final point of discussion, we connect the available student reasoning, student interview responses, and the intended curriculum. During the four interviews, the students reflected on their experiences in class indicating that these tasks were atypical. Furthermore, they shared the binary operations they knew which covered a set of standard examples: arithmetic operations, modular arithmetic, and function composition. These exemplars are unsurprising as they are leveraged heavily in abstract algebra curriculums (Melhuish, 2015). One student explained:

> The things that we were shown in modern algebra were pretty basic examples of what a binary operation does. It almost seems like we are going to explore this to the extent that helps us in other topics and just move on to more interesting topics.

In the case of each of these standard examples, most of the noncritical aspects (such as E-O-E, general rules) exist. In this sense, it is not surprising that students may not be delineating these aspects as non-essential. In both their class and their textbook (Gallian, 2012), the only type of non-examples addressed have two elements, and but lacked closure. These non-examples did not allow for contrast between unary and binary operations. The types of variation students are exposed to may prevent them from becoming focally aware of other aspects that can cause a given relation to not meet the requirements of binary operation (such as the need for two elements) or overgeneralize noncritical aspects (such as E-O-E format).

All of the interviewed students also reflected that majority of their binary operation exposure was in the group context, i.e. one of the “more interesting topics” referenced above. In this sense, most binary operations examples are well-behaved, frequently with a general named rule, and often embodying additional properties such as the associative property. One student articulated the struggle to make sense of the relationship between binary operations and some properties explaining, “I know when it was introduced to me commutative and associative were very important. I didn’t really understand the connection between them.” He went on to share his uncertainty about whether they were required or optional properties of binary operation. Another student explicitly noted a noncritical aspect as always being part of the treatment of binary operation in her class. She explained that in their notes, binary operations always looked of the E-O-E form with the usage of “something like a star and circle” whenever discussing or introducing binary operations. This consistent feature was part of her lived object of learning, and was retained through her enacted object when approaching various binary operation-related tasks.

Two of the students, unprompted, expressed direct concerns about understanding what constituted a binary operation. One student reflected, “I don’t know what is and is not the fine details of [binary operation.]” She went on to explain that she felt they were supposed to apprehend the meaning of binary operation from examples, but was unsure of the point, struggling to sort critical aspects and noncritical aspects (our language). Furthermore, she explained, “You tend to focus on one little thing that stands out and you miss another one.” Her discussed metacognition on her learning process reflects the underlying premise of learning in variation theory: becoming aware of focal features through contrast and comparison.

Conclusion

By addressing meaning and critical aspects of student conceptions of binary operation, we were able to illustrate some of the many complexities associated with a topic that is often considered straightforward. Students reasoning reflected different meanings, and leveraged different aspects of binary operation depending on task and student. The aspects and meanings used varied in a ways illustrating incoherence in students’ concept images.

The results complement earlier work looking at action, process, and object conceptions of binary operation. As illustrated in Zandieh’s (2000) work with derivatives, these conceptual levels do not necessarily map in a hierarchical way. In our study we found reasoning at the structural level often treated binary operations at the object or process level without attention to the element (or action level). However, without de-encapsulating, the responses to the tasks tended have mathematical inaccuracies. In terms of the structure sense approach to binary operation, many of the tasks corresponded to levels of recognizing familiar and unfamiliar operations, and working with properties. The levels of recognizing familiar and unfamiliar binary operations broadly addresses a complete conceptualization when many students may have partial conceptualizations with some but not all critical components needed to address whether an instantiation is a binary operation. For example, students could attend to unfamiliar binary operations in terms of closure, but could not rule out non-examples that were unary and not binary. They could engage with tasks that presented unfamiliar binary operations when the form was E-O-E, but not always with differing presentations. We see our results as providing some nuance to more hierarchical frameworks by exploring a range of properties, representations, and example-types.

In order to situate these results in the larger sphere of variation, we may want to reflect on the examples and non-examples students are exposed to in their curriculum and classes. Studies have started looking at dimensions of variability available for students in abstract algebra classes in terms of groups and rings (Cook & Fukawa-Connelly, 2015; Fukawa-Connelly & Newton, 2014). Pairing such studies with student responses can provide a powerful impetus towards consciously varying examples and non-examples to best bring attention to structural attributes. In terms of binary operation, majority of textbooks focus explicitly on certain features such as the critical aspect of closure, and the permissible variations such as the associative property commutative property. If students are never exposed to instantiations that are unary or ternary, they may not assimilate particular critical aspects of binary operation. Without an awareness of these features, students may have a weakened foundation for making sense of group theory where binary operation plays a universal and critical role.

Overall, students appeared to draw on new learnings and variations, but also prior knowledge of functions and operations when approaching these prompts. Understanding binary operation requires a coordination of a new abstract notion of binary operation with the previous understanding of functions and familiar operations. The use of a meaning and critical aspects lens appeared to be a fruitful way to dissect student reasoning, concept image, and more particularly enacted objects of learning. Students provided a number of surprising responses such as seeing multiplication and division as the same operation or identifying a function like cuberoot as a binary operation. By dissecting various critical aspects, it became clear that even after the completion of an abstract algebra class, student concept images around binary operation may not be robust. This directly challenges the expert assertions that binary operation is a trivial topic (Melhuish & Fasteen, 2016). Furthermore, in our 2016 paper, we conjectured that students may have some underlying conceptual issues with binary operation that may account for performance on tasks aimed to explore their understanding of various topics such as subgroup and the associative property. This study confirmed that incomplete conceptions of binary operation not only exist amongst this population, but were prevalent across our sample. Additionally, the student responses in our small in-depth study were consistent with the responses in the larger, representative study. While, generalizability of these results is limited because of our sample size, we garner some strength of validity based on points of intersection with the prior study (e.g. Melhuish, 2105; Melhuish & Fasteen, 2016). As instructors and researchers in advanced mathematics, we caution overlooking binary operation conceptions and the essential role they play in any number of advanced classes.

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Opportunities to Learn from Teaching:
A Case Study of Two Graduate Teaching Assistants

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Pass rates in US first-year undergraduate mathematics courses are abysmally low. However, recent studies have found that there are doctoral programs that have improved pass rates by focusing on improving instruction. In particular, these exemplary programs focus on interpreting student thinking and that this instructional shift creates a more positive experience for undergraduate students. To scale up this result, it is important to understand how instructors connect student thinking to teaching actions. The purpose of this case study is to examine how two graduate student teaching assistants developed the ability to connect student work to hypotheses about student thinking and then link these hypotheses to teaching actions. For this analysis, the researchers introduce a framework that has potential to help providers of professional development identify how to strengthen graduate students' opportunities to learn from teaching, as well identify variations in graduate students' use of undergraduate student thinking to inform teaching actions.

Key words: Post-Secondary Professional Development, Graduate Teaching Assistants, Student Thinking, First-Year Course Instruction

Future faculty members often have their first teaching experiences when they are graduate students. Assigning graduate teaching assistants (GTAs) to their first course as primary instructor – meaning, the GTA is the person most responsible for teaching the material – is something that mathematics departments should not take lightly. Many universities entrust mathematics departments with the responsibility of teaching a substantial number of freshman students and, in many cases, the first college mathematics course that undergraduate students take is taught by a graduate student. As the economy depends more and more on science, technology, engineering, and mathematics, mathematics departments must recognize that the courses they teach can be seen as 'gatekeepers' to economic opportunity (Kamii, 1990; Moses & Cobb, 2002). With this recognition, mathematics departments must acknowledge that GTA instruction is critically important at not just the departmental or university level, but also the national level. In light of this responsibility, one of the primary goals of mathematics departments should be to develop graduate students to provide high quality instruction. To meet this goal, an increasing number of mathematics departments are providing GTAs with professional development (Border, Speer, & Murphy, 2009; Deshler, Hauk, Speer, 2015; Ellis, 2015). For this reason, it is necessary and important to study not only *how* to provide professional development to graduate students, but also the *impact* that professional development has on the development of GTAs as teachers. In this case study, the researchers examine how two GTAs leveraged their professional development experiences to use observations about undergraduate student work to inform teaching actions.

Previous Literature on Professional Development

While the researchers acknowledge that there are important differences between teaching at the K-12 and post-secondary level, there is also much that can be learned from K-12 professional development (Deshler, Hauk, & Speer, 2015). Following this suggestion, this work draws on studies of effective characteristics of professional development for post-secondary and K-12 teaching.

Opportunities to Learn From Reflection on Student Thinking and Teaching

In 2009, Blank and de las Alas conducted an extensive meta-analysis of research reports between 1986 and 2007 to determine the key characteristics of effective K-12 professional development. They found that the greatest improvements in teaching could be attributed to the teacher experiencing intensive initial professional development in combination with multiple opportunities for sustained engagement, such as plan-teach-reflect cycles and reviewing student work. At the post-secondary level, similar results have been found to hold, especially when professional development opportunities allow GTAs to attend to student thinking (e.g., Kung & Speer, 2009).

It is widely accepted that teachers can and should improve their teaching by reflecting on instruction and student thinking. In addition, reflection should occur within communities of teaching. Multiple national policy documents for K-12 teacher education cite such reflection as a critical component for improvement (e.g., InTASC, 2011; NSDC, 2015). Also, reflection certainly can be a key contributor to teachers' ability to learn from their teaching experiences. However, simply providing time and a mandate for teachers to reflect on their instruction is no guarantee of improved instruction. For instance, teachers who catalogue good and bad moments without hypothesizing about factors that contribute to the quality of these moments are unlikely to increase the quality of their teaching. Also, teachers who characterize their students' conceptions without attending to how instruction shapes students' conceptions are unlikely to leverage their knowledge of student thinking. As Horn and colleagues have argued, it is not merely the presence of reflection in communities that is important, but rather the opportunities to learn that teachers create from these reflections (Hall & Horn, 2012; Horn, 2005; Horn & Kane, 2015; Horn, Kane, & Wilson, 2015).

The critical insight that the researchers gleaned from Horn and colleagues' work is the following theory: *in teachers' reflections, the more tightly linked the observations of instruction and students are to future teaching actions, the greater the opportunity to learn from reflection*. In other words, it is not sufficient to just make observations, however astute, about previously experienced instruction and student thinking. It is also not sufficient to just make claims about what one might do in future teaching actions. Rather, the greatest opportunities to learn from reflection occur when observations about previously experienced instruction and student thinking inform future teaching actions. If the model from K-12 holds, then one would expect that GTAs' opportunities to learn will also be more meaningful when their reflections link observations about instruction and student thinking to future teaching actions.

Characterizing Opportunities to Learn From Reflection

Several studies have examined variation in teachers' opportunities to learn from reflection on teaching. Using an analysis of 17 hours of secondary teachers' workgroup conversations across an entire year, Horn and Kane (2015) found that not only were opportunities to learn not equally distributed, they also represented a developmental story of "accumulated advantage" (p. 8). Where there were more opportunities to learn in the beginning, there were even more opportunities to learn later. In a comparison of two middle school teachers' workgroups, Horn, Kane, and Wilson (2015) found that how teachers interpret data about students impacts their opportunities to learn from reflection on teaching and student thinking. One group focused on increasing proficiency rates by determining how to allocate resources to particular groups of students, and thus did not focus on students' understandings. The other group attended to students' understandings but was limited by pressure to go over as many of the instances where their students might improve, rather than to discuss any instructional implication deeply.

Horn and colleagues' work on opportunities to learn contributes a frame for characterizing variation in opportunities to learn from reflection on teaching and student thinking. Namely, their work suggests that it may be beneficial to examine how well teachers are able to interpret data about student thinking in order to inform their future teaching.

Lai, Smith, Wakefield, Miller, St. Goar, Groothius, & Wells (2016) used this frame to examine GTAs' opportunities to learn from reflection on teaching and student thinking. To compare clarity of reflection, the authors modeled GTAs' reflections as arguments about future teaching actions based on data about student thinking. The clarity was conceptualized as how specific and connected the logical argument was when connecting data to future teaching action. These authors used a modified version of Toulmin's (1958) model to analyze the connections GTAs made between what they observed their students doing and their future plans for teaching. Lai et al. performed a qualitative analysis of 16 final papers written by mathematics GTAs and developed a coding scheme that categorized papers as low connectivity, medium connectivity, high connectivity with low coherence, and high connectivity. The authors concluded that even when GTAs are teaching the same course, participating in the same professional development, and completing the same task, the clarity of their reflections on the nature of student thinking varies widely.

However, no attempt was made to look at the growth that the GTAs experienced. For example, while the authors found examples of GTAs with high connectivity in their final papers, there was no indication of what growth may have occurred over the course of the professional development program. An important question that resulted from this is whether these cases of high connectivity are representative cases of individuals who entered the professional development program with these skills.

Study Description

The purpose of this intrinsic case study (Stake, 1995) was to better understand the growth experienced by two mathematics GTAs over the course of their involvement in a professional development seminar. During the seminar, GTAs collectively reflected on their teaching each week, sometimes in the context of informal discussions on readings about learning and teaching and other times in more formal writing assignments. These two GTAs selected for this case study were identified because their mathematics department had recognized them as strong teachers and pedagogical leaders. In this paper, GTA growth is conceptualized as enhanced opportunities to learn from teaching through reflection on instruction and student thinking, over time, as described in Lai et al. (2016). The central question that is guiding our inquiry is: *How did GTAs grow as learners of teaching over the course of involvement in a professional development seminar?* The central question is decomposed as follows:

1. How did the GTAs' capacity to connect student thinking to future teaching actions in their formal reflections change over time?
2. How did GTAs' capacity to use resources from the professional development seminar in their formal reflections change over time?

Using similar analytic methods as what Lai et al. (2016) used to analyze a set of papers collected from GTAs at the end of a professional development seminar, the researchers conducted this longitudinal case study to examine the growth of the GTAs over the course of the seminar. Although the context of this case study makes it hard to generalize, this analysis does help identify what GTA growth looks like and provide one possible frame from which providers of professional development can evaluate their programs.

Theoretical Frameworks

In this paper, GTA growth (as learners of teaching) is viewed as the enhanced opportunity to learn from reflection about instruction and student thinking, where the strength of an opportunity to learn is conceptualized as the connectedness from observations of student thinking and prior instruction to proposals for future teaching actions. The researchers model teachers' reflections as an argument about how observations of student thinking and instruction inform future teaching actions. They also examine the connections between student data, student thinking, hypothesis, and future teaching actions. These four components and the logical connections between them constitute our model, which is depicted in Figure 1. Our model is an adaptation of Toulmin's (1958) argumentation model, which he developed for analyzing the grounds, claims, and warrants used in legal arguments. In Toulmin's model, grounds are the evidence underlying a claim and a warrant justifies the relationship between the grounds and the claim. Recently, Toulmin's argumentation model has been applied in other fields, such as mathematics education (Inglis et al., 2007).

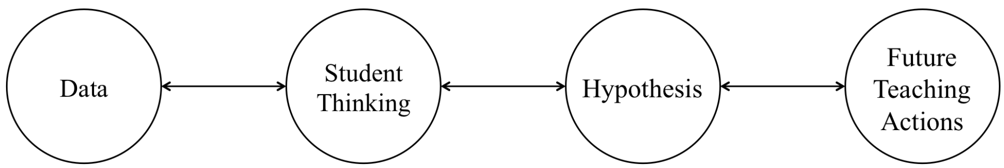
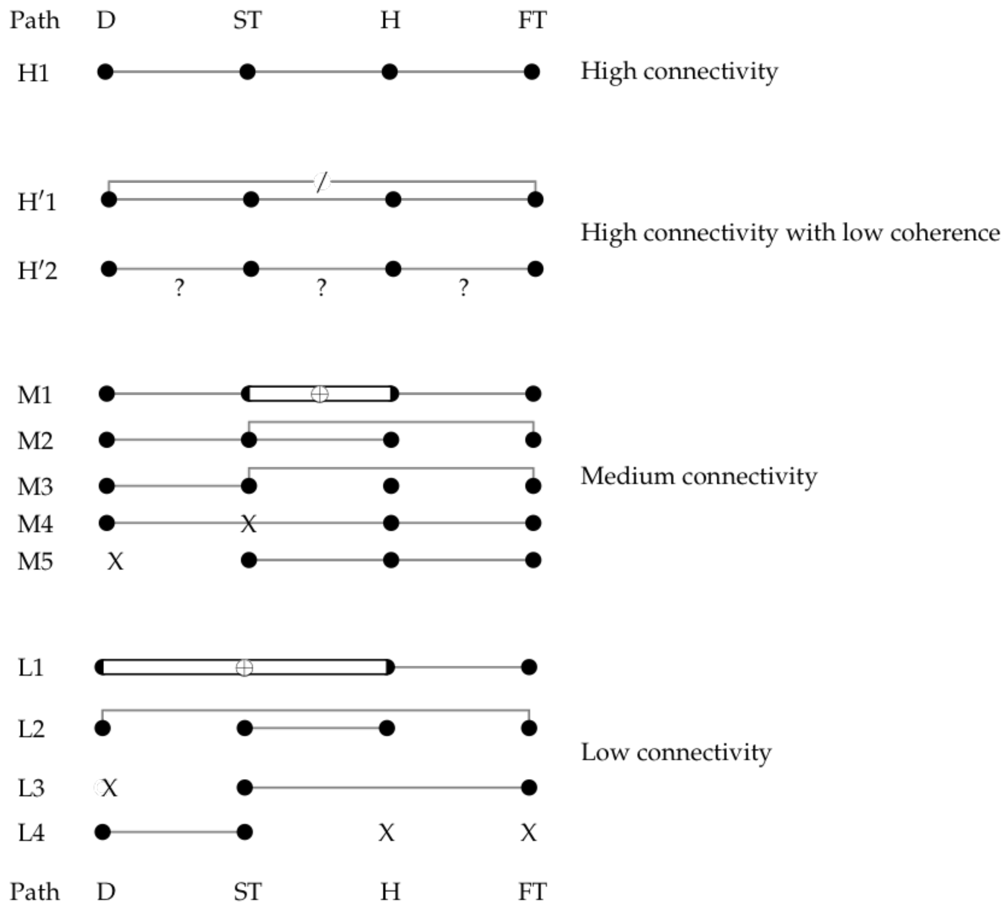


Figure 1. Framework for GTA arguments.

Following Lai et al. (2016), *data* is defined to be the written student work collected by the instructor and any memory-recalled communication observed by the instructor and recorded in their writing. In their argument, the instructor may interpret this data in the form of *student thinking.* That is, student thinking is the instructor's expression of how he or she believes the data should be interpreted in order to reflect the student's work. Also, instructors may make a *hypothesis* about the likely reasons for a student to think in the way in which the instructor has interpreted. This hypothesis depends upon the underlying origin of student thinking. Finally, an instructor may plan a *future teaching action* based upon any or all of the previous elements. These four elements, and the connections between them, form the framework the researchers used to analyze GTA work.

Using this framework, a GTA's work can be characterized by the presence or absence of any of the four elements and the connections between them. When looking at connections, coding captures not just the presence or absence of a connection but also the plausibility of a connection. Plausibility refers to the strength of a connection. For instance, if the instructor interprets the student thinking in a way that is not supported by the data, this would be coded as an implausible or weak connection. Using this coding scheme, a GTA's work is characterized as having *low connectivity* if explicit references to two or more of the four elements were missing, or the connections between these elements are not present or plausible. On the other hand, a GTA's work is characterized as having *high connectivity* if the GTA clearly articulates all four elements and makes explicit and plausible links between these elements. The tag *low coherence* is added when the four elements are articulated, but weak, implausible, or implicit links are present. A final category of *medium connectivity* is used when at least three of the four elements are present and most, but not all, of the links are explicit and plausible. A visual representation as well as an explanatory key of these various types of connectivity can be found in Figure 2.



Key: D = data, ST = student thinking, H = hypothesis, FT = future teaching actions, X = component absent, ⊕ = conflated components, •—•= link was attempted and satisfied criteria, $\bullet\overset{?}{—}\bullet$= no consensus from research team on whether link is plausible, •-/-•= future teaching actions do not plausibly address student thinking identified in data.

Figure 2. Categories of connectivity of GTA arguments.

Data and Methods

Using the framework described, the researchers examined the growth of two particularly strong GTAs by studying the written work they submitted while enrolled in a pedagogy professional development seminar course taught in a mathematics department during the fall 2015 semester.

Data Collection

Context. At the university in the study, every graduate student who is assigned to be the primary instructor of a course for their first time is required to enroll in a seminar on teaching and learning mathematics at the post-secondary level. A graduate student is a primary instructor if they have independent authority to make classroom decisions and assign grades.

In the past few years, approximately 15 students have enrolled in this class each year. This seminar meets for two hours a week in the fall semester and one hour a week in the spring semester. The graduate students in this course read educational literature, including articles ranging from foundational pieces, such as Erlwanger's (1973) discussion of Benny, to expository articles, such as Tsay and Hauk's (2013) explanation of constructivism.

While enrolled in the seminar, the graduate students are assigned to teach either intermediate algebra or college algebra, which have a typical enrollment of 34 and 40 students respectively. Small group discussions are a focus of these courses and the GTAs involved in the seminar make up the majority of the instructional team for these courses. While there are a few adjunct instructors who occasionally teach the courses, they do not participate in the seminar. The faculty member who leads the seminar may also teach one of these courses, including the term in which the data were collected for this report.

Since the graduate students enrolled in the seminar are all teaching one of two courses, the seminar provides an opportunity to discuss shared challenges and experiences. The shared experience of teaching common courses and reading common articles provides the backbone of weekly discussions for the seminar. In these discussions, the graduate students are encouraged to reflect on their own teaching utilizing the vocabulary of mathematics education.

Data. The data for this study were collected in the fall of 2015 and includes three assignments completed by the GTAs throughout the fall semester. The first assignment asked GTAs to work in teams of three and analyze student work on a quiz in order to identify an area of unproductive student thinking. In the group paper, the GTAs described the unproductive student thinking they had identified and planned an intervention for use in office hours or class to help students. Finally, the assignment called for a one-page summary explaining why the team believed the intervention would be successful. The second assignment had the same requirements, but this time the GTAs had to complete the assignment individually and analyze a different set of student work. The third assignment, which functioned as the final project for the course, followed a similar format. However, in this final assignment, the GTAs were explicitly asked, as opposed to implicitly asked, to theorize about the roots of the unproductive student thinking.

Paper Summaries

Susan. In the first group assignment, Susan analyzed a quiz question that asked students to explain how they know that a given table of data is exponential. Of the six samples of student work Susan included, three samples claimed that the data was exponential because the rate of change was not constant. Susan used the data to argue that students were thinking unproductively about exponential functions as nonlinear. As Susan reflected on possible roots of this unproductive student thinking, she referenced two hypotheses: (1) "in the semester we had spent most of our time on linear and exponential functions so they were fresh in our students' mind" and (2) "in teaching exponential functions, we emphasized the fact that they are not linear and often compared them to linear models" (Page 2). Finally, the intervention that Susan's group planned involved asking students to analyze four different functions in an attempt to demonstrate the plurality of families of functions.

In her second paper, Susan analyzed a quiz problem where students were given the graph of a piecewise-linear function and asked to graph the function $-1/2g(x)+2$. Susan analyzed two samples of student work that had identical incorrect graphs, but different steps written down to describe how they got their graphs. One student listed steps that were not only incorrect (given the formula), but also did not match the graph they drew. The other student listed steps that were correct (given the formula), but also did not match the graph they drew. The unproductive student thinking that Susan identified was that the students did not recognize the connection between the graphical and analytical representations of transformations. She then hypothesized that the root of this unproductive thinking was that the students had survived on merely a mechanical understanding of the analytical steps to transform a function. Susan intervention was designed to help students discover a step-by-step process for transforming graphs in order to tie their analytical understanding to the graphical representation. The intervention directed students to identify "important" points on a piecewise-linear graph (i.e., corners and endpoints of the piecewise function), list the analytical steps involved in the transformation, transform the "important" points at each intermediate analytical step, and then use the transformed points to sketch the transformed graph.

For the final assignment, Susan analyzed another transformation problem, but this time it came from an exam. In the problem, students were given the domain and range of $f(x)$ and asked to find the domain and range of three different transformations of $f(x)$. In analyzing the student work, Susan found that 53% of her students "used the wrong operation, the wrong order of operations, or some combination of the two" with the third transformation, which was $f(2(x+3))$. Drawing upon this data, Susan identified that students had unproductive thinking when trying to determine what order to apply the multiple horizontal transformations. Her hypothesis about the roots of this unproductive thinking was that students had attempted to memorize the order of horizontal transformations as a rule and did not understand why it worked. This caused problems when they attempted to apply the rule, because it contradicts their intuitive desire to follow the order of operations. To help students move towards a more productive way of thinking about the order of horizontal transformations, Susan developed a worksheet. The worksheet started by connecting transformations to function compositions, considered the transformation of a single point, asked the students to hypothesize about the order, and ended with a graphical application of transformations.

Karen. For the first group assignment, Karen analyzed a team quiz problem that asked students to explain how many answers there were to a system of two linear equations. Karen analyzed two sets of student work. One had the correct answer (no solutions), but an incorrect explanation, while the other one provided an incorrect answer (exactly one) and an incorrect explanation. Karen identified that both students had unproductive thinking in the explanations they gave to support their answers. In particular, the first one claimed that if all of the variables cancel, then there are no solutions. On the other hand, the other one claimed that if the lines are not the same, then they must have exactly one solution. Karen hypothesized that "if either of these groups had a better understanding of the geometric interpretation of a linear system, then they would have been able to provide better explanations as well as come to the right conclusion" (p. 21). In order to address the underlying root of the unproductive thinking, Karen designed an intervention that starts with a linear system that has infinitely many solutions. She would let the students solve it their own way, with the hope that they would come to the erroneous conclusion that there are no solutions. To show them their error, they would graph the two equations. Finally, they would consider a system of equations that has no solutions and consider its graphical representation.

For the second assignment, Karen analyzed a quiz problem on function notation. Overall, students did well on this problem, but there was one common mistake that was made. Given the function $f(x)=x^2+1$, several students were able to correctly find $f(2)$ and $2f(x)$, but incorrectly claimed that $f(2x)=2x^2+1$. Karen identified that the students had unproductive thinking when working with function compositions. Karen hypothesized that due to the fact that function compositions had previously only been done with linear equations, students had erroneously generalized properties that are only true for linear function compositions. In order to address the root of the unproductive student thinking, Karen's intervention focused on building connections between concrete and abstract algebraic operations. The intervention starts with some simple computations of $f(2)$ and $f(4)$. Then, students used their own method to find an equation for $f(2x)$. To test whether or not they got the right equation, the intervention directed them to substitute $x=1$ and $x=2$ into their new equation and see if that matches what they got for $f(2)$ and $f(4)$. Finally, the intervention ended with a discussion of how to find the correct formula for $f(2x)$.

In Karen's final paper she analyzed a problem from a quiz on algebraic fractions. Karen focused on one type of cancellation mistake that was commonly made by her students. When attempting to simplify a rational expression, Karen found that many students followed the unproductive thinking that you can cancel out factors that are common to only the denominator and one of the addends in the numerator. At the root of this unproductive thinking, Karen hypothesized that students think in terms of rules instead of concepts. In particular, they are not thinking about the mathematical idea of fraction, multiplication, and division when simplifying rational expressions. In order to help students form a more productive way of thinking, Karen developed an intervention that began by exploring the arithmetic involved in simplifying fractions. They begin by exploring two fractions, one where the denominator shares a common factor with only one of the addends in the numerator, and the other where there is a common factor in the denominator and entire numerator. Then, Karen used this familiar context to explain why cancellation doesn't work in the original rational expression.

Analysis

For each of the two GTAs in this study all three papers were collected and coded following the coding scheme developed by Lai et al. (2016). GTA names were redacted to protect the privacy of individual GTAs. Coding was completed independently by two of the researchers who then met to reconcile codes. Upon analyzing the team project submitted by one of the students, one of the researchers became aware that she had been a member of the team who had submitted that paper. To overcome any potential bias caused by having been part of the group project, another researcher assisted with triangulation for this assignment. The researchers then compared notes and reconciled differences. It is worth nothing that this particular group paper provided the research team a unique opportunity to compare codes with what the writers had intended to communicate. Differences between the intention of the paper and the coder's interpretation of the paper were taken into consideration and having a researcher reflect on her own work help strengthen triangulation.

After initially coding using Lai et al. (2016) coding scheme, the researchers noticed that there seemed to be important details that the coding scheme failed to capture. In particular, the research team all agreed that Susan had grown in ways that were not captured by the initial coding. In order to address this unexpected finding, the research team decided that a second phase of analysis was needed. To accomplish this, the research team reread all the assignments and focus on how claims were made and supported. To do this, the research team identified which articles from the seminar the GTA had used to support their arguments and how these articles were used. The reason why the researchers chose to look at the references used was driven by the research questions. While our initial analysis addressed the first research question, the researchers realized that the second analysis was needed to address the second research question.

Findings

Over the course of the professional development seminar, Karen's arguments changed from being characterized as having medium connectivity to high connectivity. Susan's arguments, however, were consistently characterized as having medium connectivity (Figure 3).

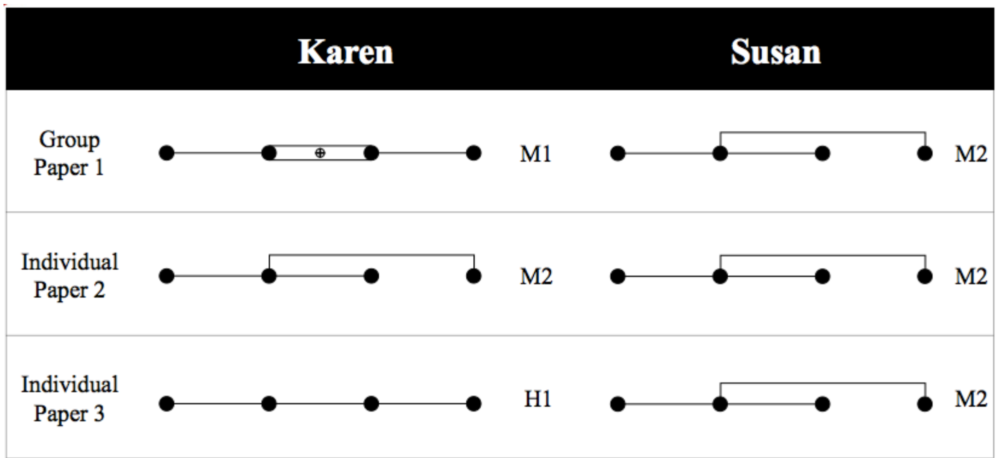


Figure 3. Classification of connectivity of Karen & Susan's arguments.

In Karen's first paper, there were strong links between the data, problem, hypothesis, and intervention. However, it was somewhat difficult to tease the problem and hypothesis apart. At first, the researchers thought that the hypothesis was missing. However, after further examination, they were able to redefine the problem in a way that disentangled it from the hypothesis. However, since it was difficult to separate the two, they felt that Karen did not make a strong case for each one individually, and therefore unintentionally conflated them together. In the second paper, Karen made strong links between the data, problem, and hypothesis. However, the link between the hypothesis and the intervention was weakly supported. Finally, Karen was able to make strong links between all four components in her final paper. In each of Susan's papers, there were strong links between the data, problem, and hypothesis, but the intervention seemed linked more to the problem than to the hypothesis.

When considering the ways claims were made and supported, a different picture is formed. In her first group paper, Susan's group did not use literature to support her claims, but instead referred to things like "well-known-facts" without supporting evidence. However, over the course of the term, Susan began to incorporate existing research into her discussions of students (Figure 4). Karen, on the other hand, did not begin directly citing existing research until the third paper. However, in her final paper, Karen synthesized multiple points of literature to support her arguments (Figure 5). It is important to note that on the rubric for the third paper, there is an item that does not appear on the previous rubrics: "Paper theorizes about the roots of the misconception drawing on literature to support the claim." The previous assignments, on the other hand, stated no requirements for citing literature. This could explain Karen's choice to use begin citing literature in her final paper. However, Karen chose to use existing literature not just for the hypothesis but also for the intervention. Furthermore, Karen did try to anchor her previous interventions in the language of constructivism. The exact effect that the change in the rubric had is unknown.

References to Seminar Papers		
Group Paper 1	Individual Paper 2	Individual Paper 3
Nothing of Interest	"As Lage and Gaisman observe in `An Analysis of Students' Ideas about Transformations of Functions,' looking at students' understanding of function transformations provides an opportunity to analyze students' ideas on functions and variables."	• "That is, in the language of constructivism, this topic will introduce a disequilibrium for the student that is not easily assimilated or accommodated." • "Not only is this a generally destructive attitude to have toward math, but the reinforcement of an external locus of control may cause the student to slip into a `Benny'-like reaction: believing that mathematics is simply a set of arbitrary rules, some of which might contradict each other." • "As one mathematically competent student in Hauk's `Mathematical Autobiography Among College Learners in the United States,' reflects, `I always try to find the answer right away and if I can't find it, I say those three words, `I hate Math.''"

Figure 4. Susan's use of literature in her arguments.

Another factor that may have contributed to the increased use of literature by both Susan and Karen is the increase in the literature available to both GTAs. The professional development seminar can be roughly broken into four themes throughout the semester: setting the stage, discussing classroom dynamics, discussing assessment, and looking at who our students are. Within each section a series of articles are used to motivate discussion. As seen in Tables 1 and 2, Susan and Karen both incorporated literature from the professional development seminar into their arguments. Moreover, Susan and Karen referenced literature that spanned multiple themes, synthesizing ideas from the literature. Figure 6 provides a temporal bipartite graph showing how literature appeared in Susan and Karen's writing.

Discussion

By taking a longitudinal approach, the researchers were able to examine how GTAs grow over the course of the professional development seminar. Using our original coding scheme, changes in the GTAs' ability to clearly reflect on the nature of student thinking were characterized. However, the researchers found that this analysis did not capture the ways in which Susan had experienced growth or provide us with a clear answer to our second research question. To address that issue, the researchers conducted a second analysis that focused on GTAs' capacity to use resources from the professional development seminar in their formal reflections by looking at how the GTAs' claims were made and supported. Based upon the analysis done in the previous study that looked at all of the final papers (Lai et al., 2016), the need for a second type of analysis was not anticipated. However, using our coding scheme as a way to analyze longitudinal data brought some of its weaknesses to the forefront.

References to Seminar Papers		
Group Paper 1	Individual Paper 2	Individual Paper 3
"Learning, in the constructivist scheme, occurs when a student is put in a state of disequilibrium and must assimilate new information, reject it, or accommodate their mental constructions to the new data. Our intervention aims to spark disequilibrium in our students so that they must reevaluate and reform their conception of solutions to systems of linear equations."	"This understanding is built by provoking disequilibrium, in the constructivist terminology.... This is the moment of disequilibrium that forces them to construct a new understanding of the relevant concepts"	• "In this way, they are similar to "Benny" in Erlwanger's study, since they think in terms of rules divorced from concepts. We shall see more commonalities with Benny shortly" • "Erlwanger identified a "behaviorist" approach to mathematics instruction as one of the root causes of Benny's misconceptions, and I believe that a similar issue can be seen here…. While the average high school math class may not be as explicitly behaviorist as Benny's "IPI," teachers nonetheless focus (unwittingly, perhaps) on obtaining answers, at the expense of developing understanding." • "I'm reminded of another article we read, about Brazilian schoolchildren who easily solved math problems when selling produce on the streets, but were unable to solve the exact same problems when presented with them in a classroom setting. Using students' existing knowledge an[d] mathematical intuition as a base off of which to build the theory would promote deeper understanding, I think."

Figure 5. Karen's use of literature in her arguments.

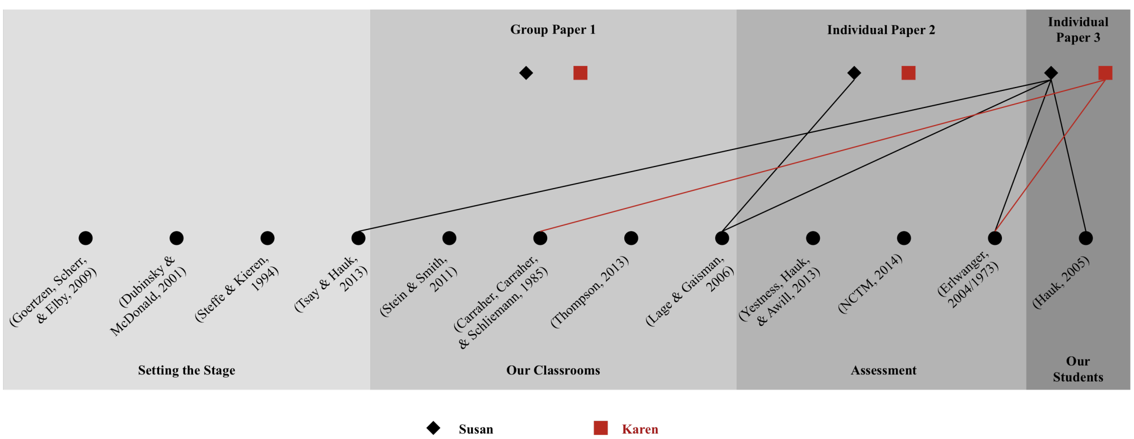


Figure 6. Temporal bipartite graph of Susan and Karen's use of literature.

While the original framework developed may not be able to capture every aspect of GTA growth, one strength that it does have is its adaptability. In particular, the researchers believe that our framework could be used as a way to analyze GTA growth in a variety of contexts. For example, there are several departments who integrate video case studies into their professional development seminars as a way to provide GTAs with the opportunity to reflect on and respond to classroom situations. Our framework would fit nicely with this type of activity, where the data is the video case study, the student thinking and hypothesis are the GTAs reflection on the video, and the future teaching actions are the GTAs plan for what they would do next. Another adaptation the researchers have considered is using this framework to analyze GTAs ability to reflect and plan future teaching actions when looking at student thinking in a way that does not focus on a deficit model. For example, GTAs could analyze student data in an attempt to identify what the students do understand, hypothesize what contributed to the development of that understanding or how they could build upon that understanding, and plan future teaching actions. In this case, the researchers believe that the hypotheses would be intrinsically connected to the future teaching actions, which would strengthen the connectedness of the GTAs arguments.

In this paper, the researchers have attempted to better understand the effects that this particular professional development program has on GTAs. Analyzing the two particular GTAs chosen has provided some insight into how GTAs might grow over the course of their involvement in a pedagogy professional development seminar course. Both GTAs did exhibit growth, but in different ways. Karen's arguments evolved to show more connectivity between data, student thinking, hypothesis, and future teaching action. On the other hand, Susan began to utilize literature from the seminar in her reflections on student thinking. However, both GTAs adopted a more constructivist approach to teaching as evidenced through their arguments. A stated goal of this particular professional development program is to support GTAs as they become evidenced-based practitioners of mathematics education. For these two GTAs, this goal appears to have been met.

Implications and Directions for Future Research

This intrinsic case study provides evidence that the professional development program discussed may have some effect on the growth experienced by two mathematics GTAs over the course of their involvement in a professional development seminar. This suggests that mathematics Ph.D. students can benefit from reading mathematics education literature, analyzing student work, and planning future teaching actions. Future research might look at similar activities across a larger sample and, in particular, analyze how GTAs who might not be identified as strong teachers grow. Secondly, it would be interesting to rework the three papers assigned in the professional development seminar to remove the apparent deficit view of student thinking and then conduct similar research from a strength-based perspective. The growth of graduate student professional development in the mathematics community is exciting and should provide a rich ground for mathematics education researchers.

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Learning to Notice and Use Student Thinking in Undergraduate Mathematics Courses

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This study evaluated the outcomes of an intervention focused on developing mathematics graduate teaching assistants' (GTAs') skills of noticing and effectively responding to instances of student mathematical thinking that have significant potential to further students' learning. Four GTAs participated in a semester-long intervention that included individual analysis and group discussion of video of undergraduate mathematics lessons. The MOST Analytic Framework (Stockero, Peterson, Leatham, & Van Zoest, 2014) was introduced to aid in these activities. The GTAs also completed a pre- and post-interview to document their real time noticing and an assessment of common content knowledge. Results indicate that the intervention was successful in improving the GTAs' noticing skills in a variety of ways and in their ability to propose student-centered responses.

Key words: Graduate Teaching Assistant Training, Teacher Noticing

Research has shown that student-centered instruction leads to more effective learning for people of all ages (National Research Council [NRC], 2005). Higher education has been slow or unsuccessful in implementing student-centered instruction (Barr & Tagg, 1995; Felder & Brent, 1996), however, with transmissive instruction (i.e., lecturing) still prominent (Ramsden, 2003; Svinicki & McKeachie, 2014). Challenges for adopting student-centered instruction include student resistance, instructor comfort level, and the time needed to see results (Felder & Brent, 1996; Seymour, 2002). Some researchers suggest promoting changes in higher education teaching methods through GTA training (e.g., Cano, Jones & Chism, 1991). Since effectiveness of GTAs typically affects undergraduate students in their early years of study, GTA training is also important in student retention (Cano et al., 1991; Speer, Gutmann, & Murphy, 2005). With a workforce shortage in science, technology, engineering, and mathematics (STEM) fields (President's Council of Advisors on Science and Technology, 2012), student retention is crucial in university STEM departments (Seymour & Hewitt, 1997; Suchman, 2014).

At the K-12 level, a teacher's ability to notice aspects of instruction as it unfolds has been recognized as important in the implementation of student-centered instruction. Many studies have found that *professional noticing of [students'] mathematical thinking*—defined to include attending to, interpreting, and deciding how to respond to students' strategies and understanding (Jacobs, Lamb, & Philipp, 2010)—can be learned and improved through teacher education (e.g., Jacobs et al., 2010; McDuffie et al., 2014; Sherin & van Es, 2009; Stockero, Rupnow, & Pascoe, 2015, 2017). Although teacher noticing interventions are not widely practiced in higher education, the gains made with K-12 mathematics teachers suggest that similar results may be possible.

Also foundational to effective mathematics teaching are the six domains of mathematical knowledge for teaching proposed by Ball, Thames, and Phelps (2008). Critical to this study is common content knowledge (CCK), the mathematical understanding and proficiency used in diverse contexts not exclusive to teaching. Without CCK, a teacher could not adequately guide students in building such knowledge. In addition, mathematics teachers would not likely be able to determine which instances of students' mathematical thinking are important to notice without a strong command of CCK. Because both noticing skills and CCK are important for effective mathematics teaching, it would be of interest to investigate if and how these factors are related.

This work examines the outcomes of a GTA training intervention focused on analyzing undergraduate mathematics lesson videos with a teacher noticing framework as a means to support GTAs' more effective noticing and use of student thinking, which has the potential to support their enactment of student-centered instruction in their classrooms. Of particular interest is measuring the effectiveness of the intervention in improving GTAs' noticing of high-potential opportunities to build on student thinking (Leatham, Peterson, Stockero, & Van Zoest, 2015) and in supporting their ability to propose student-centered responses to such instances. This work seeks to answer the following research questions: (a) How effective is the intervention in improving GTAs' noticing of MOSTs?; (b) How effective is the intervention in supporting the GTAs' ability to propose in-the-moment student-centered responses to instances they identified in video?; and (c) What is the relationship between the GTAs' CCK and their noticing skills?

Theoretical Framework

With the goal of improving GTAs' use of student mathematical thinking in undergraduate mathematics classrooms, this study used the MOST Analytic Framework (Leatham et al., 2015) to characterize instances of student mathematical thinking that are not only important to notice, but also the most fruitful to discuss in a lesson because of their potential to support students' mathematical learning. MOSTs—**M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking—are defined as "instances of student thinking that have considerable potential at a given moment to become the object of rich discussion about important mathematical ideas" (p. 90). To be a MOST, a moment must satisfy three characteristics: student mathematical thinking, mathematically significant, and pedagogical opportunity. To satisfy these characteristics, the student mathematics must be inferable and related to a mathematical point, the mathematical point must be appropriate to the learning level of the students and a central goal for student learning, the student mathematics must create an intellectual need for students to understand the mathematical point, and it must be the right time to address the intellectual need at that moment.

Methodology

Participants and Intervention

The participants were four mathematics GTAs from a Midwestern U.S. university. Three had completed their first year of graduate study and had one to two semesters of teaching experience, and one had completed two years of graduate study with six semesters of teaching experience. All four participants had previously completed training required by the mathematical sciences department: a one-week GTA orientation prior to their first semester of study, a course entitled Teaching College Mathematics in which the GTAs prepared and delivered lessons three times throughout a semester with feedback and support from peers and their instructor, and a six-week seminar during their first semester of teaching in which the GTAs read and met weekly with an instructor to discuss select chapters of *Teaching Tips* by Svinicki and McKeachie (2014). Their participation in the current study was voluntary.

The GTAs engaged in a ten-week professional development intervention facilitated by the first author in the fall 2015 semester. The goal of the intervention was to improve the GTAs' skills in attending to, interpreting, and responding to MOSTs in a student-centered manner. The design of this study was adapted from Stockero and colleagues' work with prospective secondary school mathematics teachers (Stockero, 2014; Stockero et al., 2015, 2017) focused on helping them learn to notice and respond to MOSTs that surface during a mathematics lesson. Both before and after the completion of the intervention, each GTA completed a one-on-one, video-recorded interview with the researcher/facilitator; the purpose of this interview was to capture their in-the-moment noticing. During the two interviews, the GTA watched the same short video clip from an undergraduate mathematics lesson that was recorded at the same university in which the GTAs were enrolled. The GTA was prompted by the researcher to stop the video if they thought a *mathematically important moment that the instructor should notice (MIM)* occurred. A definition of these moments was not given to the GTA to establish baseline data. When a GTA stopped the video, the researcher asked them to describe the moment they noticed, why they chose it, and what they might do if such a moment happened in their own classroom. Using Studiocode video analysis software (SportsTec, 1997-2016), the instances chosen by the GTAs were marked on a video timeline for later analysis.

In each week of the intervention, the GTAs and the researcher/facilitator individually analyzed a minimally edited video of a lesson from an undergraduate mathematics classroom that was recorded by the researcher at the university. This individual analysis was in preparation for a weekly group meeting held among the four GTAs and the facilitator to discuss the video collectively. One of the classroom videos was recorded in one of the GTAs' classrooms during a previous semester, but otherwise the videos were not of the GTAs' own classrooms.

In the first three weeks of the intervention, the GTAs used the Studiocode video analysis software to tag *MIMs* and add text to describe what they noticed and why they chose each instance (i.e., why it was important to notice). Again, the definition of such moments was left open-ended to establish baseline data. The researcher, as an experienced user of the MOST Analytic Framework, used the Studiocode software to tag and document MOSTs—the types of instances that were the goal for GTA noticing—in the same videos. The researcher reconciled any instances of uncertainty with one of two other researchers experienced in the use of the MOST framework. Before the group meeting with the GTAs, the researcher examined the participants' tagged video timelines and associated text and compared the instances chosen by the GTAs and the MOSTs identified by the researcher. Instances discussed at the group meeting were chosen with care by the researcher/facilitator, limiting the number of instances so that the meeting did not, on average, last more than one hour. Instances were selected for discussion for a variety of reasons. These included instances that one or more GTAs marked—both those that the researcher identified as MOSTs and that the researcher did not identify as MOSTs—as well as instances identified by the researcher as MOSTs that were not noticed by the GTAs.

In the group meetings, the facilitator pushed the GTAs to articulate how each moment fit the early prompt of *MIMs*; that is, what the instructor had to notice in each moment and why it was mathematically important. Through discussion with peers and guidance from the facilitator, the GTAs worked toward building a definition of *MIMs*. This early phase of the intervention was intended as an introduction to the teacher noticing construct. That is, it got the GTAs to start thinking about what might be important for an instructor to notice during a lesson and to create a need for a more formal language and criteria with which to describe such important instances (i.e., the need for a framework).

After three weeks of analyzing video and constructing a meaning for the early *MIM* prompt, the GTAs were introduced to the MOST Analytic Framework by being given a paper to read that defined MOSTs (Stockero et al., 2014). In the week that they read the paper, the GTAs reexamined two of the videos they had already analyzed and picked out instances that they believed met the characteristics of a MOST. The group meeting discussion of the reexamined videos then revolved around whether instances fit all six criteria of a MOST.

In the remaining six weeks of the intervention, the GTAs were prompted to tag MOSTs in the new classroom videos that they analyzed and to describe in text how each instance satisfied all six MOST criteria. The GTAs were provided a text template to prompt them to specifically address each MOST criterion in their written responses in the last five weeks of the intervention. In these weeks, the facilitator similarly chose with care which instances to discuss at the weekly group meetings, with the important moments now having a clear definition. The group meetings revolved around how each instance that was discussed fit or did not fit the definition of a MOST.

During the group meeting discussion of the eighth of a total of nine videos, the facilitator prompted the GTAs to propose *building* moves in response to the MOSTs that were discussed—a teacher move that engages students in collaboratively discussing the significant student mathematical thinking that is present in the classroom (Stockero et al., 2014). These building moves, if proposed and practiced effectively, would use student mathematical thinking to further the learning of all students, which aligns with effective student-centered mathematics instruction (National Council of Teachers of Mathematics [NCTM], 2014). The GTAs were then prompted to include proposed building moves as part of the text template in subsequent video analyses.

Additionally, at the conclusion of the intervention the GTAs completed the Calculus Concept Inventory [CCI] (Epstein, 2013) to measure their knowledge of calculus content. The researcher also completed the CCI and reconciled the answers with a mathematics faculty member to create an answer key. The purpose of taking the CCI was to see whether there was a relation between GTA performance on the CCI (a measure of their CCK) and noticing of MOSTs.

Data Collection and Analysis

The data for this study included the CCI results and the video timelines that indicated the instances marked and described by the GTAs in the classroom videos during the interviews and the intervention. The CCI results were scored according to the answer key. The score of each GTA was compared to the rest of the group to see whether there were any obvious differences in scores that could potentially account for differences in noticing skills.

Categorizing Instances. Each instance marked and described by the GTAs in both the interview data and in the weekly timelines was coded and analyzed by the researcher in several different ways in order to examine changes in the GTAs' noticing. First, like in the work of Stockero et al. (2015, 2017), each instance was coded according to *agent* (who or what was noticed) and *mathematical specificity* (the way in which the mathematics was discussed). Instances that had any type of student agent were also coded for *focus* (what about the student(s) was noticed). See Figure 3 for coding categories and code descriptions.

Second, like the work done by Stockero and colleagues (2017), each instance was coded according to whether it was a MOST and whether the reasoning provided by the GTA was consistent with the MOST criteria. The MOSTs for each timeline were determined by the researcher's coding since she was an experienced user of the MOST Analytic Framework. A GTA instance was coded as a MOST if it occurred at around the same time in a video as a MOST. In an instance coded as a consistent MOST, the GTA identified the characteristics of the instance that qualified the instance as a MOST according to the framework (student mathematical thinking, mathematical significance, and pedagogical opportunity). In an inconsistent MOST, the GTA captured an instance related to a MOST, but was not focused on the student mathematical thinking or was focused on non-mathematical aspects of the instance, such as student participation or motivation. A GTA instance was coded as a non-MOST if it did not correspond time-wise with a MOST in the video.

Coding Categories	Category Description	Codes	Code Description
Agent	Who or what was noticed	Teacher	The teacher is the sole object of noticing
		Teacher/ Student	Both the teacher and student(s) are noticed, with the teacher receiving more emphasis
		Student/ Teacher	Both the teacher and student(s) are noticed, with the student receiving more emphasis
		Student Group	A collection of students is the object of noticing
		Student Individual	An individual student's contribution is the object of noticing
		Math	The mathematics itself, not a person or persons, is the object of noticing
Mathematical Specificity	Whether and how the mathematics was discussed	Non-Math	The mathematics is not discussed, usually because a non-mathematical aspect of the classroom is discussed instead, like classroom management or student engagement
		General Math	The mathematics is referenced with little to no detail
		Specific Math	The mathematics is clearly stated with enough detail to recognize the mathematical topic without having to watch the video
Focus	For instances with a student agent, what about the student(s) was attended to	Affective Interaction	A non-mathematical interaction between student(s) and teacher, usually focused on something like classroom management or student engagement
		General Understanding	The nature of a student's or students' comprehension of a concept, problem, or answer
		Mathematical Interaction	An interaction between students or between students and teacher that is mathematical in nature, usually focused on the process of working together for the purpose of learning mathematics
		Noting Student Mathematics	A specific instance of student mathematical thinking is described, like a student statement or question
		Analyzing Student Mathematics	Not only is a specific instance of student mathematical thinking referenced, but an attempt to interpret is made, like reasoning why a question was asked or a solution method was proposed

Figure 3. Components of noticing coding scheme (adapted from Stockero et al., 2015, 2017).

Response Coding. Because one of the goals of the intervention was for GTAs to propose student-centered responses, the pre- and post-interview instances were additionally coded according to whether the response to each instance proposed by the GTAs was student-centered or teacher-centered. A student-centered response was one in which the teacher would involve one or more students in responding to the instances, whereas a teacher-centered response would only involve the teacher. For example, in an instance where a student shared a solution step, a GTA proposed, “Maybe ask them, 'Why would you do that? What is the purpose of that? How does that help us going forward?'” This response was coded as student-centered since the student or students in the class would be the ones thinking about and addressing the mathematics of the instance rather than the teacher explaining the significance of what the student said. In another example, a student asked a question, and the GTA proposed, “I would quickly explain.” This response was coded as teacher-centered since the teacher would be the only one involved in reacting to the student question, rather than perhaps posing the question to the class.

Coding Analysis. After the coding was complete, the data were summarized to look for changes in the GTAs’ noticing week to week throughout the intervention. The analyses focused on the components of noticing (agent, mathematical specificity, and focus) and MOSTs. Comparisons of student-centered responses from pre- and post-interviews were also made to look for changes in how the GTAs might respond to the moments they selected.

To provide a common unit of measure among all GTA coding, percentages were calculated for each code out of each GTA’s total number of marked instances in each video. These percentages were used to track changes in the GTAs’ noticing on an individual basis. Averages for all the GTAs’ coding per video were also calculated to reflect changes of the group as a whole. The intervention data was then split into three stages—early, middle and late in the intervention—and the data were summarized and average percentages were calculated for each stage of the intervention. In this analysis, *baseline* refers to Videos 1, 2, and 3, the first three videos of the intervention and before the introduction of the MOST framework. *Middle* refers to Videos 4, 5, and 6, the three videos immediately following the introduction of the MOST framework, and *final* refers to Videos 7, 8, and 9, the last three videos of the intervention when the GTAs should have had the best understanding of the framework. Data from the pre- and post-interviews, including the response coding, were analyzed separately due to the difference in the nature of the interviews. Recall that in the interviews, the GTAs engaged in in-the-moment video analysis, whereas in the individual video analysis, repeated viewings and lengthy deliberation about instances were possible.

Results

Components of Noticing

Agent. Because the goals of the intervention placed an emphasis on noticing students and their mathematical thinking, changes were examined in the GTAs’ noticing of instances in which students were the primary agent (i.e., Student/Teacher, Student Individual, and Student Group agents). Table 1 provides the percentages of such instances in each stage of the intervention and the pre- and post-interviews. In the table it can be seen that both individually and as a group the general trend was an increase in the GTAs’ noticing that had a primary student agent from baseline to middle to final. Impressively, the majority of GTAs averaged 100% and the group averaged 94% of primary student noticing in the final data.

Table 1
Noticing of Primary Student Agent by Stage and Interview

Participant	Baseline	Middle	Final	Pre	Post
GTA 1	41%	100%	100%	29%	100%
GTA 2	50%	89%	100%	41%	100%
GTA 3	63%	100%	100%	75%	100%
GTA 4	28%	67%	75%	80%	100%
Group	46%	89%	94%	56%	100%

Table 1 also shows improvement in the GTAs' noticing of instances with a primary student agent from pre- to post-interview. While GTAs 3 and 4 displayed a high level of noticing of students in the pre-interview (75% and 80%, respectively), GTAs 1 and 2 showed the most growth in this type of noticing. Most notably, 100% of the GTAs' noticing in the post data was primarily on students. This indicates that the GTAs developed the ability to focus their noticing on students, rather than on the teacher or the mathematics itself, in their in-the-moment analysis of video.

Mathematical Specificity. With student mathematical thinking and mathematical significance being two of the three characteristics of a MOST, a possible indicator of improvement in noticing of MOSTs is the ability to speak about the mathematics of an instance in a detailed manner. Thus, changes were examined in the GTAs' percentages of instances that were coded for mathematical specificity at the most detailed level, Specific Math.

Table 2 shows that the baseline percentages for Specific Math were rather high for all GTAs with the exception of GTA 4, who only had 6% of their instances coded as such. The middle data showed an increase in mathematical specificity for all GTAs, with the most considerable increase being that of GTA 4, who had a 77% increase in instances coded as Specific Math. Perhaps most important is that all of the GTAs exhibited 100% Specific Math noticing in the final data.

Table 2
Specific Math Noticing by Stage and Interview

Participant	Baseline	Middle	Final	Pre	Post
GTA 1	97%	100%	100%	100%	100%
GTA 2	85%	100%	100%	59%	100%
GTA 3	81%	100%	100%	75%	100%
GTA 4	6%	83%	100%	40%	100%
Group	67%	96%	100%	69%	100%

Table 2 also indicates improvement for all of the GTAs in instances coded as Specific Math from pre- to post-interview. While GTA 1 already discussed the mathematics in a high level of detail in the pre-interview, it is worth noting that this level of mathematical specificity was maintained in the post-interview. GTA 4 showed the most improvement from 40% in the pre-interview to 100% in the post-interview data. Like the final data, the post-interview data indicated 100% Specific Math coding for all participants, implying that the GTAs reached 100% Specific Math noticing not only when completing individual data analysis with time to reflect and write about each instance, but also when put on-the-spot in an interview setting for their video analysis.

Focus. Recall that the focus codes were only assigned to instances with a student agent and describe what about the student(s) was attended to. The two focus codes that most aligned with the goals of the intervention, due to their focus on student mathematics, are Noting Student Mathematics and Analyzing Student Mathematics. Because of their relationship to the goals, changes were explored in the GTAs' noticing with both of these focus codes. While the focus codes only apply to those instances with a student agent, the percentages calculated in the following results are out of the total instances identified by the GTAs to reflect an overall, and not a narrow, sense of their noticing.

The first focus analysis examined both Noting and Analyzing Student Mathematics in sum to capture changes in the total percentage of instances in which the GTAs were focused on student mathematics. Table 3 shows widespread improvement for all GTAs in describing and/or interpreting the student mathematical thinking in an instance. Of particular importance is that, with the exception of GTA 4, the GTAs were Noting and/or Analyzing Student Mathematics 100% of the time as soon as the middle data, immediately following the introduction of the MOST framework. While GTA 4 did not reach 100% for these noticing foci, substantial increases were still made throughout the intervention.

Table 3
Noting and/or Analyzing Student Mathematics by Stage and Interview

Participant	Baseline	Middle	Final	Pre	Post
GTA 1	32%	100%	100%	29%	100%
GTA 2	48%	100%	100%	19%	100%
GTA 3	52%	100%	100%	13%	100%
GTA 4	9%	50%	83%	20%	100%
Group	35%	88%	96%	20%	100%

The pre- and post-interview data also exhibit increases in Noting and/or Analyzing Student Mathematics for all GTAs, with all of the GTAs reaching 100% of instances with these focus codes. Thus, even for in-the-moment video analysis in an interview setting, all of the GTAs developed the ability to describe and/or interpret the student mathematical thinking in their noticing of classroom instances as a result of the intervention.

The second focus code analysis investigated changes in just the Analyzing Student Mathematics code, where the GTAs made an attempt to interpret a specific instance of student mathematical thinking. This focus code was honed in on specifically since the MOST Analytic Framework requires not only a focus on student mathematical thinking, but also that an inference be made about the student mathematics. Therefore, changes in Analyzing Student Mathematics could be an indicator of growth in Jacobs and colleagues' (2010) second noticing skill of *interpreting [students'] understandings* and in the analysis of MOSTs.

The data in Table 4 indicates that Analyzing Student Mathematics was absent or low in the baseline videos, both individually and as a group. By the middle videos, substantial increases were made by all GTAs, with GTAs 1, 2, and 3 improving in this aspect of noticing from the baseline to middle data by a range of 72% to 84%; only GTA 3 reached 100% in the final data. While improvements were made overall throughout the intervention, there were small decreases observed for GTA 1 and GTA 2 from the middle to the final data.

Table 4
Analyzing Student Mathematics by Stage and Interview

Participant	Baseline	Middle	Final	Pre	Post
GTA 1	16%	100%	94%	14%	100%
GTA 2	0%	72%	67%	0%	75%
GTA 3	24%	96%	100%	13%	100%
GTA 4	0%	33%	47%	0%	100%
Group	10%	75%	77%	7%	94%

The pre-interview and post-interview data exhibit remarkable growth for in-the-moment noticing that included Analyzing Student Mathematics. GTA 4 demonstrated the largest growth at 100% from pre- to post-interview, while increases for the other GTAs were also large, ranging between 75% to 87%. Interestingly, GTA 4's final percentage for instances coded as Analyzing Student Mathematics in the individual analysis was still quite low at 47%, but reached 100% for this focus code in the post-interview video that was analyzed in-the-moment. A possible reason for this could be that GTA 4 more completely communicated what they noticed with spoken word in the interview as compared to written word in their text that accompanied the individually-analyzed video timelines. Another possible reason is that the shorter video used for the interview was more manageable to make sense of for GTA 4. In general, these results suggest that the intervention was successful in developing the GTAs' abilities to focus on and interpret student mathematical thinking in an in-the-moment context.

MOST Analysis

While the data related to the noticing components of primary Student agent, Specific Math, and Noting and/or Analyzing Student Mathematics point to improvements in noticing aligned with the goals of the intervention, the main goal was to improve the GTAs' noticing of MOSTs. Naturally, changes were examined in the GTAs' noticing of these instances. Like the other analyses, the percentages reported are out of the total set of instances marked by the GTAs.

Inconsistent and Consistent MOSTs. The GTAs' noticing of MOSTs, both inconsistent and consistent, was first investigated. Table 5 shows that all of the GTAs improved in their noticing of MOSTs during the three stages of the intervention. That is, the percentage of the instances marked by the GTAs that aligned with MOSTs increased from stage to stage, with the group's average percentages increasing from a baseline of 19%, to 33% in the middle data, and finally to 73% in the final data. This result indicates that the intervention improved the GTAs' noticing of moments with significant potential to improve student mathematical learning, even if perhaps the GTAs were not focused on the student mathematics in those instances (making them inconsistent MOSTs).

The pre- and post-interview data showed that all of the GTAs improved in their in-the-moment noticing of MOSTs, with increases ranging from 29% to 60%. It is worth recalling that the prompt for both the pre- and post-interview was to identify *MIMs*, which may explain why there was not a higher percentage of instances that were MOSTs among those identified in the post-interview. An idea underlying the intervention was that the MOST Analytic Framework would provide a way to characterize mathematically important moments that the instructor should notice, but perhaps the connection between *MIMs* and MOSTs was not made by the GTAs, or was not internalized for in-the-moment analysis. Still, it is encouraging that improvements were made in the noticing of important student thinking as a result of the intervention.

Table 5
Noticing of Inconsistent and Consistent MOSTs by Stage and Interview

Participant	Baseline	Middle	Final	Pre	Post
GTA 1	32%	61%	87%	29%	60%
GTA 2	19%	36%	76%	19%	50%
GTA 3	25%	57%	78%	38%	67%
GTA 4	19%	33%	53%	40%	100%
Group	24%	47%	73%	31%	69%

Consistent MOSTs. The subset of MOSTs that were consistent MOSTs was then further examined; that is, those instances that both aligned with MOSTs time-wise in the video and were correctly characterized according to the MOST Analytic Framework. Table 6 shows improvement in the noticing of consistent MOSTs from stage to stage of the intervention. Of particular interest is that in comparison to the inconsistent and consistent MOST data in Table 5, the consistent MOST data have lower baseline percentages, but the middle and final data percentages match exactly in these two analyses. This signifies that all of the GTA-identified instances that aligned with MOSTs in the middle and final data were also instances that were correctly characterized according to the MOST Analytic Framework, which alludes to the framework being an important tool in not only recognizing the right moments (align with MOSTs) but also for the right reasons (consistent MOSTs).

Table 6
Noticing of Consistent MOSTs by Stage and Interview

Participant	Baseline	Middle	Final	Pre	Post
GTA 1	24%	61%	87%	14%	40%
GTA 2	19%	36%	76%	11%	50%
GTA 3	25%	57%	78%	13%	67%
GTA 4	8%	33%	53%	20%	100%
Group	19%	47%	73%	14%	64%

The pre-interview and post-interview data shows considerable increases in the GTAs' in-the-moment noticing of consistent MOSTs. As with the trends in the intervention data, the interview percentages for consistent MOSTs were lower in comparison to the combined inconsistent and consistent MOST percentages for the pre-interview, but were very similar in the post-interview (see Table 5 and Table 6). In fact, with the exception of GTA 1, the post-interview percentages for consistent MOSTs matched those of the combined inconsistent and consistent MOSTs. Like the intervention data, the interview data suggests that the intervention was successful in improving the GTAs' ability to notice MOSTs and reason about them in accordance with the MOST Analytic Framework, even in an in-the-moment noticing context.

Responses

The other main goal of the intervention was to increase the number of student-centered responses that were proposed by the GTAs to instances that they identified in the video. Table 7 displays the percentage of instances in the interview data for which the GTAs provided a student-centered response to explain what they would do if the identified instance occurred in their classroom. It can be seen that substantial increases in the percentage of such responses were made from pre- to post-intervention. In fact, with the exception of GTA 4, 100% of the responses provided by the GTAs were student-centered in the post data. While GTA 4 did not reach 100%, the percentage of student-centered responses that they proposed still increased from pre- to post-interview. These results suggest that the intervention was successful in improving the GTAs' skills in proposing student-centered responses in an in-the-moment context.

Table 7
Student-centered Responses by Interview

Participant	Pre	Post
GTA 1	0%	100%
GTA 2	57%	100%
GTA 3	83%	100%
GTA 4	33%	50%
Group	43%	88%

CCI Scores

After the intervention, the GTAs took the CCI assessment to determine whether their CCK may have had an effect on their noticing. Table 8 shows the GTAs' performances, both as a raw score and as a percentage. Overall, the GTAs' performances on the CCI were quite similar, suggesting that all of the GTAs in the study had roughly the same aptitude for calculus concepts; therefore, there was no evidence to suggest that differences in CCK accounted for differences observed in their noticing.

Table 8
CCI Scores

Participant	Score out of 22	Percentage
GTA 1	17	77%
GTA 2	17	77%
GTA 3	19	86%
GTA 4	19	86%
Group	18	82%

Discussion

This study sought to answer research questions related to the effectiveness of an intervention in improving GTAs' noticing of mathematically significant pedagogical opportunities to build on student thinking (MOSTs), the effectiveness of the intervention in supporting the GTAs' ability to propose in-the-moment student-centered responses, and the relationship between the GTAs' common mathematical content knowledge and the development of their noticing skills during the intervention. Results showed that the intervention was successful in improving the GTAs' noticing in a number of ways and in two different video analysis contexts. When analyzing video both individually and in an in-the-moment interview context, the GTAs greatly increased in their noticing of instances primarily focused on students, the percentage of instances in which they discussed the mathematics of an instance in a specific manner, their focus on describing (Noting) and/or interpreting (Analyzing) the student mathematics of an instance, and their noticing of consistent MOSTs. These results add support to the successes already seen at the K-12 level of interventions that use video and a defined framework to improve the noticing skills of preservice and inservice mathematics teachers (Santagata, 2011; Schack et al., 2013; McDuffie et al., 2014; Stockero et al., 2015, 2017) and suggest that such interventions can be successful at the undergraduate level as well.

The intervention was also successful in improving the GTAs' skills in proposing student-centered responses to instances they identified in video, the *deciding how to respond* skill of noticing (Jacobs et al., 2010). Specifically, these gains were documented from pre- to post-interview in an in-the-moment context. This finding builds upon those of existing studies (Jacobs, Lamb, Philipp, & Schappelle, 2011; Jacobs et al., 2010; Schack et al., 2013) that suggest that professional development structured around noticing students' mathematical thinking in video and/or classroom artifacts can develop teachers' abilities in not only the noticing skills of *attending to and interpreting [students'] strategies and understandings*, but also the skill of *deciding how to respond* (Jacobs et al., 2010).

While most of the GTAs showed similar improvements in their noticing skills and in the proposal of student-centered responses to instances in video, GTA 4's improvements were not as consistent. This raises questions as to why. One explanation would be a difference in CCK, as measured in this study by the CCI. The CCI scores were very similar among the GTAs, however, providing no evidence to support the idea that differences in mathematical knowledge were related to differences in noticing skills. Recall also that GTA 4's stronger post-interview results for Analyzing Student Mathematics suggested that perhaps they were better able to communicate in spoken word than in written text, another potential explanation for the lower percentages in the individual analysis data. While this participant had received the same previous departmental training as the other GTAs after admission to the university's graduate program, GTA 4 was the only international student in the study, and thus had a different cultural and educational background from the other domestic GTAs. Perhaps, then, not having a similar educational background provided a challenge to GTA 4's development in their noticing skills. This suggests that international GTAs may need additional support when engaging in a noticing intervention such as the one in this study. However, the data available from this study is limited to one international GTA, and thus further study is required.

While the results of this study suggest that similar interventions could be successful in supporting GTAs in learning to notice student mathematical thinking, there are limitations to this study and further questions to investigate. This was one, isolated study with a small number of participants. The significance of the results, therefore, must not be overgeneralized. Future work could involve replicating this study with more GTAs, with GTAs of varying cultural backgrounds, at other universities, with another set of videos, and in other subject areas to see whether there are similar results. To expand on this study further, the following questions could also be investigated: How does such an intervention affect the GTAs' classroom teaching? What role does the facilitator play in building the GTAs' noticing skills during meetings? How does the set of videos used in the intervention affect the improvement of the GTAs' noticing skills?

Limitations aside, the intervention in this study was successful in improving the noticing skills of mathematics GTAs and in the proposal of student-centered responses, both steps in the right direction for advancing student-centered instruction in undergraduate mathematics courses. The development and improvement of these skills, while achieved in this study in the structure of a professional development intervention, have the potential to improve the GTAs' classroom instruction. Thus, an intervention such as this that targets mathematics GTAs could possibly influence the instructional methods used in higher education and improve the retention of first- and second-year undergraduate students (Cano et al., 1991; Speer et al., 2005). Completing professional development focused on noticing important student ideas could not only improve mathematics pedagogy in general, but provide opportunities to practice student-centered instructional methods, which are essential for effective teaching (NRC, 2005; NCTM, 2014).

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Student Conceptions of Three-Dimensional Solids

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In the exploratory study presented in this paper, the authors aim to construct a model for the processes by which students in a multivariable calculus class conceptualize solid regions in three dimensions. We designed and recorded student work from two tasks in which students must decode a description of a solid figure and answer questions assessing the strength of their conception of the figure. Presented here are findings and common themes that emerged from the analysis of interviews and group work on two of these tasks, including five generalizable observations about how students process three-dimensional information.

Key words: multivariable calculus, spatial reasoning, three-dimensional solid

Introduction

As motivation for our study, we consider the following problem, variations of which appear in many multivariable calculus textbooks.

> **Task 0:** Let P be the solid region in $\mathbb{R}^3$ defined by the following set of inequalities.
>
> $$\begin{array}{ccccc} 0 & \leq & x & \leq & 1-z \\ z^2 & \leq & y & \leq & 1 \\ 0 & \leq & z & & \end{array}$$
>
> Find the volume of P.

In anecdotal observations of student work, the authors found that students would attempt to solve this problem, apparently without attempting to conceptualize the solid figure *E* as a subset of $\mathbb{R}^3$. In particular, students would set up the triple integral using the given bounds without attempting to form a visuospatial conception of the solid. While this observation calls into question the validity of such problems in building and assessing students' understanding of volume, it also raised the question, through what process would students decode sets of inequalities such as those presented in Task 0 as a solid figure? More generally, when working with three-dimensional spatial information, on what strategies do students rely, and what obstacles do they encounter?

In the present study, we use data from 76 undergraduate multivariable calculus students collected in individual interviews and recorded group work, to identify common strategies and obstacles in decoding, processing, and communicating representations of solid figures.

Theoretical Background

There is a rich history of studying student conceptions of three-dimensional solids in the context of spatial reasoning. Much of the work in this area relies on methodology in which subjects must interpret a two-dimensional drawing of a solid figure and then perform some spatial task such as rotation (Bodner, Guay, 1997) or identification of cross-sections (Cohen, Hagerty, 2007). However in studies where a two-dimensional drawing is used, Gorgorió (1998) warns, “Individuals’ demonstration of their spatial orientation ability depends also on their abilities for interpreting and communicating spatial information.” Indeed it has been shown that misconceptions can lead to errors both during interpretation (Parzysz, 1988; Hallowell, et al, 2015), and communication (Ben-Chaim, Lappan, & Houang, 1989; Hershkowitz, 1990). Thus, building on the paradigm set forth by Parzysz (1991), we model student responses to the tasks as information that has passed through three phases: decoding, processing, and communication.

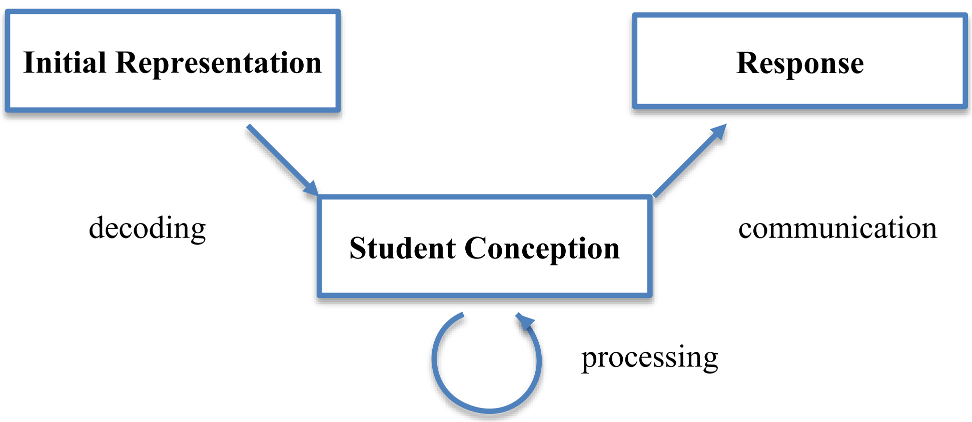


Figure 1. Model for student responses. This figure illustrates the flow of information resulting from students’ response to a task. Errors or loss of information may occur in each phase.

The *decoding* phase encompasses interpretation of the given representation. The *processing* phase includes any mathematical or visuospatial treatments the subject performs, as well as conversions between registers. Finally, in the *communication* phase, subjects encode their responses to the task. Included in the communication phases are written work, drawings, conversation with other students, and building physical models.

In a multivariable calculus class, students are asked to become fluent in interpreting descriptions of figures in $\mathbb{R}^3$ which come in a variety of forms: written descriptions, equations or inequalities in three variables, two-dimensional drawings, and combinations of these. Viewed from the theoretical framework of semiotic representation theory, this requires students to coordinate representations in several registers (Duval, 1993), and perform conversions from one register to another (Duval, 2006). Trigueros and Martínez-Planell (2010) investigate how students use this coordination of registers to perform tasks related to surfaces and equations in multivariable calculus. Our study builds on this work by applying the same theoretical framework to solid figures in three dimensions.

In particular, our study focuses on three-dimensional solid figures that are likely unfamiliar to students. Our rationale for working with such figures is threefold. First, by focusing on *unfamiliar* solids, we aimed to remove the possibility that students would use prior knowledge about the figures to answer questions. Responses to questions about the volume of a familiar solid, like a cone or sphere, we reasoned, would likely better reflect students’ knowledge of memorized formulae rather than their reasoning about the solid figure itself.

Secondly, dealing with unfamiliar solids of the type described in Task 0 requires students to comprehend the combination and interaction of primitives like planes, lines, and parabolas, with which we assume that students are familiar. Here we borrow the term *primitive* from the field of computer graphics to mean a simple geometric figure out of which more complicated structures can be built. We aimed to create tasks for which the challenge would be to conceptualize the interaction of simple parts rather than the conceptualization of the parts themselves.

Finally, we hypothesized that the lack of close representations of solid figures in traditional instruction and assessment might mean that students' framework for thinking about such figures may be entirely divorced from the three-dimensional nature of the figures themselves. Here we use the term *close* representation in the sense of Parzysz (1988) to mean that the abstract figure and its representation are in the same dimension. For three-dimensional solid figures, two-dimensional drawings (like those in Figure 3) would be *distant* representations, and three-dimensional models would be close representation. We use tasks involving close representations to force students to reckon with three-dimensional information directly.

Methodology

The present exposition is part of a larger study in which we seek to understand how students process three-dimensional information in the context of multivariable calculus. In this exploratory phase, we designed several tasks meant to elicit responses that would give insight into the strategies students use and the obstacles they face in conceptualizing solid figures. We will discuss results from Tasks 1 and 2 below.

Task 1: Let G be the solid region described by the inequalities

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 1 \\ 0 & \leq & y & \leq & 1 \\ 0 & \leq & z & \leq & x+y \end{array}$$

(a) Build the region G out of clay.

(b) Write 2–3 sentences explaining how you determined the shape of the solid region from the inequalities

Task 2: Your group will be given two models, labelled Model E and Model F.

(a) Below are two sets of inequalities. Which one describes E and which one describes F? Give reasons for your answer.

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 1-z \\ z^2 & \leq & y & \leq & 1 \\ 0 & \leq & z & & \end{array} \qquad\qquad \begin{array}{ccccc} 0 & \leq & x & \leq & z+1 \\ 0 & \leq & y & \leq & (z-1)^2 \\ 0 & \leq & z & \leq & 1 \end{array}$$

(b) Which region has a larger volume, E or F? Explain your answer in a few sentences.

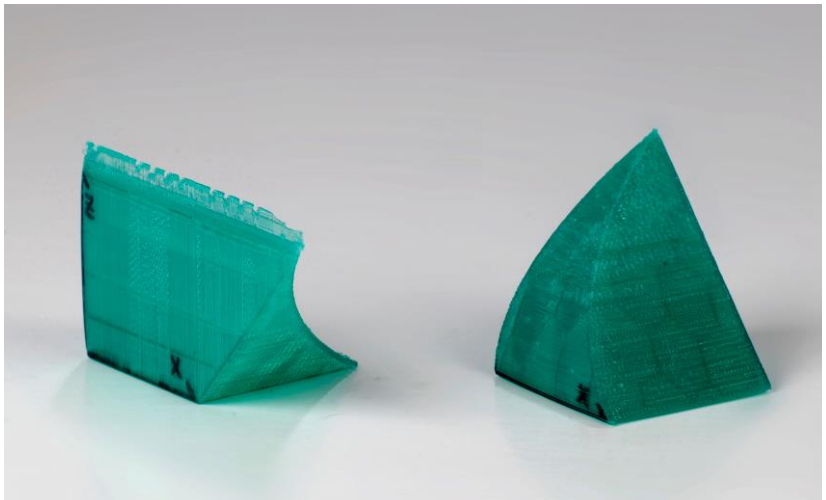


Figure 2. Photograph of the 3D printed models E and F provided to students in Task 2. The coordinate axes were drawn onto the models.

Rationale for Tasks

Our design of these tasks was motivated by the model illustrated in Figure 1. Each of these tasks pairs a close representation of a solid figure with an algebraic one. Underlying the rationale behind this pairing is the assumption that students will face little difficulty in decoding the close representation, nor in communicating a robust enough conception of the figure into the medium of a clay model. Under this assumption, errors in student work on Task 1 could be attributed to misunderstandings or miscalculations related to the decoding and processing of the sets of inequalities[1]. Likewise, in Task 2, we assume that the spatial information represented by the 3D printed models is easily decoded, and difficulties for part (a) will only arise in decoding the sets of inequalities, and the processing required for the matching task.

The figures chosen for these tasks were carefully selected so that they would likely be unfamiliar to students, so that their defining inequalities were either linear or parabolic, and so that they would be sufficiently complex. The figure for Task 1, which is described by linear inequalities only, has six planar faces corresponding to the six inequalities, of which only $z \leq x + y$ involves more than one variable. The two figures in Task 2 were chosen so as not to be easily distinguished by superficial characteristics of the figures and inequalities: each has five faces, four of which are planar, and one of which is a parabolic cylinder, and x, y, and z are all non-negative at every point in the figure. The two figures from Task 2 also satisfy Cavalieri's Principle, in that the cross-sections of the two solids by planes of the form $z = z_0$ have the same area for every constant z_0 and therefore the figures have the same volume. See Figure 2 for a photograph of the 3D models used in Task 2, and Figure 3 for drawings of the abstract figures in the tasks.

[1] This assumption was not entirely accurate since some students who demonstrated a robust conception of the solid made errors in the model the produced related to the relative scaling in the x, y, and z variables.

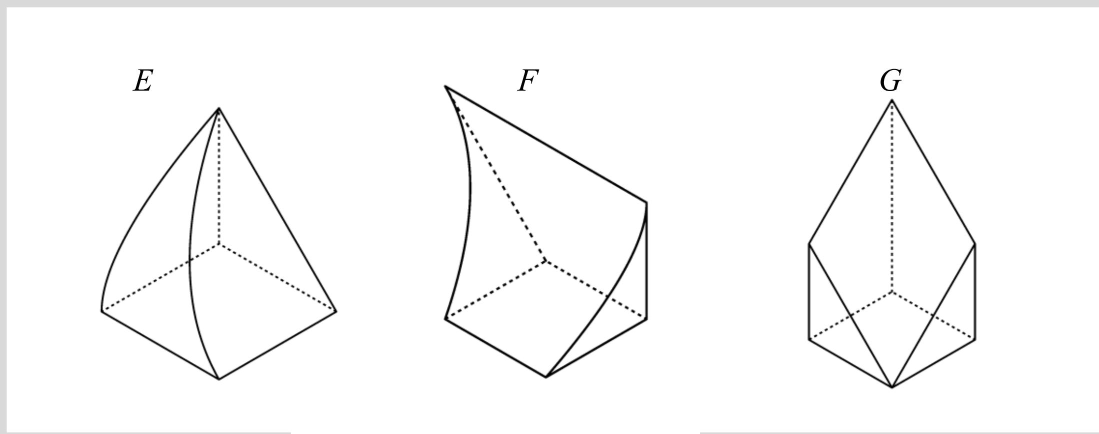


Figure 3. Isometric projections of the solid figures *E*, *F*, and *G* from Tasks 1 and 2.

Participants and Data Collection

Data collection involved videotaping students working in groups on the tasks during nine class sections from two different instructors in the spring and summer of 2016. One of these sections was a designated honors class, specifically intended to emphasize conceptual understanding, and in which students had previously interacted with 3D printed models. The rest were graduate student led discussion sections for lecture-based classes. Additionally, we conducted individual interviews with four students. In total, 76 students participated, and we grouped students into 29 response groups. We also collected students' written work and photographed the clay models produced by the students in Task 1. Because our study was exploratory and not comparative in nature, we favored variety and breadth in our data collection over rigorous control for confounding variables.

Data Analysis

We used a grounded theory approach (Glaser & Strauss, 1967) to learn what processes students used to go from a subset of inequalities in $\mathbb{R}^3$ to a robust mental image. In our early passes through the data, we transcribed the video and focused on using coding to categorize response types for each task, and to identify and classify students' task-specific strategies and difficulties. From this preliminary analysis emerged several "generalizable observations"—observations of student work that were common to both tasks, and which may be more generally applicable and observable in student work involving three-dimensional information.

Results

We organize the results of our analysis by first describing observations from each task individually, and then explaining the generalizable observations that emerged from this preliminary analysis.

Observations From Task 1

We grouped students' responses to Task 1—in the form of the clay models they built—into four categories of clay models emerged from the coding of student responses: rectangular prism

(P), tetrahedron/pyramid (T), conflicted (C), and accurate (A); examples of each are shown in Figure 4 below.

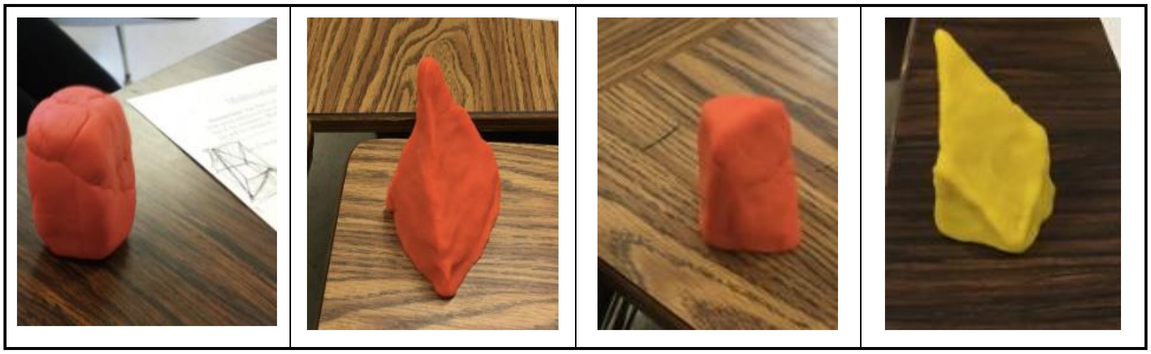


Figure 4. Examples of the four categories of responses. From left to right: rectangular prism (P), tetrahedron/pyramid (T), conflicted (C), and accurate (A).

The categories of (P) and (T) are self-explanatory. A clay model was classified as accurate (A) if it reflected all the correct faces and edges of figure described by the given inequalities; models in which the x, y, and z variables were scaled differently could still be considered accurate. Models that fell into none of the other three categories were classified as conflicted (C); in all cases, these models were some hybrid between a tetrahedron/pyramid and an accurate model.

From analysis of video and written work, we identified several distinct strategies that were used by students while completing the tasks. These are described below, along with examples of each.

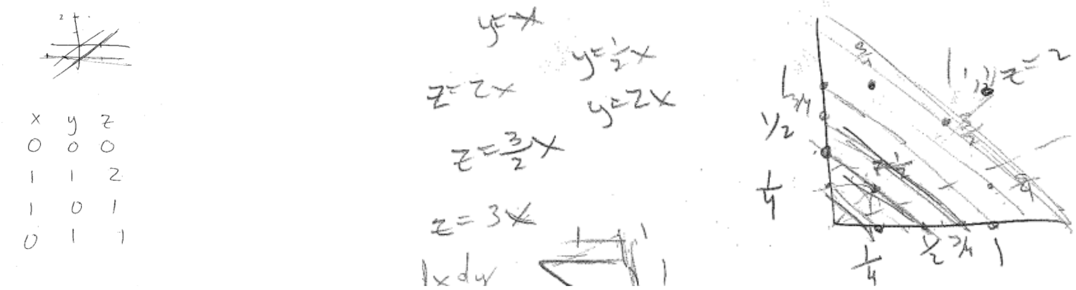
Maximum/Minimum. This strategy involved identifying the absolute maximum and minimum values for each variable; in Task 1, the only nontrivial identification is that $z \leq 2$. One student wrote, "The first two inequalities are the same, so we concluded that the base of the region is a square. The third inequality is dependent on x and y, and the largest value is 2, so the region can range from a rectangular prism to a 2d square."

Covariation. Many students invoked the observation of covariational dependence of two or more variables to explain features of the figure. In the videos, we observed students verbalizing the idea of covariation with accompanying gestures. For example, one student explained, "The z starts at zero and the origin corner at the square then goes to 2 at the (1,1) corner," indicating that the change in z is linked to the change in x and y.

Finding Vertices. Many students found it helpful to identify the extremal points of the figure, and then decide how these points would be connected. An example of a student's work can been seen in Figure 5.

Level Curves. Several students sketched level curves of the surface defined by $z = x + y$ in order to understand the top face of the figure. See Figure 5 for an example written work using this strategy.

We organized strategies by response category, as shown in Table 1. The most notable trend is that response groups that produced an accurate model seemed to use a wider range of strategies than those that produced other models.



Finding Vertices Level Curves

Figure 5. Examples of Finding Vertices and Level Curves.

Applying our model for interpreting student responses from Figure 1, we can attempt to infer in which phase information was lost or mishandled from both the response type and the strategies used. For example, we hypothesize that for students who responded model type (P), most of the information never made it past the decoding phase—that students did not correctly interpret the meaning of the inequalities taken as a set. For students who responded with model type (T), it seemed that some information was lost in the decoding phase, and that only minimal processing had taken place. For model type (C), a hybrid between (T) and (A), the mishandling of information seemed to occur in the processing phase; these students had a conception of the figure in one representation register, but failed to accurately convert it to another.

Table 1. Strategies used, organized by response category. Responses submitted by a group of students were treated as a single response. In many cases, groups reported using more than one strategy, which is reflected in the table.

Student's Strategies	**Rectangular Prism (P)**	**Tetrahe-dron/Pyramid (T)**	**Conflicted (C)**	**Accurate (A)**
Responses in this category	5	2	7	15
Maximum/Minimum	5 (100%)	2 (100%)	4 (57%)	7 (47%)
Covariation/Path			1 (14%)	3 (20%)
Finding Vertices	1 (20%)	1 (50%)		7 (47%)
Level Curves			4 (57%)	5 (33%)

Another observation from the videos that we do not include in the table above is what we will call the *knife strategy*[2]. The knife strategy refers to a specific gesture we saw repeatedly on Task 1 in which students build a model of type (R) then gesture in a slicing motion, indicating a

[2] The name derives from a video clip in which a student used a pocketknife to slice away part of his group's model.

part of the figure that is to be removed. We did not include this as a strategy because it is unclear whether this gesture indicates a particular thought process, or if it serves primarily as a means of communication.

Observations From Task 2

In this task, all of the response groups eventually arrived at the correct conclusion that the left set of inequalities corresponds to Model E and that the right set of inequalities corresponds to Model F although many groups' initial guesses were reversed. Below we list some of the common strategies we observed from video of student work.

Association of primitives across registers. We observed students seeking to find a correspondence between features of the inequalities to aspects of the models. For example, one student, pointing to the face of Model E in the xz-plane, said, "So if you just look at it from this plane [positions Model E to demonstrate], you could kinda of see this triangle shape. And this line created is the line $1 - z$ because it's in the xz-plane," referring to the inequality $x \leq 1 - z$. Another student explained, "You see this is the z [points to the point (1,0,1) on Model F], and so this is the $z + 1$ [moving over 1-unit along the edge of Model F to the point (2,0,1)]."

Covariation. Students identified covariations among variables in both the inequalities and along edges and faces of the models. In particular, they distinguished between quadratic and linear relationships in the inequalities, and associated these with parabolic and linear edges of the models. For example, the student in Figure 7 explained, "There is a quadratic relationship between the y-axis and z-axis, and we found that relationship is $y = z^2$ [positions Model E to show the face in the yz-plane] which fits with this first condition z^2 to y to 1. And along the xz-plane… [positions Model E to face the xz-plane] If you look at it at the right angle, you get a line, which is the line $z = 1 - x$ so then you get $x = 1 - z$ as a boundary."

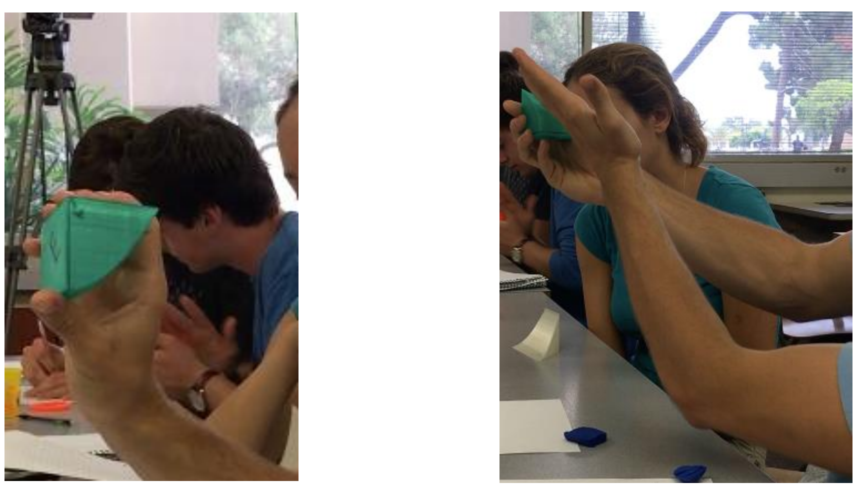


Figure 7. A student uses covariation between pairs of variables to associate features of the model to features of the inequalities.

Projecting onto a plane. We observed students frequently orienting the models so that their view of the model would be an orthogonal projection onto a coordinate plane, as in Figure 8. Each of the inequalities in the descriptions of Models E and F involves at most two of the three coordinate variables, and so seemed to allow students to temporarily disregard one variable in their analysis. Similarly, one group of students attempted to sketch orthogonal projections starting from the inequalities for comparison against the two models. This sort of thinking is also evident in the quotation above in the description of the covariation strategy.

Figure 8. A student positions Model F to give himself a view of the figure orthogonally projected onto the *yz*-plane.

Generalizable Observations

Here we make five generalizations of the observations witnessed in the two tasks, especially those that were common between the two tasks, and which may be observable more generally in students' approach to solving problems involving three-dimensional solid regions.

Use knowledge from two dimensions. In both tasks, we observed students using knowledge about two-dimensional geometry to aid in their decoding and processing of the three-dimensional figures. Students identified covariational relationships between pairs of variables in the algebraic descriptions and in the three-dimensional models. A particularly common observation in Task 2 was students' identification of the parabolic curves on the models, and the quadratic covariations described by the inequalities. Students' use of the projection strategy on Task 2 and the level curves strategy on Task 1 also fall into this category, as would the analysis of cross-sections more generally.

Absolute bounds. The maximum/minimum strategy appears in both tasks. Establishing a bounding box for a three-dimensional figure was observed as the first step for many of the response groups. In Task 1, for example, many of the groups that started by concluding that $z \leq 2$ went on to employ other strategies as well (see Table 1). In Task 2, Figure 9 shows that used similar reasoning to conclude that the set of inequalities describing Model E implies an absolute bound of $z \leq 1$, and another group of students similarly concluded that $x \leq 1$, distinguishing the inequalities from Model F.

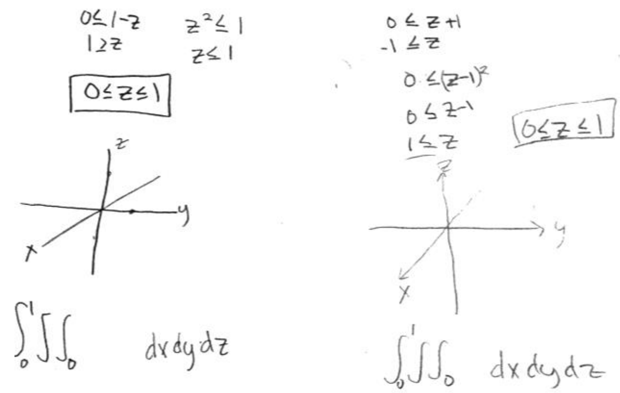


Figure 8. A student positions Model *F* to give himself a view of the figure orthogonally projected onto the *yz*-plane.

Multimodal communication. Students frequently struggled to clearly and unambiguously verbally communicate features of the figure. However, we observed extensive use of gesture and interaction with models, drawings, and algebraic statements. This mixed use of gestures and words creates what is referred to as a *multimodal* representation of mathematical objects (MacNeill, 1998), and its use by high school geometry students was studied by Chen and Herbst (2013). We hypothesize that multimodal communication is especially important for students describing three-dimensional figures because they lack an established common language for describing objects in $\mathbb{R}^3$. The knife strategy from Task 1 is a particularly salient example of this kind of communication.

Rigidity and reliance on norms. In several cases, we found that students struggled to adapt their thinking to figures and presentations that differed from examples seen in class. One student explained his difficulty with the inequalities in Task 2, "I'm just not used to seeing it in this format," referring to the fact relationships between variables did not have the familiar $z = f(x, y)$ form. Another group of students struggled to communicate with drawings while working on Task 1 because one student insisted upon orienting his axes differently than his peers.

Conversion of part-whole relationships across registers. In both tasks, we observed students seeking to associate particular inequalities with faces or other aspects of the models. This was communicated with multimodal phrases like, "This [points to parabolic cylinder face of *E*] comes from here [points to inequality $z^2 \leq y$]." Another student explained during Task 1, "Since x and y are both between 0 and 1, the base is obviously a square." In each of these statements, students identify a familiar geometric primitive with some subset of the algebraic description. However, we observed several students who seemed to suffer from a cognitive overload from the three-dimensional spatial information being presented to them (Huk, 2006), with one student initially claiming, "I don't know where to start." Considering one primitive at a time may serve as a means of coping with the complexity of the entire figure. However, the frequency of the "conflicted" response type for Task 1, which is comprised of parts that are mostly correct, shows that converting one part of the figure at a time does not necessarily lead to a correct conversion of the whole.

Implications

For instructors of multivariable calculus, awareness of the types of strategies students employ to consider three-dimensional solids may be helpful in lesson design and in identifying and addressing student difficulties. Our use of three-dimensional models in this study was, in part, meant to uncover a possible disconnect between multivariable calculus students' procedural knowledge about things like volume, and their visuospatial conception of the underlying figures. There is evidence that multivariable calculus student perceive a benefit from the use of instruction that incorporates manipulation of three-dimensional models (McGee, Moore-Russo, Ebersole, Lomen, & Quintero, 2012; Wangberg & Johnson, 2013), perhaps because the models bridge this disconnect. In addition to providing context for instructional design, we hope that cognizance of our observations will aid in the design of appropriate assessment.

While the scope of our study is limited to multivariable students' conceptions of three-dimensional solids, we have framed our generalizable observations from the previous section so that one could test our hypothesis that these strategies and difficulties will show up in student work with other objects in $\mathbb{R}^3$ from multivariable calculus, such as surfaces and space curves. While the particular solid regions chosen for analysis in this study are not especially interesting or useful, we believe that their complexity led to a wider variety of observations than a simpler or more symmetric figure would have. Also of interest would be the study of how students conceptualize functions and vector fields defined on such objects.

Finally, since the present study was largely exploratory—in the sense that we mostly wanted to see what kinds of observations we could make from administering these particular tasks—we believe there would be benefit to conducting controlled interviews with students to better understand the process students go through in working through these sorts of tasks.

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Student Mathematical Connections in an Introductory Linear Algebra Course

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In an introductory linear algebra course, students are expected to learn a plethora of new concepts as well as how these concepts are connected to one another. Learning these connections can be quite challenging for students due to the vast number of connections and student inexperience with mathematical logic. The study reported here consisted of an investigation into how inquiry-oriented teaching methods could be employed in an attempt to create opportunities for students to develop mathematical connections in an introductory linear algebra course.

Key words: linear algebra, mathematical connections, inquiry-oriented teaching

Introductory linear algebra courses have traditionally been quite challenging for students. There are several reasons for this, including the fact that students are introduced to a plethora of brand new concepts and terminology. Further, many of these concepts are connected to one another in various ways, and students are expected to learn these connections as well. While many researchers and teachers would agree that students should be able to make mathematical connections, the phrase "mathematical connection" is often loosely defined. This study considers two particular types of mathematical connection in an introductory linear algebra course: *connections between symbolic representations of a linear system* and *logical implication connections*. The symbolic representations of a linear system consist of the augmented matrix, the matrix equation, the vector equation, and the linear system itself; these representations are presented in Figure 1.

Linear System	**Augmented Matrix**	**Matrix Equation**	**Vector Equation**
$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$	$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{n2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$	$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Figure **1**: A linear system can be represented as an augmented matrix, a vector equation, and a matrix equation.

Each of these mathematical objects is connected to the other three due to the fact that they are all representations of each other; as the augmented matrix is the tool through which many linear algebraic problems are solved, particularly at the introductory level, connections between symbolic representations of a linear system are some of the first connections many linear algebra students encounter.

Relationships between various linear algebraic concepts are often summarized in theorems of logical equivalence such as the Invertible Matrix Theorem (IMT) (Lay, 2011). The statements in this theorem are all logically equivalent, meaning any statement in the theorem logically implies another (and vice versa). Thus, the logical implications present in the IMT could be described as logical implication connections. While the IMT provides a convenient presentation of logical implications in introductory linear algebra, it is somewhat restrictive due to the fact that it only applies to square coefficient matrices (as only square matrices can be invertible). However, subsets of the logical implications inherent in the IMT could be applied to non-square matrices. The IMT could actually be divided into two “sub-theorems,” which will hereby be known as the First and Second Theorems of Logical Equivalence; these theorems are presented in Figure 2.

Theorem 1: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. a) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^m$. b) A has m pivot positions; that is, A has a pivot position in every row. c) Every vector $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A. d) The columns of A span $\mathbb{R}^m$. e) The linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.	**Theorem 2:** Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. a) The equation $A\mathbf{x} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$ in $\mathbb{R}^m$. b) For each $\mathbf{b}$ in $\mathbb{R}^m$, the linear system corresponding to $A\mathbf{x} = \mathbf{b}$ does not have a free variable; that is, the linear system only has basic variables. c) A has n pivot positions; that is, A has a pivot position in every column. d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. e) The columns of A form a linearly independent set. f) No column of A is a linear combination of the other columns. g) The linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

Figure **2**: Unlike the Invertible Matrix Theorem, these theorems of logical equivalence are not restricted only to the case of square matrices.

As learning all of these mathematical connections can be challenging for students, it would be beneficial to improve the teaching of these connections. The study described in this report was part of a larger study that attempted to determine how inquiry-oriented teaching methods could be implemented in an introductory linear algebra course that, due to considerations such as large class size, limited amount of class time, and wide range of required material, would not lend itself to the traditional demands of inquiry-oriented teaching. In particular, the study explored how inquiry-oriented teaching could specifically be used toward the teaching of mathematical connections in a constrained environment. To that end, the goal of this study was to answer the following research questions:

- *What does it look like when a teacher attempts to incorporate inquiry-oriented teaching in an undergraduate introductory linear algebra class?*
- *What mathematical connections do students appear to evoke within the context of an introductory linear algebra course that employs inquiry-oriented teaching?*

This report will primarily discuss the second research question; however, the classroom context and overall goals of the study had implications for the research methodology and results that were found. Thus, it is necessary to discuss the second research question within the context of the greater study to some extent.

Literature Review

There has been research into logical implication connections in linear algebra (Selinski, N. E., Rasmussen, C., Wawro, M., & Zandieh, M., 2014; Wawro, 2014); these studies primarily considered logical implication connections within the context of the IMT. In contrast, the study reported here focuses on logical implication connections within the broader context of the three theorems of logical equivalence previously described. That said, results of IMT-focused studies are applicable to this study. In particular, Selinski et al. (2014) describe a *hub concept*, which is a concept that a student frequently uses as an intermediary concept through which other logical implication connections are evoked and justified. In adapting this construct for this study, I will refer to a *hub statement* as a statement from one of the three theorems of logical equivalence that a student frequently uses as an intermediary in evoking and justifying logical implication connections.

It is not uncommon for students to know that two linear algebraic concepts are connected but not understand *why* they are connected. This issue was well described by Harel:

> So if a student thinks of 'linear independence' to mean 'the echelon matrix which results from elimination has no rows of zeros,' without being able to mathematically justify this connection, then he or she does not understand the concept of linear independence. (Harel, 1997, p. 111)

This issue of the quality of student understanding has been previously discussed by Skemp (1987) in his description of instrumental and relational understanding. The understanding presented in Harel's example is *instrumental understanding*; that is, knowing what to do but not why. According to Skemp, true understanding of a concept involves *relational understanding*, which is "knowing both what to do and why" (Skemp, 1987, p. 153). This characterization of the quality of understanding could be applied to mathematical connections. Thus, a student has made an *instrumental connection* if the student has formed a connection but does not understand why that connection exists; similarly, a student has made a *relational connection* if the student has formed a connection and understands why that connection exists. For example, a student could present relational understanding of a logical implication connection if he or she can form a chain of logical implications beginning with one statement in a theorem of logical equivalence and ending with another. Unfortunately, students at this level often struggle with mathematical logic, and in particular, many students struggle to form these chains of reasoning (Dorier & Sierpinska, 2001).

Regarding inquiry-oriented teaching, there is a wealth of literature on inquiry-oriented teaching of linear algebra, notably through the IOLA project (Larson, C., Wawro, M., Zandieh, M., Rasmussen, C., Plaxco, D., & Czeranko, K., 2014; Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C., 2012; Wawro, M., Zandieh, M., Rasmussen, C., Larson, C., Plaxco, D., & Czeranko, K., 2014). However, as previously described, one goal of this study was to determine to what extent inquiry-oriented teaching could be developed and implemented within an introductory linear algebra course that faced constraints such as a large class size, limited class time, wide range of required material, and coordinated common sections. In light of this goal, I opted to start with a general definition of inquiry-oriented teaching and explore how that definition could be implemented in the aforementioned constrained class. There are several ways to define inquiry depending on the context or academic subject. Rasmussen and Kwon (2007) characterize student inquiry in a mathematics class through Richards' (1991) definition of mathematical inquiry, which is the mathematics of mathematically literate adults. Thus, mathematical inquiry involves participating in mathematical discussion, solving new problems, listening to mathematical arguments, and proposing conjectures. Rasmussen and Kwon (2007) also highlight the role of teacher inquiry in their definition of inquiry-oriented teaching, which refers to a teacher inquiring into student thinking. In determining how I would specifically define inquiry-oriented teaching for the purpose of this study, I decided that the teaching constraints previously described were immediate obstacles to fostering student inquiry. As my ultimate goal was discovering how opportunities for student inquiry could be provided in a constrained teaching environment, I opted to define inquiry-oriented teaching as "a practice of creating opportunities for students to engage in mathematical inquiry." While teacher inquiry is not explicitly referenced in this definition, this is not to say that teacher inquiry did not occur throughout the course of this study. Researching teacher inquiry was simply not a specific goal of this research study, while researching attempts to foster student inquiry was.

Setting and Methods of Analysis

The research question focusing on the implementation of inquiry-oriented teaching in a constrained teaching environment was born out my own desire as a teacher to explore how I could implement inquiry-oriented teaching in a specific introductory linear algebra course. Thus, this research goal is a reflection of my goal to improve my own practice. Consequently, I opted to conduct this study through an action research methodology (Mertler, 2006). This action research study began with a pilot study in the summer semester of 2015 and continued into the fall semester of 2015 and spring semester of 2016. Each of these action research cycles consisted of research in an introductory linear algebra course that I had taught. While the results of the pilot study and fall 2015 research cycle informed the spring 2016 research cycle, this report will primarily focus on the spring 2016 cycle.

In the spring of 2016, I taught an introductory linear algebra course at a large state university in the Pacific Northwest. The class consisted of sixty students; the majority of these students were engineering majors, while others were mainly math and computer science majors. The course was a two credit course, which placed considerable time constraints on the instructor. As a result of these constraints, inquiry-oriented teaching activities were largely reserved for concepts closely related to logical implication connections and connections between symbolic representations of a linear system. For example, several activities were provided as opportunities for students to have an active role in construction of the aforementioned theorems of logical equivalence; the activity presented in Figure 3 was the activity through which students developed the second theorem of logical equivalence.

Let A be an $m \times n$ matrix such that, for every vector $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has at most one solution; that is, for any particular $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ will have either no solution or exactly one solution. With this information, what else can you say about the matrix A? To aid in your discussion, provided below are some of the various concepts and terms we have seen in the course so far.

• System of linear equations	• Consistent system	• Augmented matrix
• Coefficient matrix	• Basic and free variables	• Pivot position
• Vector equation	• Linear combination	• Span
• Matrix equation	• Homogeneous equation	• Trivial solution
	• Linearly independent	

Figure **3**: This activity was designed as an opportunity for students to have an active role in the construction of the second theorem of logical equivalence.

In this activity, students worked in small groups as they discussed the various relationships between linear algebraic concepts. I would then lead a whole class discussion in which I connected student suggestions to the formal mathematics by translating those student suggestions into statements in the theorem of logical equivalence. Students also worked on several activities that focused on concepts that play prominent roles in the theorems of logical equivalence. For example, students were offered opportunities to explore span and linear independence, as these concepts play parallel roles in the first two theorems of logical equivalence; two of these activities are presented in Figure 4.

Without solving a linear system or using any elementary row operations, determine whether the following sets of vectors span the given space. For each set of vectors, formulate a conjecture about span based on that set.	Without solving a linear system or using any elementary row operations, determine whether the following sets of vectors are linearly independent or dependent. For each set of vectors, formulate a conjecture about linear independence or dependence based on that set.
• Do the vectors $\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}$ span all of $\mathbb{R}^2$?	• $\left\{\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}\right\}$
• Do the vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}$ span all of $\mathbb{R}^3$?	• $\left\{\begin{bmatrix}1\\3\\-5\end{bmatrix}, \begin{bmatrix}3\\9\\-10\end{bmatrix}\right\}$
• Do the vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$ span all of $\mathbb{R}^3$?	• $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-3\\-4\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}\right\}$
• Do the vectors $\begin{bmatrix}1\\7\\0\end{bmatrix}, \begin{bmatrix}3\\8\\0\end{bmatrix}, \begin{bmatrix}-2\\9\\0\end{bmatrix}$ span all of $\mathbb{R}^3$?	• $\left\{\begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}\right\}$
• Do the vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}$ span all of $\mathbb{R}^2$?	

Figure **4**: Each of these activities were designed as opportunities for students for explore span and linear independence in ways that would allow them to develop logical implication connections involving span and linear independence.

There were several goals inherent in the design of the span and linear independence activities. These activities were each implemented after span and linear independence had been formally defined in class. Thus, the task presented in each activity involves a question familiar to students: is the set of vectors a spanning set (or is the set linearly independent). However, each activity is presented with a specific instruction which asks the students to complete the activity without solving a linear system or using any elementary row operations. This instruction was included so that students would have to answer each question without the use of a familiar algorithm. Thus, this instruction is an attempt to take a task that would normally consist of a set of exercises and elevate that task to a set of small problems. Further, it was a goal of this activity that by forcing students to answer these questions through some alternative path, students would form and evoke logical implication connections involving span and linear independence that would assist them in their problem solving.

Data on student mathematical connections was largely collected from interviews that I conducted with nine of my students from the aforementioned class. These interviews were conducted shortly after the Invertible Matrix Theorem had been covered in class. Each interview was approximately one hour in length and consisted of students finding the solution set of a linear system, a vector equation, and a matrix equation, which are presented in Figure 5. Each task lent itself to a different theorem of logical equivalence. For example, the coefficient matrix corresponding to the linear system had a pivot position in every row, thus making every statement from Theorem 1 true for that coefficient matrix. Similarly, the vector equation lent itself to Theorem 2, and the matrix equation lent itself to the IMT. After an interviewee completed one of the problems, the interviewee was asked to describe his or her work. I would then present the interviewee with a list of vocabulary terms that had been discussed in class. The interviewees were asked to discuss as many of the vocabulary terms as they could and how they relate to each problem. I would often ask for justification of particular claims that the interviewee had made and would sometimes directly ask the interviewee whether he or she could discuss a particular vocabulary term. This was all done in an attempt to determine what logical implication connections the interviewees could evoke that incorporated some of the familiar terms and concepts involved in the theorems of logical equivalence.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & x_3 & = & 3 \\ 3x_1 & - & 6x_2 & - & 2x_3 & = & 2 \end{array} \qquad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 7 \\ -4 \\ 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} x = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

Figure **5**: Interviewees were asked to find the solution set of a linear system, a vector equation, and a matrix equation.

In analyzing the interviews, I attempted to determine what mathematically correct logical implication connections corresponding to the three theorems of logical equivalence each interviewee evoked. Evidence of logical implication connections took several forms. Many logical implications involved words such as if, then, means, because, and so. For example, “The vectors, if a linear combination of those produce every single vector in that space, then they span that space” would be considered a logical implication connection. While many logical implications were evoked entirely by the interviewees, some logical implications were evoked as a result of an interviewee responding to a question that I had asked. After determining what logical implications the interviewees evoked, I then attempted to determine which of these connections were relational connections; this was largely accomplished by determining which logical implication connections a student was able to justify.

To account for the logical implication connections that interviewees evoked, I utilized adjacency matrices, as described by Selinski et al. (2014). For each interviewee, I would construct six adjacency matrices: two for each theorem of logical equivalence. For a particular theorem, an interviewee would have one matrix consisting of evoked logical implication connections involving statements from that theorem; a second matrix would contain evoked logical implication connections involving negations of statements from that theorem. For example, consider one interviewee’s adjacency matrices for the IMT, presented in Figure 6.

	A1	B1	C1	D1	E1	F1	G1	H1	I1	J1	K1	L1	M1	N1	O1	P1
A1																
B1				2												
C1				1												
D1																1
E1		1														2
F1																
G1								1								
H1							3				2					
I1																
J1																
K1							1									2
L1								1								2
M1																
N1				1												
O1																
P1																

	A2	B2	C2	D2	E2	F2	G2	H2	I2	J2	K2	L2	M2	N2	O2	P2
A2																
B2																
C2																
D2																
E2																
F2																
G2						1										
H2											1					
I2																
J2											1					
K2																1
L2																
M2																
N2																
O2																
P2																

Figure **6**: These adjacency matrices represent the logical implication connections corresponding to the IMT evoked by one interviewee.

Each of these adjacency matrices is a representation of a digraph; the digraph corresponding to the first adjacency matrix is presented in Figure 7:

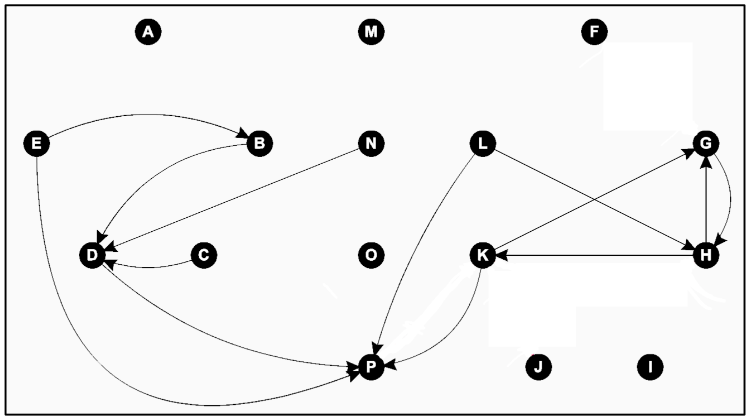

Figure **7**: This digraph represents logical implication connections that a student evoked throughout the course of an interview. Vertices A through D refer to statements from the first theorem of logical equivalence, while vertices F through L refer to statements from the second theorem. Vertices M through P refer to statements exclusive to the IMT.

Note that the digraph does not capture (and would not do so easily) the frequency of evoked logical implication connections, whereas the adjacency matrices easily do. Thus, the adjacency matrices have organizational advantages over their corresponding digraphs, and consequently, the matrices provided an effective means of analyzing the many logical implication connections evoked throughout the study.

Results

In general, the interviewees tended to evoke more connections relevant to the second theorem of logical equivalence than they did the first. This is in itself not entirely surprising; the theorems presented in Figure 2 were the versions of the theorem discussed in class, and the second theorem contains more statements than the first. As the concept of invertibility and the IMT were still new to the interviewees, they tended to evoke relatively few connections exclusive to the IMT. Due to this, the results reported here will primarily focus on connections that are not exclusive to the IMT.

Logical Implication Connections Relevant to the First Theorem of Logical Equivalence

In evoking connections relevant to the first theorem of logical equivalence, the interviewees tended to reference span, pivot positions, and linear combinations. Interestingly, several interviewees presented interpretations of span that were likely consistent with the formal definition of span, but interviewees rarely explicitly referenced the formal definition. That is, several interviewees were able to provide geometric interpretations of span or were able to describe span via linear combinations without explicitly saying the phrase “linear combinations.” For example, consider Will’s explanation of why two particular vectors span $\mathbb{R}^2$:

Will: Because these two aren't scalars of each other, they're going in different directions. They each have their own x_1 and x_2 components. If they were the same, they'd just end up looking like that.

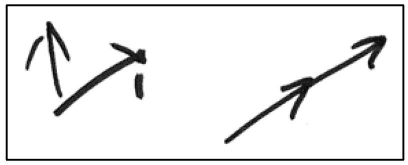

Figure **8**: Will provided a geometric description of what it means for two vectors to span $\mathbb{R}^2$. The illustration on the left represents an example Will provided of two vectors that span $\mathbb{R}^2$, while the illustration on the right represents an example Will provided of two vectors that do not span $\mathbb{R}^2$.

Seth provided an explanation that incorporated both matrix and geometric interpretations of span:

Seth: If you had a matrix, let's take this one [Seth draws the 2×2 identity matrix], then this one would span all of $\mathbb{R}^2$ because no matter how you rearrange this, you can create – uh, I'll expand it [Seth then changes his matrix to the 3×3 identity]. So uh, this one can create, because you can multiply this by infinitely many scalars outside for each row, you can create infinitely many planes, like, if you think about this geometrically, planes in any coordinate system.

Jimmy appeared to allude to linear combinations, but also referenced a geometric interpretation of span:

Jimmy: Well, for spanning, you want, uh. Every direction to be covered, every direction on the plane to be covered by some scale, er, some combination of those vectors, I think.

Jason explained that three particular vectors span $\mathbb{R}^3$ because "they go in different directions. They're not, uh, linear combinations of each other." While Jason referenced linear combinations, it was not in reference to the formal definition of span, but rather, as a description of what must be true of a set of vectors in order to span an entire space. Bill heavily alluded to linear combinations but did not explicitly reference linear combinations:

Bill: Span is having the ability to make any vector within a space. You can, like I said, manipulate any piece of the outcoming vector. You can change it by changing one of the more, one of the scalar multiples along there, not scalar multiple, scalar weights along the way you go. In this case we did at x_1, x_2, x_3. If you could change each of those to then manipulate one of the vectors in the overall value within the system, you could then change the outcome. That goes into the span. If you can do that then it does span $\mathbb{R}^3$, it does span $\mathbb{R}$ whatever. It has the ability to reach any vector, any point within that space.

Bill's description of scalar weights and manipulating vector may provide evidence that he is describing linear combinations, although he does not explicitly reference linear combinations. Thus, Bill's interpretation of span is likely consistent with the formal definition of span, even if he cannot provide the formal definition.

It should be noted that while several students provided geometric descriptions of span, geometric interpretations were not heavily emphasized in class. They were briefly referenced from time to time, but concepts were never defined from a geometric perspective. Further, prior to span being defined, the class had discussed the problem of determining whether any vector in an $\mathbb{R}^n$ space can be expressed as a linear combination of a particular set of vectors. However, when span was formally defined, it was defined more generally as the set of all linear combinations of a set of vectors. Despite this, several interviewees appeared capable of determining whether a particular set of vectors spans an $\mathbb{R}^n$ space by determining whether the vectors were linearly independent, linear combinations of each other, or go in different directions. Thus, it is likely that students developed these alternative, yet mathematically correct, interpretations of span as a result of the inquiry-oriented activity previously described.

Logical Implication Connections Relevant to the Second Theorem of Logical Equivalence

Many of the connections the interviewees evoked relevant to the second theorem of logical equivalence involved pivot positions, linear independence, and basic and free variables. The interviewees tended to refer to basic and free variables in their logical implications more than any other concept; this was particularly interesting, as the interviewees from the previous semester tended to refer to pivot positions more than any other concept. The interviewees also appeared to have understandings of the homogeneous equation inconsistent with the formal definition. For example, Fred seemed to believe that *any* homogeneous equation can only have the trivial solution:

Interviewer: If I had given you zeroes here instead of 3 and 2, would that still have a solution? That homogeneous linear system?

Fred: Yes, because homogeneous equation always have at least one solution, which is the trivial solution.

Interviewer: And what was the trivial solution again? Can you remind me one more time, what was that?

Fred: Trivial solution is $A\mathbf{x} = \mathbf{0}$, so zero is always the solution, for example $\begin{bmatrix}0\\0\end{bmatrix}$.

Jason appeared to hold a similar view:

Interviewer: Can you define homogeneous equation for me? What does that mean?

Jason: It means there's only one solution. I can't remember what it was.

Seth appeared to confound the trivial solution with the homogeneous equation:

Interviewer: Do you remember what the trivial solution is?

Seth: Uh, it's when $A\mathbf{x} = \mathbf{0}$.

Cecily, who was incredibly close to relational understanding of the connection between pivot positions and linear independence, made a similar mistake:

Interviewer: Why is it that not having a pivot position in every column tells you that these columns cannot be linearly independent?

Cecily: Because if there's not a pivot position in every column, then it can have infinitely many solutions. And for it to be linearly independent, it can only have the trivial solution.

Interviewer: Okay. So what can only have the trivial solution?

Cecily: The matrix, the linear system.

Interviewer: Okay. So what is the trivial solution?

Cecily: That's $A\mathbf{x} = \mathbf{0}$, right?

I reminded Cecily that she was describing the homogeneous equation before asking her what the trivial solution is; she claimed she did not know. As these interviewees had understandings inconsistent with the formal definition of the homogeneous equation, it is likely that any connections evoked that involved the homogeneous equation could only be instrumental; further, it suggests that these several interviewees have misunderstandings of the formal vector algebraic definition of linear independence. Some interviewees adjusted for this by providing geometric descriptions of linear independence; Bill, for example, claimed that a particular set of vectors was linearly independent "because they can all point in different directions." Others essentially appeared to instead interpret linear independence through basic and free variables instead of the homogeneous equation. For example, consider Seth's explanation of linear independence:

> **Interviewer:** How come if it has no free variables, that means it's linearly independent?
> **Seth:** Well if it has no free variable, that means that there was a pivot in every column, which would mean that it would have no free variables, because there wouldn't be, like, say a 2 out here. And, uh. This vector would always have a solution.

Seth's response was not unique. Several other interviewees tended to refer to basic and free variables often in their descriptions of linear independence, as did many students on one of the class exams.

It should be noted that when we discussed the homogeneous equation in class, we did not do this through an inquiry-oriented activity; I believed that the concept did not warrant such an activity, as students in the pilot study and fall semester appeared to understand the homogeneous equation fairly well through a mixture of lecture and whole class discussion. Looking back at the day that we discussed the homogeneous equation in the spring semester, I noticed that we concluded our initial coverage of the homogeneous equation with the following discussion of a homogeneous matrix equation that only had the trivial solution:

> **Instructor:** So, could I have free variables?
> **Student:** No.
> **Instructor:** Kay. I can't have any free variables. Why not? Why can't I have free variables?
> **Student:** You'd have infinitely many solutions.

As this was how we concluded our initial coverage of the homogeneous equation, it is possible that some students essentially replaced the concepts of trivial and nontrivial solutions with basic and free variables. That is, students made an instrumental connection between the homogeneous equation and basic and free variables, and as they did not quite understand what the homogeneous equation is and when it has nontrivial solutions, they instead considered when the homogeneous equation would have free variables. Then, when the formal definition of linear independence was provided in terms of the homogeneous equation, they tended to view linear independence in terms of basic and free variables instead of the homogeneous equation. Thus, many students in this semester relied on their instrumental connection between the homogeneous equation and free variables in order to compensate for their lack of understanding of the connection between the homogeneous equation and linear independence, thus interpreting linear independence largely through basic and free variables. This was likely exacerbated by the aforementioned linear independence activity, in which students could refer to the familiar concept of basic and free variables to determine whether the sets were linearly independent or not. Students who had not come to rely as heavily on free variables likely developed more geometric interpretations of linear independence as a result of the linear independence activity, as the sets in the activity were in $\mathbb{R}^2$ and $\mathbb{R}^3$, which can be easily visualized. Once these students had developed a more geometric interpretation of linear independence, they may have felt that the formal definition was no longer necessary for an understanding of linear independence.

Chains of Logical Implication Connections

While the unforeseen student difficulties with homogeneous equations and linear independence were disappointing, other trends appeared that suggested that students may have developed an ability to use, not just evoke, mathematical connections for the purpose of problem solving. For example, on one exam question (offered after students had learned about the IMT and subspaces), students were asked to determine, if given an invertible matrix A, what $\dim \mathrm{Nul} A$ must equal. Jimmy's response to this question is provided in Figure 9.

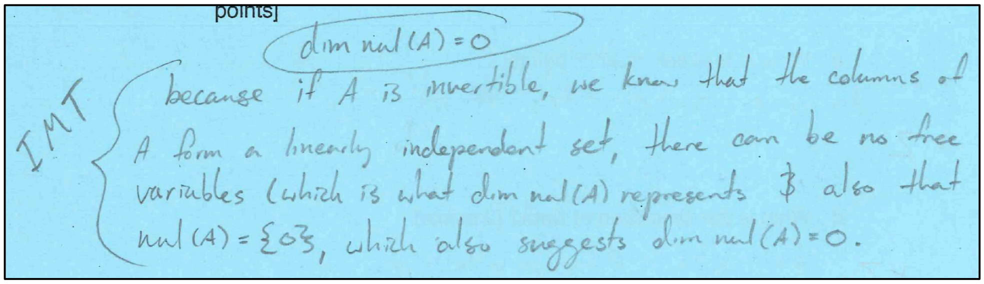

Figure **9**: Jimmy provided a chain of logical implication connections to explain why, if a square matrix A is invertible, then $\dim \mathrm{Nul} A = 0$.

Many students provided responses similar to Jimmy's in that they consisted of mathematically correct chains of logical implication connections. Students who provided such responses demonstrated not only knowledge of a number of logical implication connections, but also an understanding of how these logical implication connections could be used and chained together in order to justify another logical implication.

Connections Between Symbolic Representations of a Linear System

Several interviewees appeared to have, at the very least, instrumental understanding of several connections between symbolic representations of a linear system. Interviewees often evoked relational understanding of connections by taking a specific representation and algebraically translating it into a different representation. For example, in explaining the connection between a matrix equation and a linear system, several students used the definition of the product of a matrix and a vector to rewrite a matrix equation as a linear system.

Student understanding of the symbolic representations of a linear system appear to be influenced by their understanding of some concepts not exclusive to linear algebra. In particular, students' understanding of the term "equation" may have played a role in how they thought about vector and matrix equations. For example, when asked to describe a vector equation, several interviewees attempted to describe parametric vector form, as seen in Figure 10.

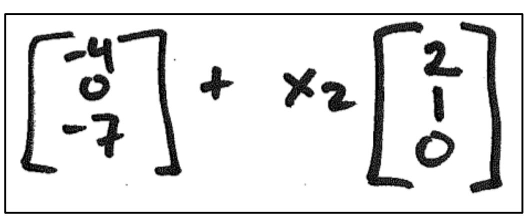

Figure **10**: Several interviewees initially referred to parametric vector form in their descriptions of vector equations. Presented here is Cecily's initial example-based description of vector equations.

Interestingly, Cecily's initial attempt to describe a vector equation did not contain an equality symbol, a necessary component of an equation. In a similar vein, several interviewees described a vector equation by providing a set of vectors corresponding to a particular coefficient matrix;

these descriptions also lacked an equality symbol despite the fact that these interviewees claimed that they were describing a vector equation. A similar trend was found in student understanding of the matrix equation. Cecily and Bobby both initially identified an augmented matrix as a matrix equation, despite the lack of any equality symbol. This was a surprisingly frequent trend outside of the interviews as well. Cecily and Bobby, in addition to several other students, identified an augmented matrix as a matrix equation on an exam problem in which they were asked to write a linear system as a vector equation and a matrix equation; Bobby's work on this problem is presented in Figure 6.

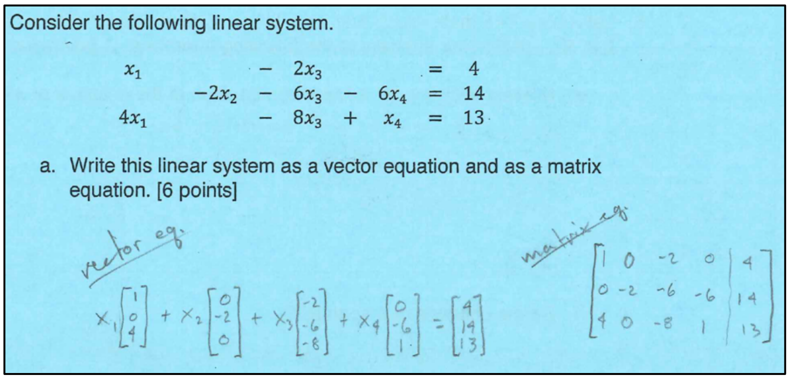

Figure **11**: Several students referred to an augmented matrix in their description of a matrix equation on this exam problem.

It should be noted that Bobby's augmented matrix does contain a bar that demarcates the coefficient matrix from the augment column, and this bar corresponds to the equality symbols in the linear system. However, several students provided augmented matrices without the bar as examples of a matrix equation. In light of this trend, it appears that student conceptions of what exactly comprises an equation may have influenced their understanding of matrix and vector equations. Had they understood an equation to contain an equality symbol, they may have been less inclined to refer to mathematical entities that do not contain an equality symbol as equations.

The Case of Will

One interviewee, Will, evoked a plethora of connections that appeared to be of a relational level of understanding. It should be noted that Will's apparent understanding was by no means representative of the rest of my students or even the rest of the interviewees. However, the connections he evoked and how he appeared to evoke those connections was particularly interesting and may shed light on how successful students understand the various connections in an introductory linear algebra course. Will evoked relational understanding of several connections between symbolic representations of a linear system, and he clearly understood that the representations of a linear system were indeed representations of each other:

Interviewer: Okay. And then what about, so you already told me how you can write [the matrix equation] as a linear system. Can you explain why you can write this matrix equation as a vector equation?

Will: Well I mean, if [the matrix equation] equals [the linear system] and this [the vector equation] equals [the linear system], then [the vector equation] would be the same as [the matrix equation].

Interviewer: Okay, okay, fair enough. Yeah. So you're saying that these are just all the same thing essentially?

Will: Yeah.

Essentially, Will appears to have formed an equivalence relation that could be described as *is a representation of*. Under this equivalence relation, Will has grouped the four representations of a linear system into an equivalence class. For example, Will appears to believe that the linear system and the matrix equation are equivalent because a matrix equation *is a representation of* a linear system. This of course is not to say that Will is consciously viewing this relationship as an equivalence relation; rather, this terminology simply provides a means to capture how Will may have organized his relational connections between symbolic representations of a linear system.

In addition to Will's representation connections, Will evoked many relational logical implication connections. Further, Will evoked connections containing versions of the statement "$A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ in $\mathbb{R}^m$" more than any other interviewee. For example, consider the following exchange with Will, which occurred after Will had found the solution set of a particular vector equation:

Interviewer: If I had given you a different vector besides 3 and 2, if I had given you any other vector, how would you know that that vector equation has a solution, no matter what vector I gave you?

Will: Because it spans all of $\mathbb{R}^2$. So any vector in $\mathbb{R}^2$ is gonna be a linear combination of these three vectors.

Essentially, I provided Will with a version of the statement "$A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ in $\mathbb{R}^m$" and asked what statements Will could connect to it. Will easily provided two such statements involving span and linear combinations. This is by no means trivial. Notice that the theorem statement, which is the version formally presented in class, contains a matrix equation. However, the version I presented to Will contained a vector equation. It is possible that Will's equivalence relation, "is a representation of," allowed him to connect to this statement so easily. Will recognizes that the matrix equation and the vector equation are equivalent. Thus, when presented with information about a vector equation, Will was able to easily replace that vector equation with the corresponding matrix equation. From there, he is able to evoke logical implication connections consistent with those presented in the theorems of logical equivalence. This suggests that there may be an important relationship between a student's understanding of the symbolic representations of a linear system and the logical implication connections that that student is capable of evoking.

Conclusions and Discussion

In light of the results from the interviews, it appears as though the inquiry-oriented activities that focused on span and linear independence were successful in creating opportunities for students to develop their own interpretations of span and linear independence. The role of geometric descriptions in the class was limited, yet several students developed interpretations of span that appeared to be more geometric in nature; further, these interpretations often heavily alluded to linear combinations while not explicitly referencing linear combinations in an algebraic sense. As many students struggled with the definition of the homogeneous equation, these students appeared to have essentially used the theorem statement "The corresponding linear system has only basic variables (has no free variables)" as a hub statement, as the concept of basic and free variables was far more accessible and easy to use. The linear independence activity allowed students to reinforce the role of basic and free variables as a hub concept, particularly as the concept pertains to linear independence and dependence. Additionally, the linear independence activity also allowed students to develop geometric interpretations of linear independence that relied on the notion that linearly independent vectors are not linear combinations of each other.

While the inquiry-oriented activities were successful in creating opportunities for students to form their own interpretations of span and linear independence and how they relate to other concepts, the implementation of these activities could be improved. Students appeared to replace their understanding of the formal definitions of span and linear independence with the understanding they developed as a result of the activities. This may have limited the students' ability to evoke logical implication connections involving these concepts and their formal definition; that is, students may be more capable of clearly evoking logical implication connections if they can connect their understandings of these concepts to the formal definitions. As an instructor, I should have devoted time to exploring how these student developments relate to the formal definitions of span and liner independence. Investigating how the inquiry-oriented activities could be improved in this regard remains an avenue for future research.

Throughout the semester, students appeared not only to form and evoke logical implication connections, but also to use these connections to evoke and justify other logical implications; this was evident in the many responses to an exam question that asked students to explain how invertible matrices and the dimension of the null space are connected. This is by no means trivial, as students often struggle to form chains of reasoning (Dorier & Sierpinska, 2001). It should be noted that, as an instructor, I would often emphasize to my class the importance of justification, particularly when providing responses to an exam question. This likely partially accounts for the numerous responses that contained a great amount of detail; however, this cannot fully account for the fact that most of these responses consisted of *mathematically correct* chains of logical implication connections. Thus, it is quite possible that the frequent opportunities to form, evoke, and justify logical implication connections through inquiry-oriented activities allowed students to develop an ability to use connections as tools in justifying other logical implication connections.

The results concerning student connections between symbolic representations of a linear system suggest that many linear algebra students may have conceptions of equations that do not contain the necessity of an equality symbol. This finding may help explain some sources of difficulty that many linear algebra students face. For example, Zandieh and Andrews-Larson (2015) describe how many linear algebra students find it easier to translate a linear system to an augmented matrix than they do an augmented matrix to a linear system. One reason for this difficulty is the reconstruction of $\mathbf{x}$, which refers to reintroducing the variables when translating from an augmented matrix to a linear system. The augmented matrix is the only representation of a linear system that does not explicitly contain reference to the variables of a linear system; it is also the only representation that does not explicitly contain an equality symbol. In light of the results of the study reported here, it is possible that students may struggle not only with the reconstruction of $\mathbf{x}$, but also with the reintroduction of the equality symbol.

Finally, while Will is by no means representative of a typical linear algebra student, his apparent understanding of various mathematical connections suggests that there may be an important relationship between student understanding of symbolic representation connections and logical implication connections. There has been previous research into symbolic representation connections (Larson & Zandieh, 2013; Zandieh & Andrews-Larson, 2015) and logical implication connections (Selinski et al., 2014; Wawro et al., 2012; Wawro et al., 2014); however, these two types of connection have been discussed largely independently of one another both within the literature and within the study reported here. Thus, a future research project emanating from this study will involve an exploration into how students use their understanding of symbolic representation connections to form and evoke logical implication connections.

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The Effects of the *Epsilon-N* Relationship on Convergence of Functions

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Much work has been done in recent years to study students' formulations of formal limiting processes. One of the most common goals is to foster a productive understanding of the relationship between the error bound epsilon and the domain of the convergence; what is called a range-first perspective. My study examines an advanced calculus student's understanding of the relationships involved in convergence of functions, and how his prior experience with limits influenced his understanding. I unpack his cognitive organization of the dependence relationships between ϵ, N, and x in functional convergence. This case study demonstrates the effects of a persistent understanding that ϵ depend on N in the convergence of sequences.

Key words: Sequences of Functions, Formal Definition of Convergent Sequences, Advanced Calculus

Introduction

Mathematics students study convergent sequences throughout their college career. Formalizing sequential convergence provides an important yet difficult stage in the development of students' reasoning about advanced mathematical processes. A vital component of formal sequential convergence is the relationship between ϵ (*epsilon*) and the critical index N. I call this the *epsilon-N* dependence relationship, which mirrors the *epsilon-delta* dependence relationship within formal limits of functions.

In advanced analysis courses, sequences take on forms beyond just real numbers. In particular, advanced calculus students encounter sequences of continuous functions towards the end of their instruction. The convergence of such functions is defined in a manner that is structurally similar to the familiar convergence of real numbers, but it has the added complexity of also accounting for variation within the domain of the functions in the sequence. Thus, the relationships between ϵ, N, and x become vitally important to understanding the different types of convergence.

While conducting a larger study involving student understanding of metric spaces, I interviewed two students to examine their concept images and definitions (Tall & Vinner, 1981) pertaining to convergence of function sequences. A case study of one student's interview demonstrates how reversal of the ϵ-N relationship may persist beyond convergence of real numbers to affect the convergence of functions. In this preliminary report, I discuss this case to demonstrate the effect of this reversal on convergence of functions. I then go into detail about some cognitive conflicts that seemed to arise within the student's concept image. I seek to answer the following research questions: 1) How does the addition of the domain value x into the definition for functional convergence affect student understanding of convergence?, and 2) How does a student's understanding of real-number convergence affect their understanding of functional convergence?

Literature Review

There is an abundance of research on student initial understanding of limits, the potential difficulties that arise and initial interpretations of limit definitions (Bezuidenhout, 2001; Cornu, 1991; Cottrill et al., 1996, Davis & Vinner, 1986; Roh, 2008; Roh, 2009; Tall, 1992; Tall & Vinner, 1981; Williams, 1991). There have also been investigations into students' intuitions about the nature of limits (Oehrtman 2003; Oehrtman 2009). Many of these studies, however, pertain to students' intuition and reasoning with informal limits rather than formal understanding.

Targeted glimpses of student understanding of formal limit definitions began with Swinyard (2011). In his teaching experiment, two students successfully reinvented the formal definition of the limit of a function. This marked the first study in which students learned (reinvented) the formal definition of a limit as opposed to studies examining student post-instructional understanding. Swinyard's reinvention study resulted multiple useful constructs for characterizing students' perspectives of the formal limit structure. Swinyard and Larsen (2012) followed up by suggesting some strategies for fostering productive generation of formal limit definitions such as a focus on the error bounds. This particular strategy is termed a range-first perspective and contrasts an unproductive domain-first perspective. These perspectives will later be explored via theoretical framing. Formal limit reinvention was later adapted to sequences by Oehrtman, Swinyard and Martin (2014). Oehrtman et. al. also guided a pair of students through the reinvention of formal sequential convergence. Their findings refined Swinyard and Larsen's (2012) domain/range-first perspectives further, and also examined the students' progression of approximation-based reasoning in the limit reinvention. These studies utilized reinvention methodology to gain detailed glimpses into the understanding of formal limit definitions.

Other efforts have also been made to investigate student understanding of the formal definition of limits with special attention paid to the relationships between the variables controlling the limiting process (Adiredja, 2015; Adiredja, 2013; Adiredja & James 2013; Adiredja & James 2014; Roh, 2009; Roh & Lee 2016; Roh & Lee, 2011; Dawkins & Roh 2016). Specifically, Adiredja (2015) utilizes the Knowledge in Pieces (diSessa, 1993) framework to make a case for equitable cognitive research in the context of formal limit quantification. Adiredja offers a case study of a single female student (Amy) and outlines useful knowledge resources she leveraged to develop a productive understanding of the temporal $\epsilon - \delta$ order. One useful distinction that came out of this is attention to statements of the "error" between functions versus utilizing an "error bound".

Roh (2011; 2016) along with Lee and Dawkins have utilized multiple interventions to foster useful intuitions in analyzing the quantification of the limit definition. The "Mayan Activity" (2011) offers a narrative of continually refining approximations that fosters useful intuitions of the limiting process. Similarly, the "$\epsilon-$strip activity" utilizes geometric visualizations of the ϵ error bound in sequential convergence. Both interventions target quantification in the formal limit definitions, and highlight attention to the logical orderings of limits along the real line.

This study builds on the existing literature by venturing into student understanding of functional convergence. Specifically, point-wise convergent sequences of functions follow a structure identical to real number sequences, with the added complexity of accounting for variation in the domain of the functions in the sequence. In this report I model a single student's understanding of functional convergence, and demonstrate how constructs mentioned above, such as domain-first perspectives and the relationship between quantifiers, offer meaningful insights into his understanding. This study also offers an initial glimpse at a student's understanding of an advanced mathematical concept that builds on previously studied concepts.

Theoretical Perspectives

Concept Image and Concept Definition

As mathematicians, we classify mathematical objects and phenomena via formal definition according to specific identified properties. Understanding such ideas, however, entails more complicated cognitive structures built through various encounters with the mathematics. This calls for a distinction to be made between the collection of cognitive objects that are brought to bear upon recollection of a particular mathematical concept, and the words that compose the definition of the concept. This is the distinction between a thinker's concept image and concept definition (Tall & Vinner, 1981).

Tall and Vinner define the concept image as the "total cognitive structure that is associated with the concept". This includes examples, graphs, images, relationships, as well as the definition of the concept. The formal language that the student uses to identify the concept is termed the concept definition (p.2). The definitions students call upon may or may not agree with a definition proposed by the mathematical community, calling for further distinction. A student's concept definition is personal when it is the "form of words that the student uses for his own explanation of his concept image" (p.2). This might contrast the definition proposed by the mathematical community, which we call the formal concept definition (p. 2). It can be useful to note when a student's personal and formal concept definitions do not agree. In particular, differences between the two statements can highlight informal understandings that might dominate a student's understanding of a concept.

Different activities may require a student to access different parts of her concept image. Which parts are activated given certain promptings may highlight important features of the student's understanding. We therefore call the "portion of the concept image which is activated at a particular time" the evoked concept image (p. 2).

Given the complexity of students' cognitive organizations, it is clearly feasible for a student's concept image to contain elements that might be at odds with each other. It is reasonable to assume that such conflicting understandings can be held concurrently unless specifically highlighted. In particular, unless simultaneously evoked, there is potentially no reason for the student to see such an issue with holding both parts of her concept image. Such contradictory parts of a student's concept image are called "conflict factors" (p. 3). A conflict factor is "potential" if it is a part of the concept image at odds with another part, and is called "cognitive" if evoked to cause actual cognitive conflict (p. 3). Such a scenario may occur if both potential factors are prompted to be simultaneously evoked. Tall and Vinner caution that potential conflict factors "can seriously impede the learning of a formal theory, for they cannot become actual cognitive conflict factors unless the formal concept definition develops a concept image which can then yield a cognitive conflict" (p.4).

Domain-First and Range First Perspectives

Previous limit studies have revealed the particularly useful constructs of domain/range-first perspectives. These perspectives identify to where students assign control of the limiting process. Standard limit notation typically manifests $\lim_{n\to\infty} a_n$ or $a_n \to a$ to denote that a sequence a_n converges to a. This is often informally stated to include a clause describing the behavior of the index increasing to infinity. In accordance with this, there is potential for students transitioning from informal to formal limit studies to assign control of the convergence to the index behavior.

This is called a domain-first perspective. Swinyard and Larsen (2012) first defined this construct as an x-first perspective to describe student understanding of limits in calculus contexts.

In an x-first perspective, the variation in the domain values x controlled the behavior of the limit. This contrasts a range-first (y-first) perspective, where deviation from a limit candidate is measured and controlled on specific domain regions. Swinyard and Oehrtman (2014) suggest that focusing first on the behavior of the range values in limiting processes will foster useful interpretations of the formal definition. The next section describes in detail the relationship between the logical formulation of the formal definition and the underlying properties of the limiting processes.

Mathematical Discussion

Mathematical insights are useful theoretical considerations. By exploring formal constructions of the explored mathematics, we as mathematicians gain valuable perspectives on the overall structure of the mathematical activity with which the students engage.

The convergence of function sequences is an advanced concept that students examine towards the end of advanced calculus instruction. There are two such types of convergence, point-wise and uniform. We will primarily engage with a student's understanding of point-wise convergence.

Sequential convergence in real analysis contexts follows a standard approximation scheme. Such a scheme is typically first introduced through convergence of real numbers. We will thus begin this discussion by examining real number convergence; a familiar context within the literature on limits. A sequence of real numbers (a_n) converges to a real number a if $\forall\epsilon>0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|a_n - a| < \epsilon$. Attention to the logical structure of this definition reveals the progression of activity used to verify convergence of such sequences. An important consideration to make when reading these logical definitions is understanding what is involved in verifying that a sequence meets its requirements. To do this, note that the "for all" symbol $\forall$ at the onset of the definitions implies that verification begins by picking a specific element from whatever set follows $\forall$. In this case, a positive real number ϵ. Similarly, the "there exists" symbol $\exists$ signifies that some element of the set in question must be found that satisfies whatever conditions the definition sets. In this case, a natural number N that acts as a particular index value for the sequence.

Thus, we begin with a positive real number ϵ. ϵ acts as a bound on the error between the sequence entries a_n and the real number a to which the sequence converges. Thus, if for every possible positive error bound ϵ, we find some index N such that past N all sequence values differ from a by less than ϵ, then we say the sequence converges to a. This highlights that the critical index N, necessary to bound the tail of the sequence, depends on the particular error bound ϵ chosen. This is succinctly explained by saying the critical index N depends on the error bound ϵ.

Convergence of functions builds on this approximation scheme while accounting for variation of the functions across their domain. This adds to the complexity of the convergence structure, and also provides the primary difference between point-wise and uniform convergence. A sequence of functions(f_n) converges point-wise to a function f on a domain D if $\forall\epsilon>0$ and $\forall x \in D$ $\exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$. Once again, attention to the logical structure gives insight into the behavior of such sequences. In particular, we reduce the convergence to that of real numbers by letting the domain values vary before verifying the accuracy of the sequence entries. Notice that once a domain value x is specified, $f_n(x)$ and $f(x)$ are real numbers. Thus, this sequence of functions converges if given a specific pair of values (ϵ, x), representing a fixed

Figure 1: Visualization of the sequence x^n

error-bound and domain value respectively, a critical index N may be found so that past N the difference between the real numbers $f_n(x)$ and $f(x)$ is smaller than the error. This implies that the sequence of functions converge on a domain D if evaluation across the domain creates convergent sequences of real numbers.

Consider the prototypical example of a point-wise convergent sequence of functions defined on the domain $[0, 1]$, $f_n(x) = x^n$. A visualization of sequence values is given in Figure 1.

Notice that by selecting $x = .4$ results a decreasing sequence of real numbers $.4^n \to 0$. This is similarly true for any value $x = 0 \leq \alpha < 1$. For $x = 1$, note that $1^n = 1$ and so we have a constant sequence converging to 1. Thus, if we define f to be the piece-wise function $f(x) = 0$ for $x \in [0, 1)$ and $f(x) = 1$ for $x = 1$, we have that $f_n(x) = x^n$ converges to $f(x)$ point-wise. A rigorous proof of this verifies that for each $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for $n \geq N$, $\alpha^n \leq \alpha^N < \epsilon$.

Finally, we have a sequence of functions converges uniformly if variation across the domain does not affect the approximation scheme. Formally, a sequence of functions $\{f_n\}$ converges uniformly on a domain D to a function f if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \epsilon \; \forall x \in D$. A potential point of confusion is that both point-wise and uniform convergence of functions include the $\forall x$ statement. Attention to the logical order in the statements clarifies the distinction. In uniform convergence, the $\forall x$ occurring at the end signifies that once the error bound ϵ is chosen and the critical index N is found, the resulting bounding condition holds across all domain values. In other words, all sequences of real numbers created by evaluation across the domain satisfy the same bound by the error ϵ.

This definition is equivalent to the definition of convergence in the metric space of continuous functions when D is compact. Here, a sequence of continuous functions converges to a continuous function f if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\sup\limits_{x \in D} |f_n(x) - f(x)| < \epsilon$. The structure of this formulation is identical to convergence of real numbers, where the objects of convergence and metric function have changed. This follows because convergence in metric spaces have the same definition despite the space and metric specified.

Methods

The episode reported on was part of a larger study exploring students' reasoning about and generalization in various metric spaces. This involved conducting teaching experiments (Steffe & Thompson, 2000) with two students, each teaching experiment consisting of six hour-long sessions with a single student.

The participant reported on was a junior mathematics and engineering major named Kyle (pseudonym). Kyle had taken the introductory advanced calculus sequence at a large mid-western university and was currently taking a course on advanced calculus in vector spaces.

In accordance with teaching experiment methodologies (Steffe & Thompson, 2000), the students were prompted to engage in mathematical activity to facilitate the exploration of novel concepts in metric spaces. During the sessions, their activity was observed and there were frequent discussions between the researcher and student wherein the student described their understanding of the specific mathematics. The sessions relevant to this report examined Kyle's understanding of functional convergence. The purpose of this examination was to motivate the generation of a metric on the space of continuous functions through manipulation of the uniform convergence structure. By observing Kyle's exploration of function convergence, I built a detailed model of his concept image.

Each interview was video recorded. The retrospective analysis (Steffe & Thompson, 2000) consisted of reviewing the video records for statements of Kyle's concept definitions as well as statements and activity that suggested Kyle's evoked concept images. Special attention was paid to Kyle's activity, definitions, statements, and examples so that a model of Kyle's concept image could be inferred. Within this model, I analyzed Kyle's potential and cognitive conflict factors so that I could identify factors contributing to his eventual shift in understanding.

Results

The goal of this interview was to explore Kyle's concept images and definitions pertaining to different types of convergent function sequences. We began by establishing Kyle's concept definition for point-wise convergence of functions. This type of convergence would have been Kyle's first exposure to functional convergence of any kind in the classroom. Kyle's initial concept definition is the first of many statements he made suggesting he had adopted a domain-first perspective. Kyle wrote on the board "A sequence of functions $\{f_n\}$ converges pointwise to f if $\forall x \in D$, $\lim_{n\to\infty} f_n(x) = f(x)$". I call this Kyle's personal concept definition, as the formal concept definition involves formalization of the limit $\lim_{n\to\infty} f_n(x) = f(x)$ as demonstrated above. Note that Kyle's statement, while technically informal, is still a correct characterization of point-wise convergence. Along with the evidence given below, this statement suggests that Kyle had adopted a domain-first perspectives of real-number convergence which he now applies to point-wise convergence. In fact, Kyle unpacks this definition by saying:

Kyle: When you're talking about point-wise convergence, depending on your x your sequence of functions may converge to a different limit. So we have to say that for all x in D we have to choose our x first, because once we run through the limit, that x is going to change what our limit function is.

This is in fact a productive distinction to make with point-wise convergence. He then continues with:

Kyle: So basically ... in the ϵ definition our ϵ ends up being dependent on x for point-wise convergence.

This statement of dependence is Kyle's first evoking of a potential conflict factor, as the logical structure of the "ϵ definition" suggests an independent relationship between ϵ and x. I then prompted him to write out the ϵ definition (formal concept definition) and he produced the correct formulation given in the Mathematical Discussion.

When asked to discuss what the definition means, he said:

Kyle: When we are trying to show point-wise convergence we are trying to find some ϵ that will always be greater than the difference between our sequence and our function, however because we are selecting our ϵ and then selecting our x then this ϵ will actually be a function of x and the N from the naturals.

As he said this he wrote out $\epsilon(x, N)$. As we began to unpack the nature of the dependence of ϵ on N and x, he made the following statement:

Kyle: N will determine what your epsilon ends up being. 'Cause N plays the same role that it always plays when we're talking about convergence. It's the last index such that every index past this you're going to have this property [$|f_n(x) - f(x)|$] be true, so in so much as that ϵ is pretty much always dependent on that natural because you want to say that past this natural N, you know this ϵ is always going to be true.

The potential conflict comes from his understanding of universal quantification and basic proof structures. For instance, he understood that $\forall\epsilon$ and $\forall x$ meant to "pick an ϵ" and "pick an x". In fact, there were multiple moments in describing his proof strategies where the conflict factors were almost simultaneously evoked. I will later describe the task that prompted these potential conflict factors to become cognitive.

As we continued discussing the nature of the ϵ dependence on N and x, we discussed some typical examples of functions that converge in a point-wise manner such as x^n, x/n, and even sequences constant in x (or rather standard real-number sequences). In each of the following examples, Kyle gave evidence that he had adopted a domain-first perspective.

For example, we first examined the sequence $1 + 1/n$ which converges to 1. Kyle justified the convergence of this sequence by saying:

Kyle: As n goes to ∞ it's going to converge to 1 ... so let n increase without bound and as you look at $[1/n]$ as n gets closer and closer to ∞ and $[1/n]$ will just go to 0 and you'll be left with 1.

Further, when examining x^n, Kyle again productively assessed that evaluation of the sequence at different x values would yield a different "limit".

Kyle: Let $x = 1$ to begin with. Then as n increases without bound, your x^n is going to equal 1 ... If you look at x between 0 and 1, then your limit as n increases without bound of x^n is always 0.

Notice that throughout these examples, Kyle is justifying convergence of the limits based on the behavior of the index (domain) values n. This further suggests that Kyle had adopted a domain-first perspective. The nature of his perspective fully came to bear when cognitive conflict resulted from evaluating x^n at $x = 1/2$. After isolating this x value, I asked Kyle to prove for me why $1/2^n$ converged to 0 using ϵ. After writing the contents of Figure 2 on the board, he said the following:

Kyle: The thing I'm confused about here is we need $[.5^n]$ to be less than ϵ, ϵ is any number greater than 0, so we need $[.5^n]$ to converge towards 0, but it doesn't converge for a finite n ... we want n to increase without bound ... Without thinking of things as limiting functions I don't know what we're trying to do. I'm so used to letting n increase without bound, well clearly if n increases without bound that's gonna be less than ϵ ... because it's getting arbitrarily small.

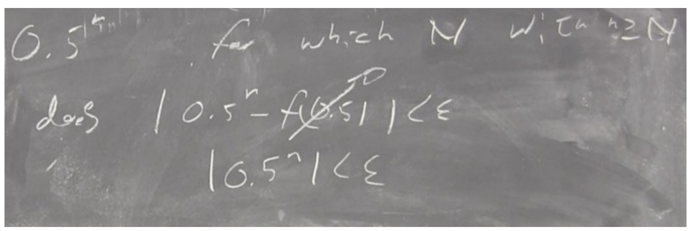

Figure 2: Kyle's initial work with x^n at $x = 1/2$

The above suggests that Kyle's evoked concept image of real number convergence involved a domain-first perspective, wherein the increasing index value controlled the behavior of the sequence. What is intriguing is how Kyle accounted for the added complexity of incorporating the domain of the function as well. As a first example of Kyle's resulting concept image, we examine Kyle's description the following *x*-*N*-ϵ relationship for the sequence of functions x/n:

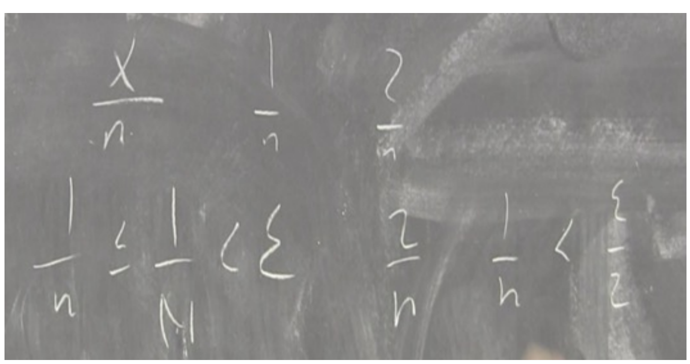

Figure 3: Kyle's work with x/n

Kyle: If you are thinking about $1/n$ as being less than ϵ, then that means $2/n$ would imply that $1/n$ is less than $\epsilon/2$. So if you're increasing the value of these $x's$, your epsilon is going to get smaller and smaller.

Consistent with his understanding that ϵ was a function of x and n, Kyle's concept image included an ϵ that was small enough to trap the arbitrarily increasing index n. Indeed, if (as above) ϵ satisfies the convergence of $1/n$, then $\epsilon/2$ would satisfy the convergence of $2/n$, and similarly $\epsilon/3$ would satisfy $3/n$ and so on. This highlights the relationship between x and ϵ, namely that variation in x results variation in the ϵ necessary to trap the sequence as n increases arbitrarily. The following demonstrates Kyle's geometric interpretation of this phenomena:

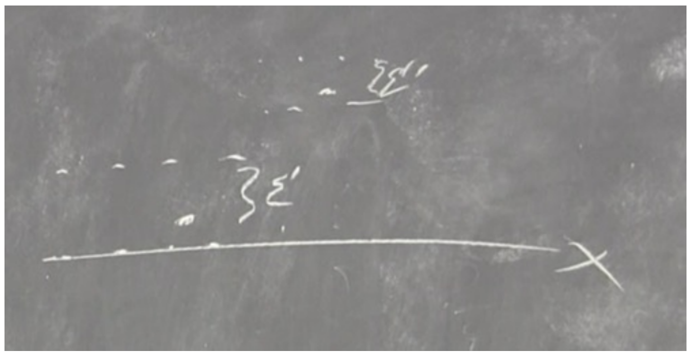

Figure 4: Changing ϵ windows

Kyle: If you have some point, and you pick some x, it might have some window that's ... we'll call it ϵ'. But if you pick another x it's possible that it has maybe a smaller window that's just like ϵ''.

Kyle's drawing of the ϵ windows demonstrates that Kyle had constructed a mathematical phenomenon that was logically consistent with his understanding of the $\epsilon - x - N$ relationship. He made use of the convention that variation in x potentially changes the limit of the resulting real-number sequence to formulate a consistent mathematical limiting process. Surely if real number convergence entails finding an ϵ window to trap a sequence of arbitrarily increasing index, then dependence of the sequence values on x implies that the ϵ window may vary. Kyle's understanding that variation in the quantifiers affects the limiting process contributed to his successful accommodation upon examining cognitive conflict factors.

Recall that evaluation of x^n at $x = 1/2$ was a cognitive conflict factor for Kyle.

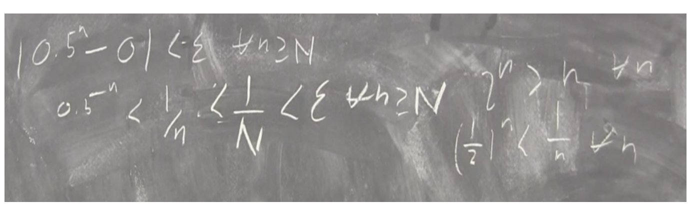

Figure 5: Kyle's work on $1/2^n$

When he could not verify the convergence in the manner he wanted, I suggested using the fact that 2^n is greater than n for all n, and so then he tried bounding $1/2^n$ by $1/n$ which could be bounded by ϵ. Thus, if $1/n$ converges to 0 then $1/2^n$ converges to 0. In showing that $1/n$ converges to 0, Kyle attempted to call upon the Archimedian property. It was through examination of the Archimedian property that Kyle resolved his cognitive conflict. I prompted Kyle to examine the structure of the Archimedian property, and he realized that for a particular number ϵ, we can find an N such that $1/N<\epsilon$. This then allowed him to complete the proof. Examination of the other examples in light of the Archimedian property revealed to Kyle the actual relationship between x, N, and ϵ. Kyle concluded by realizing that "N is doing most of the work", and re-stating the dependence to agree with the logical structure of the formal limit definition.

By resolving that the index N is found for a fixed ϵ, he immediately restructured his understanding of not only real number convergence, but also for all of his previous dependence claims. This understanding resulted in Kyle adopting a range-first perspective for real number convergence, which then transformed his understanding of point-wise convergence of functions. Kyle demonstrated a robust understanding of how variation in x affected the convergence of the function sequence. It was this understanding that was ultimately leveraged to result a fluency with the structure of point-wise convergence.

Discussion and Further Directions

This episode provides an example of the complexity that accompanies understanding advanced formal mathematical concepts. Point-wise convergence of functions utilizes the same logical structure as convergence of real numbers, with the addition of accounting for variation across the domain of the functions in the sequence. Kyle had clearly built a cognitive structure that incorporated this complexity into his existing understanding of convergence. Note that Kyle's mathematical system was logically consistent with his understanding of the fundamental structure of sequential convergence. This demonstrates Kyle's mathematical maturity, and his ability to adapt a productive understanding into his existing cognitive framework. From this case study we see that students' can develop complex and consistent mathematical systems even based on "incorrect" understandings. This conclusion is made to the credit of students' mathematical abilities, and is consistent with the idea that mathematics developed by students can be the result of productive thought despite deviation from normative understanding. At the same time, this study warrants the caution that proficiency with advanced mathematical concepts does not imply fully correct understanding of the ideas upon which they are built. This is in line with Tall and Vinner's (1981) warnings about potential cognitive factors.

Exploring the model of Kyle's concept image for functional convergence reveals a natural relationship between formal and informal knowledge when studying advanced mathematical processes. Kyle could indeed produce the formal concept definition of functional convergence, and yet his understanding of the mechanism for convergence was dominated by an informal domain-first perspective. While Kyle's concept image contained a robust understanding of how to ascertain the limit functions for point-wise convergent sequences, his understanding was primarily informal. This opens up a greater inquiry into the interplay between students' leveraging of formal and informal knowledge when exploring advanced and formal concepts. It is natural for this phenomena to occur in a real analysis setting, as students build informal understandings of the material in calculus courses. This raises an interesting question, namely when do students call upon their formal and informal knowledge when exploring advanced formal mathematics? In this particular episode, Kyle mainly attended to formality when prompted to prove claims of convergence. The logical structure of the formal definition was not as meaningful to Kyle as his informal knowledge until he was required to attend to formalism through proof. Further investigation can explore advanced mathematics students' attention to formal and informal knowledge when working in formal mathematical settings.

This episode also demonstrates the importance of logical quantification in understanding formal limit definitions, and the persistent effects of a domain-first perspective. It was evident that Kyle's domain-first perspective influenced his interpretation of the $\epsilon - N$ relationship for real number convergence. This was then compounded with the added complexity of domain variation in the functions. Thus, Kyle's interpretation of the logical structure of point-wise convergence was heavily influenced by his understanding of real number convergence. This highlights the importance of attending to logical quantification early in formal mathematics. While Kyle understood the meaning of the quantifiers and could correctly apply them in a proof scheme, the implications of such meanings came second to his informal understanding. As mathematicians, we learn to communicate greater meaning through formal symbolism and logical structure. Attending to the greater meaning of such symbolism early on may facilitate productive understanding of formal mathematics. In light of the earlier discussion of formal versus informal knowledge, further studies can investigate student's informal understandings and uses of logical quantifiers in real analytic statements.

This case study highlights the complex relationships that exist in understandings of formal and advanced mathematical concepts. By investigating point-wise convergence of functions, we were able to observe the progression of theoretical constructs developed for convergence of real numbers, and explore the nuances of a single student's understanding.

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Hypophora: Why Take the Derivative? (no pause) Because it is the Rate

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Abstract: *Part of a larger study of the development of teaching among novice college mathematics instructors, this report focuses on one participant, Disha, and her use of a questioning technique called hypophora. At the beginning of the observations, 25% of her questions were hypophora. After video-case based activities during weekly coordination meetings, her use of hypophora decreased to about 10% of questions. Although Disha rejected the idea that her teaching had changed in any way, she acknowledged that she began "breaking things into smaller pieces" to help students understand.*

Keywords: questions, calculus, hypophora, professional development

In college mathematics, from gateway classes for future teachers to advanced courses for future engineers, instructors learn about teaching almost entirely by trial and error (Kung & Speer, 2009). The result is reflected in pass rates below 50%, particularly in calculus and its prerequisite courses (Hastings, Gordon, Gordon, & Narayan, 2006; Herriott & Dunbar, 2009).

Bressoud, Carlson, Mesa, and Rasmussen (2013) report that among students who enroll in college calculus ready for the course (i.e., meeting pre-requisites and placement requirements), at least 28% fail it. If we consider the fail rate reported by Bressoud and colleagues, approximately 85,000 students will fail Calculus I each fall semester. In response to course surveys, students reported that the teaching of Calculus I was "ineffective and uninspiring… 'over-stuffed' with content and delivered at too fast a pace, assessments were poorly aligned with what was taught and the instructor lacked connection to students and the course" (Bressoud et al., 2013, p. 10). The national problem driving the research presented here is the ill-spent time, effort, and money resources when so many students arrive at college ready for calculus and fail the class.

Research has found that when teachers have a better understanding of student thinking, it improves teaching (Ball, 1997; Carpenter & Fennema, 1992; Fennema et al., 1996). By learning how to ask students questions, an instructor can open up a dialog with students and learn about the student thinking in the room. The collegiate mathematics education literature points to a need for insight into how graduate student Teaching Assistants (TAs) learn about student thinking (Speer & King, 2009). The research needed includes exploration of how novice instructors learn from their own practice and through professional development (Speer & Hald, 2008). Included in the field's identification of needed work is a call for research on the instructional practices that support learning to learn from the teaching process itself, such as the in-class use of questions (Deshler, Hauk, & Speer, 2015).

This qualitative and descriptive statistical study focused on one possible way to improve college mathematics instruction: support instructors to understand what their students are thinking. I investigated how course coordination that included video case based activities might facilitate reflection on and asking of questions in the classroom. The research questions were:

1. What is the nature of questions and change in question strategies within a semester during classroom discourse by the instructors?
2. How does video case based professional development shape the perceptions and intentions about the role of questions in teaching held by TAs?

Theoretical Framework and Researcher Stance

The conceptual framework for the work is built on the foundation of pedagogical content knowledge. Pedagogical content knowledge (PCK) is the collection of knowledge instructors have about the discipline-specific challenges students encounter, strategies for helping students, ways to listen to identify learners' thinking processes, and skills for regulating practice (Ball & Bass, 2000; Shulman, 1986). Novice college mathematics instructors acquire PCK in many ways, such as grading, examining their own learning, observing and interacting with students, reflecting on and discussing practice (Kung, 2010; Kung & Speer, 2009; Speer & Wagner, 2009). College mathematics PCK includes knowledge about formal and informal mathematical discourse, including teachers' anticipations regarding their adult students' thinking and how to turn teacher intentions into actions (Hauk, Toney, Jackson, Nair, & Tsay, 2013). These ideas were operationalized in this study by a focus on seeking and responding to student thinking through questions.

The emerging consensus in faculty development is that it is clinical work: instructors must evaluate, diagnose, and prescribe, while also developing their practice (Hinds, 2002; Persellin & Goodrick, 2012). Great success in preparing clinicians in medicine, psychology, law, and education has come through case- or story-based study (Boud & Feletti, 1997). Improving college mathematics teaching can productively start with ways to build instructional self-awareness through opportunities to compare and contrast to other people in a variety of contexts (Mason, 2010). This method has been making its way into college instructor preparation through case-based materials (Friedberg et al., 2001; 2011; Hauk, Speer, Kung, & Tsay, 2010; Hauk et al., 2013).

My epistemological stance was constructivist, both social and radical. I held to the belief that people construct their own knowledge and that knowledge is influenced by their environment. This knowledge can be a shared knowledge within a group, as in social constructivism, or the knowledge can be individually constructed, as in radical constructivism (Schunk, 2004).

Methods

Participants taught Calculus for Biological Scientists at a doctoral granting public university, referred to here as BRU. There were five participants in the larger study. All five were novice instructors. Four were graduate TAs and one was a recently graduated PhD student. I refer to those who are instructor-of-record (both TAs and other non-graduate student instructors) as "instructors." Only one of the instructors, Disha, will be reported on in this paper.

Each instructor was observed and video recorded six times during the semester. Observations focused on the type of question asked, the depth of the question, and the discourse surrounding those questions. Instructors were interviewed prior to any observations, again after two observations, and had a final interview after final exams. Instructors participated in weekly video case based course coordination meetings which focused on student thinking about mathematics. Each video case based session included activities and prompts for discussion among the instructors.

Starting with a framework I developed with colleagues (Roach, et al., 2010), four of the six original observations were coded in depth. These classes were viewed and partially transcribed as needed to provide thick, rich, descriptive detail (Merriam, 1998). After two rounds of observation I determined that an additional question category, hypophora, was needed to address a technique used by the instructors in the study. A hypophora is a question that speakers pose and then immediately answers themselves. Table 1 defines the question categories of Roach et al. (2010) and includes the additional category, hypophora, used in the coding of the observations. Table 2 illustrates the depth of the questions as defined by Roach et al. (2010).

Table 1: Question Categories

Category	Definition
Comprehension Check	To assess one or more students' declarative understanding of a topic, procedure or task (e.g., What should we do next?, Does that make sense?)
Content Check	Used to push the mathematical focus or direction of the students' attention (e.g., Should we try the chain rule?)
Elicit Student Thinking	To draw out what the students were thinking, including prompts for students to communicate their *what* they thought to other students or teacher (e.g., What do you first notice about this graph?)
Probe Student Thinking	Investigate reasoning behind or explanation for a given response or procedural work, including prompts to communicate *why* a person or group thought what they did (e.g., That's correct, but why?)
Hypophora	**A question that speakers pose and then immediately answer themselves (e.g., Why do we want to take the derivative? Because it is the rate.)**

Table 2: Relationships Among Categories and Depth of Questions

	Comprehension Check	Content Check	Elicit Thinking	Probe Thinking
Depth 0	Calls for memorization or recall			
Depth 1	Goal is procedural, without connection to concepts			
Depth 2	Purpose is connection between solution and reason/sense-making			
Depth 3	Target is "doing math": create, synthesize, make and justify conjectures			

Disha

At the time of the study, Disha was in her fifth semester of teaching (one class per semester). Disha was a doctoral student in mathematics. She grew up and went to school in India. She completed her undergraduate degree in mathematics from a major university in India. Disha saw the instructor's responsibility as presenting knowledge to students and the students' responsibility was taking that knowledge and making sense of it on their own.

In the semester of the study, Disha taught one section of Calculus for Biological Scientists with 30 students enrolled. Of these, 20 students (67%) regularly attended class meetings. Disha relied on lecture throughout the observed lessons. She indicated that she did not like group work because it seemed to her that inevitably "one person will end up doing all the work." She believed that students learn mathematics best by working individually.

Disha regularly spoke up in coordination meetings and stated in informal conversation with me, that she enjoyed coordination meetings. Her weekly logs indicated that she often used ideas from coordination with individual students, small groups of students, or in the classroom. Working with individual students, or small groups of students occurred during her office hours. She also indicated she felt that the ideas presented in coordination *sometimes* increased student confidence in mathematics, led to a deeper understanding of the mathematics, and helped increase student interest in mathematics.

Results

Disha's most common question was a "do you understand" type of question. Her next most common question type was hypophora. Across the four focus classes, Disha posed an average of 128 questions per 50-minute class period. Most of these (74%) were Comprehension Checks, the most common two questions being "Is that ok?" and "Do you understand what I am saying?"

Table 3 shows Disha relied primarily on Comprehension Check questions during the observed lessons. Disha asked few Content Check and Elicit Thinking questions and rarely asked Probe Thinking questions.

Table 3*: Disha's Question Category Percentage Per Class*

	Comprehension Check	Content Check	Elicit Thinking	Probe Thinking	Hypophora	Classroom Management
Obs A	67%	3%	5%	0%	25%	0%
Obs B	75%	0%	1%	0%	11%	12%
Obs C	76%	3%	7%	1%	12%	1%
Obs D	77%	6%	0%	0%	10%	5%
%Total[a]	74%	3%	4%	0%	14%	5%

[a] Due to rounding, rows may not add to exactly 100%.

However, the per-class distributions of these and other types of questions varied over time (see Figure1).

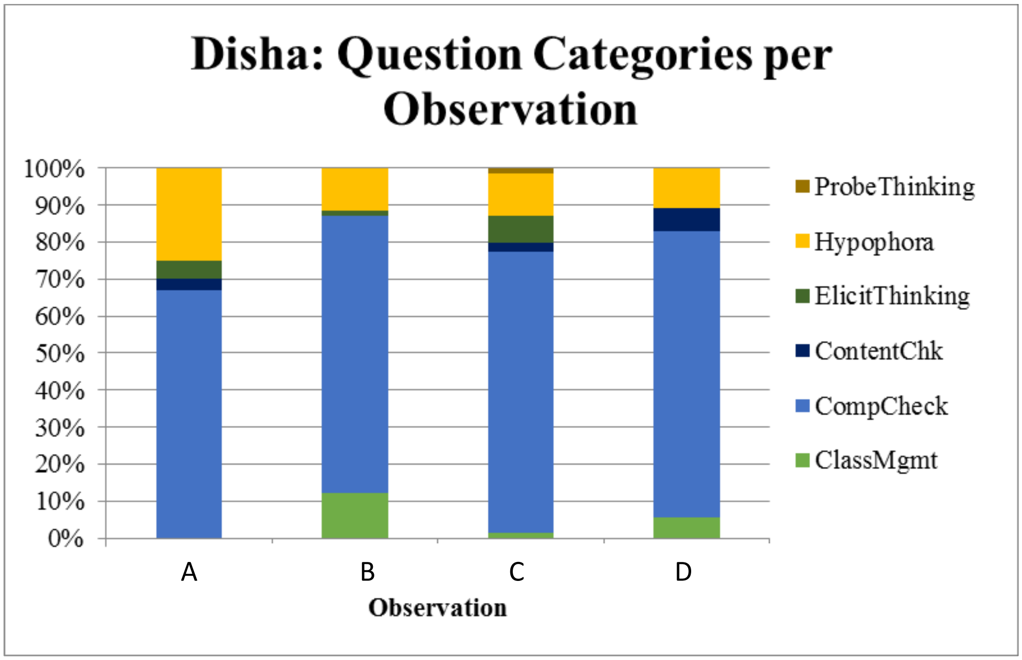

Figure 1. Relative frequencies of question codes in observations, for Disha.

During Observation C, Disha had the greatest number of Elicit Thinking questions such as, "What would you think differential equations are?" and she asked Probe Thinking questions such as "Why not?" after students responded "No" to "Will this represent the given situation?" A change in hypophora was notable across the study. In the first observation, she used the greatest number of hypophora (25% of questions). During subsequent observations, it was less likely for Disha to answer the questions she posed. Instead, she waited for students to answer. In at least one situation she asked a question and stepped away from the board and waited 30 seconds for students to respond. In Interview 3, I asked about her choice of which questions to use and why wait longer for answers on some. Disha said, "those were the questions I thought of when I [as a student] learned the material and I thought the students should think about those questions as well. I thought it would help them learn the material better." I did not gather information about how each of the instructors learned specific calculus concepts themselves. In this particular case, the questions Disha used were those she had when she was a student. I return to this, below, in discussion of what Disha valued as a "good" question.

Disha most frequently asked, "Does that make sense?" "Is that ok?" or "Do you see what I am trying to say?" The next most common questions were depth 1 Comprehension Checks and hypophoras. Below is an example of her use of hypophora, with the hypophora underlined:

> *Disha*: "<u>What do we have in stability criteria</u>? [no pause] We start with *dt*/*ds* ["dee tee dee ess"], ok. We start with *dt*/*ds*, I'm not going to do a bunch of examples here, ok. Now if *m* is the measurement, then this is how the *dt*/*ds* is represented by [pointing to "*f*(*m*)" written on the board]. Right? [no pause] then we will figure out equilibrium point. <u>Why</u>? [no pause] Because we are trying to find the stability of the equilibrium points."

This kind of reflective exchange was common for Disha during her first observation. It is important to note that Disha did not pause after asking a question and immediately continued with the answer to the question she posed. It is not clear that much cognitive demand is made of students during a chunk of hypophoric lecture. It appeared to have the same effect as a statement on the students. While it was unusual for students to attempt to answer these questions, I did observe at least one instance in which a female student attempted to answer a hypophoric question posed by Disha. Disha did not acknowledge the student in any visible way and continued with her own answer to the question.

After video case based coordination Disha began to wait after asking questions. During Observation C, Disha wrote the problem $\int 2xe^{x^2}dx$ on the board. She then asked the students what rules they had learned so far that might help with this problem. The students suggested the power rule and quickly realized that would not work because it was "weird looking." Disha continued with the following.

> *Disha*: So we cannot apply the power rule when we have the weird looking function over there. So what else can we apply? What else do we know?
> *Student*: Integral of e to the x is e to the x.
> *Disha*: [nodding] Integral of e to the x or antiderivative of e to the x is e to the x. [Looks around the room (4 seconds)] Right? (no pause) Can we make use of this rule over there? [pointing to the original problem. She waits four seconds then a student responds.]
> *Student*: No.
> *Disha*: No, Why not?
> *Student*: Because that's e to the x squared.
> *Disha*: [nodding] Because this is e to the x squared. Which function when differentiated would give me e to the x squared? [10 second pause while looking around the room] We don't know that do we? We can't make a guess out of that. Can we make a guess? [Students are quiet and then start to shake their heads no.]

After this exchange, Disha smiled and continued, reviewing all the rules for antiderivatives they had learned until this point in the semester. She pointed out that none of the current rules they had would work for this problem. She then stated, "How do we solve this? We are going to solve this with a slick trick." Then smiled, and began to discuss *u*-substitution.

This interaction was a noteworthy change for Disha. In the first observation, if Disha asked a "why" question, she would answer it herself. However in later observations, after video case based coordination, Disha waited after asking questions and the students would often respond verbally.

Discussion

The most notable change in Disha's questioning techniques was a change in her use of Hypophora, 25 (25%) in her first observation (before video case activities) and 12 (11%) in her final observation (after four video cases). Disha expressed during one of her interviews that she was impressed with the "wait time" of one of the instructors in the second video case shown during coordination (*Office Hours*). She related this "wait time" to "breaking things into smaller pieces" or scaffolding the information. After this case, Disha, declared her intention to give more time for students to answer questions. I also saw Disha giving the students more time to answer questions. In Observation A, if Disha asked a question that could be considered an Elicit Thinking or Probing question, she would immediately answer, making the question a Hypophora. However, after course coordination efforts, she gave time for students to answer questions. She stated in her final interview how impressed she was with the instructor observed in the *Office Hour* video case and that she used his methods of "breaking things into smaller pieces" and waiting for an answer, while working with students during office hours. At the same time, she rejected the idea that this video case changed her classroom teaching in any way.

Kung (2010) observed that one way TAs learned about student thinking was through interacting with students watching them work problems and listening to them discuss mathematical content, as one would during office hours. It is possible that Disha gained an understanding of student thinking while in office hours that translated to her classroom instruction. The influence of the video case may have been indirect: as a moderator of her perception of her own office hour experiences, which were in turn a moderator of her classroom practice. Disha also spent more time exploring incorrect answers with students and, in observations after video case activity began, asked questions of a greater depth. By exploring incorrect answers and asking deeper questions, it is likely that Disha was gaining further insight into student thinking (Ball, 1997; Carpenter & Fennema, 1992; Fennema et al., 1996).

Disha did not appear to be aware of an increase in wait time after asking questions during classroom instruction. One possibility is that Disha was creating "think time" not "wait time." She could have been leaving time after asking questions for the students to *think* about the question and organize their thoughts, not *waiting* for them to answer. It is possible that her intent was to answer the questions herself, but the students answered them while she was allowing time for them to think about the mathematics. If Disha was simply giving the students time to think about a question and not waiting for an answer, this could explain why Disha did not believe she had changed her teaching in any way. In a review of the literature, there did not seem to be any distinction between "wait time" and "think time." Further research could address the differences of "think time" versus "wait time" and student perceptions of these constructs.

Emerging from the qualitative analysis of Disha's questions was what Disha valued as a "good" question. She related "good" questions as questions that would have occurred to her. In one observation, after a student asked about why a certain *u* was chosen for a *u*-substitution, Disha responded with "That is a good question because that is a question that I would have thought of." In my second interview with Disha she said "I chose to ask those questions because those are the questions that I thought of when I was learning the topic." This data suggests that what Disha valued when asking and answering questions, were questions that aligned with her own way of thinking. Data was not gathered on types of questions the instructor were asked when they were students learning calculus. Future studies could examine the teaching styles of mathematics professors and how closely the professors' questioning techniques align with the instructors' questioning techniques.

Student perceptions of questions were not examined in this study. Future studies could examine the relationship of the types of questions instructors ask to the types of questions students ask--from the start of a semester until the end of the semester. Interviewing both the students and instructors about the questions asked could provide valuable feedback about student thinking and instructor response to student thinking. This would further build on Speer's (2001) work by investigating not only the instructor reason for asking questions, but also the students' reasons for asking, answering, and not answering questions as well as student thoughts about questions. Such a study would possibly include video-clip based student focus group interviews--where students watch and discuss a question-driven interaction--at least twice during a semester and a comparison of the students' perception of questions to the instructor's perception of questions. Learning more about how students think about questions could aid in identifying what types of questions can contribute to student learning. This, in turn, could shape the development of new video cases that focus on questioning.

Future studies could also compare the nature of student responses on a particular task to examine the correlation between instructional style and the nature of student understanding of a particular concept. This could be done for many concepts and one could compare the nature of student understanding from concept to concept. Such a study could inform researchers about instructional styles and how they many contribute to student learning.

Conclusion

Many questions remain. Are hypophora questions? Are hypophora useful to classroom instruction? Could hypophora be an important phase in teaching development for those who for whom English is a second language?

Hypophora seem to be an illustration of an internal thought process of the instructor. During one of the weekly coordination meetings, Disha stated "those are the questions [the students] should be thinking" when discussing questions asked by the instructors in the video case. Disha seemed to be talking about an internal dialogue that "should" be happening with students. However, while hypophora might be helpful to the instructor in organizing information, I do not believe that hypophora allow the students the time needed to process and organize thoughts. By offering the students time to think and process questions, they can construct meaning of the mathematics. More research is needed to examine the effect hypophora have on students and how it may or may not be helpful to students.

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Students' Conceptions of Mappings in Abstract Algebra

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In an effort to understand ways students approach constructing homomorphisms and isomorphisms between groups, six undergraduate math and engineering students in a lecture-based introductory abstract algebra course were interviewed. These students experienced varied success in creating isomorphisms and homomorphisms, which allowed both successful techniques for map creation and stumbling blocks to map creation to emerge from the data. Additionally, a genetic decomposition for homomorphism is outlined and students' mental constructions of both homomorphism and isomorphism are discussed. Finally, students' conceptual metaphors for homomorphic and isomorphic mappings are examined. Combining these three analyses paints a picture of the interaction between students' knowledge of properties of groups and mappings and their flexibility in creating mental images while interpreting those properties.

Key words: Abstract Algebra, Homomorphism, Isomorphism, APOS, Conceptual Metaphor

Purpose and Background

Experts have identified isomorphism and homomorphism as two of the most central topics to abstract algebra (Melhuish, 2015). Although some research has been done on how students approach isomorphism, including designing an inquiry-oriented curriculum that addresses isomorphism (Larsen, Johnson, & Bartlo, 2013), research explicitly on students' understanding of homomorphism has been scarce. Thus, a purpose of this study is to examine students' approaches to finding both isomorphisms and homomorphisms and what prevents students from finding appropriate mappings between groups.

A homomorphism between groups is defined as follows: "Let $(G,\star)$ and $(H,\boxdot)$ be groups. A map $\varphi\colon G \to H$ such that $\varphi(x \star y) = \varphi(x) \boxdot \varphi(y)$ for all $x, y \in G$ is called a homomorphism" (Dummit & Foote, 2004). Thus a homomorphism is a map that preserves the structure of the original group in the domain of the second group. It does not require the groups to have the same cardinality; group G may be larger or smaller than group H. There is always at least one homomorphism between groups; namely, the trivial homomorphism, in which every element of G is mapped to the identity in H. Further, an isomorphism between groups is defined as follows: "The map $\varphi\colon G \to H$ is called an isomorphism and G and H are said to be isomorphic or of the same isomorphism type, written $G \cong H$, if φ is a homomorphism, and φ is a bijection" (Dummit & Foote, 2004). Thus isomorphisms are a specific type of homomorphism in which the cardinalities of both groups are the same and all elements of the first group are mapped to distinct elements of the second group. For example, V, the Klein four-group, is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. However, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$ because the homomorphism property is not satisfied by any bijective map. While homomorphisms are mappings that preserve the structure of groups, isomorphisms are mappings that verify the two groups are essentially the same.

Previous studies have examined isomorphism in different ways. Early studies mostly provided students with two Cayley tables or stated two groups and asked if they were isomorphic or how they could tell they were isomorphic. The Dubinsky, Dautermann, Leron, and Zazkis

(1994) study indicated that when students considered isomorphisms between groups, they considered the cardinality of each group, but not whether the homomorphism property was satisfied. Leron, Hazzan, and Zazkis (1995) also noted students' tendency to check the cardinality of a group, but their students continued to test multiple other properties by finding the identity element (in Cayley tables), the orders of individual elements (the smallest positive integer m such that a^m is the identity for each element a), the number of elements of given orders in each group, whether a group is generated by a single element, and if it is commutative. Despite the many factors to check, students would still struggle if more than one way to construct an isomorphism existed, demonstrating a "craving for canonical procedures" (p. 168).

Other studies have considered isomorphism in the context of proof. In related studies, Weber and Alcock (2004) and Weber (2002) asked undergraduate and doctoral students to prove a number of theorems related to isomorphism and to prove or disprove specific groups were isomorphic. While both doctoral and undergraduate students were able to prove the initial simple propositions, the doctoral students continued to be successful in proving the remaining five propositions, while collectively only two (of twenty) proofs of the remaining propositions given by undergraduates were successful. Much like in Dubinsky et al. (1994), these difficulties largely related to undergraduates' tendency to form arbitrary mappings once they had ascertained that a bijection between groups could be formed. They would not apply other properties of the groups when trying to find or disprove the existence of an isomorphism.

Recent studies on isomorphism have shifted the focus to how to develop local instructional theories that could be transformed into an inquiry-oriented curriculum that included topics such as isomorphism. In the process of examining how students used their existing ways of reasoning to engage with mathematically rich tasks, other student views of isomorphism have come to light. In 2009, Larsen recorded a teaching experiment in which participants were expected to generate a definition of isomorphism. In that study, participant Jessica noted that the definition of isomorphism should include bijection because "…it has to go both ways" (p. 133). Her statement brought to light another approach to isomorphism: reversible mapping. Later, Larsen et al. (2013) noted that the homomorphism property was more challenging for students to unpack than the bijection property. Additionally, Larsen (2013) noted, "students' use of the homomorphism property is usually largely or completely implicit" (p. 722). Thus a number of tasks in his curriculum engage students in forming an explicit homomorphism in order to help students formulate the definition of homomorphism and, later, isomorphism. These studies highlight that students attend to the properties of the two groups, but that generating mappings between the groups was either ignored or very difficult for students.

In these isomorphism studies, some research has been conducted on homomorphism in the process of researching isomorphism. However, a few studies have examined homomorphism more closely in the context of proof. Nardi (2000) noted students' struggles in proving the First Isomorphism Theorem for Groups stemmed from three major sources: an inability to recall definitions or a lack of understanding of definitions, poor conceptions of mapping (such as thinking a homomorphism was an element of a group), and not realizing specifically what each part of the proof was proving. Weber (2001) observed that despite undergraduates' ability to recall relevant theorems, they struggled to move past syntactic, "definition unpacking" techniques when trying to prove theorems related to isomorphism and homomorphism, such as proving a group was abelian given the map was a surjective (onto) homomorphism. He also noted doctoral students had a tendency to use conceptual knowledge to formulate proofs more strategically and experienced more success in proving.

Theoretical Perspectives

APOS Theory

Two different theoretical lenses are used for this study. The first is APOS theory, which decomposes students' thinking into Action, Process, Object, and Schema level conceptions (Arnon et al., 2014). An action is an external transformation of objects; a process is an interiorized action; an object is an encapsulated process; and a schema coordinates objects and processes (Asiala, et al., 1996). For example, one genetic decomposition for function says a student with an action conception of function needs to take specific elements from a set and apply a rule to obtain an explicit output to make sense of a function. When students are able to internalize this procedure so they recognize elements of the domain are manipulated through a rule to produce outputs, and they can run through this procedure mentally, they are said to have a process conception. Eventually, a student will encapsulate this process into a static entity, allowing them to perform operations on functions; at this point a student has an object conception of function. Finally, indicators that a student has developed a schema conception of function are that the student can recognize whether a relationship represents a function and can determine the domain and range of a function (Arnon et al., 2014).

Previously, Leron et al. (1995) created a genetic decomposition of isomorphism based on understanding groups, functions, and existential quantifiers. Specifically, a student with an action level conception of isomorphism would likely check that individual elements of two potentially isomorphic groups act in the same way by checking the orders of elements and the cardinalities of the groups. At the process level, when a student was able to apply quantification more broadly, he or she would need to check that the homomorphism property held under a mapping, allowing the elements of one group to be mapped to another. Finally, a student able to recognize that more than one mapping from one group to another might be possible, but that the existence of one such isomorphic map guaranteed the groups were isomorphic could be described as having an object conception of isomorphism. Although Leron et al.'s (1995) decomposition was based on constructing isomorphism in finite groups, students could also be asked to determine if infinite groups were isomorphic. In this context, I conjecture that students would also need at least a process level construction of infinity to be able to generate a correspondence between infinite sets of the same cardinality or to recognize two infinite sets had different cardinalities.

While Leron et al. (1995) created a decomposition of isomorphism over twenty years ago, no one has offered a decomposition of homomorphism. This may stem from differing concept images (Tall & Vinner, 1981) of isomorphism and homomorphism. While it can be easy to describe isomorphism in a "naïve" way as "two groups (and their operations) [that are] the same" (Dubinsky et al., 1994, p. 280), it is difficult to find a similar naïve way to describe homomorphism. In fact, Leron et al. (1995) note that this might be due to the "directional (rather than symmetrical) nature" of homomorphisms, which are essentially functions (p. 172). However, to an extent we can think of homomorphisms in terms of collapsibility.

Suppose f is a homomorphism from G to G'. Then the kernel of this homomorphism is a normal subgroup of G. (The kernel of a homomorphism is the elements that are mapped to the identity in G'.) We can think of a homomorphism as taking as its domain the elements of G and producing an image within G', potentially all of G'. Sometimes each element of G is mapped to a distinct element of G'. In this case, no "collapsing" seems to occur because the kernel of f is the identity of G. However, at other times, multiple elements of G are mapped to some of the elements of G'. In this case, a number of elements are "collapsed" to each element of the image because they act in a similar way in the related quotient group. (Note that not all elements of G' are necessarily elements of the image of f, and that each element of G can be mapped to at most one element of G' because homomorphisms are functions.)

Viewing homomorphisms as a collapsing relationship provides insights into a way in which the topic can be decomposed. Namely, constructing homomorphism at an object level requires constructing function as an object, because a homomorphism is a type of function. As part of this, a student would need an object conception of group to be able to determine if the domain and image of a mapping were in fact groups, as needed to be a homomorphism. Furthermore, students need to develop an object conception of coset in order to be able to determine which elements of the first group should collapse to each element in the image. In equivalence relation terms, students need to be able to determine that the elements within each coset of the kernel of the homomorphism act equivalently.

Alternatively, one may conceive of a homomorphism as a special type of function. We can further delineate this decomposition according to how students would behave at different levels of homomorphism construction. When a student has an action conception of homomorphism, they could likely be characterized by only checking the orders of elements and the cardinalities of the "domain" group and of the desired codomain group. If the orders matched, they would likely create a mapping directly. Alternatively, they might conclude no homomorphism can be formed if the orders of elements in the two groups do not match. This is because such students would be focused on what each element is doing and how each element is being mapped according to the function form they were using. Like in Arnon et al.'s (2014) decomposition of function, students would focus on what specific elements are doing. Based on their isomorphism training, they would likely struggle in situations where the kernel of the homomorphism was not the identity. Like in the context of isomorphism, when addressing infinite groups, students would also struggle if they had not constructed a process conception of infinity.

At a process conception of homomorphism, students would be more careful to check that the homomorphism property holds in a proposed mapping from one group to another. Like in Arnon et al.'s (2014) decomposition, they would be less focused on mapping each element and more focused on the set of elements that the elements of the first group would be mapped to in the second group. Additionally, they would likely check that the homomorphism property held for a number of pairings, though they may be satisfied they had produced a homomorphism before checking all pairs of elements in the first group.

At an object conception of homomorphism, a student would finally grasp the "naïve" meaning of homomorphism; that is, a student could recognize a pattern exists between the number of elements of the kernel and the number of elements mapped to each image element. Like in Arnon et al.'s (2014) decomposition, students would be able to perform operations on the mapping, including being able to form different homomorphisms by manipulating which normal subgroup formed the kernel. Therefore, students would be able to construct (without prompting) homomorphic mappings with non-trivial kernels, such as the trivial homomorphism (in which all elements of the first group, G, are mapped to the identity in the second group, meaning the kernel of the homomorphism is G). Previous work on cosets, normality, and quotient groups by Asiala, Dubinsky, Mathews, Morics, & Oktac (1997) highlighted students' difficulty in proving that the kernel of a homomorphism is a normal subgroup, noting that students commonly assumed that the kernel equaled the identity. As the focus of the question in their study was to look at students' conceptions of normality, they simply noted this was "an issue that has to do with the concept of homomorphism" and moved on (p. 268).

Conceptual Metaphor

The second theoretical lens is the conceptual metaphor construct (e.g. Lakoff & Núñez, 2000). The properties and metaphorical expressions students used when describing homomorphic and isomorphic mappings were examined based on the characterization of students' concept images in Zandieh, Ellis, and Rasmussen's (2016) study on students' explanations of function and linear transformations. When possible, students' expressions were categorized within the five metaphorical clusters of Input/Output (InOut), Traveling (Tr), Morphing (Morph), Mapping (Map), and Machine (Mach) that were found in the Zandieh, et al. (2016) paper. A sixth cluster, Matching (Match), that did not appear in their paper emerged here.

Statements that referred to properties of a map, such as being one-to-one or bijective, without unpacking the meaning of the property were classified as Properties and the property being referenced was noted. Students' metaphorical expressions were classified as Input/Output if they implied a "putting in" and "taking out" relationship or a one directional relationship. The Traveling cluster is characterized by elements "moving", "going to", or "getting to" a specific place or element. The Morphing cluster involved "transforming" or "changing" an entity through a map. The Mapping cluster involved having a "correspondence" between groups. Although students used the verb "map" frequently, their utterance was only included in this category if there was not a reference to Traveling, Morphing, or Matching at the same time. The utterance also needed to go beyond the definition of a map to be categorized as a metaphorical expression and not a property. The Machine cluster included references to an entity that caused a change from the beginning to ending state, often with words like "produces". The Matching cluster was especially used when students were describing isomorphisms. In this cluster, students would refer to a mapping as "preserving structure", "behaving" the same, or "matching" elements. This cluster is distinct from the Mapping cluster because the Matching cluster emphasizes a quality of "sameness" whereas the Mapping cluster simply links beginning and ending sets of elements.

By using these two perspectives as well as examining student work by general approach, I will provide a multifaceted analysis of student thinking and address three research questions. (1) What approaches did students take when asked to form isomorphisms and homomorphisms between groups? (2) How can we understand the student work using APOS theory? (3) What metaphors did students draw on when speaking about homomorphisms and isomorphisms?

Methods

The participants for this study were six sophomore or junior university students in a lecture-based introductory abstract algebra course. Four were mathematics majors and two were engineering majors considering double majoring in math or transferring into the math program. Students' college math backgrounds other than introductory calculus and a proof course varied but included courses in combinatorics, discrete math, vector geometry, linear algebra, multivariate calculus, differential equations, operational methods, and real analysis.

Students were recruited from two instructors' courses with one student coming from Instructor A's section (denoted participant A1) and five from Instructor B's section (denoted participants B1, B2, B3, B4, and B5). Instructor A had taught both group and ring isomorphisms and homomorphisms earlier in the semester and his students, including A1, had been tested on that material. Instructor B had just begun teaching about group homomorphisms and isomorphisms when his students were interviewed. Four of his students were interviewed after learning about isomorphisms but before learning about homomorphisms in class (B1-B4), and one student was interviewed after learning about both isomorphisms and homomorphisms (B5).

Participants were recruited in two ways. The author asked for an announcement to be sent to Instructor A's students with interested students sending the author a message. In Instructor B's section, the author visited the class and asked interested students to provide their email address in order to be contacted. Students were given their preference of $10 or an hour of tutoring to try to incentivize students with a range of abilities to participate.

1. What is a homomorphism?
2. How do you determine if [a map or whatever used above] is a homomorphism?
3. What is an example of a homomorphism?
4. What is an isomorphism?
5. How do you determine if [a map or whatever used above] is an isomorphism?
6. What is an example of an isomorphism?
7. Is the homomorphism example you gave also an isomorphism?
8. Is the isomorphism example you gave also a homomorphism?
9. Interpret the following definitions of homomorphism and isomorphism.
   Definition: Let $(G,\star)$ and $(H,\boxdot)$ be groups. A map $\varphi\colon G \to H$ such that $\varphi\colon (x \star y) = \varphi(x) \boxdot \varphi(y)$ for all $x, y \in G$ is called a homomorphism.
   Definition: The map $\varphi\colon G \to H$ is called an isomorphism and G and H are said to be isomorphic or of the same isomorphism type, written $G \cong H$, if
   φ is a homomorphism, and φ is a bijection.

For each of the following pairs of groups in 10-17, is it possible to form an isomorphism between them? Why or why not? Is it possible to form a homomorphism between them? Why or why not?

10.

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

+	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	d	c	a	b
d	c	d	b	a

11. $\mathbb{Z}_5 \to 5\mathbb{Z}$
12. $\mathbb{Z}_5 \to \mathbb{Z}/5\mathbb{Z}$
13. $\mathbb{Z}_5 \to \mathbb{Z}_6$
14. $\mathbb{Z}_3 \to \mathbb{Z}_6$
15. $\mathbb{Z}_6 \to \mathbb{Z}_3$
16. $\mathbb{Z} \to 2\mathbb{Z}$
17. $2\mathbb{Z} \to \mathbb{Z}$
18. Does $exp(x)\colon (\mathbb{R},+) \to (\mathbb{R}^+,\times)$ where $exp(x) = e^x$ represent a homomorphism? An isomorphism? Why or why not?
19. Assume $\mathbb{Z}$ is the additive group of integers. Does $f\colon \mathbb{Z} \to \mathbb{Z}$ where $f(x) = 8x$ represent a homomorphism? An isomorphism? Why or why not?
20. Assume $\mathbb{R}$ is the additive group of integers. Does $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = 8x$ represent a homomorphism? An isomorphism? Why or why not?

Figure 1. Interview protocol for semi-structured interviews.

Each participant engaged in a semi-structured interview (Fylan, 2005) lasting approximately one hour. The interview questions were drawn from those in Figure 1, but time and students' backgrounds prevented some students from seeing certain questions. However, all students answered questions 6, 10, 11, and 13. A listing of all questions answered by each student is given in Table 1. The interviews were all audio-recorded and five of the six were video-recorded. Participants' written work was also collected. To analyze the data, the interviews were coded in multiple iterations. First, participants' interviews were transcribed and open coded for students' problem solving strategies regarding isomorphism and homomorphism. This coding generated themes, which were verified by utilizing multiple iterations of coding (Anfara, Brown, & Mangione, 2002). In this cyclic process, each task given to a student was analyzed for the problem solving strategy used. As new strategies emerged in later transcripts, early interviews were reviewed, making sure that strategies identified later in coding had not been overlooked in early interviews. After the open coding and theme generation were completed, the transcripts were reviewed again for classification according to APOS theory and for students' use of conceptual metaphors. The APOS decomposition of homomorphism was created based on previous literature and the previous analysis of problem solving strategies. The conceptual metaphor construct was used to gain insight into students' concept images of mappings.

Table 1
Questions Answered by Each Student

Student	Questions Answered
A1	1,2,3,4,5,6,7,8,10,11,12,13,14,15,16,17,18,19,20
B1	3,6,7,9,10,11,12,13,14,15,16,17,18,19,20
B2	4,5,6,9,10,11,13,14,15,16,17
B3	4,5,6,9,10,11,13
B4	4,6,9,10,11,12,13
B5	1,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18

Results

Because this study is viewed from multiple lenses, we begin with general problem solving approaches for isomorphisms and homomorphisms as well as stumbling blocks to finding maps. This is followed by students' classification using the genetic decompositions for isomorphism and homomorphism. Finally, students' use of conceptual metaphors is explored.

Problem Solving Approaches

Participants' approaches to mappings fell into three major categories: successful methods of determining that a map was an isomorphism or whether such a map could be generated between groups; successful methods of determining that a map was a homomorphism or whether such a map could be generated between groups; and stumbling blocks to success.

Isomorphisms. Students exhibited a variety of successful strategies when trying to determine whether two groups were isomorphic or whether a given map could be an isomorphism. Because a one-to-one and onto mapping must exist for a bijection to exist, groups must have the same cardinality to be isomorphic. Thus, one strategy students used was to check whether the groups were the same size. Four of the participants (A1, B2, B3, and B5) utilized this strategy at some point with three of them checking this characteristic first when faced with any of problems 11-17. For example, in response to the isomorphism part of question 11 ($\mathbb{Z}_5 \to 5\mathbb{Z}$), B2 answered, "I don't think, um, I can form isomorphism between them because, um, I don't think it is bijective, um, because I think this group [points to $5\mathbb{Z}$] is larger than this one [points to $\mathbb{Z}_5$]."

Another successful strategy was to look for a reversible map between given groups or to note that a previously stated map was reversible. A1, B1, and B2 all utilized this strategy, with B1 using this as her main approach. For example, consider this exchange when asked question 16.

B1: I believe so. Like, to get from $\mathbb{Z}$ to $2\mathbb{Z}$, you just multiply by 2. To get from $2\mathbb{Z}$ to $\mathbb{Z}$, you just divide by 2.

I: You said $\mathbb{Z}$ to $2\mathbb{Z}$, multiply by 2. Is that an isomorphism? Homomorphism?

B1: Well that in itself is, that's only a one way mapping. Its inverse also exists so paired together, isomorphism, I'd say.

When pressed on details of the "inverse" map, namely explaining why it was acceptable even though division was not the operation defined for $\mathbb{Z}$ and ½ was not an element of $\mathbb{Z}$, she said it did not matter because "1 isn't in $2\mathbb{Z}$. There is nothing that will get you to a non-integer. If it's written as 2 times something, dividing by 2 just gets rid of that." She realized that the maps she had defined would map to and from the same elements. In reference to the same problem, A1 noted $2\mathbb{Z}$ to $\mathbb{Z}$ was "the same one. Since they were isomorphic before, it works the other way." He realized that when an isomorphism exists, the mapping between the groups can be inverted so each group is the domain of an isomorphic mapping to the other.

B2 and B4 considered another potential strategy for determining whether groups were isomorphic: creating a Cayley table for each group. Additionally, B2 recognized this was only a feasible option for small groups. In question 5, when asked how to determine if a group was an isomorphism, he said he would like to make a Cayley table if possible, but if the group was too big, it would be challenging: "a Cayley table will be 16x16 and I cannot make it, but, you know, 4x4, that would be fine, so kind of [a] small example." In the reverse situation, when presented with the Cayley tables in question 10, A1 and B1 also compared the orders of elements or mapped the identities of each group to each other without being prompted to do so.

Homomorphisms. The most straightforward argument given for determining whether a mapping was a homomorphism stemmed from determining it was an isomorphism. Because all isomorphisms are homomorphisms, if a student determined an isomorphism could be formed between groups, a homomorphism could automatically be formed too. A1, B1, and B5 clearly recognized this property, making statements similar to what A1 said: "It's part of the definition to be isomorphic: you have to have homomorphism in there." The other three participants may have also recognized this concept based on their explanations of the definitions of isomorphism and homomorphism, but they never directly used this concept to answer other questions.

Closure (associated with cardinality arguments) was a property students used to check whether a map they proposed could be a homomorphism. This idea was especially used to rule out maps they had created in response to questions 11 and 13. Five of the students successfully used this strategy, with B4 making a typical response as shown in Figure 2. She used her work with Cayley tables that would not close (on the mid-right and bottom right of Figure 2) to conclude she could not map all elements in $\mathbb{Z}_5$ (pictured on the mid-left) to distinct elements in $5\mathbb{Z}$ and have a closed group result. From this, she concluded she could not create an isomorphism or homomorphism. Although her conclusion that she could not form an isomorphism was valid, her conclusion that there was no possible homomorphism was flawed because the trivial homomorphism, $\theta\colon \mathbb{Z}_5 \to 5\mathbb{Z}$ such that $\theta(x) = 0 \ \forall\ x \in \mathbb{Z}_5$, would be possible. In fact, only A1 recognized the trivial homomorphism would be a homomorphism without being prompted to consider it. He and other participants who successfully generated homomorphisms appeared to notice patterns in the orders of the groups, which allowed maps to be generated quickly. For example, when looking for a map from $\mathbb{Z}_3$ to $\mathbb{Z}_6$, B1 noticed the 0's should correspond and that $\{0,2,4\}$ acted just like $\mathbb{Z}_3$. Similarly, B2, who had found a homomorphism from $\mathbb{Z}_6 \to \mathbb{Z}_3$ before being asked about $\mathbb{Z}_5$ and $\mathbb{Z}_6$, noted a non-trivial homomorphism could not be created between them because "3 divides 6 right? So, like, it should have something important about mapping…."

Although most students attempted to use one of the strategies above first, they eventually resorted to trial and error to look for homomorphisms. Some students used pure trial and error techniques, such as B5, who attempted to map elements of $\mathbb{Z}_6$ to $\mathbb{Z}_3$ in seemingly random order

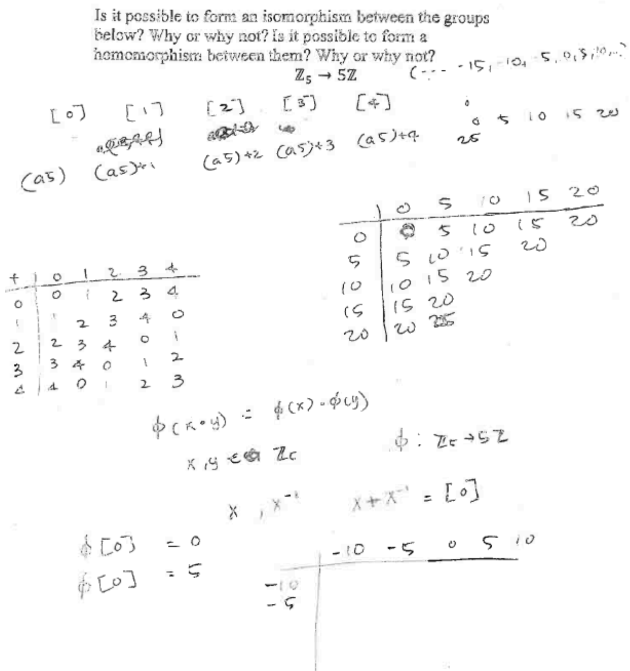

Figure 2. Work sample from B4 examining possible maps between $\mathbb{Z}_5$ and $5\mathbb{Z}$.

and then check if the homomorphism property was satisfied (i.e. mapping $\{0,3,4\} \rightarrow 0, \{1,2\} \rightarrow 1, \{5\} \rightarrow 2$). Both he and B3, users of this "random" approach, said they thought these problems were "trick questions" because there was no obvious technique to use. Participants who had more success finding suitable maps and ruling out incorrect maps tended to use knowledge of the orders of the groups to narrow possible choices, such as in B1's comparison of $\{0,2,4\}$ to $\mathbb{Z}_3$.

Stumbling blocks. A number of participants struggled to recall and utilize definitions effectively. Although most students defined an isomorphism as a bijective mapping or one-to-one and onto mapping that had the property $\phi(a * b) = \phi(a)\phi(b)$, some students struggled to unpack what "one-to-one" or "onto" meant. For example when B3 was asked what one-to-one meant, she first attempted to give the formal definition but could not recall it. When asked just to state what she thought about one-to-one and onto, she replied that in a one-to-one mapping "one [element] maps to one [element]" as she drew a set diagram illustrating a mapping. However, she still could not explain what onto meant, even in the context of her diagram. When asked to interpret the definition of homomorphism given in question 9, B2 claimed, "It should be abelian….Because you can switch these two, x and y, and then it just, isn't it automatically saying if you say yx, it'll just be y and x?" Additionally, students B2 and B5 attempted to map elements to multiple locations, thus violating the implicit function relationship in the definitions.

Although all participants tried to use formal definitions at times, some struggled to interpret them. Consider the following exchange after B4 was asked to give the definition of isomorphism.

B4: I don't even know if this is right. [Writes $(a \circ b) = a \circ b$.] I don't know what the circles are, but it's when it's preserved under the same something—I know it but I can't tell you.

I: Ok, well I feel like you're trying to give me the formal definition….Do you have just an intuitive sense of what it is?

B4: It's when two things are multiplied or something under a…say the mapping is α, if you do map of $a \cdot b$ is equal to map of $\alpha(a) \cdot \alpha(b)$ under, yeah, and it's onto and one-to-one, I think.

I: Ok. So you've given me a nice definition. Do you have any sense of, like in practical terms, what that might look like?

B4: No.

I: Ok, so if I asked you for an example you would say…

B4: Would probably give it to you in math terms. I don't know a real example. Yeah.

Some students also struggled to distinguish between the roles and the names of elements. This issue especially arose with the Cayley tables in question 10, when students often assumed they had to map element a to element a because it was the same letter rather than determining the identity of each group or using other techniques. This tendency to map elements with the same name to each other appeared in later problems too. For example, in question 13, every participant first tried to map $0 \to 0, \dots, 4 \to 4$, and all but A1 and B5 concluded that there could not be any homomorphism between the groups because the map matching similarly named elements did not work. Even when presented with the trivial homomorphism as a possible solution to question 13, B3 struggled to accept the mapping because she did not know what to do with the other elements in $\mathbb{Z}_6$ (i.e., 3+4=2 is mapped to 0, but $3+4 = 1$ in $\mathbb{Z}_6$ "isn't going anywhere").

Even when students identified the roles of elements in different groups, they did not always know how to create a mapping from this information. For instance, B2, who in response to question 5 had stated that creating Cayley tables could be useful in determining if an isomorphism existed between groups, struggled when presented with the tables of question 10. He successfully located the identity element of each group, but he was unsure what to do with this information until being prompted to consider mapping the identity elements to each other.

APOS Classification

In this section, a picture of each student is given through the lens of action-object theory. Students' perceived mental constructions are listed based on their problem solving approaches.

A1 was the strongest student in this study. Of all of the questions posed to him, he only struggled with determining if a homomorphism was possible on the Cayley table problem. Later, he observed that the trivial homomorphism is always a possible mapping between groups. He made use of the cardinalities of groups and the orders of elements, but also attended to whether the homomorphism property was satisfied while mapping. Furthermore, he had no problems with mapping from finite to countably infinite groups or between countably infinite groups. He was never explicitly asked to look for multiple isomorphisms, so it is difficult to tell whether he had constructed object level conceptions of isomorphism. However, he possessed at least process level conceptions of isomorphism, including for groups of infinite order. Because he eventually constructed mappings with non-trivial kernels, including both the trivial homomorphism and a homomorphism from $\mathbb{Z}_6 \to \mathbb{Z}_3$, he may have object level conceptions of homomorphism.

B1 was another strong student. She was comfortable working with countably infinite groups. She took the orders of groups into consideration when mapping, though she mostly based her isomorphism arguments on reversible or, as she said, "invertible" mappings. She generally would check the homomorphism property was satisfied whenever generating isomorphisms or homomorphisms, which allowed her to catch her own errors when her maps did not work. She was not asked to look for multiple isomorphisms, so it is only possible to say that she possessed at least a process conception of isomorphism. She possessed no greater than a process conception of homomorphism because she was unable to generate maps with non-trivial kernels. Although she quickly verified that the trivial homomorphism was a homomorphism when it was suggested, she did not create other non-trivial mappings, even in the $\mathbb{Z}_6 \to \mathbb{Z}_3$ problem.

B5 checked multiple elements worked before concluding his mappings were isomorphisms or homomorphisms. However, he would not check all elements and would not always happen to check the cases that failed, so he would claim a map satisfied the homomorphism property when it did not. He also was not always able to check the orders of elements; specifically, he believed that the order of each element was found by multiplying each element of a row by the identity element, thereby deciding that elements of orders 1,2,3, and 4 appeared in one of the Cayley tables. He was able to work with infinite groups without difficulty, however. Putting everything together, he had at best process level conceptions of isomorphism and homomorphism.

B4 did not make direct statements about the cardinalities of groups or elements in the context of isomorphism or homomorphism. However, she indirectly used cardinality by looking at the definition of one-to-one or by observing that the homomorphism property could not be satisfied when mapping similarly named elements to each other in groups of relatively prime order ($\mathbb{Z}_5 \rightarrow \mathbb{Z}_6$). She was always very attentive to definitions, though she did confuse the definitions of one-to-one and function. Because of her attention to definitions, she did not conclude isomorphisms and homomorphisms existed when they should not have. However, she struggled to get an intuitive sense of isomorphism and homomorphism, leading her to work very slowly. Because she was able to indirectly check cardinality and check the homomorphism property, she might have been at a process level; however, without more examples with equipotent groups that could not map easily based on the homomorphism property, it is difficult to tell if she possessed action or process conceptions of isomorphism and homomorphism. She never addressed any questions that addressed her conceptions of correspondences between infinite sets.

B2 was able to address problems when shown an issue and did recognize that he should attend to the identities of groups when mapping, but he struggled to know what to do with this information. At times, he checked the homomorphism property, but he placed more emphasis on using all of the elements of both groups, which led him to create maps that were not functions from $\mathbb{Z}_3 \rightarrow \mathbb{Z}_6$ and $\mathbb{Z} \rightarrow 2\mathbb{Z}$. Additionally, he had not taken analysis yet and struggled when asked if an isomorphism between $\mathbb{Z}$ and $2\mathbb{Z}$ could exist, as he assumed that $\mathbb{Z}$ was "larger" than $2\mathbb{Z}$. Overall, he appears to be at the action level for both homomorphism and isomorphism.

B3 struggled the most in this study. She occasionally checked the orders of elements and began to check the cardinalities of groups after she was given an interpretation of "one-to-one and onto". However, she struggled to recall and interpret definitions on her own and sometimes mapped elements at random. When looking for a homomorphism from $\mathbb{Z}_5 \rightarrow 5\mathbb{Z}$ she said, "I'm just mapping. I'm just picking" rather than checking whether her map would satisfy the property. At most, she displayed action conceptions of isomorphism and homomorphism.

Conceptual Metaphor

Students utilized a number of different metaphors and properties while defining and describing homomorphisms and isomorphisms. While all six metaphorical clusters were used by at least one person, the Mapping, Matching, and Traveling metaphors were used more commonly than the Machine, Input/Output, and Morphing clusters, both in terms of the number of participants using the clusters and the number of times each participant used the cluster. The metaphors and properties used by each student are given in Table 2 by map. Typical examples of each metaphor cluster are provided in Table 3. In addition to using metaphors, students also relied on certain properties that they either chose not to explain while using them or never explained in the interview. The properties used during explanations are also listed in Table 3.

The most successful solvers, those at the process level and above for homomorphism and isomorphism according to APOS theory, typically used Traveling metaphors. The exception was

B3, who also made use of Traveling metaphors a few times near the end of the interview. However, the interviewer made use of Traveling metaphors in some earlier explanations, so she may have simply adopted the interviewer's way of speaking. In addition, successful solvers typically integrated imagery and properties. A1, who had the most experience working with homomorphisms and isomorphisms, utilized three different metaphors while also making use of properties he had learned. The only time he struggled was when he failed to attend to properties and relied only on a Matching metaphor. Conversely, B4 struggled, especially early, when she focused almost exclusively on definitions and properties without interpretation as shown above.

Table 2
Students' Properties and Metaphorical Clusters by Mapping

Student	Homomorphism	Isomorphism
A1	C_{Map}, C_{Match}, C_{Tr}	C_{Map}, C_{Match}, C_{Tr}, P_{Gp}, $P_{1\text{-}1}$, P_{Onto}, P_{Hom}, P_{Card}
B1	C_{Map}, C_{Match}, C_{Tr}, C_{InOut}, P_{Image}	C_{Map}, C_{Match}, C_{Tr}, C_{Morph}, P_{Card}, P_{Bij}
B2	C_{InOut}, C_{Match}	$P_{1\text{-}1}$, P_{Onto}, P_{Bij}, P_{Gp}, C_{Match}, C_{Mach}
B3	C_{Map}, C_{Tr}	$P_{1\text{-}1}$, P_{Onto}, P_{Bij}, P_{Hom}, P_{Gp}, C_{Match}
B4	C_{InOut}, C_{Map}	C_{Match}, P_{Gp}, $P_{1\text{-}1}$, P_{Onto}
B5	C_{Match}, C_{Tr}	C_{Match}, $P_{1\text{-}1}$, P_{Onto}, P_{Gp}, P_{Card}, C_{Tr}

Table 3
Examples of Statements in Each Metaphorical Cluster

Name	Illustrative Example
Input/Output (InOut)	B4: "You have an equation or anything and *plug in* the value x, it would *output*, er, that *output* would be $\phi(x)$ so you *plug in* x in the group G into an equation or something of ϕ."
Traveling (Tr)	B3: "3 *goes to* 3 here, but $1 + 2 = 3$ won't *go anywhere*."
Morphing (Morph)	B1: "Say you try to add these two elements [in $\mathbb{Z}_5$]. You will get this one. Say you try to *transform* them in some way. You would have to do it $4 \to 5$, $1 \to 1$ or something else like that where you just plain don't get that when you add them [in $\mathbb{Z}_6$]."
Mapping (Map)	A1: "I don't know *what they map to* is the issue."
Machine (Mach)	B2: "I think my problem is I don't know *how this map goes* if I choose these two numbers."
Matching (Match)	B5: "The identities should be mapped *onto each other*."
Properties	One-to-one (1-1), Onto, Bijection (Bij), Homomorphism (Hom), Group (Gp), Cardinality/Order (Card), Image

B1 was striking in the variety of metaphors used. She utilized metaphors from five of the six clusters and was able to address most of the problems she was given effectively. Despite being unable to state the definition of homomorphism or isomorphism at the beginning of the study, when given the definitions, she immediately interpreted them using Mapping metaphors: "If you have any two elements, if you compose them before mapping, you get the same thing as if you compose…what their maps are for all of those elements. And here [in the isomorphism definition] it's basically that, plus you can do it backwards."

Additionally, while everyone used a Matching metaphor at some point, this image was not always helpful; in fact, it appeared to hinder some students' development. B3 was very focused on superficial sameness as she looked at Cayley tables, saying that if the groups in the Cayley tables were isomorphic, both tables would be the same; that is both should say "d,c,b,a" in the bottom row. While isomorphic groups could be relabeled in such a way as to make this happen, observing two Cayley tables of groups do not look identical is insufficient to conclude that the groups are not isomorphic. Even A1, who used the phrase "preserves structure" repeatedly and was able to address all isomorphism questions successfully, was sometimes hampered in his homomorphism efforts by trying to maintain the Matching metaphor for homomorphisms. His main problem occurred when trying to determine if a homomorphism could be established between the groups represented by the Cayley tables. Although he rapidly ascertained that no isomorphism could be formed between the groups in the Cayley tables, he confused himself by trying to continue his metaphor of "preserving structure" in the context of homomorphism. This led him to go back and forth between saying a homomorphism should not preserve the structure because that was what isomorphisms did, to saying the groups should "behave the same way" under a homomorphism. (When answering this question, he had not yet recalled that the trivial homomorphism could always be formed between groups.) He was so focused on his metaphor that he did not go back to his definition of isomorphism, in which he said that an isomorphism is a "homomorphism plus it's one-to-one and onto".

Discussion

From the problem solving approach analysis, we can see that students were more comfortable working with the bijective property than the homomorphism property, much like in Larsen et al. (2013). This seems plausible given that because the cardinality of a group can be determined rapidly, determining if a bijection exists between two groups is most likely not as difficult as determining if the homomorphism property holds for all pairs of elements of a group. Although the homomorphism property allows specific maps to be tested quickly, students had to rely on trial and error (albeit strategic trial and error) to create reasonable maps to be tested.

Additionally, students with limited concept images (Tall & Vinner, 1981) of one-to-one, onto, function, and homomorphism were at a disadvantage when creating homomorphisms and isomorphisms. Students A1 and B1, who had the most robust mental constructions of isomorphism and homomorphism, both utilized the concept of a mapping being reversible if it is an isomorphism and demonstrated the ability to use strategic trial and error to find homomorphisms quickly. These traits indicated a robust concept image of maps, much as the successful doctoral students utilized their knowledge of theorems strategically to write proofs (Weber, 2001). Like other studies, students considered the cardinalities of the groups to determine if they could be isomorphic (Dubinsky, et al., 1994; Leron, et al., 1995; Weber & Alcock, 2004; Weber, 2002). Despite literature indicating that comparing orders of elements of groups when trying to determine if a map is an isomorphism was a common technique (Leron, et al., 1995), only two participants utilized the technique in this study; however, this may be due to the limited number of questions in which both groups were finite and had the same cardinality.

From the APOS analysis, it is reasonable that students struggled more when asked to find homomorphisms from one group to another than when asked to determine whether groups were isomorphic because students need to coordinate an extra construction, namely determining the image of the homomorphism. In this study, students struggled far more when they could not map elements in the first group to distinct elements in the second group. Additionally, the "naïve" image of homomorphism is not as intuitive as the "sameness" associated with isomorphism. Because this image of homomorphism is tied to quotient groups, with which students famously struggle (Dubinsky, Dauterman, Leron, & Zazkis, 1997), students should not be expected to reach an object understanding of homomorphism until after quotient groups have been taught.

From the conceptual metaphor analysis, we see that collectively students used a variety of metaphors to describe mappings, though the most common type of metaphor was Matching. While one of the most successful solvers, B1, used a wide variety of metaphors, a better indicator of success was integrating properties with metaphors. Students who relied too heavily on properties at the expense of imagery struggled to move past computations and what they had memorized. Students who relied too heavily on word pictures at the expense of definitions could impute too many properties to their mapping (like B3 with the "same" Cayley tables) or could highlight the wrong properties (like A1 saying only isomorphisms "preserve structure").

By combining analyses, we are able to gain a characterization of what successful and struggling students might look like in abstract algebra. Successful students were able to utilize properties and definitions fluidly while also being able to characterize what was happening in a more intuitive way. By using their varied images and properties strategically, they were able to approach more problems with success (Weber, 2001). Students with limited ways of picturing mappings, who were tied to definitions and property statements, were still struggling to understand how each element would be mapped and were not in a position to create mental images of what all elements would do or how isomorphisms or homomorphisms should behave.

Future studies could use students more experienced with homomorphism and isomorphism to refine the genetic decomposition of homomorphism. Additionally, more questions with isomorphic groups should be explored to see if students utilize techniques like considering the orders of elements and creating reversible maps when faced with more groups of the same cardinality. Because of the dearth of studies on homomorphism, it would be illuminating to conduct teaching experiments examining how to enhance students' concept images of these mappings both by developing metaphorical descriptions and by grounding statements in formal properties. This might help students consider mapping elements that look different to one another despite elements of similar appearance being present in the other group (e.g., map $1 \to 2$ instead of $1 \to 1$ when generating a homomorphism from $\mathbb{Z}_3 \to \mathbb{Z}_6$). This study continues the work done by Zandieh et al. (2017) in examining students' conceptions of function in linear algebra by looking at students' conceptions in abstract algebra. Future studies could compare and contrast how students create and use mappings in other content areas, such as graph theory or analysis, generating insights into the role of mappings throughout mathematics.

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Emerging Insights from the Evolving Framework of Structural Abstraction

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Only recently 'abstraction from objects' has attracted attention in the literature as a form of abstraction that has the potential to take account of the complexity of students' knowing and learning processes compatible with their strategy of giving meaning. This paper draws attention to several emerging insights from the evolving framework of structural abstraction in students' knowing and learning of the limit concept of a sequence. Particular ideas are accentuated that we need to understand from a theoretical point of view since they reveal a new way of understanding knowing and learning advanced mathematical concepts.

Keywords: Limit Concept; Mathematical Cognition; Sense-Making; Structural Abstraction

Introduction

Theoretical and empirical research shows the existence of differences in knowing and learning concerning different kinds of knowledge (diSessa, 2002). A general framework on abstraction cannot encompass the whole complexity of knowing and learning processes in mathematics. Rather, in investigating the nature, form, and emergence of knowledge pieces, various local learning theories may be developed, which will be quite specific to particular mathematical concepts, individuals, and their respective sense-making strategies. As a consequence, the complexity of knowing and learning processes in mathematics cannot be described or explained by only one framework. Instead, we acknowledge that comprehensive understanding of cognition and learning in mathematics draws on a variety of theoretical frameworks on abstraction.

The literature demonstrates significant theoretical and empirical advancement in understanding 'abstraction-from-actions' approaches, particularly the cognitive processes of forming a (structural) concept from an (operational) process (Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991). Abstraction-from-actions approaches take account of a certain sense-making strategy, namely what Pinto (1998) described as 'extracting meaning'. However, only recently 'abstraction from objects' has attracted attention as a form of abstraction that provides a new way of seeing the complexity of knowing and learning processes compatible with students' strategy of what Pinto (1998) described as 'giving meaning'.

The purpose of this paper is to provide deeper meaning to a recently evolving framework of a particular kind of 'abstraction from objects': structural abstraction. The structural abstraction framework is evolving in the sense that the framework functions both as a tool for research and as an object of research (Scheiner & Pinto, 2016b). In more detail, we use the structural abstraction framework retrospectively as a lens through which we reinterpret a set of findings on students' knowing and learning of the limit concept of a sequence. This reinterpretation is an active one in the sense that the framework serves as a tool to analyze a set of data, while the framework is also refined and extended since the reinterpretation produces deeper insights about the framework itself. Especially, these more profound insights are what we need to understand from a theoretical point of view since they have relevance for significant issues in knowing and learning advanced mathematical concepts. Such a dynamic view that is aligned with an interpretative approach seems to be promising in responding to questions that evolve while the object of consideration is still under investigation.

We begin by providing an upshot of our synthesis of the literature on abstraction in knowing and learning mathematics. Our synthesis is to suggest an orientation toward the evolving framework of structural abstraction as an avenue to take account of an important area for consideration – that is, drawing attention to the complex knowing and learning processes compatible with students' 'giving meaning' strategy. The structural abstraction framework constitutes the foundation of the second part of the paper providing emerging insights in knowing and learning the limit concept of a sequence. These insights offer theoretical advancement of the framework and deepen our understanding of knowing and learning advanced mathematics.

Mapping the Terrain of Research on Abstraction in Mathematics Education

Abstraction seems to have gained a bad reputation after been questioned by the situated cognition (or situated learning) paradigm, and, as a consequence, has almost disappeared from recent research discourse. This criticism rests primarily on traditional approaches on knowledge transfer through abstraction that led to an understanding of abstraction as a process of decontextualization and a confusion of abstraction with generalization. The recent contribution by Fuchs et al. (2003) shows that such classical approaches to abstraction still exist:

> "To abstract a principle is to identify a generic quality or pattern across instances of the principle. In formulating an abstraction, an individual deletes details across exemplars, which are irrelevant to the abstract category […]. These abstractions […] avoid contextual specificity so they can be applied to other instances or across situations." (p. 294)

Though various images of abstraction in the mathematics education literature can be identified (see Scheiner & Pinto, 2016a), several scholars argued against the image of abstraction as decontextualization. Van Oers (1998, 2001), for instance, argued that removing context must impoverish a concept rather than enrich it. Several other scholars have reconsidered and advanced our understanding of abstraction in ways that account for the situated nature of knowing and learning in mathematics. Noss and Hoyles (1996) introduced the notion of *situated abstraction* to describe "how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed" (p. 122). Webbing in this sense means "the presence of a structure that learners can draw up and reconstruct for support – in ways that they can choose as appropriate for their struggle to construct meaning for some mathematics (Noss & Hoyles, 1996, p. 108). Hershkowitz, Schwarz, and Dreyfus (2001) introduced the notion of *abstraction in context* that they presented as "an activity of vertically reorganizing previously constructed mathematics into new mathematical structure" (p. 202). They identify abstraction in context with what Treffers (1987) described as 'vertical mathematization' and propose entire mathematical activity as the unity of analysis. These contributions demonstrate that research on abstraction in knowing and learning mathematics has made significant progress in taking account of the context-sensitivity of knowledge.

Several other contributions shape the territory in mathematics education research on abstraction. Mitchelmore and White (2004) indicate a distinction between abstraction in mathematics and abstraction in mathematics learning. They proposed to include a new meaning in the later, which seemed to be missing in the debate on the notion of abstraction, related to "formation of concepts by empirical abstraction from physical and social experience" (p. 329). Articulated to this understanding, Scheiner (2016) proposed a distinction between two forms of abstraction, namely *abstraction from actions* and *abstraction from objects*. This distinction has been further refined in Scheiner and Pinto (2014) arguing that the focus of attention of each form of abstraction takes place on physical objects (referring to the real world) or mental objects (referring to the thought world) (see Fig. 1).

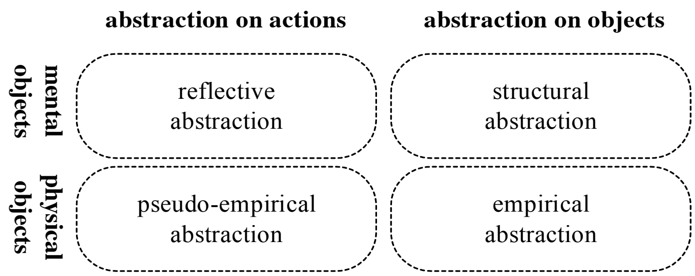

Fig. 1: A frame to capture various kinds of abstraction (reproduced from Scheiner & Pinto, 2014)

We consider this distinction as being productive in trying to capture some of the variety of images of abstraction in mathematics education (for details see Scheiner & Pinto, 2016a). It acknowledges Piaget's (1977/2001) three kinds of abstraction, including pseudo-empirical abstraction, empirical abstraction, and reflective abstraction, that served as critical points of departure in thinking about abstraction in learning mathematics. Research on abstraction in mathematics has long moved beyond classifying and categorizing approaches in cognition and learning, based on similarities of the individuals constructs. For instance, Mitchelmore and White (2007), in going beyond Piaget's empirical abstraction and in drawing on Skemp's (1986) conception of abstraction, described abstraction in learning elementary mathematics concerning seeing the underlying structure rather than the superficial characteristics. Abstraction in learning advanced mathematics, however, is almost always defined in terms of *encapsulation* (or *reification*) of processes into objects, originating in Piaget's (1977/2001) idea of *reflective abstraction*. Reflective abstraction is an abstraction from the subject's actions on objects, particularly from the coordination between these actions. The particular function of reflective abstraction is abstracting properties of an individual's action coordination. That is, reflective abstraction is a mechanism for the isolation of specific properties of a mathematical structure that allows the individual to construct new pieces of knowledge. Taking Piaget's reflective abstraction as a point of departure, Dubinsky and his colleagues (Dubinsky, 1991; Cottrill et al., 1996; Arnon et al., 2014) developed the APOS theory describing the construction of concepts through the encapsulation of processes. Similar to encapsulation is reification – the central tenet of Sfard's (1991) framework emphasizing the cognitive process of forming a (structural) concept from an (operational) process. In the same way, Gray and Tall (1994) described this issue as an overall progression from procedural thinking to proceptual thinking, whereas proceptual thinking means the ability to flexibly manipulate a mathematical symbol as both a process and a concept. Gray and Tall (1994) termed symbols that may be regarded as being a pivot between a process to compute or manipulate and a concept that may be thought of as a manipulable entity as procepts.

Scheiner (2016) revealed that the literature shows an unyielding bias toward *abstraction from actions* as the driving form of abstraction in knowing and learning advanced mathematics. This almost always exclusive view arises directly from the trajectory of our field's history; originating in Piaget's assumption that only reflective abstraction can be the source of any genuinely new construction of knowledge. While abstraction-on-actions approaches have served many purposes quite well, they cannot track detail of students' knowing and learning processes compatible with the strategy of giving meaning. The recently evolving framework of structural abstraction has attracted attention as a promising tool to shed light into the complexity of students' knowing and learning processes compatible with their strategy of 'giving meaning'.

The Evolving Framework of Structural Abstraction

The distinction between abstraction from actions and abstraction from objects reflects Tall's (2013) distinction between *operational abstraction* and *structural abstraction*. In contrast to Piaget (1977/2001) who dichotomized these two forms of abstraction, Tall (2013) argued that mathematical thinking emerges in these two forms: operational abstraction focusing on actions on objects and structural abstraction focusing on the properties (or structures) of objects. For instance, the development of geometry in the conceptual embodied world focuses mainly on structural thinking, the operational symbolic world blends both operational and structural thinking as new forms of number are introduced as extensional blends in algebra (see Tall, 2013). Obviously, mathematics education researchers used the term 'structural' in diverse ways, referring, for instance, to structural mathematics of axioms and definitions or to the properties of the structure of objects. In the following discussion, we departure from Scheiner's (2016) understanding of structural abstraction as focusing on "the richness of the particular [that] is embodied not in the concept as such but rather in the objects that falling under the concept […]. This view gives primacy of meaningful, richly contextualized forms of (mathematical) structure over formal (mathematical) structure" (p. 175). Here we focus the attention to several core assumptions that orient the evolving framework of structural abstraction (see Scheiner, 2016; Scheiner & Pinto, 2016b):

Concretizing through Contextualizing

Structural abstraction takes place on mental objects that, in Frege's (1892a) sense, fall under a particular concept. These objects may be either concrete or abstract. Concreteness and abstractness, however, are not considered as properties of an object but rather as properties of an individual's relatedness to an object in the sense of the richness of a person's representations, interactions, and connections with the object (Wilensky, 1991). From this point of view, rather than moving from the concrete to the abstract, individuals, in fact, begin their understanding of (advanced) mathematical concepts with the abstract (Davydov, 1972/1990). The ascending from the abstract to the concrete requires a *concretizing* process where the mathematical structure is particularized by looking at the object in relation with itself or with other objects that fall under the particular concept. The crucial aspect for concretizing is *contextualizing*, that is, setting the object(s) in different specific contexts. Different contexts may provide various senses (Frege, 1892b) of the concept of observation.

Complementizing through Recontextualizing

The central characteristic of the structural abstraction framework is that while, within the empiricist view, conceptual unity relies on the commonality of elements, it is the interrelatedness of diverse elements that creates unity within the approach of structural abstraction. The process of placing objects into different specific contexts allows specifying essential components. Structural abstraction, then, means attributing the particularized meaningful components of objects to the mathematical concept. Thus, the core of structural abstraction is *complementarity* rather than similarity. The meaning of advanced mathematical concepts is developed by complementizing diverse meaningful components of a variety of specific objects that have been contextualized and recontextualized in multiple situations. This perspective agrees with van Oers' (1998) view on abstraction as related to *re*contextualization instead of *de*contextualization.

Complexifying through Complementizing

The structural abstraction framework takes the view that knowledge is a complex system of many kinds of knowledge elements and structures. Complementizing implies a process of restructuring the system of knowledge pieces. These knowledge pieces have been constructed through the above-mentioned process or are already constructed elements coming from other concept images, which are essential for the new concept construction. The cognitive function of structural abstraction is to facilitate the assembly of more complex and compressed knowledge structures. Taking this perspective, we construe structural abstraction as moving *from simple to complex* knowledge structures, a movement with the aim to build coherent and compressed knowledge structures. In Thurston (1990)'s words, when the latter is achieved we "can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process." (p. 847). From the structural abstraction perspective, abstraction is acknowledged as a movement across levels of complexity (Scheiner and Pinto, 2014).

Emerging Insights from the Structural Abstraction Framework

In this section, we summarize emerging insights we gained so far by using the evolving framework of structural abstraction retrospectively as a lens through which findings on students' (re-)construction of the limit concept of a sequence were reinterpreted. The study by Pinto (1998) provided the context in which she identified mathematics undergraduates' sense-making strategies of formal mathematics. From a cross-sectional analysis of three pairs of students, two prototypical strategies of making sense could be identified, namely 'extracting meaning' and 'giving meaning':

> "*Extracting meaning* involves working within the content, routinizing it, using it, and building its meaning as a formal construct. *Giving meaning* means taking one's personal concept imagery as a starting point to build new knowledge." (Pinto, 1998, pp. 298-299)

The literature on abstraction from actions provides several accounts of how students construct a mathematical concept compatible with their strategy of 'extracting meaning'.

For instance, dynamic views of the limit of a sequence and the genetic decomposition of the limit concept of a sequence are intensively investigated by Arnon et al.'s (2014) APOS theory to respond to how students may construct the concept through the process of reflective abstraction. To mention a few recent investigations supported by the same theory, and compatible with the strategy of 'extracting meaning', Martinez-Planell, Gonzalez, DiCristina, and Acevedo (2012) focused on students' understanding of series and investigated whether students saw series as a process without an end or as a sequence of partial sums, as stated by definition. They respond how students may construct the concept, by considering a distinction amongst their understandings of the concept of a sequence as a list of numbers or as a function defined in natural numbers (McDonald, Mathews, & Strobel, 2000), and concluded that even after formal training, students often think of sequences and series as an infinite, unending process.

However, there are almost no accounts of how students construct a concept compatible with their strategy of 'giving meaning', and the structural abstraction framework has shown to be enlightening with regard to this. Students who 'give meaning' seem to develop representations of the limit concept from their concept image and use them generically (see Yopp & Ely, 2016) for constructing and reconstructing the concept (see Pinto & Scheiner, 2016). This means, such representations are not always generic in the sense of Mason and Pimm (1984) though they are used as if they were of that nature. Moreover, such representations may be productive in some, though not all contexts, in which they are needed. In spite of the striking differences in the knowledge constructions in each case study, that are made explicit by the nature of the representations construed and their use, the three case studies presented in Pinto (1998) on students' strategy of giving meaning have in common a cohesion in their sense-making and in learning the formal mathematics concept (Pinto & Scheiner, 2016; Scheiner & Pinto, 2014). Pinto and Scheiner (2016) concluded that coherence amongst students' sense-making and their (re-)construction of the formal content had been proven to be a central characteristic of those students who 'give meaning'. This does not mean that the reconstructions a student made are configured in a "satisfactory reconstruction or accommodation" scenario (Vinner, 1991, p. 70); rather, that apart of the learning scenario, a student's sense making is coherent with her or his learning of a mathematical concept.

It is important to note that the evolving framework of structural abstraction is problem driven, that is, addressing the need for bringing light into the complexity of students' knowing and learning processes compatible with their strategy of 'giving meaning', rather than filling a theoretical gap just because it exists. The reinterpretation of empirical data on students' strategies of giving meaning in the light of the theoretical framework of structural abstraction proved to be particularly fruitful – not only to provide deeper insights into the strategy of giving meaning but also as a way to deepen our understanding of the phenomenon of structural abstraction that revealed new theoretical developments (Pinto & Scheiner, 2016; Scheiner & Pinto, 2014). In the following sections, we highlight the main theoretical advancements.

The idea of complementizing meaningful components underlying the structural abstraction framework reflects the idea that whether an individual has 'grasped' the meaning of a mathematical concept is situated in specific contexts where the objects falling under the specific mathematical concept have been placed in. In the case studies revisited, these contexts include the formal mathematics one, where mathematical objects are presented as formal definitions and their properties are deducted through formal proofs. Such a diversity of situated or contextualized meanings supports Skemp's (1986) viewpoint that "the subjective nature of understanding [...] is not [...] an all-or-nothing state" (p. 43). The reanalysis of the data indicates that the object of researchers' observation should be directed to students' *partial constructions* of the limit concept. These partial constructions may be specific and productive to particular contexts but may remain not fully connected and may be unproductive in other contexts (for instance, in making sense of the formal definition). The empirical data show that, in the case of the students who give meaning, several meaningful elements and relations in understanding the limit concept of a sequence are involved, although a few elements are missing (or distorted). However, some students are able to (re-)construct some meaningful components at need by making use of their partial constructions, while others are not able to do so.

Our reanalysis indicates that some students have developed *resources* that enable them to (re-)construct the limit concept of a sequence at need. Scheiner and Pinto (2014) focused on a case where a student developed a generic representation of the limit concept of a sequence that operates well in several, although not all, contexts and situations. This particular representation, however, allows the student to (re-)construct the limit concept in other contexts and situations. The reinterpretation of the data sheds light on the phenomenon that individuals may not 'have' all relevant, meaningful components, but, rather, they may have resources to generate some meaningful components and make sense of the context at need. In that sense, the ‘completeness’ of the complementizing process cannot ever be taken as absolute.

Several researchers suggested exposing learners to multiple contexts and situations. An important insight from using the structural abstraction framework retrospectively is that exposure to multiple contexts is at least important for particularizing meaningful components: various objects falling under a particular mathematical concept have to be set into different specific contexts in order to make visible the meaningful components or mathematical structure of these objects. In so doing, the objects may be ‘exemplified’ through a variety of representations, in which each representation has the same reference (the mathematical object); however, different representations may express different senses depending on the selected representation system (see Fig. 2).

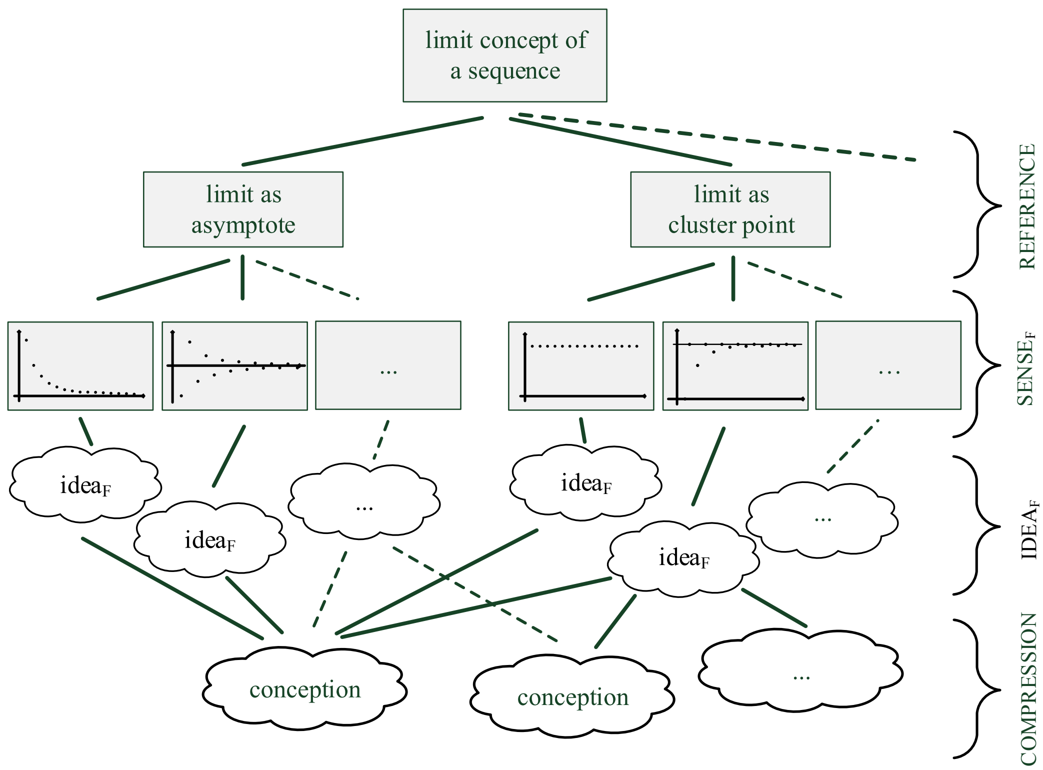

Fig. 2: Reference, sense, and idea

The distinction between *sense* and *reference* has been specified by Frege (1892b) in his work *Über Sinn und Bedeutung*, indicating both the sense and the reference as semantic functions of an expression (a name, sign, or description). In short, the former is the *way* that an expression refers to an object, whereas the latter is the *object* to which the expression refers. According to Frege (1892b), to each representation correspondents a sense; the latter may be connected with an *idea* that can differ within individuals since people might associate different senses with a given representation. Though multiple contexts and situations are needed, a new context that does not provide a new sense will unlikely be productive for concept construction.

Research also indicates that students may have difficulties with the *relationships* between the sense and the reference as well as difficulties in *maintaining* the reference as the sense changes (Duval, 1995, 2006). Thus, one might assume that these difficulties may (at least partly) be overcome by providing students a particular resource (such as a generic representation of the mathematical concept) that serves as a guiding tool in complementizing the meaningful components indicated in the different senses. From this perspective, a 'representation for' is a tool that provides theoretical structure in constructing the meaning of the concept of observation. It necessarily reflects essential aspects of a mathematical concept but can have different manifestations (Van den Heuvel-Panhuizen, 2003). Concerning the learning of the limit concept of a sequence, the reinterpretation of the data indicates that a slightly modified version of a student's representation (see Fig. 3) may support the complementizing process when the limit concept is recontextualized.

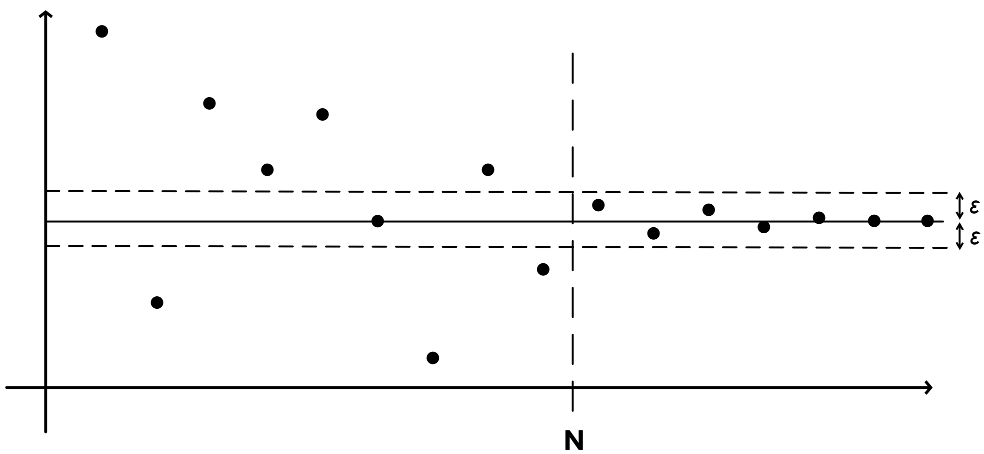

Fig. 3: A generic representation for learning the limit concept of a sequence

Notice that this generic representation for learning the limit concept of a sequence takes account of several students' common conceptions identified in the research literature, including those as (1) the limit is unreachable, (2) the limit has to be approached monotonically, and (3) the limit is a bound that cannot be crossed (see Cornu, 1991; Davis & Vinner, 1986; Przenioslo, 2004; Tall & Vinner, 1981; Williams, 1991).

The reanalysis of the empirical data gained from Pinto's (1998) study has shown that students who gave meaning built a representation of the concept and, at the same time, used it as a representation for recognizing and building up knowledge – the reconstruction of the formal concept definition, for instance. The analysis shows that these students consistently used *representations of* mathematical objects to create pieces of knowledge; or, in other words, the representations were actively taken as *representations for* emerging new knowledge and making sense of the context and situation. This shift from constituting a representation of the limit concept to using this representation as a representation for (re-)constructing the limit concept in other contexts can be described in terms of shifting from a *model of* to a *model for* (Streefland, 1985) – a shift from an *after*-image of a piece of given reality to a *pre*-image for a piece of reality to be created. Adopting this view, we may indicate variations in knowledge structures related to the possible explanations that are considered. Models may involve acceptance of other hypothesis through deduction, causality or analogy.

This mental shift from 'after-image' to 'pre-image' indicates a degree of awareness of the meaningful components and the complexity of knowledge structure that allows the transition from a 'representation of' as a result of various representations expressing specific objects set in different contexts to a 'representation for' constructing and reconstructing the limit concept, if inter alia, in formal mathematical reasoning. We suggest that a generic representation, as presented in Fig. 3, may provide an instructional tool that supports raising the awareness of meaningful components in learning the limit concept of a sequence. In other words, such a generic representation may direct students' perception of meaningful components although it does not enshrine mathematical knowledge.

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A Comparison of Calculus, Transition-to-Proof, and Advanced Calculus Student Quantifications in Complex Mathematical Statements

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Abstract: This study investigates Calculus, Transition-to-Proof, and Advanced Calculus students' meanings for quantifiers in conditional statements involving multiple quantifiers. Three students from each course participated in clinical interviews. Students were presented with the Intermediate Value Theorem (IVT) and three other statements whose logical structure was similar to the IVT except for the order of both the quantifiers and their attached variables. The results reveal that Advanced Calculus and Transition-to-Proof students made distinctions between the different statements more often than Calculus students. Several student meanings for quantification were found to be necessary for making distinctions between each of the four statements. We also address student quantifications for hidden quantifiers in the statements.

Key words: Quantifiers, Student Interpretations of Mathematical Statements, Intermediate Value Theorem (IVT), Comparative Analysis, Undergraduate Students

The teaching and learning of Calculus is important for all STEM programs. Many studies have investigated STEM students' meanings for important Calculus concepts such as limits, differentiation, and integration, and how these concepts are presented to students (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Martin, 2013; Oehrtman, 2009; Orton, 1983; Thompson, 1994; Vinner & Dreyfus, 1989; White & Mitchelmore, 1996; Zandieh, 2000). Many Calculus concepts like the ones listed above are described in mathematical statements with multiple quantifiers. However, little attention has been paid to students' interpretations of the quantifiers used to express Calculus ideas.

In this paper, we focus on students' meanings for quantifiers in complex mathematical statements from Calculus. By complex mathematical statements, we mean statements that have both if-then structure and multiple quantifiers. The Intermediate Value Theorem (IVT) is one example of such a statement: "Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a real number c in (a, b), such that $f(c)=N$" (Stewart, 2003, p. 131). Many Calculus definitions and theorems like the IVT can be classified as complex mathematical statements, even though the topic of quantification is not addressed in popular Calculus textbooks (Bittinger, 1996; Larson, 1998; Stewart, 2003).

Although some studies have investigated students' and teachers' treatment of quantifiers, these studies have dealt with a single quantifier in mathematical contexts or multiple quantifiers outside of a mathematical context (Dubinsky & Yiparaki, 2000; Piatek-Jiminez, 2010; Tabach et al., 2010; Tsamir et al., 2009). On the other hand, studies that have dealt with multiple quantifiers in mathematical contexts have placed more emphasis on Transition-to-Proof and Advanced Calculus students rather than Calculus students (Dawkins & Roh, 2011; Dawkins & Roh, 2016; Roh & Lee, 2011, 2015; Selden & Selden, 1995). This study explores the following questions for students who have completed either Calculus, Transition-to-Proof, or Advanced Calculus:

1. *How do students evaluate complex mathematical statements from Calculus?*

2. *What meanings do students at various mathematical levels have for quantifiers in complex mathematical statements from Calculus?*

Literature Review

Previous studies about student understanding of statements involving quantifiers focused on three issues as follows: students' understanding of statements with one quantifier, students' understanding of statements involving more than one quantifier, and students' understanding of statements in which quantifiers were not stated, but rather hidden.

Student Understanding of Single Quantifiers. Research in mathematics education points out that students may interpret quantifier words such as "for all" $(\forall)$, "there exists" $(\exists)$, and "there exists a unique" $(\exists!)$ differently than the mathematics community (Dawkins & Roh, 2016; Dubinsky & Yiparki, 2000; Epp, 1999, 2003). In turn, students' understanding of quantifiers in a statement plays a crucial role in their justification of a statement involving a quantifier. In particular, previous studies show that some students tended to suggest that a few examples are sufficient to show that a statement involving a single universal quantifier, in the form 'For all x, P(x),' where P(x) is a statement about x, is true (Bell, 1976; Healy & Hoyles, 2000; Moore, 1994; Selden & Selden, 1995). Some students even wished to examine additional examples even after seeing a generic proof of such a statement (Fischbein, 1982). On the other hand, when disproving a statement of the form 'For all x, P(x)', students often believed that only one counterexample is insufficient to disprove the statement (Balacheff, 1986; Galbraith, 1981) or they thought the statement must be still true as they viewed the counterexamples as outliers (Zaslavsky & Ron, 1998). In the case of a statement involving a single existential quantifier, in the form 'there exists x such that P(x),' students also rejected the notion that one example would suffice for proving such a statement (Selden & Selden, 1995). All of these issues may be rooted in the fact that students have different meanings for quantifiers other than the conventional mathematical uses of these quantifiers. Epp (2003) argues that some colloquial uses of quantifiers may influence students' use of quantifiers in mathematics. For example, the phrase "for all" in an English sentence often implies the existence of at least one element in a set. Based on such a colloquial use of the phrase "for all," students might interpret the universal quantifier in a mathematical statement as they interpret it in an English sentence whereas, in mathematics, a statement with a universal quantifier could be vacuously true. Shipman (2013) also found that students' conception of the unique existence in mathematics was not compatible with its meaning in mathematics but rather consistent with its colloquial meaning; in fact, students thought "unique" would mean *unequaled*.

Student Understanding of Multiple Quantifiers. Even if students appropriately interpret single quantifiers in mathematical contexts, they may not correctly interpret quantifiers if a statement involves multiple quantifiers. By multiple quantifiers we mean at least two quantifiers in a statement such as two quantifiers "for all N, there exists c" in the Intermediate Value Theorem. We refer to a statement of the form $\forall x(\exists y P(x,y))$ as an $\forall\exists$ statement and a statement of the form $\exists y(\forall x P(x,y))$ as an $\exists\forall$ statement. Previous studies showed that many students did not distinguish difference between $\forall\exists$ statements and $\exists\forall$ statements and more frequently misinterpreted $\exists\forall$ statements as $\forall\exists$ statements (Dubinsky & Yiparaki, 2000; Piatek-Jiminez, 2010). Other studies showed that some students switched the order of the variables attached to the quantifiers in a statement. For instance, in the case of the ε-N definition of the limit of a sequence, which is an $\forall\exists$ statement, students often considered the variable N before considering the variable ε, or determined the value of ε based on the value of N. In fact, these students did not necessitate the independence of the first variable ε and dependence of the second variable N in the statement (Dawkins & Roh, 2016; Roh & Lee, 2011, 2015).

Student Understanding of Implicit Quantifiers. Implicit (or hidden) quantifiers refer to the quantifiers that are not explicitly stated through direct phrasing such as "for all" and "there exists" (Durand-Guerrier, 2003; Shipman, 2016). The IVT, as stated in this paper, also has an implicit universal quantifier because it is a generalized conditional. The IVT is a generalized statement, and thus, "Suppose that f is a continuous function" implies that the statement has a hidden universal quantifier. Thus, f is often understood in mathematics as representing not a single continuous function, but *all* continuous functions. However, since the implicit universal quantifier is not explicitly stated in the statement, students may not take into account the arbitrariness of f. In fact, Selden and Selden (1995) found that undergraduate students in Transition-to-Proof courses were rarely able to explicitly interpret implicit quantifiers in statements in Calculus contexts. Likewise, several studies reported students' difficulties with reinterpreting a given statement with implicit quantifier into a statement including a statement that include explicit language for quantifiers, logical connectives, and structure (Durand-Gurrier, 2003; Selden & Selden, 1995).

Theoretical Perspective

In this study, we are trying to model individual students' meanings for quantifiers in mathematical statements. The discussion of meaning has permeated philosophy for centuries, particularly within the last century. Speaks (2014) discusses two major branches in theories of meaning: semantic theories and foundational theories. He claims that semantic theories specify "the meanings of the words and sentences of some symbol system" (p.2) and seek to answer the question, "What is the meaning of this or that expression?" (p.2) whereas foundational theories try "to explain what about some person or group gives the symbols of their language the meanings that they have" (p. 2). Foundational theories of meaning are more advantageous for our purposes than semantic theories in their recognition that meanings exist in the *minds* of students rather than on a page. However, foundational theories still do not address the need to analyze *individual* students who have had different individual experiences that led to their current meanings for quantifiers. In this study, we view thinking as a tool by which one constructs his meaning(s), and focus on meanings of individual*s* and not a collective group. Thus, we utilize a constructivist view of understanding and meaning throughout this paper (Glasersfeld, 1995; Thompson, 2013).

We distinguish between *understanding* and *meaning* throughout the rest of this paper. Thompson, Carlson, Byerly, and Hatfield (2014) use *understanding* to describe a student's assimilation to a scheme and a student's *meaning* to describe the mental actions or schemes that are easily triggered as a result of the understanding (assimilation). As the purpose of this study is to describe our best perception of each student's own meanings for quantifiers at different moments, we utilize the phrases *meaning* and *student meaning* throughout this paper the same way as Thompson et al. (ibid).

We utilize the phrase *student quantification* to refer to the process by which students quantify variables. Even in one student's quantification, different schemes of the student may be triggered by different tasks. Some student meanings may be stable, but other meanings may be "meaning(s) in the moment" (Thompson et al., 2014). Thompson et al. (2014) describe a meaning in a moment as "the space of implications existing at the moment of understanding" (p. 13), so students could be assimilating information in the moment and forming new meanings. A student's thoughts may begin to emerge or different meanings may be elicited in different moments. Thus, we consider several different moments and "meanings in the moment" (Thompson et al., ibid) because different moments may result in different types of student quantification.

Methods

Two-hour long clinical interviews (Clement, 2000) were conducted with nine undergraduate students during the spring and summer of 2016 at a large southwestern university in the United States. Students were placed in a category based on the highest course they had already completed. These students had various STEM majors and completed these courses with a variety of different instructors. Three students were selected from each mathematical level: Calculus I, Transition-to-Proof, and Advanced Calculus. Four researchers served in various roles for the data collection (as either interviewer, camera operator, or witness).

Interview Tasks. In the first half of the interview, students were asked to explain their understandings of the four statements shown in Table 1 and to evaluate the truth-values of each of the statements. (Only the statements in the left-hand column of Table 1 were presented to students.) The four statements in Table 1 exhaust all combinations for ordering explicit quantifiers and their attached variables. The symbolic representations of the explicit quantifiers found in the conclusion of each statement are also shown beside each statement in Table 1. Three of the statements are false. Statement 2 is the IVT and the only true statement. The variety of statements allowed us to analyze students' comparisons and contrasts among the explicit quantifiers.

Table 1
Statements Presented in Clinical Interviews

Statements	Symbolic Representations
Statement 1: Suppose that f is a continuous function on the closed interval $[a, b]$, where $f(a)\neq f(b)$. Then, for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$.	$\forall c\exists N$ such that $f(c)=N$
Statement 2: Suppose that f is a continuous function on the closed interval $[a, b]$ where $f(a)\neq f(b)$. Then, for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b), such that $f(c)=N$.	$\forall N\exists c$ such that $f(c)=N$
Statement 3: Suppose that f is a continuous function on the closed interval $[a, b]$, where $f(a)\neq f(b)$. Then, there exists a real number N between $f(a)$ and $f(b)$, such that for all real numbers c in (a, b), $f(c)=N$.	$\exists N\forall c$ such that $f(c)=N$
Statement 4: Suppose that f is a continuous function on the closed interval $[a, b]$ where $f(a)\neq f(b)$. Then, there exists a real number c in (a, b), such that for all real numbers N between $f(a)$ and $f(b)$, $f(c)=N$.	$\exists c\forall N$ such that $f(c)=N$

We presented students with graphs in the second half of the interview. Students were asked to re-examine the four statements along with each of the graphs shown in Figure 1. The interviewer asked each student to explain why they a statement was true or false, in their opinion, by using these graphs.

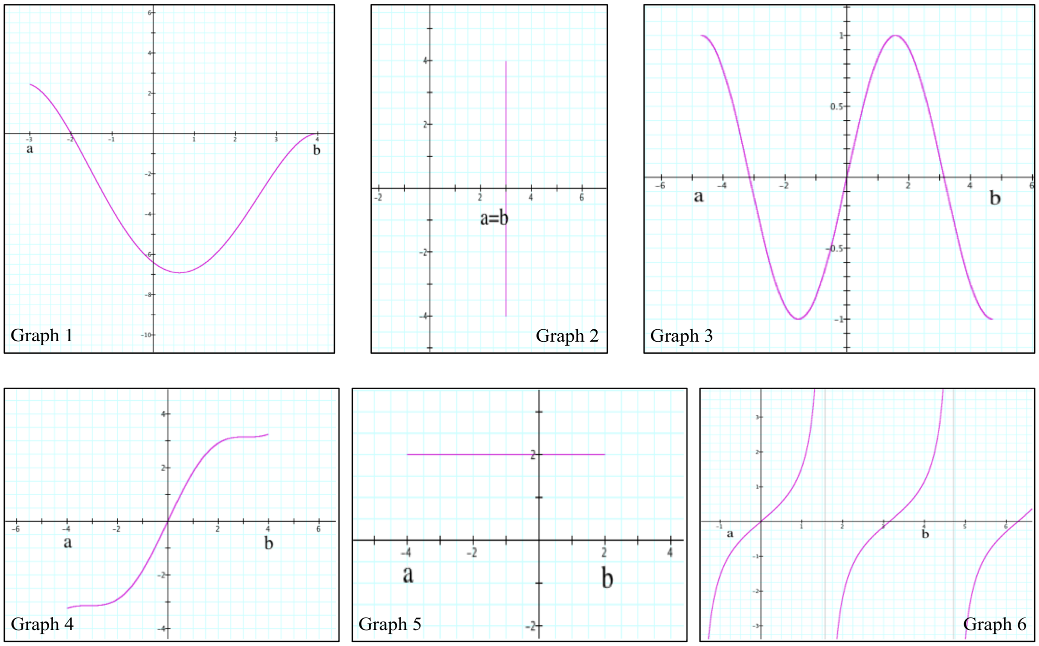

Figure 1. Graphs presented in second half of interview.

These graphs were all strategically chosen based on their properties as an example, non-example, or counterexample for each of the four statements. We used these graphs in the interview so that students could highlight the order in which they would select variables in a given statement and so that we could notice sweeping or pointing gestures that might indicate characteristics of their meanings for quantifiers.

Data Analysis. Our analysis was conducted in the spirit of grounded theory (Strauss & Corbin, 1998) using videos of the student interviews as well as the students' written work. Hence, the categories that describe students' meanings for quantifiers in this paper emerged from our data and not from previously created categories. Each interview was analyzed moment-by-moment to identify moments where distinctions could be made about a student's meanings for the quantifiers. A new moment began when a student was presented with a new question or task, changed their evaluation of a statement, or the student provided a new description of the quantifiers in a given statement. We used moments as our unit of analysis rather than students because students could exhibit various meanings for quantifiers in different moments of the interview. For example, in the second part of the interview, students were asked to explain why their evaluation was true or false given particular graphs. When students were analyzing the statements with these graphs, they would often change their evaluations of the statements or question their previous arguments.

We coded the data for characteristics of students' meanings for quantifiers. Once the difference in explicit and implicit quantifications was noticed, codes were organized and separated into one of these two categories. Categories emerged for types of student meanings for explicit and implicit quantifiers. In the final stage of data analysis, each interview was re-coded moment-by-moment for reliability. We then counted the number of moments that occurred in each category amongst all students. Some moments may not be useful for answering a particular research question. Other moments were found that lacked sufficient evidence for analysis. Thus, we only counted *relevant moments* in this study. By relevant moment, we mean any moment where sufficient evidence was available for answering a particular research question. Only relevant moments were considered in each phase of our analysis. For each research question, relevant moments were also tagged by the interviewee's mathematical level so that we could make comparisons about student meanings of quantifiers for each group.

Results

In this section, we first report how students evaluated the four statements provided (see Table 1 for the statements), and then further analyzed students' meanings for quantifiers in these statements. We also provide our comparative analysis on student meanings for quantifiers in these statements based on three different mathematical levels: Calculus, Transition-to-Proof, and Advanced Calculus.

Student Evaluations of Statements

All three groups of students correctly evaluated Statement 2 (the IVT) for more than half of all relevant moments. However, all three groups of students also incorrectly evaluated Statement 1 for more than half of all relevant moments, and Calculus students correctly evaluated Statements 3 & 4 ($\exists\forall$ statements) for less than a quarter of all relevant moments. Table 2 shows the percentage of relevant moments where students provided a truth-value for one of the statements. The unit of analysis was a moment, as detailed in the methods section. Only moments where students provided truth-values are shown in Table 2. Mathematically correct truth-values for each statement are shaded in Table 2.

Table 2
Student Evaluations for Statements 1-4 by Mathematical Level (by % of Relevant Moments)

	Calculus			Transition-to-Proof			Advanced Calculus		
	True	False	True/ False[1]	True	False	True/ False	True	False	True/ False
Statement 1 (S1): $\forall c \exists N$	6/8 (75%)	1/8 (12.5%)	1/8 (12.5%)	5/8 (62.5%)	3/8 (37.5%)	0/8 (0%)	4/9 (44.4%)	4/9 (44.4%)	1/9 (11.1%)
Statement 2[*] (S2): $\forall N \exists c$	6/9 (66.7%)	2/9 (22.2%)	1/9 (11.1%)	6/9 (66.7%)	3/9 (33.3%)	0/9 (0%)	8/9 (88.9%)	1/9 (11.1%)	0/9 (0%)
Statement 3 (S3): $\exists N \forall c$	6/9 (67%)	1/9 (11.1%)	2/9 (22.2%)	2/8 (25%)	6/8 (75%)	0/8 (0%)	0/8 (0%)	8/8 (100%)	0/8 (0%)
Statement 4 (S4): $\exists c \forall N$	8/11 (72.7%)	2/11 (18.2%)	1/11 (9.1%)	2/6 (33.3%)	3/6 (50%)	1/6 (16.7%)	0/8 (0%)	8/8 (100%)	0/8 (0%)

As shown in Table 2, while all three groups of students evaluated Statement 2 (IVT) correctly in more than 60% of relevant moments, this result does not imply that students understand the IVT properly because students' evaluations of the other three statements were often incorrect. For instance, Calculus students and Transition-to-Proof students incorrectly evaluated Statement 1 as a true statement in more than 70% and 60% of relevant moments,

[1] True/False indicates that student responses were "Sometimes True" and "Sometimes False."
[*] Statement 2 is the Intermediate Value Theorem.

respectively. Even Advanced Calculus students incorrectly evaluated Statement 1 as a true statement in more than 40% of relevant moments. We cannot claim that undergraduate students comprehend the quantification in the IVT if they cannot correctly evaluate these other statements. Also, this result indicates that even if students correctly evaluate Statement 2, they could reach a correct conclusion for the wrong reasons. We will now explain *why* these students' evaluations often differ from mathematical convention.

Student Meanings for Explicit Quantifiers

In order to understand *why* these students' evaluations of Statements 1, 3 and 4 were often incorrect, we further analyzed students' own interpretations of each of the four statements and compared whether they considered any of the four statements would have the same meaning. From this analysis process, we found that in all relevant moments throughout the interviews, Advanced Calculus students all concluded that none of the four statements have the same meaning. On the other hand, in some moments Transition-to-Proof and Calculus students said that some or all of the statements would have the same meaning. In particular, Calculus students often interpreted all four statements as Statement 1 although the order of quantifiers and the order of variables in each statement were all different from one another. We thus even further analyzed what meanings these students have for quantifiers in these statements. Three main factors emerged from this phase of data analysis that seemed to play a crucial role in and interpreting the four complex mathematical statements: (1) Having distinct meanings for single quantifiers, (2) having distinct meanings for the order of quantifiers, and (3) having distinct meanings for the order of variables attached to the quantifiers.

Distinguishing Single Quantifiers "For all" and "There exists." Some students exhibited distinct meanings that differed for each of the phrases "for all" and "there exists." Some moments involved student specification of the meanings of these phrases in their own words, such as describing "for all," as "for each one of these," or clarifying "there exists" as "I can find at least one." Other student moments were characterized by an interchange or alteration of the meanings for these phrases. Even though the phrases "for all" and "there exists" were used in the statements, students' explanations sometimes included using universal quantifier language such as "every," and "all" for the variable attached to the existential quantifier. Likewise, during some moments, students used phrases such as "I can find an" or "there is a" for the variable attached to the universal quantifier. Such student utterances indicate that in these moments, students did not have distinguishable meanings for the singular universal and existential quantifiers.

Hannah was a Calculus student who did not distinguish between "for all" and "there exists" in some moments. Hannah stated that Statement 4 ($\exists c \forall N$) had the same meaning as another previous statement. She also claimed in this moment, "anywhere you choose c to be, there is a value of N that's a real number because the function is continuous." Hannah said we could choose c-values *anywhere*, which indicates that she was treating the existential quantifier as if it were a universal quantifier. She also said that we could find *a* value of N, even though the statement says that we need to find a c that works for *all* values of N. This dialogue indicates that in this moment she was also treating the universal quantifier as if it were an existential quantifier.

Distinguishing Order of Quantifiers. Statement 1 ($\forall c \exists N$) and Statement 3 ($\exists N \forall c$) both have "there exists" with N and "for all" with c, but the order of these quantifiers is different. Statement 2 ($\forall N \exists c$) and Statement 4 ($\exists c \forall N$) also reorder in a similar fashion. Some student moments included explanations of why each statement in these pairs had a distinct meaning. Other student moments were classified by student views that these pairs were equivalent in meaning. Our findings are similar to those of Dubinsky and Yiparaki (2000), who showed that students may view reordering quantifiers, with variables attached, as inconsequential. Mike, a Transition-to-Proof student, explained why he thought Statement 1 and Statement 3 are similar:

> Mike: [Statement 3] and [Statement 1] are saying the same thing and [Statement 2] and [Statement 4] are saying the same thing. These two (*Statements 1 and 3*) are saying that there is only one output for all the inputs. But [for] Statement 1 and 3, I could assume this… for all real numbers c in the interval (a, b) I could assume that's true.

Mike acknowledged that he believed Statement 1 and Statement 3 are equivalent in meaning and Statement 2 and Statement 4 are equivalent in meaning. His classification insinuates that he did not distinguish between these quantifier distinctions. We have further evidence of his lack of distinction because he stated that both Statements 1 and 3 are about *one output* and *all* c's. Mike recognized that these pieces of the quantification are the same. However, since he emphasized the singular quantifiers and concluded that the statements are equivalent, we claim that he did not distinguish the difference in the order of the quantifiers in the statements.

Distinguishing Variable Order and Dependence. We also found a difference in how students treated the order of the variables c and N in their explanations of the statements. All but one student mentioned that the order of the variables was different in some of the statements. However, during some moments, students claimed that reordering the variables had no effect on the meanings of the statements. Zack, a Calculus student, explained why he thought Statement 2 ($\forall N \exists c$) was different, but equivalent, to Statement 1 ($\forall c \exists N$) in this moment:

> Zack: Um, so I mean obviously this [N] is now flipped with c, at least in [Statement 2]. I don't know how this [switch] necessarily affects [the statement]. So, when I explained it on the last one, I thought that N was a dependent value depending on what c is… Such that $f(c)$ is equal to N… I don't think that, I'm sorry. So it's like that… I'm interpreting [Statement 2] the same, that it's just saying that really for any real value of N's there exists a value of c, but c is still the value. Like… this (*points to c in f(c) in statement*) is the independent value versus this (*points to an N in the statement*) is the dependent value.

We claim that Zack recognized the variable order in Statement 2 in this moment, but he did not treat the first variable, N, as being chosen independently of the c-values. Zack mentioned that c is an independent variable and N is a dependent variable in the function in both statements. Zack appeared to choose c-values first because he thought that independent variables should be chosen first. However, he did not seem to connect the variable order in the quantification to his own choice for which variable should be held independent of the other.

Many student moments were different than Zack's moment; in these moments, students noticed the order of the variables and also displayed an understanding of the ramifications of this variable switch in different statements. Jay, an Advanced Calculus student, explained why Statement 1 ($\forall c \exists N$) and Statement 3 ($\exists N \forall c$) are different:

> Jay: For this one [Statement 1] all of the... like the values for individual c's can be different. So like given c, I give you an N, but that doesn't have to be the same N as some other c. But for here [Statement 3] that's not true. There exists a real... so I give you N before you give me c, meaning I know what the answer is before I even know what c is.

Jay was not only aware of the variable order in this moment, but how the variable order affects the statements' meanings. Jay contrasted the order in which the first person gave a c or N and the other person gave a variable in return. His contrast suggests that he thought of the first variable independently of the second variable. He also used the first variable's information to give him information about the second variable, which indicates that he considered that the second variable is dependent on the first variable.

Explicit Quantifier Distinctions Across Mathematical Levels. The four different meanings in Table 2 were more prevalent amongst the more advanced students interviewed. The results of this comparison of percentages from each of the three groups are shown in Table 3.

Table 3
Comparison of Students' Explicit Quantifier Meanings

Explicit Quantifier Meanings (by % of Relevant Moments)	Calculus	Transition-to-Proof	Advanced Calculus
Distinguished "for all" and "there exists"	4/17 (23.53%)	21/22 (95.45%)	17/17 (100%)
Distinguished "for all… there exists…" and "there exists… for all…"	0/8 (0%)	7/12 (58.33%)	6/6 (100%)
First Variable Independent & Second Variable Dependent	4/19 (21.05%)	14/15 (93.75%)	11/11 (100%)

Table 3 summarizes that higher-level mathematics undergraduates exhibit more moments with distinctive meanings for explicit quantifiers. The Advanced Calculus students had a greater percentage of moments where they exhibited each of these four meanings than Calculus students. Calculus student moments not only showed students' confounded meanings of "for all… there exists…" and "there exists… for all…," but they also revealed Calculus students' tendencies to confound the singular quantifiers "for all" and "there exists" as well. While the Transition-to-Proof students had a higher percentage of moments than Calculus students for distinctions between "for all" and "there exists," there were still several moments where they confounded meanings of "for all… there exists…" and "there exists… for all…" Calculus students may have recognized the change in variable order across statements, but the variable dependency moments revealed that this recognition was not associated with students' understanding for why variable ordering is important to statement meanings.

Students' Implicit Quantifications

Calculus students had fewer moments where they distinguished between different types of explicit quantifiers and their attached variables than their advanced peers. However, meanings for implicit quantifications varied amongst the groups. The phrase "Suppose f is a continuous function" elicited ambiguity in some student moments. All four statements in Table 1 are written with the intent that readers will apply each statement to *all* continuous functions. We found three ways that these students implicitly quantified f: universally, existentially, or case-by-case.

Universal Implicit Quantification. Some students did quantify the phrase "Suppose f is a continuous function" as intended. Students used words such as "arbitrary" or the phrase "*any* continuous function" to describe their meanings of the statements.

Case-by-case Implicit Quantification. Three students considered that some of the statements that we were giving them were not firmly true or false for some moments. They preferred to evaluate some statements as "sometimes true" or "sometimes false" instead of strictly true or false. Their reasoning for this choice was made apparent when we offered them graphs and they considered that the statement was true for some functions, and false for some functions. Zack claimed in one moment, “No, I still agree that this statement [Statement 1] would be sometimes true… because in my mind this graph (*points to what we would consider the counterexample graph*) proves it… proves that I can't say the statement is true one hundred percent of the time.” These students are classified as “Case-by-Case” because they were considering the if-then statement as having a variable truth-value instead of being firmly true or false. Our evidence supports Durand-Guerrier’s (2003) finding that students tend to think of a conditional as an open statement which may be true or false, depending on the case at hand.

Existential Implicit Quantification. Ron, a Transition-to-Proof student, interpreted the phrase “Suppose *f* is a continuous function” with an existential quantification. He described his lack of certainty about the intent of the statement, and described his conclusions in this moment:

> Ron: I am not sure if *f* is limited to there being an existence of a continuous function or it's “suppose that any function.” So because the wording is ambiguous in my mind I am not sure. I am just gonna keep it true for now because I am going to assume that “Suppose *f* is a continuous function” is going to be equivalent to the wording being “Suppose that *there is an existence* of a continuous function *f* on the closed interval *a* to *b*.”

Ron’s implicit quantification of “Suppose *f* is a continuous function” affected the rest of his arguments because he believed he only needed one function to make each statement true. He also thought that proving each statement false required exhausting all functions.

Implicit Quantification Across Mathematical Levels. The summary of all the students’ meanings in the moment for implicit quantifiers is shown in Table 4.

Table 4
Comparison of Students’ Quantifications for “Suppose f is a continuous function”

Implicit Quantifications (by % of Relevant Moments)	Calculus	Transition-to-Proof	Advanced Calculus
Universal	6/8 (75%)	4/8 (50%)	7/8 (87.5%)
Case-by-Case	2/8 (25%)	3/8 (37.5%)	1/8 (12.5%)
Existential	0/8 (0%)	1/8 (12.5%)	0/8 (0%)

As shown in Table 4, the majority of moments when students were implicitly quantifying *f* were distinguished by universal quantification in all three groups of students. However, there were some student moments from all three mathematical levels that exhibited a case-by-case quantification of “Suppose *f* is a continuous function.” Our only existential implicit quantification moments originated from one Transition-to-Proof student, Ron.

Discussion

This study indicates that students may correctly evaluate the IVT, but their meanings for the quantifiers in the IVT may be different than the intention of the theorem. Students who eventually evaluated all four statements correctly also provided thorough explanations of the statements and possessed distinct meanings for different types of quantifiers. These students articulated distinctions for single quantifier words "for all" and "there exists," as well as distinctions for the order of quantifiers "for all… there exists…" and "there exists… for all…" They also recognized and explained the importance of the order of c and N in the statements. On the other hand, students who incorrectly evaluated the truth values of at least one statement among the four statements often did not have distinct meanings for at least one of the three factors: single quantifiers, the quantifier order, and the variable order. There were also moments from students of all three levels who implicitly quantified "Suppose f is a continuous function" in different ways than the authors of the statements intended for the statements to be quantified.

The results of this study suggest what we may need to consider quantifiers more carefully when teaching complex mathematical statements. First of all, we may need to place more attention on students' meanings for quantifiers, the quantifier order, and the variable order in a complex mathematical statement. Calculus students in this study did not recognize the necessity of quantifiers in the statements. Compared to Calculus students, Transition-to-Proof and Advanced Calculus students had distinct meanings more frequently for different types of quantifiers. However, some Transition-to-Proof moments were characterized by a lack of distinction in quantifier order, and some of the students both Transition-to-Proof and Advanced Calculus also did not correctly interpret the implicit quantifiers in the statement. Having stable and coherent meanings for quantifiers is foundational to understanding complex mathematical statements, and we as mathematics teachers would need to reconsider how to help students recognize when variables are ordered differently across a set of statements before they focus on how this reordering alters the meanings of the statements. These results should be considered when making curriculum and instructional decisions for all mathematical courses, but particularly for Calculus and Transition-to-Proof courses.

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Exploring Undergraduates' Experience of the Transition to Proof

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We report a qualitative analysis of 14 undergraduate students' experience in a semester long introduction to proof course. Half were mathematics majors. Our research aims to characterize, conceptually and empirically, students' transition from a focus on computation to proof in mathematics. Our analysis focused on how students saw the course as different from prior courses, whether it required new or different learning activity, how they described their work in proof-writing, and how they saw their confidence and success in the course. This approach—targeting students' overall experience of the course—differs from prior research that has tracked students' challenges, focused on their work on specific proof problems, and explored how to support and improve their work (e.g., Selden & Selden, 2003). Our work has promise for informing the design of transition to proof courses and how those courses are organized and taught.

Keywords: Transition to Proof, Proof Reasoning, Students' Experience, Qualitative Analysis

The Transition to Proof and Proving

Many undergraduates experience difficulty in learning to prove mathematical propositions, including those who major in mathematics (Baker & Campbell, 2004; Moore, 1994; Selden, 2012). For many, experience with proof prior to college is quite limited (Anderson, 1994; Jones, 2000). Students' prior competence (and interest) in mathematics is typically centered on producing accurate answers to easily recognized tasks—"exercises" in Schoenfeld's terms (1992). That work involves recognizing specific types of mathematical tasks quickly and applying well-practiced procedures to solve them. These abilities do not support, and may interfere with students' work to conceptualize, write, or evaluate proofs. In addition to differences in the didactical contract between computation-heavy courses like calculus and proof-intensive courses, Selden (2012) reports that students encounter many difficulties, including mastering logic and definitions, generating and using examples and counterexamples, understanding concepts and theorems, and evaluating the arguments of others. In short, the transition to proof is complex and challenging, and much of students' prior ability appears inadequate, if not problematic for addressing these challenges.

Many mathematics departments have recognized this problem and experimented with different curricular and instructional approaches to support students' entry into proof, including dedicated "transition to proof" courses. These typically focus on precision of language and notation, reasoning, and proof and present sample accessible statements from numerous content domains to prove or disprove. But designing these courses is not a trivial task, and they have often shown limited positive effects (Selden & Selden, 2003; 2014). A crucial limitation affecting course design is that research has not systematically focused on the experience of students. Instead, course design and instructional decisions have often been shaped by assumptions about students, the kinds of challenges they face, and structures that productively support their learning. To begin to address this limitation, we explored how students describe their experiences in one such “transition to proof” course.

Conceptual Framework

The notion of a “transition” from one view and way of working in a school subject to another is as conceptually vague as it is intuitively sensible. What does a “transition” in students’ experience of mathematics mean?

To conceptualize the transition from computation to proof, we drew on prior work that conceptualized mathematical “experience” and “transition.” Smith and Star (2007) proposed that precollege students’ mathematical experience could be understood and studied as a composite of four dimensions: Achievement, disposition, differences between prior and current mathematics courses, and learning activity. *Achievement* was simply their grade performance in mathematics courses (not an assessment of what they learned to those courses). *Disposition* involved students’ attitudes toward mathematics and their beliefs about their own ability to learn it (self-efficacy). *Differences* captured students’ sense of what changed from prior mathematics courses to work in their present classroom; it embraced issues of curriculum, teaching, and assessment. *Learning activity* involved the actions that students took to learn mathematics, in and outside of class. Assessing change on these dimensions in turn supported grounded judgments of the significance/depth of students’ transitions.

In exploring students’ transition to proof, we first targeted how students saw the introduction to proof course as different from (and similar to) their prior mathematics courses (differences). We also sought to understand how students went about learning the content in the course, again both inside the classroom and outside (learning activity). Substantive differences in how students saw the introduction to proof course could drive substantial changes in how students went about their work. We narrowed the notion of disposition to self-efficacy and explored how students’ confidence to learn and achieve changed over the semester (confidence). Relative to achievement, we solicited their final grades, but also explored if and how they judged their success in the course in ways other than grades. Finally and centrally, because the transition to proof involves the development of new mathematical abilities, we explored how the students described their work in planning, writing, and evaluating proofs on assigned problems (proof reasoning).

More broadly, we approached our data collection and analysis with the constructivist focus. That perspective oriented our attention to how students used prior resources (elements of their prior mathematical success) to address the new challenges of proof writing, whether those resources proved useful and effective or not (e.g., Smith, diSessa, and Roschelle, 1993/94). For example, this perspective suggests that elements of prior competence in computing answers using practiced procedures would likely be recruited in proof writing (e.g., looking for ways to “proceduralize” the task), despite the numerous ways in which applying procedures and proving previously unknown statements differ. In a word, this general orientation expects substantial continuity in students’ thinking even as elements of prior competence “don’t work” in proof writing.

The Course & The Participants

The 14 participants in the study were all graduates of a multi-section semester course designed to introduce them—both mathematics majors and not—to proof and proving. The mathematics department hoped that the course would support greater success for both groups of students in upper-division courses that emphasize proof. The introduction to proof course focuses on appropriate syntax and notation, basic concepts in set theory and logic, and various proof methods before proving "entry-level" statements from various content domains (e.g., linear algebra, real analysis, and number theory). These methods were mathematical induction, proof by cases, working forward from definitions, contrapositive, and contradiction. The course pedagogy is not lecture-based. Instead instructors give short presentations and present proofs (or parts of proofs) before students spend substantial time working on proof tasks themselves, often in small groups. The course also includes elements of "flipped classrooms;" students read and answer basic comprehension questions prior to working on problems in class that draw on that content. Evaluation was primarily based on homework and exam grades (n = 3 before the final). Students were required to use LaTeX to post their solutions to homework tasks, but their solutions on all exams were hand-written.

We interviewed students in the summer after they completed the course in the prior spring semester. We invited all 110 enrolled students from that semester to participate; 17 (~15%) responded, and 14 (~13%) completed an hour-long interview about their experience. These interviews were audiotaped for analysis. Nine students were male and five were female; three were international students. The sample was diverse by major. Six were mathematics majors; two others were majoring in actuarial science. In addition, we interviewed students majoring in mechanical engineering, chemistry, packaging, biology, and economics. Two of those were pursuing dual majors. Most of the other "non-math" majors were pursuing minors in mathematics where the course was a requirement. That minor also requires subsequent courses that focus on proof. Table 1 provides an overview of the sample, including their final grades. Eight students received a 4.0, two received a 3.5, three received a 3.0, and one received a 2.5. Our sample was more successful than the average, relative to the course's grade distribution.

Table 1. Overview of Participants

Student	Gender	Home	Major(s)	Standing[a]	Minor[b]	Grade
1	F	US	Mathematics	1		3.0
2	M	US	Actuarial Science	2		3.5
3	M	US	Mechanical Engineering	4		4.0
4	M	US	Chemistry	3	Math	3.0
5	M	US	Actuarial Science	1	Math	4.0
6	M	Intl	Mechanical Engineering	3	Math	4.0
7	M	Intl	Computational Mathematics	3		4.0
8	F	US	Mathematics	1		3.0
9	F	US	Packaging	2	Math	2.5
10	F	US	Mathematics	2	Act Sci	3.0
11	M	US	Human Biology	3	Math	3.5
12	M	Intl	Mathematics	3		4.0
13	F	US	Economics & Statistics	2		4.0
14	M	US	Economics & Mathematics	2		4.0

[a] "Standing" reflects the amount of university coursework; in some cases, standing exceeds the number of years at the university. [b]S5 and S9 were strongly considering a mathematics minor but had not yet decided.

Our interview questions focused on understanding students' experience of the course in their terms, relative to their prior work in high school and college mathematics. We first secured basic information about them (e.g., major, standing, section/instructor, minor, and prior and intended math courses). Then we asked about their sense of the nature of the course (in contrast to their prior courses), their sense of success (or not) in it, what they did to be/become successful, and specifically how their view of and work on proof tasks may have changed in the course. As complement to seeking their verbal responses, we also asked them to graph their confidence in the course across the semester. These Confidence Graphs (Figure 1) often surfaced new information about our focal issues, as students explained the shape of their graphs and points during the semester when their confidence changed. We also asked about their sense of their instructor's view of the course, whether their course experience influenced their understanding of calculus (most had just completed that sequence), and whether they expected to be successful in subsequent math courses. Finally, we asked what they would tell incoming students about the course.

We also regularly observed (about twice per week) the classroom teaching and interactions in one section of the course for most of the semester—the section taught by the course coordinator. Observers took field notes in an observation template. These observations proved essential in preparing for the interviews, interpreting and responding to student responses, and in the analysis of the resulting data.

Figure 1. Confidence Graph (reduced in size). Each participant drew a graph of their confidence over the course of the semester. The X-axis represents time, from just before the course began through the final exam. The Y-axis represents their confidence in doing well in the course.

Analysis

Our analysis the interview data, both verbal and graphical, was qualitative, "bottom-up" (guided by our foci), and cyclical in nature. The results of initial analyses led to more detailed and focused rounds of analysis around specific issues. First, immediately after each interview, interviewers posted detailed summary descriptions of their interviewee's responses. All members of the research team then read carefully and discussed the similarities and differences evident in those summaries. Next, the audio records were transcribed verbatim and coded in the qualitative software *Dedoose* according to a coding scheme developed from the initial post-interview analyses and the categories suggested by our theoretical framework (achievement, disposition, differences between prior and current mathematics courses, and learning activity). This resulted in a coding scheme with the following top-level categories: (1) characterizing differences, (2) learning activities, (3) proof reasoning, (4) sense of success, and (5) prior mathematical experience. The coding scheme and a code book detailing criteria for assigning codes were refined through rounds of coding and discussion amongst members of the research team.

The Confidence Graphs that subjects produced were initially analyzed for their general graphical properties (e.g., initial sense of confidence and where slopes of sections were greatest, positively and negatively). A second round of analysis explored the relationship between the graphs students produced and their verbal explanations of that they drew. We paid particular attention to points in the semester when students' confidence changed and the reasons given for those changes.

Results

We report results in four main components. First, we present our summary of how the students described the course in comparison to prior mathematics courses. Second, we discuss one aspect of that comparison—how students used the terms "problem" and "answer" in describing their course work. Third, we characterize how students described their reasoning in solving proof tasks. Finally, we describe how the students viewed and evaluated their success in the course and the learning activities in which they engaged and also present the results of our Confidence Graphs analysis.

The Different Nature of the Course

Students were quite articulate about how the course differed from their prior mathematics course experiences, but their focus and emphasis varied. We found three main characterizations of difference: (a) the transition to proof course explained the mathematics, whereas prior math courses did not, (b) students' conceptions about proof and writing in mathematics changed, and most importantly, (c) the course emphasized explanation and reasoning, differing greatly from the computational and answer-focused nature of their work in past math courses. No student reported substantial prior experience with proof, in either high school or college; most reported no experience whatsoever. Those who did cite their experience with two-column proof in high school geometry were largely dismissive of that work as irrelevant.

First, students frequently stated that the course *explained* the mathematics, more so than in prior courses. Half said the course explained why mathematics is the way it is and how it came about. For example,

> *The past math courses have been a lot...you are given instructions and then you learn to kind of regurgitate them, whereas [this course] really made you think...why the rules are in place and coming up with your own rules* (S3).

Students noted that the course explained why procedures they had learned in previous mathematics worked. In particular, S8 and S6 cited limits, derivatives, and operations with matrices, respectively, as examples of this.

> *The difference is: when I learn matrices in differential equations, I only need to calculate multiplication, adding, or subtract[ing]. That's it. But [in this course], I have to know*

why is it possible to subtract two matrices. I have to show that. And that makes me to understand how's it come from when first it was invented (S6).

This pattern of response indicates that the course served an explanatory purpose for students, providing conceptual understanding over and above their learning about language, notation, and proof techniques. This is a central function of proof in mathematics (e.g., Thurston, 1994), but it was pleasing to hear students articulate it.

Second, half of the students (n = 7) talked explicitly about the new focus on proof and the dramatically increased focus on writing. S7 remarked that "calculus works with numbers and [this course] works with words," succinctly illustrating how students perceived a huge difference between these two types of courses. Students commented on changes in how they perceived mathematical writing (n = 5) and noted how much more writing was required in the course (n = 7). Some realized that proofs can include words, not just symbols (S12, S13); others came to new understandings about the importance of accurate grammar in writing compelling proofs (S2, S3). They saw that mathematical terms have much more specific meanings than do words in everyday language and communication and that that specificity was important. For example, S3 stated that "certain words mean different things in math as opposed to normal spoken English."

Last and perhaps most important, most students (n = 10) explicitly contrasted the computational, answer-focused nature of their work in prior courses (e.g., calculus) to the process- or argument-focused work in the introduction to proof course. As we note below, numerous students described the course in terms of what it was not: Writing proofs was not straightforward or step-by-step (S2-S5, S11) and did not involve applying formulas or solving equations (S6, S12, S14). Students discussed how they had to explain their thinking (n = 7), something they were not accustomed to doing in previous math courses. Nevertheless, students largely understood the rationale for the push for explanation.

Proof is actually more like explaining. So I learn how to generally explain some stuff...to justify your arguments or persuade others in order that they can understand what you wanna say (S6).

As a result, about a quarter of the sample expressed that the course valued the process of reasoning more than the answer.

Answers in proof work. As our constructivist perspective had suggested, students reused terms that were standard and relevant to computational work but seemed less appropriate to proof-writing work. All but one student used "answer" to describe products of their proof-writing work, and nine of them used that term to describe their work before the interviewer did. In some cases, students appeared to use "answer" to refer to the result to be proven (e.g., the consequent of an "if-then" statement). In other cases, "answer" appears to refer to the entire written proof. S10 explained,

Yeah but personally, me, I'm super neat so my homework I could have turned in [without conversion into LaTeX]. *But with the writing proofs you have all these ideas all over the place, and then you have one final answer where you put everything in order and it makes sense.*

S5 similarly used "answer" to refer to the general line of reasoning in his proofs. In contrast to S10, he did not seem to indicate the written details. We note the quick shift in his statement from "answer" to the more proof-consistent term "justification."

If you're struggling with a [introduction to proof] *problem, and you figure out that the answer is, that the justification by this road is extraordinarily complicated, my first instinct is not to think, oh, this is a complicated problem. My first instinct is to think* *there's an easier way to solve it.*

Overall, we found that students carried over linguistic resources (e.g., key descriptive terms) from mathematics work that focused on producing numerical and symbolic answers to proof-writing—a task that most of them saw as quite different from computational work.

Reasoning in Writing Proofs

One central goal of our work is to characterize how students see and carry out the reasoning required to produce acceptable proofs in the course. Though we did not ask participating students to solve and explain their reasoning on particular proof tasks, their responses to some interview questions produced more general views of how they came to view proof reasoning from their experience with assigned homework and exams. Most students (n = 10) provided relatively specific descriptions of their proof-writing work and thinking; more limited responses of four others left us unable to characterize their thinking. The ten students' responses emphasized five different features of proof reasoning: (a) applying proof methods, (b) looking for and using examples and models, (c) accommodating to problem solving, (d) understanding central concepts, and (e) reasoning backwards and forwards. Most students described multiple features, so our analysis is not one-to-one.

Applying proof methods. Six students (S5, S7, S9, S10, S11, S13) cited learning specific proof methods and/or described the process of applying them. That six of ten did is not surprising given the very limited prior experience with proof that students reported. Four (S7, S9, S10, S11) cited particular methods (e.g., proof by contradiction or induction). S10 indicated she saw class activity as learning proof methods and applying them to concepts she had learned in prior courses. S5, S7, S11, and S13 described proof construction as selecting an initial method and re-examining that approach if it did not work well. These students came to see that, relative to prior computational work, selecting an appropriate proof method was not a deterministic procedure; it involved judgment. For that reason, some students indicated that selecting an initial method was problematic. S9 stated that exam problems were more difficult for her than homework problems because the latter frequently included explicit hints or problem features (e.g., the phrase "for all N" in the statement) that cued specific methods (in this case, induction). Some students developed beliefs or expectations about the relationship between statements and methods. For S7, there was no shortcut to the best method for a given problem because search involved the analytic work of surveying and assessing methods. As quoted above, S5 expressed a different stance on this issue. When the solution via one method became complicated, he inferred that a simpler solution was possible via another method. By the end of the course, he judged that most all course problems could be matched to "one best proof method."

Searching for and using examples and models. Three students (S7, S11, S13) reported that they adapted examples from class or searched for proofs to homework problems outside of class. This orientation was clearly related to what had proven productive in their prior mathematics work. S13 said that the instructor gave full proofs as examples in class. She put those next to her when she worked on homework and looked for ways to adapt their features to the assigned problems. S11 indicated he searched for solutions to exact or similar problems on the internet and was frequently "successful" in finding them.

Accommodating to problem solving. Four students (S2, S7, S11, S12) explicitly stated that step-by-step procedures generally did not work in proof-writing and came to embrace the view that proving involved problem solving. S11 indicated,

> *I mean, doing like calculus, you might not know you're doing it right but at least you're going through the steps. It's like a step by step. And this* [work in the introduction to

proof course]*, there are no steps. You just kinda think about it, work your way around it and finish off. If that makes sense.*

Despite its vagueness, this statement shows S11 was coping with the essentially non-procedural nature of proof reasoning. For S2, the "problematic" nature of proof-writing followed from the greater variation proof problems compared to prior courses. He said,

> *They could still ask you the same question but like based upon like what formula they give you or what kind of function they give you,* [the proof] *could be completely different.*

Understanding central concepts. Four students (S4, S5, S6, and S12) emphasized the importance of understanding the concepts that appear the statements to be proven. They did not mean simply accurately recalling the definition; they emphasized using the definitions effectively in proof work. S4, S6, and S12 described the ways they used concepts in their proof reasoning. To prove a given statement, they needed to recall formal definitions and fit them together conceptually to support their proof writing. S4 emphasized rapid and accurate access to definitions,

> *That's like the heart of math is you have to have your definition in order to go. I kind of always forget about them in my mind when I work through it. So I know what I'm doing at this point, but just to keep them always in the back of my mind like right there, to like always have it to recall.*

He worked on homework problems with the relevant definitions at hand in pursuit of the goal of being able to recall them on exams when needed. S12 and S6 both said that they could solve all the problems in their calculus courses using only rote learning, but the introduction to proof course was different. S12 stated directly,

> *So you really have to understand what's going on and why we use it [the concept] and what, what can we apply for, yes. So it's more like understanding for memorizing concepts.....*

Reason backwards and forwards. Two students (S5 and S10) described their proof reasoning in spatial and directional terms, as reasoning backwards and forwards. S10 described her proof-writing work in stages. First she had to work forward to "the final answer"—that is, using the assumed properties in the statement to produce the desired result. Then she indicated she reversed the direction of her reasoning to write the proof.

> *This is what I think I do differently completely, is you write the proof backwards.... Because you know that it's the right answer, you just did all the math, and you can clearly see that it's the right thing. But you can't just turn that in; that is not a proof.*

S5, by contrast, described his high-level forward reasoning that told him if his selected proof method was going to work. He indicated,

> *If there's anything that I took away from this class, it's that you can usually see how the problem is gonna go prior to actually starting the problem.*

His ability to "see" forward a considerable distance in his reasoning may have contributed to his inference that one best proof method existed for most course problems.

Specific Challenges and Learning Activities

The interviews revealed three major loci of students' learning activities: (a) in-class work (e.g., mini-lectures, class discussion, group work), (b) out-of-class work that was required (e.g., completing homework sets and reading quizzes that prepared for and extended work done in class), and (c) out-of-class work that was optional such as attending the Mathematics Learning Center (MLC) and instructor or TA office hours. The students described aspects of these zones of activities that were new to the course and interesting connections between them. New expectations of students' activities in class (e.g., serious engagement of all students in group work) influenced their work outside of class (e.g., going to the MLC to work with peers they got to know in their in-class work). And as was intended in the course design, new requirements of students outside of class (e.g., completing reading quizzes before related work in class) had implications for the work done in class.

Most students described the work required to be successful in the course as more demanding than it had been in prior courses. Some were primed for challenge by characterizations offered by friends who had already taken it. Once in the course, they found that the homework carried substantial value in the grading scheme and that completing each weekly set took substantially more time than it did in prior courses—often much more time. Most students offered estimates of seven to ten hours for each set, compared to one to two hours in prior courses. This increased "time on task" combined with the students' descriptions what they did in their proof work on homework problems (above) indicates the importance of homework as a site for understanding and navigating the new work that was expected of them.

All but two students explained that they employed different practices in organizing and completing their homework than they had in prior courses. In response to the demands of frequent and challenging homework, most indicated that they responded by starting earlier in the week and/or distributing the work across days. Four communicated with peers via e-mail, worked with a classmate, or asked questions in class for the first time in mathematics. Twelve described the importance of using the MLC—many for the first time—in interesting and diverse ways. Some cited the availability of tutors to answer their questions; others simply sought a physical space to go and work alone; others emphasized collaborative work with other students in the space designated for this course. For some (e.g., S10), productive discussion and community building happened with students from other sections of the course. That said, this use of social and peer resources was not universal; some students (e.g., S5) simply worked on their own, even as they worked harder and longer.

Sense of Success and Confidence Graph

When we asked about students' sense of success in the course, most responded in terms of their final grade. Less frequently, they assessed their level of understanding the content or their sense of having mastered it. For example, S7 stated that he felt he mastered "90% of the course content." As Table 1 indicates, half of the students received final grades of 4.0, and only one student received a 2.5. S1 reported that her 3.0 grade missed being a 3.5 by a single point. So overall, despite the new demands of the course, most of the 14 students were successful by the traditional measure of final grades.

In analyzing the Confidence Graphs, five general patterns of graphical shapes emerged, with two outliers. The most common pattern (n = 4) was a "W" shape that showed an initial decrease in confidence relatively early in the semester followed by increase, then decrease again, and a final increase. An example (from S3) is given in Figure 2. Though all four graphs of this type followed this shape pattern, the distances between the peaks and valleys varied. The other groupings included two student graphs each and were characterized by (a) continuous increase, (b) concave up parabolic shaped graph, (c) initial increase followed by a sinusoidal wave for the rest of the semester, (d) initial increase followed by decrease, with a final confidence level that was lower than their initial level, and (e) other.

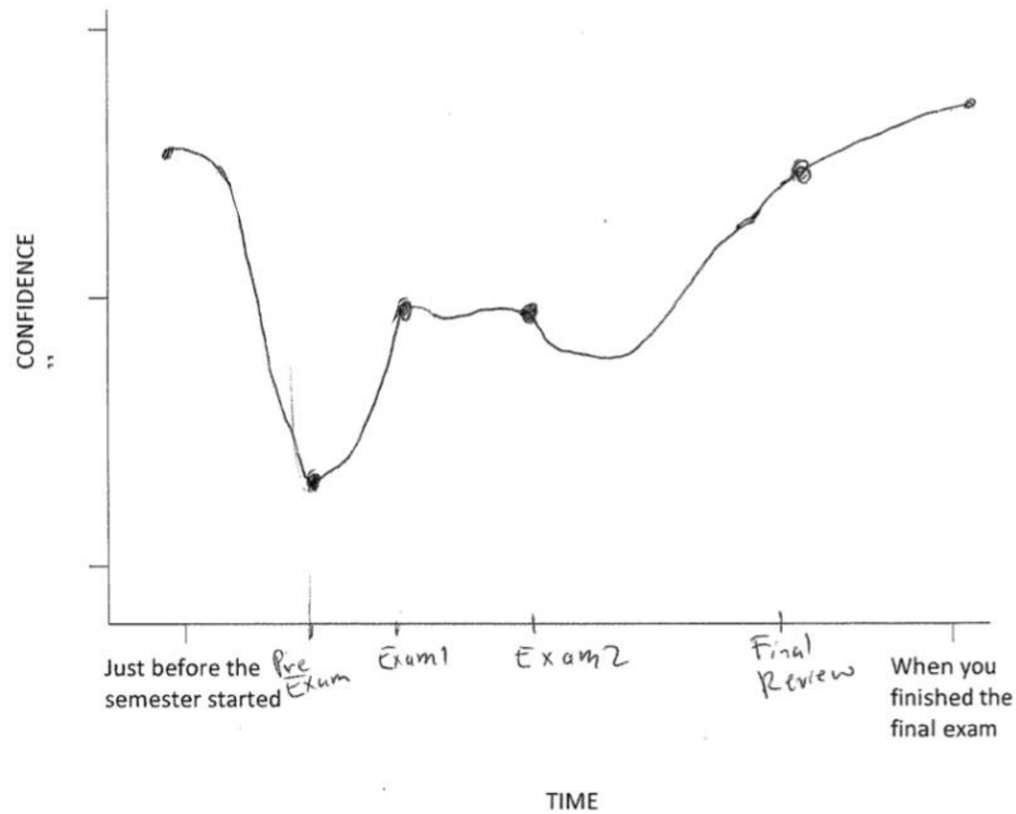

Figure 2. S3's Confidence Graph.

After analyzing the graphs by shape, we examined the drawings and the transcripts of students' descriptions of what they drew and tabulated the reasons given for initial confidence level and for factors that increased or decreased confidence thereafter. Consistent with how they described their sense of success in the course generally, most students (n = 12) identified grades (on homework and exams) as the principal causes of increases or decreases in their confidence. Many, like S3 above, located course exams as the X-coordinates of relative maxima or minima on their graphs. But when students explained what they had drawn, other factors emerged as important influences on confidence levels through our analysis of the transcripts.

Most students began at a confidence level near the middle marking on the y-axis, and described an uncertainty about what to expect in the course. Some who began at a low point or at y = 0 had heard the class was difficult. Some who started closer to the highest mark on the y-axis indicated either a strong belief in their mathematical ability or a favorable result in a prior course as orienting their initial confidence. The most commonly cited factors for increased confidence that were not grade-based were finding help in the MLC (n = 5), spending extra time studying and practicing (n = 4), and realizing a sense of understanding of the course content during the two weeks of review before the final exam (n = 4). For decreases, five students indicated that the real analysis portion of the course led them to feel less confident. Others alluded to personal reasons such as illness or missing classes, difficulty keeping up with the pace of the course, or proving in general as reasons for lowered confidence. Overall, their activity proved fruitful not only because it produced a representation of students' confidence over the duration of semester, but also because it then provided a visual focus for students to explain what aspects of the course and their work in it most affected their confidence.

Conclusions, Limitations, & Next Steps

Our research is ongoing, and we expect that subsequent steps will deepen and refine our understanding of students' experience in this course, and for those who enroll in more advanced courses, their further progress in the transition to proof. But the present analysis has been revealing, and in some unexpected ways. We first summarize our main findings.

First, we find substantial evidence that the transition to proof course "moved" students in two substantive and productive ways. It has placed them in the "space" of mathematical problem solving and thereby altered their views of competence in mathematics (augmenting, if not displacing a primarily computational view). Changing the nature of tasks from computing to proving substantially shifted mathematical work from "exercises" to real problems—that is, to tasks that for most students suggested no immediately obvious means of solution. In that sense, the course "successfully" created an environment where we could examine the birth of "new" (for students) forms of mathematical thinking and thereby students' sense of what is involved in that transition.

Second, in these initial steps to chart their transitions, we have identified aspects of cognitive continuity where students have reused or adapted aspects of their prior work in mathematics to work with proof. Many used prior computationally-oriented words to describe elements of their "new" mathematics work (e.g., "answer"), despite what many would see as strong differences in context. Some adapted prior patterns of finding solutions to work on proof tasks (e.g., finding or adapting examples and models). Many spoke to the difficulty of adjusting away from their past patterns of looking for a procedure or model solution to use to complete proof tasks on homework. These results are reminders that students try to extend the use of prior resources that have proven productive in the past even when they fit poorly or even forestall progress in mastering new challenges.

Third, where we found substantial commonality in how students saw the course as different from prior work in mathematics courses, we found considerable diversity their descriptions of their work to produce acceptable proofs. Some highlighted proof techniques (learning a set of generally-applicable proof methods); some focused on the necessary and productive focus on understanding the meaning of key concepts in the statement; some described a process of considering a top-level view of a series of steps in the proof—either forward or backward from the result. We do not suggest that course design and teaching should explicitly address one or another these top-level descriptions of proof reasoning, rather that course design and teaching practices can be usefully informed by them.

This study complements prior work that has focused on students' reasoning on specific problem-based tasks—typically proof construction and evaluation (Alcock & Weber, 2010; Selden & Selden, 2003). In targeting students' experience over the course of a semester, we have focused on more general issues, especially how students see their proof work and how they reorganize their learning to address the new challenge. These issues are important foci for all efforts to assist students in entering and succeeding in this fundamental new (for them) form of mathematical thinking. In particular, it informs the design of learning spaces that can support students' work outside of class (e.g., mathematics learning or help centers and how work is organized in these environments).

We see three main limitations in our work thus far, two of which concern our sample. Generally speaking, we do not know how well these fourteen students represent the range of experiences in the larger population of students who took the course that particular semester.

Second and more specifically, we are concerned that our sample was weighed too heavily toward successful students. Such students seemed to welcome the chance to describe their experience; less successful students, we think, are much less so. We need to find ways to reach those students—those who may have good reasons not to revisit a painful experience. Third, our interviews have sometimes posed questions that students have struggled to understand and respond to, for many reasons. The Confidence Graph activity and talk arising from it was a productive addition in that respect. It was accessible, reframed the issues under study, and produced many fluent explanations. We (and other researchers) should explore other such "stimuli" that may productively support students' efforts to recount their experience.

The most important next step in our research is to interview graduates of the course who are enrolled in subsequent proof-intensive courses, principally abstract algebra and real analysis. (An intervening linear algebra course mixes computational and proof-based work.) In the context of those two courses, we want to explore students' assessments of how well their introduction to proof course has supported their current proof writing work. We expect "mixed" and (we hope) rich stories of their continued journey as proof-writers and mathematical problem solvers. Second, we plan for more detailed analysis of the Confidence Graph data (graphical and verbal) in the goal of revealing more clearly how students experience and manage particular challenging events or periods of time in the course.

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Knowledge about Student Understanding of Eigentheory: Information Gained from Multiple Choice Extended Assessment

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Eigentheory is a conceptually complex idea whose application is widespread in mathematics and beyond. Herein we describe the development and use of an extended multiple choice assessment that gives us further insight into the ways students think about and understand eigenvectors, eigenvalues, and their related concepts.

Key words: linear algebra, eigenvector, eigenvalue, student understanding, assessment

The purpose of this report is to share results regarding student understanding of eigentheory that were gained from a multiple choice extended assessment instrument. We chose to focus on eigentheory because (a) it is a conceptually complex idea that builds from and relies upon student understanding of multiple key ideas in mathematics, and (b) its application is widespread in mathematics and beyond. Our aim to create an assessment instrument that captures nuances of students' conceptual understanding of eigentheory exists in tandem with our pursuit to frame what it might mean to have a deep understanding of eigentheory. As such, in this report we offer results both about student thinking and about possible affordances and constraints of various assessment instrument question formats.

Background and Literature

Research into people's understanding of eigenvectors and eigenvalues has had several different foci**,** such as: (a) using Tall's (2004) three worlds of mathematical thinking in conjunction with APOS theory (Dubinsky, 1991; Cottrill et al., 1996; Dubinsky & McDonald, 2001) to examine students' abilities to think about eigentheory in the embodied, symbolic, and formal worlds, and identify the various processes and objects students need to understand in eigentheory (Stewart & Thomas, 2006; Thomas & Stewart, 2011); (b) studying how mathematicians use gesture, time and motion to describe the concepts of eigenvector and eigenvalue (Sinclair & Gol Tabaghi, 2010); (c) examining how dynamic geometry software can encourage students to think geometrically about eigentheory (Gol Tabaghi & Sinclair, 2013); and (d) investigating the use of modeling problems and APOS theory to teach students the concepts of eigenvectors and eigenvalues (Salgado & Trigueros, 2015). Our current research into students' understanding of eigenvectors and eigenvalues has been influenced by the above work, but we endeavor to extend this growing body of knowledge in two ways. First, we hope to share further insights into how students think about and understand eigenvectors and eigenvalues that has not been reported on previously. Second, we are working towards the development of a framework for student understanding of eigentheory that both ties together the work others have done in this area of research, and adds our own insights on student understanding of eigentheory. We further discuss this framework within the next section.

Theoretical Framework

There exists a small collection of previous research into students' understanding of eigentheory that has worked towards developing a theoretical framework for what it means to have a deep understanding of eigenvectors and eigenvalues (Salgado & Trigueros, 2015; Thomas

& Stewart, 2011). These articles point out useful distinctions, including the processes, objects, and coordinations necessary for understanding eigentheory. As we refine our framework for student understanding of eigentheory, we aim to include the following ideas: (a) a distinction between the equations $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ and how these can influence the ways students think about and solve problems involving eigenvectors and eigenvalues; (b) the importance of eigenspaces, diagonalization, and their connection to the concepts of eigenvectors and eigenvalues; (c) how the concepts of eigentheory can be thought of within different modes of thinking (Sierpinska, 2000), modes of description (Hillel, 2000), or contexts (Zandieh, 2000); and (d) the various processes (e.g., matrix multiplication, scalar multiplication), entities (e.g., matrices, vectors), and theorems (e.g., the invertible matrix theorem) needed to understand eigentheory and the calculations involved therein. While this framework is still under development, it informed our decisions about the creation and refinement of the assessment instrument, and, cyclically, the results of the assessment continue to inform the development of the framework. We describe the assessment more fully in the following section.

Methods

In this section, we describe the development and format of our multiple choice extended (MCE) assessment instrument. We then describe the data collection and participants for this study, followed by a description of our analysis.

Instrument Development

The MCE assessment instrument for eigentheory development grows from our prior work in student understanding of span and linear independence in which we developed the MCE-style question format (Zandieh, Plaxco, Wawro, Rasmussen, Milbourne, & Czeranko, 2015). During this development, we considered literature on conceptually oriented assessment instruments in undergraduate mathematics and physics (Bradshaw, Izsak, Templin, J. & Jacobson, 2013; Carlson, Oehrtman, & Engelke, 2010; Epstein, 2013; Hestenes, Wells, & Swackhamer, 1992; Wilcox & Pollock, 2013). Questions written in an MCE style begin with a multiple choice element and then prompt students to justify their answer by selecting all statements that could support their choice, a format based on a concept inventory in Upper-division Electrostatics created by Wilcox and Pollock (2013).

To develop the assessment instrument questions for the Eigentheory MCE, we compiled a database of questions about eigenvectors, eigenvalues, and related concepts from literature on student understanding of eigenvectors and eigenvalues, online resources for clicker and classroom voting on linear algebra (Cline & Zullo, 2016), and previous linear algebra homework assignments, exams, and interview protocols used by research team members (e.g., Henderson, Rasmussen, Sweeney, Wawro, & Zandieh, 2010). The most promising questions that collectively addressed various aspects of our working framework were edited into the MCE format. The instrument has been administered in two sets of student interviews, as written homework, and as an in-class review activity, with subsequent refinements of the assessment after each administration. The most recent revision was also administered as written homework in Fall 2016. This current report relies on the administration of the assessment as an in-class review activity, described in further detail below.

Data Collection

We present data from written assessments collected from three introductory linear algebra classes taught by the same instructor at a large, research-intensive public university in the mid-Atlantic United States during Spring Semester 2016. The course utilized the Inquiry-Oriented Linear Algebra (IOLA) curricular materials. The materials, which are available at http://iola.math.vt.edu, contain three main units: Linear Independence and Span (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012), Matrices as Linear Transformations (Andrews-Larson, Wawro, & Zandieh (2017), and Change of Basis, Diagonalization, and Eigentheory (Zandieh, Wawro, & Rasmussen, 2017). The course also used Lay (2012) as its textbook.

Each class worked on one version of the MCE assessment (described in the following paragraph) during the last day of class as a review of eigentheory. Students took the assessment individually for 20-25 minutes, following which the instructor collected the assessments, and then held whole class discussions on the problems. This report focuses on the collected individual work.

All three versions of the MCE assessment consisted of the same six multiple choice question elements, with varying justification sections. Class 1 (27 students) received a version in which students indicated if each of six given justifications were true and relevant, true but not relevant, or false (see Figure 1a). Class 2 (29 students) received a version in which students only selected justification choices that were true and relevant (see Figure 1b). Lastly, Class 3 (28 students) received an open-ended version in which they wrote their own justification for their choice.

1. The matrix $A = \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix}$ has $\lambda = 6$ as one of its eigenvalues. Which of the following vectors is an eigenvector of A with corresponding eigenvalue $\lambda = 6$?

(a) $\boldsymbol{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ **CORRECT→** (b) $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Because … (indicate if the choice is true and relevant, true but not relevant, or false)

True & relevant	True not relevant	False	Statement
			(i) This vector $\boldsymbol{x}$ makes $A\boldsymbol{x} = 6\boldsymbol{x}$ a true statement.
			(ii) This vector $\boldsymbol{x}$ is the only vector in $\mathbb{R}^2$ for which $A\boldsymbol{x} = 6\boldsymbol{x}$.
			(iii) This vector $\boldsymbol{x}$ makes $(A - 6I)\boldsymbol{x} = \boldsymbol{0}$ a true statement
			(iv) Subtracting 6 from the diagonal of A yields this vector $\boldsymbol{x}$ as a column vector of the resulting matrix.
			(v) The vector $A\boldsymbol{x}$ is 6 times the magnitude and in the same direction as this vector $\boldsymbol{x}$.
			(vi) The matrix A also has $\lambda = -3$ as an eigenvalue.

(a)

1. The matrix $A = \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix}$ has $\lambda = 6$ as one of its eigenvalues. Which of the following vectors is an eigenvector of A with corresponding eigenvalue $\lambda = 6$?

(a) $\boldsymbol{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ **CORRECT→** (b) $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Because … (select ALL that could justify your choice)

(i) This vector $\boldsymbol{x}$ makes $A\boldsymbol{x} = 6\boldsymbol{x}$ a true statement.

(ii) This vector $\boldsymbol{x}$ is the only vector in $\mathbb{R}^2$ for which $A\boldsymbol{x} = 6\boldsymbol{x}$.

(iii) This vector $\boldsymbol{x}$ makes $(A - 6I)\boldsymbol{x} = \boldsymbol{0}$ a true statement

(iv) Subtracting 6 from the diagonal of A yields this vector $\boldsymbol{x}$ as a column vector of the resulting matrix.

(v) The vector $A\boldsymbol{x}$ is 6 times the magnitude and in the same direction as this vector $\boldsymbol{x}$.

(vi) The matrix A also has $\lambda = -3$ as an eigenvalue.

(b)

Figure 1. Comparison of (a) Class 1 MCE (3-Choice MCE) and (b) Class 2 MCE (Basic MCE).

By comparing performance on a measure independent of the items under investigation, namely the results of a class exam on eigentheory concepts given by the instructor (see Table 1), we can determine if the students in the three classes are of comparable ability. To do so, we ran a 1-way ANOVA looking for group variation on this data. The results of that analysis reveal there is no significant difference between the groups (F = 1.3075, p = .2761) and no post-hoc comparisons were necessary. Thus we can reasonably assume that the three sub-samples are of roughly equal ability and superior ability of a particular group will not bias the results in favor of one form of the 6-item assessment.

Table 1

Summary of Student Performance on a Class Exam on Eigentheory Material Given by Instructor (Independent of Researchers)

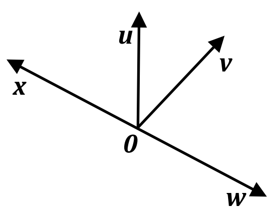
	Class 1	Class 2	Class 3	Overall
Average Score	*82.4*	75.9	80	79.4
Standard Deviation	*15.7*	13.7	16.3	15.5
Median Score	*88*	75.5	82.5	82

Analysis

After each class's written assessments were digitally scanned and grouped by question, spreadsheets were designed to enter the data from Class 1 and Class 2. Using the spreadsheets, the research group examined trends among student responses, which included looking for: (a) common sets of justifications that students selected or did not select; (b) how selecting certain justifications may have influenced students' multiple choice selection; and (c) instances where we would have expected students to select what we viewed as related justifications, but they did not. To help with comparisons and identifying these trends, some basic percentages were calculated for the justification choices of students in Classes 1 and 2.

2. Suppose the vector $\boldsymbol{x} \in \mathbb{R}^2$, in the two-dimensional sketch below, is an eigenvector of a 2x2 matrix M with real-valued eigenvalues. Which of the vectors $\boldsymbol{u}, \boldsymbol{v}$ or $\boldsymbol{w}$ illustrated below could be the result of the product $M\boldsymbol{x}$?

(a) $\boldsymbol{u}$
(b) $\boldsymbol{v}$
(c) $\boldsymbol{w}$ ←**CORRECT**
(d) Not enough information is given to know a possible result of the product $M\boldsymbol{x}$



Because … (select ALL that could justify your choice)

(i) This vector is on the same line as $\boldsymbol{x}$.

(ii) This vector and $\boldsymbol{x}$ form a linearly independent set.

(iii) This vector is in $span\{\boldsymbol{x}\}$.

(iv) This vector is a scalar multiple of $\boldsymbol{x}$.

(v) The matrix M needs to be known to determine what $M\boldsymbol{x}$ could be.

(vi) There exists a scalar c such that $M\boldsymbol{x} = c\boldsymbol{x}$.

3. Suppose A is a $n \times n$ matrix, and $\boldsymbol{y}$ and $\boldsymbol{z}$ are linearly independent eigenvectors of A with corresponding eigenvalue 2. Let $\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$. Is $\boldsymbol{v}$ an eigenvector of A?

(a) Yes, $\boldsymbol{v}$ is an eigenvector of A with eigenvalue 2. ←**CORRECT**
(b) Yes, $\boldsymbol{v}$ is an eigenvector of A with eigenvalue 5.
(c) No, $\boldsymbol{v}$ is not an eigenvector of A.

Because … (select ALL that could justify your choice)

(i) $\boldsymbol{v}$ is a linear combination of eigenvectors that have the same eigenvalue.

(ii) The set $\{\boldsymbol{v}, \boldsymbol{y}, \boldsymbol{z}\}$ is linearly dependent.

(iii) A linear combination of eigenvectors does not result in another eigenvector

(iv) $A\boldsymbol{v} = A(5\boldsymbol{y} + 5\boldsymbol{z}) = 5A\boldsymbol{y} + 5A\boldsymbol{z} = 5 \cdot 2\boldsymbol{y} + 5 \cdot 2\boldsymbol{z} = 2(5\boldsymbol{y} + 5\boldsymbol{z}) = 2\boldsymbol{v}$.

(v) $\boldsymbol{v}$ is an element of the eigenspace created by the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$.

(vi) $A\boldsymbol{v} = A(5\boldsymbol{y} + 5\boldsymbol{z}) = 5A\boldsymbol{y} + 5A\boldsymbol{z} = 5 \cdot 2\boldsymbol{y} + 5 \cdot 2\boldsymbol{z} = 5(2\boldsymbol{y} + 2\boldsymbol{z}) = 5\boldsymbol{v}$.

Figure 2. Questions 2 and 3 of the Eigentheory MCE.

The open-ended data were analyzed through multiple iterations of open coding. First, each team member individually summarized the key aspects of each student's justification process. Next, the team came together to develop a coding scheme for the student work to be used in the second iteration of coding. Each team member then individually coded the student responses with the new coding scheme before collectively determining a set of codes for each student. Lastly, the larger themes from the finalized coding were identified by examining common patterns across multiple students' solutions and justifications.

Based upon the patterns and trends found in Classes 1 and 2, as well as common student justifications in Class 3, the next step of our analysis involved the creation of a type of “conditional table” for all three classes on specific justification and multiple choice comparisons. We give further explanation of these tables in the results below as they facilitate making comparisons across classes. In this report, we focus on the first three questions from the assessment (see Figures 1 and 2), with plans to analyze all six later.

Results

Table 2 shows the general performance of the three classes on the multiple choice portion of the three questions. We make a few observations. First, classes performed similarly on all three questions, further supporting the reasonableness of comparing across them. Second, Question 1 and Question 2 may seem somewhat uninformative because a majority of the students chose the correct answer. However, in the results and discussion to follow, we share how the MCE gives more information about student thinking through the justification portion of each question. Third, as evidenced by the results on Question 3, linear combinations of eigenvectors with the same eigenvalue, in other words, elements of an eigenspace, is a difficult topic. We first share insights into students’ thinking about eigentheory we gained from each individual question, and then explain some of the overarching insights gained by looking across questions in the discussion.

Table 2
Class Percentages for Answers Chosen on the Multiple Choice Portion

		Class 1 Judgment MCE (27 Students)	Class 2 Original MCE (29 Students)	Class 3 Open-Ended (28 Students)	Overall (84 Students)
Question 1	*Choice (a)*	14.8 (4)	24.1 (7)	7.1 (2)	15.5 (13)
	Choice (b)	81.5 (22)	75.9 (22)	92.9 (26)	83.3 (70)
	No Answer	3.7 (1)	0.0	0.0	1.2 (1)
Question 2	*Choice (a)*	0.0	0.0	0.0	0.0
	Choice (b)	0.0	6.9	0.0	2.4 (2)
	Choice (c)	92.6 (25)	82.8 (24)	96.4 (27)	90.5 (76)
	Choice (d)	3.7 (1)	10.3 (3)	3.6 (1)	6.0 (5)
	No Answer	3.7 (1)	0.0	0.0	1.2 (1)
Question 3	*Choice (a)*	25.9 (7)	48.3 (14)	21.4 (6)	32.1 (27)
	Choice (b)	3.7 (1)	6.9 (2)	17.9 (5)	9.5 (8)
	Choice (c)	55.6 (15)	31.0 (9)	57.1 (16)	47.6 (40)
	No Answer	14.8 (4)	13.8 (4)	3.6 (1)	10.7 (9)

NOTE: Correct answers are shaded. Grey numbers in parentheses are the number of students.

Question 1

Recall that the main prompt for Question 1 required students to determine which given vector was an eigenvector for $\lambda = 6$ for a given matrix A (Figure 1). From among the provided justifications that support a correct answer, students were able to choose “(i) this vector makes $A\boldsymbol{x} = 6\boldsymbol{x}$ a true statement” or “(iii) this vector makes $(A - 6I)\boldsymbol{x} = \boldsymbol{0}$ a true statement.” First, we consider the open-ended responses to Question 1 from Class 3 to determine if students, of their own accord, explicitly wrote some form of $A\boldsymbol{x} = 6\boldsymbol{x}$ or $(A - 6I)\boldsymbol{x} = \boldsymbol{0}$ within their justifications. In Figure 3, we provide three examples of student responses: one that included both equations (Figure 3(a)), one that utilized $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to arrive at the solution (Figure 3(b)), and one that utilized $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ to arrive at the solution (Figure 3(c)). In Figure 3(a), the student’s response explained how solving for $\boldsymbol{x}$ in $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ could be derived from $A\boldsymbol{x} = \lambda\boldsymbol{x}$, and the goal is to find the $\boldsymbol{x}$ that makes that equation true. The approach in Figure 3(b) started with $A\boldsymbol{x} = \lambda\boldsymbol{x}$, substituted in the provided matrix A and the two options for the eigenvector $\boldsymbol{x}$, and carried out the computations to determine that the vector in option (b) was an eigenvector of A associated with the eigenvalue 6. The response in Figure 3(c) began with $(A - \lambda I)\boldsymbol{x} = 0$, converted it to an augmented matrix, found a general solution of $x_1 = 0.5x_2$, and determined that the vector in option (b) satisfied that relationship. We offer these examples as prototypes for the kinds of justifications students gave of their own accord in response to this problem, as well as illustrations of students' use of different solution strategies in conjunction with the eigen-equation they wrote down.

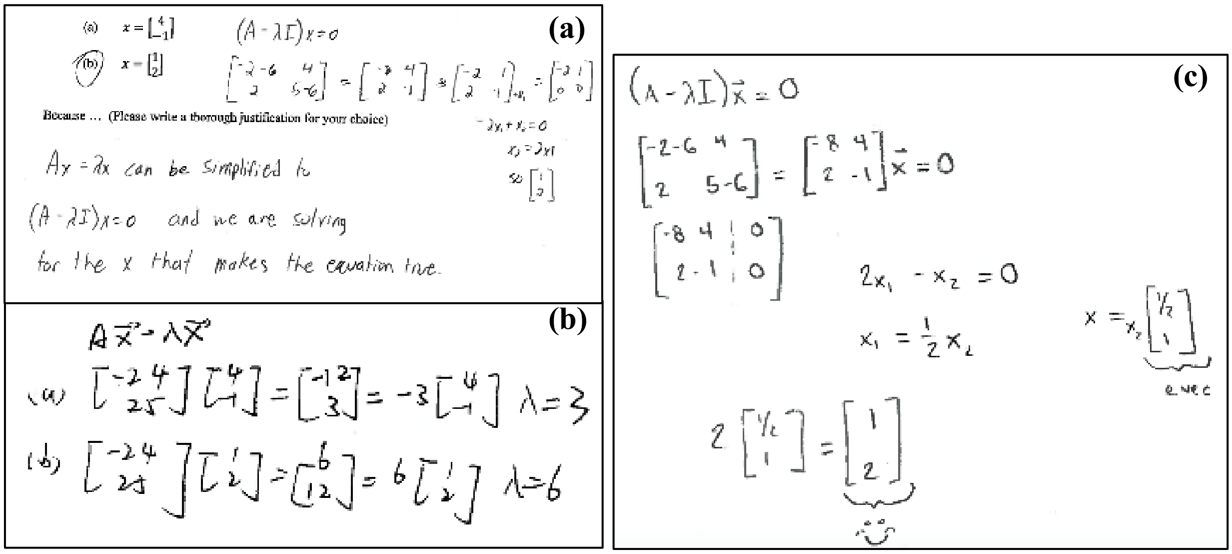


Figure 3. Three examples of open-ended justifications for Question 1.

Next, we analyze student responses from Class 1 and Class 2 in regard to their selection of justifications (i) and (iii), along with the eigen-equations Class 3 students explicitly wrote in their justifications**.** We organize this information in a format consistent with conditional tables in Figure 4, and we reuse this format throughout the paper in Figures 5, 7-9, and 11. We explain Figure 4 in detail here to familiarize the reader with the information contained in these conditional tables. Across the top of Figure 4 is justification (iii), and down the left side is justification (i). Every cell within the tables indicates the number of students who fall in that category (bold number), and gives a breakdown of their multiple choice answers (correct answer italicized). The leftmost section of Figure 4 provides information about the number of students in Class 2 that chose either, both, or neither of statements (i) and (iii). The cell in the first column and first row indicates that 23 students selected both statements as supportive of their conclusion, with 19 of them correctly choosing (b) as the solution to the main problem; 4 students selected (iii) but did not select (i), of which 1 correctly chose (b); one selected (i) but not (iii) and correctly chose (b); and one chose neither (i) nor (iii) and correctly chose (b).

(iii) This vector $\boldsymbol{x}$ makes $(A - 6I)\boldsymbol{x} = \boldsymbol{0}$ a true statement

(i) This vector $\boldsymbol{x}$ makes $A\boldsymbol{x} = 6\boldsymbol{x}$ a true statement

Class 2 (Basic MCE, 29 total)

	Selected	Not selected
Selected	**23** *19(b)* 4(a)	**1** *1(b)*
Not selected	**4** *1(b)* 3(a)	**1** *1(b)*

Class 1 (3-choice MCE, 27 total)

	TR	TNR	F
TR	**23** *19(b)* 3(a)	**3** *3(b)*	
TNR	**1** 1(a)		
F			

Class 3 (open-ended, 28 total)

	Wrote equation	Didn't write equation
Wrote equation	**4** *3(b)* 1(a)	**10** *10(b)*
Didn't write eqn	**4** *4(b)*	**10** *9(b)* 1(a)

Figure 4. Comparing students' distinction between $A\boldsymbol{x} = 6\boldsymbol{x}$ and $(A - 6I)\boldsymbol{x} = \boldsymbol{0}$ on Question 1.

The center section of Figure 4 provides information about the number of students in Class 1 that thought statements (i) and (iii) were true and relevant, true but not relevant, or false in terms of supporting their conclusion to the main problem. The cell in the first column and first row indicates that 23 students selected both statements as true and relevant, with 19 of them correctly choosing (b) as the solution to the main problem; 3 students indicated that statement (iii) was true but not relevant and statement (i) was true and relevant in supporting their solution, with all 3 correctly choosing (b); and 1 student indicated that (iii) was true and relevant, statement (i) was true but not relevant, and correctly chose (b).

Finally, the rightmost section of Figure 4 provides information about the number of students in Class 3 that explicitly wrote within their justifications either, both, or neither of the equations $(A-6I)\boldsymbol{x}=\boldsymbol{0}$ (indicated across the column) or $A\boldsymbol{x}=6\boldsymbol{x}$ (indicated across the row) in some form. The cell in the first column and first row indicates that 4 students wrote forms of both equations, with 3 correctly choosing (b) as the solution to the main problem; 10 students wrote an equation of the form $(A-6I)\boldsymbol{x}=\boldsymbol{0}$ but not $A\boldsymbol{x}=6\boldsymbol{x}$, with all 10 correctly choosing (b); 4 students wrote an equation of the form $A\boldsymbol{x}=6\boldsymbol{x}$ but not $(A-6I)\boldsymbol{x}=\boldsymbol{0}$, with all 4 correctly choosing (b); and ten students did not explicitly write some form of either equation, with 9 correctly choosing (b). We recognize, however, that students may have been thinking about one or both equations without explicitly including them in their written justification.

Comparing across the three classes produces an interesting insight. When students are asked to write their own justification (Class 3), a majority explicitly write only one of the eigen-equations, and quite a few write neither. However, when students are forced to consider the truth and relevance of the two equations contained in justifications (i) and (iii), the large majority are able to see both as true and relevant. Thus, it seems most students understand the importance and relevance of the two equivalent eigen-equations. Furthermore, the versions of the MCE that give students closed-ended justification choices (Basic MCE and 3-Choice MCE) might actually give more insight into students' understanding of eigentheory than the open-ended justification version.

(v) The vector $A\boldsymbol{x}$ is 6 times the magnitude and in the same direction as this vector $\boldsymbol{x}$

(i) This vector $\boldsymbol{x}$ makes $A\boldsymbol{x}=6\boldsymbol{x}$ a true statement OR (iii) This vector $\boldsymbol{x}$ makes $(A-6I)\boldsymbol{x}=\boldsymbol{0}$ a true statement

Class 2 (Basic MCE, 29 total)

	Selected	Not selected
Selected	**13** *12(b)* 1(a)	**15** *9(b)* 6(a)
Not selected	**1** *1(b)*	

Class 1 (3-choice MCE, 27 total)

	TR	TNR	F	NA
TR	**12** *11(b)* 1(NA)	**3** *3(b)*	**5** *4(b)* 1(a)	**1** *1(b)*
TNR				
F				

Class 3 (open-ended, 28 total)

	Wrote Geom. Idea	Didn't write Geom. Idea
Wrote equation		**18** *17(b)* 1(a)
Didn't write equation		**10** *9(b)* 1(a)

Figure 5. Comparing students' Algebraic vs. Geometric Justifications on Question 1.

We now look at students' judgment of the truth and relevance of geometric justifications on Question 1. Looking at the rows in Figure 5, we see that a large majority of students selected or wrote at least one of the eigen-equations, a compilation of the results from Figure 4. However, looking across the columns in Class 1 and Class 2, we see that a large number of students did not see the geometric justification (v) as relevant to answering Question 1 (15/29 students in Class 2, and 9/27 students in Class 1); indeed, there were even five students in Class 1 who said this justification was false. Even more striking, no students in Class 3 wrote anything geometric in their justifications. This might indicate students tend to think about eigenvectors and eigenvalues symbolically more than geometrically, or students may see symbolic reasoning as the more acceptable justification to the teacher or the broader math community.

Question 2

Students' geometric understanding of eigentheory was further explored in the results of Question 2. When looking at open-ended justifications from Class 3, two of the most common justifications students gave mentioned the result of the product being on the same line as the eigenvector $\boldsymbol{x}$, or being a scalar multiple of $\boldsymbol{x}$. We share three prototypical examples of students' justifications in Figure 6. The student in Figure 6(a) stated that the product should be a scalar

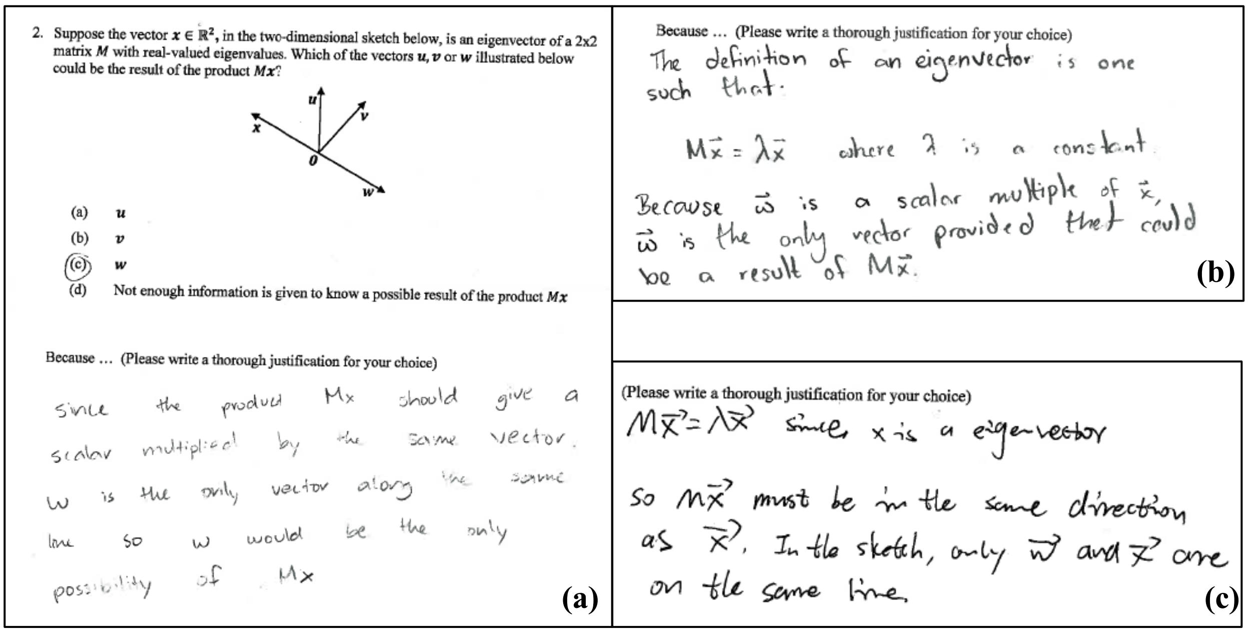


Figure 6. Example of three students' open-ended justifications for Question 2

multiple of the eigenvector, as well as on the same line; the student in 6(b) used the equation $\boldsymbol{Mx} = \lambda\boldsymbol{x}$ to argue why the product would need to be a scalar multiple of $\boldsymbol{x}$; and the student in Figure 6(c) explained how the product should be in the same direction and on the same line as the eigenvector $\boldsymbol{x}$. When we consider how students selected the related justifications (i) and (iv) on the Basic MCE and 3-Choice MCE (see Figure 7), it becomes clear that these ideas of the result being on the same line as $\boldsymbol{x}$ or a scalar multiple of $\boldsymbol{x}$ were seen by students as particularly relevant; more specifically, a majority of students in all classes selected or wrote at least one of these ideas, and in Classes 1 and 2, most students selected both justifications (i) and (iv) as true

(iv) This vector is a scalar multiple of $\boldsymbol{x}$

(i) This vector is on the same line as $\boldsymbol{x}$

Class 2 (Basic MCE, 29 total)

	Selected	Not selected
Selected	**20** *20(c)*	**3** *3(c)*
Not selected	**2** *1(c)* 1(b)	**4** 1(b) 3(d)

Class 1 (3-choice MCE, 27 total)

	TR	TNR	F
TR	**24** *24(c)*		
TNR	**1** *1(c)*		**2** 1(d) 1(NA)
F			

Class 3 (open-ended, 28 students)

	Wrote idea	Didn't write idea
Wrote idea	**3** *3(c)*	**7** *7(c)*
Didn't write	**7** *7(c)*	**11** *10(c)* 1(d)

Figure 7. Student justifications that included "scalar multiple" or "on the same line" for Q2.

and relevant. These results indicate the importance of having these justifications on the MCE instrument. Furthermore, we again wish to point out that when students write their own justifications, a large majority do not write down both ideas; but, when students are forced to consider the truth and relevance of these ideas, most select both as true and relevant. Hence, we are able to see deeper into students' understanding of eigentheory through the MCE instrument than we would through only the open-ended assessment.

An eigentheory expert might expect a similar majority of students to also select justification choice (iii) "This vector is in span$\{\boldsymbol{x}\}$" as true and relevant, since justifications (i), (iii), and (iv) could be seen as closely related. However, the results from the three classes tell a different story (see Figure 8). Looking at Class 3, we see that this concept was not something students readily volunteered as a justification, as only two students mentioned anything about span in their open-ended response. In Class 2, 16 of the 29 students did not select justification choice (iii) as true and relevant, but we cannot be sure why they did not select it. Did they think it was false, or just not relevant? The results from Class 1 may be the most illuminating, as no student said that justification choice (iii) was false, showing they understand that the result would be in span$\{\boldsymbol{x}\}$, but 10 of the 28 students said it was not relevant. Hence, although an expert might see justifications (i), (iii), and (iv) as closely related, for some students span was not relevant to answering this particular eigentheory question.

(iii) This vector is in span$\{\boldsymbol{x}\}$

(i) This vector is on the same line as $\boldsymbol{x}$

Class 2 (Basic MCE, 29 total)	Selected	Not selected
Selected	**12** *12(c)*	**11** *11(c)*
Not selected	**1** 1(b)	**5** 1(b) *1(c)* 3(d)

Class 1 (3-choice MCE, 27 total)	TR	TNR	F
TR	**15** *15(c)*	**9** *9(c)*	
TNR	**2** *1(c)* 1(d)	**1** 1(NA)	
F			

Class 3 (open-ended, 28 students)	Wrote idea	Didn't write idea
Wrote idea		**10** *10(c)*
Didn't write	**2** *2(c)*	**16** *15(c)* 1(d)

Figure 8. (Non)relevance of "span" for answering Question 2.

Question 3

As mentioned previously, students' response to Question 3 varied greatly across all three classes, suggesting this question was more difficult for students than the first two. Because of the varied response on the multiple choice portion, we looked at the relationship between students' multiple choice selection and particular justification choices. In Figure 9, we examine the relationship between students' multiple choice selection and justification (iv) "$A(5\boldsymbol{y}+5\boldsymbol{z}) = 5A\boldsymbol{y}+5A\boldsymbol{z} = 5\cdot 2\boldsymbol{y}+5\cdot 2\boldsymbol{z} = 2(5\boldsymbol{y}+5\boldsymbol{z}) = 2\boldsymbol{v}$". For Class 1 and Class 2, students who

Multiple Choice Answer

(iv) $A(5\boldsymbol{y}+5\boldsymbol{z}) = 5A\boldsymbol{y}+5A\boldsymbol{z} = 5\cdot 2\boldsymbol{y} + 5\cdot 2\boldsymbol{z} = 2(5\boldsymbol{y}+5\boldsymbol{z}) = 2\boldsymbol{v}$

Class 2 (Basic MCE, 29 total)	A	B	C
Selected	**11**		**2**
Not selected	**3**	**2**	**7**

Class 1 (3-choice MCE, 27 total)	A	B	C
TR	**7**	**1**	**1**
TNR			**10**
F			**1**
NA			**3**

Class 3 (open-ended, 28 students)	A	B	C
Wrote idea	**1**	**2**	**1**
Didn't write	**5**	**3**	**15**

NOTE: Four students in Classes 1 and 2 and one student in Class 3 did not answer, and are not included in the tables above.

Figure 9. Comparing students' multiple choice response to selection of justification (iv)

selected justification (iv) as true and relevant were more likely to select the correct multiple choice answer *A*. We note that in the open-ended responses, this was not the case; in fact, the majority of students in Class 3 did not write anything algebraic. Furthermore, Class 3 had the lowest percentage get the multiple-choice stem correct. From this result, there is a real possibility that having a correct string of algebra provided as a justification choice led the students in Classes 1 and 2 to the correct multiple choice answer. However, in some sense, it might be considered a good thing that students were able to recognize the algebraic string as correct and understand how this supports selecting the correct multiple choice answer. Considerations will need to be made on whether or not this justification is too leading for future use on the Eigentheory MCE.

To further explore why this question was particularly difficult for students, we examined the open-ended student responses to Question 3, where students used the concepts of linear combination and eigenspace in support of both correct and incorrect multiple choice answers. In Figure 10, we share examples of students using these concepts in their justification, but arriving at different conclusions. The two leftmost students both mentioned $\boldsymbol{v}$ being a linear combination of eigenvectors, but arrived at opposite answers, with one saying $\boldsymbol{v}$ would be an eigenvector, and the other saying $\boldsymbol{v}$ would not. Similarly, the two rightmost students both gave justifications that could be seen as involving eigenspace, but came to opposite conclusions, with the student on the far right mentioning that an infinite number of eigenvectors "doesn't make sense."

v is a linear combination of y and z which have the same eigenvalue.	No, because v is a linear combination of the two vectors.	v is in the eigenspace of both y and z	Multiplying the linearly independent eigenvectors by 5 makes them linearly dependent technically you could multiply the eigenvectors by any number and if you did so and another eigenvector was achieved there would be a possibility for infinite eigenvectors which doesn't make sense
Linear Combination Correctly chose (*A*)	Linear Combination Incorrectly chose (*C*)	Eigenspace Correctly chose (*A*)	Eigenspace Incorrectly chose (*C*)

NOTE: Typed versions are used here to improve readability of students' handwritten justifications

Figure 10. Example of four students' open-ended justifications for Question 3

Students' difficulty with these concepts is further supported when comparing their selection of justifications (i) and (v) found in Figure 11. Classes 1 and 2 each had a majority of students select at least one justification as true and relevant; however, students used these same justifications in support of different multiple choice answers. From Class 3, we notice that this is

(v) $\boldsymbol{v}$ is an element of the eigenspace created by the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$

(i) $\boldsymbol{v}$ is a linear combo of eigenvectors that have the same eigenvalue

	Selected	Not selected
Selected	**4** *4(a)*	**4** *3(a)* 1(c)
Not selected	**7** *4(a)* 1(b) 2(c)	**10** *3(a)* 1(b) 6(c)

Class 2 (Basic MCE, 29 total)

	TR	TNR	F
TR	**4** *1(a)* 1(b) 2(c)	**8** *3(a)* 5(c)	**1** 1(c)
TNR	**1** 1(a)	**3** *1(a)* 2(c)	**1** 1(c)
F	**3** *2(a)* 1(c)		

Class 1 (3-choice MCE, 27 total)

	Wrote idea	Didn't write idea
Wrote idea		**18** *3(a)* 3(b) 12(c)
Didn't write	**1** *1(a)*	**8** *2(a)* 2(b) 4(c)

Class 3 (open-ended, 28 students)

NOTE: Four students in Classes 1 and 2 and one student in Class 3 did not answer. Two students in Class 1 did not select either justification. These students are not included in the tables above.

Figure 11. Student use of linear combination and eigenspace for varying answers.

also the case for students that only wrote about linear combination and not eigenspace; of the 18 students that used linear combination in their justification, all three multiple choice answers were chosen, with a majority actually choosing the incorrect answer. Furthermore, in comparison to the first two questions, there is overall wider variation in students' selections for justifications (i) and (v) in Class 1 and 2. Generally, the results from this question demonstrate how the concepts of linear combinations of eigenvectors and eigenspaces are especially challenging for students.

Discussion

In this section we share insights into students' understanding of eigentheory that come from looking across the data for all three questions and how these relate to previous research on students' understanding of eigentheory. We then discuss the strengths and weaknesses of the three different MCE versions and considerations for future use of the assessments.

Insights from Looking Across Questions and Connections to Previous Research

First, there are some indications that students favor algebraic reasoning over geometric reasoning when justifying their answers to eigentheory questions, even though the class used the IOLA curriculum which specifically introduces eigenvectors and eigenvalues geometrically. As a particular example, the proportions of students who selected the algebraic justification (iv) (or wrote something similar) on the geometrically-oriented Question 2 were higher than the proportions of students who selected the geometric justification (v) (or wrote something similar) on the algebraically-oriented Question 1 (see Table 3). This may be seen as corroborating the work of Thomas and Stewart (2011), who found that "students tend to think about the concepts of eigenvalue and eigenvector in a primarily symbolic-world way" (p. 294), even after two years of linear algebra instruction. However, it should be noted that a majority of these students were able to answer the multiple choice stem of Questions 1 and 2 correctly, demonstrating some ability to reason both algebraically and geometrically about eigenvectors and eigenvalues. This is good, as many have advocated the importance of understanding eigentheory from geometric or embodied perspectives in addition to algebraic or symbolic ones (Gol Tabaghi & Sinclair, 2013, Salgado & Trigueros, 2015, Sinclair & Gol Tabaghi, 2010, Stewart & Thomas, 2006; Thomas & Stewart, 2011) and others have explained the significance of fluently translating between different modes of description (Hillel, 2000), modes of thinking (Sierpinska, 2000), or worlds of mathematics (Tall, 2004), to understanding linear algebra in general.

Table 3
Evidence of Students' Preference for Algebraic Justifications over Geometric

	Class 1	Class 2	Class 3
Proportion of students who selected the geometric justification (v) The vector $\boldsymbol{Ax}$ **is** 6 times the magnitude and in the same direction as this vector on Question 1, or wrote something similar	12/27	14/29	0/28
Proportion of students who selected algebraic justification (iv) There exists a scalar c such that $M\boldsymbol{x} = c\boldsymbol{x}$ on Question 2, or wrote something similar	17/27	18/29	12/28

Second, there is some evidence that the ways students think about and solve problems involving eigenvectors and eigenvalues are influenced by their reliance on or preference for one of the two equations $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. While we agree in the importance of students understanding both of these equations and the relationship between them, as suggested by previous research (Salgado & Trigueros, 2015; Thomas & Stewart, 2011), our results go beyond this by suggesting that the particular eigen-equation a student thinks about in the problem solving process can actually result in different ways to think about the same problem.

Some evidence for the idea of students relying on or preferring a particular equation can be seen in Class 3's results on Question 1, where 14 of the 28 students only wrote one of the two equations, suggesting students may have a "go to" equation when thinking about and solving eigentheory problems. Furthermore, as an example of a student's thinking or solution strategy being potentially influenced by one of the two eigen-equations, consider Figure 12. In this figure, we share open-ended work on Question 1 and Question 3 from a student who shows some reliance on or preference for the homogeneous equation $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. On Question 1, we can infer this student first found the matrix $(A - \lambda I)$, multiplied each vector by it, and chose the vector that was mapped to the zero vector. Then, on Question 3, although what he actually writes is not mathematically sound, we can infer he was still reasoning similarly with the homogeneous equation, imagining the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ being mapped to the zero vector by the matrix $(A - \lambda I)$, and therefore the vector $\boldsymbol{v}$ would also map to the zero vector. We recognize this is only one example, but we hope it illustrates how students' thinking and reasoning about eigentheory problems can be driven by the particular eigen-equation they adhere to at any given time. Future research plans of ours include exploring this idea further.

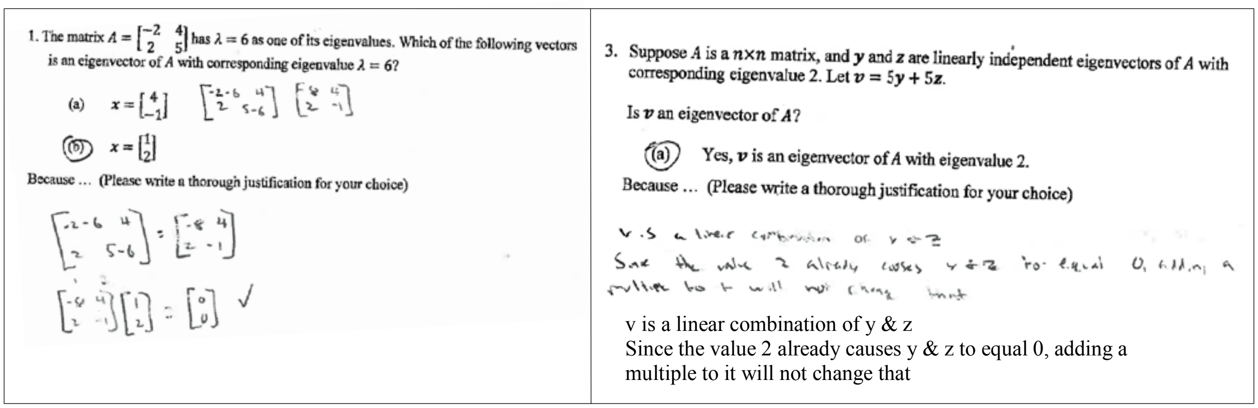


Figure 12. Evidence of student B66's Preference for Homogeneous Equation $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$

Third, the concept of eigenspace is particularly difficult for students to understand, as evidenced in the results from Question 3. This corroborates the results of Salgado and Trigueros (2015) who suggested this might be due to difficulties students have "in understanding the concepts of spanning set, basis, and space spanned, and also to the difficulties involved in comparing different geometrical objects in a three dimensional space" (p. 118). Our results go a bit further in two ways. First, Class 2's results on Question 2 above, where no student said justification (iii) "This vector is in span$\{\boldsymbol{x}\}$" was false, shows the students do have some understanding of the concept of span (at least one-dimensional span). Second, from Class 1 and Class 2's (56 students total) results on Question 3, 13 of the students said justification choice (v) "$\boldsymbol{v}$ is an element of the eigenspace created by the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$" was true, but selected (c) "No, $\boldsymbol{v}$ is not an eigenvector of A." This suggests students might not understand that any vector belonging to the span of a set of eigenvectors with the same eigenvalue (i.e., elements of the eigenspace) are themselves eigenvectors with that same eigenvalue. We suggest, along with Salgado and Trigueros (2015), further research specifically look at students' understanding of eigenspaces, and ways we might better teach this difficult concept.

Comparing the Three MCE Formats

One might be tempted to think that the open-ended version of the MCE would be the most helpful or appropriate for eliciting the ways students think about eigentheory. However, we have shown here that forcing students to think about other justifications can lead to interesting results, and gives us further insights into students' understanding of eigentheory. The question arises, which assessment (3-Choice MCE or Basic MCE) should be pursued further? We explore three considerations, namely time and cognitive demands, information gained, and scoring issues.

As both the Basic MCE and the 3-Choice MCE have a student read through six justification choices for each multiple choice question, the MCE is considerably more time consuming than a simple multiple choice assessment. This significantly affects the number of questions that can be asked using this format, and is one reason our instrument was limited to six questions. In reality, those six questions could be thought of as six multiple choice questions with 36 true-or-false sub questions. The MCE can also be cognitively taxing, as students must ponder on each individual justification choice to determine if it is true and relevant. Thus, mental fatigue is a real concern by the time students reach the end of the assessment. Furthermore, the 3-Choice MCE is more demanding in both time and cognition than the Basic MCE, because a student has to make a choice for each justification, not just circle those that are true and relevant.

On the other hand, when it comes to information gained from the assessment, the 3-Choice MCE gives more information about students' thinking than the Basic MCE. When students do not select a justification choice on the Basic MCE, we cannot know if they thought it was false, or just not relevant to answering the question. With the 3-Choice MCE, we are able to see this difference and gain better insight into their understanding of particular eigentheory concepts that the justifications are aimed towards.

Lastly, scoring of these assessments can be particularly troublesome. For example, what score is given to a student who selects all of the true and relevant justifications correctly, but chooses the wrong multiple choice answer? Or, what score is given for the exact opposite, choosing the correct multiple choice but none of the justifications correctly? (In fact, both situations have come up with the data we have). As we explore various scoring options, the question remains as to what these scores would actually mean, and how they should be interpreted or used to inform us about students' understanding.

It seems promising to focus on the use of this instrument as either a formative assessment to give instructors a general idea of their students' understanding and help inform further instruction, or a research instrument used to examine students' understanding of multifaceted concepts (such as eigentheory) similar to what we did in this current paper. Still, it would be useful if there were a more succinct way to look at individual students' results and quickly ascertain where a student's current understanding is. To this end, we hope to align a method of scoring the eigentheory MCE with our solidified framework for student understanding of eigentheory in the future. This scoring would hopefully give a summary of a student's understanding (e.g., reliance on a particular eigen-equation, ability to reason geometrically, algebraically, etc.) that would be helpful to teachers and education researchers of linear algebra.

Conclusion

In this paper we have presented insights into students' understanding of eigentheory that were obtained through the use of the Multiple Choice Extended assessment on eigentheory that we developed. These insights include students' use of the two eigen-equations and how they may influence students' solution strategies, students' tendency towards algebraic justifications when working on eigentheory problems, and the difficulty of the concept of eigenspace. We have also shown how the MCE gives us rich information about students' understanding of eigentheory. We hope that further refinement and use of the MCE, as well as developing possible scoring systems, will continue to broaden and deepen the mathematical community's understanding of the ways students reason and think about eigentheory.

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Meta-Representational Competence with Linear Algebra in Quantum Mechanics

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In this report we share analysis regarding students' meta-representational competence (MRC) that is expressed as they engage in solving quantum mechanics problems that involve linear algebra concepts. The particular characteristic of MRC that is the focus of this analysis is students' critiquing and comparing the adequacy of representations, specifically matrix notation and Dirac notation, and judging their suitability for various tasks (diSessa, 2004). With data from semi-structured individual interviews, we created categories of types of MRC elicited during students' work on an expectation value problem. We provide detail on two students who serve as paradigmatic examples of a student's power and flexibility in thinking in and using different notation systems. This work lends credence to and inspires our preliminary conjecture that strong meta-representational competence (MRC) is necessary not only to be fluent and proficient in the mathematics involved in solving quantum mechanics problems but also to develop a robust understanding of the quantum mechanics content.

Key words: *linear algebra, physics, matrix notation, Dirac notation, symbolizing*

The National Research Council's (2012) report, which charges the U.S. to improve its undergraduate STEM education, specifically recommends "interdisciplinary studies of cross-cutting concepts and cognitive processes" (p. 3) in undergraduate STEM courses. It further states that "gaps remain in the understanding of student learning in upper division courses" (p. 199), and that interdisciplinary studies "could help to increase the coherence of students' learning experience across disciplines … and could facilitate an understanding of how to promote the transfer of knowledge from one setting to another" (p. 202). Our work contributes towards this national need for basic research by investigating students' understanding, symbolization, and interpretation of eigentheory and related key ideas from linear algebra in quantum physics. One overarching research question for this work is: What are the various ways in which students reason about and symbolize concepts related to eigentheory in quantum physics?

As we examined data from individual interviews at the end of the course, one student's work in solving quantum mechanics problems was particularly striking to us because of the ease with which he moved between and explicitly discussed different notations, namely Dirac notation and matrix notation. After reviewing research literature on students' understanding of symbols, notations, and representations, we decided to align our analysis with the framework of meta-representational competence (diSessa, Hammer, Sherin, & Kolpakowski, 1991), as well as the delineation of structural features of algebraic quantum notations offered by Gire and Price (2015) which we explain further in the following section. This framing helped narrow our focus to the following sub-question: What aspects of Meta-Representational Competence exist in students' reflections on and comparisons of matrix notation and Dirac notation in quantum mechanics? In this report we share analysis of two students' reflections on explicit symbolization choices made while solving quantum mechanics problems that involve linear algebra. In particular, we inspect data of these two students solving an expectation value problem and analyze their reasons for how and why they chose a specific symbol system – either Dirac notation or matrix notation – for that particular situation. Our aim is to demonstrate how each student's rich understanding of linear algebra and quantum mechanics includes and is aided by their understanding and flexible use of different notational systems.

Background and Theoretical Framework

In this section, we give an overview of research conducted on student understanding of symbols and representations in mathematics and physics, as well as our theoretical orientation. We conclude with a brief introduction to eigentheory in Quantum Mechanics and Dirac notation.

Student Understanding of Symbols and Representations

The recognition of the importance of students having an understanding of the symbols used in mathematics and physics has grown over the past few decades. Arcavi (1994, 2005) coined this as "symbol sense," which includes (a) being "friendly" with symbols, (b) reading through symbols, (c) engineering symbolic expressions, (d) understanding different meanings based on equivalent expressions, (e) choosing which aspects of a mathematical situation to symbolize, (f) using symbolic manipulations flexibly, (g) recognizing meaning within symbols at any step in the solution process, and (h) sensing the different roles symbols can play in various contexts. Other research along this vein include: an explication of how different perspectives, such as cognitivist, situationist, and social-psychological, provide vastly different ways to understand how students make sense of and use inscriptions and symbols (Kaput, 1998); a study of how students mathematize their language from a Vygotskian perspective (Van Oers, 2002); and an exploration of how notational systems can serve as a mediational tool which triggers and sustains mathematical activity (Meira, 2002).

Research into students' competence with symbols, inscriptions, and representations is not limited to K-12 studies. For example, Harel and Kaput (2002) describe how mathematical notations play a key role in forming conceptual entities in higher mathematics. Additionally, in linear algebra research, Hillel (2000) described three modes of description (abstract, algebraic, and geometric) of the basic objects and operations in linear algebra and pointed out that "the ability to understand how vectors and transformation in one mode are differently represented, either within the same mode, or across modes is essential in coping with linear algebra" (p. 199). Thomas and Stewart (2011) found that students struggle to coordinate the two mathematical processes captured in $A\boldsymbol{x} = \lambda\boldsymbol{x}$, where A is an n x n matrix, $\boldsymbol{x}$ is a vector in $\mathbb{R}^n$, and λ is a scalar, to make sense of equality as "yielding the same result." This interpretation of the "equals" symbol is often novel and nontrivial for students (Harel, 2000). Harel also posits that the interpretation of "solution" in this setting, the set of all vectors $\boldsymbol{x}$ that make the equation true, entails a new level of complexity than does solving equations such as $cx = d$, with each taking values from the reals. Thomas and Stewart (2011) conjecture that this complexity may prevent students from progressing symbolically from $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$, which is particularly useful when solving for the eigenvalues and eigenvectors of a matrix A.

Relatedly, diSessa et al. (1991) identified students having a great deal of knowledge about what good representations are and how they are able to critique and refine them, which the authors defined as Meta-Representational Competence (MRC). In (2004), diSessa commented on the choice of term used, stating:

> In using the prefix *meta*- we do not mean to invoke the idea of metacognition. Instead, meta is used generically as it is in metascience or metaphysics (and also metacognition), purviews that transcend the mere practice of science or of physics, or, in this case, purviews that transcend the mere production and use of representations. (p. 294)

diSessa and Sherin (2000) explained that MRC includes inventing and designing new representations, judging and comparing the quality of representations, understanding the general and specific functions of representations, and quickly learning to use and understand new representations. diSessa (2002, 2004) expound upon these ideas by offering a list of critical resources students possess as part of their MRC for judging the strength of representations, such as compactness, parsimony, and conventionality. Two particular resources encompassed by MRC that we focus on in our data are "critique and compare the adequacy of representations and judge their suitability for various tasks," and "understand the purposes of representations generally and in particular contexts and understand how representations do the work they do for us (diSessa, 2004, p. 94). In this study, we align ourselves with the theory that representations are a sense-making tool, in that "the construction of representations on paper during problem solving mediates and organizes one's understanding of mathematical concepts" (Meira, 2002, p. 101). We couple this with a framing of MRC, specific to two notational systems, to investigate student reflections on their own notational preferences in quantum mechanics and what that may reveal about their understanding of change of basis and eigentheory in that context.

Extending even further, research into students' understanding in quantum mechanics has looked at how students make sense of and use a novel notation, called Dirac notation (explained in the subsequent section). Singh and Marshman (2013) showed that even after graduate level instruction in quantum mechanics, students still struggle with Dirac notation, showing inconsistencies in its use among contexts and problems. More closely related with this current study, Gire and Price (2015) looked at four structural features of three different notation systems used in quantum mechanics (Dirac, matrix, and wave function) and how students' reasoning interacts with these features. The features identified by the authors are: (a) individuation, or "the degree to which important features are represented as separate and elemental" (p. 5); (b) externalization, or "the degree to which elements and features are externalized with markings included in the representation" (p. 7); (c) compactness; and (d) symbolic support for computation. Using problem-solving interviews with students as insight into these features, Gire and Price found that students readily used Dirac notation, and that the structural features vary across the different notations as well as among several contexts within quantum mechanics.

From an expert's perspective, Gire and Price have highlighted features that could also be identifiable by a student, and articulation of some of these ideas would fit well within diSessa's framework of MRC. The work presented here attempts to develop a coding scheme that can be used to identify MRC as students discuss, describe and use matrix and Dirac notation to solve quantum mechanics problems.

Brief Introduction to Eigentheory in Quantum Mechanics and Dirac Notation

In quantum mechanics, certain physical systems are modeled and made sense of using eigentheory. To a physical system we associate a Hilbert space (such as $\mathbb{C}^2$), to every possible state of the physical system we associate a vector in the Hilbert space, and to every possible observable (i.e., measurable physical quantity) we associate a Hermitian operator (usually given in its matrix form). The only possible result of a measurement is an eigenvalue of the operator, and after the measurement the system will be found in the corresponding eigenstate.

Dirac notation, also known as bra-ket or just ket notation, is a commonly used notational system in quantum mechanics. A vector representing a possible state is symbolized with a ket, such as $|\psi\rangle$. Mathematically, kets behave like column vectors, such as $|\psi\rangle \doteq \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $a_1, a_2 \in \mathbb{C}$, and are usually normalized. The complex conjugate transpose of a ket is called a *bra*, which behaves mathematically like a row vector, such as $\langle\psi| \doteq [a_1^* \quad a_2^*]$. In addition, the eigenvalue equations for observables are central to many calculations. For instance, the eigenvalue equations for S_x (the operator measuring the x-component of intrinsic angular momentum) of a spin-½ particle are $S_x|+\rangle = \frac{\hbar}{2}|+\rangle$ and $S_x|-\rangle = -\frac{\hbar}{2}|-\rangle$, where $|+\rangle$ and $|-\rangle$ are orthonormal eigenvectors of S_x and $\pm\frac{\hbar}{2}$ are the two possible measurement results of the observable. When symbolized in terms of this eigenbasis, the matrix representation of S_x is $\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}$.

One can also measure spin along other directions, such as z; similarly, the eigenvalue equations are $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ (it is common for no subscript to be used for the z-direction). Thus, "within its own basis," the matrix representation of S_z would be identical to the aforementioned diagonal one for S_x. It is often beneficial to change between bases; for example, $|+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ and $|-\rangle_x = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$, so S_x "in the z-basis" is $\begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix}$.

An elegant use of Dirac notation involves change of basis; because the eigenvectors of one operator are often well known in terms of another, such as along the z-direction of a spin-½ particle, Dirac notation is seen to make basis change calculations efficient. Finally, inner products are involved in computing the expectation value of observable A for state psi, $\langle\psi|A|\psi\rangle$. These calculations require the bra and ket expansion to be in the same eigenbasis as the matrix representation of A. Because of the orthonormality of eigenvectors of Hermitian operators, the relevant inner products yield $\langle+|+\rangle = 1, \langle-|-\rangle = 1$, and $\langle\mp|\pm\rangle = 0$. As such, expectation value problems present a rich setting for investigating students' symbolizing of eigentheory and change of basis in a physics context.

Methods

Participants for this study were junior physics majors at a large, public, research-intensive university in the Pacific Northwestern United States. They were drawn on a volunteer basis from a class of 35 students in a Spin and Quantum Measurements course; this course met for 7 class-hours per week for three weeks and involved many student-centered activities and discussions. In addition to videotaping each class session, we conducted individual, semi-structured interviews (Bernard, 1988) with 8-13 students at the beginning and end of the course. This particular report draws on data gathered during the end-of-course interview, in which 8 students participated. The goals of the interview questions were to learn how students reasoned about linear algebra concepts (e.g., normalization, basis, and especially eigentheory), how they reasoned with these concepts as they discussed quantum mechanics concepts and solved quantum mechanics problems, and how they symbolized their work.

To begin our analysis, we viewed the videos and observed how students navigated the interview problems, while we kept in mind the overarching research questions regarding students' reasoning about and symbolizing eigentheory in quantum physics. Throughout our viewing, we noticed some students were particularly fluent in how they talked about and worked with both matrix and Dirac notations. This compelled us to investigate the literature about student use of symbols and notations, the most relevant of which were discussed above. As such, we began an analysis drawing on the work of diSessa and colleagues (diSessa et al., 1991; diSessa 2002, 2004), and Gire and Price (2015). In particular, we coded for instances of students mentioning structural features of the mathematics or students making explicit meta-commentary on the representations they chose to use. This allowed us to integrate our analysis of students'

MRC with Gire and Price's types of structural features in a way novel to the physics and mathematics education fields.

We now explain how the students for this report were chosen. One student who fluidly worked with both matrix and ket notations was student A25, and this fluidity was especially evident within his solution of an expectation value problem. Because this problem seemed to be conducive to eliciting students' understanding, critiques, and comparisons of both notations, we identified all students who had engaged in the expectation value problem in their end-of-course interview, resulting in a pool of four students to study. The second author then read the transcript or watched the video of the entire interview for all four students, indicating instances in which the student explicitly talked about either or both notations. These instances were then collated into a single document with all student identifiers removed. Using this document, each author individually coded the sections, specifically attending to ideas related to MRC (diSessa, 2002, 2004) and the characteristics of matrix and Dirac notation identified by Gire and Price (2015). We then met collectively to share our codes and decide on a final set of codes for each transcript section. After the codes were finalized, we began axial coding, placing our collective codes into categories of features and characteristics of Dirac and matrix notations students were attuned to.

In this particular report, we focus on the end-of-course interview with two students: Milan and Buzz (both names are pseudonyms). Milan was a double major in math and physics who had completed one 10-week course in linear algebra and was concurrently enrolled in a second linear algebra course, and Buzz was a double major in physics and nuclear engineering who had completed two 10-week courses in linear algebra. The purpose of focusing on Milan and Buzz was their demonstrated ability to articulate their thinking. More specifically, during the interview, they demonstrated flexibility in reasoning about the concepts we were probing, and a great deal of MRC was visible and analyzable through their explanations.

Results

Our coding of the data produced three main categories of codes: (A) Preference based on a value judgment, (B), Preference based on the problem or context, and (C) Awareness of the purposes of representations and the work they do for us. Each category is composed of 5, 6, and 4 subcodes, respectively (see Figure 1). Category (A) comprises MRC statements related to preferences students expressed related to some overarching value; statements were similar to sentiments such as one notation being preferred because it is faster to write than another [A2], more familiar to work with [A3], or easier to write than another [A5]. The MRC codes in Category (B) comprise statements students made that are more intrinsically tied to particular problems or contexts; statements were similar to sentiments such as one notation being preferred or chosen in a particular problem because of the format of the question statement (e.g., "sometimes the ket notation is nicer…if you already know the eigenvalues and you just are multiplying by operators." We note that aspects of [B1] and [B2] are consistent with Gire and Price's structural feature of "symbolic support for calculation." We also purposefully use their other three structural features as our codes for [B4], [B5], and [B6] because we saw evidence of them in our data. We do note that which main category these codes best belong in could be open for discussion. Finally, Category (C) derives its name from one of diSessa's (2004) stated functions of MRC as an overarching awareness of representations and what they do for us. Again, student statements consistent with all four (C) codes existed in our data and gave rise to the codes. We note that these codes in particular have a compatibility with Arcavi's (1994) characteristics of symbol sense but stand out as novel because of its use with MRC.

A. Preference based on a value judgment	B. Preference based on the problem or context
1. Clarity 2. Speed 3. Familiarity 4. “Likeability” 5. Ease of writing	1. Useful in calculations 2. Makes more direct use of given relationships 3. Needs less information 4. Compactness 5. Individuation 6. Externalization
C. Awareness of the purposes of representations and the work they do for us	
1. Has freedom to choose symbols 2. Has an ease with notation, writes symbols to mean what is personally desired 3. Aware of one’s own progress in notation use 4. Able to “step back” and weigh options to decide which notation system is best	

Figure 1. The list of categories of MRC that resulted from our data analysis.

We organize the remainder of our results section according to excerpts from Buzz and Milan, including aspects of their responses that particularly point to and illustrate their MRC.

Buzz

In the beginning of the interview, Buzz volunteered that he sometimes explicitly chooses between doing calculations in matrix notation or in Dirac notation:

I: So how do you feel like, using eigenvectors and eigenvalues, in spins has been similar to and different from how you've experienced those in other classes?

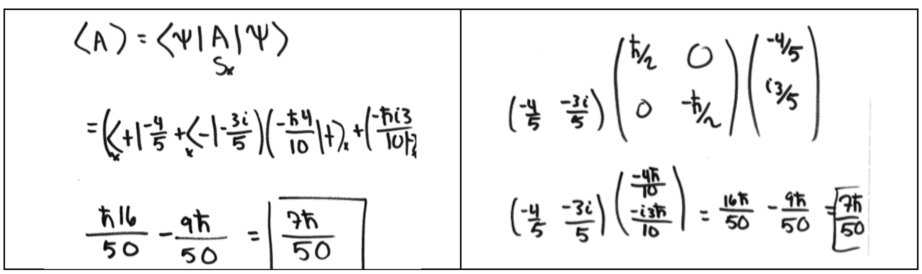
Buzz: Uh, well, it's very similar because you're doing a lot of the same math …the difference especially in physics, you're looking at kets. In, in at first it was kind of jarring, like to- to try to do the math in kets. But now, it's kinda- it's kinda easier, there's problems, there certain problems…where there's two ways to do them, they're kind of parallel, you can do it and you can expand the- the- the state in- in like as a- and expand them as- as kets in a different basis, or you can write that state as a- as a, as a vector, in that basis, and you can either do the matrix math for the like expectation values for example, you can do the matrix math or you can do the ket math, and sometimes it's, I'm finding that I, rather expand something in the ket.

From the transcript we see that Buzz was aware that there exist multiple legitimate ways to solve the problem, seemingly understanding the various mathematical nuances and implications of his notational choices. His brief explanation highlights sentiments that are consistent with Arcavi’s characteristics of symbol sense, such as being “friendly” with symbols and using them flexibly. We add, however, a metacognitive aspect of symbol sense here, noting that A25 was engaged in self-reflection rather than a researcher analyzing Buzz’s engagement with symbols. Thus, we code this as “[C4] able to “step back” and weigh options to decide which notation system is best.” We also note that Buzz was aware of his own progress, commenting that it was “it first kind of jarring…to try to do math in kets,” but that now it is “kinda easier.” We code this as “[C3] aware of one’s own progress in notation use.”

Because Buzz volunteered expectation value problems as a situation in which he could use either notation, the interviewer had him work on such a problem right away, even though it was prepared to be at the end of the interview: “Consider the state $|\psi\rangle = -\frac{4}{5}|+\rangle_x + i\frac{3}{5}|-\rangle_x$ in a spin-1/2 system. Calculate the expectation value for the measurement of S_x.” Buzz immediately worked on the problem within Dirac notation, saying, “basically to find the expectation value…

it's like denoted that way [writes $\langle A \rangle$] but really what you're doing is you're taking the, the bra of the state, and then you're putting the operator [writes $= \langle \psi | A | \psi \rangle$] in the middle of the inner product…" He continued to explain his work as he went, arriving at the correct answer of $7\hbar/50$ (see Figure 2a). Note that his work in Figure 2a involved the state's expansion and use of eigenvector equations for S_x in ket notation. In addition, this notation was first introduced to the students during this course; as such, Buzz was clearly quick to use and understand this representation (a additional quality of MRC, diSessa & Sherin, 2000).

After discussing his work and solution, the interviewer asked: "Before you were telling about bra-ket versus matrix notation, you brought up an expectation value as an example of like, either or both, so can you, now that you had this problem, kinda revisit that?" Buzz immediately solved the problem completely within matrix notation (see Figure 2b), explaining his steps. For instance, in line 1 in Figure 2b, he wrote the complex conjugate transpose of the vector representation of the state in the x basis, the matrix representation of the S_x operator in the x basis, and the vector representation of the state in the x basis. He also stated his process for computing the matrix times the column vector before he did the computation, and again in line 2 he explained "then I do it again, so, um, this time you're gonna get a number out," meaning he anticipated that a row vector times a column vector would be a number. We see this as flexibly using symbolic manipulations (Arcavi, 1995) and an anticipation of results.



(a) (b)

Figure 2. Buzz's expectation value problem, in ket notation (a) and matrix notation (b).

The interviewer then asked Buzz to reflect on his preference between the two notations:

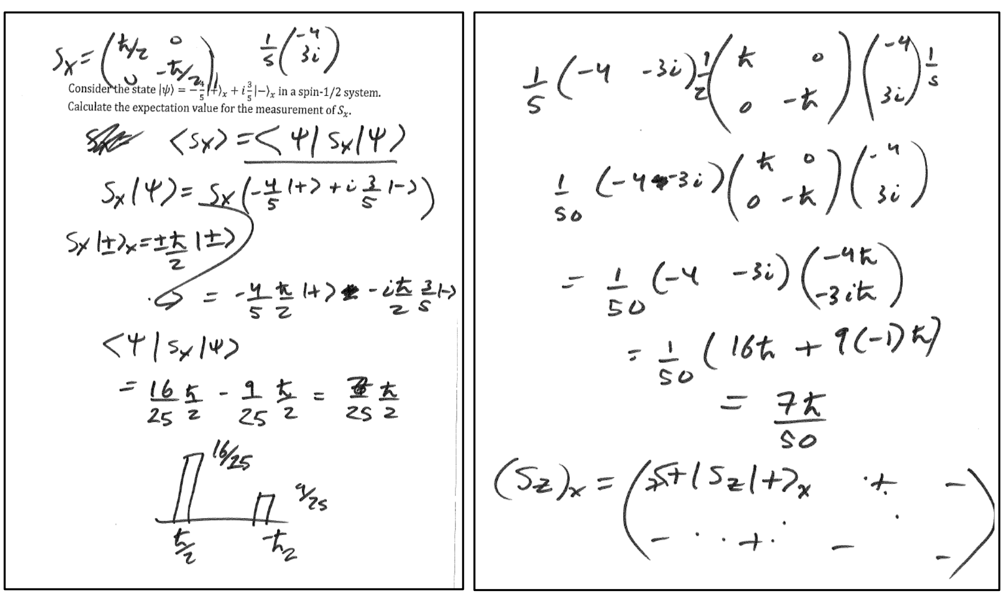
Buzz: Uh...To be honest, I don't really, I don't really know why I prefer this [Figure 2a], I think it's just because, um, I like this notation. This- this specific notation [Figure 2a line 1] like this to me is like a cleaner way of writing that [Figure 2b line 1] because that- I mean this and that [touching lines 1 in both figures] I feel like are your starting points, so you, you start here with this nice, like, looking thing [traces his finger under $\langle \psi | A | \psi \rangle$], or you start here with this big array of numbers [puts open hands around Figure 2b], and I prefer this [Figure 2a line 1], even though you have to expand this into basically the same amount of information [Figure 2a line 2]. And also, the nice thing about, about this [Figure 2a line 1], is it—actually this is really why it's better—is because you can, you can say ok S_x works- acts directly on these kets, you can just get rid of the matrix altogether...

This excerpt begins with two value statements: that Buzz prefers Dirac notation because he likes it [A4] and because it provides a cleaner way of writing the desired information [A1]. We see his use of "nice looking thing" and "big array of numbers" in comparison to one another are an example of compactness [B4]. He also compares line 1 in 2(a) and line 2 in 2(b) regarding the "amount" of information they convey, which involves reflection on the physical and mathematical content expressed in the compared notations. Finally, acting directly on the expansion in terms of the eigenstates of the operator allow him to forego the matrix calculation entirely, which speaks to Buzz's view of the utility of notation for calculation in a specific problem [B1] and his preference for Dirac in a situation that allows him to make a more direct use of the given relationships [B2].

When asked about his notation preferences if the basis expansion of a given state vector and the operator "didn't match," Buzz recalled a problem from their last homework that was "actually easier…to do the matrix multiplication," stating "you don't want to have to change these kets into different bases all over the place 'cause they're already all written in the same basis and you know what the operator is in that basis so you might as well just, do the matrix multiplication." This speaks to his awareness of using given relationships directly [B2] as well as being able to choose which notation system is best for a certain situation [C4].. Finally, when asked if the notions of basis or eigenvectors/eigenvalues come up more in one notation than the other, Buzz stated, "certainly…every time you write down a ket you're, you're very conscious of what basis you're in. In this one [points to Figure 2(b)] it's just kinda implied…all this [is] in the same basis, so you're just, you're just writing out numbers, an arrays of numbers, but here [in Figure 2(a)] you're thinking ok, this is the S_x operator, this is the x plus ket, this is the x minus bra…so I think that you're definitely more aware of what basis you're in when you're using this, because you have to be." This explanation is consistent with Gire and Price's (2015) notion of externalization [B6], in that the ket notation allows features of the problem, namely basis, to be externalized in a way that the matrix notation did not provide for Buzz.

Milan

We focus our examples of Milan's MRC on his work with the Expectation Value Problem. Like Buzz, he first completed the problem in Dirac notation but then, prompted by the interviewer because of a comment Milan had made, correctly completed the problem using matrix notation (see Figure 3).



(a) (b)

Figure 3. Milan's work on the Expectation Value problem.

Milan: So, this is very convenient because it's in the S_x basis. Um, so, basically all we need to do is put S_x in some matrix representation as — well, do we need to do matrix representation? I don't think we do. So, let's not worry about that. So, the expectation value of S_x is defined as this ψ, and the S_x, and ψ. And we're just going to include the ψ's in this — We're just going to drop the x subscript and just assume that it's, um, the x-basis….

Int: When you first started, you almost started using matrix notation, and then you decided not to. Can you talk more about that?

Milan: So, ket — because you know this eigenvalue equation, ket notation just skips all that. Um, if you wanted to, you could have written this out as, uh, 1/5(-4 3i), But, you can see that's more confusing to go through — and then you have to look at S_x and, where you can write it as ħ/2, -ħ/2, 0, 0, and you have to do matrix multiplication. Um, actually, that might be quicker, honestly…Which is, the exact same answer we got before, and it was substantially quicker. And in — I mean, I guess it depends how good you were with this kind of notation 'Cause, this is, I learned two and a half weeks ago, and this I learned almost a year ago.

As Milan began his work on this problem, he thought out loud about his approach, first stating he would put S_x in a matrix form but then deciding he did not need to do so. We coded this as [C4], an ability to "step back" and weigh options to decide which notation system is best. Once he decided to solve the problem within Dirac notation, he decided to drop the "x" subscript from the expansion of the state ψ in terms of the x basis (see lines 2, 3, and 5 in Figure 3a). This indicates Milan's ease with notation in that he can write and engineer symbols to mean what he personally desired [code C2].

After 4 minutes of work in which Milan explains his steps to correctly solve the expectation value problem (Figure 3a), the interviewer asked him to comment more on his rather explicit choice to not use matrix notation. We coded his statement, "Because you know this eigenvalue equation, ket notation just skips all that" as [B2] because of his preference for Dirac nation based on his perception that it made more direct use of the eigenvalue equations $S_x|\pm\rangle_x = \pm\frac{\hbar}{2}|\pm\rangle_x$. He also stated that one could have written the vector form of the given ψ, which is another example of his awareness of his choice in notation system [C4]. He continued by explaining how that would be more confusing in this situation [code A1, preference based on clarity] and summarized the matrix symbols and operations that would be necessary to complete. Milan then reflected out loud, saying that the matrix notation approach may actually be faster for him in this problem [code A2, preference based on value judgment of speed]. The interviewer prompted him to "do it that way," and after one minute of work (see Figure 3b), Milan concluded he got the same answer as before but substantially quicker. He then spontaneously reflected a bit more about the two notation options, noting aspects of familiarity with matrix notation [code A3] and that it takes time to make progress in using the new-to-them Dirac notation system [code C3].

Conclusion

In this report we shared our analysis of the meta-representational competence of two students as they engaged in solving a quantum mechanics problem involving linear algebra. To coalesce the ideas shared herein, we offer three reasons why this work is important to mathematics and physics educators. First, similar to one of the purposes of the work of diSessa et al. (1991), our report acts as a counter narrative to a number of articles in physics education that focus on students' difficulties or misconceptions in learning quantum mechanics (Johnston, Crawford, & Fletcher, 1998; Singh, 2001; Styer, 1996), and more specifically with the notations used in quantum mechanics (Singh & Marshman, 2013, 2016). We have shown students' abilities to critique and understand the general purposes of the various notations used within quantum mechanics, and how their meta-representational competence can even include some ideas about strengths and weaknesses of the notations that align with expert determined structural aspects of quantum representations (Gire & Price, 2015). Hence, instead of only focusing on students' difficulties with notation, educators might examine how they can build upon the meta-representational competence students already have to help deepen and strengthen their understanding of quantum mechanical and mathematical representations.

Second, as mentioned by diSessa (2004), educational research is always looking for ways we can help students develop deeper understanding of science and mathematics, and an important piece of this deeper understanding includes knowing "not only how to operate scientific apparatus (from strategies that solve problems, to representations and concepts), but also understanding how and why they work, and even being able to generate and judge alternative means" (p. 299). Thus, helping students develop deep understanding of quantum mechanics and linear algebra should include providing opportunities for students to use and improve their meta-representational competence. In fact, we noticed that the strongest students within our study in regards to their understanding of linear algebra and quantum mechanics concepts were also the students who demonstrated particularly articulate and thoughtful MRC. This begs the question, does a deep understanding of the physical and mathematical concepts create a conducive environment for developing stronger MRC, or does MRC help students develop a rich understanding of physical and mathematical concepts? We hypothesize that there is actually a cyclical relationship between MRC and physical and mathematical understanding, with development in one influencing the development of the other. For example, a student's understanding of eigentheory would influence the aspects of MRC they could attend to when using the different notations in quantum mechanics. In turn, the MRC in regard to quantum mechanical notations that student develops might strengthen their understanding of eigentheory concepts. Future research could examine and explore this cyclical relationship in greater detail, and how instructors might take advantage of it in helping students develop deep understanding of physical and mathematical concepts.

Third, we have presented codes we have used for identifying and analyzing students' MRC in their talk about eigentheory and quantum mechanics, as well as general categories for the types of MRC these codes indicate students demonstrate. This coding scheme still has the potential to be expanded, extended, and refined, especially as students' MRC within quantum mechanics and linear algebra are further explored, and as students' MRC is studied within other mathematical and physical contexts. Our hope is this coding scheme will facilitate analysis of students' understanding of representations within mathematics and physics, and gaining insight into the richness of their overall understanding of mathematical and physical concepts. Furthermore, we hope the identification of particularly useful and powerful elements of MRC, such as those found in analyzing experts or strong students, will help educators know the types of thinking and reasoning to emphasize and cultivate in teaching their students. Education researchers could then search for teaching methods, questions, and curricular materials that would support students in developing their MRC in these productive ways.

In our own future research, we plan to analyze the rest of the students from the current data set, as well as interviews with students in a Junior-level quantum mechanics course at a public university in the Northeastern United States. The latter interviews should be particularly useful for further refinement of our MRC coding scheme as we were more explicit in designing questions that might elicit students' MRC when writing the interview prompts. This analysis may also begin to give us further insight into the cyclical relationship between students' MRC and their understanding of physics and mathematics, as well as ways researchers and educators might elicit and cultivate students' MRC.

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Corequisite Remediation and Math Pathways in Oklahoma

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We examine multiple data sources to assess the current progress of implementing corequisite remediation and multiple math pathways in the state of Oklahoma. We begin by analyzing trends in national reform efforts and contrasting them with the status of current challenges and efforts in Oklahoma. We then present preliminary data from pilot sections of a corequisite College Algebra course and a new math pathway for degrees that require significant quantitative literacy but do not require engineering calculus. Finally, we consider statewide data on student course-taking patterns, degree requirements, and existing institutional efforts that will inform upcoming state-level decisions on these reforms.

Key words: corequisite remediation, mathematics pathways, reform

Introduction

The state of Oklahoma is currently in the process of reforming introductory post-secondary mathematics options and curriculum across all 27 public higher education institutions with the goal of increasing success in college mathematics courses and therefore increasing degree attainment across the state. To accomplish these goals the Oklahoma State Regents for Higher Education, the governing body for all public higher education institutions in Oklahoma, adopted the Complete College America agenda (Complete College America, 2013). The main focus of the state reforms are 1) supplementing the current system of remedial courses with a corequisite model and 2) creating multiple introductory mathematics pathways better aligned to diverse degree programs.

Traditionally, to ensure preparation of entering students, colleges assess students using various criteria (e.g. SAT scores, ALEKS, high school math GPA, etc.) then place them into college courses using these measures. Under-prepared students are placed in a remedial course sequence designed to fill in deficiencies from secondary mathematics and prepare these students for college-level courses. Once a student completes the remedial course sequence, they are allowed to take credit-bearing math courses required for a degree. This remedial system, however, often fails in its ultimate goal. In 2010, Bailey et al. (2010) found that only 31% of community college students referred to a remedial courses sequence in mathematics completed it. Only half of the students completing the remedial course sequence enrolled in the subsequent college-level course. Ultimately, only 15% of students referred to remediation passed the subsequent college credit-bearing course, which is significant as 58% of entering community college students enrolled in a remedial mathematics course (Attewell et al., 2006). Overall, 28% of college students from two or four-year colleges enroll in remediation (Attewell et al., 2006). Furthermore, delaying enrollment in college-level courses in favor of remedial courses has the consequence of extending students' time to degree, meaning both an increased cost and decreased persistence toward a degree (Complete College America, 2011).

Alternatively, in corequisite remediation, underprepared students are placed directly in college-level courses with targeted assistance. The aim of the model is to allow under-prepared students to earn college credit upon entering while still providing the students with necessary perquisite material, thereby eliminating often multiple semesters of remedial courses and enabling students to progress to through their degree programs. Corequisite remediation has been successful in several pilot programs across the country, which will be briefly discussed in the next section.

The second focus of the reforms are creating mathematics pathways beyond the standard College Algebra/Calculus sequence. Creating additional pathways provides students in non-STEM disciplines basic mathematics courses more relevant to their interests and needs. The reforms seek to increase collaboration between math departments, other academic degree programs, and employers as to the necessary mathematical knowledge and skills for students in their chosen field. For many, courses in statistics, quantitative reasoning, or mathematical modeling are more applicable to future collegiate and career needs than College Algebra. The need for increasing such diverse mathematical competencies has been highlighted by several reports from professional associations including: the *Common Vision 2025* report by the Mathematical Association and America (Saxe et al, 2015), the *Guidelines for Assessment and Instruction in Statistics Education (GAISE): College Report* endorsed by the America Statistical Association (Aliaga, 2005), and the *Beyond Crossroads* Report of the American Mathematical Association of Two-Year Colleges (Foley, 2007). Additionally, by creating clearly defined mathematics pathways at the state level, Oklahoma is aiming to increase transferability of mathematics courses between public institutions.

In this paper, we address the following research questions:

1. What are the primary national trends and lessons in corequisite remediation and math pathways relevant to the goals and structures of the Oklahoma higher education system?
2. What are the primary obstacles in implementing corequisite remediation and math pathways in Oklahoma, and what factors can local and state leaders influence to address these challenges?

A National Perspective of the Reforms

Several states have either implemented or are in the process of implementing the reforms outlined above. We briefly describe progress in two of the states to lead these reform efforts, Georgia and Tennessee.

Currently, Georgia has two pathways: the traditional College Algebra/Calculus pathway and a non-algebra pathway which focuses on either quantitative reasoning or modeling. In Fall of 2014, Georgia piloted corequisite remediation for both pathways. 67% of the 2919 students in the non-corequisite sections passed the gateway course. In the corequisite sections, there was a total 1,132 students 64% of whom passed. Comparably, only 21% of students referred to remedial education in 2010 passed their gateway course within two years (Complete College America, 2015).

Beginning in Fall of 2014, Tennessee conducted a pilot corequisite program for an introductory statistics course with 1,019 students at 9 different campuses. Tennessee saw similar results to Georgia, 63.3% of students assessed as being underprepared passed introductory statistics whereas under the previous remediation model only 12.3% completed the introductory statistics course.

Similar results can been see in other states (Complete College America, 2016). We will continue to examine the progress and challenges of the reforms across the country and how their efforts can inform the reforms in Oklahoma.

Traditional Remediation at Oklahoma Institutions

Consistent with the national picture, a large percentage of students at each of the public institutions in Oklahoma are referred to remediation. In many of the state community colleges, over 60% of the entering students are referred to some type of remediation, be it English, math, or both. Figure 1 gives the percentages of student's referred to remediation at each of the public two institutions. With stricter admissions criteria at four-year initiations, the rate at which students are placed into remedial courses when entering is lower, however, still over 40% of entering students are placed into remediation. Remedial courses have a number of effects on students who are enrolled. Remedial students pay both tuition and additional remediation fees for courses which confer no college credit. Furthermore, taking remedial courses increases the number of mathematics courses a student must take thereby increasing the potential "drop-out" points. These "drop-out" points include potential failure in the remedial courses, but also include failure to enroll in either the subsequent remedial course or the gateway course after passing the initial course. The high remediation rates are particular troubling when one considers the success of this as measured by the completion of the college gateway course as shown in Figures 1-4.

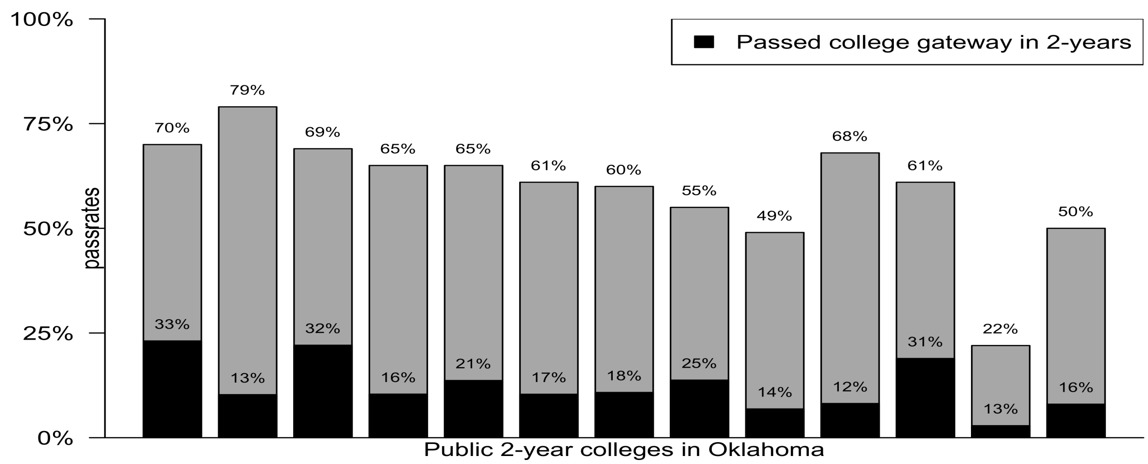

Figure 1. Entering students taking any remedial sequence.

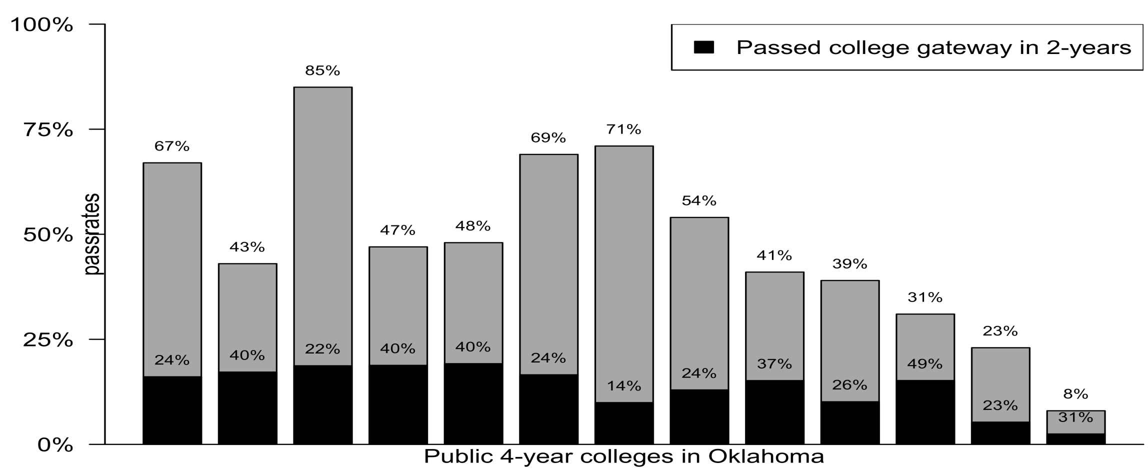

Figure 2. Entering students taking any remedial sequence.

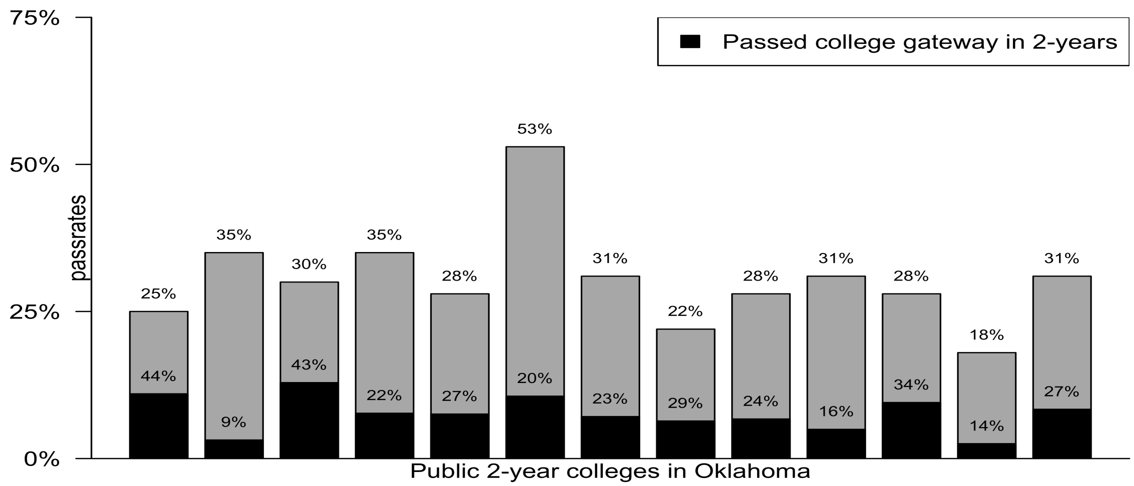

Figure 3. Entering students taking only a remedial mathematics sequence.

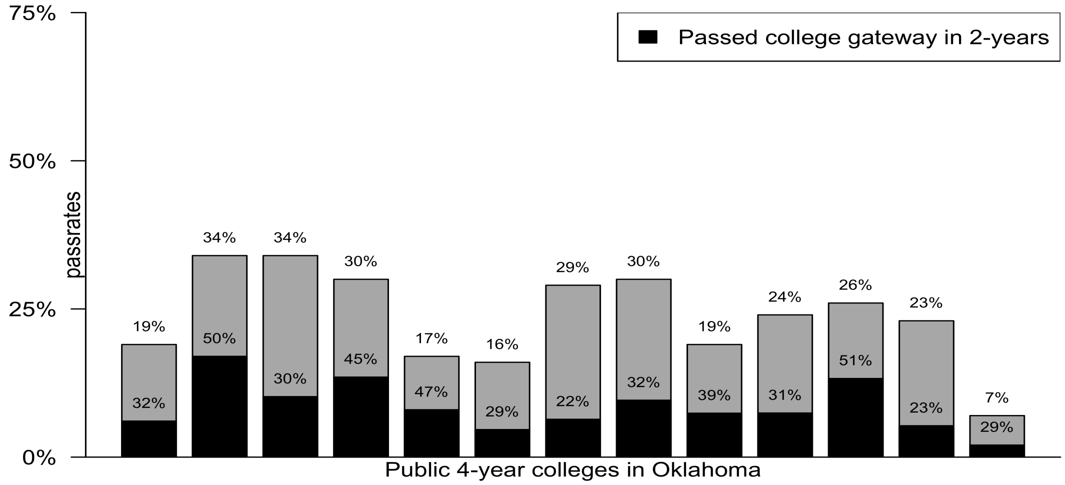

Figure 4. Entering students taking only a remedial mathematics sequence.

Figures 1-4 show that less than 35% of the students who begin college in remediation at the two-year colleges were able to complete the college gateway course within two years. The success rates improve slightly when looking at the students that besides mathematics were assessed as "college ready." Again the success rates are higher for students entering a four-year institution, but at most of the intuitions, less than half of the students pass the gateway course within two years.

As the gateway course is a degree requirement, students must pass the course in order to graduate. This is particularly relevant to the two-year colleges, as the success rate in completing the college gateway course within two-years is an upper bound for the graduation rate for these students at their starting institution. However, the possibility remains that students can transfer to another institutions to complete the college gateway course after completing the remedial sequence.

Progress in Oklahoma

Corequisite Remediation

To address these completion rates, the state of Oklahoma is in the process of implementing corequisite remediation. Several institutions have started running pilot sections, and are doubling or tripling previous success rates. In this section we report the data from the pilot sections at one of our institutions.

In fall of 2015, Oklahoma State University began piloting corequisite courses. The ALEKS PPL test is used to assess entering student college readiness. Normally, to place into College Algebra a score of 45 out of 100 is required. The pilot sections consisted of students with scores from 30 to 44 on the ALEKS test. That is, students who would ordinarily have to take a remedial course sequence were included in the pilot. The students in the corequisite sections attended class five days a week as opposed to three days in the regular course sections. Of these five days, three days were dedicated to regular instruction. During the other two days, an undergraduate learning assistant engaged students in active learning sessions designed by an experienced course coordinator to improve students' prerequisite knowledge. The learning assistants are mathematics education majors selected by the department. Students in the corequisite sections completed the same homework and took the same exams as students in regular College Algebra sections. Figure 5 shows the percentage of the students in these courses earning a D or F or withdrawing (D/F/W rate).

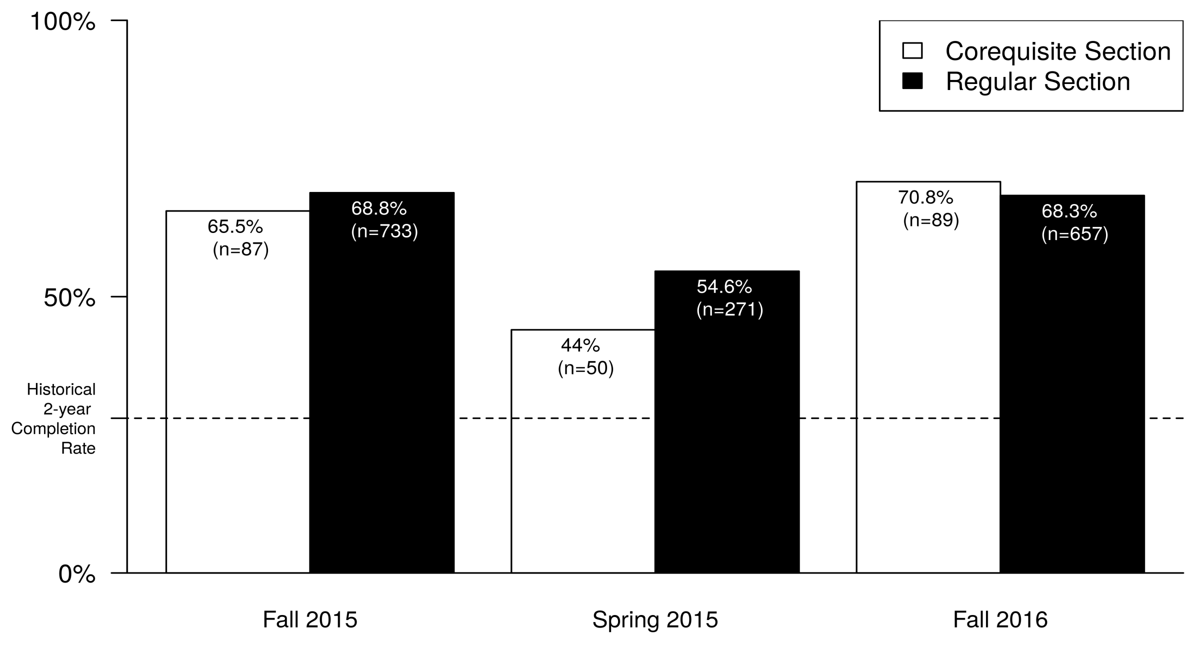

Figure 5. College Algebra Passrates.

While the success rates of the students in the pilot sections in both fall 2015 and spring 2015 were lower than the students in the regular sections, the improvement is noticeable when compared to the historic two-year completion rate in college algebra for these students. Moreover, the corequisite students have succeeded in a single semester as opposed to the minimum of two semesters that would have been normally required. Equally important is student persistence to and success in subsequent mathematics or statistics courses which is shown in Figure 6 for the Fall 2015 corequisite College Algebra cohort.

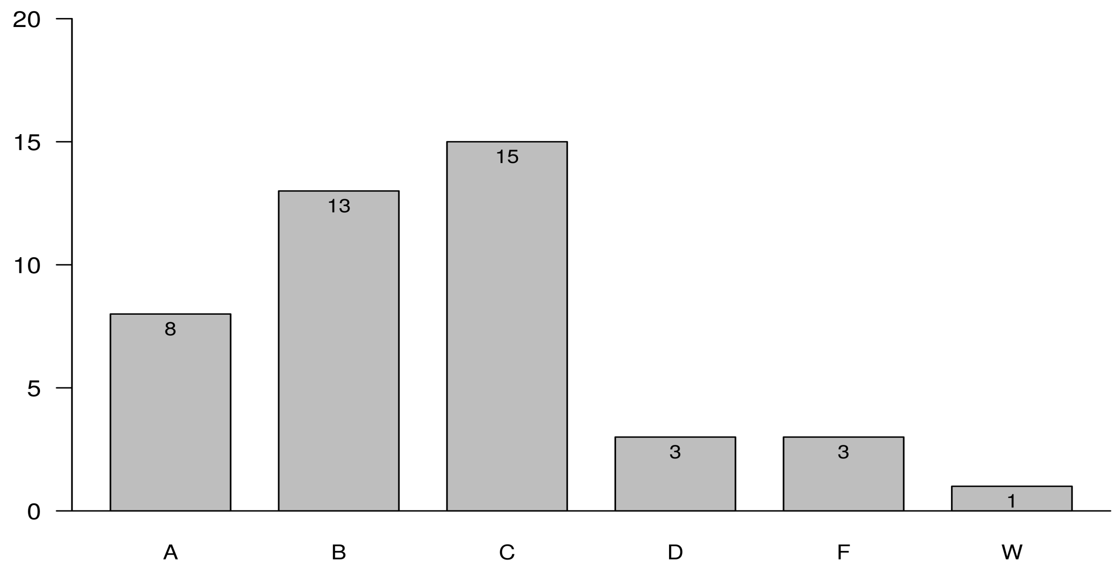

Figure 6. Fall 2015 corequisite students' grades in subsequent mathematics or statistics courses.

Half of the corequisite cohort students enrolled in a subsequent mathematics course, and over 83% of those students passed. (For reference, the typical success rates for students in these subsequent courses are around 70%.)

With regard to the statewide reforms, we have collected surveys from all public institutions in the state on their implementation of corequisite courses. We are in the process of analyzing this data, which will provide a useful baseline for comparison as the state reforms unfold.

Math Pathways

In the Common Vision 2025 (Saxe et al, 2015) the authors called for a "broader range of entry-level courses and pathways into and through curricula" (pg. 13). Of particular interest to us are moving degree programs that do not require engineering calculus away from College Algebra which is intended to prepare students for calculus. The goal is to increase relevance to students' courses of study.

Oklahoma State University has created a course *Mathematical Functions and Their Uses* (MATH 1483) which is being offered as an alternative to College Algebra for many business, social science, health, and agriculture degrees that do not require an engineering calculus course. The course emphasizes quantitative reasoning by modeling data with a calculator and/or excel and serves as an equally successful preparation for Business Calculus as College Algebra. The D/F/W rates are typically 15% and 25% for the fall and spring respectively. As degree requirements shift away from college algebra, we are in the process of analyzing D/F/W rates from subsequent gateway courses. Moreover, the shift to functions has led to an increased success in College Algebra. Additionally, in Fall 2016, the university began piloting a corequisite section of the functions course allowing traditionally underprepared students another pathway to earn college credit upon entering. The success rate in the pilot section was 86.2%, although we are hesitant to make comparisons between this pilot section and the regular sections as content and assessments were not consistent across the two.

Statewide, we collected detailed information on the mathematics requirements for every major at all 27 state public institutions and complete statewide data for student enrollment and success in math courses by major. The aim is to better understand the current math pathway options available and clusters of majors that might benefit from shifts to new math pathways.

A preliminary review identified 530 majors (out of 1790) that require college algebra but not engineering calculus. By analyzing the degree sheets for these majors we identified, 94 programs that have a path for which college algebra is not a prerequisite. The most common courses required for these majors for which college algebra is a perquisite are physics (37 programs), chemistry (100), statistics (45 programs), and trigonometry. In Oklahoma, outside of STEM degrees, college algebra is the most commonly taken mathematics course.

Discussion

Early corequisite remediation and new math pathways reforms in Oklahoma are showing promising results enabling dramatically more students to complete credit bearing gateway courses. Moreover, this data shows that these students are able to succeed in subsequent college mathematics and statistics courses. While the gains are promising, these results are preliminary. As the ultimate goal is to increase student success in college, what remains to be seen is how these efforts will affect graduation and degree completion rates. As we continue our data analysis, we have been be conducting interviews with individuals involved in the reforms in Oklahoma and individuals involved in the national reforms to better understand the data and process.

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Using Intuitive Examples from Women of Color to Reveal Nuances about Basis

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Research and surveys continue to perpetuate deficit narratives about women of color, particularly regarding their participation in and contribution to mathematics. Following the broader call for more research concerning STEM learning experiences of women of color, this study focuses on the sense making of eight women of color regarding their understanding of basis in linear algebra. We documented diverse ways that these women creatively explained the concept of basis using intuitive ideas from their everyday lives. These examples revealed important nuances and aspects of understanding of basis that are rarely discussed in instruction. These students' ideas can also serve as potentially productive avenues to access the topic. Our results also challenge the existing broader narrative about academic underachievement of women of color in mathematics.

Key words: student thinking, basis, linear algebra, equity

Most areas of science, technology, engineering and mathematics (STEM) continue to be unsupportive spaces for women of color, and people of color more generally. One indicator of this issue is the high attrition rates in mathematics among Black,[1] Latinx,[2] Native Alaskan, Hawaiian, and Native American students (NSF SEI, 2014). One way the education community has studied this problem is by exploring differences in achievement outcomes between groups of students. Outcomes of students from underrepresented groups are often compared with those of the dominant groups (e.g., White and male students). More recently, some scholars have shifted their focus to understanding ways that students are racialized and gendered in their educational experience (Martin, 2009). That is, research has begun to investigate the impact of racism on students of color, particularly Black students (e.g., McGee & Martin, 2009), and the impact of sexism on women (e.g., Herzig, 2004). Women of color as a group experience both racism and sexism, which has motivated the idea of *intersectional* feminism (Crenshaw, 1991). Ong and colleagues' (2011) literature review pointed out the dearth of intersectional research that focuses on experiences of women of color in STEM.

In this paper we operationalize racism, sexism and their intersection through social narratives (Nasir and Shah, 2011; Nasir, Snyder, Shah, & Ross, 2012). Nasir and colleagues (2012) used the term racial storyline (narrative) to describe established and shared narratives about a particular race that are enacted in social interaction in different social contexts. Existing narratives on women of color in mathematics typically frame students, particularly Black and Latinx students, as persistently underachieving and academically inferior to their White/Asian

[1] Berry (2015) uses the term "Black" to acknowledge the Black Diaspora, and to highlight the common way that Black learners, regardless of their origin are racialized in the U.S.
[2] "Latinx" is gender-neutral term to describe to describe people with Hispanic and/or Latin American origins. The term deemphasizes implicit gender binaries in "Latina/Latino," and hence more inclusive of transgender and other non-binary gender identities. Ramirez and Blay (2016) discuss the origin and different perspectives on the use of "Latinx" in scholarship, activism and journalism.

counterparts. Nasir and colleagues (2013) found that racial storylines, as an example of social narratives, limited access to educational opportunities (e.g., who gets called on in class, or who gets advised into honors classes) and added cognitive burdens for students while learning.

These deficit narratives about women of color are also supported by the dominant method to study inequities in education: achievement gap studies and comparison studies. Most comparison studies and achievement gap studies perpetuate the narrative that students (and women) of color are always lagging behind their White/Asian counterparts (Gutiérrez, 2008; Gutierrez & Dixon-Roman, 2011). Gutierrez (2008) has emphasized that comparison studies and achievement gap studies treat dominant students as the standard against which to measure. She also critiqued the implicit assumption that the validity of research on underserved populations rests on their comparison with the dominant group. Harper (2010) has also highlighted the prevalence of a deficit narrative in higher education research that discusses the participation of students of color in STEM. He found that research questions often positioned students as personally and solely responsible for their underachievement, thus positioning their underachievement as a result of personal failure and lack of motivation.

Deficit narratives about women of color intersect with deficit narratives about students' understanding and learning of mathematics. Deficit narratives about student thinking as generally naïve, full of misconceptions, and unrefined reciprocally support the narrative of academic inferiority of students (and women) of color in mathematics (Adiredja, under review). Some of these narratives about students' knowledge became more established as a result of the popular misconceptions research in the mid to late 1970's (diSessa, 2006; Smith, diSessa, & Rochelle, 1993). Nowadays, challenging the prevalence of deficit narratives about mathematical learning has been highlighted in research commentaries about equitable teaching (Bartell et al., 2017) and position statements about social justice in mathematics education (NCSM & TODOS, 2016). Thus, constructing counter-narratives about how women of color learn mathematics can help challenge the deficit narrative about women of color. This calls for researchers to be intentional about selecting research subjects and adopting an anti-deficit perspective (Adiredja, under review).

A group of Black scholars have recognized the broader deficit narratives regarding Black students, and have been intentional in shifting the narratives of research about Black students. Instead of fixating on comparing Black students to White or Asian students, scholars have focused on understanding Black students' learning experiences and offering counter-narratives about these students (e.g., Berry III, Thunder, & McClain, 2011). In the context of undergraduate mathematics, some scholars have focused instead on the experiences of successful Black mathematics and STEM majors (Ellington & Frederick, 2011; Larnel, 2016; Martin & McGee, 2009). This type of research has been able to offer counter-narratives by sharing stories of academically successful Black students in STEM. Additionally, it has revealed the resourcefulness and resilience of these students in navigating the additional burdens and obstacles embedded within the academic institution for these students. The study presented in this paper takes a similar approach in subject selection and in the way that it seeks to uncover resourcefulness of women of color in learning mathematics without comparing them to the dominant culture.

In this paper we focus on learning of women of color regarding the Linear Algebra concept of basis. The focus of this study was partly motivated by a need for more research about students' understanding of this topic. We were able to find only one study reporting on students' understanding of basis, which is an important topic in Linear Algebra. Stewart and Thomas

(2010) reported that students in their study struggled in making meaning of basis, and tended to explain it in terms of procedures. The authors also found that students struggled in explaining span and linear independence, two related concepts. In this study we explore the following questions:

1. How did the women of color in the study explain the concept of basis using ideas from their everyday lives?
2. What did their explanations reveal about nuances in the concept of basis?

These questions support the two complementary goals for this study. First, the study aims to construct a counter narrative about women of color by investigating their sense making of mathematics from an anti-deficit perspective. Second, it seeks to investigate the structure of understanding of the concept of basis in Linear Algebra. Our focus on explanations rooted in students' everyday experiences was motivated by some of the theoretical frameworks, which we turn to now.

Conceptual and Theoretical Frameworks

Sociopolitical Perspective

The design and analysis of this study are broadly guided by a sociopolitical perspective in mathematics education research (Gutiérrez, 2013, Valero, 2004). Gutiérrez (2013) explains that adopting a sociopolitical perspective involves investigating the relationship between knowledge, power, identity, and social discourses. She argues,

> [K]nowledge and power are inextricably linked. That is, because the production of knowledge reflects the society in which it is created, it brings with it the power relations that are part of society. What counts as knowledge, how we come to "know" things, and who is privileged in the process are all part and parcel of issues of power. Here, power is not a possession but it is circulated in and through discourses (p. 44).

Social discourses establish particular power relations from a sociopolitical perspective. Discourses include "institutions, actions, words, and taken-for-granted ways of interacting and operating" (Gutiérrez, 2013, p. 43). These discourses structure the world by producing "truths" about individuals and groups of people. Gutiérrez highlighted the power of discourses in deciding what counts as productive mathematical knowledge and who can be a successful mathematics student, that is, the politics of knowledge (Apple, 1992; Nasir, Hand, & Taylor, 2008).

This paper focuses on narratives as part of social discourses. The Oxford English Dictionary defines a narrative as a particular telling of a story that "reflects an overarching set of aims or values." The narrator establishes these aims and values, and they position people within the narratives. The positioning that results from these aims and values is the impact of the power that gets exercised through narratives.

Knowledge in Pieces (KiP)

Knowledge in Pieces (KiP) (diSessa, 1993) is a cognitive framework that explicitly emphasizes anti-deficit perspectives on students' knowledge and learning. KiP models knowledge as a complex system of elements. KiP rejects oversimplified deficit models of students' misconceptions, and typical efforts to replace persistent misconceptions (Smith, diSessa & Roschelle, 1993). The perspective that students have persistent misconceptions perpetuates narratives about students as unskilled, lacking resources and prone to mistakes. Instead, KiP focuses on the productivity of students' knowledge and the foundational role their prior knowledge plays in learning. KiP recognizes productivity of knowledge (elements) through its principle of *context sensitivity* (Smith et al., 1993). Misconceptions are seen as overgeneralizations of a productive use of a knowledge element in a new context.

Studies using KiP focus on understanding a student's own way of reasoning about a topic instead of assessing its correctness with respect to a normative standard (diSessa, Sherin, & Levin, 2016). More broadly, KiP is more interested in understanding aspects of the current context that might have led students to cue particular knowledge elements than in deciding if the cueing is "correct." KiP does not assume that productive knowledge has to be encoded in normative academic language. In fact, it is common for studies using this framework to uncover productive sense making in students' use of non-normative language (e.g., Campbell, 2011; diSessa, 2014).

Knowledge Analysis (KA) (diSessa, Sherin, & Levin, 2016) is a method associated with KiP that focuses on studying knowledge elements and how they are used. KA focuses on intuitive knowledge as a target of study, and assumes that forms of naïve knowledge are diverse and generative. Intuitive knowledge is seen as knowledge stemming from students' observation of their everyday experiences in engaging with the physical world. Some studies have shown the utility of intuitive knowledge in mathematics (Campbell, 2011; Pratt & Noss, 2002). This theoretical assumption motivates asking students about basis using everyday ideas.

Understanding Basis

In the same way that KiP motivated the focus of our study on students' everyday examples and our attention on students' intuitive knowledge, other frameworks motivate our organizing our study around different ways in which students understand the notion of basis.

By analyzing student understanding of a particular mathematical concept, we are following from a tradition in the mathematics education research literature of analyzing how students think about particular mathematical ideas. At the undergraduate level these include function (e.g., Sfard,1992; Oehrtman, Carlson & Thompson, 2008), limit (e.g., Williams, 1991; Roh, 2008; Swinyard & Larsen, 2012), and derivative (e.g., Zandieh, 2000).

In thinking about the various aspects involved in understanding a particular mathematical idea, we start with the notion of concept image. The term concept image has been used to refer to the "set of all mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them" (Vinner & Dreyfus, 1989, p. 356). Tall and Vinner (1981) describe a person's concept image for a particular concept as "the total cognitive structure that is associated with the concept" (p. 152). A longer term goal of our work is to create a framework for describing the details of the cognitive structure involved in understanding the notion of basis. This paper provides a first step towards describing in more detail the various aspects of students' concept images of basis.

One tradition for delineating student understanding of a particular mathematical concept is the notion of conceptual metaphor (Lakoff & Núñez, 2000). In the undergraduate mathematics education literature conceptual metaphor has been used to examine student understanding of a number of concepts including function (e.g., Zandieh, Ellis, and Rasmussen, 2017), limit (e.g., Oehrtman, 2009), and derivative (e.g., Zandieh & Knapp, 2006).

Conceptual metaphors may be used without reflection or without even awareness that a metaphor is involved. In our study we ask students to create everyday examples that may or may not be able to serve more broadly as conceptual metaphors for basis. Our intention for the purpose of this paper is to consider students' everyday examples as potential conceptual metaphors that allow insight into nuances of understanding involved in a particular student's concept image of basis and more generally into delineating a framework for what it means to understand this mathematical construct.

Methods

Data Collection

Given the desired depth and detail of analysis, this study favored the use of a small number of research subjects and videotaped individual interviews (diSessa, Sherin, & Levin, 2016). Participants were 8 undergraduate women of color at a large public research university. We invited women of color with the help of the university's mathematics advising center and by leveraging our personal contacts with students. We reached out to mathematics majors and minors who had completed an introductory linear algebra course.

The breakdown of racial and ethnic backgrounds and their past mathematics courses are presented in Table 1. This information was drawn from a student background survey that was administered at the end of the interview. All information was self-reported. The "Other Mathematics Courses" column are courses taken by the student prior to linear algebra. By the time of the interview in late Spring 2016 some students had completed additional mathematics courses post linear algebra. With the exception of Morgan, a Biomedical Engineering major, all the other students were mathematics majors or minors. Pseudonyms were selected to reflect the origin of students' names.

Each interview lasted for 90 minutes. Adiredja led the interview while Zandieh videotaped the interview and at times participated in asking questions. Students started the interview by solving four tasks that might be asked in a beginning linear algebra course. We chose tasks in which basis was not mentioned directly but for which basis might be relevant or related. This was followed by more general questions as to each student's understanding of basis including the questions that are the focus for this study:

1. Can you think of an example from your everyday life that describes the idea of a basis?
2. How does your example reflect your meaning of basis? What does it capture and what does it not?

Additional questions were asked about comparing their understanding of basis to the original four linear algebra tasks. At the end of the interview the students were often given another opportunity to share an example from their everyday life, especially if they did not offer an example when first asked.

Table 1. *Students' racial/ethnic background and mathematics course history*

Student	Racial/Ethnic Background	Linear Algebra Completion	Grade	Other Mathematics Courses
Leonie	African American	Spring 2016	A	Calculus I, II, and III
Morgan	Asian/Asian American	Spring 2016	A	Calculus I, II, and III, and Differential Equations
Annissa	Hispanic/Latinx	Fall 2014	B	Calculus I and II
Eliana	Hispanic/Latinx	Spring 2014	C	Calculus I and II
Nadia	Hispanic/Latinx	Fall 2015	A	Calculus I, II, and III

Jocelyn	Hispanic/Latinx	Spring 2015	B	Calculus I, II, and III
Stacie	Hispanic/Latinx	Spring 2016	C	Calculus I, II, and III
Liliane	Hispanic/Latinx/White	Fall 2015	B/C	Calculus I, II, and III

Data Analysis

For the purpose of this paper, we focused on student responses to the two questions listed above. We additionally considered other sections of the interview in which students introduced an example from their everyday life. We transcribed the interviews and organized the transcripts by turns, marked by changes in speaker. Transcripts use modified orthography (e.g., wanna, gonna, cus) to stay close to the actual students' utterance. Hedges (e.g., like, kinda, um) were removed from the presentation of transcripts to assist in the reading of the transcripts.

We started with open coding to capture nuances of students' understanding (Strauss & Corbin, 1994). We identified everyday contexts for each student and the details associated with that context (e.g., what is a vector, the vector space, and scalar multiplication in the context?). We differentiated utterances about the roles of the basis vectors in the larger space from utterances about the characteristics of the basis vectors. These distinctions came as we focused on students' uses of nouns, verbs and adjectives in their explanations.

We started our analysis by focusing on nouns students used to describe the context and its details. Once we identified the context, we separately analyzed the details of that context. Adiredja focused on adjectives that highlighted characteristics of basis, while Zandieh focused on verbs that highlighted the relationship between the set of basis vectors and the larger vector space. This is followed by a discussion about similarities, differences, and, at times, conflicts between our codes. We negotiated and refined the codes. For example, in comparing the codes with one another, we clarified their distinctions (e.g., *minimal* focuses on quantity vs. *essential* focuses on quality), and explored other related categories (e.g., *maximal* for *minimal*).

In our presentation of the transcript, we use bolded texts to indicate phrases that capture the codes. In cases where two codes are situated next to each other in the transcript, we differentiate them using **grey** and **black bolded texts**. We use ***bolded grey italics*** for nouns that indicate the basis vectors and the vector space. We use regular *italics* to refer to codes in the text.

Results

Students' Everyday Examples

We found that the majority of the students discussed at least one everyday context to explain the concept of basis. Table 2 provides a summary of the different contexts. In what follows we elaborate on the details of some these contexts as part of illustrating the roles of the basis vectors and their characteristics.

Table 2. *Everyday contexts used to explain basis and vector spaces*

Student	Context (for basis and vector space)
Leonie	friendship
Morgan	driving in a city (on a grid), Legos, cooking, groups of pens
Annissa	set of solutions (no actual everyday example)

Eliana	least amount of myself I need to cover the space of the room, storage room, dimension, skeleton, outline of a paper
Nadia	floor, universe and earth, syntax in programming
Jocelyn	fashion, recipe, art sculpture, collage
Stacie	walking to places in a room, floor as a plane, marching band
Liliane	religious teachings

Roles of Basis Vectors

In this section we focus on the verbs students used to discuss the role that basis vectors have in relationship to the larger vector space. Before collecting the interview data we had noticed that basis vectors can be thought of as *generating* the larger space or as *describing* the larger space. As such we designed our warm-up tasks to be of both types. For example, asking students to describe the span of a given set of vectors is a *generating* task. On the other hand, finding the eigenvectors for a given matrix may involve choices as to how to *describe* the infinite space of eigenvectors. One typical choice might be to state the basis for that eigenspace. In terms of relationships between the basis vectors and the larger space, *generating* verbs refer to creating the larger space from the basis vectors, whereas *describing* verbs start with the larger space and describe it by listing basis vectors.

Generating Illustrated

In our analysis we found that seven of the eight students used generating verbs including build, make, create, add, cover and fill. Table 3 provides examples of generating verbs with quotes from students. This table serves not only to illustrate the generating examples, but also to provide more detailed examples of the contexts that students used to talk about basis. In the quotes in Table 3, bold italics grey indicates the context the student was discussing, whereas bold non-italics black indicates a verb phrase illustrating a generating relationship between the basis vectors and the space.

Table 3. *Generating verbs within students' explanations*

Student	Verbs	Quote
Leonie	fill	I think I could describe it as my ***different friend group.*** /…/ They're all completely different. One of my friends, my best friend is, she's really serious and everything. Then my roommate is really goofy. And then my other friend, she's just really wild and crazy. So it's like, they all **fill all the gaps.**
Morgan	build, create	You're given like the 3 by 2 ***Lego [pieces]*** and you have like a 2 by 2 Lego [piece] you can just like **build on to that to create** that I guess space that you have.
Morgan	get	You could use your basis of ***one block north not south, east, west***. I think of those as like basis **to get anywhere in** ***the world.***

Eliana	cover	So it's like what's ***the least amount of myself [two arms]*** I need **to cover** ***the space of the room***. I need that direction. I need that direction, and I need that direction. So that's all you need.
Nadia	make	You need to know ***the minimum syntax*** to be able **to make** any sort of ***program*** and **make it useful**.
Jocelyn	make	The span would be all the different ***recipes*** that you **can make** with ***those ingredients***.
Stacie	create, move	Well, I remember in marching band where we have to create a circle. And you need the people **to create** ***the circle***. You need ***the field***. And you need ***the music***. You need all of three of them **to help move on to** the ***different parts of the field***.
Liliane	add to, expand, come from, reach	So I'm very religious. And so the ***teachings*** that we share with each other and that we read about and all that stuff. There are a lot of things that you can **add to** and be like, here's an application and here's the things, and this **expands** to this and this and this. But there's ***the most basic teachings*** and it all comes back to that. And this is the basic thing, like you have ***the Ten Commandments***. You have ***the Scriptures*** and you have ***the prophets*** and you have your ***connection with God*** and, all of the decisions and all of things that **come from** that and you can **reach** all of the other points with this basis.

Describing Illustrated
In our analysis we found fewer examples of verbs that indicated a student was describing a vector space by using a basis. Examples from Eliana and Nadia are listed in Table 4.

Table 4. *Describing verbs within students' explanations*

Students	Verbs	Quote
Eliana	shows	So the basis is like the ***skeleton***. You know that it, it **shows you what you need**, and it **shows you where everything is**.
Nadia	describes	I'm just thinking like the ***whole universe***, and that's like ***planets and stars*** and that like **describes** the universe. But only a planet describes earth. So to get a basis for the universe you'd need planets and stars.

In addition, a student may combine both types of verbs. Just after mentioning skeleton, Eliana discussed the outline of a paper both in terms of *generating* and *describing*,

> That's the whole point of the basis so you can **see on a smaller scale** what the rest of the space is gonna look like or what type of space here. And I could go with anything depending if you're writing a ***paper***, it would be the ***outline*** and you **build** on that.

To "see on a smaller scale" is to describe or represent the larger space using a well-chosen subset of that space. On the other hand, to "build on that" is to (re-)construct the larger space (paper) from the basis (outline).

Characteristics of Basis Vectors

Our coding of the adjectives students used to describe basis led to the list of codes below. In this section, we illustrate many of these codes through our analysis of Jocelyn's explanation using the context of fashion. The three segments we selected occurred chronologically. The analysis highlights the way that Jocelyn's explanation captures these different nuances about basis, and Jocelyn's sophistication in assessing the fit of her explanation with the formal definition.

1. **Minimal** focuses on the fact that the basis is the least number of vectors necessary.
2. **Maximal** focuses on the need to include all the basis vectors and that more would lead to redundancy.
3. **Essential** focuses on the quality of the vectors being the core and necessary.
4. **Representative** focuses on naming or identifying a smaller set as the structure or representation of the larger space.
5. **Non-redundant** focuses on not wanting extraneous information in a set, or the act of reducing or removing the extraneous information.
6. **Different** focuses on comparing items (vectors) based on their differences for the sake of keeping or removing items from the basis.

Segment 1: A focus on minimality and non-redundancy

In this episode, Jocelyn introduced the context of fashion and wardrobe to explain basis. In her explanation, she described the *generating* role that basis vectors play. Her explanation focused on the characteristics of basis vectors being the *minimal* set, and the fact that none of the vectors were *non-redundant*.

> I immediately try to think of, I thought of ***fashion and a wardrobe.*** And so, say you have all these different outfits **you want to make**. You have a **minimum number of pieces**, like **a pair of shoes or a shirt, or a pair of pants** that you need that allows you to **make all of those outfits**. But **you don't want to have two of the same pairs of shoes** cause you know that's wasteful. **You don't need two of them. You just use one.**

Jocelyn emphasized that at a minimum she needed a pair of shoes, a shirt and a pair of pants to make an outfit. She also used a situation in which she had two of the same pair of shoes as an example when the non-redundant requirement was violated.

Segment 2: A focus on difference and essential-ness

Without prompting Jocelyn also discussed a scalar multiple of a vector to emphasize the nuance that separated being *different* from being *essential*.

> When you scale a vector by a constant, it might look different. It might change its length or its direction but **its identity isn't really changed**, and so it'll be the same vector. And so **if you have different pairs of heels, they'd look different but they're still heels, they're still like formal. So they're (inaudible) still the same.**

Jocelyn was emphasizing that being different was not a sufficient condition for a vector to be part of a basis. She focused on the "identity" or the essence of a vector. Items might appear to be different (e.g., different heels), but their essence was still the same and thus both could not be part of the basis.

Segment 3: Assessment of explanation

In this episode, Jocelyn was sharing the extent to which her explanation captured the meaning of basis. In the process she summarized some of the important aspects of basis for her.

> It's **minimal**. To pick **one pair** **of heels and** **one pair** **of tennis shoes.** So when I think of my idea of a basis, my mind goes to **minimal.** Um, what doesn't it capture? Well, ok, so it's weird cause I guess you can use one pair of shoes for different outfits. But if I'm trying to make...it's harder to have a casual outfit and in a formal outfit there's **not a whole lot of overlap** **you end up having each piece in each outfit in the basis.** So it's like. How do I explain this? I feel like the basis I'm making, all of the pieces aren't as, **they're not all the same**. ***Like, you have shoes, tops and pants.*** **You can't make an outfit with just shoes.** But if **you have a basis, you can pick just some of the vectors**, **combine them, and make something,** and leave all the rest out. **Cause you can't just put on shoes and pants.** So that's where it kinda...that's one of the ways that doesn't really [fit the definition].

In her summary, Jocelyn returned to the idea of *minimal*. She needed to pick one pair of heels and one pair of tennis shoes. Each selected pair of shoes became a *representative* of its own category. The limitation of the context came when Jocelyn considered overlapping items in different types of outfits. While the idea of *non-redundancy* could be used to remove redundant vectors from a basis, Jocelyn was hoping that there were some redundancy so she could pick one representative for the redundant items. She emphasized that basis vectors had to be *different* ("not all the same"), but she saw that needing all the basis vector was a limitation of the context of fashion. Each piece in the basis is considered to be *essential*. So she could not just pick some of the vectors, like in a strictly mathematical context.

Discussion and Implications

We have two aims with this paper. The first is to construct a counter-narrative about women of color in mathematics. The second is to investigate the structure of student understanding of basis. With respect to the first goal, we have documented episodes of productive sense making by women of color in mathematics. We highlighted the creativity and breadth of the everyday contexts used to describe basis by these women of color. The range of contexts that students used was particularly interesting and useful. Most of these were not contexts we were able to generate ourselves prior to beginning the study. Although it was not the main focus of the analysis, the case of Jocelyn also showed students' sophistication with their assessment of everyday contexts they generated relative to the formal definition of basis. The fact that these women constructed these accessible contexts to explain basis positions these women of color as a resource for teaching instead of a group of students that persistently needing extra support. Taken together, these students' creativity and sophistication in assessing their examples and the potential utility of these everyday contexts challenge the narrative about women of color as persistently underachieving and academically inferior to their White and male counterparts.

With respect to the second aim, we have learned that students are capable of constructing productive explanations about basis using everyday ideas. They were able to come up with a variety of contexts in which they could describe basis. In grappling with what aspects of their contexts worked well and which did not, the students revealed many nuances of basis. This is important in two ways. First, these explanations revealed different roles basis vectors play in relation to the larger vector space, and nuanced characteristics of basis. Students' explanations delineated the relationship between basis vectors and the larger space in terms of *generating* and *describing*. As we saw with Jocelyn, nuanced characteristics such as *minimal, different, essential,* and *non-redundant* were important and interacted with one another in students' explanations. Second, methodologically, we have shown the utility of students' discussions of everyday examples in revealing their understanding of a topic. We might not have discovered these nuances using strictly formal mathematical questions. Moreover, there have been very few studies done on student understanding of basis and on students' ability to create everyday examples of mathematical constructs at the undergraduate level. For these reasons, this paper adds to the literature on student mathematical cognition at the undergraduate level.

In addition, we argue that the paper adds valuable data to the corpus of research in undergraduate mathematics education in that few studies have been written about the mathematical thinking of women of color (Adiredja & Andrews-Larson, in press). Sometimes this is because women of color have not been included in data sets (perhaps because there were not many women of color in the population from which the data was drawn). Other times we simply do not know whether or not women of color were in the data sets because, a review of the papers in recent proceeding of the Conference on Research in Undergraduate Mathematics Education (RUME) would reveal, it has not been common in the community to report data on gender and particularly on ethnicity.

With regards to findings of this study, we caution against the danger of *essentializing* these women, that is, attributing these examples as inherent to students' race and gender. It is tempting to gender or racialize these examples, which can lead to essentializing the students. Students did discuss basis in the context of fashion, cooking and religion, but they also brought up other contexts like driving, skeleton, and the universe. These contexts are likely inspired by the students' experiences, and not inherent to their ethnicity or gender. Future studies can further explore the range of contexts to explain basis, and the details of their differences. One can also investigate if there are shared learning experiences among women of color that might contribute to their flexibility to come up with these examples. Moschkovich (2012) has argued that there is nothing inherently different about the cognitive processes of students of color in mathematics, but there is a difference in their "conditions of learning" (p. 96). We can begin to explore ways that differences in conditions of learning might contribute to the examples students bring up in interviews.

We would also like to recognize the tendency to validate findings from or about an underrepresented group by comparing them to the behavior of the dominant group. We believe that our findings about this group of eight women of color are valid to stand on their own. The creativity of these women, and the productivity of their examples do not rest on how they might compare to White women, men of color, or White men. While it is interesting to find out what students from those other groups might come up with, studies with such focus are not inherently necessary to validate the findings of this paper. What we are arguing here is very much related to the critique in the literature about ways that achievement gap studies always center the achievement of the dominant group (Gutiérrez, 2008).

This work may have implications for practice and curriculum design. These everyday contexts can be used as potential entry points into basis. Moreover, as we saw with students in this study, generating explanations using everyday ideas can be a very fruitful activity in the classroom to reveal students' understanding of the topic. With respect to curriculum design, consider the experientially real starting points emphasized in the curriculum design framework of Realistic Mathematics Education (Freudenthal, 1983). Our analysis challenges us to reflect on what counts as an experientially real starting point for our students. Creating these experientially real starting points requires us to know our audience. In our past work we may have focused on certain types of students more than others in imagining what is experientially real to this audience. Making sure to interview and listen to the thinking of students who are not as often interviewed in RUME studies is vital to making sure we are reaching all students with our curriculum design, and in instruction more broadly.

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Exploration of the Factors that Support Learning: Web-based Activity and Testing Systems in Community College Algebra

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A variety of computerized learning platforms exist. In mathematics, most include sets of problems to complete. Feedback to users ranges from a single word like "Correct!" to offers of hints and partially- to fully-worked examples. Behind-the-scenes design of such systems also varies – from static dictionaries of problems to responsive programming that adapts assignments to users' demonstrated skills within the computerized environment. This report presents background on digital learning contexts and early results of a mixed-methods study that included a cluster randomized controlled trial design. The study was in community college algebra classes where the intervention was a particular type of web-based activity and testing system.

Key words: Computer-based Learning, College Algebra, Multi-site Cluster Randomized Controlled Trial

Many students arrive in college underprepared for college level algebra, despite its importance for future success in mathematics (Long, Iatarola, & Conger, 2009; Porter & Polikoff, 2012). Web-based Activity and Testing Systems (WATS) are one approach to supporting equity and excellence in mathematics learning in colleges. When it comes to technology and algebra learning in college: What works? For whom? Under what conditions? These ubiquitous questions plague educational researchers who are assessing the whats, whys, and hows of a technology intervention or addition to a course. Did the instructors have enough support to adequately implement the technology tool? Were the online materials appropriate to provide sufficient practice for each students' needs? Did instruction with the intervention equitably prepare students to pass the final exam?

This report offers early results from a large project investigating relationships among student achievement and varying conditions of implementation for a web-based activity and testing system used in community college elementary algebra classes. Implementing a particular WATS constitutes the "treatment" condition in this cluster randomized controlled trial study. As described below, there are several ways to distinguish WATS tools. Some systems, like the one at the heart of our study, include adaptive problem sets, instructional videos, and data-driven tools for instructors to use to monitor and scaffold student learning.

Research Questions

Funded by the U.S. Department of Education, we are conducting a large-scale mixed methods study in over 30 community colleges. The study is driven by two research questions:

Research Question 1: What is the impact of a particular WATS learning platform on students' algebraic knowledge after instructors have implemented the platform for two semesters?

Research Question 2: What challenges to use-as-intended (by developers) are faculty encountering and how are they responding to the challenges as they implement the WATS tool?

Background and Conceptual Framing

There are distinctions among dynamic and static learning environments (see Table 1). Though the focus of this report is a particular dynamic system, we offer information on both to situate what that means. WATS learning environments can vary along at least two dimensions: (1) the extent to which they adaptively respond to student behavior and (2) the extent to which they are based on a careful cognitive model.

Table 1. Conceptual framework of WATS environments based on adaptability and basis in a theory of learning.

		Type of Adaptivity in Design	
		Static	*Dynamic*
Is a particular model of learning explicit in design and implementation (structure and processes)?	*No*	Text and tasks with instructional adaptation external to the materials	Adaptive tutoring systems (e.g., ALEKS, Khan Academy, ActiveMath)
	Yes	Textbook design and use driven by fidelity to an explicit theory of learning	"Intelligent" tutoring systems (e.g., Cognitive Tutor)

Static learning environments deliver content in a fixed order and contain scaffolds or feedback that are identical for all users. Although often informed by a learning theory, this type of system is distinguished from others in that it is not designed to immediately adapt to individual learning needs of users. An example of this type of environment might be online problem sets from a textbook that give immediate feedback to students such as "correct" or "incorrect." Studies of college algebra student achievement and attitudes when instruction uses these tools in conjunction with face-to-face instruction (e.g., computer-based homework rather than paper-and-pencil homework) is mixed, generally indicating that use will do no harm but is not particularly beneficial (e.g., Bishop, 2010; Buzzetto-More & Ukoha, 2009; Hauk, Powers, & Segalla, 2015).

Dynamic learning environments keep track of some user behaviors (e.g., errors, error rates, or time-on-problem) and use this information in a programmed decision tree that selects problem sets and/or feedback based on estimated mastery of specific skills. An example of an "adaptive" dynamic environment might be a system such as ALEKS or the "mastery challenge" approach now used in the online Khan Academy Mission structure. For example, in working on a particular skill (e.g., the distributive property) in the Algebra Mission, a behind-the-scenes data analyzer captures student performance on a "mastery challenge" set of items. Once a student gets six items in a row correct, the next level set of items in a programmed target learning trajectory is offered. Depending on the number and type of items the particular user answers incorrectly (e.g., on the path to six items in a row done correctly), the analyzer program identifies target content and assembles the next "mastery challenge" set of items. Some studies have found correlations between adaptive-dynamic systems and student learning (e.g., Murphy et al. 2014). However, other than our own, we are unaware of any large-scale experimental studies assessing the efficacy of adaptive-dynamic systems in college mathematics.

Above and beyond responsive assignment generation, programming in a "cognitively-based" dynamic environment is informed by a theoretical model that asserts the cognitive processing necessary for acquiring skills (Anderson et al. 1995; Koedinger & Corbett, 2006). For instance, instead of specifying only that graphing is important and should be practiced, a cognitively-based environment also will specify the student thinking and skills needed to comprehend graphing (e.g., connecting spatial and verbal information), and provide feedback and scaffolds that support these cognitive processes (e.g., visuo-spatial feedback and graphics that are integrated with text). In cognitively-based environments, scaffolds themselves can also be adaptive. For example, more scaffolding through examples can be provided early in learning and scaffolding can fade as a student acquires expertise (Ritter et al., 2007). Like other dynamic systems, cognitively-based systems can also provide summaries of student progress, which better enable teachers to support struggling students. The efficacy of early computer versions of such an approach has been documented in some large-scale studies in high school and college settings (Koedinger & Sueker, 1996; Koedinger et al., 1997). However, no fully tested cognitively-based web-based activity and testing system currently exists for college students learning algebra.

As mentioned, several adaptive dynamic systems do exist (e.g., ALEKS, Khan Academy "Missions"). The particular WATS investigated in our study is accessed on the internet and is designed primarily for use as replacement for some in-class individual seatwork and some homework. ***Note***: We report here on data collected from the first of two years of study. The second year of the study – which repeats the design of the first – is currently underway. Hence, we purposefully under-report some details.

Methods

The study we report here uses a mixed methods approach that combines a multi-site cluster randomized trial with an exploration of instructor and student experiences. Half of instructors at each community college site were assigned to use a particular WATS in their instruction (treatment condition), the other half taught as they usually would, barring the use of the Treatment WATS tool though other WATS might be used (control condition). Faculty participated for two semesters in order to allow instructors to familiarize themselves with implementing the WATS with their local algebra curriculum. Specifically, the first term in Fall was a "practice" semester to field-test the intervention and the second semester of the same academic year was the "efficacy" study from which data were analyzed.

Sampling Strategy

Rather than recruit a sample by convenience, which is likely to result in poor generalizability, we utilized a stratified sampling approach developed by Tipton (2014). This method is a way of recruiting a sample that is compositionally similar to the target population for which the results of the study are meant to generalize. The target population for this study was defined as students at all community colleges in semester-long elementary algebra courses (also known as "developmental" or "beginning" algebra, the equivalent of a first year of algebra), in the U.S. state where the study took place. This population was selected in part because the state is large and diverse, and in part because we sought to decrease variability that may result from differing high school mathematics standards and graduation requirements across multiple states.

To recruit a sample that was compositionally similar to the target population, we first created a database that included information about all eligible community college sites (more than 100 across the state). We included information on college-level characteristics that existing research suggests will correlate with the study outcome (e.g., the average age of students at the college, the proportion of adjunct faculty, the proportion of students enrolled in remedial math courses). We conducted a cluster analysis on these potential covariates with all of the eligible colleges. The analysis resulted in a five-cluster solution that explained 29% of the variance between colleges. Examination of the characteristics that were unique to each cluster yielded the following descriptive observations:

Cluster 1. Represented 25% of colleges. These are colleges with a total student enrollment near the average (across all community colleges in the state) whose students tend to take more credits in the evening relative to colleges in other clusters. Cluster 1 colleges have more Temporary Faculty, and more Hispanic students, African American students, and students over 40 years old.

Cluster 2. Represented 15% of colleges. These colleges serve primarily students aged 25 and above who take fewer credits and more commonly are evening students.

Cluster 3. Represented 22% of colleges. These are colleges with a total student enrollment near the state average where students are more commonly Asian, younger, and enrolled full time during the day.

Cluster 4. Represented 23% of colleges. Cluster 4 represents smaller colleges that have a higher proportion of white students that tend to be younger, mostly full-time, and take fewer evening courses.

Cluster 5. Represented 15% of colleges. These are larger colleges that have more Hispanic and younger students. Students tend to take more daytime courses, with more fulltime loads and many remedial mathematics courses and high remedial math enrollment.

Our recruitment efforts aimed to include a proportionate number of colleges within each of the five clusters. Recruitment for the first cohort of participants yielded a study sample of colleges similar to the overall distribution across clusters that was the target for the sample. Due to attrition (instructors leaving the study), the representation shifted away from the target slightly for Clusters 1 and 4 by the end of the second term (see Figure 1).

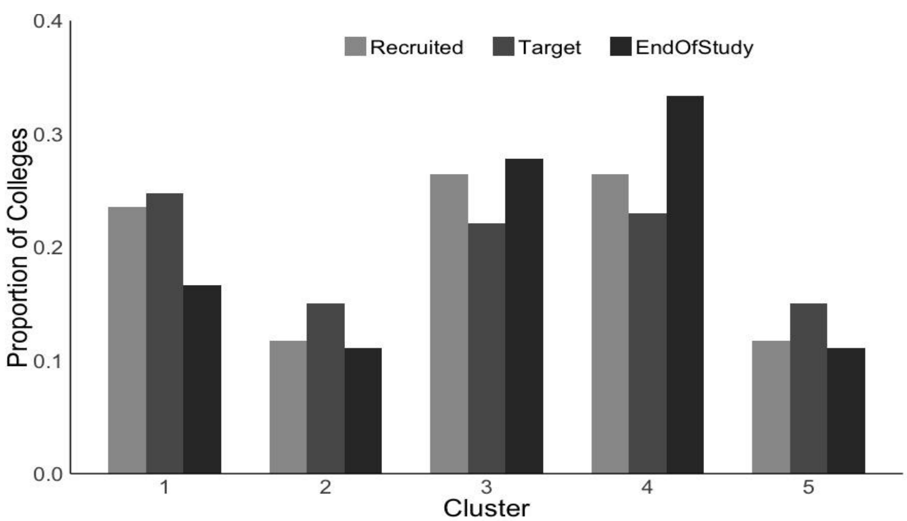

Figure 1. Recruited, target, and end of spring sample proportions across clusters.

Sample for this Report

Initial enrollment in the study included 89 instructors across 38 college sites. Attrition of instructors from initial enrollment to the end of the spring efficacy data semester was significant (68%). For this report, we analyzed the data from 510 students of 29 instructors across 18 colleges. Student and instructor numbers related to the data reported on here are shown in Table 2 and characteristics of the teachers and colleges are presented in Table 3.

Table 2. Counts of Instructors, Students, and Colleges in the Study.

Condition	*Instructors*	*Students*	*Colleges**
Control	17	328	13
Treatment	12	182	11
Total	*29*	*510*	*18*

* Note: there were multiple instructors at some colleges.

Table 3. Descriptive statistics for the student and instructor populations across the colleges in the study, by condition.

		Treatment		Control	
		M	*SD*	*M*	*SD*
		Student Characteristics			
	Enrollment	26,520	10,240	25,300	18,200
Proportions	U.S. Citizens	0.88	0.05	0.88	0.12
	Math Basic Retention*	0.80	0.03	0.82	0.06
	African American	0.04	0.03	0.06	0.03
	Asian	0.11	0.07	0.15	0.14
	Hispanic	0.49	0.21	0.41	0.19
	Native American	0.00	0.00	0.01	0.02
	White	0.27	0.17	0.30	0.16
	Below 25	0.61	0.05	0.58	0.07
	25 and Above	0.39	0.02	0.42	0.03
	Day Students	0.76	0.04	0.72	0.11
	Evening Students	0.18	0.04	0.16	0.04
		Instructor Characteristics			
	Part Time Faculty	0.45	NA	0.33	NA
	Years Experience Teaching Math	15.78	8.86	15.54	6.59
	Semesters of Algebra Teaching	18.60	11.99	15.36	13.82

* Proportion retention in remedial mathematics courses

Measures

A great deal of textual, observational, and interview data were gathered last year and will be gathered again for the second iteration of the study. These data allow analysis of impact (Research Question 1) and careful analysis of the intended and actual use of the learning environment and the classroom contexts in which it is enacted – an examination of implementation structures and processes (Research Question 2). Indices of specific and generic fidelity derived from this work also will play a role in HLM generation and interpretation in the coming year. The instruments are summarized below. With the exception of the observation and interview tools, all measures were administered online.

Instructor Instruments

Technology and Teaching Survey. This survey measures teachers' self-reported ability to use technology for teaching.

Perspectives Survey. This survey consists of questions related to teachers' background (e.g., years of experience teaching algebra, demographic information) as well as their attitudes and perspectives about teaching.

Measures of Effective Teaching – Algebra Test. This test was developed, piloted, and validated by the Educational Testing Service as part of the *Measures of Effective Teaching* (MET) project. It assesses instructors' pedagogical content knowledge in developmental algebra.

Weekly Instructor Logs. After extensive pilot testing, weekly logs were developed that ask about course format, topics, and resources used for that week's instruction.

Observation & Interview. The observation protocol captures a variety of information, including frequency of mention of WATS use, work completed in a WATS (treatment or other), teacher in-class use of WATS tools, as well as amount of time spent in whole class, group, and individual work. The interview focus is on the successes and challenges teachers face in using a WATS as part of instruction.

Student Instruments

Mathematics Diagnostic Testing Project (MDTP) Assessment. The MDTP serves as the study's primary student outcome measure. The *Algebra Readiness* form is the pre-test administered at the start of the semester and the *Elementary Algebra* form is the end-of-semester post-test. The MDTP tests have been shown to be valid and reliable measures of students' algebraic understanding (Gerachis & Manaster, 1995).

Student Background Questionnaire. This survey asks students about academic and demographic information such as academic history in mathematics, eligibility for financial aide

Motivated Strategies for Learning Questionnaire (MSLQ). This questionnaire measures students' motivation and attitudes towards mathematics.

Student Evaluation of Teaching Survey. The evaluation survey asks students to assess their experience in the course using Likert-scale questions.

The way performance is calculated is a non-trivial issue in educational measurement. One way to estimate student achievement on the MDTP tests is to calculate the raw percentage correct (i.e., summing the number of correct scores, and dividing by the total possible score). However, such a calculation does not take into consideration other parameters of interest, such as item difficulty, that provide added information that can be used to estimate student ability. To address this issue, we used a multilevel extension of the two-parameter logistic item response theory model to compute student pre- and post-test scale scores (Birnbaum, 1968). Specifically, we computed response-pattern *expected a posteriori* estimates (EAP scores; Thissen & Orlando, 2001) for each student. Similarly, we created EAP average scores for each classroom (a teacher-level score). We used individual and classroom aggregate student EAP scores in the analytic model described below.

Results

Quantitative Analysis

The study employed Hierarchical Linear Modeling (HLM), controlling for students' pretest MDTP EAP scores, to estimate the impact of WATS use on student achievement. The hierarchical modeling approach accounts for the nested structure of the sample (Raudenbush & Bryk, 2002), specifically the nesting of students within instructors. Preliminary analysis revealed that the HLM choice was justified, as the intra-class correlation in the unconditional model was 0.36, suggesting that the observations were not independent (i.e., student scores varied based on their classroom – statistically, the teacher mattered – so other approaches, such as single-level regression, would be inappropriate). The specific HLM model we used:

$$Y_{ij} = \beta_{00} + \beta_{01}(WATS)_j + \beta_{10}(StuPre)_{ij} + \beta_{02}(InstructorPre)_j + \xi_{0J} + \epsilon_{I0} \quad \text{(Equation A)}$$

In the equation above,

- Y_{ij} is the MDTP post-test EAP score for the *i*-th student of the *j*-th instructor;
- β_{00} is the grand mean of EAP scores across all students;
- $(WATS)_j$ is a dichotomous variable indicating instructor assignment to use the particular treatment WATS or not;
- $\beta_{10}(StuPre)_{ij}$ is the student MDTP pre-test EAP score;
- $\beta_{02}(InstructorPre)_j$ is the MDTP pre-test EAP estimate for all students in the class of instructor *j*;
- ξ_{0J} and ε_{I0} represent a random effect term for instructors and a random error term, respectively.

All covariates were grand-mean centered to achieve the desired model interpretation (i.e., covariates were transformed to be centered on a mean of zero). Importantly, the impact of the treatment WATS use is captured by β_{01}.

Baseline equivalence. The What Works Clearinghouse (2014) considers baseline differences with a Hedges $g \leq .25$ to be within the range of statistical correction. However, differences of Hedges $g > .25$ are considered not amenable to statistical correction. As can be seen in Table 4, both situations occurred. The differences between Instructor mean EAP scores (i.e., classroom average) and student pre-test raw scores were moderate between the two conditions. However, the difference between student pre-test EAP scores was substantive across conditions (g = 0.30). The EAP pretest difference for students is large enough that the analytic sample might be considered non-equivalent at baseline on this variable (below, we discuss details that attempt to address this difference).

Table 4. Baseline equivalence analysis on the analytic sample.

	Effect Size Hedges g	*WATS*		*Control*	
		M	*SD*	*M*	*SD*
Student Pre (Raw Scores)	0.25	30.58	8.27	28.54	7.82
Student Pre (EAP Scores)	0.30	0.45	1.10	0.14	0.99
Instructor Pre (EAP Scores)	0.08	0.22	0.53	0.18	0.39

Intervention impact. The aim of the impact analysis was to address the question: After controlling for student and classroom-level average pre-test scores, what is the impact of the WATS intervention on students' elementary algebra knowledge, as measured by the MDTP? To address this question, we use Equation A to estimate the average impact of going from the control to the treatment condition. Ideally, what we are interested in is this: what would a control students' algebra achievement be if his/her instructor, in an alternative universe, were assigned to the treatment group? Because students cannot participate to both conditions simultaneously, our randomized trial is a proxy for this counterfactual scenario. The results of random and fixed effects in the model are presented in Tables 5 and 6, respectively.

The random effects (Table 5), tell us that the amount of variance that the instructor-level accounts for (i.e., the intraclass correlation) is about 28% (from Table 5 and a quick calculation, we see instructor variance divided by the total variance = **0.16**/(**0.40**+**0.16**)=**0.28**). This means that student level values are not independent. Put another way, students within classrooms were more similar to each other than students between classrooms. The intraclass correlation justifies our hierarchical analytic approach over single level regression. More generally (and in future work), we want to look at what instructors are doing to see how the instructor-level activity is shaping student achievement. The fixed effect model estimates are provided in Table 6. Controlling for students' pretest EAP scores, we found that using this particular WATS platform corresponded to a **0.35** increase in students' post-test EAP scores. This difference is considered a statistically significant positive effect ($p < .05$). The Hedges g value for this effect is 0.32, which is judged to be substantively important for educational research studies of this type (WWC, 2014). The 95% confidence interval around the effect estimate was 0.14 - 0.50, which is large, but spans an exclusively positive range.

Table 5. Random effects of the model.

	Variance	*Standard Deviation*
Instructor ξ_{0J}	**0.16**	0.40
Level-1 Error ε_{I0}	**0.40**	0.63

Table 6. Fixed effect results of the model.

	Estimate	*St. Error*	*p-value*
Intercept β_{00}	-0.10	0.10	0.34
WATS β_{01}	**0.35**	0.16	0.04
StudentPre β_{10}	0.73	0.03	< .001
InstructorPre β_{02}	0.30	0.19	0.13

Using raw MDTP scores (instead of EAP estimates) as outcomes and covariates in the model, we obtained similar results. In the raw score model, the impact of WATS was estimated to result, on average, in a 2.57 point increase in student raw score. This was a statistically significant positive effect ($p = 0.04$, $SE = 1.18$, *Hedges* $g = 0.32$). The control group mean was estimated at 22.04 (out of 50 points), thus, the 2.57 point difference corresponds to nearly 12 percentage points increase in post-test scores relative to the control group (2.57 / 22.04 * 100 = 11.66). Since baseline differences between treatment and control group student raw scores were within the range of statistical correction, the similarity between the two *models* (raw score and EAP score models) is important, providing more confidence in the estimates of positive impact.

The effect size across both analyses was estimated at 0.32. This result can be interpreted as the WATS group of students would have scored an estimated 0.32 standard deviations higher, on average, than the control group of students on the MDTP, had the groups been fully equivalent prior to the intervention. However, to interpret the effect size of 0.32 in a more meaningful way, we converted the effect size using properties of the normal distribution. In a normed sample, a one standard deviation increase from the middle of the distribution corresponds to a 34 percentile point increase in scores. Thus, an effect size of .32 would correspond to an approximate 11 percentile point increase in scores (i.e., .32 * 34 = 10.88). Therefore, if students in the control condition perform at the 50th percentile in a normed sample, the students in the WATS condition would perform at the 61st percentile in the normed sample (50 + 10.88 = 60.88).

While these results suggest that WATS has a positive impact on students' elementary algebra achievement, it is important to note that this study suffered from high instructor attrition. This fact, coupled with moderate to large baseline differences at pretest, warrant caution in interpreting the results. In order to determine whether the results of the present study are robust, we are repeating the study with a second cohort of instructors and their students in the 2016-17 school year. Pooling the results of these two studies will help to determine the extent to which the findings replicate with different samples and will lend more confidence in the study conclusions (Cheung & Slavin, 2015).

Qualitative Analysis

As in many curricular projects, developers of the WATS in our study paid attention to learning theory in determining the content in the web-based system, but the same was not true for determining implementation processes and structures. The pragmatic details of large-scale classroom use were under-specified. Developers articulated their assumptions about what students learned as they completed activities, but the roles of specific components, including the instructor role in the mediation of learning, were not clearly defined. Thus, there was an under-determined "it" to which developers expected implementers (instructors and students) to be faithful.

Fidelity of implementation is the degree to which an intervention or program is delivered as intended (Dusenbury, Brannigan, Falco, & Hansen, 2003). Do implementers understand the trade-offs in the daily decisions they must make "in the wild" and the short and long-term consequences on student learning as a result of compromises in fidelity? As Munter and colleagues (2014) have pointed out, there is no agreement on how to assess fidelity of implementation. However, there is a growing consensus on a component-based approach to measuring its structure and processes (Century & Cassata, 2014). Century and Cassata's summary of research offers five components to consider in fidelity of implementation: Diagnostic, Procedural, Educative, Pedagogical, and Student Engagement (Table 7, next page).

The components in Table 7 are operationalized through a rubric, a guide for collecting and reporting data in our implementation study. A rubric articulates the expectations for a category by listing the criteria, or what counts, and describes the levels of quality from low to high.

Each component has several factors that define the component. The research team has developed a rubric for fidelity of implementation that identifies measurable attributes for each component (for example, see Table 8 on the next page for some detail on the "educative" component). Data for assessing each row come from the survey, observation, and interview measures described earlier.

Table 7. Components and Focus in a Fidelity of Implementation Study.

Components	*Focus*
Diagnostic	These factors say what the "it" is that is being implemented (e.g., what makes this particular WATS distinct from other activities).
Structural-Procedural	These components tell the user (in this case, the instructor) what to do (e.g., assign intervention *x* times/week, *y* minutes/use). These are aspects of the *expected* curriculum.
Structural-Educative	These state the developers' expectations for what the user needs to know relative to the intervention (e.g., types of technological, content, and pedagogical knowledge needed by an instructor).
Interaction-Pedagogical	These capture the actions, behaviors, and interactions users are expected to engage in when using the intervention (e.g., intervention is at least *x* % of assignments, counts for at least *y* % of student grade). These are aspects of the *intended* curriculum.
Interaction-Engagement	These components delineate the actions, behaviors, and interactions that students are expected to engage in for successful implementation. These are aspects of the *achieved* curriculum.

Table 8. Example Rubric Descriptors for Levels of Fidelity, Structural-Educative Component.

Educative: These components state the developers' expectations for what the user (instructor) needs to know relative to the intervention.

	High Level of Fidelity	Moderate Fidelity	Low Level of Fidelity
Users' proficiency in math content	Instructor is proficient to highly proficient in the subject matter.	Instructor has some gaps in proficiency in the subject matter.	Instructor does not have basic knowledge and/or skills in the subject area.
Users' proficiency in content (CK), pedagogical (PK), and technological knowledge (TK)	Instructor regularly integrates content, pedagogical, and technological knowledge (TK) in classroom instruction. Communicates with students through WATS.	Instructor struggles to integrate CK, PK, and TK in instruction. Occasionally sends digital messages to students using WATS tools.	Instructor CK, PK, and/or TK sparse or applied in a haphazard manner in classroom instruction. Rarely uses WATS tools to communicate with students.
Users' knowledge of philosophy behind the intervention	Instructor understands philosophy of WATS resources (practice items, "mastery mechanics," analytics, and coaching tools),	Instructor is aware of it, but understanding of the philosophy of WATS tool has some gaps.	Instructor is not aware of or does not understand philosophy of WATS resources.
*Users' knowledge of requirements of the intervention**	Instructor understands the purpose, procedures, and/or the desired outcomes of the project (i.e., "mastery")	Instructor understanding has some gaps (e.g., may know purpose, but not all procedures, or desired outcomes).	Instructor does not understand the purpose, procedures, and/or desired outcomes. Problems are typical.

* Note: Disagreeing is okay, this is about instructor knowledge of it.

Defining and Refining Measures for the Fidelity of Implementation Rubric

The ultimate purpose of a fidelity of implementation rubric is to unpack and articulate the conditions of implementation and the relationship between those conditions and impact on student achievement. In addition to allowing identification of alignment between developer expectations and classroom enactment, an examination of implementation provides the opportunity to discover where productive adaptations may be made by instructors, adaptations that boost student achievement beyond that associated with an implementation faithful to the developers' view.

In using the rubric, we assign a number to each level of fidelity for each teacher across the year of data collection. This can be as simple as the approach shown in Table 8, a 3 for a high level of fidelity, 2 for a moderate level of fidelity, or a 1 for a low level. The general score for a teacher-level index of implementation fidelity will be the total number of points assigned in completing the rubric as a ratio of the total possible. At a more detailed level, once we have completed rubric analysis to create the row by row scores for each instructor, these scores will be used as a vector of values in statistical modeling of the impact of the intervention as part of a "specific fidelity index" (Hulleman & Cordray, 2009).

We are at the beginning of addressing Research Question 2: What challenges to use-as-intended (by developers) are faculty encountering and how are they responding to the challenges as they implement the WATS tool? To date, analysis of observations, interviews, and weekly logs has provided the opportunity to discover instructional orientations. Several orientations are emerging from analysis now and include a "denial" orientation in which instructors see the WATS as no different from themselves as a teacher, a "polarized" orientation where an instructor is either indifferent (no/low expectations for success) or enthusiastic (high/excessive expectations for success) about the power of student engagement with the WATS, a "cautious optimism" in which the instructor sees the WATS as one tool in a collection of resources to be used strategically in designing instruction, or an "adaptation" orientation in the sense that the instructor sees the WATS as a resource for which appropriate instructional use is negotiated with and through the students' goals for interaction with the software in the context of the algebra course. In addition to the fidelity scoring of alignment between developer expectations and classroom enactment, these orientations may serve to explain the relationship between implementation and impact, getting at how and for whom WATS are most effective.

Next Steps

As indicated above, we will continue this study with a second cohort of new participants in the 2016-2017 academic year. Our specific objectives in the coming six months are to complete the second cohort's efficacy semester, generate fidelity indices for each instructor in each cohort, and complete separate and collective statistical modeling explorations.

Implications for practice. Though the study is ongoing, the early results might be considered promising. If the question is: Should I use a WATS? The answer is: It depends. Taking into account the potentially biased statistical impact results to date and the exploration of variation in instructor implementation, it appears likely that an orientation of "cautious optimism" or "adaptation" may be required for a dynamic WATS tool like the one in the study to have significant impact on student learning.

Implications for research. A mixed-methods study like the one reported here is large and complex. We note here that there were significant challenges in recruiting and retaining community college mathematics faculty for the project. To build community and assist in future research efforts in two-year colleges (and as part of our dissemination about the work) we have targeted outlets read by community college faculty (e.g., MathAMATYC Educator – a journal of the American Mathematical Association of Two Year Colleges). It is important for practitioners and potential participants in studies on research in undergraduate mathematics education to be aware of research and the enormous contributions they can make to it. Secondly, a major implication for research (for us) was the work in managing all the data generated by the project. The reader is encouraged to review the piece by our colleague Aleata Hubbard that also was presented at the conference, *Data Cleaning in Mathematics Education Research: The Overlooked Methodological Step*.

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"Explanatory" Talk in Mathematics Research Papers

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In this paper we explore the ways in which mathematicians talk about explanation in their research papers. We analyze the use of the words explain/explanation (and various related words) in a large corpus of text containing research papers in both mathematics and physical sciences. We found that mathematicians do not frequently use this family of words and that their use is considerably more prevalent in physics papers than in mathematics papers. In particular, we found that physicists talk about explaining <u>*why*</u> *disproportionately more often than mathematicians. We discuss some possible accounts for these differences.*

Key words: corpus linguistics, mathematical language, mathematical explanation.

The notion of explanation in mathematics has received a lot of attention in both mathematics education and the philosophy of mathematics. In mathematics education, scholars have been particularly interested in proofs that explain mathematical theorems (i.e. proofs that provide an insight into why a mathematical claim is true) and their role in the mathematics classroom (e.g. Hanna, 1990). Philosophers of mathematics have discussed at length possible equivalents for mathematics of existing philosophical theories of scientific explanation (e.g. Steiner, 1978). Some of these discussions bring to bear the extent to which explanation is relevant to the actual practice of mathematicians and often cite individual mathematicians' views on mathematical explanation. In this report we explore the extent to which mathematicians talk about explanation in their research papers, and the ways in which they do so.

Literature review

In an influential paper in mathematics education, de Villiers (1990) argued that proof serves several different roles in mathematics, that proof is not only used in mathematics as a way to verify results, to provide conviction of the truth of those results (see also Bell, 1976). One of those other functions of proof was to *explain* mathematical results, to provide an insight or understanding into why these results were true, as opposed to just evidence in support of that result. Hanna (1990) made a similar distinction in the context of the teaching and learning of mathematics, discussing the idea that certain proofs fulfilled this explanatory function better than others, to the point that among the set of all proofs one could identify proofs that explain why a theorem is true, while others simply demonstrate that a theorem is true. Mathematics educators have generally suggested that in the mathematics classroom, mathematical explanation should be an important, if not the primary role of proof (de Villiers, 1990; Hanna, 1990; Hersh, 1993).

This distinction between proofs that explain and proofs that demonstrate has a longer history in the philosophy of mathematics. Steiner (1978) put forward a model of mathematical explanation, arguing that an explanatory proof could be better defined in terms of what he called *a characterizing property* of a concept in the theorem, as opposed to other alternative defining characteristics such as the abstractness or the generality of the proof. Steiner's top-down approach to modeling mathematical explanation by providing a general definition of explanatory proof (and thus creating an absolute distinction between explanatory and non-explanatory proofs) has been criticized by other philosophers of mathematics. In particular, Hafner and

Mancosu (2005) argued that ascribing explanatoriness to specific proofs should be done based on practicing mathematicians' evaluations, not philosophers' own intuitions (such as Steiner's). The extent to which practicing mathematicians not only agree with philosophers' characterization of mathematical explanation, but simply talk about explanation in their practice plays an important role in the general argument for the *existence* of explanation in mathematics (which not all philosophers believe). As such, it is not uncommon for philosophical discussions of mathematical explanation to mention how much mathematicians talk (or do not talk) about it. For example, Steiner claimed that "mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain" (p.135). However, there is sharp disagreement regarding the extent to which mathematicians explicitly discuss such distinctions, or the notion of explanation in general. For instance, Resnik and Kushner (1987) claimed that mathematicians "rarely describe themselves as explaining" (p. 151), and Hafner and Mancosu (2005) responded stating that "[c]ontrary to what Resnik and Kushner claim (p. 151), mathematicians *often* describe themselves and other mathematicians as explaining" (p. 223, emphasis on the original). Hafner and Mancosu (2005) supported this claim by presenting several examples of what they called *"explanatory" talk* in mathematical practice: passages of research mathematics papers in which the authors explicitly discuss the role of explanation in their own work. However, this kind of evidence is not sufficient to settle the disagreement. Indeed, the specific cases discussed by Hafner and Mancosu have been interpreted in significantly different ways:

> "I believe that detailed case studies, such as those by Hafner and Mancosu (2005), decisively refute Resnik's and Kushner's [claim]" (Lange, 2009, p. 203, our emphasis).

> "Though philosophers have lately been pointing out some exceptions, the examples tend to be rather exotic (e.g., in Hafner and Mancosu 2005). There has been no systematic analysis of standard and well-discussed texts illustrating any pattern of mathematical explanations." (Zelcer, 2013, 179-180)

We agree that a systematic analysis of the type suggested by Zelcer (2013) is necessary to address this issue. Overton (2013) performed one such analysis in the context of scientific discourse. He analyzed all regular articles published in the journal *Science* in a one year period (a total of 781 papers and approximately 1.6 million words), searching for all "explain" words (defined to be one of the following words: explain, explains, explained, explaining, explainable, explanation, explanations, unexplained, unexplainable, explicate, explicates, explicated, explicable, inexplicable) and comparing their frequencies to those of other types of words. Overton (2013) found that approximately 45% of the 781 papers contained at least one "explain" word (with a total of 368 explain words per million words in his sample of papers) and concluded: "The numbers for "explain" are perhaps surprisingly low if scientific journals are vehicles for explanations. […] The observed frequencies of "explain" words suggests that explanation is only moderately important in science." (p. 1387)

Our goal in this paper is to report a similar analysis in the context of mathematics. One method of studying mathematical discourse at such a scale is to use the techniques of corpus linguistics, a branch of linguistics that statistically investigates large collections of naturally occurring text, known as corpora. Methods developed by corpus linguists can be used to investigate many different types of linguistic questions. Here, we report a study that employs some of these techniques to address the following questions: to what extent do mathematicians discuss explanation in their research papers, how does it compare to the extent to which they discuss other important related notions (such as *proving* theorems), and how does it compare to discussions about explanation in other types of scientific discourse?

Theoretical perspective

Discussions about mathematical explanation tend to differentiate between explanations of other mathematics (i.e. mathematics X explains mathematics Y, or X is an explanatory proof of theorem Y), and explanations of physical phenomena (i.e. mathematics X explains physical phenomenon Y). Colyvan (2011) refers to these two types of explanation as *intra-mathematical* and *extra-mathematical*, respectively. Here we focus on intra-mathematical explanations.

Hafner and Mancosu (2005) further differentiated between two uses of intra-mathematical explanations: those that are "instructions" on how to master the tools of the trade (as in explaining how to employ a certain mathematical technique), and those that "call for an account of the mathematical facts themselves, the reason why" (p. 217). While Hafner and Mancosu considered the latter to be a "deeper" use of mathematical explanation, which is also the focus of the larger philosophical discussion around explanatory proofs, others have emphasized the importance of the former type of explanation in mathematical practice. For instance, Rav (1999) insisted that one of the main reasons mathematicians read proofs is because of all the mathematical know-how embedded in them, emphasizing the mathematical methodologies and problem solving strategies/techniques contained in proofs. According to Rav, "proofs are for the mathematician what experimental procedures are for the experimental scientist: in studying them one learns of new ideas, new concepts, new strategies—devices which can be assimilated for one's own research and be further developed." (p. 20) Indeed, there is empirical evidence (from both small scale interview studies and large scale surveys) that mathematicians maintain that one of the main reasons they read proofs is to gain insights into how they can solve problems that they are working on (Weber & Mejía-Ramos, 2011, Mejía-Ramos & Weber, 2014).

An interesting question related to the specific ways in which mathematicians talk about explanation in their papers (to the extent that they do), relates to these two types of *"explanatory" talk*: to what extent do mathematicians discuss explanations of *why* a certain mathematical statement is true, compared to their talk about explanations of *how* to do something in mathematics?

Methods

One of the main ways in which mathematicians around the world communicate about mathematics is through research papers stored in the ArXiv. The ArXiv is an online repository of electronic preprints of scientific papers in the fields of mathematics, physics, astronomy, computer science, quantitative biology, quantitative finance, and statistics. These papers constitute a large corpus of scientific text that can be used to analyze mathematical discourse.

We downloaded the bulk source files (mostly TeX/LaTeX) and converted the source code to plain text, which we could then analyze using standard software packages for corpus analysis (all analyses reported in this paper were performed using CasualConc, version 2.0.3). We then sorted these articles based on their primary and secondary subject classification (Alcock et al., 2017, discussed the details about the processing of these source files). The analyses reported in this paper are based on a proper subset of this corpus, containing all mathematics and physics articles (based on their primary subject classification) uploaded in the first eight months of 2009. In the first part of the paper we focus on articles uploaded in the first four months of 2009. After analyzing these articles, we performed these same analyses on the papers uploaded in the following four months of 2009 (May through August), in search of further evidence supporting our findings.

Results

Table 1 presents the number of physics and mathematics papers (as well as the number of words[1] in each set of papers) uploaded in January-April and May-August of 2009. We notice that researchers uploaded to the ArXiv approximately 2.4 times as many physics papers as mathematics papers in these eight months. We also notice that on average physics papers contained approximately 5000 words, whereas on average mathematics papers contained roughly 6200 words.

	January-April 2009		May-August 2009	
	#papers	#words	#papers	#words
Mathematics	5,087	30,892,695	4,970	31,289,569
Physics	11,787	58,859,660	12,370	62,807,075

Table 1. Number of papers and words in the physics and mathematics corpora

Frequency of explicit *"explanatory" talk* in the January-April 2009 papers

Following Overton (2013), we defined explain-words to be one of 18 words linguistically related to the word *explain* (henceforth *explain-words*)[2]:

Explain-words: explain, explains, explained, explaining, explainable, explanation, explanations, explanatory, unexplained, unexplainable, explicate, explicates, explicated, explicating, explicable, inexplicable, explication, explications.

Table 2 shows the frequencies of explain-words in our corpus of 5087 mathematics papers and 11787 physics papers uploaded between January and April of 2009. Explain-words showed up a total of 4910 times in the mathematics papers (around 159 times per million words), an average of 0.97 times per paper, with 1898 of mathematics papers (approximately 37%) in this sample containing at least one explain-word. While this certainly provides an existence proof of explicit *"explanatory" talk* in this corpus, it is not very surprising (it would very rare if no word based on the word explain showed up in these many mathematics papers). In comparison, explain-words showed up 21345 times in the corresponding set of physics papers (around 363 times per million words), an average of 1.81 times per paper, with 6499 of these papers (roughly 55% of the physics papers) containing at least one explain-word. We see that the number of explain-words per million words in the physics papers is around 2.28 times that of the mathematics papers.

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[1] A word here is any string of characters between spaces. Importantly, as discussed in Alcock et al. (2017), for these analyses we opted to replace all occurrences of inline mathematics with the string "inline_math" and count it as one word. For instance, the string "Let $f\colon X \rightarrow Y$ be a bijection" in a paper, coded in LaTeX by the authors as "Let $f:X\rightarrow Y$ be a bijection", would have been translated to text as "Let inline_math be a bijection" and coded as having 5 words.

[2] We added *explanatory, explication, explications,* and *explicating* to the 14 words in Overton's (2013) analysis.

	Mathematics		Physics	
	frequency	per million	frequency	per million
explain	1827	59.14	7768	131.97
explained	1690	54.71	6513	110.65
explanation	498	16.12	3564	60.55
explains	484	15.67	1601	27.20
explaining	175	5.66	914	15.53
explanations	119	3.85	675	11.47
explanatory	51	1.65	62	1.05
unexplained	22	0.71	177	3.01
explication	13	0.42	4	0.07
explicated	10	0.32	15	0.25
explicate	6	0.19	5	0.08
explicating	5	0.16	0	0.00
unexplainable	4	0.13	8	0.14
explications	4	0.13	2	0.03
explainable	1	0.03	23	0.39
explicates	1	0.03	1	0.02
explicable	0	0.00	9	0.15
inexplicable	0	0.00	4	0.07
Total	4910	158.94	21345	362.64

Table 2. Frequency and frequency per million words of explain-words appearing in the January-April mathematics and physics papers

In order to get a sense of the extent to which these frequencies were high or low in this type of mathematical discourse, we compared them against the frequencies of words related to other important mathematical activities. Table 3 presents the frequencies of words linguistically related to the notions of conjecturing, defining, modeling, proving, showing, and solving.

Conjecture-words:[3] *conjecture, conjectured, conjectures,* conjectural, conjecturally, conjecturing.

Define-words: *defined, define, definition, defines, definitions,* defining, definable, undefined, redefine, redefined, definability, redefinition, definably, redefining, welldefined, definedness, interdefinable, predefined, redefinitions, interdefinability, redefines, definitional, definitionally, undefinability, undefinable.

Model-words: *model, models,* modeled, modeling, modelled, modelling, countermodel, submodel, submodels, modelized, modelization, modelisation, modelize, modelizing, countermodels, premodel, remodeled.

[3] For each group, words are listed in order of frequency, with the most frequent words in the group listed first. The italicized words in each group make up 95% of all instances of words from that group appearing in the mathematics papers uploaded in the first four months of 2009.

Prove-words: *proof, prove, proved, proves, proofs*, proving, proven, provable, reprove, disprove, provability, provably, reproved, disproved, unprovable, unproven, reproving, disproving, reproves, prover, unproved, subproof, disproof, disproven, disproves, reproven.

Show-words: *show, shows, shown, showed*, showing.

Solve-words: *solution, solutions, solve, solving, solvable, solved, solves, resolvent,* solvability, subsolution, resolved, resolving, supersolution, resolve, solver, resolvents, unsolved, resolves, solvers, nonsolvable, supersolutions, subsolutions, unresolved, nonsolvable, unsolvable, cosolvable, equisolvable, supersolvable, unsolvability.

We note that measured against these other frequencies, mathematicians used explain-words rather infrequently. For instance, mathematicians used explain-words in their papers approximately 12 times less frequently than show-words and nearly 23 times less often than prove-words.

	frequency	per million	per paper	in #papers	in %papers
define	124129	4018.07	24.40	4838	95%
prove	111838	3620.21	21.99	4710	93%
show	59359	1921.45	11.67	4691	92%
solve	53013	1716.04	10.42	3073	60%
model	23658	765.81	4.65	2013	40%
conjecture	8362	270.68	1.64	1413	28%
explain	4910	158.94	0.97	1898	37%

Table 3. Frequencies (including frequencies per million words and per paper) of words related to *explaining, conjecturing, defining, modeling, proving, showing,* and *solving* in the January-April mathematics papers. The last two columns provide the number of papers containing at least one word in that group and the percentage of such articles

Finally, the search for explain-words may be thought of as requiring an extremely explicit discussion of explanation, one that would leave unnoticed a significant amount of the *"explanatory" talk* in these papers. Hafner and Mancosu (2005) offered a list of eight expressions that they had found to be commonly used in the mathematics and philosophy of mathematics literature to describe the search for explanations. Table 4 presents these expressions along with the specific concordance search we made to investigate their prevalence in both the mathematics and physics papers, and the frequencies with which these alternative expressions appeared. We note that the total number of occurrences of these expressions is only about 10% of the total amount of explain-words in each set of papers (with disproportionately more occurrences of these expressions in the physics papers than the mathematics ones) and thus this analysis does not affect the finding made by only investigating the use of explain-words.

Based on these results, we also conclude that Hafner and Mancosu (2005) may have grossly overestimated how common these expressions are in mathematics research papers. We are also left wondering to what extent such *common* alternative expressions exist.

Alternative expression	Concordance search	Mathematics	Physics
"the deep reasons"	deep* reason*	5	16
"an understanding of the essence"	understand* the essence	0	5
"a better understanding"	better understand*	161	767
"a satisfying reason"	satisfy* reason	0	0
"the reason why"	reason* why	312	924
"the true reason"	true reason	3	1
"an account of the fact"	an account of the fact	0	0
"the causes of"	cause* of	16	609
	Total	497	2322

Table 4. Frequencies of alternative expressions of related to *"explanatory" talk* in the January-April mathematics and physics papers

Explaining why vs. explaining how

In order to investigate mathematicians' discussion of *explanations of why* a certain mathematical statement is true (Hafner and Mancosu's "deep" explanation), in comparison to their talk about *explanations of how* to do something in mathematics (related to Rav's notion of mathematical know-how), we created a concordance of the corpus of papers and identified every instance an explain-word had been immediately followed by the words why or how (e.g. unexplained why, explanation how). We did this by searching the concordance for *expla* why and *expla* how, and checking that all results were indeed uses of explain-words. We then repeated the process with the corpus of physics papers. As, shown in Table 5, there is a clear difference between the ways that explain-words show up in the mathematics and the physics research papers.

	Mathematics		Physics	
	frequency	per million	frequency	per million
expla why	247	7.99	952	16.17
expla how	458	14.83	353	6.00
Total	705	22.82	1305	22.17

Table 5. Frequencies and frequencies per million words of explain-words immediately followed by the words *why* or *how* in the January-April mathematics and physics papers

We note that when taken together the total of *expla*-why and *expla*-how expressions were roughly as common in math papers as they were in physics papers, with approximately 22 of these expressions showing up per million words in each set of papers, and also a relatively small subset of the wider use of explain-words (roughly 14% and 6% of explain-word usage in mathematics and physics, respectively). However, the distribution of these two different types of expressions in the two sets of papers was significantly different (Fisher's exact test, $p < .001$), with mathematicians using nearly twice as many *expla*-how expressions as *expla*-why expressions, and physicists on the other hand using between two and three times as many *expla*-why expressions as *expla*-how expressions.

Frequency of explicit *“explanatory” talk* in the May-August 2009 papers

A valuable outcome of having access to such large corpora of scientific writing, and the means to analyze them, is that we can parse these corpora and use certain parts for exploratory and other parts for confirmatory purposes. As such, having conducted the previously reported analyses on the mathematics and physics papers uploaded to the ArXiv between January and April 2009, we then performed these analyses on the papers uploaded between May and August of that year. Table 6 presents the frequencies of explain-words appearing in this new set of mathematics and physics papers, while Table 7 presents the frequencies of explain-words immediately followed by the words *why* or *how* in these set of papers.

	Mathematics		Physics	
	frequency	per million	frequency	per million
explain	1881	60.12	7974	126.96
explained	1841	58.84	6596	105.02
explanation	537	17.16	3788	60.31
explains	525	16.78	1694	26.97
explaining	166	5.31	954	15.19
explanations	98	3.13	740	11.78
explanatory	36	1.15	78	1.24
unexplained	19	0.61	159	2.53
explication	1	0.03	5	0.08
explicated	6	0.19	12	0.19
explicate	6	0.19	12	0.19
explicating	0	0.00	5	0.08
unexplainable	0	0.00	1	0.02
explications	2	0.06	5	0.08
explainable	1	0.03	24	0.38
explicates	0	0.00	0	0.00
explicable	0	0.00	24	0.38
inexplicable	1	0.03	12	0.19
Total	5120	163.63	22083	351.60

Table 6. Frequency and frequency per million words of explain-words appearing in the May-August mathematics and physics papers

	Mathematics		Physics	
	frequency	per million	frequency	per million
expla why	277	8.85	970	15.44
expla how	526	16.81	464	7.39
Total	803	25.66	1434	22.83

Table 7. Frequency and frequency per million words of explain-words immediately followed by the words *why* or *how* in the May-August mathematics and physics papers

Table 6 reveals the same pattern of frequencies explain-words as those presented in Table 2. Indeed, the same five explain-words (explain, explained, explanation, explains, and explaining) made up 95% of all explain-words in each set of papers. Furthermore, the number of explain-words per million words was very similar in each set of papers (going from 158.94 to 163.63 in mathematics, and from 362.64 to 351.60 in physics), with the number of explain-words per million words in the physics papers being around 2.15 times that of the mathematics papers.

Similarly, the frequencies presented in Table 7 are consistent with those in Table 5: there was a similar number of *expla*-why and *expla*-how expressions per million words in each discipline (25.66 in mathematics, and 22.83 in physics paper), but there was a significantly different distribution of these two different types of expressions in the two sets of papers (Fisher's exact test, $p < .001$), with mathematicians using nearly twice as many *expla*-how expressions as *expla*-why expressions, and physicists on the other hand using a little over twice as many *expla*-why expressions as *expla*-how expressions.

Discussion

Our analysis of *"explanatory" talk* in a large sample of mathematics papers does not offer support for a claim often made in the philosophy of mathematics: that this type of talk is prevalent in mathematical discourse. When compared to explicit discussion of other related mathematical practices (e.g. showing results, solving problems, and proving theorems), mathematicians do not seem to discuss explanation nearly as much. Furthermore, when compared to another scientific discourse, we found that mathematical discourse contains only a fraction of *"explanatory" talk* as research papers in physics. Indeed, we believe these findings suggest that the prevalence of *"explanatory" talk* in mathematical discourse has been widely exaggerated.

Furthermore, by analyzing the frequency with which variations of the expressions *explain why* and *explain how* occur in mathematics and physics research papers, we found that, to the extent to which they engage in *"explanatory" talk*, mathematicians seem to be much more interested in discussing explanations of how to do something in mathematics, than in explanations of why things are the way they are in mathematics. In physics we found the situation to be the opposite. This is particularly interesting given mathematics educators' and philosophers' of mathematics preoccupation with the type of intra-mathematical explanations of the form X explains why Y (where X and Y are mathematical assertions), and particularly with the notion of explanatory proofs (in which proof X explains why theorem Y is true). This focus may have been inherited from the more traditional study of the notion of scientific explanation, which is not only naturally concerned with this type of explanations (the desire to explain the real world is full of why-questions), but according to our findings may also be more commonly discussed in scientific discourse in terms of answers to why-questions. However, our findings suggest that this focus may also be misguided for those interested in studying the notion of mathematical explanation as it more commonly occurs in the discourse of professional mathematicians. Indeed, as suggested by Rav (1999), it seems that when it comes to proofs and explanations, mathematicians are primarily interested in learning how to solve other problems, possibly over learning the reasons why some mathematical results hold true.

Now, one must be careful about several inferential jumps made in this kind of analysis. First, while the ArXiv may well be the largest, most widely used repository of this type of preprints and postprints in the world, we have analyzed a very specific type of mathematical discourse, leaving open the possibility that studies of mathematical discourse in others settings (conversational or other digital communications) could lead to contrasting findings. Second, we have analyzed these research papers for a limited type of *"explanatory" talk*, one required to contain explain-words or a limited number of alternative, related expressions. While this was an obvious place to start to investigate *"explanatory" talk* in mathematical discourse, it is certainly possible that the analysis of other expressions related to mathematical explanation may skew our results. These limitations of the present study indicate clear avenues for future empirical research on mathematical explanation.

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How Limit can be Embodied and Arithmetized: A Critique of Lakoff and Núñez

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In Where Mathematics Comes From*, Lakoff and Núñez (2001) describe how the notions of infinity and limit can be constructed through metaphorical extensions of embodied experiences. This paper will critique their historical and psychological analysis, revealing an unresolved tension between a simplified, geometric "approaching" conception and the arithmetization of calculus by Weierstrass. A proposal of how to rectify this conflict through acknowledging how novices can metaphorically tie these concepts together is discussed.*

Key words: Embodied Cognition, Limit, Infinity

Lakoff and Núñez created a detailed guide of how humans learn mathematical ideas through the principles of embodied cognition in *Where Mathematics Comes From* (2001). From constructing a foundation of grounding metaphors directly connected to embodied experiences, to extending mathematical ideas beyond what can be directly experienced (such as the concept of infinity), the authors created a new theoretical paradigm for mathematics education research. The work, however, has also been criticized most notably for the inadequacies of their "mathematical idea analysis" methodology (Schiralli & Sinclair, 2003). It is long overdue that we follow the advice of these critics and attempt to solidify the metaphorical construction of the mathematical concepts that form the building blocks of their work, so that we as a field can continue to conduct embodied cognition mathematics education research on the firmest foundation possible.

Limit Conceptualization: A Brief History

In particular, an unresolved tension exists between how Lakoff and Núñez connect the two different ways a student could conceptualize limit, through either a dynamic conception or a static conception (Cornu, 1992). The dynamic conception is a motion-based idea of limit, frequently expressed using dynamic language such as "approaching". Graphically, this is demonstrated by moving a point along a graph towards another point for the limit of a function. The static conception, on the other hand, forgoes all motion in order to describe limit in terms of closeness through the coordination of intervals. This is expressed through static language such as "is close to" or "within the range of". Graphically, this is demonstrated through intervals on each axis for the limit of a function.

The underlying difficulty with Lakoff and Núñez's approach is that they conflate these two possible conceptions of limit. The following sections explore in detail the how and why they do this, and the difficulties associated with this approach, but there are a couple of general problems that should be addressed at the outset.

First, this approach is historically inaccurate. Limits were originally conceived of in terms of fluxions, or what we now think of as infinitesimals, which is a different sort of hand-waving than approaching, but it caused similar problems in terms of formality (Kleiner, 2001). Cauchy and Weierstrass later formalized the work of Newton and Leibniz (Grabiner, 1992a) by eliminating the crutch of fluxions and requiring the coordination of two small intervals for the limit of a function at a point. One interval's radius, signified by the Greek letter delta (δ), surrounds the input or x-value, while the second interval's radius, represented by the Greek letter epsilon (ε), surrounds the output or result of the function applied to the x-value. The coordination of the two intervals is accomplished through the logical quantification portion of the definition: for any epsilon, there exists a delta, such that if the input value is within the delta interval, then the output value must be within the epsilon interval. These intervals are symbolized through the distance interpretation of the absolute value, while the logical quantification uses common logical symbols for "for any," "there exists," and the "if …, then …" structure. Conflating the dynamic and static conceptions of limit misses the historical context of how the static conception was the solution to the informalities of another conception.

Second, and more important in terms of a theory of cognition, conflating these conceptions is also problematic because, if these two conceptions were one and the same, we'd see evidence of that in student thinking, and we don't. The trajectory of mathematics education research of limit began with an initial focus on establishing misconceptions in undergraduates' understanding of limit (Bezuidenhout, 2001; Davis & Vinner, 1986; Tall & Vinner, 1981), which then evolved into proposals on how students might build appropriate conceptions of limit (Cottrill et al., 1996; Lakoff & Núñez, 2001; Williams, 1991, 2001). These proposals brought about several more studies yielding successful interventions into students' limit conceptions (Boester, 2010; Oehrtman, 2009; Roh, 2008, 2010; Swinyard & Larsen, 2012). While there are unresolved questions concerning the relationship between the dynamic and static conceptions of limit, altogether, the entire body of mathematics education limit research on limit has this difference between the dynamic and static conceptions as their most basic finding. Students simply treat these conceptions as fundamentally different.

Certainly, a robust conception of limit most likely contains both a dynamic aspect and a static aspect, but that does not mean that they are the same. Conversely, this doesn't necessarily mean that embodied cognition is flawed, simply that it has been misapplied to explain how we conceptualize limits. The difficulties encountered by conflating these conceptions of limit will hopefully provide a guide to appropriately apply embodied cognition to motivate and ultimately coordinate both conceptions into a solid metaphorical framework.

Conceptualizing Limit Through the BMI

Before introducing the concept of limit, Lakoff and Núñez first discuss how we as humans can conceptualize infinity. All mathematical ideas that are not taken directly from embodied experiences must be built through metaphorical extensions. This is particularly important here, because infinity does not actually exist in the universe. Since we can't directly experience infinity, we have to use metaphors in order to conceptualize it.

Infinity is commonly thought of as an infinitely continuous process, one which goes on forever. In order to motivate this specific type of process, which cannot actually occur, Lakoff and Núñez propose a particular metaphorical leap in our thinking: finite iterative processes can be thought of as infinite iterative processes, which in turn can be thought of as infinitely continuous processes. For example, we can begin with "jump", an inherently "perfective" process, which means it has an end, a completion. Each jump has an end state. But if you describe someone as "jumping and jumping and jumping", we tend to assume this means an indefinite number of jumps. So while "jump" is a finite iterative process, "jumping and jumping and jumping" is an infinite iterative process. We can then extend this thinking further to "swimming", an imperfective process which has no designated end. Thus "swimming and swimming and swimming" is an infinitely continuous process.

Now of course, a person cannot go on swimming forever, but Lakoff and Núñez argue that we can create a metaphorical leap to think about what that would look like, as swimming is a continuous activity, and what could happen at the end of such an infinite process, which moves us from a potential infinity to actual infinity.

The Basic Metaphor of Infinity

Source Domain Completed Iterative Processes		*Target Domain* Iterative Processes That Go On and On
The beginning state	→	The beginning state
State resulting from the initial stage of the process	→	State resulting from the initial stage of the process
The process: From a given intermediate state, produce the next state.	→	The process: From a given intermediate state, produce the next state.
The intermediate result after that iteration of the process	→	The intermediate result after that iteration of the process
The final resultant state	→	**"The final resultant state" (actual infinity)**
Entailment *E:* The final resultant state is unique and follows every nonfinal state.	→	**Entailment *E:* The final resultant state is unique and follows every nonfinal state.**

Figure 1. The basic metaphor of infinity (Lakoff and Núñez, 2001, p. 159).

With the basic metaphor of infinity (BMI) in place, Lakoff and Núñez are ready to introduce limit by first discussing a one-dimensional limit: an infinite sequence along a number line that approaches a particular number (the limit). The traditional, formal definition of limit in terms of sequences is: the sequence $\{x_n\}$ has L as a limit if, for each positive number ε, there is a positive integer n_0 with the property that $|x_n - L| < \varepsilon$ for all $n \geq n_0$ (pp. 189-90). Unfortunately, this definition cannot use the BMI directly, because there is nothing being iterated. That's important because the basic metaphor of infinity relies on iteration (the jumping). Because the basic concept of limit includes infinity, the authors need to use the BMI here, but because the BMI relies on iterative processes, they need a definition of limit where something iterates.

Thus, the definition they create relies on nested sets, whose iterative quality can be directly connected to the BMI: $0 < r < |x_n - L|$, where R_n is the set of all values r bounded between zero and $|x_n - L|$. As $|x_n - L|$ gets smaller, the range of values in R_n gets smaller. Since the size of the sets R_n is decreasing, these sets can be nested: $R_{n+1} \subset R_n$. This chain of nested sets is then used in the BMI to obtain the limit (where the "last" nested set, R_∞, would be the empty set). Lakoff and Núñez take the expression $0 < r < |x_n - L|$ to be synonymous with $|x_n - L| < \varepsilon$ so that we can then extend the BMI to cover the standard formal definition.

In summary, Lakoff and Núñez are required to use the BMI to explain the concept of limit, because limit utilizes the concept of infinity. Because infinity does not exist in reality, this presents a critical challenge to a theory of mathematics that is entirely based on real-world, embodied experiences, hence the necessity of the BMI. But the BMI uses indefinite iterative processes, and the standard formal definition of sequences does not. So the authors need to formulate a new definition which does, hence nested sets. These nested sets can be plugged into the BMI as an indefinite iterative process, and thus be used to explain our conception of limit.

Simple Approaching and the Nonnecessity of Nested Sets

To demonstrate the process of approaching a limit, and how this utilizes the BMI, Lakoff and Núñez give an example sequence $\{x_n\} = n/(n+1)$. As n increases, the value x_n gets closer and closer to 1 (p. 187).

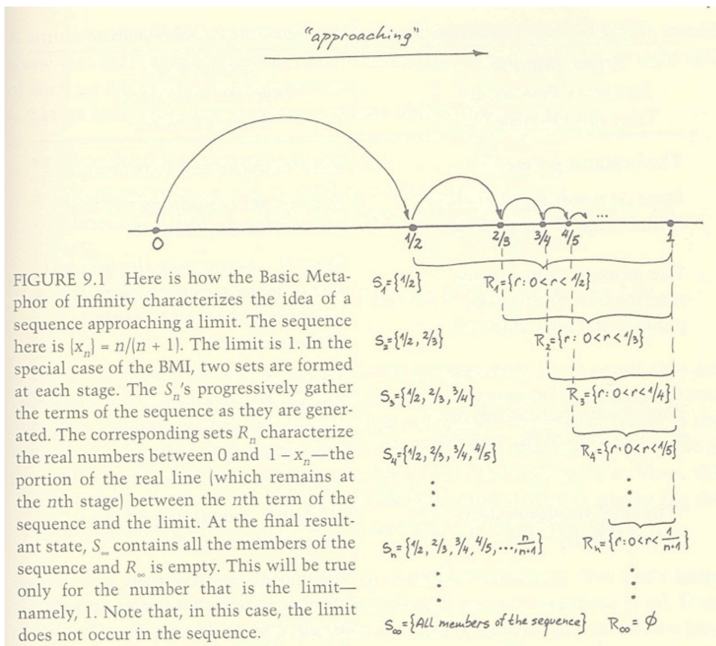

Figure 2. Example monotonic sequence (Lakoff and Núñez, 2001, p. 189).

This example is strictly monotonic, in that it creeps up on the limit in one direction, always getting closer and closer to 1, never farther away. This matches the smooth, motion-based "approaching" conception of limit. Taking smaller and smaller steps towards a wall would be a physical interpretation of the steps of this sequence.

Notice that Lakoff and Núñez are using a very formal, very complicated argument here to justify not just limit, but the concept of approaching, labeled at the top of the above diagram. Unfortunately, there is no evidence in the limit research literature to support the conclusion that students think of approaching in this complicated, set-theoretic way. One could make the argument that perhaps this is what students are doing under the surface, and we just haven't looked hard enough. Or that what Lakoff and Núñez describe is how experts think, but not students (although in that case, they don't describe what students might be thinking or the process to go from a novice to expert perspective).

The real problem here is that Lakoff and Núñez only have one definition for limit, and it must encompass both the dynamic and the static conceptions. Thus, it has to contain all of the formal pieces of the static conception, even when those pieces are overkill, like in the above example. In their attempt to mathematize the concept of "approaching", they are completely ignoring the natural, physical-ness of approaching. What they are trying to do is characterize the distance between you and the limit: the distance becomes smaller as you approach the limit. However, they use a complicated mathematical concept (nested sets) encoded in a complicated mathematical notation, when a description of "approaching" as simply going towards something would suffice.

Overextending Approaching to Cover Complex Limits

Not all sequences behave as nicely as the above example, however. Sequences (and functions) can approach limits in far more complicated ways than simply monotonically. Another example Lakoff and Núñez provide is what they call a teaser sequence.

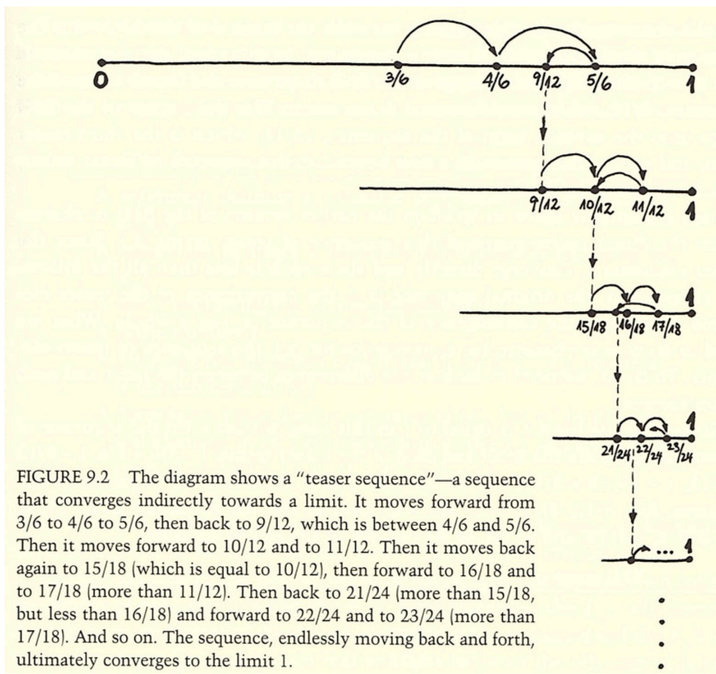

FIGURE 9.2 The diagram shows a "teaser sequence"—a sequence that converges indirectly towards a limit. It moves forward from 3/6 to 4/6 to 5/6, then back to 9/12, which is between 4/6 and 5/6. Then it moves forward to 10/12 and to 11/12. Then it moves back again to 15/18 (which is equal to 10/12), then forward to 16/18 and to 17/18 (more than 11/12). Then back to 21/24 (more than 15/18, but less than 16/18) and forward to 22/24 and to 23/24 (more than 17/18). And so on. The sequence, endlessly moving back and forth, ultimately converges to the limit 1.

Figure 3. Example teaser sequence (Lakoff and Núñez, 2001, p. 193).

This sequence converges indirectly but still has a limit. In terms of nested sets, one needs to pick particular x_n terms to ensure that $R_{n+1} \subset R_n$. In this case, every other term would work (corresponding to the dotted lines in the above diagram). This improves their coverage of possible types of limit convergence, but does stretch the metaphorical interpretation of limits as "approaching". This would correspond to a physical interpretation where, for each step taken towards a wall, a step a fraction of that size is taken going away from the wall (remembering that the steps overall are getting smaller and smaller).

The physical interpretations of both monotonic and indirect convergence also illuminate another way that students may think of limit, as a barrier that cannot be crossed (Davis and Vinner, 1986). Mathematically, this is called a bound. Both examples presented thus far are bounded at the limit because they only approach from one direction.

Lakoff and Núñez do not provide an example like an alternating sequence such as $\{x_n\} = (-1)^n / (n+1)$, whose limit is neither an upper or lower bound. (The limit of this sequence is zero, but one could consider any sequence which "alternates" above and below a non-zero limit.)

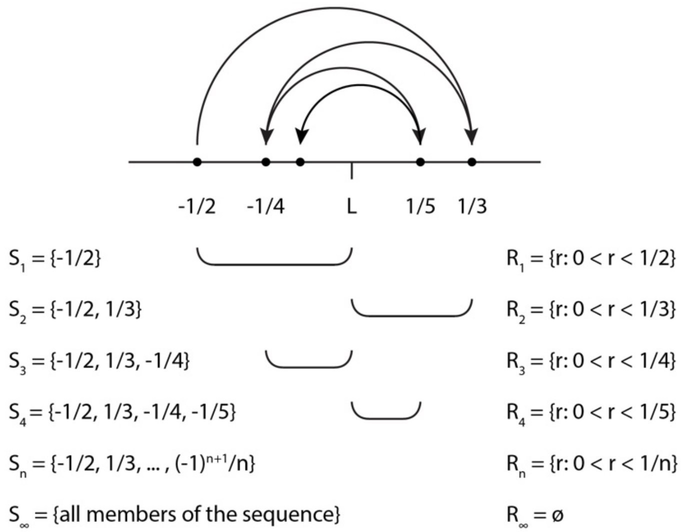

Figure 4. Example alternating sequence.

Having terms of the sequence on both sides of the limit is covered by the absolute value portion of the original formal definition of limit of a sequence: by making a range around *L*, it doesn't matter if you are above *L* or below it, as long as you are no more than ε away. Lakoff and Núñez also cover this possibility in their definition when they utilize absolute value notation for nested sets. Remember that the nested sets R_n are defined as the sets of the values *r* can take when $0 < r < |x_n - L|$. However, none of the examples presented in the book necessitate the usage of absolute value in the definition, as all instances of x_n are less than *L* (thus simply writing $0 < r < L - x_n$ would suffice). The above alternating sequence finally requires us to use the absolute value portion of the definition Lakoff and Núñez propose. But how does this example affect the meaning of "approaching"?

While it is still possible to stretch this motion-based concept to include approaching from both sides, there are some important implications in doing so. First, as one will be approaching the limit from both sides, this example is difficult to conceptualize using the types of physically-based interpretations used to ground the concept of "approaching": it would be like trying to approach a wall from both sides simultaneously. Second, one would need to abandon the idea that the limit is a barrier or bound, because you would be (repeatedly) passing through it. You could not conceptualize the wall in our example as a barrier. Third, the sequence may equal the limit while it is "approaching" the limit. In taking the limit of the sequence $x_n = \left(\sin\left(\pi n / 4\right)\right) / n$, there are many times where the sequence yields values that are zero. In fact, we don't even need to turn to something as complicated as this: a constant sequence will equal its limit everywhere. This really challenges the concept of "approaching" the limit, since, for a constant function, you don't actually move at all.

These three implications push the "approaching" metaphor away from the natural, motion-based conception of limit mentioned earlier. Of course, one might be able to create a convincing argument on how we can take a physical sense of approaching and stretch it or reinterpret it to cover the case of an alternating sequence, but then I'd just propose another sequence with more obscure "approaching" behavior. (This situation gets even more unacceptable with the definition of limit for a function at a point, as we can consider examples that misbehave in spectacular ways, such as a function that is discontinuous everywhere but still has a limit at a particular point). We'd have to keep pushing and pushing on this physical sense of approaching when confronted with worse and worse sequences, making more and more special cases and reinterpretations for those sequences, until the concept of "approaching" is a litany of exceptions and really doesn't connect with our embodied experiences much at all anymore.

Conflating Approaching With the Formal Definition

If Lakoff and Núñez want to stay with the embodied, motion-based "approaching" conception of limit, they need to stick with simple examples which are not stretched too far from normal experiences. However, in mathematizing these simple examples, they should not use the absolute value, because it is not necessary. This means that whatever mathematical definition they try to establish, based on the examples, it cannot use the absolute value, which means that it cannot duplicate the formal definition (Figure 5, argument one).

If they want to motivate the formal definition, then they must use the absolute value as it appears in the formal definition when using the BMI. In order to motivate the usage of the absolute value, they need more complex examples which approach the limit from both sides. In doing this, they then must use much more complicated explanations of "approaching" than those which naturally arise because of the implications discussed above (Figure 5, argument two).

The authors try to do both. Their bottom-up argument originates with the desire to motivate the formal definition using the BMI. They cannot do this without using the absolute value. However, working top-down they examine the "approaching" conception of limit and use examples which naturally tie into this conception. Unfortunately, there is no reconciliation between the simple examples and their usage of the absolute value (Figure 5, Lakoff and Núñez).

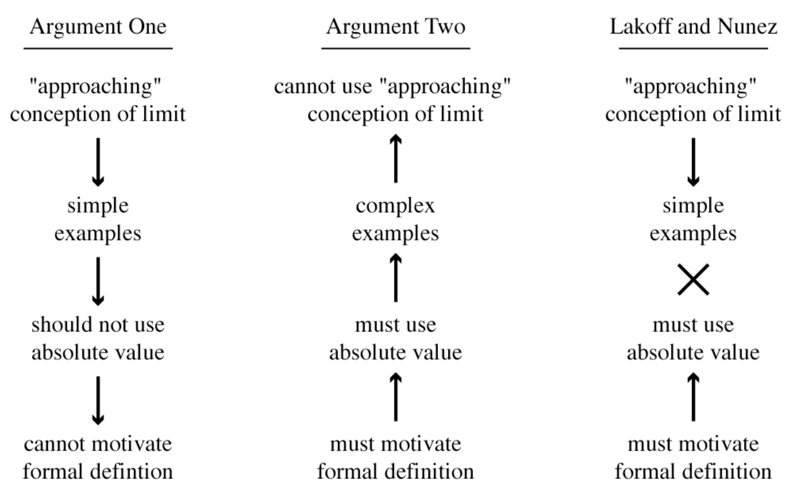

Figure 5. Linking the informal, motion-based conception of limit with the formal definition.

The way to fix this problem is to acknowledge that these two conceptions of limit are fundamentally different, they come from different things, they have different consequences, which means they should be built through separate metaphorical constructs. Traditionally, students first learn the dynamic conception, and we should base this on a natural metaphor of motion along a path. Next, students typically learn the static conception, which should be based on its own metaphor for closeness. Lakoff and Núñez's BMI-produced subset metaphor, one which is motion-free and emphasizes sets, is suitable for sequences, and a similar metaphor could potentially be used for the input-output relationship for functions. Then, students can use a final metaphorical association to combine both the static and dynamic conceptions into an overall conception of limit.

Approaching as Motion Along a Path

At the beginning of their book, Lakoff and Núñez discuss grounding metaphors, those which directly access our physical, embodied experiences. One of those utilizes motion along a path, an experience that we have all had many times, starting from an early age, to build what is illustrated below, the Source-Path-Goal schema. A schema is a coordinated combination of several concepts, which here includes a source, a path, a trajector (you moving along the path), and the goal. I propose that this is the schema that students base their conceptual metaphor for approaching a limit, for both sequences and functions.

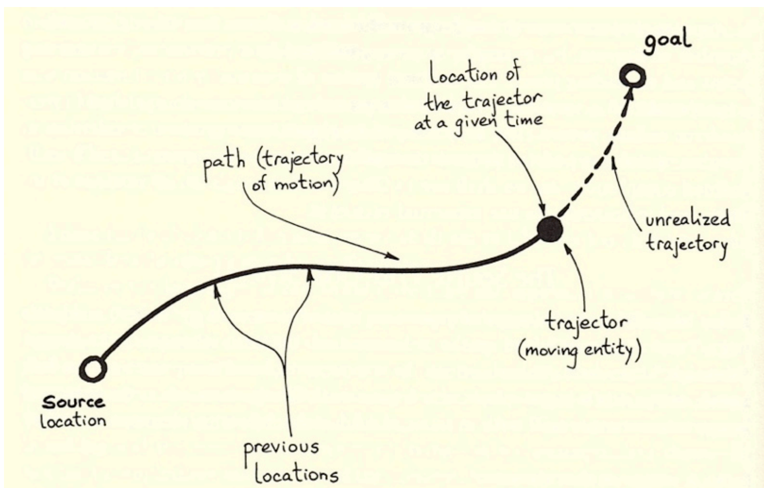

FIGURE 2.2 The Source-Path-Goal schema. We conceptualize linear motion using a conceptual schema in which there is a moving entity (called a trajector), a source of motion, a trajectory of motion (called a path), and a goal with an unrealized trajectory approaching that goal. There is a logic inherent in the structure of the schema. For example, if you are at a given location on a path, you have been at all previous locations on that path.

Figure 6. Source-Path-Goal schema (Lakoff and Núñez, 2001, p. 38).

Creating a metaphorical mapping from this schema to the limit of a function is more straightforward than a mapping to the limit of a sequence, as motion along a two-dimensional, continuous function is a more natural extension of motion along a path. If we think of a nicely behaved, prototypical function that has a limit at a point, then we can see how the parts of the Source-Path-Goal schema (left, Figure 7) can be mapped to the parts of "approaching" a limit of a function at a point (right, Figure 7). Your position on the function is the trajector, the goal is the x-value of the function being approached, the route is the path of the function's graph, and the direction is getting closer to the limit. Your starting point is a little nebulous, as that is not clearly defined, but you still have one, it is just somewhere away from the limit (wherever you started). The most important part is the last line, as the final location, the goal, is the y-value of the function the path is approaching regardless of what happens when you actually get there (since that's the whole point of a limit). This is the sort of metaphorical entailment that is possible through embodied cognition: the actual goal for the source domain may not be realized in the target domain (as the function may not equal the limit at the x-value of interest).

Approaching a Limit is the Source-Path-Goal Schema

Source Domain Source-Path-Goal Schema		Target Domain Approaching a Limit
A trajector that moves	→	Your position on the function
A source location (the starting point)	→	A coordinate on the function away from the limit
A goal – an intended destination for the trajector	→	The x-value being approached
A route from the source to the goal	→	The path (graph of) the function
The position of the trajector at any given time	→	Your (moving) position on the graph
The direction of the trajector at that time	→	Moving towards the limit
The actual final location of the trajector	→	The y-value the function tends toward, regardless of what the function does at the actual x-value being approached

Figure 7. Approaching a limit at a point on a function is the Source-Path-Goal schema.

Approaching a Limit is the Source-Path-Goal Schema

Source Domain Source-Path-Goal Schema		Target Domain Approaching a Limit
A trajector that moves	→	Your position on the number line
A source location (the starting point)	→	The first term of the sequence
A goal – an intended destination for the trajector	→	The "last" (infinite) term of the sequence
A route from the source to the goal	→	The steps from each term to the next
The position of the trajector at any given time	→	The n^{th} term of the sequence
The direction of the trajector at that time	→	Moving towards the limit
The actual final location of the trajector	→	The value the sequence tends toward

Figure 8. Approaching a limit of a sequence is the Source-Path-Goal schema.

Because a sequence is a set of discrete values, building a metaphorical mapping from this schema to the limit of a sequence is somewhat more complicated, as the “path” isn’t so obvious. Instead of the two-dimensional path present in a function, we now have a one-dimensional path with discrete steps. But the same type of metaphorical mapping from the Source-Path-Goal schema (left, Figure 8) to the target domain of a sequence (right, Figure 8) works here as well. The trajector is your position on the number line, there is a clear starting point with the first term of the sequence, the goal is the last (infinite) term of the sequence, the route is the jumps from one term to the next (even when the jumps are complicated), and the direction is still (generally) moving towards the limit. Again, the final location is the value the sequence tends towards, even if the sequence never attains that particular value.

Formal Definition as Closeness

Once students have built a dynamic conception of limit for sequences or functions, I propose that students build a separate, static conception of limit that supports the formal definition. For sequences, Lakoff and Núñez provide the details for how students metaphorically map iterative processes to the limit of infinite sequences. As long as this metaphor is not trying to support both the static and dynamic conceptions of limit simultaneously, then this mapping has none of the problematic issues detailed above. If it is only used to provide a sense of closeness divorced from the conception of approaching a limit, then the notation and rigor of nested sets is entirely appropriate.

THE BMI FOR INFINITE SEQUENCES (PROTOTYPICAL VERSION)

Target Domain ITERATIVE PROCESSES THAT GO ON AND ON		*Special Case* INFINITE SEQUENCES WITH A LIMIT L
The beginning state (0)	⇒	The Sequence and Limit frame
State (1) resulting from the initial stage of the process	⇒	S_1 = the set containing the first term of the sequence
The process: From a prior intermediate state (n–1), produce the next state (n).	⇒	From S_{n-1} containing the first n–1 terms of the sequence, form S_n containing the first n terms of the sequence.
The intermediate result after that iteration of the process	⇒	The set S_n. The set R_n containing all positive real numbers r such that $0 < r < \|x_n - L\|$. $R_n \subset R_{n-1}$.
“The final resultant state” (actual infinity “∞”)	⇒	**The set S_∞ containing all the terms of the sequence**
		There is no positive real number r such that $0 < r < \|x_i - L\|$ for all x_i in S_∞. Hence, $R_\infty = \varnothing$. L is the limit of the sequence.
Entailment *E*: The final resultant state (“∞”) is unique and follows every nonfinal state.	⇒	**Entailment *E*: L is the unique limit of the sequence.**

Figure 9. The BMI for infinite sequences (Lakoff and Núñez, 2001, p. 190).

Lakoff and Núñez do not provide a similar metaphorical mapping from the BMI to the limit of a function at a point. They suggest that the input and output of a function can be thought of as two infinite, coordinated sequences, and therefore the output of the function is close to the limit of the output sequence when the input is close to the limit of the input sequence (p. 198). However, this coordination of the two sequences is nontrivial. In an earlier paper, Lakoff and Núñez (1997) attempt to characterize this coordination:

> Many students of mathematics are falsely led to believe that it is the epsilon-delta portion of these definitions that constitutes the rigor of this arithmetization of calculus. The epsilon-delta portion actually plays a far more limited role. What the epsilon-delta portion accomplishes is a precise characterization of the notion of "correspondingly" that occurs in the dynamic definition of limit where the values of $f(x)$ get "correspondingly" closer to L as x gets closer to a. That is the only vagueness that is made precise by the epsilon-delta definition. (Lakoff & Núñez, 1997, p. 71).

While it is true that mathematicians must formalize the concept of "correspondingly", that, in and of itself, does not formalize "closeness". And by creating two infinite, coordinated sequences, it is the "closeness" of the input and output to their respective limits that Lakoff and Núñez have formalized, not the "correspondingly".

For this reason, I propose a different metaphorical mapping for the formal definition of limit at a point on a function, taken from an earlier study (Boester, 2010). The bolt problem was introduced to students to give them a real-world context for the logical quantification involved in the formal definition (the "correspondingly"). It basically asks the students to state the relationship between the error tolerances on the input (raw materials) and the output (length of the bolts produced by the machine).

The bolt problem is the formal definition of limit

Source Domain Bolt Problem		Target Domain Formal Definition of Limit
For any bolt length tolerance	→	$\forall \varepsilon > 0$
There exists a raw materials tolerance	→	$\exists \delta > 0$
Such that	→	Such that
If ..., then ...	→	... → ...
the amount of raw materials put into the machine is within the raw materials tolerance	→	$0 < \lvert x - a \rvert < \delta$
the length of the bolt the machine produces will be within the bolt length tolerance	→	$\lvert f(x) - L \rvert < \varepsilon$

Figure 10. The bolt problem is the definition of limit at a point on a function (Boester, 2010).

Once students could explain the solution to the problem (left, Figure 10), the notation of the formal definition and the canonical delta-epsilon picture were given, and the pieces of all three were mapped to each other.

Resolution: Connecting Approaching With the Formal Definition

Finally, once students have both the dynamic and static conceptions, how could they ultimately combine both conceptions? It could potentially occur in a few different ways, but data collected from the earlier study (Boester, 2010) suggests that students put the static conception into motion (rather than freezing their dynamic conception to make it static). These students still preferred the dynamic conception over the static (as shown through their language choices). When describing the static conception, they no longer restrict themselves to static language, and instead use a mix of static and dynamic language. (The same students use solely dynamic language when asked to explain how a limit "approaches" a number.) Eventually, some people will flip this relationship, so that the static conception will be the dominant conception and the dynamic conception is a special case.

For example, in an interview, I asked Lisa to explain the formal definition of limit at a point. She drew the following diagram while giving the explanation below:

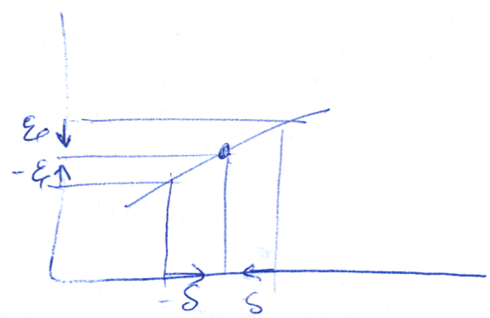

Figure 11. Lisa's limit graph (Boester, 2010).

30. L: I don't, not really, 'cause I can't really remember. But um, like, so you're like *going towards* a point on a graph…
31. T: Mmm-hmm.
32. L: … like this is [unintelligible]. And then, you have like here, here, so this is your um, this is your, uhh, **delta**, and this is your other **delta**…
33. T: Mmm-hmm.
34. L: … and as you're like *getting closer* this way …
35. T: Ok.
36. L: … then you're *getting closer* that way, I think.
37. T: Ok.

Even though she's describing delta (and the "other delta", as she had forgotten the word "epsilon" – she's referring to the ranges), she's using dynamic language to do it. The only static language pieces (bold) are those referring to the ranges. The dynamic language pieces (italics) describe those normally static ranges being put into motion. She's coordinating the delta and epsilon intervals, and is expressing that as the epsilon interval gets smaller (arrowheads), then the corresponding delta interval has to get smaller too, depending upon the function. She's leveraging the dynamic aspects of approaching in order to help her understand and connect the formal, normally-static definition of limit of a function at a point.

This suggests that students initially connect these distinct metaphors by assuming the "approaching" conception is the correct one, and that the static conception of the formal definition is a special case, as shown by putting the synchronized ranges created in the definition into motion. This would imply that students are using the pieces of the "approaching" conception as the source domain and the static conception as the target domain, thus mapping the dynamic aspect of motion along a line to the static (now dynamic) aspect of the range. As students' conception of limit evolves through this connection, recognizing that the formal definition conception is the correct one and that the "approaching" conception is the special case, they take on a conceptual structure analogous to an expert view. This would imply that the students are using the static conception as the source domain and the dynamic conception as the target domain, thus mapping the static ranges to the dynamic (now static) aspect of approaching.

Conclusion

The dynamic, "approaching" conception of limit and the static, formal definition cannot be reconciled the way Lakoff and Núñez propose. Instead of the argument presented, the authors could have simply kept the motion-based conception and the formal definition initially separate, instead of conflating the two. First, they could have presented the motion-based, "approaching" conception of limit, the examples which match this, and a conceptual metaphor which does not use the absolute value. Then later, they could have shown how the motion-based, "approaching" conception of limit breaks down when faced with more complicated types of limit convergence, how this ultimately lead Cauchy (Grabiner, 1992a, 1992b) to move to a new conceptual metaphor of proximity or range, which was later refined by Weierstrass (Lakoff and Núñez, 2001, p. 308) and how this leads to the absolute value being used in the formal definition.

By mixing a motion-based definition with an static-based definition, Lakoff and Núñez cloud the real issue: how do people move from the intuitive, grounded, dynamic conception of limit to the formal, static definition? This is the central pedagogical question that we need to be asking, but through their casual use of absolute value, they gloss over it. The authors attempt to begin with the "approaching" conception of limit, which students use, and try to directly connect it to an understanding of the formal definition. However, while the formal definition is an important concept for students to understand, it is based on a completely different metaphorical foundation than the motion-based conception of limit. This implies that, when it is time for the transition from the "approaching" conception to the formal definition to take place, a new metaphor needs to be introduced.

The formal definition was created to solve and explain cases of limit convergence that the "approaching" conception cannot explain, thus no amount of twisting the informal conception of limit will suffice. This has happened with other prominent scientific concepts as well. For example, Einstein's theory of general relativity was created to explain not only the normal cases explained by Newtonian mechanics, but also cases where those ideas break down (particularly situations concerning the very large and the very small). Thus, a new metaphor was created which encompasses the old one, but does not build off of it. Newtonian mechanics cannot be sufficiently extended in order to link to the general theory of relativity; however, the general theory of relativity can still be used in situations where Newtonian mechanics works. The same principle should be attempted here: because the "approaching" conception simply cannot be extended to cover all cases of limit convergence, a new metaphor which explains the formal definition needs to be introduced to students, making clear that, while the formal definition works in all cases, the "approaching" metaphor still works in simple cases. The informal dynamic conception should not be abandoned, as there is value in supporting such dynamism, even if the expert view needs to contain the formal definition (Marghetis & Núñez, 2013).

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Alice Slowly Develops Self-Efficacy with Writing Proof Frameworks, but Her Initial Progress and Sense of Self-Efficacy Evaporates When She Encounters Unfamiliar Concepts: However, It Eventually Returns

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We document Alice's progression with proof-writing over two semesters. We analyzed videotapes of her one-on-one sessions working through the course notes for our inquiry-based transition-to-proof course. Our theoretical perspective informed our work and includes the view that proof construction is a sequence of mental and physical, actions. It also includes the use of proof frameworks as a means of getting started. Alice's early reluctance to use proof frameworks, after an initial introduction to them, is documented, as well as her subsequent acceptance of and proficiency with them by the end of the real analysis section of the course notes, along with a sense of self-efficacy. However, during the second semester, upon first encountering semigroups, with which she had no prior experience, her proof writing deteriorated, as she coped with understanding the new concepts. But later, she began using proof frameworks again and regained a sense of self-efficacy.

Key words: Transition-to-proof, Proof Construction, Proof Frameworks, Self-efficacy, Coping with Abstraction, Working Memory

This case study focuses on how one non-traditional mature individual, Alice, in one-on-one sessions, progressed from an initial reluctance to use the technique of proof frameworks (Selden & Selden, 1995; Selden, Benkhalti, & Selden, 2014) to a gradual acceptance of, and eventual proficiency with, both writing proof frameworks and completing many entire proofs with familiar content. We also consider how this approach to proof construction helped this individual gain a sense of self-efficacy (Bandura, 1994, 1995) with regard to proving, but later evaporated upon encountering unfamiliar abstract concepts. However, after some time she was able to return to using the technique of proof frameworks and regained a sense of self-efficacy. This study also further illuminates the well-known, documented tendency of students to write proofs from the top-down, who consequently are often unable to develop complete proofs.

Theoretical Perspective

In our analysis and in our teaching, we consider proof construction to be a sequence of mental and physical actions, some of which do not appear in the final written proof text. Such a sequence of actions is related to, and extends, what has been called a "possible construction path" of a proof, illustrated in Selden and Selden (2009). For example, suppose that in a partly completed proof, there is an "or" in the hypothesis of a statement yet to be proved: "*If A or B, then C.*" Here, the situation is having to prove this statement. The interpretation is realizing the need to prove C by cases. The resulting action is constructing two independent sub-proofs; one in which one supposes A and proves C, the other in which one supposes B and proves C.

When several similar situations are followed by similar actions, an *automated link* may be learned between such situations and actions. Subsequently, a situation can be followed by an action, without the need for any conscious processing between the two (Selden, McKee, & Selden, 2010). When a situation occurs together with an action well beyond the simple forming of a link through associative leaning, it may be "overlearned" and the action will then occur automatically in the presence of the situation (Bargh, 2014). In our course, we have observed that, with sufficient practice, many proving actions can become the result of the enactment of linked, and sometimes automated, situation-action pairs. We have called these automated situation-action pairs *behavioral schemas* (Selden, McKee, & Selden, 2010; Selden & Selden, 2008). Linking proving actions to triggering situations and automating those actions can considerably reduce the burden on working memory, a very limited resource, and this tends to reduce errors (Baddeley, 2000).

Related Research and Concepts

While studies of students' proving have been made before, they have *not* often focused on students' use of proof frameworks. For example, Selden and Selden (1987) examined errors and misconceptions in undergraduate abstract algebra students' proof attempts. They found instances of assuming the conclusion, proving the converse, improper use of symbols, misuse of theorems, and trouble with quantifiers. Similarly, Hazzan and Leron (1996) in their study of students' misuse of Lagrange's Theorem found that students often used the converse of a theorem as if it were true or invoked the theorem where it did not apply. Selden, McKee, and Selden (2010) reported instances of students' tendencies to write proofs from the top down and their reluctance to unpack and use the conclusion to structure their proofs. (See the case of Willy, who focused too soon on the hypothesis in Selden, McKee, & Selden, 2010, pp. 209-211).

The Formal-Rhetorical and Problem-Centered Parts of Proofs

Previously, we (Selden & Selden, 2013) introduced the idea of the *formal-rhetorical* part of a proof as the part of a proof that comes from unpacking and using the logical structure of the statement of the theorem, associated definitions, and previously proved theorems. In general, this part *does not* depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p .74). However, the *problem-centered* part of a proof *does* depend on genuine mathematical problem solving, intuition, and a deeper understanding of the concepts involved. A major portion of the formal-rhetorical part of a proof can consist of a proof framework.

Proof Frameworks

An early version of the idea of proof frameworks was introduced by us (Selden & Selden, 1995):

> By a *proof framework* we mean a representation of the "top-level" logical structure of a proof, which does not depend on detailed knowledge of the relevant mathematical concepts, but which is rich enough to allow the reconstruction of the statement to be proved or one equivalent to it. A written representation of a proof framework might be a sequence of statements, interspersed with blank spaces, with the potential for being expanded into a proof by additional argument. (p. 129).

We went on (Selden & Selden, 1995) to connect the ability to unpack the logical structure of mathematical statements with the ability to construct proof frameworks and with proof validation. We also pointed out that mental skills were involved. The learning and mastering of such mental skills can involve much mental energy and considerable working memory. (p. 132). While we did not state this explicitly at the time, in the sample validation in the Appendix, we did note that sometimes checking a sufficiently complex part of a proof might overload working memory and potentially lead to error. (p. 146).

The First and Second Levels of a Proof Framework

Later, we further developed the idea of proof frameworks, including that there are often both first-level and second-level proof frameworks. A *proof framework* is determined by just the logical structure of the theorem statement and associated definitions. The most common form of a theorem is: "If P, then Q", where P is the hypothesis and Q is the conclusion. In order to construct a proof framework for it, one takes the hypothesis of the theorem, "P", and writes, "Suppose P" to begin the proof. One then skips to the bottom of the page and writes "Therefore Q", leaving enough space for the rest of the proof to emerge in between. This produces the *first level* of a proof framework. At this point, a prover should focus on the conclusion and "unpack" its meaning. It may happen that the unpacked meaning of Q has the same logical form as the original theorem, that is, a statement with a hypothesis and a conclusion. In that case, one can repeat the above process, providing a *second-level* proof framework in the space between the first and last lines of the emerging proof. (For some examples, see Selden, Benkhalti, & Selden, 2014).

Finally, we do not claim that mathematicians should write proofs using this technique, but only that doing so will be helpful for novice students and that their mathematics professors will accept the results. As long as the logical flow and clarity of a proof submission is correct, it does not matter (and is impossible to recover) in which order the sentences were written. We now return to the problem of overloaded working memory that can occur when a proof construction, or a proof validation, is sufficiently complex.

Working Memory

It has been said that the "two major components of our cognitive architecture that are critical to [thinking and] learning are long-term memory and working memory" (Kalyuga, 2014). *Working memory* makes cognition possible but has a limited capacity that varies across individuals. It is associated with the conscious processing of information within one's focus of attention. However, working memory can only deal with several units, or chunks, of information at a time, especially when working with novel information (Cowan, 2001; Miller,1956). In contrast, *long-term memory* can be thought of as a learner-organized knowledge base that has essentially unlimited capacity and can be used to help alleviate the limited capacity of working memory (Ericsson & Kintsch, 1995). However, when working memory capacity is overloaded, errors and oversights are likely to occur.

Coping with Mathematical Abstraction and Formality

While the mathematics education research literature does not seem to have considered working memory overload during learning per se, there are a few studies of coping with abstractions. These could be reinterpreted as related to working memory overload causing confusion. For example, Hazzan (1999) investigated how Israeli freshman computer science students, taking their first course in abstract algebra in a "theorem-proof format", coped by "reducing the level of abstraction". Specifically, she found that they tended "to work on a lower level of abstraction than the one in which the concepts are introduced in class" (p. 75). For example, when doing homework on abstract groups, a student might actually be thinking of a familiar group like the integers under addition.

Further, Leron and Hazzan (1997) pointed out that students in mathematical problem-solving situations "often experience confusion and loss of meaning." (p. 265), and that students attempt to make sense of a problem situation "in order to better *cope* with it." (p. 267). While this coping perspective occurs at all levels, they stated that "the phenomena of confusion and loss of meaning are even more pronounced in college mathematics courses." (p. 282). They also suggested that more work on the coping perspective in mathematics education is needed. Indeed, somewhat similarly, Pinto and Tall (1999) considered two different university students' coping mechanisms when confronted with formal definitions and proofs in real analysis. These were the ideas of *giving meaning* to definitions using concept images versus *extracting meaning* from the formal definition via deduction. However, not many university level mathematics education studies have specifically considered students' coping perspectives.

Methodology: Conduct of the Study

We met regularly for individual 75-minute sessions with a mature working professional, Alice, who wanted to learn how to construct proofs. Alice followed the same course notes previously written for our inquiry-based transition-to-proof course[1] used with beginning mathematics graduate students who wanted extra practice in writing proofs. The sessions were almost entirely devoted to having Alice attempt to construct proofs in front of us, often thinking aloud, and to giving her feedback and advice on her work. The notes had been designed to provide graduate students with as many different kinds of proving experiences as possible and included practice writing the kinds of proofs often found in typical proof-based courses, such as some abstract algebra and some real analysis. The notes included theorems on some sets, functions, real analysis, and algebra, in that order.

Alice had a good undergraduate background in mathematics from some time ago and also had prior teaching experience. She only worked on proofs during the actual times we met. While she usually came to see us twice a week to work on constructing proofs. Sometimes, when her paid work got a bit overwhelming, she would take a week off. Thus, unlike the graduate students who took the course as a one-semester 3-credit class, Alice worked with us on our course notes for two semesters at her own pace and did not want credit.

We met in a small seminar room with blackboards on three sides, and Alice constructed original proofs at the blackboard, eventually using the middle blackboard almost exclusively for her evolving proofs. After several meetings, she began to use the left board for definitions and the right board for scratch work. She did not seem shy or overly concerned with working at the board in front of us, and from the start, we developed a very collegial working relationship. She seemed to enjoy our interactions as she worked through the course notes. Thus, we gained

[1] A description of the course and course notes can be found in Selden, McKee, and Selden (2010, p. 207).

greater than normal insight into her mode of working. We videotaped every session and took field notes on what Alice wrote on the three boards, along with her interactions with us. For this study, we reviewed the first and second semester videos and field notes several times, looking for progression in Alice's approach to constructing proofs.

Alice's Progression Through the First Semester

Our First Meeting with Alice

We introduced Alice to the idea of proof frameworks and explained in detail how and why we use them. We also introduced her to the idea of unpacking the conclusion and mentioned that proofs are not written from the top down by mathematicians. With guidance, she was able to prove "If $A \subseteq B$, then $A \cap C \subseteq B \cap C$." In addition, she worked three exercises on writing proof frameworks--one on elementary number theory and two on set equality. Near the end of this meeting, Alice produced a proof framework for the next theorem in the notes. We felt that she not only understood our rationale for using proof frameworks, but also how to construct them.

Our Second Meeting with Alice—Her Reluctance to Use Proof Frameworks Surfaces

At the beginning of the second meeting, Alice went to the middle board and produced the same proof framework that she written five days earlier at our first meeting (Figure 1).

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof: Let A, B, and C be sets. Suppose $A \subseteq B$. Suppose $x \in C - B$. So $x \in C$ and x is not an element of B.	Thus $x \in C$ and x is not in A. Therefore x is in $C - A$. Therefore, $C - B \subseteq C - A$.

Figure 1. A proof framework that Alice produced on the middle blackboard.

Then Alice stopped and after a long silence of 65 seconds, much to our surprise, said, "I have a question for you. I find it very difficult to see the framework. Let me show you how I do it, because somehow I get confused with the framework." We asked her what it was about the framework that was confusing, but she seemingly could not put it into words. So we encouraged her to write a proof the way she preferred. Thus, on the left board, Alice began to write the proof in her own way in top down fashion (Figure 2).

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof. Let A, B, C be sets. Suppose A is a subset of B. We need to prove that $C - B$ is a subset of $C - A$. Suppose $x \in C - B$. We need to prove that $x \in C - A$.

Figure 2. Alice's attempt at constructing a proof in her own way.

Then she then paused for 15 seconds, and said, "We need to have one more," and wrote into her proof attempt, **"and $x \in A$"** immediately below **"$x \in C - B$"**, indicating with a caret that **"and $x \in A$"** was also part of her supposition (Figure 3). Then, after a 35-second pause, she added to her proof attempt, “**Since $x \in A$ and A is a subset B. Then $x \in B$.**” Shortly thereafter, Alice quietly said, “Oh, a contradiction”. This was followed by, “Yeah, ‘cause x doesn’t belong to B. Yeah, problem here.” Then, after a ten second pause, Alice said, “The problem is right here, isn’t it?” pointing and underlining “$\boldsymbol{B}$” and the statement “**and $x \in A$**.” We asked, “And what do you think that problem is?” Alice replied, “I assumed that [pointing to **“and $x \in A$”**], but I do not know. I only know this [pointing to **“A is a subset of B”**]. We replied, “So that’s a good point you’ve made.”

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof. Let A, B, C be sets. Suppose A is a subset of $\underline{\boldsymbol{B}}$. We need to prove that $C - B$ is a subset of $C - A$. Suppose $\boldsymbol{x \in C - B}$. We need to prove that $x \in C - A$. **<u>and $x \in A$</u>.**

Figure 3. Alice’s adjustments to her proof attempt, done in her own way.

After that, for a few minutes, we talked about the structure of proofs, and why we use proof frameworks. Then we asked Alice to elaborate on why **“and $x \in A$”** is a problem. She said, “I didn’t write it right. I should have said here [pointing to the blank space to the left of **“and $x \in A$”**] I’m going to make an assumption like **‘Suppose x belongs to the A’**, and then since x belongs to the A and I know that A is a subset of B, **then the x will belong to the B**.” She continued, “I also know that x belongs in the $C - B$, because I said it earlier. Then x belongs to the C but **x does not belong to the B**.” To which one of us replied, “And then you said something. I thought I heard you say the word **‘contradiction’**.” Alice explained, “Yeah, I got a contradiction because then I’m saying here [pointing to the board] the x belongs to the B, and the x doesn’t belong to the B.” We agreed, and she offered, “That assumption [pointing to **“and $x \in A$”**] was bad.” We then reiterated why proof frameworks are structured the way they are, and suggested that we could take Alice’s original framework (Figure 2) and what Alice had written on the left board (Figure 3), and change the order to write a proof. We proceeded to help Alice do this. For the rest of the semester, Alice seemed more inclined to attempt to use proof frameworks.

Alice’s Way of Working

By midway through the first semester, Alice had developed her own pattern of working. She would:

1. First write the statement of the theorem to be proved on the middle board.
2. Then look up in the course notes the definitions of terms that occurred in the theorem statement and write them exactly as stated on the left board.
3. Next underline the relevant portions of definitions to assist with writing the proof framework. However, we did not teach her to use this strategy. (See Figure 4).
4. Use the right board for scratch work as needed.

Alice continued this pattern of working into the second semester.

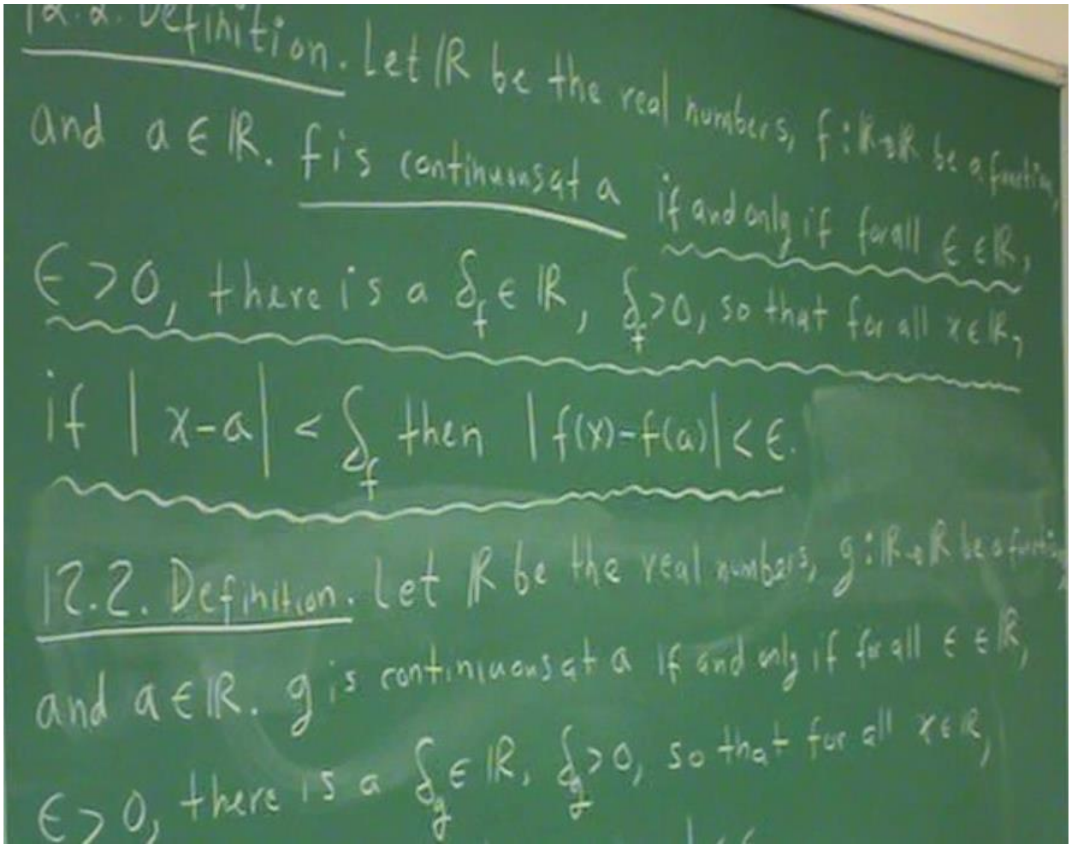

Figure 4. Alice's underlining of relevant parts of a definition.

Subsequent Meetings with Alice

As the first semester meetings went on, we observed that Alice became very methodical in her approach to proving, and also somewhat more accustomed to writing proof frameworks. We hypothesize this was because of her technical work experience and perhaps because of her natural tendencies. By the 12th meeting, Alice had developed the following pattern of working: She would write the statement of the theorem to be proved on the middle board, then look up in the course notes the definitions of terms that occurred in the theorem statement, write them exactly as stated on the left board, and use the right board for scratch work. Indeed, during the 12th meeting, when she got to the theorem, "*Let X, Y, and Z be sets. Let f: X→Y and g: Y→Z be 1-1 functions. Then g○f is 1-1*," she wrote the first- and second-level frameworks ostensibly on her own, and with some guidance from us, completed the proof and read it over for herself aloud.

By the 19th meeting at the end of the first semester, Alice was more fluent with writing proof frameworks than on the 12th meeting, and she had adopted the technique of writing definitions on the left board and changing the variable names to agree with those used in the theorem statement – all without prompting from us. This is remarkable as our experience has been that many students do not change variable names in definitions even when we suggest doing so, and this can often lead to difficulties. At this 19th meeting, Alice proved that the sum of two continuous functions is continuous (Figure 5). This proof has a rather complicated proof framework that necessitates leaving three blanks spaces -- one for using the hypothesis appropriately, one for specifying a δ, and one for showing that the chosen δ "works" (by showing the relevant distance is less than ε.)

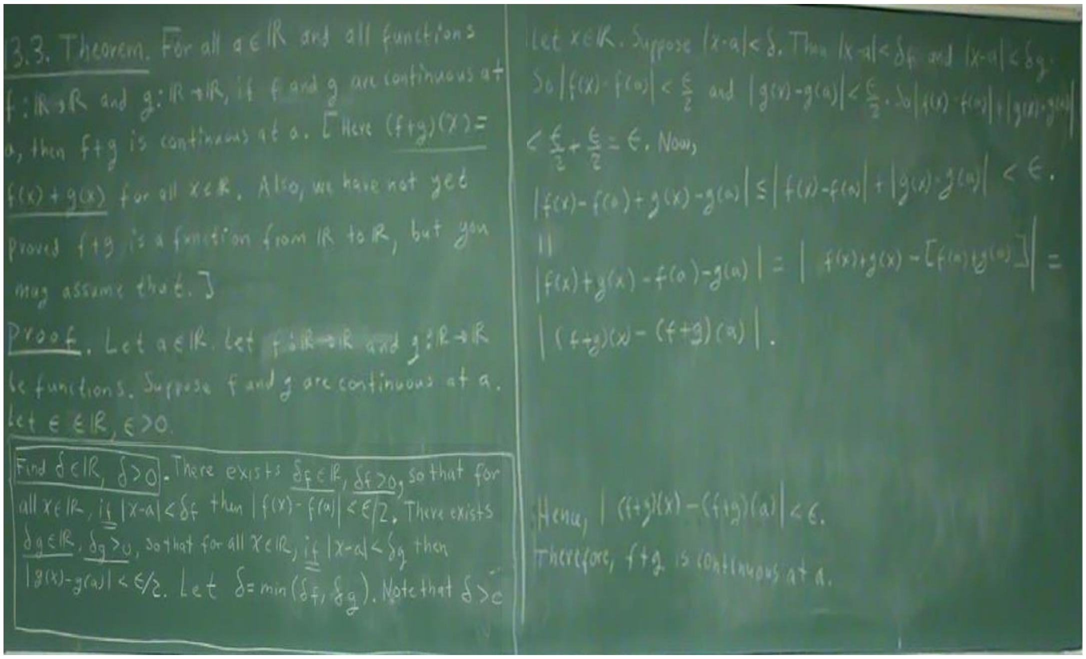

Figure 5. Alice's proof of that the sum of two continuous function is continuous.

Alice continued meeting with us and working on the course notes at her own pace during the second semester.

Alice's Progression through the Second Semester

Alice Continues Proving Real Analysis Theorems

Upon resuming in the second semester, Alice continued proving real analysis theorems, attempting to prove that the product of two continuous functions, f and g, is continuous in our first three meetings (i.e., our 20^{th}-22^{nd} meetings). She set up the proof framework correctly and explored the situation in scratch work. During this proving process, Alice made some astute observations, for example, having gotten to $|fg(x) - fg(a)| = |f(x)g(x) - f(a)g(a)| \le |f(x)|\,|g(x) - g(a)| + |g(a)|\,|f(x) - f(a)|$, and having dealt with term involving $|g(a)|$, she noted that the former term was the "hard part" because $|f(x)|$, unlike $|g(a)|$, is not a constant. Somewhat later, Alice exhibited some self-monitoring, noting that she needed to move her sentence about the bound on $|f(x)|$, prior to setting δ equal to the "minimum of [the] three" deltas she had found. (See Figure 6). She also noted, in the 22^{nd} meeting, that it seemed "weird" to write the restrictions on $|f(x) - f(a)|$ and $|g(x) - g(a)|$ without immediately explaining why she had chosen the bounds $\frac{\varepsilon}{2|g(a)|}$ and $\frac{\varepsilon}{2M_f}$, respectively, when applying the definition of continuity at a point to f and g.

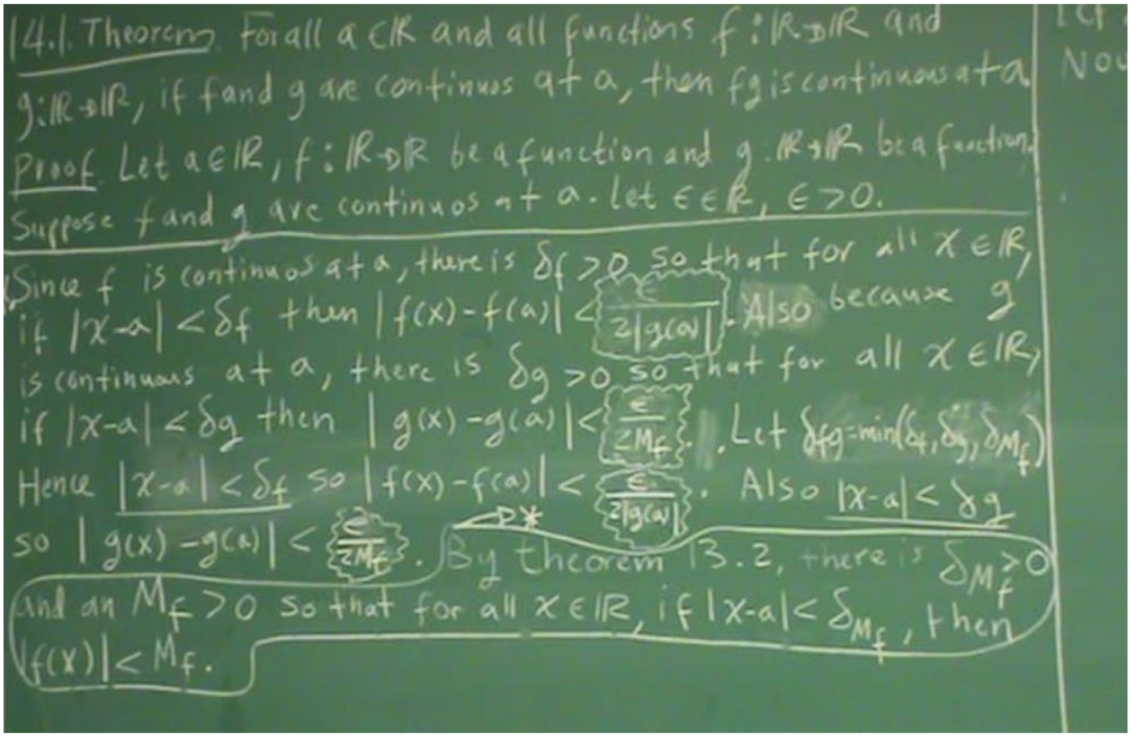

Figure 6. The first half of Alice's proof that the product of continuous functions is continuous, showing the "minimum of the three deltas".

Figure 7 shows the remainder of Alice's proof that the product of continuous functions is continuous, that is, that the her chosen δ "works".

Alice's Encounter with Semigroups Begins

By our 25th meeting, Alice had completed the real analysis section of the notes, and was ready to begin the abstract algebra (semigroups) section that starts with the definitions of binary operation and semigroup, followed by requests for examples. She provided only the most obvious of examples, such as the integers under addition or multiplication, and when asked for something "stranger", she said she could use the real numbers. When asked for another "strange" example "with no numbers at all", she suggested union as the binary operation, and with help, wrote up the example of the power set of a set of three elements. Next, when it came to providing examples of semigroups, she suggested the natural numbers with subtraction, but had to be prodded to check associativity; for this she considered (3 - 7) - 2 versus 3 - (7 - 2) and correctly inferred this was not a semigroup. To provide examples of left and right ideals, Alice needed to come up with a noncommutative semigroup, but she drew a blank. We suggested the semigroup of 2×2 real matrices under multiplication, and for an ideal, the subset of matrices of the form $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$. After some calculation, Alice correctly concluded the subset is a right ideal, but not a left ideal.

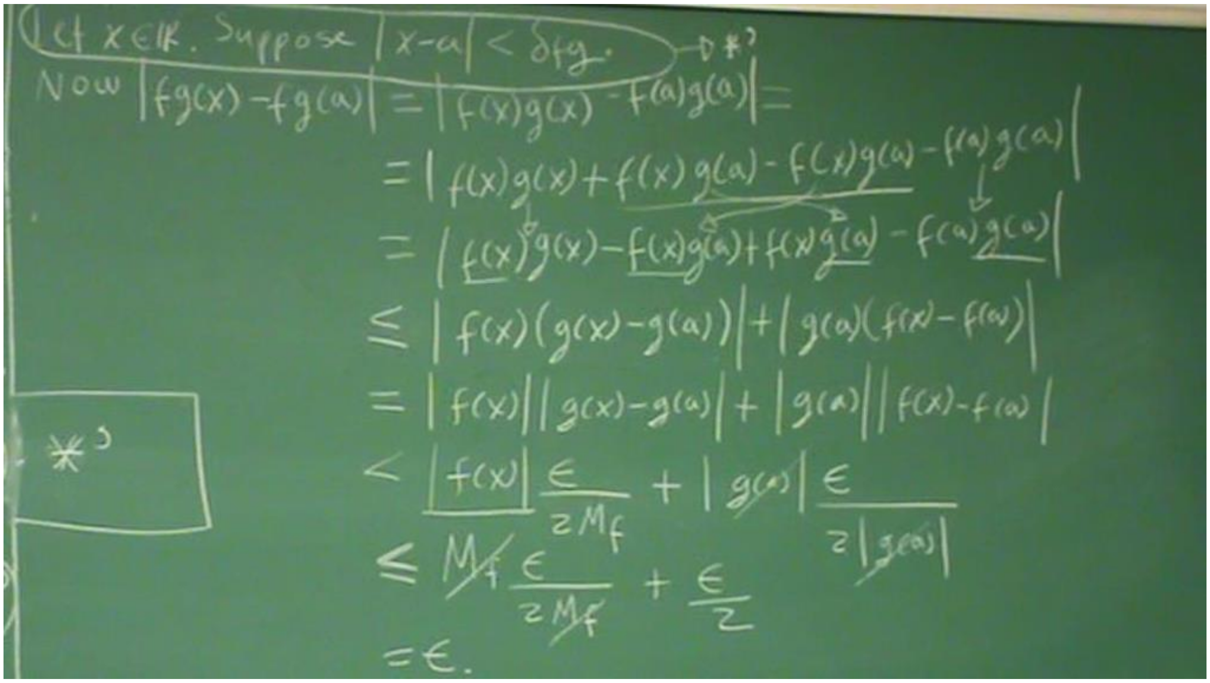

Figure 7. The remainder of Alice's proof that the product of continuous functions is continuous, showing that her chosen δ "works".

Alice Hits a "Brick Wall" with the First Semigroup Theorem

Alice continued considering examples for the first 35 minutes into the next (26th) meeting, after which she came to the first semigroup theorem to prove: "*Let S be a semigroup. Let L be a left ideal of S and R be a right ideal of S. Then* $L \cap R \neq \emptyset$." She first wrote the definitions of semigroup, left ideal, right ideal, and ideal on the left-hand board, as she had done many times before. Then she wrote the first-level framework on the middle board, after which she went to the right-hand board and began doing some scratch work, which included drawing a Venn diagram of two overlapping circles, L and R, with an arrow pointing to the intersection. She wrote in her scratch work "$L \cap R = \emptyset$" and "there exists an element $a \in L \cap R$". With this, it seemed that Alice was trying to clarify the theorem statement for herself. However, she had not yet attended to the second-level framework. We pointed this out.

During the rest of her proving attempt, we seemed to need to remind Alice of relevant actions, such as considering what she knew about ideals (i.e., that they are nonempty), and hence, concluding that each of L and R contains an element, which she labeled l and r respectively. Then, using those, she tried to "explore" to find an element in $L \cap R$, in order to conclude it was not empty. With our guidance, Alice finished the proof, but her sense of self-efficacy seemed shaken.

Indeed, at the next (27th) meeting, Alice wanted to reprove the theorem about the intersection of left and right ideals before continuing. (See Figure 8).We now feel that she had been somewhat overwhelmed, or confused, by the new content, perhaps causing working memory overload. She had tried to cope as best she could by concentrating on the new concepts, while “forgetting” her prior proof writing skills. Alice’s hesitant behavior continued for eight more meetings.

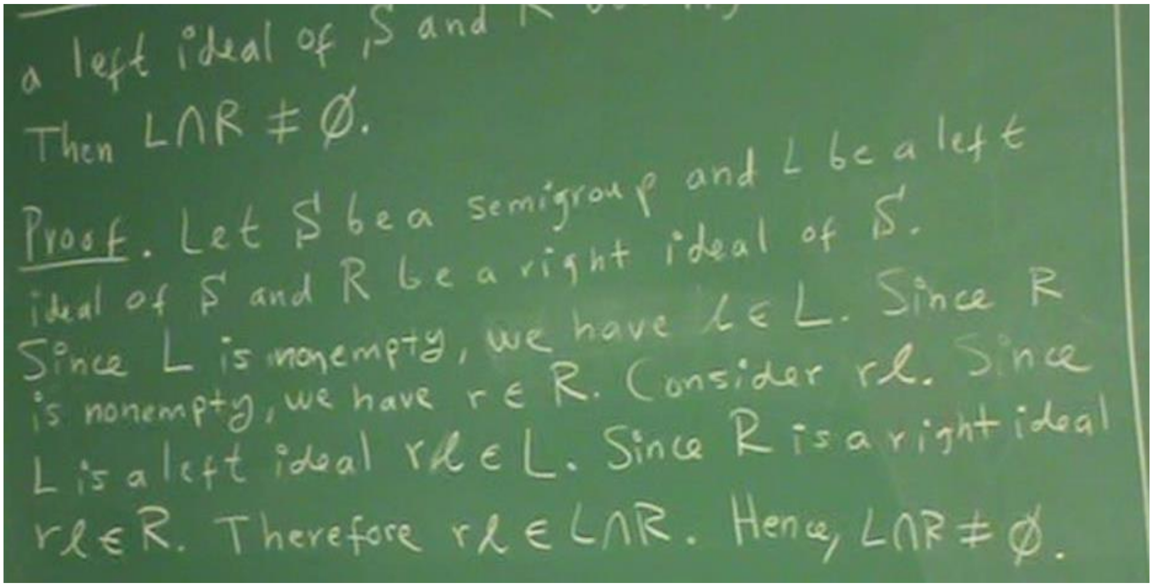

Figure 8. Alice’s proof that the intersection of a left ideal and a right ideal is nonempty.

Alice Regains Her “Footing”

Then, during the 35th meeting, Alice considered the theorem, “*If S and T are semigroups and f: S→T is an onto homomorphism and I is an ideal of S, then f*(*I*) *is an ideal of T*”. She wrote the definition of homomorphism on the left board, wrote “What I know” on the right board, constructed the first-level proof framework, unpacked the conclusion, wrote the second-level proof framework, and decided to do a two-part proof – one part for left ideals and a second part for right ideals. (See Figure 9). With this, Alice seemed to have regained “her footing”. At the next meeting, she finished the proof, with some help from us. She continued proving semigroup theorems for the rest of the semester, exhibiting increasing proficiency and self-efficacy.

Discussion

Working with Proof Frameworks

Alice came to us with a reasonable undergraduate mathematics background, some of which she had forgotten. At the first meeting, we explained the use of proof frameworks and our rationale for using them, and she practiced producing several of them. However, at the second meeting she told us that she found this way of working confusing. When she attempted her own alternative method of proving, she got into difficulty, and as a result, was more willing to try using proof frameworks again. Over the course of our subsequent meetings during the first semester, Alice became fluent with writing both first- and second- level proof frameworks and adopted her own methodical way of working. As the first semester went on, she was able to complete proofs with less guidance from us. Indeed, she often mainly required some help with the problem-centered parts of proofs.

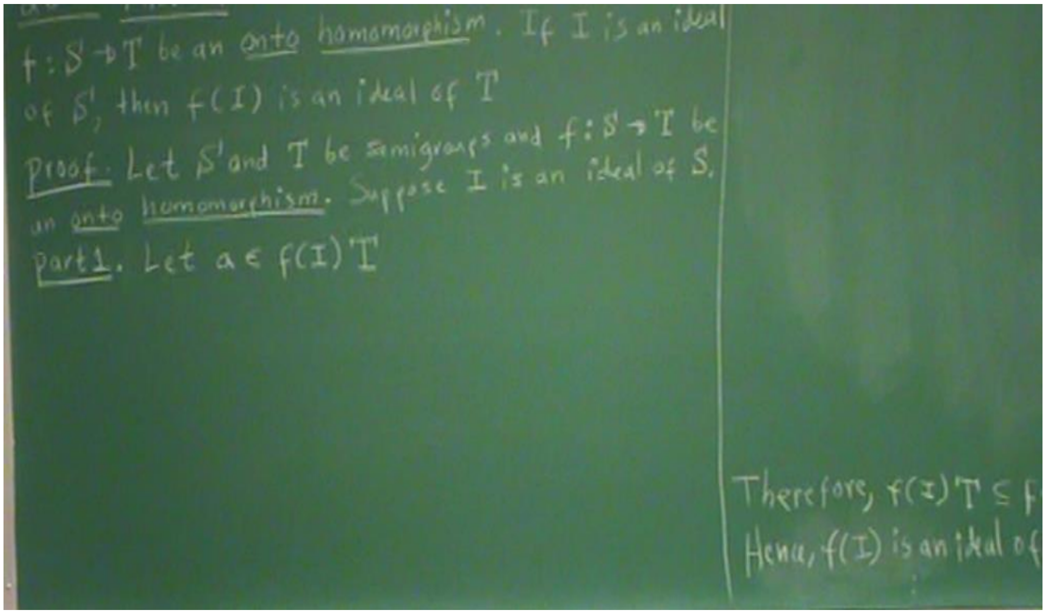

Figure 9. Alice's proof framework for the theorem about homomorphic images of ideals.

During the second semester, Alice continued meeting with us and working on the course notes. She began the second semester with the construction of additional real analysis proofs and seemed to be making very considerable progress, both with writing proof frameworks and with the harder problem-centered parts of proofs. By the end of the real analysis section of the course notes, we felt that she had developed greatly in her proving ability and had developed a sense of self-efficacy (Bandura, 1994, 1995) about proving.

Difficulties Surface When Encountering Unfamiliar Content

However, the subsequent introduction of unfamiliar, abstract content in the form of several definitions and a theorem about semigroups at the 25th meeting seemed to cause her confusion. She constructed only the most obvious examples somewhat hesitantly. Also, when asked to prove the first theorem about semigroups, she did not begin by producing a proof framework, as she had previously consistently done with the real analysis proofs, but rather began writing what she knew or could find in the notes, on the right-hand blackboard. Her proof construction, while not top-down, seemed to consist of first trying to gather as many semigroup ideas as she could, followed by trying to arrange them into a final proof. We feel that concentrating on understanding the unfamiliar abstract content was Alice's initial way of coping, that her working memory may have been overloaded, and that she wasn't able to deal with the additional onus of constructing a proof framework. It was not until the 35th meeting, almost at the end of the second semester, that Alice seemed to have regained her sense of self-efficacy, and she again constructed proofs using the technique of proof frameworks that she had learned and perfected previously with real analysis proofs.

Implications

It seems that coping with newly introduced abstract concepts is not easy, even for someone as experienced as Alice. It also seems that one cannot expect, having learned the skill of constructing proof frameworks in more familiar settings, that this skill will be easily invoked while new abstract content is being learned, perhaps due to working memory overload.

Fragility of Recently Acquired Proving Skills Can Be Overcome

Amongst other things, this case study illustrates the fragility of recently acquired proving skills, in the context of the acquisition of new abstract mathematical concepts. It also suggests that, with persistence, the difficulty due to such fragility can be overcome. Our own experiences as mathematicians suggests that one can (implicitly) learn not to be greatly disturbed by the introduction of several new abstract ideas at once. However, some school curricula avoid certain introductions of concepts, such as the Bourbaki definition of function, because they are considered too abstract (Tabach & Nachlieli, 2015). Further, as Hazzan (1999) found, students sometimes cope by "reducing the level of abstraction." Yet Alice's case suggests that, with time, effort and persistence, students can learn to cope with abstraction.

Eventual Successful Use of Proof Frameworks, along with Persistence and Self-efficacy

The initial tendency of many university students to write proofs in a top-down fashion tends to fade after sufficient exposure to writing proof frameworks. One might ask where this tendency comes from. According to Nachlieli and Herbst (2009), it is the norm among U.S. high school geometry teachers to require students, when doing two-column proofs, to follow every statement immediately by a reason. This implies top-down proof construction. However, as noted previously (Selden & Selden, 2013), automating the actions required to write the formal-rhetorical part of a proof (i.e., writing first- and second-level proof frameworks) can allow students to "get started" writing a proof and exposes the "real problem" to be solved in order to complete the proof. For this, persistence and self-efficacy are needed.

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Questioning Assumptions about the Measurability of Subdomains of Mathematical Knowledge for Teaching

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One common approach to assessing mathematical knowledge for teaching (MKT) is designing items to measure individual subdomains of MKT as specified by a theoretical framework, with factor analyses confirming (or disconfirming) hypothesized subdomains. We interpret this approach as adhering to a "compartmentalized" view of MKT, as opposed to a "connected" view of MKT. We argue in this paper that a compartmentalized view of MKT is embedded in the ways that frameworks are represented, discussed, and used in the field, but that this view of MKT may unintentionally undermine understanding of MKT, and in turn, how to measure and cultivate it in teachers. Using an analysis of nine items previously shown to assess MKT, we illustrate that the tendency for practice-based items to capture multiple subdomains is a common issue across frameworks, and perhaps one that is a necessary result of their design.

Key words: mathematical knowledge for teaching, content knowledge for teaching

We define mathematical knowledge for teaching (MKT) in broad terms as the content knowledge used in recognizing, understanding, and responding to mathematical situations, considerations, and challenges that arise in the course of teaching mathematics. In the 1980s, Lee Shulman advanced a framework to describe a professional knowledge base for teaching (e.g., Shulman 1986), and since that time there have been persistent efforts to describe that knowledge base in greater detail, with particular interest among mathematics educators in the dimensions relating to mathematical knowledge. Scholarly attention to the construct of MKT emerged from work in the 1990s led by Ball and Bass at the University of Michigan in which a close qualitative analysis of video records of teaching allowed researchers to build a descriptive account of the *mathematical* work of teaching, documenting ways in which the work of teaching demands that teachers invoke mathematical knowledge in ways that are quite different than the ways their students do so (Ball & Bass, 2003). In the 2000s, Ball and colleagues described a resulting framework for MKT that subsumes Shulman's content related categories of *subject matter knowledge* and *pedagogical content knowledge* (PCK) and in which particular attention is called to the components of MKT that are both purely mathematical and "unique to teaching" (Ball, Thames, & Phelps, 2008, p. 400).

Another product of this line of research has been the development of assessments to measure MKT and its sub-components, the most well-known of which, the Learning Mathematics for Teaching (LMT) assessments, was developed by the Michigan team beginning in 2000 (LMT Project, n.d.), and has been recognized in the field as an example of the state of the art in terms of assessing MKT (National Research Council, 2013). As interest in both the construct and its measurement has increased, a number of competing theoretical frameworks have been introduced, often based loosely on Ball et al. (2008) but differing in focus or intended use (Heid, Wilson, & Blume, 2015; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012), or focused on differing areas of content by extending the framework to secondary or post-secondary levels (Baumert & Kunter, 2013; Krauss, Baumert, & Blum, 2008) or narrowing to particular bands of content (Herbst & Kosko, 2014).

In this paper, we call attention to two ways one might think about these frameworks – *compartmentalized* or *connected*– and discuss the implications for assessment of and teacher learning of MKT. One view, the *compartmentalized* view, is that frameworks map out distinct knowledge domains, with clear boundaries, each of which can be conceptualized, organized, taught, and measured individually, with separate tests for each domain, and each of which might rely on separate coursework to help novice teachers learn the content of the domain. A second view, the *connected* view, is that these frameworks describe important features of MKT in ways that help us understand it better, but that these features may not correspond to clearly non-overlapping portions of the knowledge, and that the overlap may necessarily play out during the work of teaching as teachers draw on connected knowledge. We argue in this paper that the first view is embedded in many of the ways that various MKT frameworks are represented, discussed, and used in the field, but that in adopting this compartmentalized view of MKT we may unintentionally undermine our understanding of MKT, and in turn our understanding of how to measure it and cultivate it in teachers.

The study we present here is not a critique of MKT frameworks, but rather an attempt to step back from the ways in which those frameworks have come to be used and consider this key distinction, between compartmentalized knowledge domains and connected knowledge domains. We designed the study to examine instances of knowledge application, using brief MKT assessment items to which respondents must apply their MKT in solving a problem of practice, and we explore the extent to which these instances of knowledge application involve applying compartmentalized or connected knowledge domains of MKT, looking across a set of five frameworks. We offer the analysis as an illustration of the degree to which MKT knowledge domains may be naturally connected in situations that approximate actual teaching. We suggest that the compartmentalized view may be incompatible with how assessments are designed, especially when the design is based on a connected view of MKT. We suggest that the connected view has implications for teacher education, including how to support to practice-based approaches that engage pre-service teachers in the work of teaching rather than attempting to provide coursework tailored to distinct domains of knowledge.

Theories of MKT and MKT Measurement

In this section, we discuss MKT frameworks in the literature. We define practice-based item design, a widely shared approach to measuring MKT, and provide an example MKT assessment item. We discuss the emergent empirical research base supporting claims that practice-based assessment items do indeed measure MKT as intended, and revisit our distinction between compartmentalized and connected views of MKT.

Theoretical Frameworks for MKT

In recent years, a number of frameworks for MKT have emerged, building on Shulman's (1986) theorized knowledge classifications of subject matter knowledge and pedagogical content knowledge (PCK). The predominant framework, particularly in the US, is that of Ball et al. (2008). Other frameworks generally overlap with, draw on, or complement the Ball et al. framework, and for the most part they likewise maintain Shulman's high-level distinctions but differ in whether and how they define any distinctions below that level. For example, the Cognitively Activating Instruction (COACTIV) framework (Baumert & Kunter, 2013) draws directly on the Ball et al. theory, but COACTIV only distinguishes between content knowledge and PCK in their accompanying assessments, essentially ignoring some of the subdomains that Ball et al. consider most crucial from a theoretical standpoint.

Several frameworks focus directly on subsets of content or particular grade levels. Examples include Knowledge of Algebra for Teaching (KAT: McCrory et al., 2012), focused on Algebra; COACTIV and Teacher Education and Development Study in Mathematics (TEDS-M: Tatto et al., 2008), focused on upper and lower secondary level content respectively; and MKT-Geometry (Herbst & Kosko, 2014), focused on secondary level geometry. KAT also differs from a number of other frameworks in its underlying organizational structure, which diverges from Ball et al. (2008) by categorizing knowledge according to where in a typical university trajectory a prospective teacher is likely to have learned that knowledge. And a number of frameworks are organized to support quite different aims than assessment development, including the Mathematical Understanding for Secondary Teaching (MUST) framework, which is designed to support professional development and teacher education (Heid et al., 2015), and the Knowledge Quartet, which focuses on identifying situations in which mathematical knowledge is used in teaching (Turner & Rowland, 2008).

Practice-based Items: A Different Kind of Theory

One could argue that part of the reason early theories of MKT were compelling was their grounding in analysis of the work of teaching, a kind of job analysis technique examining the work teachers need to do, generating descriptions of what they called the *mathematical work of teaching* (Thames, 2009), and working back from there to determine the knowledge teachers need to have in order to do that work. Early work by Ball and Bass (2003) described this as "practice-based theory" in that it was developed through the study of teaching practice. The term "practice-based" has also been used to describe assessment items developed out of this theory, and while it is often left unspecified what characteristics make an item practice-based, those who do describe it focus on close approximation of the cognitive work of teaching as the defining characteristic. Phelps, Howell, Mikeska, Kirui, and Mislevy (in press) describe the early LMT items as practice-based in that they "are designed to situate the test taker in a context of practice" (p. 9). Bridging multiple content areas, Gitomer, Phelps, Weren, Howell, and Croft (2014), Selling, Garcia, and Ball (2016), and Mikeska, Phelps, and Croft (in press) describe item development efforts across content areas, all of which are guided by frameworks organized around the work of teaching are used to develop practice-based assessment items. Hill (2016) also cites the use of such frameworks as critical.

Implicit in the cited assessment development efforts is an approach we call a *practice-based item design theory*, in which the design of such items is intended to engage the test taker in some approximation of the cognitive work of teaching, asking the test taker, for example, to analyze a student work sample or choose an appropriate problem for a particular instructional goal. By doing so, items are intended to prompt test takers to apply their mathematical knowledge as they would in such a teaching situation. By requiring test takers to apply their knowledge to a teaching situation, the item is intended to more accurately capture MKT, which be described as a form of applied knowledge (Bass, 2005; Stylianides & Stylianides, 2014). Perhaps because of the widespread influence of the LMT assessment as a model, most MKT assessment items currently available in the field show evidence of a practice-based item design theory approach (Hill, 2016), although there are differences in the ways that various efforts provide context to situate the test taker (Phelps et al., in press).

What is more subtle about practice-based item design may be the underlying way in which it defines MKT according to the contexts in which it is used. MKT, as these items assess it, is defined by the work of teaching that invokes it. The items are *practice*-based in that the organizing unit is work done in teaching *practice*, and what is assessed is what we will call the associated package of knowledge required to engage in that work. It has been argued elsewhere that this organization around the work of teaching is a natural organization, in that it makes sense to teachers and allows them to apply their knowledge in ways that they would in their teaching (Phelps et al., in press), and it "brings the development of measures closer to the actual practices we hope teachers will successfully master" (Hill, 2016, p. 5). Because items are organized around the work of teaching, they are designed to require application of knowledge, including recognition of what to pay attention to and the ability to bring relevant information to bear as well as simply knowing that information. And this plays out in response patterns. Howell, Phelps, Croft, Kirui, and Gitomer (2013) highlighted an interview response pattern in which respondents discussing an MKT item clearly hold propositional knowledge about a common student error pattern but are unable to recognize an instance of that error pattern in student work. A test designed simply to measure knowledge of student error patterns might miss critical information that the practice-based item, by requiring applied knowledge, reveals. One can easily appreciate the argument that this applied knowledge is critical to the work teachers do.

We discuss the practice-based item design theory in some detail for three reasons. First, it will help the reader to understand the items that we analyze in this paper, which are written to a practice-based item design theory. Second, authors of each of the frameworks examined in this study describe their theoretical development as practice-based. Third, we point out that this is a fundamentally different type of theory than the knowledge theories described in the prior sections, one that does not necessarily compete with or contradict knowledge frameworks. One goal in this paper is to illustrate that practice-based item design theory and frameworks of knowledge domains within MKT are not incompatible unless one assumes a direct mapping between assessment items and knowledge domains, as the compartmentalized view invites.

During a lesson near the end of a unit on quadratic functions, Mr. Swift asked his students how many ordered pairs are needed to determine a quadratic function. Some students in the class responded as follows.

Andrew: Well, I could draw lots of different parabolas that are functions through two points by changing where the maximum or minimum is located, but once I have a third point only one of those parabolas would work.

Bria: But what if all three points are on the same side of the vertex? You can determine the function with three points, but at least one point needs to be on each side of the vertex.

Cynthia: That isn't right. All you need are the two roots of the function. Like if the two roots are a and b, you can just do $(x-a)(x-b)$ and you know what the function is.

Daniel: But you wouldn't be able to tell where the maximum or minimum is if you only have the roots. You gotta have three points, and they gotta be the two roots and the vertex.

Which of the students demonstrated the *best* understanding of the ordered pairs needed to determine a quadratic function?

(a) Andrew (b) Bria (c) Cynthia (d) Daniel

Figure 1. Swift item: An example of a practice-based item.

Figure 1 presents an example of practice-based item that we analyzed in the present study, the Swift item. This item requires the test taker to decide which of four students has demonstrated, in his or her comments, the best understanding of the ordered pairs needed to determine a quadratic function. To do so requires a subtle mix of understandings. At stake, mathematically, is the fact that the students are attending to both the number and the nature of the points that determine the function. The test taker must first figure out, on the basis of students' statements, the claim each student is making. He or she must then evaluate the accuracy of each statement, taking into account that each statement may simultaneously demonstrate both strengths and weaknesses. The test taker has been asked to evaluate the relative strengths of the students' understandings, not just determine mathematical correctness.

The Swift item exemplifies practice-based item design. It asks the test taker to engage in a constrained version of the work of teaching. Analyzing student statements in terms of their mathematical strengths and weaknesses is something that teachers must do routinely, which requires not only a deep mathematical knowledge, but also an ability to apply that knowledge to a situation in which multiple ideas may be on the table, and those ideas may be expressed inchoately. In this case, the best understanding is that shown by the first student, Andrew, who indicates that three ordered pairs are needed and that they need not be any particular pairs, and who gives a reasonable if incomplete justification in terms of a visualization of the possible graphs.

Evidence that Practice-based Measures Capture MKT and Its Subdomains

There is an emerging research base supporting the conclusion that practice-based measures such as those cited above capture the overall construct of MKT as intended. Studies have examined records of the reasoning elicited by assessment items seeking qualitative evidence of MKT being used as theorized (Howell et al., 2013), or observing differences in how members of known groups such as teachers, mathematicians, mathematics doctoral students, or mathematics majors reason about the items or perform on practice-based measures (Buschang et al., 2012; Hill, Dean, & Goffney, 2007; Krauss et al., 2008). A number of studies link performance on MKT measures to student achievement (Baumert et al., 2010; Hill et al., 2005; Rockoff et al., 2011). These studies are often cited as evidence that MKT is important to teaching, and they also provide additional evidence that the measures capture MKT as theorized, because part of that theorization is that it is knowledge that underlies effective instruction and, in turn, student learning.

Across efforts to assess MKT, a common practice has been to apply factor analysis to determine whether items hypothesized to measure distinct subdomains of the MKT construct do so. Factor analysis, generally speaking, is a method in which data gathered from assessment items are used to describe the measured construct(s) in terms of a manageable number of hypothetical variables, or factors (Lester, Inman, & Bishop, 2014). Factor analysis can be confirmatory or exploratory, and researchers often use both together. In confirmatory factor analysis, researchers pre-specify factors so that they can "test some theory about the number and nature of the factor constructs needed to account for the intercorrelations among the variables being studied" (Comrey & Lee, 2013, p. 4). In exploratory factor analysis, researchers use intercorrelations to "explore the underlying dimensions of a given data set" (Lester et al., 2014, p. 61) and generate conceptual explanations, after the fact, for the factors that emerge. Applications of factor analysis can be useful for theory testing (Hill, 2016), as they permit generalization to theory from the item pool, but only to the extent that items can be successfully produced to capture the theorized knowledge.

Results of studies applying factor analysis to assessments of MKT have been mixed. For example, Hill, Schilling, and Ball (2004) found that some items designed to measure attributes of PCK loaded instead as content, some as intended as PCK, and some as both. Similarly mixed findings have been reported in other studies spanning mathematics and other content areas (e.g., Floden & McCrory, 2007; Herbst & Kosko, 2014; Phelps & Schilling, 2004). Our perspective is that this set of analyses, across multiple practice-based measures, suggests that practice-based items are likely to measure multiple subdomains of the construct.

The Critical Distinction between Compartmentalized and Connected MKT

We return here to the critical distinction between compartmentalized and connected views of MKT. We argue that the compartmentalized view is supported in many ways by the manners in which framework authors and other scholars have presented and used the frameworks. Frameworks are often represented in graphics as area models in which non-overlapping sets are joined together to make up the whole of MKT. Ball et al. (2008), as an example, call careful attention in textual descriptions to what they call the "boundary problem" (p. 403), that knowledge domains are often closely coordinated in the work of teaching and so it is difficult to discern the precise division between domains. However, the powerful heuristic of the graphical representation by Ball et al., with its unambiguous divisions, tends to be the reader's main takeaway, rather than fuzzy overlap among domains; the representation strongly suggests mutually exclusive categories with clear divisions. And it is not incorrect to emphasize distinctions; in fact, the distinctions between knowledge types such as common and specialized content knowledge are what make them conceptually powerful ideas. The fact that specialized content knowledge exists as a construct that is distinct from both PCK and common content knowledge is an important and influential idea. Nonetheless, the distinctiveness of specialized content knowledge does not imply that it can be applied, learned, or measured entirely separately from common and pedagogical content knowledge.

Factor analysis approaches clearly adopt a compartmentalized framework view, taking as given a test design logic that maps the items based on a framework directly onto subdomains of that framework. Researchers of these studies often design items for the explicit purpose of assessing subdomains, sometimes on separate tests. These researchers then use factor analysis in attempts to answer underlying questions to the effect, are subdomains empirically distinguishable? However, test design logic may simply be incompatible with practice-based item design theory because one relies on a compartmentalized view and the other on a connected view of MKT. The mixed evidence resulting from factor analysis across multiple assessment efforts may reflect this incompatibility.

Our analysis takes a different approach to the question by asking, in responding to a set of nine practice-based items, how knowledge domains across multiple frameworks are elicited, and whether, under any of these frameworks, they are invoked in isolation from one another. Our logic is as follows: by selecting strong items known to exemplify practice-based item design and to measure MKT, we create a common comparison point across the frameworks. For each item under each framework, we examine which subdomains of that framework the item measures, noting in particular how many subdomains are measured. Our goal is to illustrate that the tendency for practice-based items to capture multiple subdomains is a common issue across frameworks, and perhaps one that is a necessary result of practice-based item design. In the following section we describe the items and frameworks we selected for the analysis and describe our coding procedures. We then present our results table and discuss the implications with respect to the frameworks and with respect to assessment item design.

Methods

We examined a set of nine assessment items designed to measure mathematical knowledge for teaching (Howell, Lai, Miller, & Phelps, 2016). These items were selected for several reasons. The items are typical representations of practice-based item design. A prior study provided evidence that this set of items captured MKT as they were designed to and demonstrated no significant design flaws (Howell et al., 2016). Additionally, as part of this prior work, researchers generated knowledge maps for each item specifying the knowledge measured, and these knowledge maps were validated by comparison to responses captured in think-aloud interviews. In the present study, the items themselves became the data, which we then coded by subdomains of each of the five specified frameworks. For the purposes of this study, the responses to each item were used to validate that the knowledge maps were accurate. The analysis presented in this study is not of responses to the items, but rather of the knowledge maps of the items. An example of the analysis is given below for the Swift item. Before the example analysis, we discuss our selection process for MKT frameworks to use and the coding process.

Framework Selection

In this study, we considered five theoretical frameworks for MKT: Ball et al.'s (2008) MKT framework; KAT (McCrory et al., 2012); COACTIV (Krauss et al., 2008); TEDS-M (Tatto et al., 2008); and MUST (Heid et al., 2015). We restricted our analysis to frameworks that are fully developed in available, seminal documents, which directly claim to describe the MKT construct, and for which the framework describes the entire construct. We did not restrict framework selection to those explicitly designed for secondary level MKT, but did exclude those that could not reasonably be taken to describe the MKT measured in our item set. For example, we excluded the MKT-Geometry framework because our item set did not include geometry items.

Operational Definitions

We examined seminal documents for each of the five frameworks to extract operational definitions of key subdomains. We added clarifications throughout the coding process where authors' original language was insufficient to inform coding decisions, and did so for all frameworks. A list of source documents for each framework and subdomains coded is provided in Table 2. We illustrate this process for one framework (Ball et al., 2008).

The article, "Content knowledge for teaching: What makes it special?" (Ball et al., 2008), lays out clear definitions for six subdomains of MKT and discusses the authors' sense of the relative clarity of each. Ball et al. first subdivide all of MKT into CK (content knowledge) and PCK (pedagogical content knowledge), following Shulman's (1986) distinctions. They then further decompose each of these. CK is decomposed into *common content knowledge*, *specialized content knowledge*, and *horizon content knowledge*. Common and specialized knowledge are distinguished from one another by context of use, and the authors stress that this is a key distinction, suggestive of a body of professional mathematical knowledge needed by teachers alone. Each is a form of "pure" mathematical knowledge and explicitly excludes knowledge of students or teaching, but *common* is knowledge that is needed in settings other than teaching, whereas *specialized* is knowledge that is only needed for the teaching of mathematics. *Horizon* content knowledge is defined somewhat more ambiguously, as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403), but the authors also specify that it is a less robustly theorized knowledge type than the other two and, in fact, that it might be a different way of describing a component of PCK or a type of specialized content knowledge.

Given the strength of the claim made by Ball et al. that the distinction between common and specialized CK is important and their hesitance about the construct of horizon content knowledge, we decided to code for common and specialized but not horizon content knowledge. We planned, in the course of coding, to flag any cases which suggested the missing code might be problematic, but no such cases were noted. An ambiguity we noted in the distinction between "common" and "specialized" is the under-specification of what it means for knowledge to be needed in "settings other than teaching," (Ball et al., 2008, p. 399). We operationalized this distinction as *professional work other than teaching that requires similar levels of educational preparation; for secondary level topics, this is work that requires a math degree or equivalent*.

We note here that early in this process we were faced with the need to make a choice about coding subdomains of PCK. Several frameworks included a subdomain called PCK, making explicit reference to Shulman's (1986) definition of the term and delineating similar subdivisions within PCK, but two of these frameworks (COACTIV and TEDS-M) elected not to differentiate subdomains of PCK in their assessment design. And while the Ball et al. (2008) framework does define subdomains of PCK, most assessment efforts based on this framework have focused on differentiating between common and specialized knowledge, with less focus overall on measuring PCK and its subcomponents. Given these considerations, we elected not to code subdomains of PCK.

Table 2. MKT frameworks examined.

	Ball et al. (2008)	**Knowledge of Algebra for Teaching (KAT)**	**Cognitively Activating Instruction (COACTIV)**	**Teacher Education Study in Mathematics (TEDS-M)**	**Mathematical Understandings for Secondary Teaching (MUST)**
Subdomains examined (subdomain most similar to PCK listed in same row)	Specialized content knowledge Common content knowledge	School algebra Advanced mathematics	Content knowledge	Mathematical content knowledge	Mathematical proficiency Mathematical activity
	Pedagogical content knowledge	Algebra-for-teaching knowledge	Pedagogical content knowledge	Pedagogical content knowledge	Mathematical context of teaching
			General pedagogical knowledge		
Key Citation	Ball et al. (2008)	McCrory et al. (2012)	Baumert & Kunter (2013) Krauss et al. (2008)	Tatto et al. (2008)	Heid et al. (2015)

Coding

As described above, each item had a previously produced knowledge map, which included a summary of the key knowledge, skills, and reasoning required to respond to the item. Once an initial code list was developed from seminal documents of each framework, these codes were in turn applied to each of the identified pieces of knowledge, skill, or reasoning for each item..A codebook was created and used in order to maintain adequate rigor (MacQueen, McLellan, Kay, & Milstein, 1998; Saldaña, 2012), and, following Creswell's (2009) suggestion to document as many steps as possible, the coding sheet included a note section for the coders to keep a record of their reasoning for assigning certain codes to an item, any questions they had, the decisions made, and additional observations worthy of attention. For each item and each framework, researchers coded pieces from its knowledge map for the subdomain(s) of the framework that the piece represented. Coding was done independently, with two coders per item, and a third coder brought in to arbitrate discrepancies as needed. At least one author coded each item; two colleagues familiar with research on MKT served as additional coders as needed. Initial inter-rater reliabilities were generally adequate, ranging from .69 to .97, but our goal in this coding was less to establish reliability in coding and more to produce a consensus around accurate final coding. Because we drew on codes extracted from the selected frameworks, we were limited in how far we could develop our code list without compromising fidelity to each framework's author's intentions. Given this, the inter-rater reliability in this study may reflect more than anything else the degree to which each framework's provided definitions were amenable to use as codes. What is more salient to our study is the intercoder agreement, a term introduced by Creswell (2009) to represent the final agreement among researchers after reconciling discrepancies in initial coding. Gibbs (2007) suggested that researchers can minimize bias by first individually coding the same set of data using the same codes, and then discussing the results of the coding with a goal of reaching a common agreement about the meaning and application of each code. Following this method, we attained 100% agreement on final codes.

From coding to results: The Swift example. We illustrate our coding process using the Swift item example presented in Figure 1. The previously generated knowledge map for this item called out the following four distinct components of knowledge, skills, and reasoning that a test taker might draw on in responding to the item:

1. Knowledge that any three ordered pairs is sufficient to determine the equation of a quadratic.
2. Knowledge that the complete formula for a quadratic expression is $c(x-a)(x-b)$.
3. Ability to construct the general mathematical claim implicit in a student's responses in the context of discussion.
4. Ability to evaluate the generalizability and validity of a mathematical claim.

Under the Ball et al. (2008) framework, the first and second points were coded as common content knowledge, and the third and fourth coded as both specialized content knowledge and PCK, reflecting a certain ambiguity in the framework that we were not able to resolve because, as described in Hill et al. (2008), specialized content knowledge and PCK may be used toward similar ends. For example, constructing the general mathematical claim implicit in a student's response might be accomplished from a purely analytical standpoint, attending only to the mathematics represented in the stated claim independently of the idea that it was presented as student work. But it also might be informed by knowledge of common ways that students approach such problems or represent their thinking in words. Under this framework, then, the item was coded as measuring three of the subdomains of the framework: *common content knowledge*, *specialized content knowledge*, and *PCK*.

Under the COACTIV framework, the first point was similarly unambiguously coded as content knowledge, but the coders indicated that the second point did not fall into the framework at all, due to the COACTIV's conceptualization of content knowledge as a "deep understanding of the secondary school mathematics curriculum" (Krauss, Baumert, & Blum, 2008, p. 876). COACTIV's content knowledge category, unlike that of the prior framework, explicitly excludes "the school-level mathematical knowledge that good students have" (p. 876), effectively excluding from the MKT construct portions of what Ball et al. (2008) term common content knowledge. The same definitional distinction shifted the coding of the third point, which was less ambiguous under this framework and was coded simply as PCK. And the fourth point was again double coded as content and PCK but for a different reason, as deep mathematical knowledge as defined by COACTIV would include the ability to evaluate mathematical claims in general. Under this framework the item was coded again as measuring two of the subdomains: *content knowledge* and *PCK*. Similar patterns of coding occurred across the remaining frameworks.

Results and Discussion

Number of Subdomains Assessed by Framework and Item

Table 1 shows the number of subdomains of each framework assessed by each of the nine items. There was some variation by item. For example, the Hillyard item measured a single knowledge subdomain (PCK) under four of the five frameworks, and measured multiple subdomains under only one, the MUST framework (Heid et al., 2015), where it was coded as measuring both *mathematical activity* and *mathematical context*. By way of contrast, the Swift item measured multiple subdomains under all five frameworks, indicating that it measures a blend of knowledge regardless of which framework is selected, and in fact measured three subdomains in all but the COACTIV framework (which contained only two content-specific subdomains) and the TEDS-M framework (which contained only two subdomains).

One explanation for our results is that this set of items was designed to capture MKT by following a practice-based item design theory. From the practice-based perspective, what makes an assessment task effective in measuring MKT is how closely it represents the work of teaching and, hence, the item is considered successful based on how well it represents "the actual practices we hope teachers will successfully master, rather than the more-slippery notion of the 'kinds of knowledge' teachers should possess" (Hill, 2016, p. 5). And because these tasks of teaching are designed to approximate the work of teaching, it stands to reason that, like teaching, a strong assessment item might call on multiple types of knowledge and ask the test taker to coordinate them in application to the work of teaching. Because the set of items we examined were designed to be practice-based, we cannot claim that our findings are likely to generalize to assessment items that follow a different design. However, as Hill (2016) points out, many current assessments (and all of those explicitly based on the frameworks we examined) follow some degree of this practice-based design in which an item focuses on engaging the test taker in key tasks of teaching mathematics. This suggests that the findings may generalize, at least in part, to many current MKT assessment efforts.

Additional Results

We note here one minor result that may be of interest. When multiple frameworks used the same terminology, we found a number of items that were coded inconsistently with respect to that terminology. For instance, three frameworks contain "PCK" as a subdomain. However, coding for PCK in one framework did not guarantee that the item was coded as PCK in all other frameworks, due to how PCK was defined. Moreover, as reflected in the Swift coding examples, there were also some significant differences between frameworks in how mathematical content knowledge was defined. While in the case of mathematical content knowledge, the framework authors did not use identical terminology, the similarity of terminology around content knowledge, which one might reasonably expect to be fairly unambiguous, arguably creates the potential for misunderstanding. These observations mirror those made by Kaarstein (2014), who found similarly divergent categorizations analyzing a different set of items across a subset of the frameworks we considered, and substantiated concerns more broadly that the field may be suffering from an unproductive lack of coherence with respect to our work around MKT (Hill, Sleep, Lewis, & Ball, 2007). While different frameworks may serve different purposes, certainly it is potentially problematic when common terms are used differently in ways that undermine clear communication, and our results provide additional evidence that this may be the case.

Table 1. Number of Subdomains Measured by Item by Framework

	Ball et al.	KAT	COACTIV	TEDS-M	MUST
# subdomains coded	3	3	3	2	3
Item name					
1. Allen	3	2	1	2	2
2. Hillyard	1	1	1	1	2
3. Morgan	3	3	2	2	3
4. Watkins	2	2	1	2	2
5. Carlies	3	2	2	2	3
6. Swift	3	3	2	2	3
7. Rose	3	2	1	2	3
8. Swain	3	2	2	2	2
9. Williams	2	2	1	1	3

Limitations

We caution here against potential overgeneralizations of our results, which we would characterize as more illustrative than generalizable. Our set of items is clearly quite small, and while we intentionally selected items to represent strong examples of practice-based item design theory for which we had evidence of success in capturing MKT, our observations may not generalize to all practice-based items and almost certainly do not generalize to differently designed assessment items. That a practice-based approach is prevalent in the field provides some evidence that it is a fruitful design method, but other approaches may emerge that are equally able to measure MKT and better able to capture its subdomains, or more careful domain analysis and framework development might produce a set of knowledge distinctions that can be measured in clear isolation from one another.

We also note that the evaluation or comparison of frameworks on the basis of our numerical results is inappropriate given the purpose and method of the coding. Frameworks did not have equal numbers of categories to begin with, and our coding decisions may have exaggerated these differences or obscured them. Some frameworks afforded clearer coding decisions than others, but none were designed to support this type of coding, and many of the authors of the frameworks acknowledge some ambiguity in the boundary cases between knowledge types. Not all the frameworks were explicitly designed to support assessment design, and it is also worth noting that none were utilized to design the set of items we analyzed, and it is possible therefore that our items may distinguish subdomains less well than items designed for that purpose under a particular framework might do.

We also call attention to the two items that seemed to measure distinct subdomains best: the Hillyard item, which was coded as measuring one subdomain under most frameworks, and the Williams item, which was coded as measuring one subdomain under three frameworks. These cases illustrate that is clearly possible, even within the set of items we analyzed, for an item to measure a single subdomain under a framework, and it is not our claim that practice-based items measure multiple subdomains, with no exception. We simply claim that measuring isolated subdomains is unlikely to be the typical pattern.

Conclusions

We conducted an analysis of the study items in which we coded each item for the knowledge subdomains measured using five different frameworks that have been proposed in the field. Our results illustrate that this set of items does not measure subdomains of any of these frameworks distinctly, suggesting that the types of MKT elicited by the items are interconnected. We discuss three implications of these results. One implication is that these frameworks, each designed with different and particular intent and potentially useful with respect to that purpose, may not be useful for distinguishing measureable subdomains of MKT. To some degree this is a divide between approaches to theory and approaches to assessment. If one takes a compartmentalized view of an MKT framework, assuming that subdomains must be distinctly measureable in isolation from the larger construct in order to establish the value of the theory, this result could be viewed as problematic. We hold, however, that identification of distinctly measurable subdomains is not the only or even the most desirable purpose of theory; a theorized subdomain need not be distinctly measureable in isolation from the larger construct to be useful in informing the field's thinking, designing policy, or as a heuristic for organizing teacher supports. For example, Ball et al.'s (2008) framework, and in particular the subdomain *specialized content knowledge*, has pushed the field substantially by drawing attention to difficult mathematics that elementary teachers need to know and are not likely to learn in conventional preparation; this attention has changed policy for the mathematical education of elementary teachers (e.g., Conference Board of the Mathematical Sciences, 2001). The KAT framework helps us think about where prospective teachers are likely to learn particular types of knowledge, and by tying their categorization closely to existing course structures for secondary teacher preparation they directly inform what might be the content of such courses. In other words, different frameworks can be powerful in calling attention to different aspects of MKT and of its importance, and while there are many reasons to measure important constructs, it is not obvious that measurability is a necessary indication of a theory's importance or validity. However, much assessment development work in this area has focused on doing precisely this: producing items to measure specific subdomains of MKT in an effort to validate the theory. We hold that this approach is not necessarily optimal if the goal is either to produce valid and reliable assessments of MKT or to generate useful theory, unless the explicit intended use of that theory is to support assessment development.

A second implication is that items written to a practice-based framework are unlikely to measure isolated subdomains of MKT well. Items organized around the idea of capturing a moment in the work of teaching, where the work of teaching of necessity demands that teachers draw on and coordinate complex pieces of various types of knowledge simultaneously, are likely to be successful only insofar as they measure complex and coordinated knowledge. In other words, under this design theory the key to an item's "success" in measuring the MKT domain is its authentic approximation of teaching practice, which may undermine its ability to capture distinct knowledge domains. We nevertheless propose that, given the field's current knowledge, the benefit of a practice-based item design in successfully measuring the MKT construct far outweighs any drawbacks associated with such items not being able to distinguish among theoretical subdomains particularly in cases where there is no clear rationale for the necessity to distinguish subdomains.

Finally, we call attention to the implications for teacher education. Frameworks for MKT effectively called the field's attention to aspects of teachers' content knowledge that are needed but which traditional teacher preparation provides inadequate opportunity to learn, prompting attempts to organize coursework to support that learning (e.g., CBMS, 2001; CBMS, 2012). We would argue, however, that it is crucial to attend to the way MKT occurs in the work of teaching, and not just to particular subdomains of MKT, however important they might be. While the content knowledge that can be called specialized content knowledge, for example, may be important for novice teachers to learn, there is no evidence that courses can be organized to teach specialized content knowledge directly in isolation from other components of MKT, or that the mathematics that makes up specialized content knowledge makes up a mathematically coherent topic area. Programmatic organizations that attempt to do this may be, in essence, missing the point of a practice-based approach. The work of teaching may form a natural organizing structure for teacher learning, just as it does in practice-based items, because the work of teaching is, in fact, the work we want to help teachers learn to do. We would argue that it makes sense to organize teacher preparation around that work in ways that support learning of the whole of MKT, and that we use theory not to decompose what is to be learned into distinct pieces but rather to help us appreciate the depth and complexity of the knowledge needed to do that work.

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Those Who Teach the Teachers: Knowledge Growth in Teaching for Mathematics Teacher Educators

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This theory-based report gives evidence and builds a conceptual framework for a construct called "mathematical knowledge for teaching future teachers" (MKT-FT). Mathematics teacher educators construct MKT-FT as they teach courses for pre-service teachers. Connections to mathematical knowledge for teaching (MKT) are discussed, with an emphasis on the complex relationships among aspects of pedagogical content knowledge in MKT-FT and MKT.

Key words: Mathematical knowledge for teaching, Discourse, Teacher educators

In the 30 years since Shulman's (1986) seminal speech on the importance of pedagogical content knowledge, a variety of theories about such knowledge have emerged (Depaepe, Verschaffel, & Kelchtermans, 2013). Among the most well known in the U.S. is at the heart of a primary-grades-focused model of *mathematical knowledge for teaching* (MKT) introduced by Hill, Ball, and Schilling (2008). The subject matter knowledge (SMK) and pedagogical content knowledge (PCK) components of Ball and colleagues' model of MKT are illustrated in Figure 1.

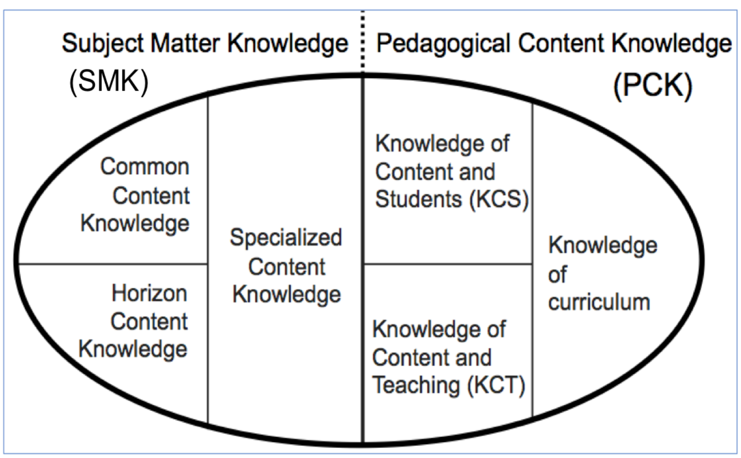

Figure 1. Model of mathematical model for teaching (Hill et al., 2008)

In the context of more advanced mathematics, others have explored how the idea of MKT may be productively refined for use in research and development in secondary and post-secondary settings (Hauk, Toney, Jackson, Nair, & Tsay, 2014; Speer, King, & Howell, 2015). Speer and colleagues considered college instructional questions such as: What are the types of specialized, horizon, and common mathematical knowledge for teaching calculus? While Hauk and colleagues have tackled: How does one unpack the aspects of PCK – knowledge of content and curriculum, content and teaching, content and students – when the teaching is in a college, the students are adults, the collections of mathematics experiences brought to the classroom are larger, and the sociocultural relationships among students, teacher, and institutions are quite different from those assumed in the K-8 foundations of the initial framing of MKT? Hauk and colleagues (2014) developed an expanded model for college teacher PCK. They used the PCK components in Figure 1 as the vertices of the base of a tetrahedron and added a fourth vertex which they called *knowledge of discourse* (knowledge about the nature of communication, including context and valued forms of inquiry, socio-mathematical norms, and language in, for, and through mathematics in post-secondary educational settings) – see Figure 2.

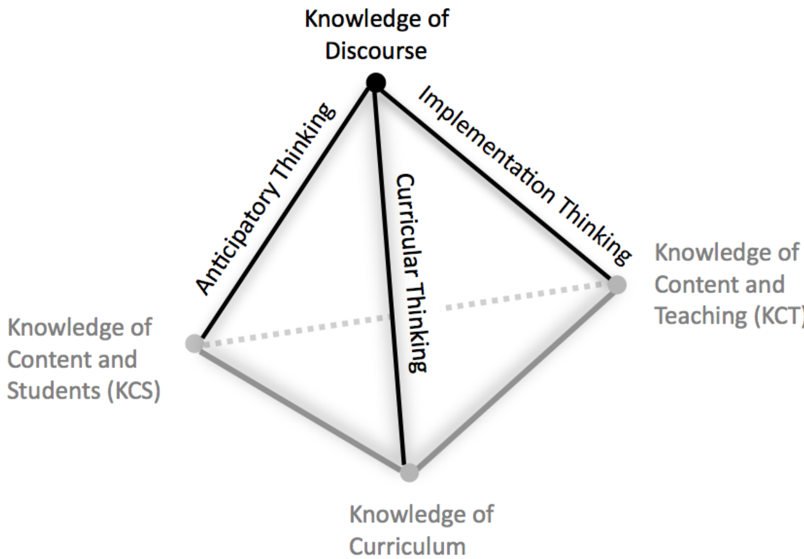

Figure 2. Tetrahedron model of PCK (Hauk et al., 2014).

Context and Researcher Stance

Here we consider an MKT-related question in undergraduate mathematics education: What is the nature of "mathematical knowledge for teaching" for college instructors who teach mathematics for pre-service elementary teachers? Such instructors are teaching adults in post-secondary settings where the mathematical content is in the context of elementary school mathematics (rather than advanced) *and yet* the content is itself linked to MKT for grades K-8.

Indeed, Gallagher, Floden and Gwekwerere (2012) note that we know little about what skills are required to be an effective teacher of future K-8 mathematics teachers nor do we know much about how those skills develop. Here, by *mathematics teacher educator* we mean anyone who provides guidance, mentoring, or professional learning opportunities to prospective or in-service teachers. The current paper focuses on the subpopulation of mathematics teacher educators who teach mathematics-content-rich courses where the learners are pre-service K-8 teachers.

For us, an important aspect of the context of this work are the experiences we bring to it. Below, we outline background that has informed our thinking.

Shandy Hauk. My experience in the last 25 years includes being a classroom teacher in K-6, middle, and high school. The latter as both an English and a mathematics teacher. Others perceive me as a white woman though my early experience growing up in and around Los Angeles was as a "transcultural kid" (ours was one of two pink families in a neighborhood of more than 30 brown families; all were poor). I was raised and have worked in socio-economically and culturally diverse communities. Along the way, I completed a PhD in mathematics (chaos theory and global climate modeling) and a post-doc for research in undergraduate mathematics education. The work I have done with mathematics teacher educators has included research about the professional practices of district-level coaches and teachers on special assignment. Most closely related to the work reported here is a recently completed project where we spent five years looking at the practices and professional development of in-service high school teachers and the mathematics faculty who teach them. In that project, Billy and I, with several others, began our journey of model-development for how to describe the aspects of knowledge (and thinking) about teaching we saw (e.g., Hauk et al., 2014). Billy, JenqJong and I are just starting a new project that develops and researches professional learning about MKT among college mathematics faculty who teach future K-8 teachers.

Billy Jackson. I grew up in the southeastern U.S. as a member of a typically privileged group (white and male). Like Shandy, my PhD is in mathematics and I did a post-doc in mathematics education. As part of my post-doc years, I spent a year as a high school mathematics teacher. An eye-opening experience. My passion in understanding how to unpack MKT and communicate that to other people has grown to include how to represent and communicate MKT to mathematics teacher educators and to people who do research with teacher educators. I have typically had success in mathematics and know the culture of mathematics and now am working in undergraduate classrooms where my professional values are not shared by my students. Being a male mathematician when 90% of the student audience is female (pre-service elementary school teachers) is a challenge. One bias I face is that as a white male mathematician I am not perceived as a "nurturer" by my students. In the classroom, on the first day of class a student told me "I get the sense you are a no nonsense kind of guy and get straight down to business." I have found I am more accepted as a nurturer in a one-on-one environment like office hours, when we have more time and space for negotiating the cultural differences. In a lot of ways, I am the antithesis of what my students perceive about my School of Education colleagues. One of the biggest rewards in working with K-8 future teachers is the great feeling to see learners have aha! moments and ask me "Why weren't we taught this way? It makes so much more sense!"

JenqJong Tsay. I was raised, educated, and initially taught elementary school in Taiwan and completed a PhD in mathematics education in the US. I have spent the last 15 years working in and developing skill in a new culture. It means I can draw on my learning experience about what teaching and classrooms are like in the U.S. and in Taiwan. This comparison has been part of the perspective I have brought to this work. Billy noted that being male when 90% of the preservice teacher student audience is female was a challenge. For me, the more pertinent challenge comes in the perceptions and assumptions students have of me because of my appearance and accent. Most of the time I teach middle and high school pre-service teachers. This year I am teaching the K-8 Number and Operations content course for the first time. My perception is that most of my preservice elementary teacher students do not like, okay to be honest, hate mathematics. This is a challenge for me. My instruction is inquiry-based, so I do not use direct instruction or create procedural guidelines. For most of the students in the K-8 fundamentals of math course, the inquiry-based approach is met with impatience and frustration. My students are not used to the idea of spending time to contextualize mathematics and make sense with mathematics for 20 minutes on one idea. The first few weeks, we spent a lot of time building socio-mathematical norms with some reinforcement every class period of this negotiation. In Taiwan, undergraduates in general, and preservice teachers in particular, are more persistent and concentrated in their study, in part because of their parents' expectation and financial support. Nevertheless, what I have learned from my teaching and research in the U.S. seems to be that conceptual understanding is emphasized more in the preparation of teachers in the U.S. than Taiwan.

We are still building our personal experiences in research with mathematics teacher educators. We rely heavily on the work of others, including the policy and research literature on people who teach in-service teachers (see for example AMTE, 2017 and Bitto, 2015). We also pull from the limited research and practice literature on the nature of professional knowledge among instructors responsible for the collegiate mathematics education of pre-service teachers. It is a strength of our theory and model development work that we have these three distinct sets of practical experiences. At the same time, we may still have blind spots about the theory and we look to the reader and other people in the field to contribute to the triangulation needed in rigorous research.

Importance to RUME Community

A natural question arises: Why should the Research in Undergraduate Mathematics Education (RUME) community have an interest in examining the knowledge required of mathematics teacher educators to perform their jobs effectively? First and foremost, much of the mathematical preparation for teaching among future K-8 teachers happens in colleges, most at the undergraduate level (Masingila, Olanoff, and Kwaka, 2012). In fact, Masingila and colleagues found that 88% of the teaching of "mathematics for elementary teachers" courses happens in mathematics departments. Across institution types (2-year, 4-year, and master's and doctoral degree granting) most faculty teaching these courses have advanced degrees in mathematics (e.g., more mathematics PhDs teach the courses than those with doctorates in mathematics education).

We know that early learning experiences are formative and that children who learn to see themselves as mathematical agents do better in secondary school and beyond (Aud, et al., 2013, Shim, Ryan, & Anderson, 2008; Woodward et al., 2008). We know teaching that supports children in building skills with mathematical process, practices, and content is socio-culturally rich and responsive to community experiences and needs (Aud & KewalRamani, 2010; Gay, 2010; Khisty & Chval, 2002; Téllez, Moschkovich, & Civil, 2011). We know that future teachers have greater resources to draw on and are more likely to offer children what they themselves have experienced as learners (including the undergraduate learning experiences that are most proximal to their launch as teachers; e.g. Ball and Bass, 2000; Conference Board of the Mathematical Sciences, 2012; Hodgson 2001). There is a need for mathematics faculty who are prepared to teach mathematics content courses for pre-service elementary teachers (PSETs) in ways that resonate with the kinds of classrooms those future teachers are expected to sustain.

In the U.S., the current population of instructors for such courses includes adjuncts, graduate students, and full time tenure- and non-tenure-track track mathematics faculty (Masingila, et al., 2012). Large segments of this instructor population have difficulty teaching courses for PSETs (Flahive & Kasman, 2013; Greenberg & Walsh, 2008). Though instructors in mathematics departments usually have a deep mathematical background, they often face challenges teaching content that is relevant and has utility for PSETs, unaware of the "cognitive and epistemological subtleties of elementary mathematics instruction" (Bass, 2005, p. 419).

Given this state of affairs, Masingila, Olanoff, and Kwaka (2012) advocate for the design and implementation of professional development for mathematics teacher educators. Indeed, Masingila and colleagues note that many faculty who participated in their study asked the researchers where they could find professional learning resources!

The RUME community includes experts on such matters. Any design and implementation of effective professional development for mathematics teacher educators must involve attention to identifying the types of knowledge that faculty use and need. In this case, the research and subsequent implications for practice involve the teaching of college students who are on a professional path to teaching K-8 students.

Mathematical Knowledge for Teaching Future Teachers (MKT-FT)

For these reasons, we propose that college instructors possess a specialized constellation of knowledge to be studied: *mathematical knowledge for teaching future teachers* (MKT-FT). We posit that like MKT, MKT-FT is largely individually constructed by mathematics teacher educators while being heavily socially mediated. Seaman and Szydlik (2007) discussed the necessity but insufficiency of the early model of MKT for college mathematics instruction, particularly in the context of teaching future teachers. Several authors have noted the existence of what we see as components of MKT-FT. Zopf (2010) and Olanoff (2011) argued that effective teaching of future teachers requires mathematical knowledge of the work of teaching K-8 mathematics and awareness of the complexities of K-8 MKT itself.

According to Rider and Lynch-Davis (2006) and Smith (2003), the mathematical knowledge needed for teaching future teachers attends to the fact that one is teaching adult learners who have some familiarity with the mathematics (as opposed to teaching children who may be learning content for the first time). And, we note, there is a perceived autonomy of the learner in the post-secondary setting that is largely absent in K-8 and high school contexts. Smith (2005) has claimed that faculty who work effectively with future teachers have some (perhaps implicit) knowledge of educational theory and K-12 practice, as well as knowledge resources for connecting ideas and concepts in ways that prepare pre-service teachers to review, select, and engage with the wide array of curricular decisions that must be made by a schoolteacher (e.g., decisions regarding which resources, worksheets, texts, and activities to use or avoid, decisions about how to orchestrate classroom mathematical discussions). Olanoff (2011) points out that Deborah Ball herself considers MKT to be the analog of "common content knowledge" for faculty "mathematical knowledge for teaching" teachers.

Research and development on the preparation of teacher educators has long assumed a nesting of types of knowledge. One representation of that can be seen in Carroll and Mumme's work (2007). Figure 3 represents the nesting of mathematical content as subject matter knowledge (orange disk), linked to (future) teacher and elementary student within the larger context of the classroom (yellow disk).

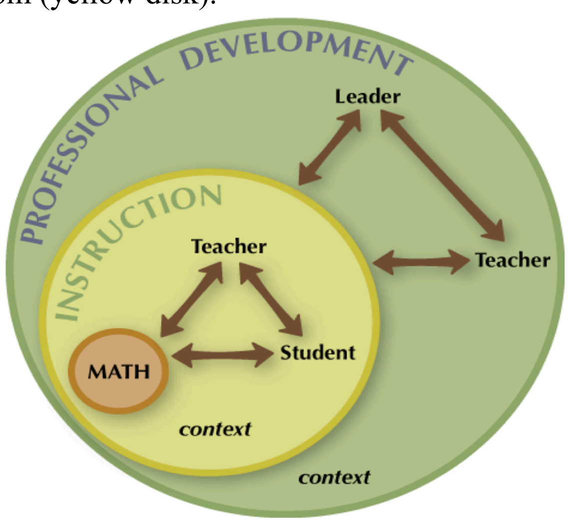

Figure 3. Nesting of teaching and learning connections (Carroll & Mumme, 2007).

Similarly, in Figure 3, mathematical knowledge for teaching (the stuff in the yellow disk) is linked to (future) teacher and mathematics teacher educator ("leader") within the larger context of teacher professional learning (green disk).

Clearly, this is a multi-dimensional situation. For each disk in Figure 3 there is an associated set of specifications for what counts as the context and for what constitutes the content about which one has "pedagogical content knowledge." Perks and Prestage (2008) made the case that an additional aspect they called "professional traditions" connects to the categories in Figure 3. Like Carroll and Mumme, they asserted that knowledge for teaching and for teaching teachers operates on several levels with a partially-nested self-similar design. Their model for *teacher-educator knowledge* is shown in Figure 4. On the left in Figure 4 is the model they associated with Instruction. Unlike Figure 3, Perks and Prestage focused on kinds of knowledge brought to a classroom (rather than the interactions among people and content foregrounded by Carroll and Mumme). So, though both models involve a self-similar nesting, the nature of what is nested and of what is represented by arrows/edges differs.

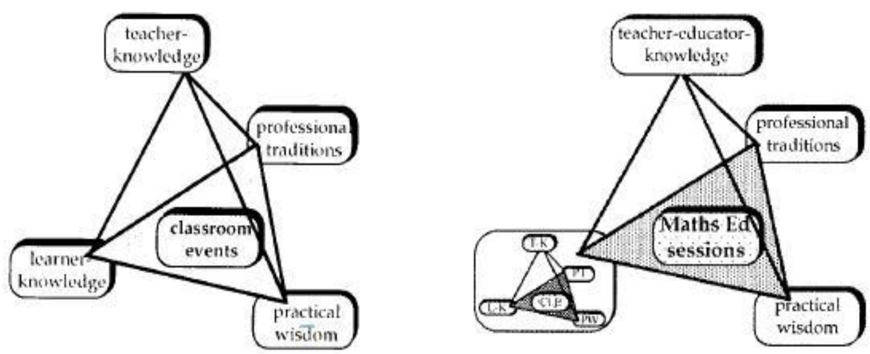

Figure 4. Tetrahedron models for teacher-educator knowledge (Perks & Prestage, 2008).

Model of *Pedagogical Content Knowledge* for MKT-FT

In our own work since 2006, we have used a tetrahedron to represent the relationships among types of knowledge (4 vertices) and types of thinking (6 edges) associated with the development of mathematical knowledge for teaching. As discussed above for Figure 2, the model for MKT-FT in Figure 5 (next page) foregrounds attention to pedagogical content knowledge. At each vertex in Figure 5 are the mathematical content in the college class *and* "content" that is the mathematical knowledge for teaching K-8 students (illustrated for just the KCS vertex as magnified and highlighted, lower left, in Figure 5). We claim a similar cascade of knowledge structures, related to Content & Teaching, Curriculum, and Discourses in K-8 are embedded in the vertex knowledge of college mathematics instructor MKT-FT (illustrated by similar "mini" tetrahedra at each of the other vertices in Figure 5, next page).

While the nesting of knowledge structures within others is represented as geometrically self-similar, a fractal structure, the knowledge and thinking represented in the "big" and "small" tetrahedra are not identical. Each vertex of the "big" tetrahedron representing MKT-FT has a ten-to-one mapping (i.e., the 4 vertices and 6 edges of the "small" tetrahedron). For instance, suppose the MKT-FT vertex for *knowledge of content and students* (KCS) is defined, as by Ball and colleagues, as "content knowledge intertwined with knowledge of how students [who are future teachers] think about, know, or learn this particular content" (Hill et al., p. 375).

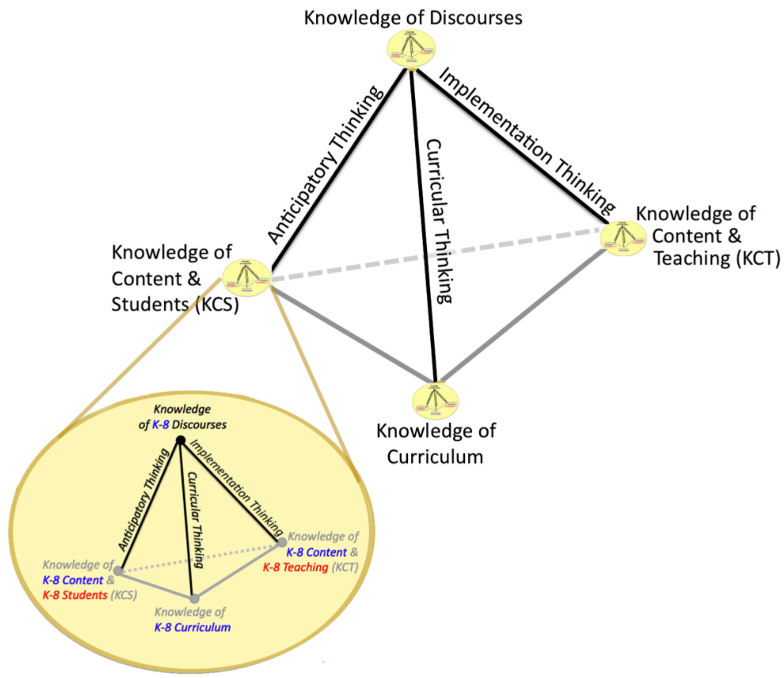

Figure 5. Model of MKT-FT pedagogical content knowledge – each vertex itself contains knowledge structures related to K-8 MKT.

In Figure 5, the KCS vertex represents teacher educator knowledge of (a) how to create instruction that will engage college students with acquiring the profound understanding of mathematics they will need in their future work as teachers as well as (from the small tetrahedron) (b) instruction that engages college students in learning to encapsulate, do, and unpack mathematical ideas in anticipation of their future work with children in K-8 mathematics lessons (i.e., K-8 MKT). In the MKT-FT model, knowledge of content and teaching includes a knowledge of teaching mathematics in a college course for future teachers as well as attention to K-8 MKT in making instructional choices in that college course.

MKT-FT and Task Design

Example 1. As a math teacher educator I incorporate my knowledge about my students and about their future work as teachers into the design of tasks we do in the college classroom. Using my knowledge of content and future teachers, I know that they know how to use the distributive property of multiplication across addition, but may not understand what it means (their mathematical knowledge) or how to unpack and describe that meaning to teach others (K-8 MKT). So, one task I have created starts like this:

> The situation: You are teaching a 5th grade class about division, talking about the properties of division. Daisy, one of your students, raises her hand and says “shouldn’t division distribute across addition just like we say multiplication does since division is the opposite of multiplication?”
> The goal of this activity is to investigate Daisy’s claim.

In the task, future teachers make groups of objects and circle groups of objects. Among other things, they compare 30 divided by 10 to 30 divided by 5 plus 30 divided by 5.

Example 2. In an activity for college faculty new to teaching future teachers, to illustrate how a task for pre-service teachers might be designed, we have used video of Sybilla Beckmann's college classroom (Beckmann, 2013) and a fourth grade classroom (from the Teaching Channel). In the college video, the instructor asked preservice teachers to come up with scenarios in which multiplication is not a correct operation for the multiplicative phrase "four times as many as." At first glance, this multiplicative phrase indicates a number sentence of the form $b = 4a$ (i.e., product (b) = 4 (multiplier) times a (multiplicand). Stated this way, a and b have a multiplicative relationship, but if b is given and a is the unknown to be found, an appropriate operation would be division. When problem solvers take a direct translation approach to convert a verbal sentence into a number sentence, an inappropriately stable piece of knowledge – about the operation – can be adopted. In the fourth grade classroom video, the majority, if not all, of the children demonstrate such difficulties with the problem "Maria saved $24. She saved 3 times as much as Wayne. How much did Wayne save?" This, in spite of the teacher's effort to show them otherwise. When a mathematics teacher educator is aware of the in-the-wild demands of elementary school teaching and the fractal structure of PCK, a task for their preservice teachers can be designed to build future teachers' knowledge to tackle this obstacle. A simple task could be similar to Beckmann's questioning of undergraduates to find different scenarios. A more complex task would emphasize the syntax and semantics of linguistics and introduce attention to analytic translation and compare it to direct translation.

These examples illustrate an important aspect of what mathematics teacher educators do with their MKT-FT. Teacher education has long noted that task design is a significant component in the development of knowledge for teaching teachers. For example, Stylianides and Stylianides (2006) asserted that it is crucial for activities to be *teaching-related mathematics tasks,* so future teachers learn important school mathematics while at the same time making connections between how learning the content relates to its teaching. For Seaman and Szydlik (2007), well-designed tasks deepen the "mathematical sophistication" of future teachers, which they define as occurring as a result of enculturation into the mathematics community (e.g., this might be signified by teachers exhibiting as their own the ways of knowing and values of mathematicians). A broader example comes from the *Journal of Mathematics Teacher Education* issue devoted to the important topic of task design (Zaslavsky, Watson, & Mason, 2007). Papers in this special issue discussed different aspects of good task design in courses for teachers. For instance, according to Chapman (2007), effective tasks facilitate new understandings of familiar concepts and prompt reflection, while Bloom (2007) argued that quality tasks enhance mathematical habits of mind among college learners who are future teachers.

Yackel, Underwood, and Elias (2007) demonstrated the profound effect that attention to task design and reflection on task implementation can have on MKT and MKT-FT development of those who teach future teachers. One of their mathematics teacher educators commented,

> I found it interesting that adult students also go through some of the same progressions that children do. In particular, I often noticed that many students initially needed to use [iconic representations] to perform calculations, such as explicitly drawing boxes, rolls, and pieces…Having never taught young children, I had never seen this first hand. *Base 8 gave me the opportunity to experience this part of children's learning* [emphasis added]. I think this is valuable to college instructors because most, like myself, will never have an opportunity to work with elementary school children closely. (p. 364)

Hence, we see that experiences with task design can support teacher educator growth of Knowledge of Content and Teaching in MKT-FT by creating and leveraging a knowledge of content and K-8 students. This example also illustrates the nonlinearity needed for an MKT-FT model. The instructor in Yackel and colleagues' work mentioned building Knowledge of Content and Students (both for child and adult learners) by thinking about Knowledge of Content and Teaching.

As another example of the multi-dimensional nature of MKT-FT, consider what constitutes the "subject matter knowledge" in the mathematical knowledge needed for teaching future teachers. For MKT-FT, the "subject matter" is a combination of mathematics and mathematics education As Ball noted, MKT (everything in Figure 1) becomes common content knowledge for mathematics teacher educators. Simultaneously, "specialized content knowledge" for mathematics teacher educators includes an awareness of educational policy and literature as a means of supporting future teachers' awareness about why certain mathematical or pedagogical practices are favored (e.g., in the Common Core Standards). Horizon knowledge for teacher educators includes recognition of pending consequences of district, state, and national mathematics standards. So, in MKT-FT, specialized content knowledge is rich in educational as well as mathematical knowledge.

And what about the knowledge of discourse discussed earlier? Well, for mathematics teacher educators, MKT-FT knowledge of discourse subsumes the same knowledge of discourse that teachers have, and draws on knowledge of communicating about MKT and the teaching of mathematics in different instructional situations.

As Hauk, et al. (2014) point out, the literature on PCK includes both stable and dynamic features. The edges in the tetrahedron represent the *ways of thinking* about teaching mathematics used in planning for, implementing, and reflecting on practice. These ways of thinking are enacted as teachers adapt to varying sociomathematical and cultural contexts that arise over time. In like manner, effective mathematics teacher educators also possess ways of thinking about teaching mathematics and about teaching MKT that change as the social, mathematical, and cultural climates change in their courses for future teachers. The edges in our fractal tetrahedron also represent these dynamic ways of thinking for faculty who teach future teachers.

Concluding Remarks

It is worth noting that much of what we have presented here as knowledge required to teach future elementary mathematics teachers is documented in the recent *Standards for Preparing Teachers of Mathematics* issued by the Association of Mathematics Teacher Educators (AMTE) in February of 2017. For instance, the AMTE document stresses (as we have done here) the importance of task design and implementation:

> In such [active learning] sessions, learners [future teachers in this case] are typically provided challenging tasks that promote mathematical problem solving and are provided opportunities to discuss their thinking in small group and full group discourse, thus promoting important mathematical practices. (p. 31)

This quote also underlines the importance of discourse knowledge that is required of teacher educators. Facilitating active learning in a collegiate classroom requires the teacher educator to navigate across the many social, cultural, and socio-mathematical norms present in the room.

Discourse knowledge is further amplified in paying attention to building reasoning and justification skills among future teachers, which the *Standards* address by noting that "without explicit classroom attention to mathematical argumentation, effective programs risk diminishing the opportunities candidates have to engage in the mathematical practice of constructing viable arguments" (p. 32). The fractal nature of MKT-FT is also reflected in the *Standards:* "In effective programs, mathematics teacher educators explicitly identify and address mathematical practices" (p. 31). Ultimately, faculty teaching future teachers must possess sufficient knowledge of mathematical content and of the mathematical nuances of K-8 MKT to offer instruction the uses and is aware of valued mathematical practices (e.g., those in the Common Core State Standards).

Acknowledgement

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Raising reasoning through revision: A case study of an inquiry-based college geometry course

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Geometry is the subject where U.S. students are weakest on international assessments, but college geometry is an area of proof that is understudied. Since geometry is secondary students' only exposure to proof, it is vital our secondary teachers can prove effectively in this content area. The purpose of this case study, drawn from a larger project, was to understand how, if at all, pre-service teachers' proof schemes became more axiomatic throughout a one-semester inquiry-based college geometry course. Participants in this study, Alexis and Lindsey, were pre-service teachers enrolled in an inquiry based college geometry course. Although Alexis and Lindsey had differing experience with proof at the start of the course, the change to revise their proofs and discuss problems with their peers helped both students advance to more axiomatic geometric thinking.

Key words: College geometry, inquiry based learning, proof

Introduction and Motivation

Compared to other nations, the students of the United States of America are floundering in geometry. The Trends in International Mathematics and Science Study (TIMSS) evinced that twenty-one educational systems, including China, Japan, Israel, and England, have higher geometry scores than the U.S. (Mullis, Martin, Foy, & Arora, 2012). This deficiency is partly because pre- and in-service teachers possess an inadequate understanding of the structure of geometry. Despite this difficulty for students and educators, college geometry remains severely under-researched (Speer & Kung, 2016).

A possible solution to this problem is to reassess our preparation of pre-service teachers for geometry content and proof. Research suggests that instead of teaching proof in instructor-centered environments, proof courses should consist primarily of "student-student and student-teachers interactions regarding students' proof attempts" (Selden & Selden, 2008, p.19). One way this can be achieved is through implementation of inquiry-based learning (IBL) pedagogies. In an IBL environment, instruction is content- and student-centered, it is driven by students' exploration, and it is concentrated on acquisition of the material (Padraig & McLoughlin, 2009).

The research question of this study was: to what extent do students' proof schemes became more axiomatic throughout a one semester inquiry-based college geometry course? Through this study, we can determine whether or not this type of instructional environment is a potential solution to pre-service teachers' shallow understanding of proof and geometry. Mathematicians and teacher educators can then make more informed decisions on how to structure college geometry courses. We argue that the revisions present in an inquiry based classroom were vital to help students develop from a perceptual to a more axiomatic proof scheme.

Literature Review

Geometry arises from a set of undefined terms and axioms through which all other theorems and definitions are constructed. Hence, a thorough understanding of geometry involves a deep understanding of axiomatic proof; yet, pre-service teachers are not necessarily equipped with the geometry content knowledge required for teaching after merely majoring in mathematics (Jones, 1997). Furthermore, teachers possess a narrow understanding of proof. Studies indicate that pre- and in-service teachers believe proof only helps explain ideas used in mathematical concepts, and they do not recognize the ability of proof to systemize results (Mingus & Grassl, 1999; Knuth, 2002b). Pre- and in-service teachers are prone to accept pictures and other types of empirical arguments as proof (Knuth, 2002a). Also, research depicts teachers disregarding or accepting an argument as proof based on its format (Dickson, 2008).

Constructing proofs at the post-secondary level, compared to proofs at the secondary level, requires student to possess a larger knowledge base to apply in the proving process. The final products are expected to have a more complex structure as well as expected to be precise and concise (Selden, 2012). However, Jones discerned that undergraduates do not recognize "…mathematics as a field of 'intricately related structures'…" (Jones, 2000, p. 59). Hence, students fail to establish the connections between ideas required to formulate a proof. Undergraduates are also presumed to enter into a proof course with some strategic knowledge, knowledge of which proof techniques and helpful theorems, to apply in the proving process; yet, research depicts that undergraduates do not have these understandings (Weber, 2001). Therefore, undergraduate students are not entering into proof courses with the background required to confront proofs at the post-secondary level.

A multitude of proof research has devoted itself to unearthing the reasons students do not enter a post-secondary proof course with the required background knowledge. Alarmingly, students do not have an adequate understanding of what arguments qualify as mathematical proof (Weber, 2001). Harel and Sowder (1998) attribute this to the fact that teachers often present proof as an obvious proposition instead of engaging students in the process of discerning which arguments are convincing. Furthermore, students lack the important comprehension of the mathematical language and concepts necessary to construct proof. For example, students may not recognize the difference in certain terms when used in everyday vernacular and when used in a mathematical argument (Selden, 2012). Another difficulty associated with mathematical language for students is the poor comprehension of quantifiers – students misunderstand the implications and the importance of universal and existential quantifiers (Selden, 2012). Research also depicts that students possess shallow understandings of definitions and theorems (Weber, 2001; Selden, & Selden, 2008). Hence, students may misapply or misinterpret ambiguous terms, quantifiers, or the pieces of knowledge essential in constructing or validating a proof.

If educators assume students will develop proficient proof skills without any feedback, research manifests students will fail and most likely cultivate ineffective strategies (Weber, 2001). These ineffective strategies are typically proof schemes dependent upon external and empirical convictions. A proof scheme "consists of what constitutes ascertaining and persuading" for a particular person (Harel, & Sowder, 2007, p. 7). Rather than rely simply on external or empirical convictions to formulate geometry proofs, students need to be able base arguments on axioms and logical deductions. Table 1 depicts the proof schemes observed in this study as well as their definitions. Proof schemes dependent upon external and empirical justifications are the authoritarian, ritual, and perceptual proofs schemes. Proof schemes that utilize axiomatic and deductive justifications are the deductive, intuitive axiomatic, and structural axiomatic proof schemes.

Methods

The theoretical perspective used in this project was the reduced Toulmin model of argumentation. In mathematics education literature, there are two formats in which the model appears. We will distinguish them as the reduced Toulmin model (Figure 1) and the extended Toulmin model. The reduced model consists of three types of statements that represent different pieces of the argument, and the extended model consists of six statements. The three statements in the reduced Toulmin model are as follows: The data (D) is the foundation on which the argument is based. The conclusion (C) is the statement the arguer intends to convince. The warrant (W) justifies the relationship between the data and the conclusion. When students are asked to prove a claim, the conclusion is correct because it is given to the students. However, the data is typically a mixture of right and wrong. This amalgamation occurs because of the warrant the student applies – their reasoning from a piece of information to the conclusion. Although a warrant is specific to an argument, a warrant-type is a category of warrants with similar properties.

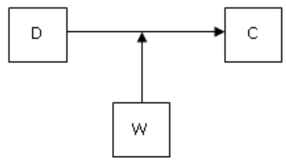


Figure 1. Reduced Toulmin model. Adopted from (Inglis, Meija-Ramos, & Simpson, 2007).

This study took place at a midsized, rural, research university in the South, and the students who participated were those enrolled in a college geometry course based upon Miller (2010) geometry course notes. The data collected was part of a larger study; this study is a case study of two students – Alexis and Lindsey. Both Alexis and Lindsey are Caucasian females whose majors were math education. Alexis was classified as a senior and had one prior proof courses and Lindsey was classified as a freshman with no prior proof experience. Students in the course were provided with course notes that presented open-ended problems related to a specific learning goal. For each new assignment, students were assigned a specific problem from the provided course notes and a group. If a group appeared to be making little progress or moving in an unproductive direction, the teacher would use guided questioning to redirect students' thoughts. If multiple groups stopped progressing, the teacher would initiate a whole class discussion.

To determine students' proof comprehension, researchers examined the assignments students turned in. Students were allowed to revise and resubmit all assignments, and these were analyzed as well. Researchers also used observations to gain further understanding of students' proof comprehension. As students discussed their ideas, a researcher sat behind them listening and taking notes on their interactions.

The submissions were analyzed by assignment, and all the drafts from an individual participant were analyzed at the same time. After this initial reading of blinded assignments, researchers would journal their impressions of the coding and the trajectory exhibited in the multiple submissions. These journals were used to operationalize the proof schemes in Harel & Sowder (1998) and Harel (2007), and to construct the standards of evidence (Table 1). Although it is not standard to give numerical values to the proof schemes, we did so in this case because of the nature of the assignments in the course. Students were always given a situation where they had to collect data and use that data to draw a picture and make a geometric conjecture before writing a proof of their findings. For a complete solution, students needed to give the data, the geometric conjecture based on their data, and a proof. As a result, we privileged axiomatic proof schemes for the proof portion of the assignment, since a student using an axiomatic proof scheme also had to use perceptual proof schemes to form their conjecture.

Table 1. Standards of Evidence (Harel, 2007; Harel & Sowder, 1998; Harel & Sowder, 2007).

Code	Definition	Standards of Evidence
Authoritarian (1)	Acceptance of an argument as proof is based mainly on if the argument is presented by an authority source.	○ Argument produced after intentional scaffolding or direct instruction from the teacher. ○ Student did not participate in the group's reasoning, and/or writing process. ○ The correct pieces of reasoning were discussed in class, and the argument does not arise from the students' data.
Ritual (2)	Acceptance of an argument as proof is based mainly on the appearance or format of the argument.	○ Student misapplies multiple axioms and theorems. ○ Argument mirrors the format of a known correct argument, but the argument does not arise from the axiomatic system. ○ Student restates the axiom or theorem as it is originally written despite the fact that the terminology does not relate to the context of the proof.
Perceptual (3)	Acceptance of an argument as proof is based on perceptions – rudimentary mental images formed without forming or considering the results of transformations.	○ Student refers to or provides only a diagram as justification for reasoning. ○ The argument is driven by students' perceptual observation of the figure he or she drew and not by the implications of axioms and theorems.
Deductive (4)	Acceptance of an argument as proof is based on validating conjectures by logical deductions.	○ The argument follows the correct deductive process, but the student does not establish the definitions, theorems, or axioms the process utilizes. ○ The argument follows the correct deductive process, but there is one instance in which the student relies on another lower warrant. ○ The argument follows the correct deductive process, but at least one clarification statement is necessary for validity. ○ The argument follows the correct deductive process, but an axiom or theorem is misinterpreted and misapplied.
Intuitive Axiomatic (5)	Acceptance of an argument as proof is based on justifying conjectures using intuitively grasped undefined terms and axioms.	○ The argument is built from undefined terms and axioms, definitions, and theorems of an axiomatic system that is intuitively grasped such as Euclidean geometry.
Structural Axiomatic (6)	Acceptance of an argument as proof is based on justifying conjectures from different realizations using an understood common structure determined by a permanent set of axioms.	○ The argument is logical and made up of systematic application of axioms and theorems. ○ If any portion of the argument could be clarified, the clarification is not necessary for the argument's validity.

Findings

After analyzing the data, we mapped Alexis' and Lindsey's warrants throughout the course. Figure 2 presents a chronological summary of their proof scheme warrants. On all assignments, except Midterm Problem 3biii, students worked on the solutions with a group during class.

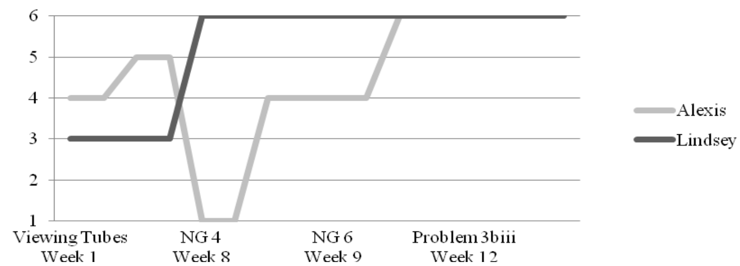
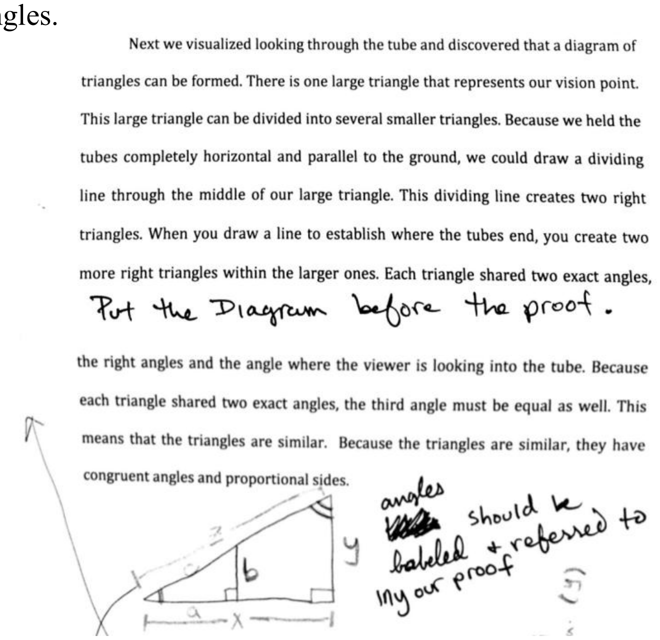


Figure 2. Proof schemes used throughout the semester

Viewing Tubes. The Viewing Tubes was the first assignment of the course and the required the only Euclidean proof. In this assignment, students were asked to determine the formula for the area of the wall that can be seen through a tube. To accomplish this, students needed to establish that the two triangles resulting from the situation are similar, and they struggled with the justification that both triangles are right triangles.

Although Alexis turned in three drafts for the Viewing Tubes assignment, her second draft addressed formatting and not argumentation issues. The proof scheme warrant in the first draft is deductive, and the proof scheme warrant in the third draft is intuitive axiomatic. Since Alexis' draft two had only formatting changes, in the following section, we will discuss Alexis' first and third drafts.

The first draft from Alexis claims each triangle has a right angle, and her data is the explanation of the actions she took in constructing the situation. Alexis' warrant is the deductive proof scheme because she follows the correct deductive process, but she lacks the important clarification that her actions produce perpendicular lines which establishes right angles.


Next we visualized looking through the tube and discovered that a diagram of triangles can be formed. There is one large triangle that represents our vision point. This large triangle can be divided into several smaller triangles. Because we held the tubes completely horizontal and parallel to the ground, we could draw a dividing line through the middle of our large triangle. This dividing line creates two right triangles. When you draw a line to establish where the tubes end, you create two more right triangles within the larger ones. Each triangle shared two exact angles,

Put the Diagram before the proof.

the right angles and the angle where the viewer is looking into the tube. Because each triangle shared two exact angles, the third angle must be equal as well. This means that the triangles are similar. Because the triangles are similar, they have congruent angles and proportional sides.

Figure 3. Alexis Viewing Tubes Draft 1

Alexis' third draft employs the same data and claim from draft one. However, her warrant is the intuitive axiomatic proof scheme because in draft three Alexis clarifies that the lines formed through her actions are indeed perpendicular lines. With this correction, she produced a valid argument based of Euclidean definitions and theorems, and it was achieved prior to the scaffoled class discussion.

Because we held the tubes completely horizontal and parallel to the ground, we could draw a dividing line through the middle of our large triangle (AE). This dividing line creates two right triangles. When you draw a line to establish where the tubes end, it is perpendicular to the tubes' midline and you create two more right triangles within the larger ones

Figure 4. Alexis Viewing Tubes Draft 3

In the Viewing Tubes assignment, Alexis was able to map the correct deductive process for justifying the claim and her warrant was the deductive proof scheme. Receiving feedback allowed her to reassess and amend her argument to include an important clarification, and ultimately, Alexis was able to reason at the intuitive axiomatic level.

While Lindsey drafted three different versions of the Viewing Tubes assignment and alterations were made between each of the drafts, the argument portion of the paper remained unchanged. The proof scheme warrant used in this argument is perceptual. Lindsey claims that each triangle is a right triangle, and her data is her diagram. In her justification, Lindesy provides the diagram in Figure 4 and states, "On the triangle, angle BAC and angle DAE were the same angle. Angles ACB and AED were both right triangles. This made the last angles ABC and ADE the same…" To convince herself that the triangles are right triangles, she relies only on her observations. Each of her revisions presented this same argument even after a scaffoled class discussion. Hence, Lindsey never grasped that her argument lacked the necessary reasoning to determine the truth of her claim.

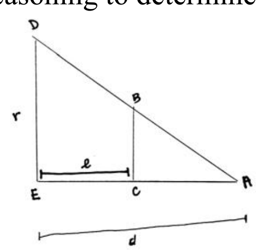


Figure 5. Lindsey Viewing Tubes Draft 1

Neutral Geometry Worksheet 4 (NG 4). NG 4 was assigned after students completed their revisions of the Viewing Tubes assignment. Using the NG axiom system (Figure 6) students were asked to prove each point belongs to at least two different lines. This proof requires the consideration of multiple scenarios in different spaces that arise from application of the axioms, and students struggled to foresee the axiom system in a non-Euclidean context.

"Point" and line are undefined. Lines will consist of points.
Axiom 1. If L is a line, then there exists at least two points belonging to L.
Axiom 2. If L is a line, then there exists at least one point not on L.
Axiom 3. There exists at least one line.
Axiom 4. If A & B are distinct points, then there is at least one line containing both A & B.
Theorem 1. There exists at least three different points.
Theorem 2. If P is a point, there is a line containing P.
Theorem 3. If P is a point, then there exists two different lines containing P.
Axiom 4'. If A & B are different points, then there exists one and only one line which contains both A & B.

Figure 6. Neutral Geometry Axiom System and Theorems Proven in the Course

Alexis' warrants for the Viewing Tubes assignment was deductive on draft one and intuitive axiomatic on draft three. For the NG 4 assignment, Alexis turned in four drafts, and her proof scheme warrant starting this assignment is authoritarian. Her second draft consists primarily of formatting changes, but her third draft utilizes the deductive. Alexis' fourth draft contained only formatting changes, so in the following section, we will discuss her first and third drafts.

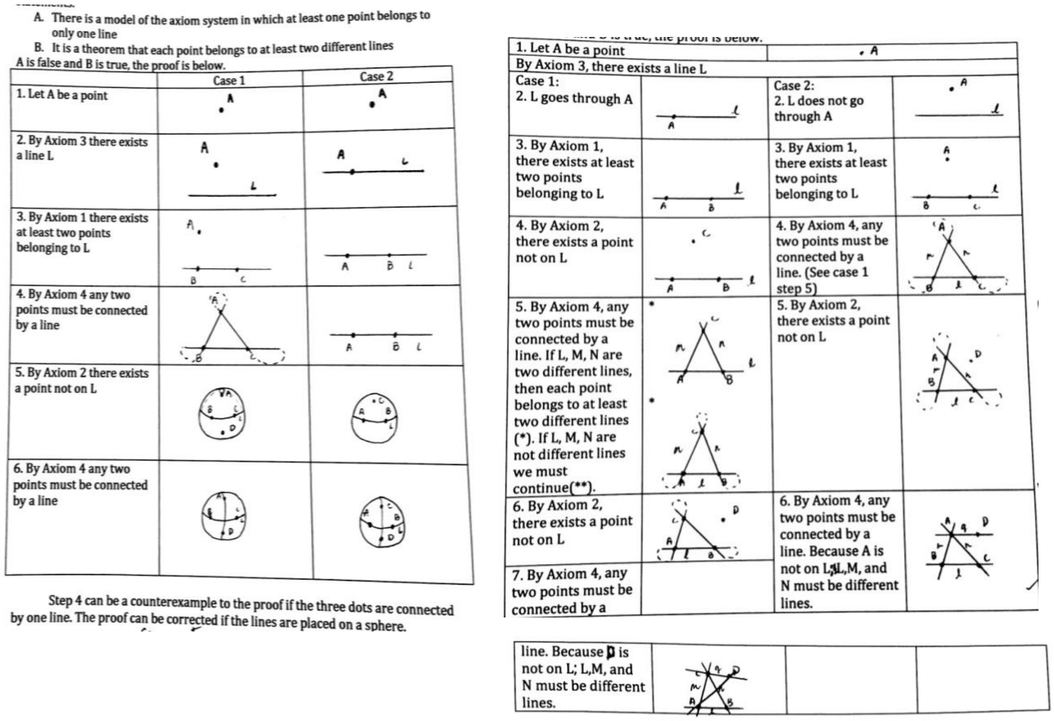


Figure 7. Alexis NG 4 Draft 1 (left) and Draft 2 (right).

The first draft provided by Alexis claims that each point belongs to at least two distinct lines, and her data is her written portion of the argument. When students worked on this assignment during class, Alexis was absent, so she received notes from her group members. Her first draft portrays some correct pieces of reasoning such as her consideration of two cases and the sphere; however, these correct ideas are not applied appropriately. For example, Alexis states that step 4 could be a counterexample to a proof, so it is apparent she does not understand the implications of what she is trying to argue. Hence, her warrant is the authoritarian proof scheme because although there are correct portions, they are not formed from her data.

On her third draft, Alexis' claim remains the same, and she clarifies and completes her data. Unlike her previous drafts, the warrant for this argument is the deductive proof scheme. She follows a generally correct deductive process and considers situations that could arise on different surfaces. However, Alexis forgets the truths established by her applications of the axioms such as in step 5 of case 1. In this step, she connects a point using line L, but that point was established to not be on line L. This misuse of the axiomatic system detains her from reaching the structural axiomatic proof scheme, but her consideration of the generality of the axioms places her warrant at deductive.

Lindsey's proof scheme warrant for all of her Viewing Tubes drafts was the perceptual proof scheme. On the NG 4 assignment, Lindsey was partnered with Martin and she turned in two drafts of this assignment. The proof scheme warrant used in both drafts is structural axiomatic. Lindsey's second draft consisted only of formatting alterations, so in the following section, we will discuss the first draft submitted by Lindsey.

For her first draft, Lindsey's data is the written argument she provides for both cases. Her claim is that each point belongs to at least two distinct lines,

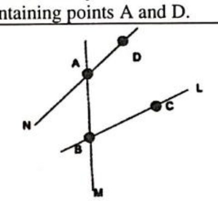
Assumption: Let there be a point, A.	
By Axiom 3, there exists at least one line, L.	
CASE 1: If the point A lies on Line L, then at least point is not on L, point B, by Axiom 2.	CASE 2: If point A does not line on Line L, then by Axiom 1, at least two points, points B and C belong to Line L.
Since points A and B are two different points, then by Axiom 4, another line, Line M contains points A and B.	By Axiom 4, since A and B are two different points, there exists a line containing both A and B, line M.
Because B does not lie on L, line L and line M are two different lines. Therefore, point A belongs to two different lines, Line L and Line M.	By Axiom 2, there exists another point not on Line M, point D.
CASE 2: If point A does not line on Line L, then by Axiom 1, at least two points, points B and C belong to Line L.	Since Point A and Point D are two different points, then by Axiom 4, there exists a line containing points A and D.
By Axiom 4, since A and B are two different points, there exists a line containing both A and B, line M.	
By Axiom 2, there exists another point not on Line M, point D.	Because points A and D are two different points, and point D does not belong to line M or Line L, then Line M, N, and L are different lines. Therefore, point A belongs to 2 different lines. Line N and Line M.

Figure 8. Lindsey NG 4 Draft 1

and her warrant is the structural axiomatic proof scheme. Although Lindsey needs to clarify why she continued her argument after establishing line M, the validity of her argument is not hindered by the lack of this statement. Lindsey also considers the generality of the axioms, and she applies the axioms to form a deductive argument.

While her arguments on Viewing Tubes employed only the perceptual proof scheme as a warrant, Lindsey moves past this type of reasoning in this assignment. Between these assignments, students explored other arguments and the Neutral Geometry Axiomatic System. Lindsey was able to garner a better understanding of the arguments that constitute as proof. She then applied this knowledge on the NG 4 worksheet, and her warrant on this assignment was the structural axiomatic proof schemes.

Neutral Geometry Worksheet Problem 6 (NG 6). The NG 6 assignment took place after students had completed at least their first NG 4 drafts. This assignment asked students to prove there exists a line not containing a given point on the NG Axiom System; however, for this problem, axiom 4 is replaced with axiom 4' which establishes the uniqueness of a line containing two points.

Alexis' warrants for the NG 4 assignment started as the authoritarian proof scheme, and through revisions, her warrants became the deductive proof scheme. For the NG 6 assignment, Alexis was partnered with Jeremy, and she turned in two drafts. The proof scheme for the first draft is deductive, and the proof scheme warrant for the second draft is structural axiomatic.

Alexis claims on her first draft that for a given point, there exists a line not containing that point, and her data is the written argumentation she provides. Her warrant is the deductive proof scheme because she applies axioms and theorems to develop a correct deductive process; however, there is a portion of her argument that lacks clarity. Although she states that the line constructed by axiom 4' "cannot curve to connect non-collinear points," she needs to clarify why such a line would violate axiom 4'.

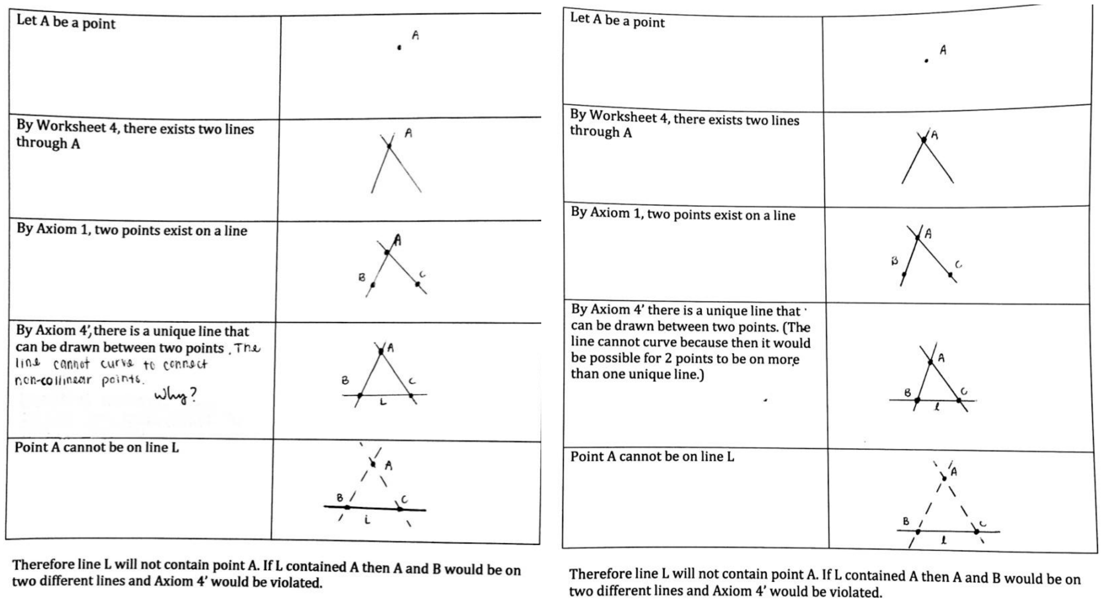


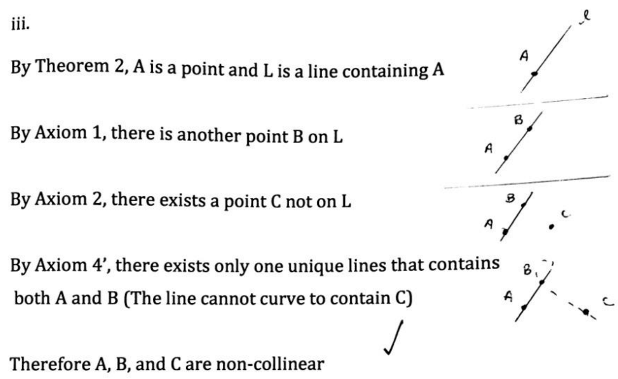
Figure 9. Alexis NG 6 Draft 1 (left) and Draft 2 (right).

The second draft turned in by Alexis has the same data and claim as her draft. On this draft, however, she clarifies the contradiction with axiom 4' that would occur if point A was on line L. Following this amendment, her warrant becomes the structural axiomatic proof scheme. After spending some time relying on the reasoning of others and processing the information, Alexis progressed her reasoning to the deductive proof scheme. She continued to improve her reasoning as she spent more time working in the Neutral Geometry Axiom System. On the NG 6 assignment, the warrant started at the deductive proof scheme and progressed to the structural axiomatic proof scheme.

Although her warrant for her first assignment was perceptual, once Lindsey started working in the Neutral Geometry Axiom System, her proof scheme warrant became the structural axiomatic proof scheme. For the NG 6 assignment, Lindsey turned in two drafts, and her first draft is comparable to the second draft produced by Alexis. The data for Lindsey's first draft is her written argumentation, and the claim is that there exists a line not containing a given point. Lindsey crafts a correct, deductive argument based on the axioms. While her explanation for why line M does not contain point A could be clearer, the first wording she produces does not hinder the understanding or validity of her argument. Her warrant for this draft is then the structural axiomatic proof scheme.

Midterm Problem 3biii (Problem 3biii). This assignment took place during week 12 of 15. Students were asked to prove, using the NG axiom system including axiom 4', that for any given point there exist two other distinct points such that the collection of points is non-collinear. While all other assignments were allotted class time and students worked in groups, on the Midterm, students worked individually outside of class.

Alexis' data for Problem 3biii is her written argumentation, and she claims that for a given point there exist two other different points such that the points are non-collinear. In this argument, her warrant is the structural axiomatic proof scheme. Alexis produces a clear argument formed from logically applies theorems and axioms.


iii.

By Theorem 2, A is a point and L is a line containing A

By Axiom 1, there is another point B on L

By Axiom 2, there exists a point C not on L

By Axiom 4', there exists only one unique lines that contains both A and B (The line cannot curve to contain C)

Therefore A, B, and C are non-collinear

Figure 10. Alexis Problem 3biii Draft 1

On the NG 6 assignment, the warrant for Alexis' first draft was the deductive proof scheme, and the warrant for her second draft was the structural axiomatic proof scheme. Continuing to work in the Neutral Geometry Axiom System, Alexis maintained a warrant of the structural axiomatic proof scheme in Problem 3biii.

Like Alexis, Lindsey produced a correct deductive argument based on the axioms for the Neutral Geometry Axiom System. Hence, her warrant for this argument is the structural axiomatic proof scheme. The warrant Lindsey used on the NG 6 assignment was the structural axiomatic proof scheme. Working independently of her group, Lindsey was able to maintain her reasoning at the structural axiomatic proof scheme on Problem 3biii.

Discussion

The purpose of this study was to determine to what extent students' geometry proof schemes became more axiomatic throughout a one-semester college geometry course. By assigning each of the proof schemes a number rank based on the level of axiomatic reasoning utilized and creating time series graphs to analyze the changes in students' proof schemes through the course, we discerned three proof scheme trajectories: the static participant, the peripheral participant, and the central participant. These proof scheme trajectories emerged as a reaction to the revision component of the IBL course design (Figure 11). The revision component forced students to reanalyze and revise their reasoning. Static participants resisted this revision process; peripheral participants withdrew from group participation but used this time to process ideas; and central participants invested more time analyzing ideas within their groups.

Regardless of prior proof experience all but one student in this study started with a perceptual proof scheme. This suggests that students may have had initial trouble linking prior formal proof training to a more familiar content area. After receiving extensive feedback on their initial attempts of the viewing tubes lab, participants had three different responses to the need to revise their work, two of which were successful.

The static participants were Jeremy and Kayla, and they relied on prior ideas such as their perceptions or beliefs about what a proof should look like to form their reasoning. Jeremy and Kayla struggled with beliefs many pre- and in-service possess such as an inclination to accept perceptual arguments as proof, and the belief that the validity of an argument depends on its formatting (Knuth, 2002a; Goetting, 1995; Dickerson, 2008). However, since they were reluctant to reassess these prior beliefs, their proof reasoning improved minimally.

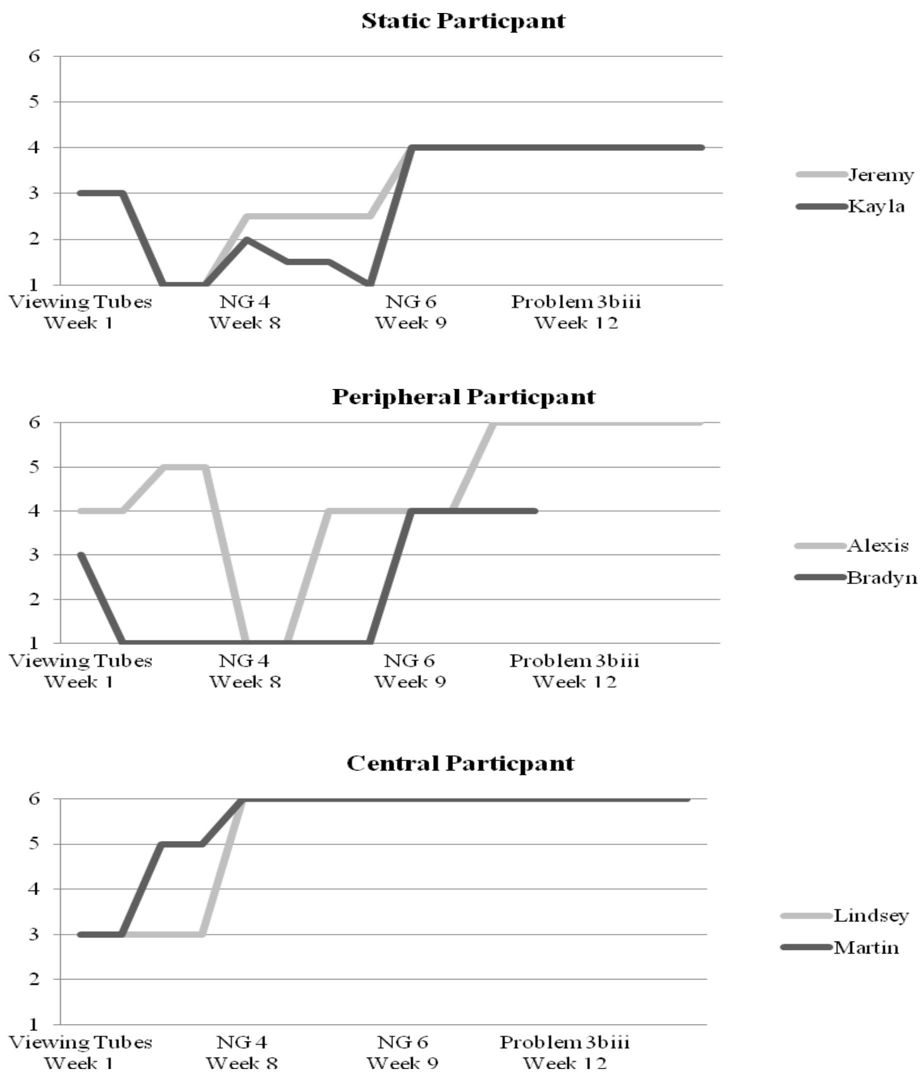


Figure 11. Proof Scheme Trajectories

Alexis and Bradyn were the peripheral participants, and their response to revising arguments that were outside a familiar context was to become seemingly passive group members. This passivity occurred when these students had limited knowledge of the content and context they were asked to prove in because teachers exhibit difficulty in proof when they lack content knowledge (Ko, 2010). On the Viewing Tubes assignment, Alexis had a strong understanding of the content within the Euclidean context and was able to exhibit axiomatic reasoning. When the proofs left the Euclidean context and moved into unfamiliar content in NG 4, Alexis' reasoning relied solely on others and she employed the authoritarian proof scheme. Bradyn did not possess a strong understanding of the content in Viewing Tubes, so this move to the authoritarian proof scheme occurred earlier in the semester for him. Since Bradyn did not have a thorough understanding of the Euclidean content before moving into other contexts, he utilized the authoritarian proof scheme longer than Alexis. Observing this period where Alexis and Bradyn relied on the authoritarian proof scheme, they appeared to play the role of a spokesman paraphrasing the ideas of others (Krummheuer, 2010). However, the data analysis revealed that these students were successful after watching their peers. This indicates that they were actually partaking in what research refers to as legitimate peripheral participation (Krummheuer, 2010). Hence, peripheral participation proved to be a successful reaction to revision for improving proof reasoning.

The central participants, Lindsey and Martin initially encountered difficulties experienced by Jeremy, Kayla, and other teachers – the willingness to accept empirical arguments in the form of diagrams as proof (Knuth, 2002a; Goetting, 1995). This is apparent in both students initial use of the perceptual proof scheme in Viewing Tubes. However, instead of clinging to their prior beliefs, Lindsey and Martin responded to the need for revisions by investing more time discussing ideas and the questions posed by the instructor within their groups. By NG 4, both students were employing the structural axiomatic proof scheme. Unlike Jeremy and Kayla who resisted the IBL course design, Lindsey and Martin fully embraced it. They essentially took full advantage of the availability of student-student and student-teacher interactions built into the course design. These are the interactions literature suggests would be successful for inducing logic and proof techniques in students (Selden & Selden, 2008; Harel & Sowder, 1998). Since they centered their time on investigating and critiquing one another's ideas, their proof reasoning improved quickly. It should, however, be noted that Lindsey and Martin were also quite talented as students.

These proof scheme trajectories were not a phenomenon of group structure or proof training. Groups typically contained the following pairs: Kayla and Bradyn, Jeremy and Alexis, and Lindsey and Martin. Although the central participants were paired together for the majority of the assignments, the static and peripheral participants were seldom in the same group. The trajectories exhibited by the static and peripheral participants occurred across the groups. Furthermore, these proof scheme trajectories were not a result of proof training. Kayla was the most experienced with proofs and started the class having completed two proof courses; however, Kayla struggled to utilize deductive reasoning to formulate her arguments. Particularly, Kayla was hindered by the belief that a valid proof had to adhere to a particular format. This is an idea that was perhaps even cultivated by her prior proof training. Lindsey, conversely, had no prior proof training, and she rapidly moved into the structural axiomatic proof scheme. Hence, these proof scheme trajectories that emerged were neither a result of the group structure nor a product of proof experience.

The success exhibited by the peripheral participants following their passivity was unexpected. Since these students relied solely on the authoritarian proof scheme for a time, it did not appear they were developing their own logical and deductive reasoning. Yet, their sudden improvement in proof reasoning demonstrates that they were actually partaking in what literature refers to as peripheral participation (Krummheuer, 2010).

Although completing other direct instruction proof courses was not sufficient for pre-service teacher to be adept at geometry proof, the findings of this study indicate that developing a proof course consisting primarily of student-student and student-teacher interactions helped all students improve their geometry proofs. This course design was achieved through an inquiry-based learning environment, and the revision component of the course was pivotal. The way students chose to revise and reanalyzed their arguments in these forced revisions is what formed the three proof scheme trajectories and informed students' successes. In analyzing the proofs formed in an IBL course such as a modified Moore method course, it is imperative to gather both observation data and the documents students produce. Since the proof scheme trajectories were informed by students' reactions to the revisions, it is essential to observe how students participate in their groups as well as the actual arguments they produce.

Limitations to this study arise from the fact that there were a small number of students enrolled in the course. Due to the limited number of participants, there was little ethnic and first language diversity as well as minimal diversity in prior proof experience. There may also be other proof scheme trajectories that were not observed in the small sample of students.

While teachers and students have long struggled with geometry and proof, geometry is under-researched in mathematics education (Speer & Kung, 2016). However, this study provides insight into how to structure a course aimed at instructing geometry and proof. Findings depict the impact of an IBL geometry course on the proof schemes utilized by students as they form geometry proofs. As the revision element of the course design proved to be an integral component due to students' reactions, this study also portrays some reactions students could exhibit and their successfulness.

In future studies, we suggest copying students' drafts as they are turned in throughout the course. We had students assemble a portfolio that we made a copy of at the end of the semester. However, some students did not turn in a complete portfolio and this diminished the amount of usable data. Copying drafts as students complete them would also reduce confusion about what was on the draft when it was turned in versus what may have been a note that the student added at a later time.

There are several ways that the results of this study may be extended in further inquiry. Since it was observed that completing direct instruction proof courses did not translate to successful proof reasoning in this study, further research should follow these students through their next proof course to determine the usefulness of this type of course design. The standards of evidence table developed in this study could also be used to analyze the proof scheme trajectories of students in other courses differing in structure and content. Applying this type of analysis to other courses could identify other possible proof scheme trajectories as well. Furthermore, due to the rapid transition to the structural axiomatic proof scheme for the student with no prior proof experience, Lindsey, this type of course design should be applied to an introductory or transition to proof course that follows the participants through following proof courses.

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Developing Student Understanding: The Case of Proof by Contradiction

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Proof is central to the curriculum for undergraduate mathematics majors. Despite transition-to-proof courses designed to facilitate the transition from computation-based mathematics to proof-based mathematics, students continue to struggle with all aspects of mathematical proof. In particular, research suggests that proof by contradiction is an especially difficult proof method for students to construct and comprehend and yet, there are no satisfactory instructional models for how to teach the method. The purpose of this paper is to discuss preliminary results of a teaching experiment on student comprehension of proof by contradiction within a transition-to-proof course. Grounded in APOS Theory, this paper will illustrate that a student's conception of mathematical logic, and quantification in particular, plays an important role in their comprehension of proof by contradiction.

Keywords: Proof by Contradiction, Teaching Experiment, Transition-to-proof course

Proof is central to the curriculum for undergraduate mathematics majors. Indeed, the *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences* contains the following recommendation: "Students majoring in the mathematical sciences should learn to read, understand, analyze, and produce proofs at increasing depth as they progress through a major" (Schumacher & Siegel, 2015, p. 11). Despite transition-to-proof courses designed to facilitate the transition from computation-based mathematics to proof-based mathematics, students continue to struggle with all aspects of mathematical proof (Samkoff & Weber, 2015). In particular, research suggests that proof by contradiction is an especially difficult proof method for students to understand and produce (Antonini & Mariotti, 2008; Brown, 2011; Harel & Sowder, 1998). In order to address students' understanding of proof by contradiction, we examined the literature for what proof by contradiction means and what is known about how students develop an understanding of proof by contradiction. A short summary to these questions is provided below.

Proof by contradiction is based on the *law of the excluded middle*: either a statement is true or the negation of the statement is true. By showing the negation of the statement cannot be true (i.e., leads to a contradiction), the statement must be true. However, some scholars do not attribute the law of the excluded middle as the foundation of proof by contradiction. Indeed, Lin, Lee, and Wu Yu (2003) state that "the conceptual knowledge of proof by contradiction is the law of contrapositive" (p. 4-446). Other scholars do not always distinguish between proof by contradiction and proof by contraposition when describing how students understand indirect proofs. For example, Antonini and Mariotti (2008) state that the commonality of negating the thesis in both methods is enough to consider cognitive aspects of students' difficulties with both methods simultaneously. We consider the two indirect proof methods as having distinct underlying concepts: logical equivalence of an implication and its contrapositive (Contraposition) and the law of the excluded middle (Contradiction). These distinct underlying concepts require cognitive models to consider how students develop an understanding of each method separately.

The only current cognitive models of how students understand proof by contradiction have been proposed by Lin, Lee, and Wu Yu (2003) and Antonini and Mariotti (2008). Yet, neither of these cognitive models are satisfactory as they do not focus on how students develop an understanding of how the law of the excluded middle validates the contradiction proof method. Our larger study aims to address this deficiency in the literature by first modeling how students could come to understand proof by contradiction and then, using this model, develop instructional materials to teach proof by contradiction. The purpose of this paper is to discuss preliminary results on a part of this larger study; that is, to discuss preliminary results of a teaching experiment on a single student's understanding of proof by contradiction. In particular, this paper will address the following question: How does a student's conception of mathematical logic affect their understanding of proof by contradiction? What follows is a detailed discussion of the theoretical framework we will use to answer this question.

Theoretical Framework

Our theoretical framework of choice is APOS Theory – a cognitive framework that considers mathematical concepts to be composed of mental Actions, Processes, and Objects that are organized into Schemas. What follows are definitions of the mental constructions, along with examples within the context of proof by contradiction.

An *Action* is a transformation of mathematical objects by the individual requiring memorized or external step-by-step instructions on how to perform the operation. In the context of proof by contradiction, this may entail a list of steps (either written down or memorized) that a proof by contradiction of an implication statement follows. As an individual reflects on an Action, he/she can think of these Actions in his/her head without the need to actually perform them based on some memorized facts or external guide; this is referred to as a *Process*. Staying within the context of proof by contradiction, a student may interiorize the external list of steps that a particular type of proof by contradiction follows so that these steps become generalized to prove any statement by contradiction. In this way, the student would not need to write the proof in order to talk about what a step may look like in a specific proof. As an individual reflects on a Process, they may think of the Process as a totality and can now perform transformations on the Process; this totality is referred to as an *Object*. Again in the context of proof by contradiction, this could entail comparing proof by contradiction to similar methods such as proof by cases. Finally, a *Schema* is an individual's collection of Actions, Processes, Objects, and other Schemas that are linked by some general principles to form a coherent framework in the individual's mind (Dubinsky & McDonald, 2001). In the case of proof by contradiction, this may include an individual's general proof Schema, mathematical knowledge Schema, and mathematical logic Schema, as well as any steps (external and specific or internal and general) required to write or identify a proof by contradiction.

One of the central goals when using APOS Theory is to develop a series of hypothetical constructions students may need to make in order to understand a concept, referred to as a *genetic decomposition*. This cognitive outline is constructed based on an analysis of the historical development of the concept in question, a literature review of the concept, and the conception of the instructor or researcher. In addition to outlining the mental constructions a student should make in order to understand a concept, a genetic decomposition may include a description of prerequisite knowledge a student should possess in order to begin developing a concept (Arnon et al., 2014). A description of the prerequisite knowledge for proof by contradiction follows.

First, a student should have a general understanding of the propositional logic that underscores all proofs. In particular, a student should be able to perform and understand negations of the following types of statements: without quantification, implications, and single-level quantified statements. This is necessary as every proof by contradiction requires a negation of some statement. In addition to understanding the mathematical logic, a student should be able to move between semantic, symbolic, and algebraic representations of mathematical statements. For example, a student should be able to represent the statement 'If a is even, then $a + 1$ is odd' as the symbolic implication '$P \rightarrow Q$' as well as the algebraic representation 'If $a = 2n$ for $n \in \mathbb{Z}$, then $a + 1 = 2m + 1$ for $m \in \mathbb{Z}$'. Moving between representations of mathematical statements is necessary in any mathematical proof and thus is necessary before developing a notion of proof by contradiction. Finally, a student should be familiar with the direct proof method in a mathematical context as this is the basis of understanding any proof.

With this prerequisite knowledge in mind, we developed a preliminary genetic decomposition. We should be clear that a genetic decomposition need not be unique and so the genetic decomposition below may be one of several different ways in which students may develop an understanding of proof by contradiction.

Preliminary Genetic Decomposition for Proof by Contradiction

1. Students outline the propositional logic of a given proof to develop specific, step-by-step instructions to construct proofs by contradiction for the following types of statements: (i) implication, (ii) single-level quantification, and (iii) property claim.
2. Students interiorize each of the Actions in Step 1 into individual Processes by examining the purpose of statements of given proofs. These Processes become general steps to writing a proof by contradiction for statements of the form (i), (ii), and (iii).
3. Students coordinate the Processes from Step 2 by comparing and contrasting the general steps to determine the necessary steps for *any* proof by contradiction. This Process becomes general steps to writing any proof by contradiction and identifying a proof as a proof by contradiction.
4. Students encapsulate the Process in Step 3 as an Object by utilizing the law of excluded middles to show proof by contradiction is a valid proof method. Alternatively, students encapsulate the Process in Step 3 as an Object by comparing proof by contradiction to other, similar proof methods. Students can now comprehend proofs on a holistic level.
5. When necessary, students de-encapsulate the Object in Step 5 into a Process similar to Step 3 that then coordinates with a Process conception of quantification to prove multi-level quantified statements.

First, note that students' initial conception of proof by contradiction is through the specific steps to construct a proof by contradiction. That is, students think that each of these steps are necessary for a proof to be by contradiction. Students may also need the external cues from specific steps, such as seeing the word "contradiction" near the end of the proof. Students then reflect on these specific steps to considering the general procedure of proof by contradiction. Still, students conceptualize a proof by contradiction as a dynamic procedure that must be followed and thus their conception closely matches the steps to writing a proof by contradiction. It is only after encapsulating the dynamic procedure as a static proof method, by comparing the main idea of the method to similar methods, that students do not rely on the steps of a proof by contradiction to describe the method.

Once developed, a genetic decomposition is used as a guide to develop all instructional material. One pedagogical approach aligned with APOS Theory is the ACE teaching cycle; an instructional approach that consists of three phases: Activities, Classroom discussion, and Exercises. In the *Activities* phase, students work in groups to complete tasks designed to promote reflective abstraction. These tasks should assist students in making the mental constructions suggested by a genetic decomposition. In the *Classroom discussion* phase, the instructor leads a discussion about the mathematical concepts that the Activities focused on. For example, during this phase the instructor may formally state the theorem that was central in the Action phase and together with students write its complete proof. In the *Exercises* phase, students work on standard problems designed to reinforce the Classroom discussion and support the continued development of the mental constructions suggested by the genetic decomposition. The Exercises also provide students with the opportunity to reinforce and apply what they have learned in the Activities and Classroom discussion phases to related mathematical concepts (Arnon et al., 2014). We utilized the proof comprehension assessment model by Mejía-Ramos et al. (2012) to develop standard proof comprehension questions for the Exercises. As this assessment model was pivotal in the development of instructional materials, we will briefly introduce it below.

Mejía-Ramos et al. (2012) presented a multidimensional model for assessing proof comprehension in undergraduate mathematics. This model contains seven different aspects of proof split into two categories: local and holistic. *Local* types of assessment focus on only one, or a small number, of statements within a proof, whereas *holistic* types of assessment focus on student understanding of a proof as a whole. These seven types of assessment are reproduced below:

1. *Meaning of terms and statements:* items of this type measure students understanding of key terms and statements in the proof;
2. *Logical status of statements and proof framework:* these questions assess students' knowledge of the logical status of statements in the proof and the logical relationship between these statements and the statement being proven;
3. *Justification of claims:* these items address students' comprehension of how each assertion in the proof follows from previous statements in the proof and other proven or assumed statements;
4. *Summarizing via high-level ideas:* these items measure students' grasp of the main idea of the proof and its overarching approach;
5. *Identifying the modular structure:* items of this types address students' comprehension of the proof in terms of its main components/modules and the logical relationship between them;
6. *Identifying the general ideas or methods in another context:* these questions assess students' ability to adapt the ideas and procedures of the proof to solve other proving tasks;
7. *Illustrating with examples:* items of this type measure students' understanding of the proof in terms of its relationship to specific examples. (Mejía-Ramos et al., 2012, p. 15-16).

These assessment questions revolve around the reading and understanding of a presented proof. While the authors caution that all types of assessment questions may not be appropriate for some presented proofs, such as if the proof was too short to warrant breaking into modules, we choose proofs such that each type of assessment question could be applied.

In summary, we used APOS Theory to develop a hypothetical model (genetic decomposition) of possible mental constructions students may need to make in order to develop an understanding of proof by contradiction. We then introduced the ACE teaching cycle, an instructional method closely aligned with APOS Theory, which we used to develop lessons on proof by contradiction. We used the proof comprehension assessment model by Mejía-Ramos et al. (2012) to develop the routine exercise questions for the Exercise phase of the ACE teaching cycle. The methodology section will provide a detailed explanation of the lessons on proof by contradiction as well as provide the context of the study.

Methodology

In order to assess our preliminary genetic decomposition for proof by contradiction, we conducted a teaching experiment (Steffe & Thompson, 2000). Unlike a typical instructional sequence of the ACE teaching cycle that, in a regular classroom, usually lasts for a week, this teaching experiment consisted of 5 shorter, consecutive teaching sessions each mimicking the ACE teaching cycle. That is, each session consisted of: students working on the Activity worksheet focusing on a particular component of the genetic decomposition for proof by contradiction (A); a discussion about the concepts from the worksheet (C); and a typical series of proof comprehension questions related to the content of the worksheet (E).

Our teaching experiment was conducted at a large, public R1 university in the southeastern United States with students from a transition-to-proof course. This course, *Bridge to Higher Mathematics*, is the first course in which students at this university are formally introduced to mathematical proofs and their accompanying methods of proof. *Bridge to Higher Mathematics* has the following general learning outcomes:

- Basic Logic (e.g. truth tables, negation, quantification),
- Proof Methods (e.g., direct, contradiction, induction),
- Introductory Set Theory (e.g., union, intersection, power set, cardinality), and
- Introductory Analysis (e.g., least upper bound and greatest lower bound, open/closed sets, limit points).

We began with teaching episodes two weeks into the course, just after a review of basic logic and an introduction to direct proofs. *Bridge to Higher Mathematics* is a primarily lecture-based course where proof and proof methods are taught primarily through proof construction (i.e., asking students to replicate proofs). Thus, in both teaching style and assessment, the designed teaching experiment deviated from the regular classroom. However, the teaching episodes did correspond to the content and proofs that were normally covered in the course.

Data for this report was collected during Summer 2016 from teaching episodes with Chandler. Chandler was an untraditional student: he had already graduated once with a bachelor degree (in an unspecified topic) and was unsure as to whether or not he wanted to complete a bachelor's degree in mathematics. As a senior, Chandler had already taken courses beyond the prerequisites for *Bridge to Higher Mathematics,* including *Differential Equations, Probability Theory*, and *Vector Calculus*. However, none of these additional mathematics courses required proof writing and thus *Bridge to Higher Mathematics* was still Chandler's first experience learning how to read and write formal proofs. While Chandler initially consented to completing all five teaching episodes, he dropped the course before mid-semester. As such, Chandler only completed the first three teaching episodes.

We choose to focus on Chandler for this preliminary report for two primary reasons. First and foremost, Chandler's progression of understanding proof by contradiction is similar to the majority of other participants and thus can be considered representative of a general participant's understanding. In addition, Chandler was open to communicating, answering, and asking questions without any reservation. This provided us with rich data on his thought process. The next section will describe how we analyzed Chandler's thought process through the three teaching episodes he completed.

Data Analysis

All three teaching sessions were video recorded and then transcribed by the teacher/researcher. Transcripts of these three sessions went through multiple passes of analysis. First, we identified Chandler's conception of proof by contradiction, according to APOS Theory, for each teaching episode. Next, we identified Chandler's local and holistic understanding of proof by contradiction and proof in general, according to the proof comprehension assessment model by Mejia Ramos et al. (2012), during each teaching episode. We then open coded for other themes that affected Chandler's understanding of proof by contradiction, during which quantification emerged. Finally, we examined quantification's role in Chandler's understanding per teaching episode. What follows are the results from this analysis.

Results

For each episode, we will first present the statement and proof Chandler was asked to read as well as provide the comprehension questions he was then asked to complete. Afterwards, we will provide some of Chandler's responses that best illustrate his understanding of proof by contradiction in general and his understanding of the presented proof. When applicable, we will include a discussion of how mathematical logic, and quantification in particular, affected Chandler's proof comprehension.

Pre-Teaching Episodes

Before the first teaching episode, we asked Chandler a few basic questions to ascertain his background with mathematical proof. In particular, we were focused on determining whether Chandler possessed the following prerequisite knowledge necessary to begin developing a conception of proof by contradiction: mathematical logic (especially negating statements), how to transition between representations, and proof in general. Chandler exhibited at least an Action conception of mathematical logic as he was able to describe rules to use in order to negate propositional statements. Chandler was also able to answer questions that required him to transition between mathematical statements, their propositional representations, and their algebraic representations. Finally, with respect to his understanding of proof in general, Chandler gave the following definition:

An effort to show something is true or false based on real evidence and facts.

We see his definition as a dynamic procedure, i.e. an effort to show something, and so infer Chandler possesses a Process conception of proof. Based on his responses, we believe Chandler had the prerequisite knowledge to begin developing a conception of proof by contradiction.

We also wanted to know what conception, if any, Chandler had of proof by contradiction before the first teaching episode. When asked for a definition of proof by contradiction, he stated:

Demonstrating the opposite of a statement is false, so the statement is true.

Note there is no mention of an assumption or contradiction in terms of "demonstrating the opposite of a statement is false" and thus this is not a complete definition of proof by contradiction. At most, this is a definition for a general type of indirect proof that utilizes the tautological equivalence of a double-negated statement and the statement itself. We therefore do not believe Chandler has a conception of a mathematical proof by contradiction.

Teaching Episode 1

Teaching episode 1 began with Chandler converting a series of statements into propositional and predicate logic on his own. After checking his answers to these conversions, it was clear that Chandler was able to convert mathematical statements with a single quantifier into a logical form. Chandler was then asked to read the following statement and proof.

Statement 1: The set of primes is infinite.
Proof 1: Suppose the set of primes is finite. Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$. Let n be one more than the product of all of them. That is, $n = (p_1 p_2 p_3 \ldots p_k) + 1$. Then n is a natural number greater than 1, so n has a prime divisor q. Since q is prime, $q > 1$. Since q is prime and $p_1, p_2, p_3, \ldots, p_k$ are all of the primes, q is one of the p_i in the list. Thus, q divides the product $p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k$. Since q divides n, q divides the difference $n - (p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k)$. But this difference is 1, so $q = 1$. From the contradiction, $q > 1$ and $q = 1$, we conclude that the assumption that the set of primes is finite is false. Therefore, the set of primes is infinite.

Chandler was then prompted to answer the following comprehension questions on his own.

1. Please give an example of a set that is infinite and explain why it is infinite.
2. Why does n have to have a prime divisor?
3. Why exactly can one conclude that q divides the difference $n - (p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k)$?
4. What is the purpose of the statement "Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$."?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of the proof?
7. In the proof, we define $n = (p_1 p_2 p_3 \ldots p_k) + 1$. Would the proof still work if we instead defined $n = (p_1 p_2 p_3 \ldots p_k) + 31$? Why or why not?
8. Define the set $S_k = \{2, 3, 4, \ldots, k\}$ for any $k > 2$. Using the method of this proof, show that for any $k > 2$, there exists a natural number greater than 1 that is not divisible by any element in S_k.

Considerable attention was given to local comprehension questions 1-4 as Chandler was not able to discuss questions 5-8. We highlight his most enlightening responses to the Activity and these comprehension questions below.

After reading the statement and its proof, Chandler was asked to describe or define proof by contradiction. He said:

> *Aren't you supposed to show not the conclusion, or try to prove not the conclusion and try to come up with something that is?*

While this definition discusses the final steps of a proof by contradiction, it focuses on "showing not the conclusion" rather than discussing how the initial assumption and contradiction relate to the original statement. Thus, we believe Chandler exhibited a PreAction conception of proof by contradiction.

Chandler was able to describe the meaning of key words and phrases, such as when he described infinity as:

> *Every time it [a real number] goes up, you can still go up by 1.*

However, Chandler was unable to describe the purpose of statements and provide justification for lines in the proof. For example, he initially wrote that the purpose of the statement 'Let $p_1, p_2, p_3, \dots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$' was:

> *To construct an infinite number of possibilities for number* n*, since numbers are composites of primes* p *and* n *is not a particular natural number.*

When the teacher asked Chandler to clarify his written response during the discussion, he said:

> *Infinity. Uh, well the set of primes is infinite. [long pause] Is it,* n *is composed of a finite number of parts?*

In both cases, Chandler did not recognize the statement as defining the assumed finite set of primes so that they may be used to produce a new prime. When the teacher asked Chandler to provide justification for the line 'q divides the difference $n - (p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_k)$', Chandler realized that he wrote multiple answers and even expressed confusion on what he was thinking:

> *I wonder what I decided… there's several things boxed in.*

All of the answers Chandler initially provided involved rewriting statements in the proof without further clarification. Even after an extended discussion considering the statement with a small number of primes, Chandler was unable to provide justification for the line. These examples suggest that while Chandler could understand what individual statements meant, he did not know how these statements logically or mathematically followed from one another. Thus, we believe Chandler showed some local understanding of the presented proof.

When asked to summarize the main idea of the proof, Chandler restated the statement. When asked what the key steps of the proof were, he said:

> *Supposing the contradiction is true, that the set of primes is finite and showing how to prove that by reducing an infinite number of primes down to the number 1… I don't know what that means.*

When asked whether the proof would still work if we changed $n = (p_1 p_2 p_3 \dots p_k) + 1$ to $n = (p_1 p_2 p_3 \dots p_k) + 31$, he said that it would "Because the difference is 31, so q is 31 and q is still greater than 1. I don't know." In the questions above, Chandler does not exhibit any holistic understanding. Yet, Chandler does show some holistic understanding in his attempt to show that for any $k > 2$, there exists a natural number greater than 1 that is not divisible by any element in $S_k = \{2, 3, 4, \dots, k\}$. Chandler was able to identify the previous construction of n as being necessary for the new proof and began his proof by appropriately modifying the new number $n = (2 \cdot 3 \cdot 4 \cdot \dots \cdot k) + 1$. Rather than showing that this number is not divisible by any element in S_k, Chandler tried to mimic the presented proof. It is likely that Chandler was unsuccessful in writing the proof because he did not understand the multi-level quantified statement and, in particular, did not know how to negate this statement. Consider the following discussion between Chandler and the teacher about his proof:

T: Any idea how to negate this statement?
C: And I thought of that but I just, it didn't seem following along, and I sure didn't [know] how to get it following along like this one. I think the contradiction would be: for all natural numbers greater than 1, wait [pause, mumbling to self] for all natural numbers greater than 1… I don't know.
T: Okay, let's try to rewrite this in symbols, maybe.
[After prompting, Chandler writes $\forall k > 2, \exists n > 1 \in \mathbb{N}\ s.t.\ s \nmid n \ \ \forall s \in S_k$*]*
T: Can you negate this now?
C: I think, piece by piece. There's a whole string of these, I don't know. I think for all n less than or equal to 1, or yeah, is that right?
T: We also do the, for every k greater than 2, so maybe it's "there exists a k greater than 2" so there is actually one of them.
C: Or, wait, which part? You don't negate this [points at $n > 1$*], you negate this [points at* $\forall$*]. The quantifier?*
T: Yes, yes.
C: So you don't negate the [trails off].
[After more discussion on negating parts of the statement.]
C: I'm trying to remember [pause] such that for some s, s divides n, for s in the set. Is that the entirety?

While Chandler was unable to negate the multi-quantified statement (either as a mathematical statement or in algebraic notation), he did say

> *And I thought of that [negating the statement] but I just, it didn't seem following along, and I sure didn't how to get it following along like this one…*

which indicates that he may have been able to complete proof had he been able to negate the multi-quantified statement. We also see phrases such as "I'm trying to remember…" and errors such as "I think [the negation would be] for all n less than or equal to 1…" that indicate Chandler may need an external guide or table to negate quantified statements even single-quantified statements. At least in this case, it is possible that Chandler's conception of quantification was inhibiting his holistic comprehension of the proof.

Teaching Episode 2

Teaching episode 2 began with Chandler examining the following statement and proof, after which he was asked to provide a definition of proof by contradiction based on the presented proof.

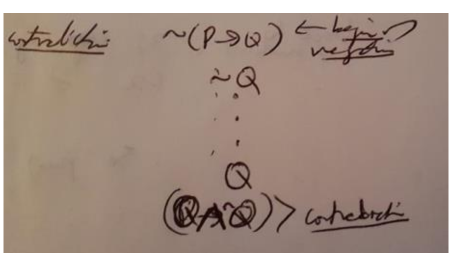
> *Statement 2:* If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.
> *Proof 2:* Assume that every even natural number greater than 2 is the sum of two primes and that it is not the case that every odd natural number greater than 5 is the sum of three primes. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k. Then $k = 2n + 1$. Since $k > 5$, $2n + 1 > 5$ and so $n > 2$. Rewriting k, we have $k = 2(n-1) + 3$. Since $n > 2$, $2(n-1) > 2$ and so is the sum of two primes: p and q. Thus $k = p + q + 3$. This is a contradiction as we assumed k was not the sum of three primes. Therefore if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

After a discussion on the underlying structure of the presented proof and how this related to proof by contradiction in general, Chandler was asked to answer the following comprehension questions on his own.

1. Please give an example of a prime number and explain why it is prime.
2. Why is every even natural number greater than 2 the sum of two primes?
3. Why exactly can one conclude that $k - 3$ is the sum of two primes?
4. What is the purpose of the statement "Since $k > 5$, $2n + 1 > 5$ and so $n > 2$."?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of the proof?
7. In the proof, we rewrite $k = 2(n-1) + 3$. Would the proof still work if we instead rewrite $k = 2(n-2) + 5$? Why or why not?
8. Using the method of this proof, show that: if every odd natural number greater than 5 is the sum of three primes and 3 is one of those primes, then every even natural number greater than 2 is the sum of two primes.

Unlike teaching episode 1, Chandler was able to talk more about the holistic questions than he could previously. We highlight his most enlightening responses below.

After discussing the specific structure of the presented proof, Chandler came up with the following step-by-step guide for a general implication statement:



<u>Contradiction</u> $\sim (P \to Q)$ ← <u>begin negation</u>

$\sim Q$

$\vdots$

Q

$(Q \wedge \sim Q)$ > <u>contradiction</u>

Figure 1: Chandler's written structure (left) and his recreated structure (right)

First, note that the logical connection between the assumption, $\sim (P \to Q)$, and the next line, $\sim Q$, is omitted as well as the logical connection between the assumption and contradiction in order to conclude the proof. Also, there is no mention of the role of P in the proof. These omissions indicate that the structure above is meant to *write* a complete proof by contradiction and not to outline the logical structure of the method. In other words, Chandler focus is on the procedure and not in understanding why the procedure works. Thus, we believe he exhibits an Action conception of proof by contradiction.

After reading the first line of the proof and before attempting any of the comprehension questions, Chandler said:

> *I'm not sure when something contradicts, if you begin with 1, step 1. The assumption, or the given, was that P and [sic] Q. If you begin with P and not Q, it's already a contradiction, isn't it? No?*

Once the teacher explained the logical equivalence between the hidden initial assumption step, $\sim (P \to Q)$, and the first statement of the proof, $P \wedge \sim Q$, Chandler was able to understand the logical relation between the statement and the first line of the proof. However, rather than include this equivalence in his proof by contradiction steps above, he only included $\sim Q$, the consequent of $P \wedge \sim Q$ the proof uses to move forward and end up with a contradiction. While his steps may now avoid attending to local understanding of the statement and assumption, we see an instance in which Chandler, unprompted, attended to the local understanding of a particular proof by examining the propositional logic.

When writing his proof for comprehension question 8, Chandler had difficulty negating statement Q (a key step in his current conception for proof by contradiction) without first translating the statement into propositional logic and using rules to negate the propositional logic. An excerpt of what transpired is provided below:

> *T: Do you have any ideas on how to approach this proof? [Long pause]*
> *C: Not Q would be 'every even natural number greater than 2 is not the sum of two primes'? No, all? I don't know.*
> *[Teacher prompts Chandler to write the statement in propositional logic]*
> *T: Alright. So for every n here, that's how you said it to me. So parentheses for all n, in the natural numbers, if n is greater than 2, then n is p + q. So what would be the negation of this statement?*
> *C: There is a natural number…*
> *T: There is a natural number.*
> *C: n greater than 2 that is not equal to p + q.*

We see Chandler utilizing his general procedure for implication statements by starting with negating statement Q. Once he wrote the statement in propositional logic, Chandler was able to recognize the negation of a 'for all' statement as 'there exists' and continue using his procedure to complete the proof. This suggests that Chandler's conception of quantification still inhibited his holistic understanding of the proof.

Teaching Episode 3

Teaching episode 3 began with Chandler examining the following statement and proof, after which he was asked to provide a definition of proof by contradiction based on the presented proof.

Statement 3: There is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k.

Proof 3: Assume it is not true that there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. Then there is an odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. Let n be that integer; that is, $\exists n \in \mathbb{N}$ such that $n=4j-1$ and $n=4k+1$ for $j,k \in \mathbb{Z}$. Then $4j-1=4k+1$ and so $2j=2k+1$. Note that $2j$ is an even number and, since $2j=2k+1$, $2j$ is an odd number. A number cannot be both even and odd and thus this is a contradiction. Therefore, it is not true that it is not true that there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. In other words, there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k.

Chandler was then asked to compare and contrast Proof 2 and Proof 3. After a discussion on the similarities and differences of the underlying structures of these proofs how this related to proof by contradiction in general, Chandler was asked to answer the following comprehension questions on his own.

1. Please give an example of an integer that is odd and explain why it is odd.
2. What kinds of numbers can be expressed in the form $4j-1$?
3. Why exactly can we assume "$\exists n \in \mathbb{N}$ such that $n=4j-1$ and $n=4k+1$ for $j,k \in \mathbb{Z}$." ?
4. What is the purpose of the statement "Note that $2j$ is an even number and, since $2j=2k+1$, $2j$ is an odd number."?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of the proof?
7. In the statement, we have $4j-1$ and $4k+1$. Would the proof still work if we instead have $4j-3$ and $4k+3$? Why or why not?
8. Using the method of this proof, show that there is no odd integer than can be expressed in the form $8j-1$ and $8k+1$ for integers j and k.

The teacher/research focused Chandler's attention on the holistic comprehension questions as Chandler was able to give quick, confident answers to the local comprehension questions. We provide some of his most enlightening responses below.

As mentioned above, Chandler was asked to compare Proof 2 and Proof 3 in their presented form. An excerpt of this comparison is provided below, along with a recreation of Chandler's two proof structures that he eventually compared in place of the semantic representations of the proofs.

C: They are both contradiction, proof by contradiction, and [pause] I don't know.
T: Well, maybe… let's see. You have the structures of both proofs, right?
[Teacher prompts Chandler to examine the structures he already developed.]

T: So you were saying something before you started talking about the $\sim Q$; you were pointing at [steps] 5-7.
C: Yeah. Those are similar – [steps] 4-7 are similar to this, [steps] 4-6, of course.

Table 2: Recreation of Chandler's proof structures for implication (left) and nonexistence (right) statements.

$P \rightarrow Q$	$(\nexists x)(P(x))$
1. $P \wedge \sim Q$ 2. $\sim Q$ 3. $\sim Q \rightarrow R_1$ 4. $R_1 \rightarrow R_2$ 5. $R_2 \rightarrow R_3$ 6. $R_3 \rightarrow Q$ 7. $Q \wedge \sim Q$ 8. $\sim (P \wedge \sim Q) = P \rightarrow Q$	1. Assume $\sim (\nexists x)(P(x))$ 2. $\sim (\exists x)(P(x))$ [sic] 3. $P(n)$ 4. Algebra until contradiction: $4j - 1 = 4k + 1$ means $2j = 2k + 1$ 5. $\sim (\sim (\nexists x)P(x))$ 6. $(\nexists x)P(x)$

What Chandler is talking about here is the role of parts of the proof. In particular, he is considering how each structure arrives at the contradiction and uses this contradiction to complete the proof (Note: although he initially excludes step 8 in the implication structure as seen in Table 2, he clarifies the role of this step later in the discussion). Chandler uses this insight when he developed a general set of steps for any kind of proof by contradiction, replicated below:

1. *Assume the claim is false.*
2. *Rewrite the assumption.*
3. *Do algebra until contradiction.*
4. *The assumption is false, so the original claim is true.*

Chandler leveraged the propositional and quantified logical structure of the proofs to produce a general, non-logical list of steps for *any* proof by contradiction and thus we believe he developed a Process conception of proof by contradiction.

Chandler used of the general steps above to identify the purpose of single statements in the presented proof. When asked why the proof can assume 'there is an odd integer n such that $n = 4j - 1$ and $n = 4k + 1$ for $j, k \in \mathbb{Z}$', Chandler responded:

Because we are claiming the negation of "there is no odd integer n such that $n = 4j - 1$ and $n = 4k + 1$" which is "there is an odd integer n…"

We see that Chandler recognized 'there is an odd integer n*…'* as a rewrite of the assumption without needing to refer to the logical equivalence of the statements. In addition, Chandler identified that the purpose of the statement 'Note that $2j$ is an even number and, since $2j = 2k + 1$, $2j$ is an odd number' was to identify the contradiction. In both cases, we see Chandler relating the logical status and justification of statements to the general steps he has developed for a proof by contradiction. In doing so, he no longer needed to refer to the underlying structure of the proof in order to locally comprehend the presented proof.

Unlike teaching episodes 1 and 2, Chandler was able to describe the key steps of the presented proof and stated:

C: Rewording it, I think. Rewording the claim into a negation and the algebra.
T: That makes sense. Alright.
C: For this one. I mean… I don't know.

What Chandler is describing as the key steps for this proof are steps 2 and 3 for general proofs by contradiction that he constructed during the Discussion of this teaching episode, though he clarifies that these may only be the key steps for this proof. Based on the previous steps he has listed as key steps, it seems that Chandler conceptualized the key steps of a proof as the steps in which he found to be most difficult. With this in mind, his clarification that the key steps of another proof may be different likely refers to the idea that the *difficult steps* of another proof may be different and not that the *general steps* of another proof by contradiction may be different. Thus, we believe Chandler exhibits holistic understanding of the presented proof based on his general understanding of proof by contradiction.

Discussion

It was evident that logic, and quantification in particular, played an important role in Chandler's understanding of particular proofs and proof by contradiction in general. First, he leveraged propositional logic to develop specific steps for a particular kind of proof by contradiction as well as considered the logical connection between the statement and assumption of a particular proof. We also observed that Chandler's initial difficulties with negating quantified statements inhibited his ability to transfer the methods of a particular proof into a similar context. By the end of the teaching experiment, Chandler developed a list of general steps any proof by contradiction should contain through the comparison of two different logical structures of particular proofs by contradiction. These steps not only aided Chandler in transferring the method of one proof to similar proof, but they aided Chandler in recognizing the logical status and justification of statements in a particular proof.

Chandler's conception of mathematical logic seemed to, at times, inhibit his proof comprehension. In particular, Chandler exhibited a need to convert statements into propositional or predicate logic in order to either negate the statement or recognize the logical relation between pairs of statements. This is especially important to highlight for proof by contradiction as students are not convinced of the validity of the proof method (Brown, 2011). However, by exploring the logical relation between the assumption, contradiction, and original statement, students may become convinced and thus more accepting of the method. At the very least, we found that prompting Chandler to explore the logical relation between statements in a proof and to develop key steps aided him in developing a robust understanding of proof by contradiction.

Future Plans

Overall, the teaching experiment seemed to aid Chandler in developing a robust understanding of proof by contradiction by focusing on the logical relationships within proofs. These results further a growing belief that attending to the logical relationship of statements may greatly improve proof comprehension (Brown, 2013; Hodds et al., 2014). Before we consider the teaching experiment useful to a general student at the university, we need to consider more case studies. Five more students from *Bridge to Higher Mathematics* volunteered and participated in at least 3 teaching episodes during Fall 2016. We plan to analyze these students' understanding of proof by contradiction in terms of APOS Theory and their understanding of specific proofs via the proof comprehension assessment model by Mejía-Ramos et al. (2012) in a similar fashion to Chandler. Positive results from these cases would strengthen our claim that the teaching experiment would be useful for general transition-to-proof students and allow us to examine the role of mathematical logic in understanding proof by contradiction.

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Using Women of Color's Intuitive Examples to Reveal Nuances about Basis

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Research and surveys continue to document the underrepresentation of women of color (WOC) in mathematics. Historically, their achievement in mathematics has been framed in a deficit way. Following the broader call for more research concerning WOC's learning experiences in STEM, we interviewed eight WOC about their understanding of basis in linear algebra. We documented diverse ways that these women creatively explained the concept of basis using intuitive ideas from their everyday lives. These examples revealed important nuances and aspects of understanding of basis that are rarely discussed in instruction. These students' ideas can also serve as potentially productive avenues to access the topic. Our results also challenge the existing broader narrative about the underachievement of women of color in mathematics.

Key words: student thinking, basis, linear algebra, equity

Women of color continue to be underrepresented in most areas of science, technology, engineering and mathematics (STEM), and more research is needed to understand the experiences of women of color in those areas (Ong, Wright, Espinosa, & Orfield, 2011). Their underrepresentation is also situated in the national call for more graduates in those fields in the U.S. (PCAST, 2012). In this study, we centralize the mathematical sense making of these students to counter the common colorblind approach to studying cognition. This focus has the potential to construct counter-narratives about women of color's achievement in STEM (Adiredja, in preparation). Research has historically positioned students of color as struggling or underachieving (Harper, 2010).

Research in post-secondary mathematics education has uncovered useful insights into the process of learning of advanced topics by focusing on students' individual cognition. However, scholars have noted the tendency of cognitive studies to deemphasize equity concerns (Martin, Gholson, & Leonard, 2010). Studies of mathematical cognition often take a colorblind approach in which students' background information is omitted (Nasir, 2013). There is a broader call for research at the post-secondary level to focus on addressing inequities, which include exploring ways that studies of student thinking can engage with issues of equity (Adiredja & Andrews-Larson, under review).

Theoretical perspectives on the nature of knowledge and how it develops directly impact the way we assess students' understanding and their contributions in mathematics. Studies of cognition share the power to determine what counts as productive knowledge, how learning is supposed to happen, and what kinds of students benefit in the process (Adiredja, 2015; Apple, 1992; Gutiérrez, 2013). For example, if we believe that mathematical knowledge can only be built upon prior formal mathematical knowledge, then it would be reasonable to privilege such knowledge in learning. However, one implication of this stance is that students who do not have the requisite formal knowledge would then be positioned as "not ready" or "less able," while students who do are seen as "smart," and are allowed to move forward (Gutiérrez & Dixon-Roman, 2011; Herzig, 2004). This deemphasizes the reality that such knowledge has been more available to some groups than others (Oakes, 1990). Cognitive studies can challenge some of these assumptions, and broaden what counts as productive ways of thinking and who counts as successful learners.

One way this is occurring at the undergraduate level is through research that focuses on building from students' informal knowledge and intuitions. For example, the instructional design theory, Realistic Mathematics Education (RME) (Freudenthal, 1983) has inspired some researchers to design curricula that build from experientially real start points (e.g., abstract algebra, Larsen, 2013; differential equations, Rasmussen & Kwon, 2007; geometry, Zandieh & Rasmussen, 2010; linear algebra, Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). Others focus on students' intuitions or conceptual metaphors to make sense of formal mathematics (Adiredja, 2014; Oehrtman, 2009; Zandieh, Ellis, & Rasmussen 2012). Most of these studies focus on using students' intuitions/ informal knowledge as building blocks for the more formal mathematical knowledge. In this paper, we explore ways that students' intuitive explanations can reveal nuances about a formal mathematical topic.

We explore students' explanations about the concept of basis in linear algebra using everyday ideas. Linear Algebra is a critical course for engineering and mathematics majors, and the concept of *basis* is a central topic. There has not been much research done about the concept of basis, though some researchers suggest that it is challenging for students (Stewart & Thomas, 2010). Stewart and Thomas (2010) found that students struggle in identifying span and linear independence in their description of basis. Moreover, they also struggle to define each of those terms, and tend to explain those concepts in terms of procedures. However, the authors found that students who received instruction that emphasized "geometry, embodiment and linking of concepts" (p. 177), were much better in describing the concepts compared to those whose instructions solely focused on symbolic algebra and isolated concepts. Students from the former group were also able to draw richer concept maps about basis than students from the more traditional class. These findings further support the approach of building from experientially real starting points (geometry and embodied ideas) to support students' understanding.

In this paper, we want to explore the diversity of ideas used by eight women of color to describe the concept of basis. We are less interested in identifying students' struggles with the concept of basis. Later we argue for the importance of adopting an anti-deficit perspective in analyzing students' ideas. In particular we explore the following research questions:

1. What everyday contexts do these women use to explain the concept of basis?
2. What do their explanations reveal about nuances in the concept of basis?

In this paper, we position these women as informants into their mathematical thinking and what it reveals about the nuances of the concept of basis.

Conceptual and Theoretical Frameworks

The "Anti-deficit Achievement Framework" from higher education research (Harper, 2010) guides the design and analysis of this project. Instead of perpetually focusing on examining deficits or struggles of students of color in STEM, Harper's framework focuses on understanding the success of these students despite existing inequities. For example, instead of asking the question, "Why are Black male students' rates of persistence and degree attainment lowest among both sexes and all racial/ethnic groups in higher education?" This deficit-oriented question can be reframed with the Anti-deficit Framework as, "How did Black men manage to persist and earn their degrees, despite transition issues, racist stereotypes, academic underpreparedness, and other negative forces?" (p. 68). The Anti-deficit Framework focuses on challenging a particular narrative about underachievement of students of color in STEM, which is also attached to women *of color* (e.g., Why are women dropping out of computer science?). While the author's work focuses on inequities in STEM higher education, some researchers in undergraduate mathematics education share similar perspectives in their study of students (e.g., successful Black mathematics majors, Ellington & Frederick, 2010).

In this paper we focus on countering the narrative of underachievement of women of color in mathematics, particularly with regards to their participation. The principle of "centralizing without essentializing" experiences (Bell, Orbe, Drummon, & Camara, 2000) provides an alternative to *essentializing* experiences of women of color (e.g., "all women of color struggle with basis in this way"). Centralizing instead leads to a focus on understanding the richness of their knowledge and sense making without attributing particular ways of sense making to all students in these groups.

The two authors of this paper come from different theoretical traditions, which guide our joint analysis. The first author comes from Knowledge in Pieces (KiP) (diSessa, 1993) perspective, and the second author has used RME (Freudenthal, 1983) and conceptual metaphors (e.g., Lakoff & Johnson, 1980) to productively analyze student thinking. The commonality of the different perspectives is that they view students' knowledge in an anti-deficit way. These perspectives highlight the use of intuitive ideas to explain mathematical and scientific ideas, and ways that they can be productive in building on students' existing understandings. In laying out our perspectives we position ourselves as being open to students' intuitions and non-normative language.

Methods

Given the desired depth and detail of analysis, this study favored the use of a small number of research subjects and videotaped individual interviews (diSessa, Sherin, & Levin, 2016). Participants were 8 undergraduate female students of color at a large public research university. The university's mathematics advising center shared contacts of mathematics majors and minors who identified as women of color. We invited students via email, and through personal contacts of the authors of this paper. The breakdown of racial and ethnic backgrounds and their past mathematics courses are presented in Table 1. This information was drawn from a student background survey that was administered at the end of the interview. With the exception of one student who is a Biomedical Engineering major (Morgan), all the other students were mathematics major or minor. All pseudonyms were selected to reflect the origin of students' names.

Each interview lasted for 90 minutes. We developed the protocol to explore student understanding of basis. Students started the interview by solving four linear algebra problems that did not mention basis but for which basis could be relevant. We then asked them about basis, which included the way they would define it, and everyday ideas that were useful to explain the concept. In this paper, we focus specifically on students' discussion of everyday examples. These occurred most often in response to questions Q2a and Q2b below, but also sometimes in response to other questions later in the interview.

Q2a. Can you think of an example from your everyday life that describes the idea of a basis?
Q2b. How does your example reflect your meaning of basis? What does it capture and what does it not?

Table 1. *Students' Racial/Ethnic Background and Mathematics Course History*

Student	Racial/Ethnic Background	Linear Algebra Completion	Grade	Other Mathematics Courses
Leonie	African American	Spring 2016	A	Calculus I, II, and III

Morgan	Asian/Asian American	Spring 2016	A	Calculus I, II, and III, and Differential Equations
Annissa	Hispanic/Latin@	Fall 2014	B	Calculus I and II
Eliana	Hispanic/Latin@	Spring 2014	C	Calculus I and II
Nadia	Hispanic/Latin@	Fall 2015	A	Calculus I, II, and III
Jocelyn	Hispanic/Latin@	Spring 2015	B	Calculus I, II, and III
Stacie	Hispanic/Latin@	Spring 2016	C	Calculus I, II, and III
Liliane	Hispanic/Latin@/White	Fall 2015	B/C	Calculus I, II, and III

We transcribed the interviews following guidelines from Ochs (1979). Transcripts were organized by turns, marked by changes in speaker. Transcripts use modified orthography (e.g., wanna, gonna, cus) to stay close to the actual students' utterance. Our analysis first focused on identifying the everyday context and the details associated with that context (e.g., how does the student think about a vector, a vector space, or scalar multiple?). We then differentiated between utterances that had to do with characteristics of the basis vectors, and those that had more to do with roles of the basis vectors in relation to the larger space. The next step of the analysis is developing codes through open coding (Strauss & Corbin, 1994) to capture nuances of students' understanding of basis.

Results

Students' Everyday Examples

We found that the majority of the students discussed at least one everyday context to explain the concept of basis. Table 2 provides a summary of the different contexts. We elaborate on the details of some these contexts in a later section.

Table 2. *Everyday contexts used to explain basis and vector spaces*

Student	Context (for basis and vector space)
Leonie	friendship
Morgan	driving in a city (on a grid), Legos, cooking, groups of pens
Annissa	set of solutions (no actual everyday example)
Eliana	least amount of myself I need to cover the space of the room, storage room, dimension, skeleton, outline of a paper
Nadia	floor, universe and earth, syntax in programming
Jocelyn	fashion, recipe, art sculpture, collage
Stacie	walking to places in a room, floor as a plane, marching band
Liliane	religious teachings

Characteristics and Roles of Basis Vectors

Students discussed characteristics of the basis vectors in their everyday examples. While many of the ideas they brought up can be associated with the notion of linear independence, their ideas reveal nuances about linear independence and its role in defining basis. The first set of codes below capture these characteristics of basis:
1. Minimal/maximal focuses on the required number or amount of vectors needed for the basis. Minimal focuses on the fact that the basis is the least amount of vectors necessary. Maximal focuses on the need to include all the basis vectors and that more would lead to redundancy.
2. Essential focuses on the quality of the vectors being the core and necessary.

3. Representation focuses on naming or identifying the smaller set as the structure or representation of the larger space.
4. Non-redundant focuses on not wanting extraneous information in a set, or the act of reducing or removing the extraneous information.
5. Different/sameness focuses on comparing items (vectors) based on their difference/similarity for the sake of keeping or removing items from the basis.

Prior to conducting the interviews we had noticed that basis vectors are sometimes emphasized in mathematics lessons as a way to *generate* a space of vectors. This is an emphasis on the basis vectors as a spanning set. Other situations might emphasize the role of basis vectors to describe the vector space, e.g., the basis for the null space of a non-invertible matrix serves as a short hand to *describe* that space. We asked students directly about this idea of generating and describing through Q5 and Q6, which ask, "Can you see a basis as a way to [generate/ describe] something?" But even before we asked students directly, they spontaneously brought up the idea of generating and describing in their description of basis, suggesting resonance of this idea. Lastly, related to the code of representation, students also brought up the role of choosing the particular vectors to represent the larger space.
1. Generating: To create the larger space from the basis vectors.
2. Describing: To describe the space using the basis vectors.
3. Sampling: To choose the particular basis vectors as representatives of the larger space.
In the next section we share our analysis of one student, Jocelyn to illustrate how we operationalize these codes.

Illustrative example of analysis. We illustrate our analysis and codes with one student, Jocelyn. Asking students to assess the validity of their everyday example turned out to be very informative of aspects of basis to which students were attending. Jocelyn's case illustrates ways that some of the codes emerged in our analysis. Jocelyn described basis in the context of fashion/creating an outfit. Jocelyn saw basis as minimum number of clothing pieces that "allows you to make all those outfits." In this turn, she was explaining what aspects of basis her example captured. We used bold texts to mark the codes in the write up of our analysis.

Interviewer: In the way you think about this sort of outfit idea to describe basis, um, what aspects of your understanding of basis is captured with your example and what part of it is not captured?

Jocelyn: Um it's minimal. To pick one pair of heels and one pair of tennis shoes. So when I think of my idea of a basis, my mind goes to minimal. Um, what doesn't it capture? Well, ok, so it's weird cause I guess you can use one pair of shoes for different outfits. But like if I'm trying to make...it's harder to kind of have like a casual outfit and in a formal outfit there's not a whole lot of like overlap you end up having each piece in each outfit in the basis. So it's like. How do I explain this? I feel like the basis I'm making, all of the pieces aren't as like they're not all the same. Like you have shoes, tops and pants. You can't make an outfit with just shoes. But if you have a basis, you can pick just some of the vectors, combine them and make something and leave all the rest out. Cause you can't just put on shoes and pants. So that's where it kinda...that's one of the ways that doesn't really [captures it].

Jocelyn attended to the **minimal** aspect of a basis. In addition to using the word minimal, she explicitly identified the need for the specific quantity of one pair of heels and tennis shoes. The particular pair of shoes serves as a **representation** for formal shoes and casual shoes respectively, and you end up with "each piece in each outfit in the basis." Earlier in the interview she discussed the necessity of one pair of heels for a formal outfit and one pair of tennis shoes for a casual outfit. It was also important to Jocelyn that the basis vectors were **different** ("they're not all the same"), and as a set was **non-redundant** (lack of "overlap" in the pieces). She also highlighted the **essential** nature of each element of the basis. She explained this in the way that an outfit needs shoes, pants *and* tops. She asserted, "you can't make an outfit with just shoes," or "just put on shoes and pants."

In addition to attending to particular aspects about the basis vectors, Jocelyn also attended to the roles of the basis vectors in the larger space. In particular she attended to the role of the basis vector in **generating** the larger space, and the necessity of choosing or **sampling** the particular basis vectors to represent the larger space. We coded the phrase "combine them and make something" as highlighting the **generating** relationship. We coded "to pick one pair of heels and one pair of tennis shoes" as also highlighting the **sampling** relationship. This is not to be confused with the representation code, which focuses on the basis vectors as representatives of the larger space.

Jocelyn's assessment also revealed a unique concern that we did not observe with other students about the ability to choose some of the vectors in the basis but not all of them. Jocelyn argued that with basis vectors, one could use a subset of them to generate a vector in the space. In the context of fashion, she could take a subset of the pieces to generate an outfit. The fact that she needed a top, a bottom and a pair of shoes meant that her example required that she had one piece from each category of clothing. She saw this as a limitation to her example. In the next section we explore the extent to which students in the study brought up roles of the basis vectors.

A focus on the generating relationship. In the process of analyzing students' everyday examples we found a wide range of examples of generating, a smaller number of examples of describing, and some examples of sampling. We focus now on most common code on the role of the basis vectors: **generating**. Each student had at least one example of what we label as generating.

Common verbs for the generating code were variations of "to make" or "to build." For example, Morgan talked about building in terms of Legos, "you're given like the 3 by 2 Lego [pieces] and you have like a 2 by 2 Lego [piece] you can just like *build* on to that to create that I guess space that you have." Nadia spoke about computer programming syntax, "Syntax is like stuff so you can *make* a program that doesn't give you an error." Other variations include "add," "expand", and "come from that," which Liliane used in her description of basis using religious teachings:

> So I'm very religious and so the teachings that I that we share with each other and that we read about and all that stuff. Like, there are a lot of things that you can *add* to and be like here's an application and here's the things, and this *expands* to this and this and this. But there's like the most basic teachings and like it all comes back to that. And this is the basic thing like you have the Ten Commandments. You have the Scriptures and you have like the prophets and you have your connection with God and, like all of the decisions and all of things that *come from that* and you can reach all of the other points with this basis.

These examples illustrate the different ways students brought up the notion of generating, and these women's creativity with the everyday contexts.

Discussion and Implication

The two main components of our results illustrate (1) the creativity and breadth of the everyday contexts used to describe basis by these female students of color, and (2) the nuances in understanding of basis that have come out of our open coding of this student data. The range of examples that students used was particularly interesting and useful. Most of these were not examples we had thought of ourselves prior to beginning the study. We are mindful of not gendering or racializing these examples, which would lead to essentializing the students. Students did discuss basis in the context of fashion, cooking and religion, but they also brought up other contexts like driving, skeleton, and the universe. These contexts are likely inspired by the students' experiences, and not their background characteristics. Future studies can further explore the range of contexts to explain basis, and the details of their differences. One can also investigate if there are shared learning experiences among these women that contributed to their flexibility to come up with these examples. Moskovich (2012) asserted that there is nothing inherently different about the cognitive processes of students of color in mathematics. However, there is a difference in their "conditions of learning" (p. 96). We conjecture that the different conditions for learning might have contributed to the creativity of these students.

These women were also fairly sophisticated in judging their own examples in terms of what aspects of the examples worked well for their understanding of basis and what aspects of the context were harder to line up with their understanding of basis. For example, Jocelyn did not think her outfit example captured the idea that you can create vectors in the space by just using one or two of the basis vectors, whereas in her outfit example, one would need to use all three basis vectors (shoes, tops and pants) to create a wearable outfit. In grappling with what aspects of their context worked well and which did not, the students revealed many nuances of basis that we might not have discovered using strictly formal mathematical questions. Together, these students' creativity and sophistication in assessing their examples challenges the narrative of underachievement of women of color that Harper (2010) has noted.

We argue that this paper makes contributions both to research on student cognition in addition to equity research. From a cognitive point of view, this is the only study that we know of that focuses on students' everyday examples of basis. In fact, there have been very few studies done on student understanding of basis and also few studies on students' ability to create everyday examples of mathematical constructs at the undergraduate level. For these reasons, this paper adds to the literature on student mathematical cognition and reasoning at the undergraduate level. In addition, we argue that the paper adds valuable data to the corpus of research in undergraduate mathematics education in that few studies have been written about the mathematical thinking of women of color. Sometimes this is because women of color have not been included in data sets (perhaps because there were not many women of color in the population from which the data was drawn). Other times we simply do not know whether or not women of color were in the data sets because, as one can see from a review of the papers in recent proceeding of the Conference on Research in Undergraduate Mathematics Education (RUME), it is not common in the RUME community to report data on gender and particularly on ethnicity.

This work may have implications for curriculum design. As an example, consider the experientially real starting points emphasized in the curriculum design framework of RME. Our analysis challenges us to reflect on what counts as an experientially real starting point for our students. Creating these experientially real starting points requires us to know our audience. In our past work we may have focused on certain types of students more than others in imagining what is experientially real to this audience. Making sure to interview and listen to the thinking of students who are not as often interviewed in RUME studies is vital to making sure we are reaching all students in instruction and also with our curriculum design.

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Comparing Expert and Learner Mathematical Language:
A Corpus Linguistics Approach

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Corpus linguists attempt to understand language by statistically analyzing large collections of text, known as corpora. We describe the creation of three corpora designed to enable the study of expert and learner mathematical language. Our corpora were formed by collecting and processing three different genres of mathematical texts: mathematical research papers, undergraduate-level textbooks, and undergraduate dissertations. We pay particular attention to the method by which our corpora were created, and present a mechanism by which LaTeX source files can be easily converted to a form suitable for use with corpus analysis software packages. We then compare these three different types of mathematical texts by analyzing their word frequency distributions. We find that undergraduate students write in remarkably similar ways to textbook authors, but that research papers are substantially different. These differences are discussed.

Key words: corpus linguistics, mathematical language, proof

Understanding the nature of mathematical language is a goal for at least three research communities. Sociologists and philosophers have long been interested in the practices of intellectual communities, and mathematics, with its uniquely deductive mode of inquiry, has attracted particular attention (e.g., Larvor, 2016). Mathematicians increasingly recognize that novices need to learn not only the content of mathematics but also the practices of mathematicians. Transition-to-proof courses are therefore common, and a growing number of textbooks directly address logical norms of mathematical communication (e.g., Vivaldi, 2014). Mathematics educators at all levels aim to support learners in developing sophisticated modes of thinking: a commonly-stated goal is that learners should engage in authentic mathematical activity that involves reasoning, proving, and communicating their arguments with others in the classroom and in written work (e.g., Stylianides, 2007).

These communities – sociologists and philosophers, mathematics educators, and mathematicians – therefore share an interest in understanding the norms of mathematical practice and communication. To date, however, there are relatively few empirical studies of this practice, and those that exist indicate less homogeneity among mathematicians than is typically portrayed in introspective accounts (e.g., Inglis, Mejía-Ramos, Weber & Alcock, 2013). In particular, to our knowledge, there have been few large-scale attempts to study the authentic mathematical communication of research mathematicians, or to compare this to the communication of undergraduates.

One method of studying language is to use the techniques of corpus linguistics, a branch of linguistics that statistically interrogates large collections of naturally occurring text, known as corpora. Methods developed by corpus linguists can be used to investigate many different types of linguistic question, and have revealed important and surprising findings (e.g., McEnery & Hardie, 2011).

Our goal in this project was to compare three distinct types of mathematical written language: that used by mathematicians when writing research papers, that used by mathematicians when communicating with undergraduates for pedagogical purposes, and that used by undergraduates when writing for assessment. By collecting and processing naturally occurring mathematical texts of these three types we aimed to understand the similarities and differences of these three genres of mathematical language. A subsidiary goal was to compare all these versions of mathematical language with general (non-mathematical) written English. We first discuss the process involved in creating our three corpora.

Collecting the Texts

The first task for a researcher who wishes to create a corpus is to collect examples of the language that they wish to study. We adopted two largely pragmatic criteria:

1. We collected only text in LaTeX format to enable consistent processing (discussed below).
2. We collected only text that had been published non-commercially or, in the case of student projects, where the author agreed to assign us copyright.

Subject to these criteria, we collected texts for the three corpora in different ways.

To create the learner corpus we invited undergraduate students to submit their final-year projects or dissertations. Such dissertations are common in the UK (where students specialize to study three subjects at age 16 and to one or two when at university) and degree programs vary in their dissertation criteria. But students commonly have the opportunity to undertake an individual project – which might be expository or applied– accounting for approximately one sixth of their final-year credit. We invited submissions from such students via project coordinators at 15 universities, who sent on an email directing interested students to a Facebook page that explained how to process their dissertation LaTeX file to remove personal identifiers, and how to submit this along with a copyright transfer form. Each student who submitted received a £5 (approximately $7) Amazon voucher and was asked to encouraged their friends to submit too. By this method we collected 50 student dissertations, which contained a total of 419,965 words.

To create the pedagogic text corpus we located online undergraduate-level open textbooks using the Open Textbook Library, the College Open Textbooks site, and the American Institute of Mathematics Approved Textbook list. Topics included abstract algebra, analysis, linear algebra, complex analysis, and textbooks designed to support the transition to proof. If the textbooks were not available in LaTeX format we contacted the author and asked for permission to access their source files. This approach left us with the source files for 21 complete undergraduate textbooks, which contained a total of 1,518,932 words.

To create the expert corpora we first downloaded all papers that had been uploaded to the arXiv in the first four months of 2009. The arXiv is an online repository that is routinely used by research mathematicians to share their research articles. The majority of articles on the arXiv are available in LaTeX format, and can be bulk downloaded using a command line tool. We then sorted the articles using their primary subject classification (e.g., mathematics, physics, etc.) and further sorted them using their secondary subject classification (e.g., algebraic geometry, algebraic topology, etc.). This left us with a total of 5087 mathematics articles, containing 30,892,695 words.

Processing the Texts

Collecting mathematical language and converting it into a form that can be processed using the standard software packages used by corpus linguists presents a challenge. Unlike most texts, mathematical language contains numerous atypical characteristics, such as inline mathematical notation. Most mathematics is written using the LaTeX markup language, not plain text. Our first goal was therefore to create a method of converting LaTeX source code to plain text in a way that preserved the natural sentence structure of the language, but which removed non-linguistic features of the source code (the code for bold or italic text for instance).

An important question for the would-be creator of a mathematical corpus concerns how to deal with inline mathematical notation. For instance, a typical mathematical sentence might be "Let $f: X \to Y$ be a bijection." How would we want the "$f: X \to Y$" to appear in a plain text corpus? One approach would be to leave the LaTeX source code intact and to analyze the code as if it were natural language. The difficulty with adopting this option is that there are several different ways in which one could encode "$f: X \to Y$" in LaTeX. For instance, `$f:X\rightarrow Y$` and `\(f:X\rightarrow Y\)` produce identical output, and `$f\,:\,X\longrightarrow Y$` only differs stylistically. We therefore felt that this approach would be unhelpful for the majority of questions a researcher would wish to answer using a mathematical corpus (although our code does allow this approach as an option, as a researcher who wished to primarily focus on the semantic content of papers might wish to retain these markup codes).

A second option would be to delete all mathematical code entirely, and simply record the example above as "Let be a bijection". We rejected this option as it seemed not to preserve the logical structure of sentences, which would influence certain analyses (those that investigate the collocation of words, for instance). Instead we opted to replace all occurrences of inline mathematics with the string "inline_math" (although this decision can be altered by the researcher if desired). The scripts we used to convert LaTeX to analysis-ready plain text are freely available for the research community at:
`https://github.com/sangwinc/arXiv-text-extracter`

As our corpus of general written English we used the combined Lancaster-Oslo/Bergen corpus (commonly referred to as the LOB corpus; Johansson, 1986) and Brown corpora. The Brown University Corpus of Standard American English (commonly referred to as the Brown corpus; Francis & Kucera, 1961) is formed of 1 million words of American English from texts published in 1961. The LOB corpus consists of written British English created to mirror the structure of the Brown corpus (i.e. texts were taken from similar sources in similar proportions). Thus our combined Brown/LOB corpus consisted of 2 million words of British and American written English.

Analyzing the Corpora

Having created the corpora, our primary goal was to understand the extent to which they were similar: is it the case that the language used in mathematical textbooks, mathematics research papers, and undergraduates' final year projects is consistent? If not, where are the differences between these genres, and how can these differences be characterized?

Kilgarriff (2001) proposed a variety of measures that aimed to assess the similarity of different corpora. All his approaches relied upon the so-called 'bag of words' model of text construction. This model ignores the order in which words occur and instead focuses on understanding texts by assessing their distributions of word frequencies. The basic idea is that two texts are likely to be a similar genre, and focus on a similar topic, if they have broadly similar word frequency distributions. Of the measures he studied, Kilgarriff concluded that a chi-squared approach performed best. Suppose one wishes to calculate the similarity of corpora A and B. Kilgarriff proposed determining the most frequent n words in the supercorpus formed of $A \cup B$, and then calculating the test statistic for a chi-squared test of goodness of fit. Since these n words were selected to be the most frequent, and not sampled randomly from the population of words, it would be inappropriate to actually perform the chi-squared hypothesis test, but Kilgarriff reasoned that the test statistic would serve as a suitable measure of similarity (with lower values represent more similarity).

Unfortunately Kilgarriff's (2001) chi-squared method would not suffice for our purposes, as it requires that we are comparing corpora of the same size. We therefore modified his proposal as follows. We first determined the 100 most frequent words across our four corpora, where each corpus was weighted as representing 25% of the supercorpus (we needed to perform this weighting because our expert arXiv corpus was considerably bigger than the others). We did not include "inline_math" as a word for this analysis, as clearly it did not appear at all in the Brown/LOB corpus. We then calculated the proportion of each corpus consisting of each word. For instance, the word "the" represented 6.08% of the arXiv corpus, 6.72% of the textbook corpus, 6.62% of the learner corpus, and 6.69% of the Brown/LOB corpus.

For each pairwise combination of corpora, A and B, we then calculated

$$S_{AB} = \sum_{i=1}^{100} \left(\frac{(a_i - b_i)^2}{a_i} + \frac{(a_i - b_i)^2}{b_i} \right)$$

where a_i represents the proportion of corpus A formed of word i, and b_i represents the proportion of corpus B formed of word i. While this is not a true chi-squared value (which would be calculated with frequencies rather than proportions) it fulfils a similar role. Therefore if $S_{AB} < S_{AC}$, we can conclude that corpora A and B are more similar than corpora A and C. The S_{AB} values for each pairwise combination of our four corpora are given in Table 1, and plots of the frequencies of the top 100 words are shown in Figure 1 (so, a point at (x,y) in the bottom left graph indicates that the same word formed x% of the arXiv corpus and y% of the textbook corpus).

The results shown in Table 1 and Figure 1 paint a consistent picture. We found that the textbook and learner corpora had remarkably similar word frequency distributions, that the arXiv corpus formed of mathematical research papers was somewhat different, and that all three mathematical corpora were substantially different to the regular written English of the Brown/LOB corpus. Before exploring the differences between the arXiv and textbook corpora below, we first make some remarks on these findings.

	Textbook	Learner	Brown/LOB
ArXiv	0.105	0.097	1.215
Textbook		0.011	1.317
Learner			1.320

Table 1: The similarity measures, S_{AB}, for each pairwise combination of our four corpora.

Although our analysis was exploratory in the sense that we did not have strong hypotheses about the results in advance, we were somewhat surprised by these findings. We anticipated there would most likely be a gap between the language used by experts and novices. After all, mathematicians have typically had many years of enculturation into the discipline, whereas the undergraduates who provided the texts for our learner corpus had only had three or four years of university-level study. However, we found something quite different. Our learners seemed to produce very similar written language to that found in textbooks written by experts, at least in the sense that their word frequency distributions were close to identical. One hypothesis that might account for this similarity would be if the two corpora had similar balances of mathematical topics. For instance, two corpora focused on linear algebra might be expected to have similar word frequencies for “kernel”, “matrix”, and so on. But we do not believe that this suggestion can account for our data. Because we only considered the 100 most frequent words, few were highly domain specific: in fact, only “theorem” and “proof” were words in the overall top 100 which had fewer than 100 occurrences in the Brown/LOB corpus.

Instead our conclusion is that the undergraduate students who provided the texts for our learner corpus did successfully produce written mathematics that was consistent with that found in undergraduate textbooks written by expert mathematicians. At least in the sense that it shared a similar distribution of word frequencies.

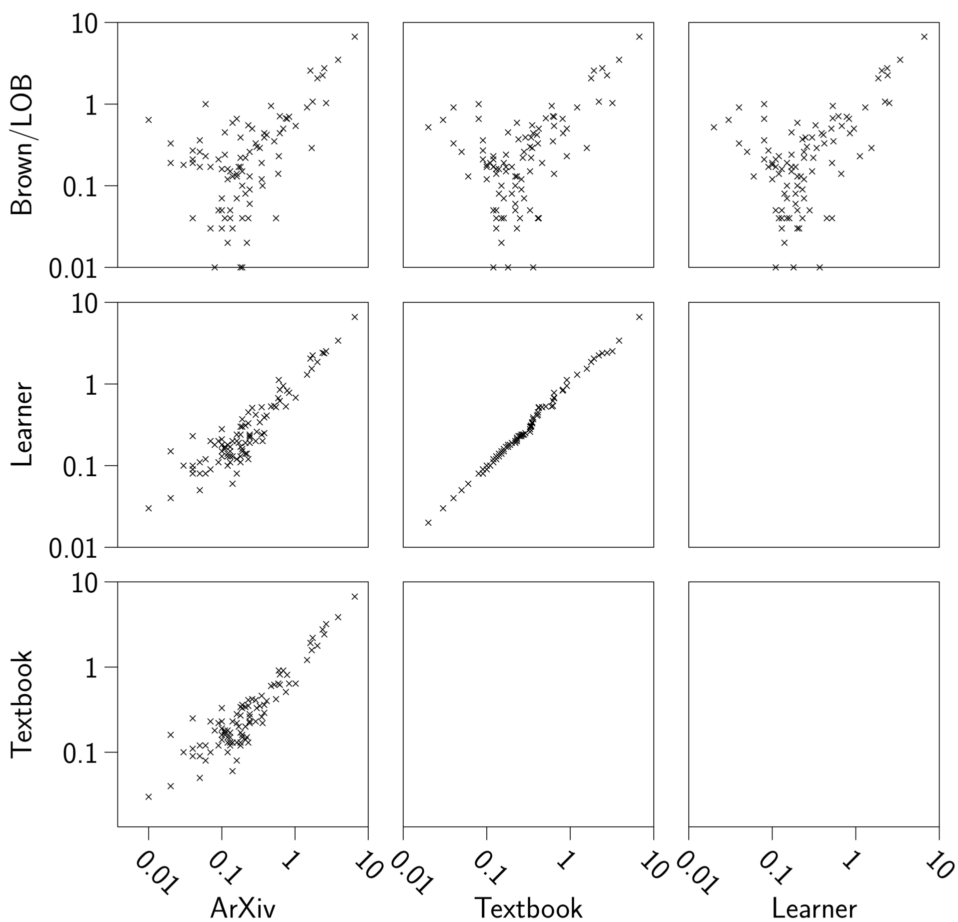

Figure 1. Scatterplots showing the frequencies of the top 100 words (as percentages) for each pairwise combination of our four corpora. Axes have logarithmic scales (therefore words with zero frequency in one corpus are not shown).

The main difference we found between the mathematical corpora concerned the word frequency distributions of the textbook and arXiv corpora. We can explore this difference in more detail by considering the keywords for each corpus – those words which occur disproportionately in one corpus compared to the other. These are shown in Table 2, which is ordered by chi-squared value (i.e. the contribution of the word to S_{AB} defined earlier).

Some of these key words are unsurprising: for instance, ‘example’, and ‘solution’ occur proportionately more often in the textbook corpus than the arXiv corpus. The textbook corpus also contains proportionately more instances of verbs such as ‘find’, ‘show’, ‘do’, and ‘prove’ – than the arXiv corpus. Indeed, the only verbs appearing in the right-hand side of Table 2 are ‘let’ and ‘see’. Although one might attribute this to the inclusion of exercises in textbooks, this explanation would not account for the extremely similar frequencies for these words found in the textbooks and the undergraduates’ final-year projects: although clearly textbooks normally contain exercises, student projects do not.

One further difference between the mathematical corpora concerned the frequency of mathematical notation. The arXiv corpus had considerably more instances of “inline_math” per 100 words (11.1%) compared to the textbook (8.8%) or learner (7.6%) corpora.

Further analyses are required to understand the significance of some of the other differences between the corpora. For instance, ‘by’ occurs disproportionately often in the arXiv corpus (1.02% of words) compared to the textbook corpus (0.64%) , but why? Investigating the most common clusters of words that include ‘by’ in the arXiv corpus indicates that the word is used to both name (“defined by”, “denote by”) and assert (“given by”, “obtained by”, “generated by”). By systematically studying such cases we can begin to understand the differences between research-level and undergraduate-level mathematical language.

Conclusion

Our main goal in this paper has been to describe the creation of three mathematical corpora designed to aid researchers understand mathematical language. The tools we used to construct these corpora are freely available for the research community to use. Having constructed the corpora we presented an analysis of word frequency distributions which suggested that undergraduate students are, by the end of their courses, surprisingly successful at writing in a manner consistent with the language used in undergraduate textbooks. The developmental trajectory by which students develop mathematical language skills would be a worthy topic of future study. In contrast to the similarity observed between textbooks and final year dissertations however, the language mathematicians use in research papers is different to both.

In this paper we have focused on comparing the word frequency distributions of four different corpora, but there are a great many other techniques that can be used to analyze corpora which go well beyond this approach (e.g., McEnery & Hardie, 2011). Given the interest shown by mathematics educators and other researchers in mathematical language, we believe that corpus linguistics is a potentially useful, but currently under used, research technique.

More frequent in the textbook corpus	
Word	χ^2
find	0.01279
what	0.01103
number	0.00689
example	0.00477
if	0.00266
must	0.00265
use	0.00265
show	0.00242
function	0.00232
set	0.00220
that	0.00218
or	0.00218
about	0.00212
is	0.00201
solution	0.00181
do	0.00167
two	0.00163
not	0.00159
each	0.00141
than	0.00139
so	0.00130
prove	0.00124
this	0.00123
are	0.00113
a	0.00113

More frequent in the arXiv corpus	
Word	χ^2
by	0.00367
on	0.00190
where	0.00144
case	0.00120
i	0.00120
with	0.00100
for	0.00094
let	0.00071
in	0.00061
see	0.00058
following	0.00055
such	0.00054
space	0.00050
which	0.00049
proof	0.00033
theorem	0.00028
also	0.00019
we	0.00015
now	0.00013
given	0.00010
our	0.00007
and	0.00007
since	0.00003
from	0.00002
every	0.00002

Table 2: The left-hand table shows the top 25 words that occur in the textbook corpus that differentiate it from the arXiv corpus. The right-hand table shows the equivalent words for the arXiv corpus.

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Mathematics instruction leadership in undergraduate departments

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Many universities have begun to coordinate their introductory mathematics courses to handle multiple sections of the same course, a situation necessitated by the large numbers of students taking precalculus and calculus. Robust coordination systems consist of two major elements: uniform course elements (e.g., common text; exams) and regular instructor meetings. These regular meetings may turn calculus instruction into a joint enterprise, potentially engendering a community of practice. Of particular importance are those who act as leaders (formally and/or informally) within these coordination systems – these people have the potential to influence and "nudge" instructors towards improving their practice (Rasmussen & Ellis, 2015). This study combines case study findings with social network data to investigate instructional leaders at five diverse institutions, considering both formal and informal coordination phenomena, and hypothesize about their potential to influence practice in their departments.

Key words: Coordination, instructional leadership, social networks, change

Many universities have begun to coordinate their introductory mathematics courses to handle multiple sections of the same course, a situation necessitated by the large numbers of students now taking precalculus and single-variable calculus (Precalculus to Calculus 2, P2C2). Robust coordination systems consist of two major elements: uniform course elements (e.g., common text; exams) and regular instructor meetings. These regular meetings can have the effect of turning calculus instruction into a joint enterprise, potentially engendering a community of practice (Rasmussen & Ellis, 2015). P2C2 instruction has been identified as one of the culprits in the mass exodus of college students from STEM majors, and many calls for improvement hint at the need for collaborative work: "all faculty, including tenure-track faculty and the teaching/pedagogy faculty, should engage collaboratively in undergraduate curriculum and teaching reform" (TPSEMath, 2015, p. 10). Coordination systems provide a venue for this collaboration to take place. Of particular importance are those who act as leaders (formally and/or informally) within these coordination systems – these people have the potential to influence and "nudge" instructors towards improving their practice (Rasmussen & Ellis, 2015).

To better understand the nature of coordination systems and instructional leaders, this study combines case study findings with social network data to investigate mathematics departments at five diverse institutions, considering both formal and informal coordination phenomena. This work is guided by the following questions:

1. Who in the department influences instructional practice?
2. To what extent do leaders identified using social network analysis coincide with those in formal leadership positions (e.g., coordinators; chair)?
3. How do leadership roles and influences vary?

Theoretical Perspective and Literature Review

Leadership is a concept with many definitions, though all involve influence. A review of definitions over the last century emphasizes that it is relational, situated, and can be formal

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and/or informal (Carter, DeChurch, Braun, & Contractor, 2015). By *relational*, we mean that leadership involves relationships between people: for someone to lead, their influence must be felt by others. By *situated*, we mean that a person who leads in some situations may follow in others, it is not an unchanging characteristic of an individual. A key element of the situational nature of leadership is that different characteristics and expertise have different value in different contexts – a person who is considered a leader in one domain (e.g., pedagogy) may not be seen as a leader in another domain (e.g., research). Finally, leadership involves formal and/or informal influence. That is, leadership can originate from positional power (e.g., manager, chairman), or from personal power (e.g., respect, trust), and the two are not exclusive (Carter et al., 2015).

In this study, we focus on *instructional leaders*, whom we conceptualize as those with the ability to influence the practice of instructors in the classroom. This influence may be explicit (e.g., providing advice) or implicit (e.g., conversation partners), and may relate to particular elements of a course (e.g., exams, homework) or be a more diffuse influence on general approaches to instruction. Those in formal positions of leadership can certainly influence the choices others make, but many without formal titles have informal influence. While in some organizations it is possible to mandate actions, instructors at institutions of higher education enjoy a remarkable amount of autonomy, particularly those with tenure. However, even those with complete independence can be nudged toward making certain decisions.

The idea of *nudging*, increasing the chances of a certain decision being made, comes from the work of behavioral economists Thaler and Sunstein (2009). They describe how the moves of choice architects, those who organize the context of decision making, can strongly influence the actions of others without removing their ability to make their own choices. The general principle they recommend is to make it easy for individuals to make "healthy" decisions, and more difficult (though not impossible) to select alternatives. This often comes in the form of setting desired options as a default, providing feedback to choosers, mapping out courses of action, and providing information about others' actions. As detailed by Rasmussen and Ellis (2015), many of these roles are within the purview of course coordinators. This report builds on that previous analysis to include a more diverse set of institutions, some with less formal coordination systems, and instructors of a more diverse set of introductory mathematics courses.

The notion of leadership also necessitates conceptualizing the community of those involved. Toward this end we draw on the communities of practice literature (Wenger, 1998). In particular, we take the point of view that norms and practices related to instruction are part of a continuously developing, shared repertoire which develops through mutual engagement surrounding a joint enterprise. Central members of a community of practice are those perceived as the most expert, and exert stronger influence over the group than newer or more peripheral members. While the departments being studied are not necessarily true communities of practice, they (at least officially) refer to undergraduate instruction as one of the purposes of the department, and one of the responsibilities of instructors and faculty.

Bringing these literature bases together directs our attention to the nature of instructional leadership, the scope of its potential to influence others through social interactions, and the conduits through which people engage and negotiate normative practice and beliefs. We next turn to data sources and how the methods of social network analysis offer an approach for using these data to assess leadership and identify conduits for influence.

Data Collection Methods

Data for this report comes from two research projects, *Characteristics of Successful Programs in College Calculus* (CSPCC) and *Progress through Calculus* (PtC). The CSPCC project surveyed faculty and students across the nation to identify those whose Calculus 1 implementations were more relatively more successful, where success was defined in terms of affective changes, passing rates, and persistence rates. Case studies of these more successful programs led to the articulation of common features, at least at the research universities (Bressoud, Mesa, & Rasmussen, 2015; Bressoud & Rasmussen, 2015). Follow-up surveys and interviews were also conducted at several case study sites, when possible, to further explore the themes and surrounding context. These, and other project findings, influenced the design of the PtC census survey, which investigated P2C2 courses at departments that offer a graduate degree in mathematics (Apkarian et al., 2016; Rasmussen et al., 2016). The five departments investigated in this report come from sites that were selected in the CSPCC study, had follow-up investigations, and participated in the PtC census survey.

Of particular interest to this study are data and findings related to course coordination. From the original CSPCC case studies come interviews with faculty and course coordinators of Calculus 1. From the PtC survey comes information about the coordinated elements, coordinators, and instructor meetings for each P2C2 course. From the follow-up social network surveys comes information about social interactions and relationships between instructors and faculty at each department. Social network surveys are used to ascertain the ties, and strength of those ties, between people by asking participants to identify others with whom they have a particular relationship (Daly, 2010; Kadushin, 2011; Scott, 2012). Our follow-up survey asked about five types of interaction: seeking advice about teaching, seeking instructional materials, discussing instructional matters, friendship, and instructional influence, supplemented with demographic information and Likert scale questions to characterize the actors between whom ties exist or do not exist (Coburn & Russell, 2008; Daly, 2010; Scott, 2012).

Data Analysis Methods

Graph theoretic techniques common to social network analysis were used to identify those with influence over instructional matters (Borgatti, Everett, & Johnson, 2013; Daly, 2010; Kadushin, 2011; Scott, 2012). Table 1 explains the interpretations of participant A selecting participant B for each of the four instruction-specific networks. This interpretation was used to determine what network measures were appropriate for each relationship. This proposal discusses only degree though other more complex measures are being used to ensure the validity of interpretation and to identify more nuanced differences. *Degree* refers to the number of ties that an actor in the network has. An actor's *in-degree* refers to the number of nominations they receive, while their *out-degree* refers to the number of nominations they provide, in a given network. High and low degree values are determined based on the distribution of degrees across all actors in a particular network, rather than in absolute terms.

Table 1. Interpretation of social network survey responses.

If A chooses B for…	…this means that:	The instructional influence being passed is:	An influential actor is identified by:
Advice	A believes B has worthwhile advice	Specific; explicit; primarily from B to A	High in-degree
Materials	A believes B has useful materials	Specific; implicit; primarily from B to A	High in-degree
Influence	A has been influenced by B	Nonspecific; explicit; primarily from B to A	High in-degree
Discussion	A and B have discussed a topic	Nonspecific; implicit; direction indeterminate	High total degree

In addition to identifying actors with high degree values, it is informative to compare an actor's in- and out-degrees to determine their level of involvement (higher degree values = more involved) as well as their perceived function. When someone receives many nominations but provides very few, they function as an authority; when they provide many but receive few, they function as an apprentice (Borgatti et al., 2013).

The social network data was used to determine who in these departments has functional influence on their colleagues' instructional practice. This information was then combined with data from our other sources to understand where positional leadership overlapped with instructional leadership, where it did not, and (to a certain extent) why. Actors with positional authority were identified through their titles, either from official records or interview data. The formal setup of coordination systems (see Table 2) was identified through the PtC census survey data. Interview data from CSPCC case studies reveal nuances about the coordination system, including characterizations of coordinators, attitudes towards the coordination system, and the extent to which cooperation with the coordination systems is enforced.

Results

A first step in this project was to identify basic facts and features of each institution, department, and P2C2 coordination system. Table 2 gives a brief overview of each site, including institutional descriptors, the individuals identified as having positional power related to undergraduate instruction (other than chairpersons), and selected information about the coordinated elements and meetings of P2C2 courses.

Table 2. Characteristics of each institution, department, and P2C2 coordination system.

	Large Public University	Public Tech. University	Private Technical Institute	Public University 1	Public University 2
Institution type*	R1	R2	R2	M1	M1
Undergraduate enrollment*	30,709	7,099	6,381	9,501	12,503
Participant pool	61	56	26	24	23
Coordinators	3 course coord.	3 course coord; 1 lab coord	1 course coord; calc cmmte. chair	Calc cmmte. chair	Calc cmmte. chair
Course content and schedule	Yes	Yes	Topics for all; pacing for Calc 1	Topics, not pacing	Topics, not pacing
Uniform homework	Online homework	Online homework	Not required	Not required	Not required
Uniform exams	All calculus exams	Yes	Finals only	1/3 of items on final	Calculus 1 final
Uniform grading	Calculus: exams and overall	Exams and overall	Calculus 1; exams only	Not required	Not required
Instructor meetings	2-4 per term	Weekly	Calc 1 weekly; 1-4 per term for other	1-4 per term	1-4 per term for pre-calculus(es)

**Carnegie Classifications (Indiana University Center for Postsecondary Research, 2015)*

As we begin to describe the results of this project, we remind the reader that leadership is *situated*, and that someone with influence in one domain may not have the same level of influence in other contexts. We reinforce this idea to clarify that if we identify a chair as *not* exerting instructional leadership, this does not mean that there are no contexts in which they are highly influential. Bearing that in mind, we first consider some common features across all five institutions, followed by a more detailed treatment of three sites. Full reports of the remaining two sites have been omitted from this proposal due to space constraints.

General results

At each of the five institutions examined in this study, a few individuals emerge from the social network analysis as instructional leaders with regard to undergraduate mathematics courses. Their formal roles vary, as do the networks in which they appear important, but across all institutions it appears that these individuals are not simply the "most popular" members of the department. Justification comes from usage of the friendship network question – as their position in the friendship network is not extreme, it appears that they are sought out for advice because of perceived expertise or authority, not simply because they are nice to talk to.

In these five institutions, which were identified in the CSPCC study as having relatively successful Calculus I programs, there is overlap between those with formal leadership titles and those who function as leaders via the social analysis. This may be an indication of a "healthy" environment, wherein department members trust in the expertise of those in power. While trust was not one of the explicit questions participants were asked to comment on, looking at those who are sought out for advice and those who are identified as influential can serve as a proxy in the context of instruction.

Large Public University - LPU

This institution's mathematics department has three course coordinators, each specializing in a particular course. While their specific assignments rotate through Precalculus, Calculus 1, and Calculus 2, these three are permanent coordinators and work together as a unit to manage the courses. These three individuals emerged as instructional leaders from the social network analysis. They function as authorities in advice, material, and influence networks, and are highly involved in the discussion networks. In practice, this means that these three are the most likely to be asked for advice or sample materials (thus influencing specific practices explicitly); very likely to be involved in informal conversations about instruction (thus influencing practice implicitly); and considered by many instructors to be the most influential on their teaching practice. Thus we see that the positional authority of all three coordinators is supplemented with real instructional influence, indicating that at LPU the coordination system is functioning as intended. We note that at LPU, the department chair had very low involvement in every network, implying that his/her positional authority does not extend to matters of undergraduate instruction. No other actors appeared to exert significant influence over instructional matters.

Interview data are consistent with the network-based findings. These coordinators are appreciated by other instructors as reliable resources for advice about content as well as teaching. They maintain websites to host instructor resources and sample materials, which are widely used by others. While these three were viewed by their colleagues as leaders, it was noted that they do not insist on every detail, thus allowing instructors independence, even within a coordinated system. At this institution, P2C2 courses are only coordinated during the on-sequence terms of instruction, but many instructors note that they use the coordinated materials no matter which term they are teaching a course, freely choosing to act under the coordinators' influence.

Public Technical University - PTU

This institution has three course coordinators and a lab coordinator, each a multi-year, semi-permanent position. One course coordinator, the lab coordinator, and (to a lesser extent) the department chair emerged from the social network analysis as instructional leaders. The coordinators function as authorities with regards to advice, materials, and influence and are highly involved in the discussion networks; the chair is an authoritative figure in the influence network but does not appear highly involved in advice or materials. In practice, this means that these two coordinators are the heaviest influences on explicit elements of practice (via advice and material sharing); are involved in many conversations about instruction (a more general, implicit influence); and the chair joins them in generally influencing instruction. These three actors appear to have positional power backed up with real influence over instruction, representing a good match between theory and practice. Two other faculty members in the department had high involvement in discussion networks, but not in any of the direct influence networks. They also appeared highly in the friendship network, perhaps indicating that these two are friendly, but not perceived as having any particular expertise in instructional matters.

The CSPCC interview data provides some context and explanation for the result that only one of the three coordinators has real influence over instruction. The influential coordinator frequently takes the helm and provides default options to those working under her; the other coordinators project less authority and consider themselves to be “organizers” rather than supervisors of the coordinated courses. Their decision to operate “on the same level” as the other instructors is apparent from their interviews as well as interviews with instructors, and may explain why they do not exert significant influence over instructional practice in the department.

Private Technical Institute - PrTI

PrTI has one course coordinator and a calculus committee to oversee introductory courses, and the calculus committee is managed by a chairperson. This institution, which is smaller than LPU and PTU, has more distribution of degree than some other places – a sign that leadership is somewhat distributed. However, instructional leaders do emerge from the analyses, coinciding with the course coordinator and the calculus committee chair. Neither appear to function as authorities, but they are the most influential in terms of advice and discussion networks. The committee chair was also identified as a major source of advice. No actors emerged as particularly influential with regards to course materials; no other actors emerged in leadership roles; and the department chair has low involvement in the instructional networks. In practice this indicates that while instructional materials are not influenced by particular members of the department, the positional authority invested in the coordinator and chair of the calculus committee is reinforced by personal power and influence – another instance of a good match between title and role. As with LPU, the department chair’s positional authority does not appear to extend to influence over instructional practice.

It is worth noting that the network analysis identified the committee chair as having influence before his/her position was known. Follow-up investigation of the available data revealed that they did, in fact, have a position which carries some inherent power. The fact that no actor, not even the coordinator, is a major source of instructional materials is made less surprising when we note that this department does not have many uniform course elements – homework and non-final exams are at the discretion of the instructor. This is further supported by the CSPCC interview data, which indicated that the coordinator organizes meetings and provides feedback to instructors, but does not insist on the usage of pre-specified elements.

Public Master’s Universities – PU1 & PU2: A Brief Overview

At both PU1 and PU2 the calculus committee chair and department chair have instructional influence as well as positional authority. Also at each, instructors do not feel particularly restrained by the recommendations of the committee, but there is evidence of collaboration, cooperation, and some informal coordination through the influence of leaders. Moreover, at each the networks are somewhat distributed and the leaders, though they have influence, do not stand out as much as they do at the doctoral institutions we studied.

Discussion

One overarching message from our work is that, despite having significant pedagogical autonomy, university instructors are influenced by their colleagues. Note that pedagogical autonomy is distinct from academic freedom – the former refers to instructional practice in the classroom while the latter refers to research. Our results lend themselves to two related implications: one for the study of leadership in undergraduate departments and one for the potential of social network analysis to inform the change process. We present some hypotheses and questions for future research in this area.

We have discussed the nature of leadership as being relational, situated, and both informal and formal. The theoretical premise and empirical results of this paper lead us to extend these ideas to consider larger systems. The relational nature of leadership lends weight to the consideration of *people* and their interactions within a department, and we highlight again that leadership comes from and is embedded in relationships – it is not an inherent trait of particular individuals. While our work identifies relationships between instructors and faculty members, we have little evidence of relationships connecting members within the department to those outside. We see that centers for teaching and learning are not heavily utilized (when they exist), and there is not much coordination between departments. While administrators (e.g., deans) have some positional power over department activities, they did not appear to be involved in interactions related to instruction. The situated nature of leadership implies that in different contexts, or when considering different domains, the traits that coincide with influence are varied. Our work here focuses on instructional leadership rather than administrative or research leadership, but there are other contextual factors to consider. How does institution size and/or mission affect the nature of leadership? What about institutional or departmental culture? Considerations of departmental culture lead to the nature of leadership as both informal and formal. While formal meetings are held with some regularity in many departments, we do not know the extent to which instructors meet informally with each other. Having identified influence and interactions related to instruction at schools without regular instructor meetings, we believe that informal interactions have some impact on the development of influencing ties. Furthermore, the literature and our data support the idea that titles are neither necessary nor sufficient conditions for instructional leadership.

We believe that this work has implications for those interested in studying or planning change at the undergraduate level. Understanding the readiness of a department for change involves understanding the context and community, without which efforts may well fail (Lee, Hyman, & Luginbuhl, 2007). Identifying those with influence and those with positional authority may go some way to explaining the success or failure of a particular effort – when the two do not coincide there is the potential for subversion via misaligned influences. We may go so far as to recommend social interventions prior to the implementation of change efforts, in the event that the organization (in this case department) is fractured or heavily influenced by an opposition leader – something which can be revealed through social network analysis (Carolan, 2014; Daly, 2010; Lee et al., 2007). We see great promise in coordination systems for instructional improvement. Not only do these systems provide more uniform experiences for students, but a coordinator with both positional and personal instructional influence is in a position to be a choice architect. That is, the default options they set for their colleagues is likely to be accepted and their recommendations are likely to have real implications for instructional practice. Their ability to nudge others makes them a high-priority ally for change agents interested in making a difference (Kezar, 2014; Rasmussen & Ellis, 2015; Thaler & Sunstein, 2009).

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Using learning trajectories to structure teacher preparation in statistics

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As a result of the increased focus on data literacy and data science across the world, there has been a large demand for teacher preparation in statistics. Project-SET constructed two hypothetical learning trajectories for teacher learning and subsequently used the hypothetical learning trajectories to structure a professional development curriculum. We illustrate how the utilization of learning trajectories to design professional development allowed participating teachers to develop several aspects of Statistics Knowledge for Teaching (Groth, 2013).

Key words: Learning trajectories, statistics, professional development, teacher preparation

Large-scale research provides some indication of the key characteristics of effective teacher training (Doerr, Goldsmith, & Lewis, 2010; Garet, Porter, Desimone, Birman, & Yoon, 2001; Heck, Banilower, Weiss, & Rosenberg, 2008). These include a focus on content knowledge, opportunities for active learning, and coherence with other learning activities (Garet et al., 2001).

The purpose of this article is to report on how using teacher learning trajectories (TLTs) to design teacher training offers a structure to develop teacher content knowledge in deep and meaningful ways. This study is based on the implementation of Project-SET, a project funded by the National Science Foundation to develop professional development curriculum to enhance teachers' content knowledge of two statistics topics – sampling variability and regression. Project-SET first developed TLTs for these two topics and then designed a professional development curriculum around the trajectories. In exploring the outcomes of the implementation, we recognized the important role that the TLTs played in creating opportunities for teacher participants to achieve different aspects of Statistical Knowledge for Teaching (SKT).

We aim to answer the following research question: *How did the use of learning trajectories to design professional development curriculum support the development of teachers' statistics knowledge for teaching?* Our findings suggest that TLTs offer a promising structure for aiding professional development design.

Learning Trajectories

The idea of a learning trajectory (LT) was initially introduced by Simon (1995) as a way to conceptualize how students might progress through a learning sequence. He stated that a LT included "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (pp. 133). Originally grounded in constructive theory, LTs connect students' thinking and learning for specific mathematical content with a conjectured pathway to move students through a developmental progression (Clements & Sarama, 2004). While all learning trajectories essentially organize learners' thinking and learning, how a learning trajectory is built and the scope for which it is used differs in the literature.

For example, Confrey and Maloney (2010) describe LTs for a several of topics as "a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms, of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey, 2008; Confrey et al., 2008, 2009). Clements and Sarama (2004) instead focus on early childhood mathematics and describe LTs for narrow sequences of topics.

The LTs constructed by Project-SET differ that those discussed in the literature as they are focused on teachers. In other words, the Project-SET LTs were build and designed to conceptualize *teacher* content knowledge in statistics. Project-SET adapted the definition of a learning trajectory as that of Clements and Sarama (2004): "learning goal, developmental progression of thinking and learning, and sequence of instructional tasks." The learning goal for the trajectories was to address teacher content knowledge on the topics of sampling variability and regression. This goal was coupled with a progression and instructional tasks.

Theoretical Framework

The Project-SET teacher learning trajectories (TLTs) served to guide and develop a professional development for teachers. In applying the LTs to teachers, we intersect LTs with frameworks of teacher knowledge. In particular, we draw upon the Mathematical Knowledge for Teaching (MKT) framework and the Statistical Knowledge for Teaching (SKT) framework (Groth, 2013).

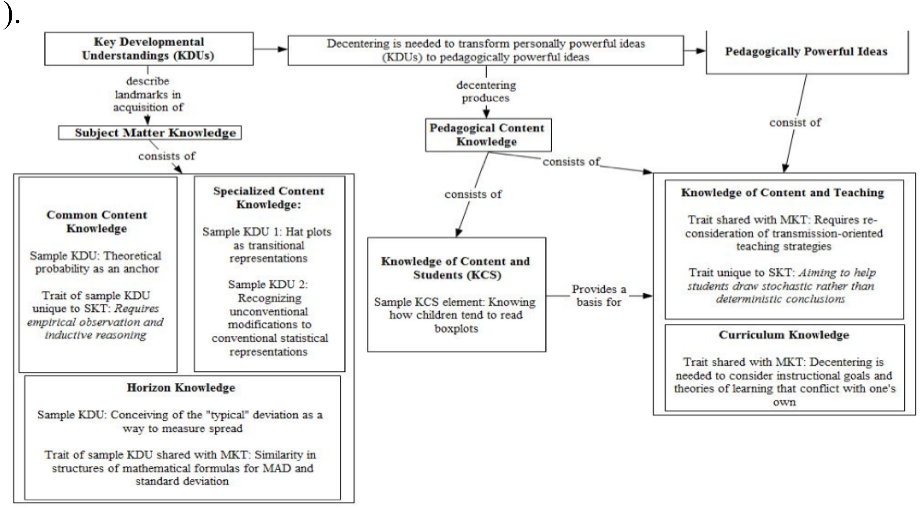

Figure 1 SKT Framework (Groth, 2013, pp. 143)

Groth (2013) identified Key Developmental Understandings (KDUs) as landmarks in the teachers' development of subject matter knowledge. Building from the work of Simon (2006), Groth describes KDUs as significant conceptual shifts. According to Groth, these landmarks or conceptual shifts can occur in each of the three types of subject matter knowledge in his framework (common content knowledge, specialized content knowledge, and horizon content knowledge). Groth's SKT framework also incorporates ideas outlined by Silverman and Thompson (2008) regarding the development of pedagogical content knowledge. In particular, Silverman and Thompson assert that teachers' development of KDUs with regard to subject matter knowledge are a necessary, but perhaps insufficient, first step with regard to improving student learning.

The purpose of the Project-SET professional development was to develop teachers' content knowledge. These TLTs served as guides for the structure of the professional development. Figure 2 represents the relationship between teacher learning trajectories, professional development, and SKT. This study focuses on understanding how this process might work.

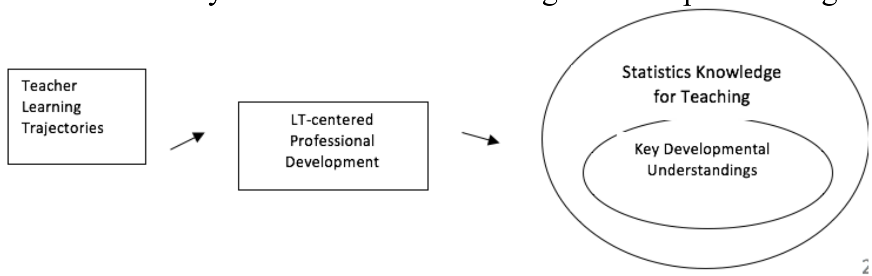

Figure 2. Project-SET Conceptual Framework

Methods

Participants

Nine secondary teachers completed the first implementation of the professional development. Seven of the 9 teachers taught in the local public school district. Two of the teachers taught in a private school within the city. Their average number of years teaching statistics was 2.4.

Data Sources

End-of-Loop Assessment Tasks. Assessment tasks were completed by the teachers at critical points of the LT in order to measure understanding with respect to the content included in the LT. The scoring of each part was modeled after the AP Statistics scoring of: E (Essentially Correct); P (Partially Correct); or I (Incorrect). The assessment tasks were scored each week by two scorers who were part of the research team but not present during the professional development session. The scorers graded the papers separately and then discussed their scores to come to a consensus on the final scores.

End-of-LT Assessment. At the completion of the content of each LT, teachers were assigned as homework an assessment intended to bring together the content of the entire trajectory.

Video of Class Sessions. Each class session of the professional development was videotaped. Outlines of the videos were created and portions of the videos were transcribed. The videos provide a means to confirm and elaborate on the observed patterns of teacher learning documented from the teachers' written work.

Analysis

A two-phase process was used to investigate how the use of TLTs supported the development of SKT, with a particular focus on KDUs. The first phase took place during the analysis of teachers' written work on the End-of-Loop Assessment Tasks and the end-of the LT projects. This analysis provided insight into which ideas were pivotal to teacher understanding thus permitting the research team to identify a preliminary list of KDUs.

In the second phase of the data analysis, these prospective KDUs were then examined through the analysis of classroom interactions. During this phase, the videos were examined in order to determine how teachers' SKT developed as they progressed through the LTs.

Results

We present illustrative examples of the KDUs that we identified along with supporting evidence consisting of samples of teacher work or transcript segments.

Example 1 Common Content Knowledge KDU: Sample Size and the Sampling Distribution

One of the most persistent ideas that surfaced in teachers' work and discussion involved the relationship between sample size and the shape and spread of the sampling distribution. We have identified this as a KDU reflecting Common Content Knowledge in Groth's framework insofar as this is not a concept specific to the domain of teaching.

Teachers repeatedly made statements alluding to the fact that when repeated samples were taken and a sample mean was computed, then the shape of the sampling distribution should become more bell-shaped and the variability of the sampling distribution should decrease. For example, an assessment task for sampling variability asked teachers to compare three different approximate sampling distributions taken with samples of n=5, 15, and 30 according to their shape, variability, and center. There is evidence at this point that the nine teachers developed an understanding of the effect of the sample size on the spread, even if they were not yet clearly articulating the relationship to shape. Three samples of teacher responses are provided:

Table 4. Sample Answers to Loop 3 Task
Assessment task question: Compare the three distributions that you constructed. What can you say about the shape of the distribution as the sample size, *n*, increases? What can you say about the mean? What can you say about the standard deviation?
Teacher Example 1: As *n* increases, the data gets more "compact" around the population mean $27,000 – thus the variation decreases. The mean income was closer to the population mean when n=15 than when n=30, but still both close to $27,000. The standard deviation decreases as *n* increases.
Teacher Example 2: As the sample size increase, the distributions are becoming less spread out. There is less variability in the distributions as the sample size increases.
Teacher Example 3: The shape of the distribution is more unimodal and symmetric, becoming more approximately normal. There is less variation as the sample size n increases. The mean got smaller and then bigger as the sample size increased. However, the mean stayed in the same interval between 25,000 and 30,000. The standard deviation decreased as the sample size n increased. I would expect that trend to continue, but the standard deviation was decreasing at a slower rate as n increased.

The videos of the class sessions provide further support for the assertion that this was a KDU for teachers that the loop design of the LTs fostered. After investigating these ideas over a period of time, teachers expressed different "aha" moments around the effect of the sample size. For example, the activity for Loop 3 engaged the teachers in sampling from four populations with vastly different distributions (bimodal distribution, skewed distribution, roughly normal distribution, and scattered distribution). The teachers took samples of size n=5, n=10, and n=25 and generated approximate sampling distributions for each sample size. They compared the sampling distributions of different sizes and noted the similarities in the effect of the sample size on the sampling distribution.

Instructor: So, tell us what you've got there from Jamal [the bimodal distribution] and what happened with the samples of different sizes? Teacher 6: As you can see, as you increase the sample size, the variability gets smaller and smaller, if you look you are going from 20 to about 110, versus the spread going from 20 to about 80 where for size 10 you are going about 52 to 66, your variability is decreasing.

As the discussion continued, several teachers also compared the sampling distributions to the corresponding population distributions, noting the way in which the distributions with the larger sample sizes behaved in a similar manner, regardless of the distribution of the population. In particular, even when the population had a non-symmetric or bimodal shape, as the sample size increased, the variability of the sampling distribution decreased. During this comparison of the different distributions, one of the teachers, focused on the behavior of sampling distributions of the bimodal population distribution, began to talk through the reasons behind what she had observed.

Teacher 5: No matter what the population looked like, there was a mean. And our data, or our samplings, were samplings of the average. So, they all should have been near the average of the population. No matter what [the population] looked like.

Another teacher builds on this idea and offers an argument for why the variation of the sampling distribution should logically decrease with an increase in the sample size:

Teacher 7: I guess they can't do this because they are obviously...cards, but if we had done N=60, a.k.a. all the cards, it would have just been a straight line at 60... So that like N = 60 literally is just 60, 60, 60, 60 [the mean of the sample] over and over again, just a straight line of 60, *but that would have been a good thing to compare to n = 4, n=10, n = 30*. [Note: The example to which she was referring had a population mean of 60 and a population size of 60.]

In this segment, the teachers appear to not only recognize the effect of the sample size on the spread of the sampling distribution (as with their written work, their descriptions of the shape are not as explicit at this point). They also appear to be creating corresponding mental images of why this makes sense, no matter what the shape of the population is.

The significance of this idea as a KDU is reflected in their own comments a few minutes after the observations from Teacher 5 and 7 described above.

Teacher 3: You know...I'm not sure that I ever understood that...I'm serious. Teacher 5: The light did come on, in terms of understanding what was happening in this activity.

Example 2 Specialized Content Knowledge KDU: Line of Best Fit Counterexamples

Specialized content knowledge, defined in the SKT framework as knowledge of content needed in the practice of teaching, may include teachers' ability to comment on student work and strategize ways they can address student errors. One way to illustrate to students their errors would be to provide students with examples for which their solution will not work. The ability to develop such counterexamples is knowledge specific to teaching.

In the regression LT, the teachers were asked to examine a scatterplot of drop heights versus bounce heights of a golf ball and place a piece of spaghetti on the scatterplot in such a way that they believed represented a line of best fit. Teachers also had to explain what their criterion was for the placement of the line and why they chose to place it there. This same activity had been given to 8^{th} grade students. A second component to the teacher activity was then for the teachers to comment on the 8^{th} grade students' work and, if the work showed a misunderstanding, then provide a counterexample scatterplot that would illustrate to the student that their placement criterion would not be successful in general.

During part one of the activity, all of the teachers created criteria that matched that of the previously collected student data. For example, one teacher asked "do you assume it [the spaghetti] goes through (0,0)?" She noted that in the context of the problem, dropping balls, if you dropped the ball from 0 height, you would get a 0 bounce height. She thus concluded that her line of best fit must go through the origin. This same reasoning was also seen in the student work. Another teacher stated that she placed her line in such a way that "there are 4 dots above and 4 dots below and so it is in the middle." Again, similar reasoning was uncovered in the student work with a student stating that they wanted to "split" the points.

When teachers were given the student results to analyze, they were asked to evaluate whether the criteria the students used to place their line was one that would work for *any* data set. If not, then the teachers were to give an example scatterplot for which their student criteria would not work. This proved difficult for the teachers. For example, one student had the origin criteria similar to one of the teachers. Looking at the student work, she stated: "I think that is a good idea." However, teacher 8 responded by saying "in this case, it [going through the origin] is ok but not all the time." At this point, a conversation emerged as to whether the criterion the student applies must work for any set of linear data or just the golf ball data in front of them. After a short deliberation about what defining a criterion means and how it should be applied, it was accepted that a criterion must work for any set of data. Then, the teachers created the counterexample of a data set that had a negative association and thus would require a line to have a negative slope so it would not enable the line to go through the origin. Although the teachers were able to develop counterexamples to help guide student misunderstandings in the context of the line of best fit, the work was non-trivial.

Discussion & Conclusion

As noted by Simon (2006), for someone to develop a KDU, one must have repeated exposure to the concept. Additionally, according to Simon, students without a KDU "do not tend to acquire it through explanation or demonstration" (p. 362); instead a KDU must emerge through discovery. In this way, a person would be able to shift their understanding and gain a Key Understanding. We gathered evidence to show that the Project- SET LTs offered a platform for teachers to develop KDUs by scaffolding more complex ideas and repeatedly looping for each topic. Due to limitations in space, only two examples were presented above. We assert that the design of the Project-SET activities to progress teachers through the TLT facilitated the development of SKT and the emergence of KDUs. In addition, the TLT also allowed for the conceptual unpacking necessary to develop teachers' knowledge.

The TLTs' mapping created clearly-defined conceptual boundaries that allowed us to recognize when inadequate connections were begin made to horizon content knowledge. Thus, although this is a small study with nine teachers, we see promise in the use of TLTs in the design of teacher preparation curriculum to support the growth of teachers' knowledge. The SKT construct asserts that teacher statistical knowledge for teaching consists of both content knowledge and pedagogical knowledge. We found evidence that building professional development using TLTs can help teachers advance both their content knowledge and their pedagogical knowledge. In addition to the development of subject matter knowledge, we evidenced the translation of this subject matter knowledge into pedagogically powerful ideas.

The goal of this paper was to analyze the affordances of TLTs in the design of professional development. In particular, we sought to understand how the use of TLTs might support the development of SKT, with a particular focus on the KDUs that emerged. Doerr et al. (2010) have identified this type of small-scale study as making important contributions to our understanding of professional development. In particular, through the description and analysis of the "critical elements" (Borko et al., 2008) of the program, it is possible to better understand the teacher learning process and the potential of the program for sustained success.

We view the findings in this paper contributing to the advancement of knowledge and literature base in three ways. First, this study provides small-scale evidence that learning trajectories can not only be used to map student curriculum and learning, but also can be used as maps for teacher curriculum and learning.

A second contribution of this study is the connection of learning trajectories to existing teacher learning constructs such as SKT, KDUs, and Pedagogically Powerful Ideas. This study provides evidence that TLTs offer a means to observe and develop such constructs with teachers.

The analysis of the use of TLTs as a "critical element" of a professional development program suggests that TLTs can offer similar structures for teacher learning that mirror those previously documented for student learning. In particular, the TLTs offered a framework for identifying and achieving KDUs and making instructional decisions based on the KDUs. The TLTs gave the research team a way to see how KDUs were directly related to the development of SKT. Furthermore, the TLTs provided a means for teachers to achieve KDUs due to their repeated exposure while moving through the loop structure of the TLTs. By construction, the TLTs provided scaffolding for KDU development. The repeated exposure illustrated when cognitive shifts were occurring in teachers' knowledge. In addition, this repetition allowed teachers to transform KDUs into pedagogically powerful ideas.

While Project-SET has a specific focus on teachers' statistics knowledge, we submit that the implications for mathematics teacher training are broader than this teacher population. In particular, by focusing on a "critical element" of Project-SET– the use of LTs for teacher knowledge – we assert that the model has potential for other content within mathematics teacher professional development. The use of LTs for teacher learning offers a potentially powerful strategy for developing teachers' knowledge of other concepts within the larger mathematics curriculum.

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Interaction, Activities, & Feedback: A Taxonomy of GTA Professional Development

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In this report we present a taxonomy of mathematics graduate student teaching assistant (GTA) professional development (PD) programs. This taxonomy is based off of the characterization of GTA PD programs from 120 mathematics departments, and is informed by the framework developed by Ellis (2015) based on case studies of four GTA PD programs. A cluster analysis revealed nine distinct models of GTA PD within the 120 programs. These nine models vary with respect to the amount of interaction the GTAs have through the PD, the amount of activities involved in the PD, and the level of feedback given to GTAs involved with the PD. We present a characterization of one of the nine models using Ellis's framework.

Key words: GTA, professional development, taxonomy, institutional change

Graduate teaching assistants (GTAs) are playing an increasingly integral role in the education of undergraduate students at major universities in the United States, serving as both primary instructors of introductory courses and as lab or recitation assistants. Although many of the GTAs coming in to these teaching roles in mathematics departments are likely well-versed in the content of the course they are teaching, this may not be enough (Ball, Thames, & Phelps, 2008). Shulman (1986) and Ball, Thames, and Phelps (2008) argue that teaching also requires an additional type of understanding and knowledge, that of the "organizing principles and structures and the rules for establishing what is legitimate to do and say in a field" (Ball, Thames, & Phelps, p. 391, 2008). Thus, it is essential that GTAs receive adequate training in order to help them to be successful in their roles as teachers.

There are numerous case studies and journal articles (such as Alvine et al., 2007; Belnap, 2005; Barry & Dotger, 2011; Harris, Forman & Surles, 2009; Luft, Kurdziel, Roehrig & Turner, 2004; Wayne et al, 2008) about improving GTA training programs at single universities across the nation. This is further corroborated by a recent national survey in which about 33% of the 210 PhD or Masters granting institutions that responded said that changes to their GTA teaching preparation program are either being discussed or changes have recently/currently been implemented (Ellis, Speer, & Deshler, 2016). One difficulty in evaluating current GTA professional development programs is that there is not an agreement on how this training is defined (Shannon, Twale, & Moore, 1998). Consequently there is a need to create a taxonomy of GTA professional development (PD) programs and an accompanying metric for determining the effectiveness of these programs. The overarching question that guides the analysis in this paper is: *What models of GTA PD are currently being implemented in US mathematics departments where GTAs are tasked with teaching?*

In order to help us characterize the existing GTA training programs, we can gain insight from the PD programs at the secondary level. Graduate students often receive training before they teach (similar to pre-service teachers) and ongoing training while they teach (similar to in-service teachers). In order to develop a characterization of the current GTA PD programs we will also elaborate on the framework given by Ellis

(2015). Ellis's framework "can help direct attention to important components to consider [for a GTA training program], as well as provide a visual representation of the many components" of such a program (p. 12, 2015). Figure 1 provides an outline of the framework and the main components of the framework.

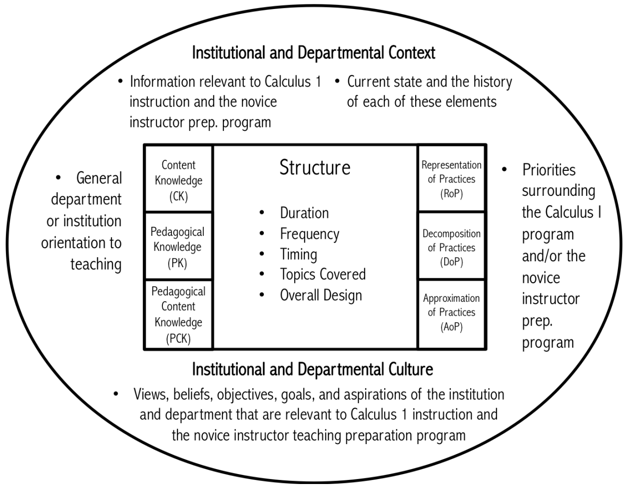

Figure 1. GTA PD framework (Ellis, 2015)

It will be beneficial to test and expand (if needed) Ellis's proposed framework to GTA PD programs across the nation as this framework is currently based on models used by only four universities. The process for collecting the data used to characterize these models was very intensive. In the characterization that we propose, the information that went into categorizing the training programs came from national survey data and did not require site visits and interviews, although these could be helpful in further classifying GTA PD programs. In the discussion section we consider the applicability of the framework based on the data available through the survey.

Methods

The aim of this research is to investigate how graduate-degree granting mathematics departments are training their GTAs, and more broadly refine a systematic classification of such programs. To understand what models of GTA PD currently exist in graduate degree granting mathematics departments, we analyze survey data from the Progress through Calculus (PtC) Census Survey. This survey was sent to every masters and doctoral degree granting mathematics department in the US, and was comprised of questions related to many aspects of the Precalculus through Calculus II (P2C2) sequence, including student placement, what courses the school offers in the sequence, and who teachers these courses and how these courses are taught.

We focus on the section of the survey dedicated to graduate student teaching involvement in the sequence, and the GTA PD involved. The PtC research group (NSF DUE-1430540) and the College Mathematics Instructor Development Source (CoMInDS) (NSF DUE-1432381) research group jointly developed this section of the survey. Overall, the response rate for the survey was 68%. There were 223 schools that completed the GTA section of the survey. Of the 223 schools, we focus primarily on schools who had a department wide training. Only 148 of the 223 schools responded that they had department wide training that was required. Of the 148 schools with department specific training, 120 indicated that their GTAs primarily served as the sole instructor of a P2C2 course. The breakdown of the 120 schools can be found in Table 1.

Table 1. Overview of institutions involved in analysis

Degree Awarded	Undergraduate Population	Number of Schools	% of Schools	% within Degree
Masters	<20,000	29	24%	83%
	>20,000	6	5%	17%
PhD	<20,000	55	46%	65%
	>20,000	30	25%	35%

To understand the main components of the GTA PD programs currently being implemented, we started by looking at each school's responses to five questions on the survey (as shown in Figure 2) related to the structure of the GTA PD.

1. WHEN do GTAs participate in the department's teaching preparation program? Mark all that apply.
- ❑ Before teaching for the first time (e.g., pre-term orientation)
- ❑ During their first term of teaching
- ❑ During their second term of teaching
- ❑ At some point later (e.g., an on-going series of teaching seminars, activities later in the graduate program)
- ❑ Other (please explain)

2. Which of the following best describes the FORMAT of your main activity in the GTA teaching preparation program? Mark all that apply.
- ❑ Short workshop or orientation (1-4 hours)
- ❑ One-day workshop
- ❑ Multi-day workshop
- ❑ Term-long course or seminar
- ❑ Occasional seminars or workshops
- ❑ Other (please explain)

3. Which of the following activities, related to providing feedback on GTA's teaching, does your program FORMALLY include? Mark all that apply.
- ❑ GTAs practice teaching and receive feedback on their teaching
- ❑ GTAs are observed by an experienced instructor while teaching in the classroom and receive feedback on their teaching
- ❑ New GTAs are observed by experienced GTAs while teaching in the classroom and receive feedback on their teaching
- ❑ New GTAs teaching in the classroom are videotaped for review and discussion with a mentor or experienced instructor.
- ❑ GTAs are paired with a mentor to discuss teaching
- ❑ Other (please explain)

4. Which of the following activities, related to evaluating GTAs' teaching, does your program FORMALLY include? Mark all that apply.
- ❑ GTAs are observed by a faculty member while teaching in the classroom
- ❑ Student evaluations required by the institution or department
- ❑ Student evaluations gathered specifically for the purpose of evaluating GTAs (in addition to or separate from those required by the institution or department)
- ❑ Other (please explain)

5. Which of the following other activities does your program FORMALLY include? Mark all that apply.
- ❑ GTAs watch or read cases of others teaching and discuss the teaching
- ❑ Experienced GTAs are observed by new GTAs while teaching in the classroom
- ❑ GTAs develop lesson plans
- ❑ GTAs learn classroom assessment methods
- ❑ GTAs learn about what research tells us about how students learn mathematics
- ❑ Other (please explain)

Figure 2. Survey questions used for analysis

To prepare for the analysis, some of the response options were combined. For instance, we combined responses from schools that indicated their GTAs participate in training before teaching and during their first term of teaching with schools that indicated their GTAs participate before teaching and also at some point later. We felt these two groups could be combined because they both had a pre-teaching training and follow up training. Similar combinations were made to reduce the large number of variables. Our main goal in the analysis was to group schools together based off of their responses to the five questions, with essentially 25 binary yes or no questions.

To help guide us with the grouping we used SPSS to perform a hierarchical cluster analysis. We used Ward's method and, since the data was binary, a squared Euclidean distance and focused our analysis on the dendrogram. From this dendrogram we were given initial groupings of schools. One thing that became quickly apparent was that across the groups the type(s) of evaluation did not significantly vary. We concluded this would not be a deciding factor of our groupings, and thus question four (concerning evaluation) was dropped from our cluster analysis to help make more clear groupings.

Running the analysis again resulted in more distinct groups that we felt we could describe based on the institutions' responses to the remaining four questions. Nine distinct groups emerged from the cluster analysis and were illustrated by the dendrogram. After identifying these nine groups, we determined that the important factors were the amount of interaction (based off of question one and question two) and the amount of activities they were conducting (question five), and that the amount of feedback (question three) seemed to be a function of the level of interaction and the amount of activities involved in the training.

Results

The cluster analysis produced nine groups based off the amount/type of interactions and the amount/type of activities involved in the GTA PD programs. Table 4 has the breakdown of those groups based on the interaction level (low, medium, or high) and the amount of activities involved in the professional development (low, medium, or high).

Table 2. Nine models of GTA PD delineated by interaction level and activities

Model Name	Interaction Level	Activities	Number of programs
Model 1	Low	Low (0-1)	9
Model 2	Low	Medium (2)	13
Model 3	Low	Medium (3)	11
Model 4	Medium-Mixed	Low (0-1)	15
Model 5	Medium-One Semester	Low (0-1)	27
Model 6	Medium	Low (1-2)	15
Model 7	Medium	High (4-5)	9
Model 8	High	Medium (2-3)	11
Model 9	High	High (4-5)	10

The programs that are categorized as *Model 1* have GTAs participate in professional development before teaching their first course, with this training primarily consisting of a short workshop or orientation (1-4 hours) or a one-day workshop. The GTAs do not participate in any of the activities listed on the survey during training. This is considered a model with a low level of interaction and a low level of activities. The type of feedback that the GTAs receive was split between practice teaching with feedback and being observed by an experienced instructor with feedback.

The programs categorized as *Model 2* primarily have their GTAs participate in professional development before teaching their first course with this training either lasting one day or multiple days. Schools using this model include two activities (on average) into the training with the creation of lesson plans and assessments being the most common activities. Model 2 is considered to have a low level of interaction, but a medium level of activities. The feedback for GTAs primarily came from practicing teaching and being observed by an experienced instructor and/or an assigned mentor.

The programs that are categorized as *Model 3* have GTAs participate in training before teaching their first course and lasting either multiple days or continuing for one semester, thus having a low interaction level. Schools using this model incorporate an average of three activities (medium level of activities). The predominant activities are creation of lesson plans, creation of assessments, and the discussion of case studies. Model 3 incorporates several ways of providing feedback to GTAs, including feedback from practice teaching and being observed by an experienced instructor while teaching.

The programs that are categorized as *Model 4* have GTAs participate in training before the first course they teach with ongoing training during their first semester of teaching. Over half of the schools using this model also have training taking place during the GTAs second semester of teaching. The format of these trainings varied across schools ranging from only a few hours of PD and then follow up training that met as a semester course, occasional seminars/workshops, or a meeting schedule that went longer than one semester. Schools using this model are considered as having a medium level of interaction, but use a minimal number of activities as identified by the survey. All of the schools in this group gave feedback to their GTAs by having observations by an experienced instructor and/or an assigned mentor.

The programs that are categorized as *Model 5* have GTAs participate in training before the first course they teach with the training continuing for one semester. This model has a medium level of interaction, with minimal activities conducted during the training. Feedback for the GTAs is primarily from practice teaching and receiving feedback from an experienced instructor and/or an assigned mentor.

The programs that are categorized as *Model 6* have a very similar structure of GTA training and style of feedback as those categorized as Model 5; however, the schools using Model 6 incorporated at least one activity (often two) into the training. The most prominent activities used are the creation of lesson plans and the creation of assessments.

The programs categorized as *Model 7* have GTAs participate in training before teaching their first course and during their first and second term of teaching which is categorized as a high level of interaction. During these trainings, GTAs are involved in a high number of activities (4-5) with the most common among schools categorized in this model being creating lesson plans, creating assessments, and reading research on how students learn mathematics. The amount of feedback given is also high with it coming in the form of feedback from practice teaching, teaching while being observed by an experienced instructor, and one of the following: being videotaped while teaching, being observed by experienced GTAs, or having a mentor.

The interaction level of GTA training of schools categorized as using *Model 8* is similar to that of Model 7, but the level of activities incorporated is considered medium (2-3 activities). The major activities included are creating lesson plans, creating assessments, and discussing case studies or research on how students learn mathematics. The main method of feedback used in Model 8 is practice teaching and being observed by an experienced instructor, with the level of feedback classified as medium.

The schools that are categorized as using *Model 9* have GTAs participate in training before the first course they teach with most school having the training continue through their first semester of teaching. GTAs engaged with a high number of activities (4-5) with all school having GTAs create lesson plans, create assessments, discuss case studies, and observe experienced GTAs and/or discuss research about how students learn.

In Table 4 we present a graphical depiction of the nine models, with the x-axis representing the amount of activities involved in the GTA PD program, and the y-axis the amount of interaction involved. We represent the amount of feedback involved in the PD by the size of the circle in this graphical depiction. This visualization helps to illuminate the apparent positive correlation between the amount of feedback and the amount of interaction and activities. In other words, GTA PD programs that have the GTAs participate more often in the professional development and where the PD involves more activities (represented by the programs in the upper rand hand corner) also provide GTAs more feedback on their teaching. This is worth noting as feedback is beneficial for improving teaching skills, and is thought to be correlated with more effective teaching (Shannon, Twale, & Moore, 1998).

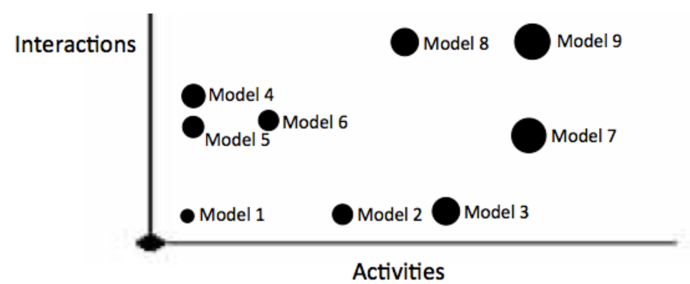

Figure 3. A graphical representation of the nine GTA PD models

Figure 3 helps to identify clusters of models. Model 1 is the least robust, while models 7-9 are the most robust. Models 4-6 have more interaction but less activities involved than Model 1, while Models 2 and 3 have less interaction but more activities. Based on these nine models, it appears that the amount of activities involved may be more related to the amount of feedback provided to the GTAs compared to the amount of interaction (as evidenced by Model 3 compared to Model 4).

Discussion

In this study we set out to determine how graduate-degree granting mathematics departments are currently preparing their GTAs for teaching. We approached this problem scientifically by developing a taxonomy of the characteristics of GTA PD programs. This taxonomy was informed both by Ellis's (2015) initial work to this end, and by the structure that emerged through the cluster analysis illustrated through the dendrogram. After identifying the nine models of GTA PD that currently exist across the country in graduate-degree granting mathematics departments based on this taxonomy, we can begin to explore a metric for comparing GTA PD programs and to aid in the development and improvement of such programs. In this discussion, we briefly explore the applicability of Ellis's (2015) framework to the models identified through the cluster analysis and based on survey data.

When Ellis (2015) initially began work on the taxonomy she created a framework to characterize GTA PD programs based on in-depth case studies of four programs. These case studies provided a large amount of information, but were time and resource intensive. We wondered if the information provided by the census survey would be sufficient to characterize our nine models using her framework. To do this, we identified relevant questions in the census survey and conducted basic descriptive statistics to determine the average responses for each cluster of schools per model. We attended to the structure of the program, the types of knowledge and pedagogies of practices emphasized in the structure, and the institutional and departmental context and culture, as described in Figure 1. We were happy to find that we were able to characterize each aspect through survey questions, though there is inherent bias in the survey based on the subjectivity of who answered the survey, whereas with case study data collection it is possible to triangulate findings. Due to space limitations, we characterize only Model 8 in this representation, as shown in Figure 4.

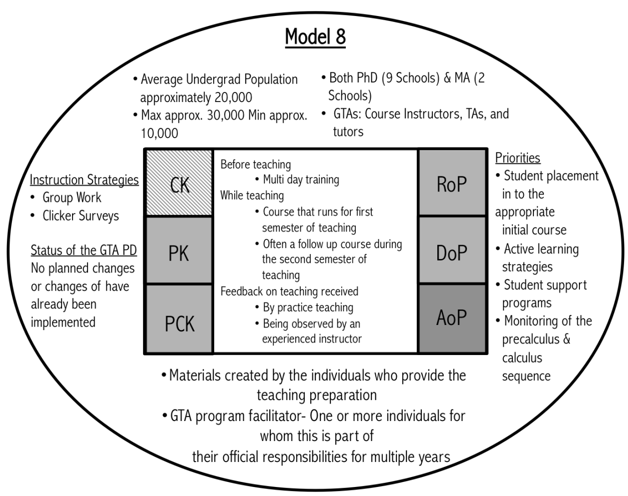

Figure 4. Characterization of Model 8 using Ellis's (2015) framework

In the presentation we will explain this characterization in depth, including the shading for the types of knowledge and practices, and compare this model to another using the oval representation. We will also explore what the taxonomy tells us about a metric for evaluating and comparing programs, and what the direct implications of this work are for mathematics departments seeking to improve their GTA PD program.

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Difficult Dialogs About Degenerate Cases: A Proof Script Study

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Abstract: The purpose of the reported study is to explore students' reasoning about the "within argument contradictions" that arise from logically degenerate cases by analyzing the problematics noticed in students' proof scripts. The work proposes a framework for students' noticed proof problematics and explores the viability of the proof script methodology as a mechanism for identifying difficulties experienced by students but unseen by experts. In the case of logically degenerate cases, findings indicate students held conceptions of proofs by cases that inhibited students' reasoning about the encountered contradictions.

Keywords: Proof by contradiction; proof scripts; proof comprehension; proof by cases

Research on students' understanding and production of proofs by contradiction indicates that this form of proof is exceptionally difficult for students (Robert, & Schwarzenberger, 1991). In a reflective account of multiple teaching experiments, Leron (1985) proposed students' difficulties with proof by contradiction are rooted in a preference for constructive as opposed to destructive arguments, in part because the latter require learners to reason against that which is "real" to them. Harel and Sowder (1998) reported on multiple teaching experiments with undergraduates and observed that students don't use proof by counterexample and are not convinced by proof by contradiction. Like Leron, they argued students' difficulties are tied to ways of thinking in which ascertainment is reliant on constructive approaches; that is, students' reasoning with a constructive proof scheme. In contrast, in their study of young children playing the game Set, Reid and Dobbin (1998) reported observations of children spontaneously reasoning with informal contradiction arguments when convincing others of the validity of their solutions. And, Maher and Martino (1996) reported similar findings from their longitudinal studies of children's combinatorial reasoning. Hence, studies on students' views of indirect proof have offered disparate views. In work that bridges these conflicting reports, Antonini and Mariotti (2008) analyzed instances of students producing informal contradiction arguments but experiencing difficulties linking those arguments to formal proofs by contradiction. Thus, the conflicting reports may be related to the level of formalism expected.

Attending to the specific demands of indirect proofs, researchers have noted particular sources of difficulty. Taking a more syntactical approach, Wu, Lin, and Lee (2003) found that students were rather unsuccessful negating quantified statements. In relation to inferential practices, Antonini and Mariotti's (2006) analyses of interview protocols illustrated how, when students assume a true statement is false, they may experience difficulties understanding the ramifications of this assumption for they assume that other true statements may be false in this new "absurd world." And, Antonini and Mariotti (2008) documented how students' difficulties with indirect proofs (proofs by contradiction and proof by contraposition) can be tied to students' difficulties accepting the logical "theorems" that allow for indirect proofs; e.g., the tautologies $(P \Rightarrow Q) \equiv (\sim Q \Rightarrow \sim P)$ and $(P \Rightarrow Q) \equiv \sim (P \wedge \sim Q)$. Taken together these studies show that students may encounter a myriad of difficulties that are specific to proofs by contradiction.

One issue with the existing research, however, is that much of the research on students' understanding and production of proofs by contradiction has been generalistic in nature rather than focused. By this I mean that while researchers have argued that students dislike and are not convinced by proofs by contradiction, there is little empirical evidence related to the specific features students attend to which foster students' reactions and their subsequent lack of conviction or dislike of proof by contradiction, with the exception of the work of Antonini and Mariotti (2006, 2008). Indeed, their work has demonstrated connections between: (1) students' difficulties at the level of the logical theory and their lack of acceptance of proofs by contraposition; and (2) students' lack of conviction and students' confusion regarding which mathematical statements hold true and can be reasoned with when assuming a true statement is false. Yet, despite this progress, further research is needed on students' understanding and production of proofs by contradiction. There are two reasons for this claim. First, outside of research in geometric contexts, existing research has primarily focused either on basic conditional statements, "If n^2 is even then n is even" (Antonini & Mariotti, 2008) or non-compound statements, "there exists infinitely many primes" (Leron, 1985) and "$\sqrt{2}$ is irrational" (Tall, 1979). Yet, contexts that recurrently call for reasoning with contradictions have not been studied; such as, proofs of existence, nonexistence, disjointness, or those involving logically degenerate cases. Second, research on proof by contradiction has primarily involved interview protocols and assessments in which students' attention to and reasoning about proofs by contradiction are elicited by a knowledgeable other, whose questions provide a focusing effect. While both approaches are commonly taken by researchers and are of value, there is reason to question the types of difficulties students "see" when they are not engaged in communications that direct students' attention (e.g., structured interviews or comprehension tests). Stated another way, little is known about students' *problematics*; that is, "what students themselves see as issues of difficulty" (Koichu & Zazkis, 2013, p. 364). Indeed, as Koichu and Zazkis argue:

> As a rule, students' difficulties with constructing and understanding proofs are exposed by means of documenting and interpreting their (often poor) performance when coping with various proving tasks. This research approach implies that students' understanding of proofs and their difficulties are mainly examined from an expert point of view (p.364).

To move away from methodologies that necessarily employ preselected foci and assume the difficulties determined *a priori* by experts will coincide with those experienced by students, researchers have begun to build on the "lesson play" methodology developed by Zazkis, Liljedahl, and Sinclair (Zazkis, Liljedahl, & Sinclair, 2009; Zazkis, Sinclair, & Liljedahl, 2009; Zazkis, Sinclair, & Liljedahl, 2013) and the dialog methodologies of Gholamazad (2006, 2007) to develop techniques for examining students' proof "problematics" through students' *proof scripts* (Koichu & Zazkis, 2013; Zazkis & Zazkis; 2014). At the most basic level, the proof script methodology involves asking participants to write a dialog between characters in which the two characters discuss points of difficulty and resolve problematic issues. Building on Sfard's (2008) theoretical work, which views cognition as a form of interpersonal communication, researchers employing these methodologies have argued proof scripts provide an avenue by which participants may make "personal thinking salient."

The Study

The purpose of the reported study was to explore students' reasoning when confronted with within argument contradictions that arise from logically degenerate cases. A *logically degenerate case* is a case in which the case's assumption lead to a contradiction with the constraints of the mathematical statement or a definition, axiom, or property of the reference theory.[i] Such cases are not only common when proving basic properties of the real numbers (e.g., in proofs that use the Law of Trichotomy) but also occur in a variety of divisibility and topology proofs. Moreover, it was observed during prior teaching experiments that there are theorems for which one may produce a simple proof by contraposition or a direct proof involving logically degenerate cases. Among those seeking to avoid the Law of Contraposition, difficulties occurred interpreting the consequences of the contradictions encountered, when the overall structure (and the student's intention) was that of a direct proof. Building on prior work, the reported study aimed to document students' ways of reasoning about contradictions arising from logically degenerate cases by examining the *problematics* evident in students' proof scripts.

Data Collection

This research is part of a larger study on students' reasoning about and production of indirect proofs. For the purposes of this paper, we report on the *proof scripts* produced by 20 students randomly selected from two, inquiry-based "Introduction to Proof" courses at the end of the academic term. Students enrolled in these courses were either mathematics majors or minors, at a designated Hispanic-serving institution, where the majority are first generation college students and are eligible for need-based financial assistance. Collected proof scripts were blinded of students' names and scanned into a database, prior to the instructor's markings.

The scripting task instrument included instructions for the dialog and a line-numbered proof of the theorem "For every integer m, if $3 \nmid (m^2 - 1)$, then $3|m$". The scripting task (see Figure 1) was assigned as homework, so students had access to their texts, course notes and other resources.

Instructions: Create a 1-2 page dialogue that introduces and explains the theorem and its proof. Highlight the problematic points in the proof with questions and answers. In your submission:

- The dialogue should occur between two students, you and a mathematics student named Gamma, which you can denote with either Γ or γ.
- Start by reading the proof and identifying what you believe are the "*problematic points*" for a learner when attempting to understand the theorem/statement or its proof. List these "*problematic points*" in a bulleted list.
- **Write a dialogue** between you and Gamma in which you explain the theorem and the proof to Gamma, paying special attention to the *problematic points* you identified and listed in your bulleted list. In the dialogue you should both pose questions to Gamma and answer any questions Gamma might ask. (THIS IS THE MAIN PART OF THE ASSIGNMENT)
- You may add comments at the end or within the dialogue using [ ], explaining your questions or answers, if these are not obvious to the reader.

Theorem: For every integer m, if $3 \nmid (m^2 - 1)$, then $3|m$

1) *Proof*: Assume $3 \nmid (m^2 - 1)$.
2) Since $(m^2 - 1) = (m + 1)(m - 1)$, we can say $3 \nmid (m + 1)(m - 1)$.
3) By a previous theorem we know,
4) if $3 \nmid (m + 1)(m - 1)$ then $3 \nmid (m + 1)$ and $3 \nmid (m - 1)$.
5) Since $3 \nmid (m + 1)$, it follows from the Division Algorithm, that
6) $(m + 1) = 3k + 1$ or $(m + 1) = 3k + 2$ for some integer k.
7) *Case 1.* Let $(m + 1) = 3k + 1$
8) Then $m + 1 - 1 = 3k + 1 - 1$
9) $m = 3k$
10) Since k is an integer, $3|m$.
11) This is our desired result.
12) *Case 2.* Let $(m + 1) = 3k + 2$
13) Then $m + 1 - 1 = 3k + 2 - 1$
14) $m = 3k + 1$
15) $m - 1 = 3k + 1 - 1$
16) $m - 1 = 3k$
17) Since k is an integer, $3|(m - 1)$,
18) This is a contradiction.
19) It follows from *Case 1* and *Case 2*, that $3|m$. □

Figure 1. Proof Script Assignment [ii]

This particular proof was chosen for two reasons. First, one can either prove the statement directly, with the possibility of logically degenerate cases[iii] (as shown above) or one can write an indirect proof using use the Law of Contraposition. Since researchers have argued that students' prefer direct proofs, the selected proof afforded an opportunity to explore students' reasoning about the outcomes of their potential proving proclivities. Second, as mentioned above, the proof was produced by a group of students during a previous teaching experiment. Thus, it was an "authentic" student proof, by which I mean that it arose from students' mathematical activities, as opposed to an expert's. In that experiment, the authoring students asked to share the proof with the class, for they were uncertain of the proof's validity. Thus, there was reason to posit not only that students' might produce such proofs but that they might find them problematic. Indeed, during the prior teaching experiment the proof fostered an extensive debate about how to interpret the contradiction encountered. Thus, the proof was viewed as providing a fruitful venue for exploring students' reasoning about contradictions arising from logically degenerate cases.

Analytic Methods

Taking the perspective that asking students to produce dialogs around observed *problematics* is akin to asking students to share and elaborate on the difficulties noticed, the analytic methods employed in the study were heavily influenced by the construct of *noticing* (Mason, 2002; Jacobs, Lamb, Philipp, & Schappelle, 2011; VanEs, 2011). Building on Mason's work (Mason, 2002; 2011) it was hypothesized that noticing requires "a movement or shift of attention." Drawing on the work of Jacobs, Lamb, Philipp, and Schappelle (2011), it was posited that attending to noteworthy aspects and potential complexities is a sign of expertise. Thus, one can interpret "difficulties attending to difficulties" as indicative of one's need to grow expertise rather than as indicative of expertise. Drawing on the work of VanEs (2011), it was argued that noticing of complex phenomena is marked not only by *what* is noticed but *how* one notices; in particular, the extent to which one can accurately describe, focus on, or extend an observation is indicative of one's emerging level of expertise.

To analyze students' proof scripts, VanEs's (2011) "framework for learning to notice," which was developed to characterize teachers' noticing of classroom lessons, was adapted to examine two dimensions of students' noticing: what proof problematics students noticed and how students noticed those proof problematics. Specifically, three levels, which are defined in Table 1, were used to characterize the foci of students' dialogs and the depth of engagement with those foci.

FRAMEWORK FOR STUDENTS' NOTICED PROOF PROBLEMATICS			
Dimension	**Level 1**	**Level 2**	**Level 3**
1. What students noticed (i.e., students' identified problematics and other dialogical foci).	(*a*) Identifying (perceived) rudimentary errors, wording refinements, instances in which the naming or stating of definitions or theorems could add clarity, or (*b*) failing to attend to the proof by dismissing the proof in its entirety.	Addressing level 1 issues <u>*and*</u> potential points of confusion that are in need of clarification due to the application of content, theorems, or linkages between consecutive statements.	Addressing level 2 issues <u>*and*</u> identifying potential points of confusion across sets of statements or related to potential consequences of a sequence of statements or the logical architecture of a proof.
2. How students noticed (i.e, students' ways of attending to problematics and other dialogical foci).	Remarks are limited to correcting rudimentary errors, reiterations of lines of the proof in a "lecture-oriented" dialog or a single speaker monolog, or to assertions. (Note: Corrections may contain errors.)	Engaging with *problematics*, beyond the rudimentary level, to explore potential rationales or implications but in a perfunctory manner. (Note: Rationales or implications may include errors.)	Attending both holistically <u>and</u> locally to *problematics* so as to link the content and structure of the proof. (Note: Linkages may include errors.)

Table 1. The Framework for Students' Noticed Proof Problematics

What students' noticed (Dimension 1) was determined by comparing the student's bulleted lists to the students' written dialog, so as to produce a list of attended to proof problematics. *How* students' noticed (Dimension 2) was determined through analyses of the rationales, claims, and backing (if any) the student explicitly provided when attending to proof problematics.

Prior to data collection, it was anticipated that students would: (1) remark that a variable *m* is introduced but not defined; and, (2) experience difficulties determining the appropriate number of cases, i.e., incorrectly argue additional cases arise from the statement $3 \nmid (m - 1)$. In relation to the logically degenerate case, it was anticipated that students' might view the contradiction in Case 2 as implying the entire proof was a proof by contradiction and, therefore, that the original theorem was false. No other difficulties were anticipated prior to data collection.

Results

The coding of the students' noticed *proof problematics* indicates the majority of novices experienced a great deal of difficulty interpreting and attending to the given proof, for nearly half of the students (9 out of 20) produced *Level 1* proof scripts; that is, proof scripts focused on rudimentary errors or wording refinements, without attention to conceptual issues, linkages across statements, or the logical architecture of the proof. With regard to the given proof, typical characteristics of the Level 1 scripts included: (1) an extended discussion on the need to define the variable m; (2) incorrectly claiming cases were missing for the statement $3 \nmid (m - 1)$; and, (3) a verbatim reiterations of lines 12 – 18 coupled with a lack of attention to the role/ramifications of the logically degenerate case, which I will refer to as the *within-case contradiction*. The following written dialog excerpt illustrates a Level 1 response with characteristic (3):[iv]

<u>*Example 1*</u>

Student 1: (written dialog reiterates lines 13 – 18 verbatim)

Gamma: Damn, we got it wrong.

Student 1: No we didn't. This was our desired result, so in the end we got it right.

Gamma: Cool.

Student 1: So once we proved the cases, we can write this 'Since both cases had the desired results, we can say that $3|m$, therefore the statement is true.

Level 2 dialogs were produced by 8 of the 20 students. These responses differed from the *Level 1* dialogs in that beyond addressing rudimentary issues, the students' dialog clarified content and explained the application of theorems and/or linkages between consecutive statements. With regard to the given proof, common characteristics of *Level 2* scripts included: (1) specifically stating and clarifying the application of the "previously proved theorem;" (2) explaining how to apply the division algorithm and/or the meaning of "$a|b$"; and, (3) addressing the within-case contradiction in a perfunctory manner either by merely identifying the statement that was contradicted or by suggesting alternative proofs. Indeed, several students' remarks were limited to asking Gamma to find the contradicted statement; namely, $3 \nmid (m - 1)$. And, many either informed Gamma that the within-case contradiction implied the entire proof was a proof by contradiction (see Example 2) or told Gamma that "a stronger argument" would have been a proof by contraposition, with several providing the proof by contraposition.[v]

<u>*Example 2*</u>

Gamma: In Case 1, the statement is manipulated to show that $m = 3k$, then by definition 27 we write $3|m$, which is the desired result.

Student 2: We can see that the right idea is being established. In Case 2, the same process is used but results in a contradiction.

Gamma: Case 1 gives us the desired result but case 2 does not.

Student 2: Therefore, since Case 1 results $3|m$ but Case 2 results $3|m$-1, the original statement is false. [- *End of dialog* -]

Level 3 dialogs were produced by 3 of the 20 students. These responses differed from the *Level 2* dialogs in that beyond addressing rudimentary issues and clarifying definitions or theorems, *Level 3* students attended to the conjuctive structure (i.e., the logical "and") of the statement "$3\nmid(m+1)$ and $3\nmid(m-1)$" and attempted to address (either successfully or unsuccessfully) questions concerning the meaning of the within-case contradiction. Regarding the unsuccessful attempts, these students ($n = 2$) argued that the Division Algorithm resulted in "*or* cases" and, consequently, one case could be "false" or omitted. Example 3 illustrates responses of this form. Moreover, it should be noted that brief remarks indicative of similar reasoning were present in some *Level 2* dialogs (see Example 4), but these dialogs did not meet the criteria of the *Level 3* responses.

<u>*Example 3*</u>
Gamma: Case 2 has proven that $3|(m-1)$ which is the contradiction of $3|m$. Then does that mean the theorem is false?[vi]
Student 3: From line 6, it is stated that the remainders are 1 or 2, that means only one case is needed to be true for the $3|m$ to be true. Moreover, Case 2 is not quite necessarily needed when Case 1 is already proven to be true to make the theory true. [- *End of dialog* -]

<u>*Example 4*</u>
Gamma: … Case 2 ends in a contradiction, does that mean that the whole proof is wrong?
Student 4: Not at all! We were looking for only 1 case to be true, in terms of the value of *r*. We obtained our desired result in Case 1 already.

Regarding the successful attempt, the student argued that the contradiction arose from an earlier statement in the proof "$3\nmid(m-1)$" and that $m + 1 = 3k + 2$ could not occur under the theorem's given constraints. In so doing, the student moved beyond the Division Algorithm's disjunctive structure to make connections across statements and (correctly) conclude the case was logically degenerate (i.e., can't happen), under the conditions $3\nmid(m+1)$ *and* $3\nmid(m-1)$.

Discussion

The purpose of the reported study was to explore students' reasoning when confronted with *within argument contradictions that arise from logically degenerate cases* by examining the proof problematics students identified and attended to in their scripts. The aim of employing proof scripts, rather than clinical interviews or comprehension assessments, was to create a context within which students could make their "personal thinking salient" and articulate the *problematics* visible to students but potentially unseen by experts. Certainly, the findings suggest the methodology afforded these opportunities. Neither the students' beliefs related to "or cases" and nor the students' general lack of attention to the within-case contradiction were anticipated.

In regard to students' efforts to account for the contradiction by arguing it was a consequence of the disjunctive result of the Division Algorithm, it appears that many of the students incorrectly reasoned $[(A\vee B) \Rightarrow C] \equiv [(A\Rightarrow C)\vee(B\Rightarrow C)]$ rather than reasoning $[(A\vee B) \Rightarrow C] \equiv [(A\Rightarrow C)\wedge(B\Rightarrow C)]$. In other words, the students appear to have extended the logical "or" linking the statements derived from the Division algorithm ($m + 1 = 3k + 1$ or $m + 1 = 3k + 2$) to the (implicit) connective linking the cases. Consequently, the students viewed the cases as "*or*-cases." This response may explain why so few students attempted to relate the contradiction to the concomitant constraint $3\nmid(m-1)$. While difficulties making connections broadly across statements within a proof might not be unexpected when studying novices, and have been discussed by others (Selden & Selden, 2003), the students' views of proofs by cases (if prevalent) are unexpected. Moreover, they could suggest a serious misconception. Consideration of the ways such reasoning might manifest itself leads one to imagine a similar scenario in which when asked to prove, "if n is an integer, then $2|(n^2 - n)$," these students might (incorrectly) argue:

> By the Division Algorithm, $n = 2k$ or $n = 2k + 1$. If $n = 2k$, then $(n^2 - n) = 4k^2 - 2k = 2(2k^2 - k)$. Since $(2k^2 - k)$ is integer, $2|(n^2 - n)$. The result follows.

And, the instructor might note "you forgot the case $n = 2k + 1$" thinking the student was being absentminded, when in fact the student was acting on a conception of or-statements in proofs by cases. Indeed, such difficulties would be difficult to recognize from the experts' point of view.

In regard to the students' general lack of attention to the *within-case contradiction*, it is surprising that nearly half of the students produced *Level 1* dialogs in which the student avoided engaging with the contradiction. How might such low level responses be accounted for? One reason may be that the responses are reflective of the students' classroom experiences and the students are accustomed to *problematics* arising that are either ignored or are only given cursory explanations by their instructors. In such circumstances, the student may simply be attempting to follow the *modus operandi*. It is also possible that, despite the fact that the dialogs were graded (or perhaps due to their being graded), students avoided issues they felt inadequately prepared to address even though, in the students' eyes, they were *problematics*. In such situations, a lack of attention would indicate a higher degree of confusion rather than understanding. Such an account of the dominance of *Level 1* responses is reasonable, for as researchers (Mason, 2002; Jacobs, Lamb, Philipp, & Schappelle, 2011; VanEs, 2011; Sherin, 2001) studying noticing have argued, noticing of complexities requires expertise – a point cleverly illustrated by Sherin (p. 75):

> Imagine that you are standing at the site of an archeological dig. On your left, you see a large rock with a dent in the middle. Next to it you see a pile of smaller stones. Aside from this, all you see is sand. An archeologist soon appears at the site. What looked like just a rock to you, he recognizes as the base of a column; the small stones, a set of architectural fragments. And where you saw only sand, he begins to visualize the structure that stood here years before.

Thus, the prevalence of *Level 1* proof scripts might speak to the students' need to further develop their expertise. Indeed, the predominance of *Level 1* scripts coupled with the prevalence of students' incorrect conclusion related to the Division Algorithm and students' belief that additional cases were required for $3\nmid(m - 1)$, lead to significant concerns about the student reasoning that occurs when such proofs are presented in lectures and the opportunities required (but likely not afforded) for students to develop appropriate interpretations. Indeed, recent work by Lew, Fukawa-Connelly, Mejia-Ramos, and Weber (2016) on advanced mathematics lectures suggests that direct approaches (e.g., stopping to remark on such issues) might not be effective. And, as noted by Hemmi (2008), "Teachers' intentions of focusing on certain things in their presentation do not necessarily imply that these aspects become visible to students" (p. 424). Thus, future research is needed on how, within the various didactical constraints, students' proof *problematics* might not only be elicited but also productively addressed in classrooms, which seek to develop students' understanding of proof in advanced mathematics.

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[i] In mathematics the phrase *degenerate case* is used to refer to a case that is qualitatively different from and often simpler than the other cases, for it belongs to a simpler class of objects or has some other features that distinguishes it from the class of objects. In line with this definition, the phrase *logically degenerate case* is being used to refer to cases that result in contradictions and are, therefore, qualitatively distinct from those cases by which one produces a desired result.

[ii] The previously proved theorem was "For any integers x and y, if $3 \nmid xy$ then $3 \nmid x$ and $3 \nmid y$.

[iii] I have used the phrase "with the possibility" because one may avoid degenerate cases by arguing after line 4 that the integers $m - 1$, m, and
$m + 1$ are consecutive integers and then prove that for any triplet of consecutive integers one member of the triplet must be divisible by three.

[iv] Due to space limitations, the discussion of students' dialogs will be focused on the primary research question of how students reasoned about the within-case contradiction.

[v] Examples of this type of response have been excluded due to their length.

[vi] Earlier in the dialog the student had argued that 3 could not divide two consecutive integers.

Stages of development for the concept of inverse in abstract algebra

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In this study, we conducted a teaching experiment with two students to investigate the development of a generalized concept of inverse in abstract algebra. In particular, we document the stages through which the students' reasoning progressed, initiating with an understanding of the additive inverse of an element as the result of a procedure applied to that element, and concluding with a generalized understanding of inverse that was broad enough to identify instances of inverses in various and unfamiliar algebraic structures. Of critical importance was the development of and coordination with a corresponding concept of identity.

Key words: student thinking, abstract algebra, Realistic Mathematics Education, teaching experiment

Introduction

The concept of inverse is prevalent in abstract algebra and is used to define such foundational algebraic structures as ring and field. Inverses are a key concept in secondary algebra as well and have been stated as a direct connection that pre-service teachers should make between their advanced coursework in abstract algebra and the algebra they will be teaching (CBMS, 2001). Accordingly, as abstract algebra is intended to be "the place where students might extract common features from the many mathematical systems that they have used in previous mathematics courses" (Findell, 2001, p. 12), the inverse concept – like the other properties that characterize rings and fields – is certainly familiar to abstract algebra students from previous courses. What makes inverse unique, perhaps, is the wide variation it exhibits across different contexts. Consider, for instance, the following examples of inverse from the undergraduate curriculum that reappear in abstract algebra:

- Inverse of a complex number: if $a + bi$ is nonzero, then $(a + bi)^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$.
- Inverse of an integer modulo n: if $\gcd(a, n) = 1$, then $a^{\phi(n)-1} \equiv a^{-1} \pmod{n}$.
- Inverse of a matrix: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

These few examples are widespread and varied, each largely dependent on its mathematical context. Without coordinating the binary operation with these examples of inverse and their respective identities, there are few surface-level indications that these are all direct implementations of the same overarching concept. The key notions of the generalized inverse concept – that the inverse of a given element is the element that, when combined with the given element, yields the identity – might be obscured within specific examples of inverses. Though students would likely recognize such examples of inverse within each context, research suggests that it might be challenging to recognize commonalities amongst these examples that reflect a generalized inverse concept. In particular, familiarity with algebraic properties prior to abstract algebra does not guarantee proficiency with these properties in abstract algebra (e.g. Larsen, 2010). Additionally, when working with examples, students attempt to overgeneralize in order to reduce the level of abstraction (e.g. Hazzan, 1999) and struggle to focus their attention on productive facets of the example structure (e.g. Simpson & Stehlikova, 2006). Thus, it is unclear if and how introductory abstract algebra students might identify these examples as instances of a general notion of inverse. This observation motivated our central research question: how might students in abstract algebra come to understand a generalized notion of inverse in a way that enables them to recognize instantiations of inverse within various structures? We propose that it is the coordination of the inverse concept with a corresponding concept of the *identity* that is critical to using and recognizing inverses in abstract algebra. Analyzing results from a ring theory teaching experiment, we document three stages of development for the inverse concept through which two undergraduate students progressed as we guided their reinvention of the concept of ring.

Literature

Much of the research pertaining to inverses provides characterizations of students' understanding of particular kinds of inverses, such as the reciprocal of a rational number (e.g. Tirosh, 2000), inverse functions (e.g. Even, 1992), and inverse matrices (e.g. Wawro, 2014). We focus on those papers describing student activity with inverse (1) across multiple contexts and (2) in abstract algebra.

Use of the Superscript -1 Symbol Across Multiple Contexts

Consistent with our observation above that the varied manifestations and contexts in which inverses arise might prevent students from noticing that each is an instance of the same underlying concept, Zazkis and Kontorovich (2016) pointed out a discrepancy related to potential interpretations of the superscript -1 symbol in a study of students' lesson scripts. For example, 5^{-1} can be interpreted as the reciprocal of the rational number 5, whereas f^{-1} refers to the inverse of the function f, a concept that itself admits various interpretations. For example, students might interpret f^{-1} algebraically (switching x and y and solving for y), graphically (reflecting the graph of f about the line $y = x$), or as the result of reversing or undoing the function f (Carlson & Oehrtman, 2005; Even, 1992). In accordance with these contextually varied interpretations, many students (15 of 22) described the superscript -1 as "the same symbol applied to different, unrelated ideas" (p. 103). All other students viewed the superscript as having "different but related" (p. 107) meanings, each depending on the context in which it is used. One of these students wrote that "we have an example of using the same symbol or same word in different contexts. Both are inverse … a multiplication inverse means 1 over, and an inverse function is when you switch x any y." There are no substantial commonalities amongst her descriptions; her use of the word "inverse" seems to be the only link between "1 over" and "switch x and y." Notably absent from any of the above interpretations of inverse is any mention of an identity element.

While Zazkis and Kontorovich noted the extent to which the students attempted to account for the apparent variation in meaning across different instances of inverse, their study left for future research the question of how students' might develop a generalized, coordinated understanding of the concepts of inverse and identity that enables them recognize instantiations of these concepts within various structures.

Students' Use and Understanding of Inverse in Abstract Algebra

Even though inverses are a familiar concept from school algebra, there is reason to believe that employing the inverse axiom in abstract algebra can be challenging for students. Research on student thinking about inverses delineates two lines of thinking that might explain these challenges. First, students might not carefully attend to closure and the binary operation(s) of an algebraic structure (Nardi, 2001). This could lead to determining inverse pairs by overextending the familiar operations of the real numbers (Hazzan, 1999) or identifying a potential inverse for an element but not verifying that the inverse is in the given set (Brown et al, 1997, p. 207). Second, students might not carefully attend to the algebraic properties on which their reasoning hinges. Indeed, there is evidence that students fail to verify or directly acknowledge inverses (Brown et al., 1997), even if they are implicitly invoking inverses by cancelling adjacent inverse pairs in a calculation (Larsen, 2013).

Theoretical Perspective

We adopted the perspective of Realistic Mathematics Education (RME) to guide our inquiry into how students might develop a generalized notion of inverse because of its view that mathematics can and should "link up with the informal situated knowledge of the students" (Gravemeijer, 1998, p. 279) in order to "enable them to develop more sophisticated, abstract, formal knowledge" (ibid). We leveraged the RME principle of *guided reinvention*, the goal of which is to design tasks that encourage students to "formalize their informal understandings and intuitions" (Gravemeijer, Cobb, Bowers, & Whitenack, 2000) en route to developing formal mathematical concepts themselves. The reinvention principle shaped both the overarching objective of the teaching experiment (to investigate how students might be guided to reinvent the concepts of ring, integral domain, and field) and also our specific objectives in this paper (to investigate how students might develop generalized, coordinated concepts of identity and inverse from their own intuitive understandings).

We propose that a robust understanding of the inverse concept necessarily involves coordination with the relevant binary operation and a generalized concept of identity. This is based upon suggestions in the literature (e.g. Brown et al., 1997) and also the hypothesis that such a coordination elucidates the common inverse structure across different binary operations and algebraic structures (as opposed to comparing the formulas or procedures to compute inverses across contexts). By *generalized concept of the identity*, we mean that a student should understand that an identity, if it exists, can only be conceptualized with respect to a particular binary operation (this is particularly important when studying rings, which have 2 binary operations and, therefore, two potential identity elements). An identity element, then, if it exists, is an element of that structure such that, when combined (in any order) with any element of that structure (under the relevant binary operation), leaves that element unchanged. That is, given an algebraic structure R with binary operation $*$, an identity is an element $I \in R$ for which $a \cdot I = a = I \cdot a$ for any $a \in R$. This understanding should be broad enough to accommodate instances of identity elements beyond the familiar 0 and 1 (such as an identity matrix or the element 8 in the ring $4\mathbb{Z}_{12}$). It should also be broad enough to employ in symbolic arguments for a general (unspecified) algebraic structure. Accordingly, a *generalized concept of inverse* depends not only upon a particular binary operation but also the aforementioned understanding of identity. Moreover, an inverse should be understood as an *inverse element* of the algebraic structure in question (as opposed to an inverse *operation*), and therefore the student needs to understand that, if an element a of an algebraic structure R with binary operation $*$ has a corresponding inverse element a^{-1}, then a^{-1} is also an element in R, and combining a with a^{-1} (in any order) yields the identity, i.e. $a * a^{-1} = I = a^{-1} * a$. This understanding should be broad enough to accommodate unfamiliar instances for which there is not a canonical procedure or formula that determines the existence of an inverse or produces the inverse element itself, and should also be broad enough to employ in symbolic arguments for an unspecified algebraic structure. A student must also understand that some elements in certain algebraic structures might not have a well-defined inverse, and that some elements might have an inverse that exists only outside the algebraic structure under consideration.

Freudenthal (1973), the founding father of RME, provided suggestions about how students might be guided to reinvent such generalized notions of algebraic structure. Arguing that there is a hierarchy of levels of mathematical activity, he noted that "the means of organization of the lower level become a subject matter on the higher level" (1973, p. 123). He defended this claim by describing the historical development of the concept of group. In the 19th century, he noted, groups were only implicit in mathematicians' intuitive reasoning, which led to explicit formulation of the group properties and, eventually, to the axiomatic abstraction of the definition of group. This characterization provides an operational, hypothesized model of how a student might abstract algebraic properties by leveraging his/her own activity:

(1) The property is implicit in the student's activity with an example structure;
(2) The property appears explicitly as a student's general description of his/her activity with the structure;
(3) The student uses the property as a lens to classify and investigate other structures.

Larsen (2013) verified that a learning trajectory proceeding in such a manner could indeed be leveraged to support students' reinvention of the group concept, and earlier studies provided similar evidence for the efficacy of this approach for reinventing rings (Cook, 2012). In this study, we hypothesized that students would be able to achieve the desired, coordinated understanding of identity and inverse by engaging in a task sequence informed by Freudenthal's characterization of the emergence and gradual formalization of an algebraic property in a student's activity.

Methods

We adopted the teaching experiment methodology (Steffe & Thompson, 2000) in order to construct models of students' thinking about the concepts of inverse and identity. We also sought to discern how their understanding of these concepts might evolve to a coordinated, generalized understanding of these concepts as they engage in mathematical activity in response to our teaching actions. It should be noted that such explanatory models of student thinking "may not, and probably cannot, account for students' mathematics" (Steffe & Thompson, 2000, p. 268), and instead reflect a researcher's best attempts to provide a rational frame of reference for a student's observable behaviors in response to a mathematical scenario. Thus, we deemed the models of student thinking that we constructed as valid insofar as they provided a viable explanation for students' utterances and written work.

Participants and Data Collection

Two undergraduate mathematics majors – Josh and Meagan (pseudonyms) – at a large Midwestern research university participated in this study. Both Josh and Meagan were juniors and had completed courses in linear algebra and number theory but had no prior exposure to concepts in abstract algebra. We selected these students for participation not only because their respective stages of mathematical preparation and abilities (as indicated by their own reports of their mathematical course experience and by a survey administered prior to participation) were suitable for this study, but also because of our perception of their enthusiasm for participating and willingness to openly articulate their mathematical thoughts. The teaching experiment consisted of 7 sessions of up to 2 hours apiece. The first author served as the interviewer and the second author served as an observer. We conducted two sessions per week to allow sufficient time for ongoing analysis (explained below). Each session convened on the campus of the students' institution. We recorded students' verbal utterances and written work using Microsoft Surface tablets, which recorded video-like capture of their written work with synchronized audio.

Data Analysis

We employed two methods for data analysis: ongoing and retrospective. The ongoing analysis occurred during and between sessions as we attempted to construct more stable models of student thinking in situations involving the concepts of inverse and identity. Ongoing analysis involved the first author's intuitive responses to students' activity in the form of asking questions or introducing tasks that would provide additional insight into their thinking and test the viability of our proposed explanations for their mathematical behavior. The models constructed during sessions were necessarily fluid and in need of additional refinement. Therefore, between sessions, we both viewed the videos from the previous session and, in particular, looked for evidence (in the form of the students' observable behaviors) that would either confirm or disconfirm the hypothesized models that we had developed during that session. Any additional hypotheses that arose would inform the instructional tasks administered in the next session. At the conclusion of data collection, we engaged in a retrospective analysis (Steffe & Thompson, 2000) in order to look for conceptual change across the entire teaching experiment.

Results

While it is beyond the scope of this proposal to comprehensively document the evolution of Josh and Meagan's understanding of the inverse concept throughout the entire teaching experiment, in this section we simply describe the first few stages of development:

Stage 0: Entrance Survey

Prior to selection for this study, Josh and Meagan, working independently of each other, answered the following questions as part of an entrance survey (the questions that appear here are only those that involved inverses; they have also been renumbered for easier reference):

- Consider the set $\{0, 1, 2, \ldots, 11\}$ with addition and multiplication modulo 12. To familiarize yourself with this set and its operations, construct an operation table for both addition and multiplication modulo 12, and then answer the following questions.
    - Q1: Does this set contain an identity with respect to addition? Explain.
    - Q2: Does this set contain an identity with respect to multiplication? Explain.
    - Q3: Does this set contain an additive inverse for 4? If so, identify it. If not, explain.
    - Q4: Does this set contain an additive inverse for 5? If so, identify it. If not, explain.
    - Q5: Does this set contain a multiplicative inverse for 4? If so, identify it. If not, explain.
    - Q6: Does this set contain a multiplicative inverse for 5? If so, identify it. If not, explain.
- Consider the set $\{0,4,8\}$ with addition and multiplication modulo 12. To familiarize yourself with this set and its operations, construct an operation table for both addition and multiplication modulo 12, and then answer the following questions.
    - Q7: Does this set contain an identity with respect to addition? Explain.
    - Q8: Does this set contain an identity with respect to multiplication? Explain.
    - Q9: Does this set contain an additive inverse for 4? If so, identify it. If not, explain.
    - Q10: Does this set contain a multiplicative inverse for 4? If so, identify it. If not, explain.

The results from this survey indicate that Josh and Meagan's understanding of inverse seemed more tied to notions of inverse as an *operation* (i.e. the inverse of addition is subtraction) or a *procedure* (i.e. take the reciprocal), as their additive inverses were initially negative numbers (Q3, Q4, and Q9), the multiplicative inverses were reciprocals (Q5, Q6, Q10). It appears that they both strongly associated the number 1 with multiplicative identity, and, moreover, there was no evidence that they coordinated these inverses with the respective identities. Had they been coordinating with an identity, we would have expected their explanation in response to Q10, for example, to focus on their assertion in Q8 that a multiplicative identity did not exist in the set.

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Josh	Yes; 0	Yes; 1	Yes; -4=8	Yes; -5=7	No; 1/4 not in set	No; 1/5 not in set	Yes; 0	No; 1 not in set	Yes; -4=8	No; 1/4 not in set
Meagan	Yes; 0	Yes; 1	Yes; -4=8	Yes; -5=7	No; 1/4 does not make sense	No; 1/5 does not make sense	Yes; 0	No; 1 not in set	Yes; -4=8	No; 1/4 does not make sense

Stage 1: Additive Inverse of a as $-1 \cdot a$

Notions of inverse first appeared in the teaching experiment in response to the first task, which prompted them to use addition and multiplication modulo 3 to determine the total number of distinct elements that could be generated, starting from $1 + 2i$. The task permitted them to add and multiply $1 + 2i$ by itself as many times as desired; they could also add and multiply results of any calculations they performed. The idea behind this task was that the algebraic properties that define a field would emerge implicitly in their activity as they acclimated to the elements and operations of what we knew to be the field $\mathbb{Z}_3[i]$. Soon after they generated the element -1, Meagan concluded that the existence of -1 enabled them to include the "opposite" of every element in their list because they could multiply any element by -1:

Meagan: So, like … if we … Isn't negative one, is negative one one of our numbers?
Josh: Yeah, because since negative one is one of our numbers …
Meagan: It's always going to be there.
Josh: Every one will have an opposite.

This characterization persisted unchanged into session 3, at which time they were completing operation tables to keep track of their calculations. In doing so, Josh and Meagan noticed that the addition table was a Latin Square (every element appears exactly once in each row and column). Following Larsen's (2013) recommendation regarding the role of proof in furthering students' reasoning, we prompted Josh and Meagan to prove this result, which essentially amounts to justifying (1) existence (that each equation of the form $a + x = b$ has a solution) and (2) uniqueness (that $ax = ab \Rightarrow x = b$). Their existence proof involved the cancelling of additive inverses. At this point, we prompted Josh and Meagan to articulate their "opposite" rule. Josh wrote:

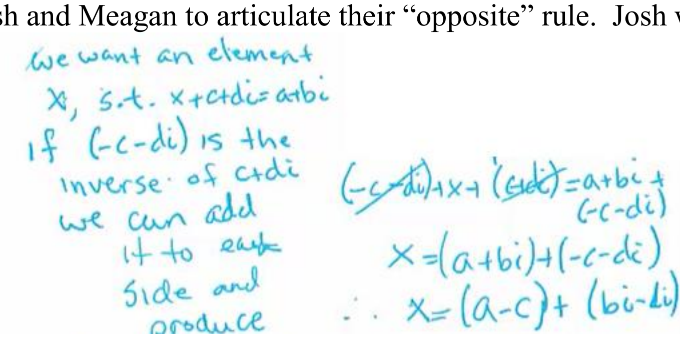

Figure 1: Josh and Meagan's initial characterization of additive inverse.

Notice that their demonstration of their use of this rule involves a cancellation, but does not acknowledge the role of the identity (or any of the other algebraic properties involved). We asked them to develop a "test for inverses" so that we could determine if cancelling featured prominently in their characterization of inverses or if it was just a consequence they derived from a stable $-1 \cdot a$ conceptualization:

Meagan: Multiply them by the negative, and then according to our mod three, um, determine if they're, equal. And you can do that with any equation.

Thus, their operational characterization of additive inverse still centered on $-1 \cdot a$.

Stage 2: Beginnings of a Coordination of $-1 \cdot a$ with the Additive Identity

It was not until Josh and Meagan proved the "uniqueness" component of the Latin Square property that the additive identity began to emerge in their discussions of additive inverses.

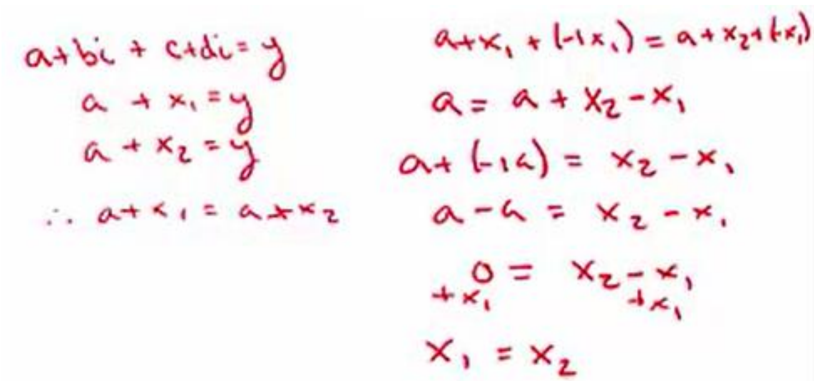

Figure 2: Josh and Meagan's proof of uniqueness for the Latin Square property.

Upon writing out the proof (Figure 2), I asked them to justify that $a + (-1 \cdot a) = 0$:

Researcher: Josh, what you've written there seems to be minus one times a?
Josh: Right.
Researcher: Ok, how do you know that minus one times a gives you zero? Is that true?
Meagan: I mean, it might be another rule. An inverse plus itself should always equal zero.
Right?
Josh: Yeah.

We asked Josh and Meagan to update their statement of the inverse rule:

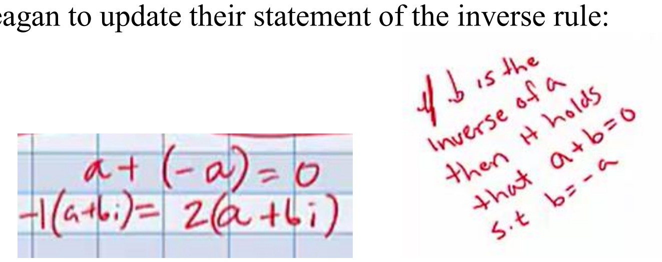

Figure 3: Josh and Meagan's statement of their additive inverse rule includes 0.

This concludes the results section, which is admittedly brief due to space constraints. The conference presentation (pending the acceptance of this proposal) will provide a much more comprehensive and detailed analysis of Josh and Meagan's understanding of inverse and identity and how it evolved throughout the teaching experiment.

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Characterizing the Nature of Introduction to Proof Courses: A Survey of R1 and R2 Institutions across the U.S.

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A number of institutions with mathematics programs offer introduction to proof courses in order to ease mathematics students' transition from primarily calculation-based courses to proof-centered courses. However, unlike most tertiary mathematics courses, whose mathematical content is directly implied by their course titles, introduction to proof courses may vary in terms of the mathematics content discussed. In this study we document the variation in content of introduction to proof courses. This is achieved by examining recent syllabi and other relevant course documents from introduction to proof courses at 179 R1/R2 universities across the United States. The various types of content used in these courses are discussed. We describe the 15 categories of ITP courses that emerged from the course information we collected and offer our categories as a framework for classifying ITP courses or students in future studies.

Key words: Introduction to Proof Courses, Syllabi, Survey, Categorization

Introduction

Undergraduate students taking introduction to proof (ITP) courses (also commonly called transition to proof courses), are a commonly studied population in mathematics education research on proof. However, what the courses that these students are drawn from look like across the United States is not well-known for several reasons. Unlike Calculus, Linear Algebra, Group Theory, Graph Theory, Topology, Combinatorics etc., ITP courses are one of the few undergraduate mathematics courses where the mathematical content covered is not specifically implied by the title of the course. Additionally, the possible mathematics content used to introduce students to proof is not well-defined. So, a number of mathematical topics may be the basis of an intro to proof course and the specific curriculum/texts used have not been documented. In spite of this lack of transparency, researchers interested in beginning proof students commonly make content-general claims from data of students working within a specific mathematics context (Dawkins and Karunakaran, 2016). Furthermore, students from ITP courses are often treated as if they are comparable regardless of the mathematics content used to introduce these students to proof. For example, the use of phrases like, "We present data on the reasoning of two students in a transition-to-proof course" (Alcock & Weber, 2010), without any further description of the course are fairly common. This language creates a false sense that intro to proof courses are compatible enough that the results of one study of ITP students is likely applicable to other ITP students. Such treatment also generates the impression that ITP courses are ubiquitous across mathematics programs. Both the ubiquity and comparability of ITP courses can be tested, rather than tacitly assumed, which is our goal in this manuscript.

Our goal in this study is to simply document the variability in ITP courses in order to facilitate researchers' descriptions of the particular ITP student population they are drawing from. This documentation will allow researchers to increase the specificity of the descriptions of the populations they study and will allow for comparison of courses across institutions. We achieved this by gathering syllabi and other relevant documentation from all 215 US universities that a) offer an undergraduate degree program in mathematics and b) are categorized by the

Carnegie foundation as either high research output (R2) or very high research output (R1). These syllabi provide a lens into what curricula ITP courses cover across the US.

Methods

The goal of this study was to collect and classify information on the nature of Introduction to Proof (ITP) courses offered at the undergraduate level across the United States. In order to gain a clear and accurate picture of the nature of these courses, the 215 of 221 R1 and R2 universities that offer undergraduate mathematics degrees were considered for this study. By examining a university's course catalog, math department website, and/or a list of course offerings, we determined whether an institution offered an ITP course. When this information was not available through online sources, we emailed instructors or administrators to acquire the needed information to make an ITP determination. For our purposes, a course was labeled an ITP course if the course description indicated an explicit emphasis on easing the transition to proof, bridging the gap between Calculus and higher level mathematics, and/or developing written mathematical arguments. To acquire an ITP label these emphases needed to be central to the course's content, as opposed to covered during a brief review period during the initial stages of the course.

Some courses functioned as ITP courses, but centered on a single mathematical topic, such as analysis or abstract algebra. In order to distinguish between these ITP courses and introductory courses on a mathematical topic that are not ITPs, we used schedules of content coverage in addition to course descriptions and listed objectives to make our ITP determinations.

For each of the 179 R1 and R2 schools that we determined offered ITP courses, we set out to collect one recent course syllabus per school through Internet searches and departmental websites. These Internet searches were typically of the form: "Course Code, Course Title, University Name, course syllabus." (Ex: MAT 300 Mathematical Structures Arizona State University course syllabus). When multiple syllabi were found, preference was given based upon precedence in search results, most recent date, and level of detail, in that order. If a course syllabus was not available through Internet searches or departmental sites, we emailed the department's administrative staff or a professor who had recently taught the course (when known) requesting a recent copy of the course syllabus. In all, 164 course syllabi were collected from 164 of the 179 ITP-offering institutions. Twelve of the remaining 15 schools had highly detailed course descriptions on their webpages. These descriptions were collected in lieu of course syllabi. Three schools with ITP courses did not respond to emails or provide enough detail online to sub-categorize them in terms of the mathematical content they covered. These three schools were excluded from the remainder of the analysis.

From these 176 ITP courses, (164 with syllabi, 12 with detailed online course information) the following information was collected in a spreadsheet: institution, location, course code, course title, textbook/author used, date, and course description/content covered. The vast majority of syllabi collected were from courses taught within the last 3 years (Fall 2013-Spring 2016) and only 6 syllabi were dated earlier than 2011. The results section below describes our analysis and sub-categorization of these courses based on their approach to introducing proof.

Results

Once we compiled all of the relevant information from each course, we looked for trends in types of courses, textbooks used, and topics covered in order to categorize these courses. From the beginning, there was striking variation, perhaps most noticeably in the number of textbooks used. Across the 176 courses, 63 different textbooks were used and 16 courses relied entirely on instructor or departmental lecture notes. The two most commonly used texts: *Mathematical Proofs: A Transition to Advanced Mathematics*, by Chartrand, Polimeni and Zhang, and *A Transition to Advanced Mathematics*, by Smith, Eggen and St. Andre were each used by only 16 and 15 of 176 of the courses surveyed, respectively. From the spreadsheet data collected for each course, we examined the types of courses in terms of topics covered and two broad categories of courses emerged: courses that followed a highly similar order of topics and courses that introduced proof in the context of a particular advanced mathematical topic. Of the 176 ITP courses, 144 (82%) of the courses were of the first type, which we categorized as Standard ITP courses and the remaining 32 (18%) we categorized as Topic-Based ITP courses. Once we separated the courses into these two categories, we looked for further trends and subcategories within each category. We will proceed to explain our categorizations, sub-categorizations, and provide examples of the two most common course type in each to illustrate how we made our determinations, as well as the frequency of each type of ITP course.

Standard ITP Courses

The 82% (144/176) of ITP courses we categorized as Standard ITP courses typically covered the following topics: symbolic/formal logic, truth tables, propositions, quantifiers, methods of proof (including contradiction and induction), number systems, sets relations and functions, infinite sets, and cardinality. Some of the Standard ITP courses covered additional topics from advanced mathematics towards the end of the course such as analysis or discrete mathematics topics. Courses that only covered these standard topics and did not include additional topics we categorized as Standard only ITP courses, which comprised 50% (72/144) of the Standard ITP Category. The other 50% (72/144) of Standard ITP courses contained introductions to topics beyond the standard topics, which we categorized as Standard+ Topic ITP courses.

Standard only: Within the Standard only ITP subcategory, which consists of 41% (72/176) of all ITP courses surveyed, 29 different texts were used, many of which are used for ITP courses in other categories. The most commonly used text, used by only 15% (11/72) of courses in this category, was *Mathematical Proofs: A Transition to Advanced Mathematics*, by Chartrand, Polimeni and Zhang. Additionally, 8% (6/72) of Standard only ITP professors use their own lecture notes or notes developed by their department. As described earlier, these courses cover what we take as the standard set of topics as a means of introducing proofs and do not introduce any other mathematical topics. We provide an example of an overview of topics from one such course:

Topics:

1. Symbolic logic (propositional calculus)
   - Logical Connectives and truth tables
   - Quantifiers
   - Tautologies
   - Relation to mathematical statements and proof
2. Foundations of mathematics
   - Set theory: union, intersection
   - Cartesian products, relations, partitions

Figure 1. Howard University, Mathematical Reasoning: Writing and Proof, Standard only ITP Course

Standard+ Topic: Any course that focused primarily on the same topics as a Standard only ITP course, but also introduced a certain mathematical topic or topics for a small portion of the course, we categorized as Standard+ Topic, according to which topic was introduced. These courses made up another 41% (72/176) of the ITP courses surveyed. Typically these courses covered formal logic, sets, functions, relations, etc. (the "standard" material) for the majority of the semester and then spent a few weeks on introducing a particular topic from a certain branch of advanced mathematics, often towards the end. The most common Standard+ Topic courses added on 2-4 weeks of discrete mathematics topics or introductory analysis topics. The breakdown of all of the Standard+ Topic ITP course subcategories that emerged is as follows: Standard + Discrete 35% (25/72), Standard + Analysis 15% (11/72), Standard + Combination 11% (8/72), Standard + Algebra 10% (7/72), Standard + Number Theory 8% (6/72), Standard + Algebra & Analysis 7% (5/72), Standard + Sampler 7% (5/72), and Standard + Other 7% (5/72).

Standard+ Discrete: Making up 35% (25/72) of all of the Standard+ Topic ITP courses, Standard+ Discrete courses usually spent the last few weeks of the course on several Discrete Mathematics ideas such as counting techniques, permutations, combinations, the Binomial Theorem, probability, graph theory, or possibly elementary number theory. Among the courses of this type, the most commonly used text, used by only 20% (5/25) of the category, was *Discrete Mathematics and its Applications* by Kenneth Rosen. We provide a sample course overview from a course in this category:

Overview:
This course is a rigorous introduction to many of the tools useful in higher mathematics and computer science. The major topics to be covered will be logic and proof techniques. Also included will be a brief introduction to set theory, number theory, relations, functions, complexity, combinatorics, and graph theory.

Figure 2. San Diego State University, Discrete Mathematics, Standard+ Discrete ITP Course

Standard+ Analysis: The second most frequent, Standard+ Analysis courses made up 15% (11/72) of the Standard+ Topic category. After the standard material, these courses covered introductory topics within Analysis such as sequences, limits, and continuity. Among these courses, 8 different texts were used. The most commonly used text was *A Transition to Advanced Mathematics*, by Smith, Eggen and St. Andre, and was used by only 27% (3/11) of Standard+ Analysis ITP courses. We provide two examples from course syllabi from this category. Note that while the additional Analysis topics are part of the course, the entire course is not centered on Analysis. Such ITP courses were categorized as Topic-Based ITP courses and will be discussed later in this manuscript.

Learning Outcomes

A successful completion of the course should permit the student to achieve the following skill levels:

- Develop a framework of widely used fundamental mathematical concepts;
- Understand the basic facts about sets, relations, functions, and structures;
- Formulate and prove mathematical statements using standard proof techniques;
- Learn the introductory concepts associated with the branch of mathematical analysis;
- Develop experience and confidence in the presentation of abstract mathematics.

Figure 3. George Washington University, Introduction to Mathematical Reasoning, Standard+ Analysis Course

Standard+ Algebra: Likewise, the courses that added several weeks of Algebra topics to the standard material in their introduction to proofs were categorized as Standard+ Algebra and made up 10% (7/72) of the Standard+ Topic category. The concluding topics for courses in this category include topics such as groups, subgroups, homomorphisms, cosets, rings, and fields. Each course in this category used a distinct textbook from other courses in this category, though some used the same text as courses in other categories.

Standard+ Number Theory: Six of the 72 courses in the Standard+ Topic category (8%) added number theory topics to the standard set of material. These courses are distinct from the Standard+ Discrete category since they only added number theory concepts specifically, and did not cover any other type of discrete mathematics such as counting, graph theory, or probability.

Standard+ Algebra & Analysis: Five of the 72 (7%) Standard+ Topic ITP courses added introductory ideas from Algebra and Analysis. These courses typically only added a few topics from each of Algebra and Analysis and do not have as much time to devote to each topic.

Standard+ Combination (other than Algebra & Analysis): While Algebra and Analysis were the most common two topics to add into standard material in an ITP Standard+ Topic course, 11% of courses (8/72) found in the Standard+ Topic category added different combinations of topics from two branches of mathematics to their course. The different combinations observed include: Discrete and Analysis, Geometry and Number Theory, and Algebra and Topology.

Standard+ Sampler: Five of 72 (7%) Standard+ Topic ITP courses added topics from more than two branches of advanced mathematics, which we classified as Standard+ Sampler ITP courses.

Standard+ Other: Five of the 72 (7%) Standard + Topic ITP courses did not fall under one of the previous categories were labeled Standard+ Other. These courses delved into more tangentially related topics, such as intermediate logic or certain software such as LaTex.

Topic-Based ITP Courses

Unlike courses in the Standard category, Topic-Based ITP courses introduce proof in the context of either a single mathematical topic, or occasionally through several topics. Such courses devote little, if any, time to the "standard" material that is the focus of Standard only and Standard+ Topic courses. These courses, through an introduction to a particular mathematical topic or topics, emphasize proof-writing and formal mathematical argumentation as a major goal of the course. Much less frequent, they comprise 18% (32/176) of the ITP courses surveyed and consist of the following subcategories, listed in order of prevalence: Analysis 38% (12/32), Discrete 25% (8/32), Algebra 16% (5/32), Sampler 9% (3/32), Combination 6% (2/32), Number Theory 6% (2/32). The following is a description of the characteristics of each subcategory.

Analysis*:* The most prevalent of the topic-based ITP courses, Analysis ITP courses have the goal of teaching mathematical proof in the context of Analysis topics and make up 38% (12/32) of the topic ITP category. Typically, these courses cover the real number system, limits, continuity, differentiation, integration, sequences, all in the context of single-variable functions. The most commonly used textbook for courses in this category, used by 25% (3/12) of Analysis ITP courses, was *Analysis with an Introduction to Proof*, by Steven Lay. Below is a description of one such course we placed in this category:

Strongly recommended for future math majors, this course covers calculus more thoroughly and more theoretically, giving an introduction to important mathematical techniques and results that give a foundation for further work in analysis. It serves as an introduction to the rigorous proofs and formal mathematical arguments needed in all upper division math courses.

Figure 4. Princeton University, Honors Analysis in a Single Variable, Analysis ITP Course

Discrete: These courses, which make up 25% (8/32) of the Topic-Based ITP courses have the goal of learning mathematical proof in the context of various topics in Discrete Mathematics including graph theory, probability, and combinatorics. These courses may include graphs, trees, Boolean algebras, cryptography, divisibility, permutations, combinations, binomial coefficients, and may include a sampling of number theory topics, although courses situated entirely in the context of number theory were divided into their own subcategory. Among these courses, the most frequently used text was *Discrete Mathematics with Applications* by Epp, employed by 50% (4/8) of the courses in this category. The following course description illustrates the nature of these types of courses:

2513 Discrete Mathematical Structures. *Prerequisite: 2423 or concurrent enrollment. A course for math majors or prospective math majors. Provides an introduction to discrete concepts such as finite sets and structures, and their properties and applications. Also exposes students to the basic procedures and styles of mathematical proof. Topics include basic set theory, functions, integers, symbolic logic, predicate calculus, induction, counting techniques, graphs and trees. Other topics from combinatorics, probability, relations, Boolean algebras or automata theory may be covered as time permits. (F, Sp, Su)*

Figure 5. University of Oklahoma, Discrete Mathematical Structures, Discrete ITP Course

Algebra: The third most frequent type of Topic-Based ITP course at 16% (5/32), the Algebra ITP courses have the goal of understanding mathematical proof in the context of abstract algebra topics. Typically, these courses cover groups, subgroups, normal subgroups, quotient groups, rings, homomorphisms, polynomials, subrings, cosets, isomorphism theorems, and sometimes include fields, integral domains, prime numbers and the division algorithm. Every Algebra ITP course surveyed used a different textbook.

Sampler: Three of the 32 (9%) topic-based ITP courses had the goal of learning mathematical proof in the context of a wide-array of mathematical topics, which we labeled as "sampler" courses. Often the goal of these courses is to introduce students to many different subjects in advanced mathematics. Courses in this category used the context of a minimum of four topics.

Number Theory: Distinct from Discrete ITP Courses, which typically sample various discrete mathematics topics, these courses, though rare, have the goal of learning mathematical proof in the context of Number Theory. In our study we only came across 2 such courses.

Combination: Courses in this category are topics-based intro to proof courses in the context of a combination of two different topics, and make up a very small portion (6% or 2/32).

The following table summarizes all of the ITP categories that emerged from the course information, as well as their prevalence.

Table 1
ITP Course Category Data Summary

Course Category (Percent of ITP Courses)	**Description**	**Subcategories (Percent of Category)**

Standard 82%	*Standard + ∅ ITP 41%*	Introduce proof through the following topics: symbolic logic, truth tables, propositions, quantifiers, methods of proof, number systems, sets, relations and functions, and cardinality. No additional topics covered.	N/A
	Standard+ Topic ITP 41%	Cover the same topics as a Standard ITP course for the majority of the course, but also include an introduction to a specific advanced mathematical topic or topics near the end, which typically lasts 2-4 weeks.	35% Standard+ Discrete 15% Standard+ Analysis 11% Standard+ Combination 10% Standard+ Algebra 8% Standard+ Number Theory 7% Standard+ Algebra & Analysis 7% Standard+ Sampler 7% Standard+ Other
Topic 18%	*Topic-Based ITP 18%*	Grounded in the context of either a single or several topics in combination. Majority of time on the topic and devote little to no time to formal logic and other "standard" topics.	38% Analysis 25% Discrete 16% Algebra 9% Sampler 6% Combination 6% Number Theory

Discussion

Our chosen focus on syllabi facilitated the collection of data across a fairly complete sample of R1 & R2 universities. This allowed for breadth rather than depth; syllabi offer a fairly accurate look into the material covered and textbook used but have little information regarding day-to-day classroom dynamics. Even with these limitations we were able to discover a wide variation in ITP courses. As discussed in the results section, ITP courses are not necessarily offered at all universities, meaning the universality of ITP student populations should not be assumed to exist within any tertiary mathematics program. We observed that no single textbook had more than 9% market share and most textbooks designed for ITP courses were used at only 2-3 institutions. Furthermore, when categorizing ITP courses we generated 15 different categories to describe the differences in mathematical content covered. These results point to the high variability in how students across the United States are introduced to proving, and highlight the importance of taking this variability in content into account when studying students enrolled in ITP courses. Since we only chose one representative syllabus per institution, in an effort to keep the data set manageable, we did not account for variability within institutions, which likely exists at many schools, as we found at our own institution. In other words, the categories and counts we've compiled are likely a simplification of the true degree of variability among these courses. Unlike Dawkins and Karunakaran (2016), we are not necessarily encouraging all researchers interested in proof to take mathematical content into count when analyzing student data, although we do believe doing so adds depth to data analysis. Instead, we encourage researchers to be more specific when describing the ITP population they are drawing from in order to help situate their chosen population relative to the various types of ITP courses that exist. We offer the 15 categories and descriptions provided here as a framework for ITP course categorization for future studies. Our categories serve to provide additional information and may add depth to a highly-studied population in mathematics education. Identifying and categorizing the variability within these courses lays the foundation for further studies, which might include studies within certain categories, or comparisons of students across categories of ITP courses.

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The Role of Visual Reasoning in Evaluating Complex Mathematical Statements: A Comparison of Two Advanced Calculus Students

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The purpose of this study is to examine the role of visual reasoning while students evaluate complex mathematical statements about real-valued functions. We conducted clinical interviews with nine undergraduate students from mathematics courses at different levels. In the interviews, we asked these students to evaluate several mathematical statements alone and then using various graphs. In this paper, we focus on the cases of two students who had completed Advanced Calculus to highlight the contribution of their visual reasoning about several graphs. We found that student's graphical interpretation of "between" in these statements affected their evaluation of the statements. Even at advanced levels, students' visual cues dominated their reasoning about the statements. Our findings indicate that students' visual reasoning contributes to their evaluation of mathematical statements and helps to account for differences between students' meanings of statements.

Key words: Visual Reasoning, Complex Mathematical Statements, Graphical Interpretations, Undergraduate Students, Intermediate Value Theorem

Introduction

Mathematical statements, including definitions, theorems, and mathematical claims, which may be true or false, are a central part of elementary through postsecondary mathematics curriculum. Mathematics education research has shown that students often struggle with both understanding mathematical statements (Roh, 2010; Selden & Selden, 1995) and evaluating whether mathematical statements are true or false (Bubp, 2014; Dawkins & Roh, 2016; Selden & Selden, 2003). In particular, studies have shown student difficulty with statements involving multiple conditionals such as "if $p \rightarrow q$, then $r \rightarrow s$" (Zandieh, Roh, & Knapp, 2014) or multiple quantifiers such as "$\exists x \forall y\ P(x, y)$" (Dubinsky & Yiparaki, 2000). In this paper we focus on students' understanding of statements involving both conditional structure and multiple quantifiers, which we refer to as *complex mathematical statements*. Although these statements look complicated, statements with such structure are common in Calculus and Real Analysis. For example, the Intermediate Value Theorem (IVT) can be stated as a complex mathematical statement as follows: "Suppose that f is a continuous function on $[a, b]$ with $f(a) \neq f(b)$. Then, for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b) such that $f(c)=N$."

In order to help students understand these types of statements, instructors may often provide a graphical illustration to introduce the idea of the statement. Several textbook authors and researchers have cited the benefits of incorporating visual representations of theorems and/or proofs in instruction (Alcock & Simpson, 2004; Arcavi, 2003; Davis, 1993; Guzman, 2002; Hanna & Sidoli, 2007). While research has looked at students' understanding of multiple quantifiers (Dawkins & Roh, 2016; Dubinsky & Yiparaki, 2000; Epp, 2003; Roh & Lee, 2011) and conditional structure (Durand-Guerrier, 2003; Zandieh et al., 2014), few studies have looked at the role of students' reasoning about the graphs or images that correspond to a statement about a real-valued function. Although instructors may consider such explanations clear representations of the statement, we do not know what sense students make from these graphical illustrations. While these visual representations have the potential to be valuable to student understanding, little research has been conducted to examine the role of students' visual reasoning in understanding and evaluating complex mathematical statements. This study addresses the following research question: *How do students' visual reasoning contribute to their evaluations of complex mathematical statements about real-valued functions?*

Theoretical Perspective

Our study is grounded in a constructivist perspective and builds on Moore and Thompson (2015) and Moore (2016)'s theories of students' graphical activity. We adopt von Glasersfeld's (1995) view that students' knowledge consists of a set of action schemes that are viable given their experience, and that we, as researchers, do not have direct access to their knowledge. This perspective implies that we as researchers can only model student thinking based upon their actions. Thus, our analysis reflects our best attempt at creating a hypothetical model of student visual reasoning grounded in evidence found in their words, gestures, and markings on paper.

In this study, we reference Moore and Thompson's (2015) distinction between *static shape-thinking* and *emergent shape-thinking* as well as Moore's (2016) constructs of *figurative* and *operative* thought in the context of graphing as a means of situating our constructs that describe students' visual reasoning we observed. Moore and Thompson (2015) describe *static shape-thinking* as conceiving of a graph as an object in itself, in which mathematical objects and actions are subordinate to visual perception. In contrast, they describe *emergent shape-thinking* as conceiving of a graph as a trace which emerges from the coordination of two varying quantities. To develop this distinction, Moore (2016) aligns static shape-thinking with *figurative thought* and emergent shape-thinking with *operative thought*, consistent with Piaget's (2001) and Steffe's (1991) use of the terms. Moore (ibid) explains that figurative thought is dominated by visual properties of a graph, such as aspects of the shape of the curve which overrides considerations about relationships of the quantities being represented. In contrast, for students engaged in *operative thought*, "figurative elements of their activity are subordinate to that coordination [of covarying quantities]" (Moore, 2016, p. 3). Students engaged in operative thought may still perceive visual properties of a graph, but their thought is guided by the relationships represented in the graph. Furthermore, Moore (ibid) explains that these modes of thought are not exclusive of the other. Students may engage in both, although one may dominate the student's activity, depending on which mode of thought the students' actions are subordinate. While these constructs help to distinguish certain student graphing behavior, they do not account for important nuances of the graphing behavior we observed in this study. Thus, we define the constructs *location-thinking* and *value-thinking* to distinguish between the relevant aspects of the graphing behavior we observed, as explained in the results section.

Methods

For this study, we conducted two-hour clinical interviews with nine undergraduates at three different levels: students who had just completed Calculus, an Introduction to Proof course, and Advanced Calculus, respectively. Due to the complex nature of the interview, we decided to have one researcher serve as the interviewer with the other three researchers as witnesses, two of whom were in the interview room, and the third who watched the interview live-streamed from elsewhere. We recorded the interview with three cameras to capture the entire frame, student work on the table, and to live-stream the interview to the third witness. All four researchers had laptops, whose screens the participant could not see, to communicate their current models of the participant's thinking in real-time via group chat. This set-up allowed all four researchers to offer clarifying questions for the interviewer to pose to the participant to test these models.

In the interview, the participants were asked to evaluate four complex mathematical statements about real-valued functions and provide justification for their evaluation. One of these statements was the Intermediate Value Theorem (IVT), which we presented in the introduction. The remaining three statements were variations on IVT with the re-ordering of the quantifiers (for all, there exists) and/or variables reversed (N, c). For example, another statement read, "Suppose f is a continuous function on $[a, b]$ where $f(a) \neq f(b)$. Then there exists a real number N between $f(a)$ and $f(b)$, such that for all real numbers c in (a, b), $f(c)=N$." This altered statement was made from a reversal of *both* the quantifiers *and* variables in IVT.

Once the participants had evaluated each of the four statements, the interviewer asked them to compare the statements and explain whether any of the statements had the same meaning, in their interpretation. The interviewer later asked students to look at each statement along with six graphs we created, with the chance of changing their evaluation in order to gain insight into the effects of students' reasoning about various graphs on their evaluation. These graphs, which were intended to represent a spectrum of possible functions and relevant counterexamples, included: a constant function, a monotone increasing function, a vertical line, a polynomial with extrema beyond the endpoints of the displayed function, the tangent function, and the sine function. The participants were also asked to explain how they interpreted various aspects of the graph and to label relevant points and values on the graphs where appropriate.

After conducting the interviews, our research team closely analyzed the video data for patterns in student thinking that could explain their statement evaluations. Our data analysis was consistent with Corbin and Strauss's (2014) description of grounded theory, in which categories of student visual reasoning emerged from the data, as we did not begin our study with the specific intent of investigating visual reasoning. Through analyzing the student interviews, students' interpretation of the phrase "between $f(a)$ and $f(b)$" emerged as highly relevant to their reasoning about the graph and subsequent evaluation of the given statement.

Results

In analyzing the interviews, a distinction emerged in students' interpretation of the phrase "between $f(a)$ and $f(b)$," which stemmed from their meaning for $f(a)$ and $f(b)$. As we analyzed student meanings for this phrase, students fell into one of two categories in terms of their interpretation of this phrase, and we labeled each student as a *value-thinker* or *location-thinker,* accordingly. In this categorization, *value-thinkers* interpret "between $f(a)$ and $f(b)$" as referring to output *values* between the *values* of $f(a)$ and $f(b)$. In contrast, *location-thinkers* interpret "between $f(a)$ and $f(b)$" as *locations* between the *locations* of $f(a)$ (at $(a, f(a))$) and $f(b)$ (at $(b, f(b))$). In other words, value-thinkers distinguish between the value of the output of the function and the resulting location of the coordinate point on the graph, whereas location-thinkers do not make this distinction. Students who were labeled as value-thinkers were seen as engaging in operative thought and emergent shape-thinking, as their visual perception of the graph was subordinate to their meanings for output value and they coordinated the values of the varying quantities. Location-thinkers, on the other hand, did not clearly fall into the category figurative thought, and by extension, static shape-thinking. The visual cues from the graph dominated their thinking in some aspects of the graph and informed their interpretation of "between $f(a)$ and $f(b)$." Although location thinkers engaged in figurative thought, they did not conceive of the graph statically. Instead, for these students, a graph emerged from the coordination of input values with locations of the points on the graph. In this way, we consider our constructs of value-thinking and location-thinking as novel in describing students' graphical activity. We summarize the characteristics of each way of thinking in the following table.

Table 1. *Comparison of location-thinking and value-thinking characteristics*

<table>
<tr><th rowspan="2"></th><th colspan="2">Location-Thinker</th><th colspan="2">Value-Thinker</th></tr>
<tr><td>Visual Reasoning</td><td>Evidence</td><td>Visual Reasoning</td><td>Evidence</td></tr>
<tr><td>Output of Function</td><td rowspan="2">Confounds the output value of a function with the location of the coordinate point</td><td rowspan="3">▪ Marks output on the curve

▪ Speaks about points as the resulting output of the function (e.g. “an input maps to a point on the curve”)</td><td>The resulting value from inputting a value in the function</td><td>Speaks about output values</td></tr>
<tr><td>Point on Graph</td><td>The coordinated values of the input and output represented together</td><td rowspan="2">▪ Labels points as ordered pairs

▪ Speaks about points as the result of coordinating an input and output value</td></tr>
<tr><td>Graph</td><td>A collection of geometric points associated with input values</td><td>A collection of coordinates relating the value of two quantities</td></tr>
<tr><td>“Between f(a) and f(b)”</td><td>All points on the curve located between (a, f(a)) and (b, f(b))</td><td>Sweeps along entire curve between (a, f(a)) and (b, f(b))</td><td>All values between the values f(a) and f(b)</td><td>Marks the interval of values between f(a) and f(b) on the output axis</td></tr>
</table>

Based on our analysis of the interview data, four students were identified as location-thinkers and five students were classified as value-thinkers. In this paper, we choose to highlight the cases of two students, Jay, a value-thinker, and Nate, a location-thinker, to illustrate the difference in these visual reasoning and their subsequent impact on the student's understanding and evaluation of the given statements. Both Jay and Nate had recently completed Advanced Calculus and earned an A in the course. Additionally, both students showed evidence of having strong meanings for multiple quantifiers and conditional structure. However, their evaluation of the truth values of these statements differed. Jay correctly evaluated each of the four statements and offered justifications that were consistent with the mathematical community. Nate paraphrased one of the statements[1] as, "for every single *c* there is a point on the curve that it maps to," indicating that he correctly interpreted the multiple quantifiers in the statement. While Jay correctly evaluated this statement as false, Nate evaluated it as true. Jay and Nate's evaluations of these statements serve as a contrast due to the significance of visual reasoning in their evaluations of the statements.

Jay: A value-thinker

The excerpt of Jay's interview below provides evidence for Jay's interpretation of "between" as value-thinking. In this excerpt, Jay has just looked at the IVT and evaluated the statement as true. He used the graph he drew (*Figure 1, left*) to explain why this statement, in his interpretation, is true.

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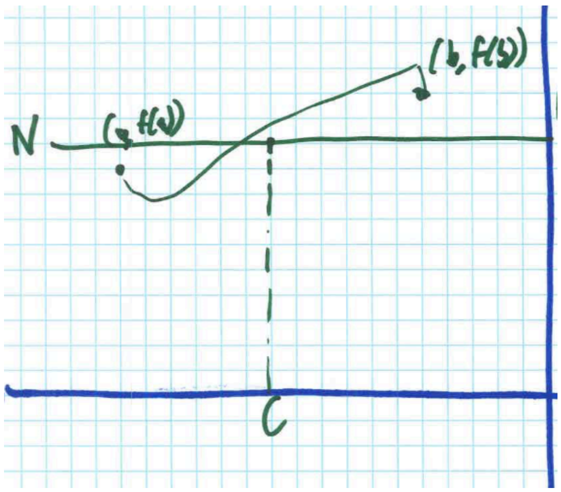
[1]Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$ such that $f(c)=N$.

… So this right here is *y* equals *N*, this line is *y* equals *N* (*points to horizontal line which he labeled N*). If *f* crosses this line (*points to horizontal line he just drew*), then there exists a real number c in the open interval such that *f*(*c*) is *N*. Because if I cross these lines... lines say here, this (*marks c on x-axis*) is my value of *c*, right? Because that's where *f*(*c*) is *N*. Okay, so the only way for like this not to be true is if you can draw a continuous graph through it or a continuous graph from *a* to *b* with *f*(*a*) being this (*marks f(a) below the line drawn), f*(*b*) being this (*marks f(b) above the line drawn*), such that you skip *N*.



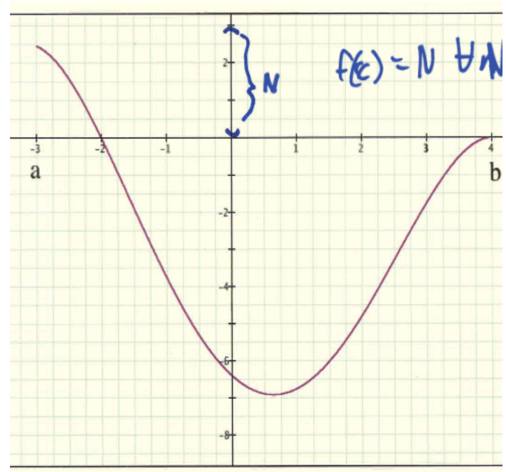


Figure 1. Jay's hand-drawn explanation for why the IVT is true (left) and Jay's possible values of N (right)

Both Jay's graph and his explanation above of why IVT is true reveal how he made sense of the phrase "between *f*(*a*) and *f*(*b*)" in the statement. Jay drew a horizontal line between these output values that intersected the *y*-axis, which indicated that he attended to the *values* of the output, rather than the *location* of the ordered pair point in space. Later, Jay described that the horizontal line he drew in his diagram is arbitrary, indicating that he imagined all of the possible horizontal lines passing between the horizontal line through *f*(*a*) and the horizontal line through *f*(*b*). For Jay, the phrase "between *f*(*a*) and *f*(*b*)" was connected to this image that he shared. Not only did Jay consider the values of *f*(*a*) and *f*(*b*), he visualized the graphical meaning of IVT, which is that a continuous function will always intersect any horizontal line drawn between the output values of *f*(*a*) and *f*(*b*). Further evidence of Jay's visual reasoning for "between" was found later in the interview, when Jay evaluated another statement and provided graph, and marked off the relevant interval for *N* along the *y*-axis (*Figure 1, right*). Jay's interval of possible *N* values between *f*(*a*) and *f*(*b*) in Figure 1 indicates he again considered the *values* between the value of *f*(*a*) and *f*(*b*) rather than the spatial locations of these points.

Nate: A location-thinker

In contrast with Jay's value-thinking, Nate's reasoning was labeled as location-thinking. Evidence for Nate's thinking can be seen in the following excerpt. Nate had already evaluated

Statement 1 (which is false)[2] as true, and below explained why it is true using the same provided graph that Jay used above (which, from our perspective, is a counterexample to Statement 1).

> Nate: So for all these *c*'s (*sweeps pen along x-axis between a and b)* you can see that it mapped to a point on the curve. For every single *c* there is a point on the curve that it maps to…So after that *c, N* would be here (*marks c around 1 on x-axis, N at ordered pair location of (c, f(c))*. So it maps to that. And this *c* would be in here (*marks c around 2 on x-axis*). And this *c* would be like, *N* right here (*marks corresponding N's for each c on graph*).
>
> Interviewer: Okay. Lets say we picked a point over here (*points to a point beyond (a, f(a)), (b, f(b)), rightmost marked point on Figure 2*). Could we say that the output would be between *f*(*a*) and *f*(*b*)?
>
> Nate: I would not say its between *f*(*a*) and *f*(*b*). Even though the, yeah. This is the, the actual numbers 2.5 and 0 (*marks the points 2.5, 0*). This would be… If you are looking at numbers 2.5, 0, this would be in between that interval. But it's in between that number interval. But it's not in between the functional interval in this case. So *f*(*a*). The interval will refer, refers to all these points between *f*(*a*) and *f*(*b*) (*sweeps pen along curve)*. All points of the function. That's what I am interpreting.

In the transcript above, when asked about the far right point the interviewer selected, if the output would be between *f*(*a*) and *f*(*b*), Nate explained, "it's in between that number interval, yeah, but it's not in between that functional interval in this case." The "it" Nate referred to here is the point in question, which he confounded with the output of the function at that point. Nate further clarified that "the interval refers to all the points between *f*(*a*) and *f*(*b*)" and illustrated what points he is talking about by sweeping his pen along the curve between the points *(a, f*(*a*)) and (*b*, *f*(*b*)).

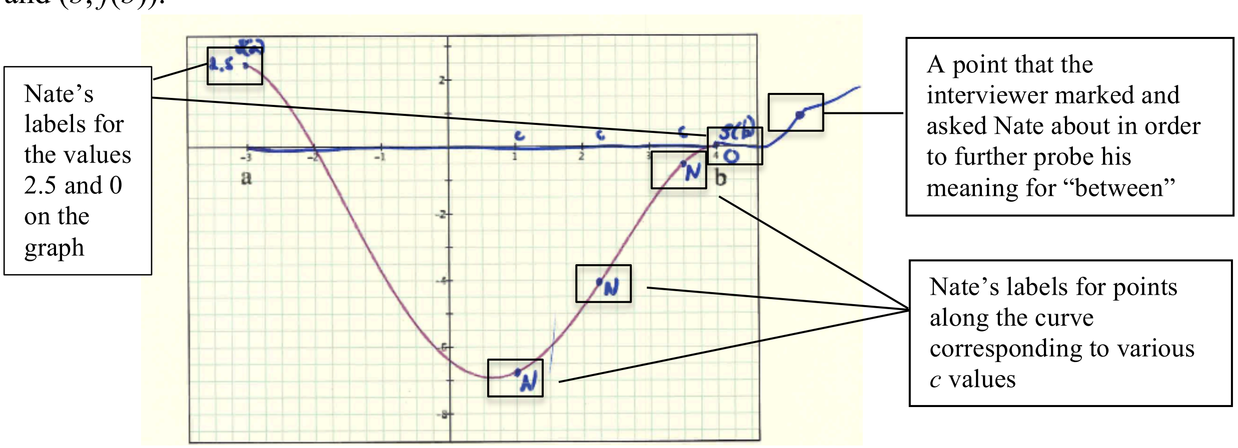


Figure 2. Nate's work on provided graph with his possible N values labeled on the curve, outside the range of values between f(a) and f(b).

Nate's response indicates that his perception of the graph guided his interpretation of and evaluation of Statement 1. Nate labeled *f*(*a*) and *f*(*b*) not on the *y*-axis, but at the location of the point (*a*, *f*(*a*)) and (*b*, *f*(*b*)), which indicates that he conceived of the output of the function as the point itself. These two labeled points became visual boundaries for Nate, between which lay all

[2] Statement 1: Suppose that *f* is a continuous function on [*a*, *b*] such that $f(a) \neq f(b)$. Then, for all real numbers *c* in (*a*, *b*), there exists a real number *N* such that $f(c)=N$.

of the relevant N values. In other words, he interpreted the relevant values of N "between $f(a)$ and $f(b)$" as *all* of the points on the path of the curve between the starting point that he labeled as $f(a)$ and ending at the point in space that he labeled $f(b)$. Nate's response to the question of the curve extending beyond the point $(b, f(b))$, but including values that are numerically between $f(a)$ and $f(b)$, is further evidence that Nate conceived of "between $f(a)$ and $f(b)$" as all of the points along the path between $(a, f(a))$ and $(b, f(b))$. While Nate said the point in question was not between $f(a)$ and $f(b)$, he acknowledged that this point was numerically between 2.5 and 0 (the values of $f(a)$ and $f(b)$). The fact that Nate did not ignore the numerical value of the outputs, and even acknowledged the confusion surrounding this meaning indicates that he recognized multiple possibilities for the meaning of between. Nate's interpretation of "between" in terms of the location of the points, prompted by his visuospatial perception of the graph, reveals the power of his visual perception to override his numerical evaluations.

Together, both Jay's and Nate's meanings for "between" and their subsequent evaluations of the given statements highlight the role of visual reasoning in evaluating such statements. Since both participants understood the quantifiers and conditional structure of Statement 1 the same way, but had different truth-value evaluations, the difference in their visual reasoning became the focus of our analysis. For Jay, the phrase "between $f(a)$ and $f(b)$" was visually connected to horizontal lines between the *values* of $f(a)$ and $f(b)$. Jay interpreted the outputs of the function as *values*, distinct from the coordinate points on the graph and thus was classified as a *value-thinker*. For Nate, "between $f(a)$ and $f(b)$" referred to every point along the path of the graph between the *location* $f(a)$ (at $(a, f(a))$) and the *location* $f(b)$ (at $(b, f(b))$). Since Nate did not distinguish between the outputs of the function and the locations of the coordinate point on the curve, Nate was labeled a *location-thinker*. In summary, Nate and Jay, both successful Advanced Calculus students consistently interpreted "between" differently, which led to their different understandings, and subsequent opposing truth-value evaluations of Statement 1.

Discussion

Based on our findings that almost half of the students whom we interviewed were identified as location thinkers, visual reasoning about mathematical statements plays a significant role in students' understanding. These results highlight and explain some important aspects of students' graphical activity that were not previously accounted for by current theories and studies on visual reasoning (Moore & Thompson, 2015; Moore, 2016). Thus, the use of our constructs of value-thinking and location-thinking could progress the depth of analysis in the field of student graphical activity, especially with regard to ideas from Calculus and Analysis. As illustrated through the cases of Nate and Jay, different interpretations for "between," both rooted in visual reasoning, contributed to different evaluations of a complex mathematical statement about real-valued functions. While in our study, value-thinking helped students to understand versions of IVT, in other contexts, such as Geometry, location-thinking may be preferable. Ideally, students should possess the ability to think in both ways, as well as the discernment for when to use each. Overcoming various perceptual cues found in graphs, beyond conceiving of the graph as a static shape, is a nontrivial achievement, even for advanced students. Teachers utilizing such representations should support students in overcoming adherence to visual cues. In the classroom, instructors should be aware of the various ways in which students may interpret information from a visual representation. Further research on this topic may include developing instructional tasks to address this topic, whether in the context of the IVT or other statements commonly associated with graphs, such as the formal limit definition of continuity at a point.

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Features of Explanatory Proofs: An Exploratory Study

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In mathematical research as well as pedagogy, mathematicians rely on proofs to convey mathematical knowledge. Both mathematicians and mathematics educators have argued that a proof is more valuable to students when it explains why a theorem is true. In this contributed report, I discuss attributes of explanatory proofs that eleven doctoral students in mathematics described. Doctoral students in this study interpreted the nature of mathematical explanation in the context of a proof in a wide range of ways. In particular, these participants expressed that they are more likely to consider a proof more explanatory when it succeeds in providing (a) insight into the derivation of certain formulas, (b) intuition as to why the theorem is true, or (c) insight into how the author or the reader could have discovered the proof in practice.

Keywords: Proof, Mathematical explanation, Roles of proofs, Abstract Algebra.

In 1976, two mathematicians—Kenneth Appel and Wolfgang Haken—jointly supplied a computer-assisted proof of the four-color theorem, which roughly states that four colors are sufficient to color the regions of any map so that no two adjacent regions have the same color. Soon after its publication, the proof generated controversy within the mathematics community. The controversy was less about the validity of the proof but more about absence of insight gained from the method the proof employed. For some prominent mathematicians such as Paul Halmos, the proof failed to provide a sense of illumination as to why the theorem is true, since it relied on a computer program verifying thousands of cases (Thurston, 1994). The controversy surrounding the proof of the four-color theorem suggests that for a mathematician a proof is far more than a certificate of truth and that mathematicians have considerable interest in the reasoning used in a proof. Furthermore, we have evidence lending support to the claim that mathematicians do not always need a proof in order to believe a claim or proposition. In a recent survey study of mathematicians, Weber et al. (2014) argued that mathematicians do obtain conviction based on quasi-empirical or heuristic evidence. de Villiers (1990) also noted this when he wrote: "proof is not necessarily a prerequisite for conviction—conviction far more frequently is a prerequisite for proof" (p. 18).

Furthermore, a careful look at the history of mathematical research suggests that mathematicians continue to search for proofs for conjectures that are widely believed to be true. Consider, for instance, the Riemann hypothesis; it is believed that most mathematicians consider it to be true, yet mathematicians continue the search for a proof. This is perhaps because a proof, if found, will be a rich source of insight into why Riemann's conjecture is true. In their book The Mathematical Experience, Davis and Hersh (1983) made the following observation on why mathematicians continue to search for the proof:

> …It is interesting to ask, in a context such as this, why we still feel the need for a proof…It seems clear that we want a proof because…if something is true and we can't deduce it this way, this is a sign of a lack of understanding on our part. We believe, in other words, that a proof would be a way of understanding why the Riemann conjecture is true, which is something more than just knowing from convincing heuristic reason that is true (p. 368).

The quote above underlines the fact that verification of the certainty of a theorem is not the only role a proof plays in mathematical scholarship. Several authors such as de Villiers (1990) have written extensively about the various roles a proof can play in conveying mathematical understanding. This paper aims to explore the explanatory role a proof plays in conveying mathematical understanding. In other words, this study seeks to expand our understanding of what it really means for a proof to explain by collecting and analyzing interview data from doctoral students in mathematics with a wide range of research interests.

Conceptual Framework and Relevant Literature

According to de Villiers (1990), a proof can function as a means of verification, explanation, systematization, discovery, or communication. Hersh (1993) maintains that in mathematical research conviction is an essential aspect of any valid mathematical proof. Hersh (1993), in fact, defines a proof simply as: "a convincing argument, as judged by competent judges" (p. 389).

For a mathematician, a proof—beyond convincing—also could function as an explanation as to why a theorem is true (de Villiers, 1990; Hersh, 1993; Knuth, 2002; Thurston, 1995; Weber, 2002; Weber, 2008). Roughly speaking, a proof serves as an explanatory argument when it can lead one to "acquire an intuitive understanding of the main ideas of the proof" (Fukawa-Connelly, 2013, p.2). Moreover, in mathematics instruction, researchers have suggested that the primary purpose of a proof in a mathematics class should be to provide complete explanations why a given theorem is true (Hersh, 1993; Hanna, 1995; Weber, 2012; Yopp, 2011). Hersh (1993) writes: "proof can make its greatest contribution in the classroom only when the teacher is able to use proofs that convey understanding" (p.7). Harel and Sowder (2007) also noted this when they said: "…mathematics as sense making means that one should not only convince oneself that the particular topic/procedure makes sense, but also that one should be able to convince others through explanation and justification of her or his conclusions" (p. 808-809). Mathematics professors interviewed in Weber's (2012) and Yopp's (2011) study also maintain that they present proofs to students to convey understanding of why theorems are true.

The consensus view from research cited above is that the primary role of a proof—at least in the teaching of mathematics both at the K-12 and undergraduate levels—should be to convey mathematical explanations (Weber, 2012; Hersh, 1993). That is to say, a proof should help students develop understanding of why something (be it a claim, proposition, lemma, theorem, corollary, and so forth) is true. However, the relatively few existing studies on this topic do not provide a full account of what it means for a proof to explain or when a proof explains. To reiterate, one of the goals of this study is to expand our understanding of what it means for a proof to explain.

In addition to verifying that a theorem is true and explaining why it is true, a proof functions as a means of systematization of known results (de Villiers, 1990). According to de Villiers (1990), proofs are our only tool "in the systematization of various known results into deductive system of axioms, definitions, and theorems" (p.20). For example, the proof of the intermediate value theorem for continuous functions; de Villiers (1990) asserts that the primary function of this proof is basically a systematization of continuous functions. Systematization, among other things, provides global perspective, simplifies mathematical theories, and enables us to identify inconsistencies, circular reasoning, and hidden assumptions (de Villiers, 1990).

Finally, proofs can convey established or novel techniques that can later be used to tackle other problems (Thurston, 1994; Weber, 2012; Yopp, 2011; Lew et al., 2015). For example, mathematicians interviewed in Weber's (2010) study stated that when reading a proof, they would hope to learn new techniques that might eventually help them prove conjectures or problems they have been thinking about. Also, mathematicians interviewed in Mejia-Ramos and Weber (2013) said that the main reason they read published proofs is to gain techniques that they can later apply to other problems. The notion that mathematicians use proofs to convey proving methods or techniques is not limited to research publication; indeed, a significant number of mathematicians interviewed in both in Weber's (2012) and Yopp's (2011) study claimed that they presented proofs to students to illustrate proving techniques. A recent study by Lew et al. (2015) also maintains that mathematics professors present proofs to convey new proving methods.

Methods

Participants

The data to address the research questions are obtained from interviewing eleven doctoral students in pure mathematics from a PhD granting public university in the northeastern United States. All participants have passed their PhD qualifying exams. The participants' research interests include, but are not limited to, algebra, category theory, algebraic topology, analysis, and operator algebra. After completing their PhD degree, nearly all participants (nine out of eleven) indicated that they wanted to pursue an academic career in a more teaching focused institution. This attribute of these participants is of particular importance to this study as they are near experts and likely to be teaching undergraduate mathematics courses in the near future. Also, researchers have argued that for mathematics research to advance, we should examine the beliefs and practice of mathematicians (Harel & Sowder, 2007).

Research Procedure

I met each participant individually for a videotaped, semi-structured task-based interview that resulted in about 63 minutes of footage. One of the goals of these interviews was to gain insight into characteristics of proofs that some consider explanatory. During the interview, I gave participants four tasks. Two of the tasks are given in appendixes A and B. In each task, participants were asked to read two different proofs proving the same result. Participants were told that they can write on and/or highlight the text of the proofs. Additionally, they were encouraged to think out loud while reading each proof.

Participants were asked to read each proof for understanding as opposed to validating the proofs, and they were told that these proofs are taken from either an undergraduate textbook or a research journal. The methodological rationale for incorporating two proofs for each result was to elicit the multiple perspectives on the nature of explanatory proofs. For each task, I asked participants to compare and contrast each proof. In particular, after participants read both proofs in each task, I asked them the following questions:

- Does one proof show something different than the other proof? If so, how so?
- Do you prefer one proof to another? If so, why? If not, why not?
- Do you find one proof more explanatory than the other? Why or why not?

Materials: The Tasks

During the interview, rather than simply asking participants if they hope to gain explanation from proofs they read, I opted to elicit some narrative as to how participants view the nature of mathematical explanation in the context of proofs. To that end, I took proofs deemed explanatory in the literature and asked participants questions to probe their views on the method employed in the proofs in tasks A and B. A subset of proofs in tasks A and B has been presented in the proof literature as being conceptually more beneficial to students because these proofs succeed in showing not only that something is true—convincing or verifying that the claim is true—but also, they provide an explanation of why it must be true (cf. Hanna, 1990, Knuth, 2002, Cellouci, 2008). Additionally, participants were asked questions to elicit some narrative about their philosophical disposition about proofs. Of particular interest to this research was whether or not they distinguished proofs that only verify from proofs that, in addition to verification, also establish why a theorem is true.

Data Analysis

Each interview was transcribed fully and imported to a qualitative software, MAXQDA, for analysis. Participants were assigned pseudonyms G1-G11. I then analyzed interview transcripts using thematic qualitative text analysis (Kuckartz, 2002). In the first pass through the data, participants' responses were organized by topic: explanatory role vs. non-explanatory role. I coded participant's response as explanatory role when the response followed a prompted question about whether a participant found a given proof explanatory or not. Furthermore, a participant's response was coded as explanatory role when it suggested any of the following comments:

- A proof establishes a cause or reason for the veracity of the theorem (Yopp, 2011)
- A proof provides an intuitive understanding of the key ideas so that from the proof it is clear why the theorem must be true (de Villiers, 1990; Steiner, 1978; Yopp, 2011; Fukawa-Connelly, 2013)
- A proof provides intuition about mathematical objects or the topic involved in the proof

Furthermore, I wrote analytic memos while carefully reviewing each interview transcript. During this process I documented emerging themes; in particular, those relating to how participants talked about the meaning of explanation in proofs.

Results

Participants described how a proof can explain in a wide range of ways. For example, for G11, a proof is more explanatory when it succeeds in showing why a certain result is true by showing how it can be derived. According to this interpretation of mathematical explanation in proofs, participants such as G11 drew distinction between proofs that only verify from those that, in addition to verification, also provide a reason as to why something is true. When I asked G11 which proof in task B he finds more explanatory, he commented:

> Obviously the first one…because...so we're trying to show that the sum of the first n integer is given by this formula. And uh so it actually shows you where this formula comes from. It exactly shows if you do it this way then it exactly shows where this formula comes from, but now in the second proof you are starting with this statement that let p_n be this number n times n plus one over two so where did

you get this from…

In the excerpt above, note that G11 does think that proof 2 of task B, the induction argument, is valid and does verify that the formula for the sum is correct; however, he pointed out that it does not provide a reason or insight into why $\frac{n^2+n}{2}$ is the formula for the sum of the first n positive integers. In short, proof 1 shows how one could have obtained or derived the formula, whereas proof 2 first assumes the formula and then it shows that it is correct.

Participants stated that while some proofs such as proof 1 in task B do make the distinction between showing something is true—verification—and why it is true more apparent, other proofs are less successful at doing so, as the following excerpt from G1 illustrates:

> Why in mathematics is a tricky thing. Uh so for some of them like the sum of k where k is 1 to n [referring to the first proof of task B], I feel like the why there is the first proof where it's gauss's proof. It's self-evident, it's nice.

Some participants, such as G5 and G7, also spoke about mathematical explanation in proofs in terms of the extent to which a proof displays the author's thought process. That is to say, these participants find a proof that disguises the thinking behind the discovery of the proof less explanatory. And, they consider proofs that enhance one's clarity by providing some sense of understanding as to how the author originally interpreted the theorem to be more explanatory. Consider, as an illustration, the various ways participants described the absence of this phenomena—mathematical explanation—in proofs of task A. G7 said the following:

> Okay so here we say note that 641 has this expression. Okay now 641 has many expressions. Okay and so it seems funny to say note that well I don't know. I guess maybe it's…and it doesn't say like why are we considering the number 641…where does this number come from you know? Did we just try the you know 200 and however many odd numbers or odd numbers before that or I don't know did we try the prime numbers up to 641 and finally that's the one that worked? So you know this is very like opaque to me [emphasis G7's]

Likewise, G7 did not find the second proof of task A any more explanatory than the first, as the following excerpt illustrates:

> Well it's conclusive but it doesn't teach me anything. Okay and this one um so I mean there are two important parts to this proof and one of them is hidden. Okay so one important part of this proof is um we should identify, somehow identify this number 641 with which to attack this the 2 to the 2 to the 5th plus 1. So I mean we're going to attack 2 to the 2 to the 5th using 641. Okay. So some part of this proof is coming up with 641 is the useful number we should apply. Okay and then all right and the rest of the proof is this modular arithmetic argument to get you know to get that it works. You know I sort of lost interest.

In both excerpts above, we observe that for G7, the mysterious appearance of 641 made both proofs less explanatory. Both proofs fail to provide some sense of understanding as to why or how the authors thought of the number 641 to be important in the first place. To put it simply, both proofs were less successful in conveying how the author came up with those factors of the number in question, and as such, G7 found them less explanatory. Another participant, G5, also found proof 1 of task A less explanatory. He commented:

> I mean okay I will say this at least the second proof uses techniques of number theory to convince me that 641 divides this number but it's still kind of coming off like they know 641 is going to divide this number. I mean where's the lead up to knowing that 641 is even going to be a candidate? You know that would be

nice. I mean I look at both of these and I still just go okay uh how would I ever recreate this.

In this excerpt, G5 stated that proof 1 of task A failed to provide an explanation for the sudden appearance of the number 641. He asked, "where's the lead up to knowing that 641 is going to divide this number." According to G5, proof 1 masks the thought process that led to its discovery and the proof would have been more exploratory had it described why the number 641 is crucial or how one could have discovered it initially.

Discussion

Hanna (1995) writes: "in trying to define the true function of proof in mathematics education it is helpful to look at the function of proof in mathematics itself" (p.14). This study contributes to the mathematics education community in several ways. First, it expands our understanding of the roles a proof can play in conveying mathematical understanding; in particular it sheds light on how proofs can convey mathematical explanation.

Much of the scholarship on proof and proving (e.g., Hanna, 1990; Weber, 2012; Yopp, 2011; Hersh, 1993; De Villiers, 1990) has suggested that a proof is pedagogically more valuable when it is used during instruction as explanatory argument, yet the notion of mathematical explanation in proofs—despite being one of the most cherished roles of a proof—is rarely articulated in mathematics education research. Several authors such as Hanna (1990) and de Villiers (1990) discuss the explanatory role of a proof, but neither of these studies present interview or survey data with students or mathematicians; instead, the discussion of explanatory proof in previous studies is merely based on the authors' reflections and hypotheses based on their own experience. Moreover, even though both Weber (2012) and Yopp (2011) present the explanatory role of a proof based on interviews with mathematicians, still the lack of concrete examples in these studies makes the distinction between the verification and explanation role of a proof subtle.

While participants in this study did not display the same interpretation of mathematical explanation in proofs, their responses provided a rich source of perspective on the nature of proofs that are deemed explanatory. Further study, with more participants and a wider range of proofs, is called for so that we can continue clarifying our understanding of what it means for a proof to explain.

The current study also provides data that suggest that certain presentations of a proof have the potential to enhance the proofs' explanatory power. For example, I conjecture that a proof is more likely to be explanatory when it clearly displays to its audience how its key ideas are linked to prove the theorem. In other words, the degree to which a connection between a proof's key ideas are made transparent to the reader has some bearing on the level of mathematical explanation conveyed in the proof; in some cases, this entails including more detail or explanation of assertions within a proof. Sometimes explanation of assertions within a proof can be made using visual aids such as pictures or diagrams. However, the extent to which students can actually extract a mathematical explanation from proofs that incorporate a picture or a diagram is mainly untested. Future research can examine the pedagogical effectiveness of proofs that include pictures, heuristic approaches.

Appendix A: Task A

Note: The following two proofs are borrowed from Avigad (2006).

Claim: $2^{2^5}+1$ is not prime.

Proof 1

A calculation shows that $2^{2^5}+1=(641)(6700417)$. Q.E.D.

Proof 2

First, note that $641=(5)(2^7)+1 \Longrightarrow (5)(2^7)\equiv -1(mod\ 641) \Longrightarrow (5^4)(2^{28})\equiv 1(mod\ 641)$. On thoe other hand, we have that $641=5^4+2^4 \Longrightarrow 5^4\equiv -2^4(mod\ 641)$. Then $(5^4)(2^{28})\equiv -2^{32}(mod\ 641)$. So we have that $1\equiv -2^{32}(mod\ 641)$. Q.E.D.

Appendix B: Task B

Note: The following two proofs are borrowed from Hanna (1990)

Claim: $\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$.

Proof 1

We write out the terms first forward (as in *) and then backward (as in **)

1 + 2 + 3 +… + (n-1) + n …………(*)

n + (n-1) + (n-2) +… + 2 + 1 ………..(**)

The conclusion follows from the fact that the sum of the two terms in each column is (n+1) and we have n columns. Q.E.D.

Proof 2

We will proceed by induction. Let $P(n)$ be a statement that says the sum of the first n natural numbers is $\frac{n(n+1)}{2}$. We will show that $P(n)$ holds for all natural numbers n. The case for $n=1$ is immediate. For the inductive step, assume that for some $k, P(k)$ is true, so $1+2+\cdots+k=\frac{k(k+1)}{2}$. We must show that $P(k+1)$ is also true. Since $1+2+\cdots k+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}$, we conclude that $P(n)$ is true. Q.E.D.

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Undergraduate Students' Holistic Comprehension of a Proof

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In this paper I explore eleven undergraduate students' comprehension of a proof taken from an undergraduate abstract algebra course. My interpretation of what it means to understand a proof is based on a proof comprehension model developed by Mejia-Ramos, et al. (2012). This study in particular examines the extent to which undergraduate students are able to summarize a proof using the proof's higher-level ideas. Additionally, eleven doctoral students in mathematics were asked to provide a summary of the same proof that the undergraduate students received. Undergraduates' holistic comprehension of the proof was then analyzed in light of summaries that the doctoral students provided. The main finding of the study is that undergraduates' comprehension of the proof was overall inadequate—notably, they demonstrated limited skills in summarizing a proof via the proof's key ideas. Moreover, undergraduates failed to recognize the scope of the method used in the proof.

Key words: Proof, Proof Comprehension, Abstract Algebra.

In advanced undergraduate mathematics courses, students are expected to spend a significant portion of their time reading proofs. Although proof comprehension is a fundamental aspect of undergraduate mathematics education, studies by Conradie and Frith (2000), Rowland (2002) and Weber (2012) suggest that mathematicians rarely measure their students' comprehension of proofs. For example, Weber (2012) reports that mathematicians assess their students' comprehension of proofs by asking students either to reproduce the proof or prove a similar or a trivial consequence of proofs they presented in class. However, Conradie and Frith (2000) suggests that asking students to reproduce a proof may not be a pedagogically useful way of assessing students' understanding of a proof because students can and do correctly reproduce a proof by simply memorizing it word for word, with virtually no understanding at all.

Despite its importance in undergraduate mathematics education, research on undergraduates' comprehension of proofs is limited. In fact, much of the proof literature focuses on students' aptitude to construct or validate proofs and less on their ability to comprehend proofs (Mejia-Ramos et al., 2012; Mejia-Ramos & Inglis, 2009). Mejia-Ramos and his colleagues (2009) systematically investigated a sample of 131 studies on proofs and they found that only three studies focused on proof comprehension. They hypothesize that the scarcity of the literature on proof comprehension is perhaps due to the lack of a model on what it means for an undergraduate student to understand a proof. In this study, I adopt an assessment model for proof comprehension that was developed by Mejia-Ramos, et al. (2012) to explore undergraduates' comprehension of proofs. In particular, this study seeks to address the following research questions.

To what extent do undergraduates comprehend a proof? More specifically, to what extent do undergraduates:

- summarize a proof using its high-level ideas,
- recognize and appreciate the scope of a method used in a proof?

Theory: Assessment Model for Proof Comprehension

Mejia-Ramos, et al. (2012) proposed that one can assess undergraduates' comprehension of a proof along seven facets. These seven facets are organized into two overarching categories: local and holistic. A local understanding of a proof is an understanding that a student can gain "either by studying a specific statement in the proof or how that statement relates to a small number of other statements within the proof" (p.5). Alternatively, undergraduates can develop a *holistic* comprehension of a proof by attending to the main ideas of the proof. Below, I will elaborate on what it means to understand a proof holistically.

Assessing the Holistic Comprehension of a Proof

According to the proof comprehension model, the holistic understanding of a proof consists of being able to: (1) summarize the proof using the proof's main ideas, (2) identify the modular structure of the proof, (3) recognize and extend the method used in the proof, and (4) illustrate the method of the proof using a specific example or diagram. They developed these four facets of holistic comprehension of proofs based on: (a) mathematicians' perspectives on how and why they read and present proofs and what it means for them to understand a proof; (b) the proof literature on the role of proof; and (c) the recommendations by mathematicians and mathematics educators on proof presentations that would presumably improve students' proof comprehension. Below, I elaborate on (1) and (3).

Summarizing a proof via its high-level ideas. Mejia-Ramos, et al. (2012) state that "one way that a proof can be understood is in terms of the overarching approach that is used within a proof" (p.11)." Being able to summarize a proof via its high-level ideas entails understanding the proof's "top-level overview" or "big idea". A good summary of a proof may include what Raman (2003) describes as a proof's *key ideas.* Raman (2003) defines *key ideas* as "heuristic ideas which one can map to a formal proof with appropriate sense of rigor" (p. 323). *Key ideas* provide "a sense of understanding and conviction why a particular claim is true" (Raman, 2003, p. 323). Mathematicians can evaluate their students' understanding of this aspect of a proof in at least two ways. They can, for instance, directly ask students to provide a brief summary of the proof that includes the proof's higher-level ideas. Alternatively, they can provide students with a few summaries of the proof and ask them to choose which summary best captures the main ideas of the proof (Mejia-Ramos et al., 2012). In this study, I asked undergraduates to provide a proof summary using the proof's main ideas.

Transferring the general ideas or methods to another context. Mejia-Ramos, et al. (2012) suggested that identifying the scope of a method or technique used in a proof is an important aspect of proof comprehension. Mathematicians can assess this aspect of proof comprehension by asking students to (a) identify methods or techniques without which the proof would have collapsed, or (b) prove new claims by applying methods similar to those used in the original proof. In this study, undergraduates were asked questions to elicit their understanding of the scope of a method used in a proof.

Review of the Literature

As noted earlier, educational research on proof comprehension in undergraduate mathematics has received little emphasis. Osterholm (2006) was among the first to look into student's comprehension of mathematical texts. He conducted a quantitative study of reading comprehension of abstract algebra students (he compared texts with one including symbols and another one not). He concludes that "mathematics itself is not the most dominant aspect affecting the reading comprehension process, but the use of symbols in the text is a more relevant factor" (p.325). His contention is based on the fact that the group of students who had almost no symbols and notations in their reading assignment outperformed, in a reading comprehension test, those whose reading assignments involved mathematical symbols and notations. Although Osterholm (2006) does point to difficulties student encounter while reading mathematical texts, it should be noted that his study only asked students to read mathematical texts and not proofs specifically.

Research also suggests that undergraduates are not successful in gleaning understanding from the proof they see during lecture (Selden & Selden, 2012; Lew et al, 2015). For example, students interviewed in Lew et al.'s (2015) study fail to comprehend much of the content the instructor desired to convey, including the method used in the proof. Students interviewed in Selden and Selden's (2012) study also failed to understand a proof holistically since they were fixated on verifying each line and put little emphasis in attending to the overarching methods used in the proof. One purpose of this study is to build on the growing body of research on proof comprehension.

Methods

Research Settings

This study took place in a large public university in the northeastern United States. The content of the proof used in this study come from an introductory abstract algebra course. In the chosen research setting the standard textbook used is *Abstract Algebra: An introduction* by Hungerford (2012). The goal of the course (as stated in the syllabus) is to introduce students to the theory of algebraic structures such as rings, fields, and groups in that order.

Participants

Undergraduate student participants. Since the main purpose of this study is to explore undergraduates' comprehension of proofs—in particular, proofs that appear in an introductory abstract algebra course—I personally approached undergraduates who had taken or were enrolled in an introductory abstract algebra course. Eleven undergraduates agreed to participate in this study and were assigned pseudonyms S1-S11. At the time of the study, six of the eleven undergraduate participants (S3, S5, S6, S7, S8, and S9) were enrolled in an introductory abstract algebra course. Seven participants—S1, S2, S3, S5, S6, S7, S8, and S9—were pursuing a major in secondary mathematics education and said they intended to be high school mathematics teacher. The remaining four students were mathematics majors. Furthermore, each participant had taken a minimum of three proof-based courses and all but three (S1, S3, and S6) said they received an A or B in their introduction to proof course. Participants' responses on a background survey suggest that each participant spent at least two hours per week reading proofs outside of class.

Doctoral students. Eleven doctoral students at the aforementioned research site agreed to participate in this study. I used doctoral students to analyze undergraduates' summaries of a proof. At various times, I asked the doctoral students to provide, in writing, a summary of a proof using the proof's key or main ideas. To avoid confusion, in the remainder of this paper I will refer to these doctoral student participants as experts.

Materials and Research Procedures

In this study undergraduates were asked to read a proof that shows that any finite integral domain is a field. This proof is given in appendix 1. I chose this proof because (a) it nicely draws connection between two important topics covered in abstract algebra: integral domains and fields, and (b) it uses a proof technique that I speculated most participants probably have not seen.

The principles I employed to write this pedagogical proof—a proof that is geared toward undergraduates for the purpose of pedagogy—is based on the Lai, Weber, & Mejia-Ramos (2012) study. Lai et al. (2012) report that mathematicians valued pedagogical proofs that (1) made assumptions and conclusions of the proof explicit, (2) centered on important equations to emphasize the main ideas, and (3) did not contain "true but irrelevant statements." (p.94). Also, when writing the proof, I consulted mathematics professors and made appropriate modifications.

Participants were asked to read the proof until they felt they understood it and were encouraged to write and/or highlight on the proof paper as well as to think out loud while reading. Once a participant finished reading the proof, she/he was asked to:

- to provide a good summary of the proof including the proof's main or higher-level ideas
- to indicate assertion(s) in the proof that would fail if R was infinite.

Analysis

Recall that eleven doctoral students in mathematics were asked to write a summary of the proof using what they think are the main ideas of the proof. Doctoral students were also asked to describe the key ideas of each proof. First, I carefully studied doctoral students' summaries of the proof. I then developed a synthesized summary for the proof. This synthesized summary, which will hereafter be referred to as the expert's summary, also incorporated all the key ideas that doctoral students described for each proof. That process resulted in the following summary of the proof.

> The proof shows that any finite integral domain R is a field by showing that any non-zero element of R has a multiplicative inverse. Let a be any non-zero element of R. The absence of zero divisor in R taken together with the fact that R is finite implies that left multiplication by a defines a bijective map from the integral domain to itself (In other words, $f_a: R \to R$ given by $f_a(x) = ax$ is a bijection). Surjectivity of this map guarantees that a has a multiplicative inverse.

The key ideas identified in the expert's summary of the proof:

- overarching method: given an arbitrary non-zero element $a \in R$, one can show that there exists $b \in R$ such that $ab = 1_R$ (R is a finite integral domain)
- define a left multiplication by a non-zero element $a \in R, f_a: R \to R, x \mapsto ax$. Use the kernel of f_a to show that it is injective.
- the finiteness of R coupled with the injectivity of f_a shows that f_a is a surjective map from R to R.

The above expert's summary of the proof was eventually verified by two experienced researchers in mathematics education as to whether or not it indeed incorporated all the *key ideas* of the proof that doctoral students discussed; modifications were then made to the expert's summary, as needed.

Finally, using a rubric, two researchers, both with a master's degree in mathematics, independently conducted a comparative analysis of undergraduates' summaries of the proof against expert's summaries. When disagreement emerged, w engaged in discussion until a consensus was reached.

Results and Discussion

Results on Undergraduates' Summaries of the proof

Recall that undergraduates' summaries of the proof were analyzed in comparison to the expert's using a rubric that is omitted here for a shortage of space. Nearly all participants, nine out of eleven, provided a summary of the proof that suggested a limited proof comprehension. In particular, their responses indicated that they either *poorly* or *very poorly* understood the proof. The results of students' summaries of the proof is given in Table 1. Note, in table 1, that a majority, six undergraduates, provided a *very poor* summary of the proof, which implies that their summary failed to highlight the main ideas of the proof that was described in the expert's summary.

Table 1 Undergraduate students' summaries of the proof

Evaluation	Undergraduate students
Very poor	S3, S5, S6, S8, S9, S11
Poor	S1, S2
Satisfactory	S4, S7, S10
Good	None

In order to give you a sense of a *very poor summary*, I will provide a detailed description of summaries given by S5, S8, and S9. When asked to provide a good summary of the proof including the proof's main ideas, S5 offered the following summary:

> R is an integral domain and field. It is both surjective and injective to the kernel of the function that defines it. There is also $1_R \in R$ that allows it to have multiplicative inverse, thus units. I know a field's non-zero elements all make units, a field.

S5's summary consists of incomplete sentences and phrases that are either mathematically incorrect or appear to have been copied from the proof word for word. For instance, S5 begins by supposing the thing that needs to be shown—that R is a field. Also, there is evidence of a misunderstanding about injectivity and surjectivity of a map. This is evident when he asserts that R, as opposed to f_a, is both surjective and injective. Finally, S5's summary doesn't mention how crucial assumptions in the proof such as the finiteness of R are used in the proof. Thus, S5's summary is deemed to be *very poor*, which means it was considered to be very different from the expert's as it did not highlight main ideas of the proof. S8, likewise, provides a *very poor summary*:

> The proof basically gave a less textbook traditional explanation of a way to prove that all finite integral domains are fields. It used the understandings of kernals [sic], bijections, injection, surjections in order to prove facts about rings, where usually you learn the facts about rings before being introduced to functions

Her summary above makes no mention of the proof's high-level or *key ideas* and thus does not suggest a satisfactory comprehension of the proof. While she enumerates topics that are used in the proof, her summary does not illustrate how they were employed in the proof. S8's summary, for example, indicates that the concept of bijectivity is employed in the proof, but she does not explain how it is used. Along the same lines as S5, and S8, S9 also supplied a summary of the proof that did not suggest a satisfactory comprehension of the proof. For example, S9 provided the following summary:

> Given a finite integral domain, you can prove that it is a field by showing it has a multiplicative inverse, no zero divisors, injective and surjective, kernels, and if there is a multiplicative inverse such that $ax = 1_R$ and $a \neq 0_R$ then R is a field.

S9's summary above says very little beyond restating the claim. S9 essentially repeats phrases that appeared in the proof verbatim. He does not draw any connection between *key ideas* described in the expert's summary of the proof. Also, S9's summary states that R is first shown not to have zero divisors, but this is neither necessary nor true; R is assumed to be an integral domain and therefore it does not have zero divisors. Overall, S9's summary fails to mention the proof's key ideas and consequently shows limited comprehension of the proof.

While the majority, six out of eleven students, provided what is considered to be a *very poor summary* of the proof, S1 and S2 supplied a *poor summary* of the proof. S2, for example, provided the following summary of the proof:

> The aim of the proof is to show that if R is a finite integral domain, then R is a field. It then wants to show that R has a multiplicative inverse, then that the kernel of $f_a: x \to ax$ to be trivial. The prof then shows that $x = 0$ since if $ax = 0$ then a or x is 0 but a is not, thus ker $f_a = \{0\}$ so f_a is injective. Thus, $|R| = |f_a(R)|$, which shows it is surjective. It then proves a has a multiplicative inverse, so R is a field.

S2's summary is incoherent and appears to duplicate some parts of the proof word for word. Moreover, her summary includes way too much unnecessary information; for example, she repeats the argument for the triviality of f_a. While S2's summary does mention some *key ideas* that are noted in the expert's summary, it doesn't make the logical connection between those *ideas*. In fact, S2 appears to have the logic of the proof backward, as she seems to think the existence of multiplicative inverse is what guarantees the triviality of the kernel of f_a.

While no one provided a *good summary* of the proof, three students—S4, S7, and S10—provided a *satisfactory summary* of the proof. S7, for instance, summarized the proof as follows:

> First it is important to show each nonzero element of R has a multiplicative inverse. Then I consider a nonzero element $a \in R$ and the map of f_a. I use the kernel of f_a to prove f_a is injective. Then from the fact that f_a is injective and therefore $|R| = |f_a(R)|$, f_a is also surjective. Finally, I show $f_a(x) = 1_R$ hence a has a multiplicative inverse and therefore R is a field.

Evidently, S7's summary above has significant resemblance to the expert's summary of the proof. In particular, S7 does mention some *key ideas* of the proof. However, she did not indicate the fact that the surjectivity of f_a depends on the finiteness of R, which was a crucial idea that was noted in the expert's proof. Furthermore, the last line of her summary is incorrect in the sense that f_a is not identically equal to the identity 1_R. Also, based on what is stated at the very end of her summary, S7 does not seem to have understood how a (the fixed nonzero element) has a multiplicative inverse. However, S7's summary overall does bear some resemblance to the expert's summary and suggests a satisfactory understanding of the proof. On the whole, while nearly all students provided a *poor* or a *very poor* summary of proof, no one provided a *good summary.* Indeed, only S4, S7, and S10 managed to provide a satisfactory summary of the proof.

Results on Undergraduates' Ability to Recognize or Appreciate the Scope of a Proof's Method

No undergraduate demonstrated why R must be finite for the proof to be valid. When asked why the method of the proof would fail if R was assumed to be infinite, six out of eleven undergraduates offered no response or said "I don't know…" The other five students pointed incorrectly to an assertion that would fail if R is infinite. Table 2 below illustrates the various responses they provided:

Table 2 Reasons undergraduates provided for why R must be finite

Reason why R must be finite	Participants
R would not be commutative (line two)	S1
There wouldn't be exactly be the same number of elements in each	S3
f_a would not be injective (line seven)	S2, S8, S5
No response/I don't know/Not sure	S4, S6, S7, S9, S10, S11

Table 2 shows that participants failed to pinpoint a specific assertion in the proof that would fail—the argument in line eight—if R was infinite. That is to say, if R is infinite, $f_a(R) \subseteq R$ and $|R| = |f_a(R)|$ taken together would not guarantee that $f_a(R) = R$, which would not make f_a surjective.

To summarize, Mejia-Ramos and colleagues (2012) maintain that being able to provide a good summary of a proof is a key indicator of comprehension. However, undergraduates in this study showed a limited comprehension of the proof. In particular, in their proof summary undergraduates failed to highlight the proof's main idea. For a large number of participants, their proof summary was essentially a replica of a few sentences that appeared in the proof. Further research is needed to identify why undergraduates fail to summarize a proof using the proof's main ideas.

Appendix 1

Claim Let R be a finite integral domain. Then R is a field.

Proof. 1. Let R be a finite integral domain whose multiplicative identity is 1_R and whose additive identity is 0_R.

2. Since R is a commutative ring, it suffices to show that every nonzero element in R has a multiplicative inverse.

3. Let a be a fixed nonzero element of R $(a \neq 0_R)$. Consider the map $f_a: R \to R$ defined by $f_a: x \to ax$. I first show that the kernel of f_a is trivial.

4. Note that kernel of $f_a = \{x \in R: f_a(x) = 0_R\} = \{x \in R: ax = 0_R\}$.

5. Since R has no proper zero divisors, $ax = 0_R \Longrightarrow a = 0_R$ or $x = 0_R$. But, $a \neq 0_R$ thus $x = 0_R$.

6. Therefore kernel of $f_a = \{0_R\}$ and so f_a is injective.

7. Next, note that $|R| \geq |f_a(R)|$. Since f_a is injective, it follows that $|R| = |f_a(R)|$.

8. Because $f_a(R) \subseteq R$ and $|R| = |f_a(R)|$, I have that f_a is surjective.

9. Finally, since $1_R \in R$, I have that $\exists x \in R$ such that $f_a(x) = ax = 1_R$. So a has a multiplicative inverse. Therefore, R is a field.

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Generalising Univalence from Single to Multivariable Settings: The Case of Kyle

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A function is defined as a mapping from one nonempty set (the domain) to another nonempty set (the co-domain or range) such that each element of the domain maps to exactly one element of the range. Algebra curricula typically include classification tasks in which students determine if a relation violates the univalence criterion – the condition that each element in the domain corresponds to exactly one element of the range. This paper provides a longitudinal case study of how one student generalised the univalence criterion from single- to multivariable functions. For f(x), Kyle primarily thought of univalence in terms of the vertical line test and the variables x and y. He generalised univalence for the multivariable function f(x,y) by thinking about input, output, independence, and dependence. Kyle's story provides an example of how a student might generalise facets of the function concept in normatively correct ways.

Key words: multivariable functions, generalisation, univalence

Introduction

Functions are fundamental to mathematics. Introduced in algebra, functions appear throughout calculus, real and complex analysis, transformational geometry, and many other areas of mathematics. It follows that understanding function is critical for student success in mathematics. While high school curricula have traditionally focused on functions of one variable, the Mathematical Association of America has proposed adding multivariable topics as a way to increase mathematical competence for all students (Ganter & Haver, 2011; Shaughnessy, 2011). Supporting this, research findings indicate that seventh graders can accurately model real-life situations of two variables (Yerushalmy, 1997). However, at the undergraduate level, we know students struggle with multivariable calculus topics (Dorko & Weber, 2014; Jones & Dorko, 2015; Kabael, 2011; Trigueros, & Martínez-Planell, 2010). Hence research about how students come to understand multivariable functions can help inform instruction for a broad band of students.

Understanding how students think about multivariable functions necessarily includes how they determine what is and is not a function. This involves evaluating whether or not a relation violates the univalence criterion, and students tend to use the vertical line test in such evaluations (Leinhardt, Zaslavsky, & Stein, 1990). However, we do not know how students think about what is and is not a function, and how they generalise univalence, in the multivariable case. Hence this paper focuses on the following research question: *How do students generalise univalence?* I investigated this via a longitudinal study of five students who completed a series of interviews throughout their differential, integral, and multivariable calculus sequence. In this paper, I present a case study of a single student who successfully generalised univalence.

Background

Univalence is the condition that each element in a function's domain is mapped to exactly one element of the range. For a single-variable function f: R $\rightarrow$ R such that $y = f(x)$, univalence means that each x maps to exactly one y. Beginning algebra instruction typically includes 'classification' tasks in which students are given a relation and asked if it represents a function (Leinhardt, Zaslavsky, & Stein, 1990). For a graphical representation, students typically evaluate univalence with the vertical line test. A graph is univalent, and hence a function, if any line $x = c$ for some constant c and x in the domain will intersect the graph exactly once. For a multivariable function g: R x R → R such that $z = g(x,y)$, the univalence condition means that each (x,y) maps to exactly one z. The vertical line test generalises such that any vertical line through a point (x,y) in the domain will intersect the graph of $z = g(x,y)$ exactly once.

Student understanding of univalence for single-variable functions

Researchers have documented that students commonly solve graph classification tasks with the vertical line test (Clement, 2001; Thomas, 2003) and that some students will convert a non-graphical representation to a graph so they can use the vertical line test (Kabael, 2011; Thomas, 2003). Other students solve classification tasks by looking for *x*'s and *y*'s (Thomas, 2003). For example, Thomas (2003) observed students say the set of points $(x, 2x)$ for $x \in \mathbb{R}$ was not a function because there were no *y*'s. Finally, students may solve classification tasks by comparing graphs to prototypical examples such as $x = y^2$ (Clement, 2001).

Students commonly conflate univalence with other properties, such as one-to-one-ness (Bakar & Tall, 1991; Clement, 2001; Even, 1993; Markovits, Eylon, & Bruckheimer, 1986; Vinner & Dreyfus, 1989). Students may not include the univalence criterion when defining function, or they may state it backward, saying each element in the range is paired with exactly one element in the domain (Even, 1993; Kabael, 2011). Similarly, Thomas (2003) found that some students used a horizontal line test instead of a vertical line test on graphical classification tasks.

Student understanding of multivariable functions

Kabael (2011) found after instruction about multivariable functions, 21 of 23 students solved graphical classification tasks by converting the graphs to algebraic formulae. In a test at the end of the multivariable calculus course, 13 of the 23 solved graphical classification tasks with the vertical line test. On classification tasks in which the students were given algebraic formulae, 5 of the students drew a graph and used the vertical line test. Four students evaluated univalence by determining algebraic formulae for all classification tasks on both tests. These findings suggest by the end of the course, some but not all students had generalised univalence across representations (Kabael, 2011).

Researchers have found that thinking of functions in terms of inputs and outputs allows students to generalise in normatively correct ways (Dorko & Weber, 2014; Kabael, 2011). In contrast, focusing on specific variables leads some students to believe that the domain of a function $f(x,y)$ is the set of all possible x values and the range is the set of all possible y values, as is often true for the $f(x)$ case (Dorko & Weber, 2014). Kabael (2011) also observed that students struggled with domain and range, giving a particular (x, y, z)-tuple as an example of an element of the domain or an element of the range. Difficulty with domain and range may play a role in students' ability to generalise univalence, as testing for univalence requires identifying an element of the domain and checking that it is paired with exactly one element of the range.

Theoretical Perspective

I interpret students' thoughts and generalisations from an actor-oriented perspective (Lobato, 2003; Ellis, 2007). This perspective focuses on what connections students identify across situations, even if their perceived similarities are not normatively correct. In the actor-oriented perspective, generalisation is defined as "the influence of a learner's prior activities on his or her activity in novel situations" (Ellis, 2007, p. 225). Students may make connections across situations by identifying similarities, or by "discerning differences and modifying situations" (Ellis, 2007, p. 225). Hence in exploring how students generalised univalence in graphical and tabular contexts, I looked for similarities and differences in how they handled the single- and multivariable tasks.

The actor-oriented approach is an alternative to traditional transfer (e.g., Gick & Holyoke, 1980; Judd, 1908), in which research questions tend to focus on whether or not participants identify pre-determined similarities. By attending to only what is normatively correct, such a perspective often fails to capture the sense participants make of situations. One reason the actor-oriented perspective is appropriate for the design of this study is that in the multivariable pre-interview (see below), students were looking at a novel context (R^3) with significant differences from the original (R^2) context. Hence the actor-oriented perspective allowed us to capture what students saw as similar, what they saw as different, and what adaptations they made between the two contexts.

Research Design

Data Collection

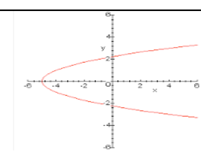
I conducted task-based clinical interviews (Hunting, 1997) with each of five students at three separate instances over the span of their differential, integral, and multivariable calculus courses. This paper focuses on a subset of tasks from the interview in which students were given graphs and tables and asked which represented functions. The interviews were recorded with audio, video, and Livescribe technology, and I subsequently transcribed them for analysis.

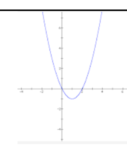
The first interview, conducted at the beginning of students' single-variable calculus course, contained R^2 graphs and tables (Figures 1). The second interview, conducted during integral calculus, contained R^3 graphs (Figure 2). Asking students about graphs in R^3 before instruction about multivariable functions allowed me to capture students' initial sense-making about multivariable functions and how the univalence criterion might generalise. Before starting the tasks I presented students with a picture of R^3 axes, pointed out the axis labels, and explained that the *xy* axis made a flat plane like a tabletop and the *z* axis was vertical and orthogonal/perpendicular to the *xy* plane. I demonstrated this using the corner of a table and a pen held orthogonal to it. Some of the students had plotted points on R^3 axes in high school algebra, but that was the extent of their experience with multivariable topics.

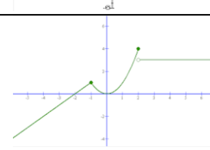
Which of the following represent functions?

i	*x*	-2	0	3	3	5	6
	y	9	2	4	5	7	7
ii	*x*	-2	0	3	5	6	7
	y	-1	1	4	6	7	8
iii	*x*	-2	0	3	5	6	7
	y	0	0	8	15	31	-5

Which of the following graphs represent functions of *x*?

a


b


c


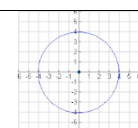
d


Figure 1. Interview 1 classification tasks

Which of the following graphs represent functions? (from Kabael, 2011)

A B C D E F

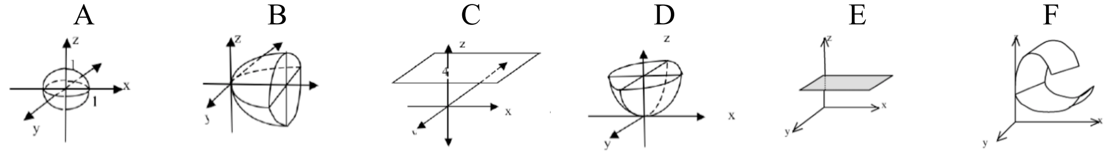


Figure 2. Interview 2 classification tasks

Do the following represent functions?

I II III IV

X→ Y	-1	0	1	2	3
-1	1	0	-1	-2	-3
0	0	0	0	0	0
1	-1	0	1	2	3
2	-2	0	2	4	6
3	-3	0	1	6	9

V

X→ Y	-1	0	1	2	3
-1	-1	6	-1	-2	-3
0	-1	6	0	0	6
1	-1	0	1	2	3
2	-2	0	2	4	6
3	-3	0	1	6	9

b2

X→ Y	-1	0	1	1	2	3
-1	-1	6	-1	8	-2	-3
0	-1	6	0	-4	0	6
1	-1	0	1	10	2	3
2	-2	0	2	15	4	6
3	-3	0	1	-7	6	9

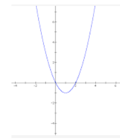
a2

X→ Y	-1	0	1	2	3
-1	-1	6	-1	-2	-3
0	-1	6	0	0	6
1	-1	0	1	2	3
2	-2	0	2	4	6
3	-3	0	1	6	9
3	-4	9	0	7	14

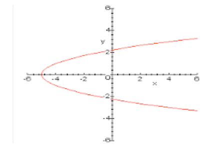
c2

X→ Y	-1	0	1	1	2	3
-1	-1	6	-1	8	-2	-3
0	-1	6	0	-4	0	6
1	-1	0	1	10	2	3
2	-2	0	2	15	4	6
3	-3	0	1	-7	6	9
3	-4	9	0	2	7	14

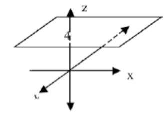
A2

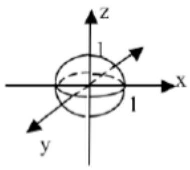


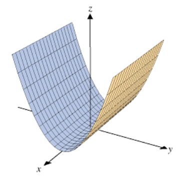
G2



F2







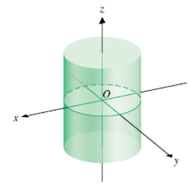


Figure 3. Interview 3 classification tasks, listed in the order they appeared to students. Tasks c2 and A2 from Kabael (2011)

The phrasing "do the following represent functions?" in the Interview 2 and 3 tasks was purposefully ambiguous. Since findings from research indicate that some students generalise domain and range by thinking about inputs and outputs and/or independence and dependence (Dorko & Weber, 2014; Kabael, 2011), I wanted to see how students would determine which variables were the input(s), output, independent, and dependent variables.

The third interview, conducted at the end of students' multivariable calculus course, contained both graphs and tables in R^2 and R^3 (Figures 3). I explained how to read the tables and gave the example 'in Table I, when x is 0 and y is -1, z is 0.' These post-instruction tasks captured the sense students made of normative ideas and how they connected those ideas to prior knowledge. In short, the longitudinal nature provided unique insight into the generalisation phenomenon.

Data Analysis

I used a constant comparative analysis (Strauss & Corbin, 1998) to code students' reasoning. Findings from literature influenced my coding in that I looked for x, y, input, output, independent, and dependent, as these are common ways students think about multivariable functions (Dorko & Weber, 2014; Kabael, 2011). However, I was careful to include other phrases not within that list. After coding students' responses for particular phrases, I compared the students' reasoning from interview 1 to 2 to 3. Because of the longitudinal nature of the study, this allowed me to compare how students thought about what a function is pre- and post- instruction, and to identify ideas that stood out to them. In particular, I looked for ways of thinking that were the same across interviews, and ways of thinking that were different. This allowed me to address the question of how students generalised from the single- to multivariable case.

Results

In thinking about the questions, students used words like x and y, input, output, independence, dependence, no dependence, repeated x values, pattern, and vertical line test. When answering the multivariable questions, many first stated the definition of univalence for a single-variable function. They then applied this criterion directly to the multivariable setting (example below), and/or worked on how to adjust it for R^3.

Kyle was the only student who answered all of the questions correctly. I present some of his thinking as an example of how a student successfully generalised univalence.

Single-variable tables and R^2 graphs

Kyle answered all of the single-variable table and single-variable graph questions correctly. He searched for *x* values paired with multiple *y* values in both contexts and used the vertical line test for the graphs (Excerpts 1 and 2).

Excerpt 1. SV Interview, SV Tables i, ii, and iii

Kyle: Okay, so for every *x* there is only one *y*. Well this one is not a function, so *i*, *i* is not a function… Because there are two values for *x* of 3 that have different *y* values. The *y* is 4 and this one, 5 on this one.… *ii* is a function. And *iii* is a function… I determined that because … in *ii*, for every *x* value, it has a *y* value. It has one *y* value, not multiple *y* values. And same with *iii*. There's no repeated *x* values that have different *y* values.

Kyle's work with the single-variable tables revealed that he understood that each element in the domain must map to something in the range ("for every *x* value, it has a *y* value"). I take the statements "for every *x* there is only one *y*" and his example of $x = 3$ mapping to $y = 4$ and $y = 5$ as evidence that Kyle understood the univalence criterion. His responses to the R^2 graph tasks provided further evidence that he understood a function must be univalent (Excerpt 2).

Excerpt 2. SV Interview, SV Graphs b and c

Kyle: So *b* is a function of *x*, and this is also, *c* is also a function of *x*. So you determine whether a graph represents a function if for every *x* value there's only one *y* value that is present. And so for both these graphs, that's the only, that's the case. There's, there's like a test you could do where you make lines going straight down and if the line only goes through one point in the curve…then it's a function.

Here, Kyle described the vertical line test and restated his criterion of each *x* mapping to one *y*. One point of note is that while Kyle stated the letters *x* and *y*, he did not use any other descriptors (e.g., input, output, independence, dependence). This may be an artifact of the tasks, which were all posed with *x*'s and *y*'s.

Multivariable graphs (pre-instruction)

In classifying MV graphs A, B, C, D, E and F, Kyle sought to generalise the vertical line test and his notion that each *x* has exactly one *y*. In doing so, he appeared to draw on the particular symbols, coordinate points, and the notion of an output.

Symbols and coordinate points

Kyle reasoned with symbols and the form of coordinate points for graph A and graph C (Excerpt 3 and 4).

Excerpt 3. MV Pre-interview, MV Graph A

Kyle: [Graph A]…it's a function, I think…for every *x* there's a *y*, for every, for every *x* there could be multiple *y*'s, so it's not a function…

Excerpt 4. MV Pre-interview, MV Graph C

Kyle: [Graph C] I mean I guess it can be a function, because it could have a different… at every, maybe…for 3D if it's a value of *x*, it has to have a different *y* and *z* [writes *(x,y,z)*] instead of just for every value of *x* there's a *y*, as long as something's different in the *y* or the *z*, then it's okay… for some value of *x* there has to be a value of *y* and a value of *z*.

Kyle's first attempt at generalising seemed to be to evaluate the R^3 graph with the same criterion he used for R^2 graphs. That is, his first statement was that the graph was not a function because "for every *x* there could be multiple *y*'s", which was his evaluation criteria for graphs in R^2 (Excerpts 1 and 2). In further thinking, Kyle sought to change the way he evaluated the R^3 graph. For instance, he considered that maybe a multivariable function would have 'something different' in the *y* or the *z*.

I interpret Kyle's statement "for some value of *x* there has to be a value of *y* and a value of *z*" as a generalisation of his statement in the single-variable tables "for every *x* value, it has a y value." That is, he seemed to try to generalise the idea of each element in the domain mapping to an element of the range. I take as evidence the phrases "for every x" (single variable) and "for some value of x" (multivariable), suggesting that Kyle was attempting to pair x with something in each case. He generalised x having "a y" (single-variable) to x having "a value of y and a value of z" (multivariable). In these statements, Kyle seemed primarily focused on the particular symbols themselves rather than what they might represent.

As Kyle thought about this, he wrote the three-tuple (x, y, z). This seems salient for two reasons. First, it is further evidence of Kyle's attention to symbols. Second, he appeared to use it as a tool to help him generalise. The evidence for this is how he alternated between talking about the single variable case to the multivariable case (Table 1).

Table 1. Using the coordinate point structure as a tool for generalisation

Statement	SV or MV
"maybe…for 3D if it's a value of x, it has to have a different y and z"	Multivariable
[Writes (x, y, z)]	Multivariable
"instead of just for every value of x there's a y,	Single variable
as long as something's different in the y or the z, then it's okay."	Multivariable

In short, the coordinate point seemed to help him generalise the 'mapping' idea from single- to multivariable. Kyle then wrote $f(x, y, z)$. The interviewer explained that for a function in R^3, the notation is "f(x,y) equals something." This helped Kyle discern that z was an output (Excerpt 5).

Excerpt 5. MV Pre-Interview, Graph C

Kyle: Right, that equals something, and that equals the z value. [writes $f(x,y) = z$] then z would be what we're, the output of it… $f(x) = y$. For every x value there's only one y value. So my thought now is that every x and y value there's only one z value for it to be a function.

Kyle appeared to see the $f(x) = y$ and $f(x,y) = z$ forms as similar, allowing him to realise that z was the output. He then correctly stated the univalence criterion for $f(x,y)$. I infer that seeing z as the output afforded this generalisation.

Vertical line test

Kyle's approach to the sphere task included generalising the vertical line test. Like the coordinate points, he alternated between talking about the single- and multivariable cases (Excerpt 6).

Excerpt 6. MV Pre-interview, MV Graph A

Kyle: [Graph A]…there's parts that don't pass the vertical line test, then it's not a function… So for something to be a function, for every x value there can only be one y value…. And you can test that with the vertical line test because what it's doing is it's, it's keeping x the same but…if there's more than one value of y, then it, then it doesn't pass… But I think that, I think that works for not just, if it's x and y, like this, then that's the vertical line test [draws R^2 example]. But if it's x, y, and z, it could be, it could either be the vertical line test for z [gestures to indicate lines parallel to the z axis], or the I guess the same principle applies if it's going that way [draws lines on the xy plane, parallel to the y axis].

I take Kyle's statement "I think that works for not *just*, if it's x and y" [emphasis mine] as Kyle generalising the vertical line test to R^3. Specifically, I infer that the word 'just' as meaning Kyle thought the vertical line test would work *not only* for x and y, *but also* for x, y, and z. However, although he thought the vertical line test generalised to R^3, he was unsure which direction the lines should go, as evidenced by his drawings of lines both parallel to the z axis and parallel to the y axis. He stated later he thought lines parallel to z would be correct.

Multivariable tables and graphs (post-instruction)

Kyle answered all the multivariable table tasks correctly. In contrast to his focus on *x*'s and *y*'s in the first interview, here he thought about inputs, outputs, and having one *z* value "for each pair of independent variables" (Excerpt 7).

Excerpt 7. MV Post-interview, Table 1 and Graph G

Kyle: [Table 1] So my definition is like for each pair of independent variables, there can only be one *z*…it is a function… like for a point *x* is 0, *y* is 0, *z* is, there's one value for that.

Kyle: [Graph G] So if it's *f*(x,y) equals *z*, yeah, the output would be *z*. For every *f*(x,y) there could only be one *z*. And this is a function because, because yeah there's only one *z* value for every *x* and *y*.

I infer that Kyle leveraged the ideas of independence and output to aid generalisation. He stated that "the output would be *z*", then returned to variable-based language to state his generalisation "there's only one *z* value for every *x* and *y*." Like the second interview, he appeared to draw on the symbols *f*(x,y) = *z* to conceptualise *z* as an output, which afforded generalising univalence.

Discussion

In the first interview, Kyle spoke only about *x* and *y*. This may have been an artifact of the task design. He continued to reason primarily with symbols, and it is notable that all of his statements of univalence were in terms of variables (e.g., "there's only one *z* value for every *x* and *y*"). This does not mean that Kyle had a weak conception of function. On the contrary, he talked about independence and dependence, and outputs, and in particular these ideas helped him generalise. However, symbols served as tools as Kyle generalised. one of his first attempts at generalising included thinking about the coordinate point *(x, y, z)* and what such a set of points would have to look like to make something a function. He seemed to attend to the similar forms form $f(x) = y$ and $f(x,y) = z$ to think about which variable represented the output. Others have documented relating objects as a way students generalise (Dorko & Weber, 2014; Ellis, 2007).

One answer the research question *How do students generalise univalence?* is students generalise univalence by thinking about inputs, outputs, independence, dependence, specific coordinate points, using examples and counterexamples, and attending to symbols. Another way is that students often first seek to apply their criterion from R^2 directly to R^3. For example, Kyle's first attempt at generalisation was to say the sphere was not a function because there existed multiple *y'*s for a single *x*. He also considered that the vertical line test in R^3 might use a line parallel to the *y* axis. After applying these R^2 ideas, Kyle seemed to try to adapt them for R^3.

Suggestions for Instruction and Future Research

My findings support others' that function machines, input-output, and independence-dependence provide powerful ways for students to think about and generalise functions (Dorko & Weber, 2014; Kabael, 2011). Hence I suggest that we continue to emphasise these ways of thinking in instruction. One way to do this is to decrease our use of *x* and *y*, asking questions like

m	-4	9	0	0	3
n	1	4	-6	8	2

Could *m* be a function of *n*? Could *n* be a function of *m*?

$r = q^2$

Could *r* be a function of *q*?
Could *q* be a function of *r*?

There is still much work to be done regarding how students generalise multivariable topics. One question concerns the cognitive mechanisms underlying such generalistion. For instance, how can we explain students' tendency to apply an idea from R^2, and then alter it? Such work can contribute toward developing theory about generalisation and transfer.

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SCNI: A Robust Technique to Investigate Small-Group Learning at College

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The Stimulated Construction of Narratives about Interactions (SCNI) technique for data collection, introduced in this paper, enables robust investigations of small-group learning at college. The SCNI technique consists of promptly soliciting participants' perspectives on their recent joint activity using video records thereof. Thus the SCNI technique creates a space to network the narrative discourses that shape how participants understand their world, and the pragmatic forces that shape participants' interactions in a practice. Data reported in this paper are collected from a number theory class comprised of ethnically diverse students. In this paper, I will report three cases to illustrate the advantages of SCNI data over data collected by video records and unmediated interviews in elucidating, nuancing, and expounding what matters for group work. Through these cases, I will use three different analyses appropriate for SCNI data. Limitations and recommendations for efficient conduct of SCNI are discussed as well.

Key words: Small-Group Learning, Narrative Identity, Interaction Analysis, Discourse Analysis.

Since the 1970s, qualitative and quantitative research have been constantly emphasizing and confirming the positive effect of small-group work on students' learning and achievements (Astin, 1977, 1993; Cockrell, Caplow, & Donaldson, 2000; Laursen, Hassi, Kogan, Hunter, & Weston, 2011; Springer, Stanne, & Donovan, 1999; Tinto, 1997). Simultaneously, an ever growing body of scholarship has been reporting debilitating processes in group work, such as lack of social skills (Barnes & Todd, 1977), poor mathematical knowledge (Webb, Ing, Kersting, & Nemer, 2006), weak coordination (Barron, 2003), undue influence (Engle, Langer-Osuna, & McKinney de Royston, 2014), communication problems (Sfard & Kieran, 2001; van de Sande & Greeno, 2012), problematic identities (Heyd-Metzuyanim & Sfard, 2012; Langer-Osuna, 2011) and others (see a review by Webb, 2013). To investigate the differential processes taking place in group work, researchers have been investing less in achievement studies and more in the study of social interactions in small-group learning, as noted by Cohen (1994).

Additionally, recent research (Esmonde, 2009a, 2009b; Esmonde & Langer-Osuna, 2013; Gresalfi, 2009; Langer-Osuna & Esmonde, In Press) has emphasized the need to study learning ecologies constructed through ongoing interactions with an emphasis on how socio-emotional and power relationships play out in the moment-by-moment of group work. Such a research program, I claim, requires new techniques for data collection. To this date, video records and/or unmediated interviews have been the most common data sources used to investigate small-group learning. Socio-emotional and power forces at play through ongoing interactions do not sufficiently manifest through videos or unmediated interviews for the following reasons:

- During ongoing interactions, participants often act without manifesting the forces underpinning or motivating their actions.
- In unmediated interviews about interactions, participants often sustain a handful of strong impressions from group work and become oblivious of fleeting but otherwise significant ones.
- Although data from unmediated interviews can elicit participants' individualized discourses by which they narrate self, others and the world, it cannot help determine which discursive forces are at play in situ.

This paper will show how video records of group work and interviews of participants are not sufficiently robust to investigate socio-emotional and power relationships in small-group learning. The *Stimulated Construction of Narratives about Interactions* (SCNI) technique for data collection, to be introduced and explicated in this paper, attempts to address the aforementioned challenges to an ecological study of small-group learning. The SCNI technique consists of promptly soliciting participants' perspectives on their recent joint activity as they watch a video of the group work.

In this paper, I will first lay out the conduct of SCNI interviews, then explicate the underpinning theoretical frame of this technique. Second, I will present three cases of SCNI data, which will illustrate how SCNI data complement data from video records. I will also use different analytical methods to treat the SCNI data as illustration of the diverse appropriate analyses that can be used with these data. The underpinning belief of this paper is that combined data from SCNI and video records afford robust understanding of processes that shape small-group learning.

The SCNI Technique

The SCNI technique consists of probing participants to construct narratives about (comment on) their social interactions by watching a video of the activity in which they participated within 24 hours prior to the SCNI interview. The SCNI technique builds on the cognitive technique commonly known as Stimulated Recall (Gass & Mackey, 2000) and video viewing practiced by interactional analysts (Jordan & Henderson, 1995). Contrary to the heavily cognitive SR technique, the SCNI technique aims at eliciting not only cognitive but also fleeting and enduring social processes individualized by participants. While SR and video viewing interviews seek to understand what happens in the studied group work, SCNI interviews are also interested in knowing who participants are and what frameworks they have individualized to make sense of their world. This distinction will become evident as I explicate the theoretical grounding of the SCNI technique.

Conduct of SCNI Interviews

An SCNI session consists of (i) the preparation of the *medium* (i.e., the video of actual activity); (ii) a low-probing video viewing section; and (iii) a proactive open interview section. I started recording the SCNI sessions with an audio recorder but soon came to realize that they are best recorded as videos to facilitate future analyses.

During the classroom activities, stable and unmonitored video cameras (wirelessly connected to microphones placed in the middle of the group table) captured the entire interactional space of each group. The resulting videos are used to mediate the following two interview styles.

The SCNI sessions, conducted individually in this study, take place within 24 hours of the end of the videotaped group activities. The interviewer launches the interview with this probe: "In my study I try to understand the interactions between people. Today, I would like you to help me see through your eyes to understand what happened in your recent group session. You will watch a video of it to help you recall what happened. You can pause the video at any time you recall your significant mathematical reasoning and your feelings about yourself or your groupmates at the moment of the interactions. Try your best not to confuse your current thoughts and emotions with those you experienced when you were working in group."

The interviewee and interviewer sit facing a screen placed in front of the interviewee and streaming the recently videotaped group session. This seating foregrounds the video and backgrounds the interviewer to the informant. During this phase, the interviewer may use gentle probes, such as pausing the video and asking whether the informant can recall what s/he was thinking or feeling at that time, or if s/he understood a groupmate's explanation. If informants drift away from talking about the moment of interactions, the interviewer may gently orient him/her back to the task by asking, "Is this what you were thinking/feeling at the moment?" The interviewer must also note the time stamps when the video is paused for commenting.

At the end of the session, the interviewer follows up on the participant's significant comments by rewinding the video to those moments and asking open questions such as, "What did you mean when you said you were frustrated here?"; "Is it common that you feel/think like this in such circumstances?"; or "At time *t*, you said Fred is smart. Why is that?"

People tend to tailor their talks to their perception of their interlocutors' knowledge. Hence, to trigger talks about the mathematics involved in the group work, the interviewer must be knowledgeable in the subject matter and exhibit his relevant mathematical knowledge to interviewees. In my data collection in an upper-division course, I introduced myself as someone who had earned a Master degree in mathematics and assisted students during group work and tutoring sessions.

Theoretical Grounding of SCNI

The SCNI technique creates a space to network the narrative discourses and the pragmatics that govern participants' interactions in a practice (Figure 1). As participants in SCNI talk about their interactions in a recent activity, their comments will be constrained and enabled by two forces: on the one hand, the narrative discourses that construct and regulate their subjectivities (Kramsch, 2010) and, on the other, the pragmatic forces that have just regulated their interactions. Hence, in SCNI sessions, interviewees are positioned in a space negotiating two realms: (1) their individualized discourses that shape the narrations of self, others, and activities and (2) the pragmatic forces that shaped the recent interactions they watched in a video.

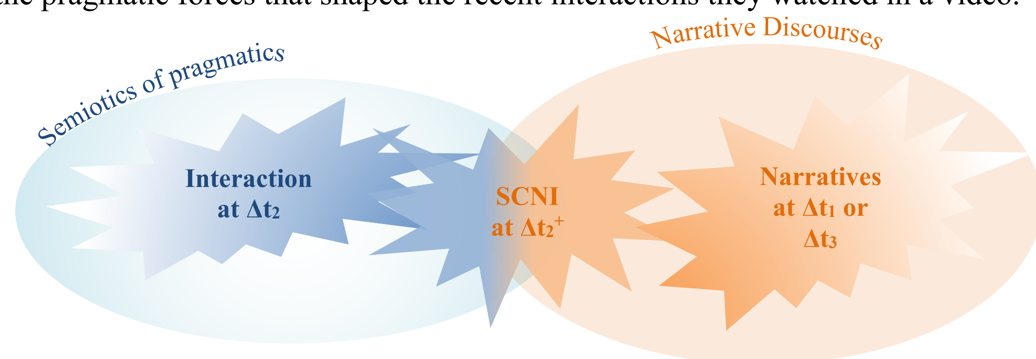

Figure 1. The SCNI conducted at time Δt_2^+ on social interactions at time Δt_2 produces data shaped by (a) the semiotics of pragmatics that govern social interactions at time Δt_2 and (b) discourses that frame unmediated narratives -at time $\Delta t_1 < \Delta t_2$ and $\Delta t_3 > \Delta t_2$.

Following interaction analysis methods (Jordan & Henderson, 1995), the SCNI technique provides participants' perspectives, that is, the pragmatic frames that participants activate to make sense of interactions *during ongoing activity*. Are the narratives produced by interviewees in SCNI sessions pure recall of what really happened in recent activity? Certainly not. But neither are they pure constructions disconnected from the reality of recent activity. Narratives of past events are partly faithful and partly unfaithful to actual events. By placing interviews, like in

SCNI sessions, soon enough to the focal activity and constraining narratives by strong stimuli like videos of activity, we increase the reliability of recruited data.

Informants' responses in SCNI sessions ought to be treated as narratives. Like all narratives, they draw on *remembered* impressions that are *narrated* employing internalized genres and styles (Baynham, 2014; Dervin & Risager, 2014). The discontinuity of SCNI data with actual interactions does not threaten its reliability. On the contrary, the internalized narratives by which informants narrate their recent interactions regulate their enduring identities (Kramsch, 2010). To put it in a way pertaining to small-group learning, by talking about their recent group work in SCNI sessions, participants implicitly or explicitly give off signs of their entrenched understanding of themselves, their groupmates, mathematics, and group work activity. Whether in group work or in SCNI sessions, participants act by the same *habitus* (Bourdieu, 2003; Bourdieu & Thompson, 1991) that gives each one of them an enduring existence through different contexts and activities.

As noted in the introduction of this paper, researchers of small-group learning must invest in robust studies of ongoing interactions in group work and all that they involve (hidden social norms, emotions, and power). Yet ongoing interactions are fleeting. Traditionally they are accessed through either records thereof, such as videos, audios and fieldnotes, or perpetrators of actions. Researchers have attempted to capture ongoing interactions through video records and thus made them objects of analyses. Although video analysis is a powerful tool to investigate ongoing interactions, it presents limitations and biases, such as the orientation of the camera (transforming the 3-D reality into 2-D video), and the limitation of capturing good sound quality in a noisy environment (see Erickson, 2006; Hall, 2000). In some way, videos are one narrative about reality. In addition to videos, past interactions can be accessed through participants who engaged in them. The uniqueness of SCNI technique resides in investigating ongoing interactions by revealing the subjectivities/habitus/identities of participants in relation to the studied interactions.

The SCNI technique has its own limitations. First, interviewees must have agility in talking about interactions and verbalizing their emotions and experiences. Although the SCNI technique resulted in rich data for all participants in my research (youngest age is 19 years old), it may not be equally informative with elementary and middle school students. Second, crucial for rich SCNI data, the interviewer and interviewee must construct a safe environment as much as possible, allowing the interviewee to speak his/her mind and say what s/he conceals in her/his ongoing interactions with groupmates because of face-saving and politeness processes (Goffman, 1967; Pearson, Kreuz, Zwaan, & Graesser, 1995).

Results

Data reported is collected from a number theory class at a Northern California college. The course was solely based on small-group work throughout the semester. Students (10 females and 13 males) composed their groups at their will. There were 5 groups of 4 or 5 students each. Four groups (G1, G2, G3 & G4) consented to be videotaped (all their group sessions were videotaped) and 15 students of these groups participated in individual interviews (early and late unmediated interviews + SCNI). Groupmates participated in SCNI interviews on the same day and every other week. I conducted 45 SCNI interviews in total.

Case 1: Friends, yet not alike

Izabelle, Nawal, Leila, and Gaia wanted to be in the same group, G1, because they had been friends for at least a year. On September 22nd (henceforth 9/22) the instructor Hoffmann started the class by explaining the definition of primitive roots for about 10 minutes then set the groups to work. The group G1 was still in the middle of a worksheet that they could not finish during the last session. While groupmates were opening their notebooks and getting ready to work together, Gaia called the instructor. We observe in the video of this group session that Leila was surprised (evidence: eyes wide open, head turns swiftly, then she frowns) when she looked up and found the instructor at the group table. The video does not show significant reactions from Izabelle and Nawal. In the individual SCNI interviews that followed this group session, Izabelle did not comment on Gaia's behavior, but Leila and Nawal did.

Leila: [*pauses the video at the moment when Gaia calls the instructor*] I was a little annoyed by what happened there ... haha because we were all like kind of like trying to get like going and then Gaia she's been like really on top of stuff but I just felt like as a group like ... we should have ... kind of like went over stuff together and then she just like jumped in and like asked Hoffmann when we like I or at least I didn't feel like I was ready to like kind of ... ask the questions about that yet. Because I hadn't even really had the chance to look at it so like as far as like working as a group I was like a little like ... frustrated like I hadn't even had the chance to look at it and we were already going [*cough*s] going for it.

Nawal: [*pauses the video slightly before the moment when Gaia calls the instructor*] I felt kind of annoyed with Gaia when she did what was coming up. Cuz she called over the professor right away instead of like asking us the question she had and I thought it was kind of like not very like group like because I feel like if you have a question and you're sitting with your group you should like ask the rest of your group members first because like maybe someone gets it too or maybe they have the same question. So I remember feeling kind of like annoyed and I thought it was kinda rude how she like called him [Professor] over and asked him directly instead of like talking to us about it first.

While Izabelle seemed to not be affected by Gaia's calling the instructor, Nawal and Leila found Gaia's act annoying. However, Nawal and Leila narrated their frustrations differently. Leila was frustrated because she "hadn't even had the chance to look at" the problem and thus not yet ready to follow any conversation about it. Nawal felt insulted ("it was kinda rude") that Gaia did not consider her groupmates capable of answering her question, because Gaia "called and asked [Hoffmann] directly instead of like talking to [them] about it first." Three women who had associated with each other in the past two years and shared a Hispanic background had different reactions and understandings of the same act. The diversity of individualized discourses employed to make sense of group interactions would not have been revealed without SCNI interviews. Video analysis is not enough to study group interactions.

Case 2: My personal space

For the purposes of this paper, I will merely sketch the argument for cases 2 & 3 (the full set of data will be included in the extended paper, if the proposal is accepted). In the early unmediated interview (general questions about students' experiences with mathematics classrooms and small-group work), Leila (on 9/12) said that she needed her "personal space" and became irritated when groupmates, particularly Gaia, started writing on her notebook and got physically close to her. Indeed, on the group session of 9/22, Leila gave Izabelle, who moved her body close to Leila, a look from the side of her eyes and immediately Izabelle backed off and smiled. In her SCNI on 9/22, Izabelle commented, “Leila likes her own space.” Nonetheless, on 9/22 Gaia stood twice (for about 1 minute 30 seconds each time) over Leila’s shoulders and wrote on her notebook. Surprisingly Leila did not seem irritated by Gaia in the video.

In her SCNI interview on 9/22, Leila repeatedly commented that she was struggling to make sense of the mathematical conversation of the instructor with her group. She also mentioned that she was masking her lack of understanding by nodding every time the instructor looked at her. When the instructor left the group, Leila was totally confused about the task and what they were supposed to prove and how. But her groupmates started working on the problem without noticing her struggle, for which reason she was even more irritated. She ended up asking her groupmates a couple of questions, and it was Gaia who immediately volunteered to explain matters to Leila by standing over her shoulder and writing on Leila’s notebook. In her SCNI interview, she commented on this moment of group session.

Excerpt SCNI-Leila on 9/22. [*Leila pauses the video of group session at 28:10 when Gaia is standing behind Leila and explaining to her*].

Leila: […] at this point I was like understanding like when Gaia explained to me [*clears throat*] how you get from like our what we're trying to show like how you would factor out one part or whatever [*sniffs*] and then like basically like how we got to like what our conclusions were supposed to be and then this is where it clicked for me, I was like, Oh that makes sense now … So I was feeling good about myself. [*laughs*] for once.

[*The video is running at 28:50. Events in the video: Izabelle remarks that the difference is reversed; Gaia moves close to Izabelle to explain to her that* $if\ a-b\ \equiv 0\ then\ b-a\equiv 0$]

Leila: Yeah see Gaia is so teachery [*laughs*]

Facilitator: Ok, umm, why are you saying this?

Leila: Gaia is like very teachery like sometimes it's good like I like like in cases like this I like it but sometimes it's a little too much for me ... like it's a little sometimes I feel like it's a little overbearing. But like in this situation like the way that she like approached helping me and showing me like, Oh I was really appreciative of it because she like really helped me and explained it really well.

Leila did not seem irritated by Gaia’s invasion of her personal space as she was explaining the exercise to her in this group session (9/22). She backgrounded her sensitivity about her personal space because other contextual issues were more pressing at that moment. Leila was desperate to understand the mathematics discussed and to catch up with her groupmates’ work. Because Gaia’s explanation was helpful to Leila and thus responding to a contextual pressing need, the latter could tolerate Gaia’s intrusion into her personal space. This case cautions us about using data from unmediated interviews to explain ongoing interactions, which are subject to contextual forces that can qualify or compromise what people think about themselves and others in absolute terms.

Case 3: Two types of passive engagement

Boutros, a member of group G3, and Tito, a member of group G2, were only passively engaged in group-work for several sessions. Both of them sat in their group silently, listened to what groupmates were discussing, took notes, sometimes detached themselves from the group to do individual work, and almost never shared their mathematical thoughts with the group. They would engage sometimes in off-topic conversations with groupmates.

In their first SCNI interviews (Tito on 9/24 and Boutros on 10/15), Tito and Boutros commented on their respective group sessions in two distinct ways. Tito exhibited evidence that he was following the group discussion, even when he was not actively participating. He accurately commented on group activity using "we" (31 times) instead of "they" (5 times), despite his rarely participating in the group activity. On the contrary, Boutros spoke little about group activity, which he could rarely identify, using the pronouns "they" (7 times) and "we" (4 times). While Tito identified himself with the collective activity (e.g., "we solved the problem"), Boutros narrated a gap between his and his groupmates' actions. Following are two samples of Tito's and Boutros's comments.

Tito: Um, clearing it up. Cause we already finished it <u>we</u> were um, <u>we</u> concluded that Tom's part was right, so <u>we</u> just so everyone's writing it down.

Boutros: Uh, yeah I was looking at the problems on the worksheet and tryna figure out what they were working on. Yeah like I think I was listening to them a little bit, so I knew what problem <u>they</u> were doing […] It was the second one? So <u>they</u> finished the first one and then <u>they</u> started working on the second which, which was, which was can't remember right now. [*underlines are mine*]

I also coded the subjectifications and objectifications (as presented in Heyd-Metzuyanim & Sfard, 2012) in Tito's and Boutros's styles of narrations. Each one of them produced 100 subjectifying units, i.e., attributing social and socio-mathematical actions to agents. Note that for Tito, 31% of subjectifying units were attributed to the collectivity ("we"), compared to only 4% for Boutros. Boutros produced 20, as opposed to Tito having produced only 4, objectifying units.

According to the sfardian learning theory (Sfard & Prusak, 2005), Boutros is more prone to maximize his learning gain than Tito, because he is aware of a gap between his and his groupmates' actual actions. Although working in small-group, Tito may have fallen into the same illusion of students attending lectures, who misperceive the neat proofs laid out on the whiteboard by the instructor as representations of their own state of understanding and thus remain heedless of their actual state of knowledge. Future work will investigate this prediction and conjecture.

Conclusion

When working in small-groups, students draw on their social and socio-mathematical habits, which they internalize through their prior experiences in other or outside classrooms, to act and interpret groupmates' acts. As the field of small-group learning is moving toward investigating interactions and their underpinning power and socio-emotional forces, the SCNI technique is well suited to the task. Data from unmediated interviews may not be reliable for contextual analysis (case 2) and video analysis of group sessions may not be sufficiently informative (case1). The SCNI technique affords nuanced data in rapport to ongoing interactions, mainly due to participants' perspective *in situ*, and opens up the possibility for new analytical methods (for example, case 3) to enhance our understanding of small-group learning.

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Infinitesimals-based registers for reasoning with definite integrals

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Abstract: Two representation registers are described that support student reasoning with definite integral notation: adding up pieces (AUP) and multiplicatively-based summation (MBS). These registers were developed in a Calculus I class that used an informal infinitesimals approach, through which differentials like dx directly represent infinitesimal quantities rather than serving as notational finesses or vestiges. Student reasoning reveals how the AUP register supports modeling with integral notation and how the MBS register supports sense-making with and evaluation of integrals.

Keywords: calculus, integral, register, semiotics, infinitesimal, differential

My goal is to examine and illustrate two registers for interpreting and working with definite integral notation, registers that are particularly useful for supporting student modeling and sense-making with integrals. These registers—adding-up-pieces (AUP) and multiplicatively-based summation (MBS)—are situated in a Calculus I course that uses an "informal infinitesimal" approach to calculus. I briefly summarize this general approach to calculus before describing these registers and how students reason with them.

Informal Infinitesimals Approach to Calculus

For nearly two centuries, Calculus was "the infinitesimal calculus." For its inventors, G. W. Leibniz and Isaac Newton, it was a set of techniques for systematically comparing infinitesimal quantities in order to determine relationships between the finite quantities that they comprised (and vice versa). In the 19th century, calculus was reformulated in terms of limits rather than infinitesimals, and 20th century calculus textbooks have followed suit. Yet calculus textbooks and courses still use Leibniz' notation, dx and $\int$, but without the meanings Leibniz assigned to these: dx is an infinitesimal increment and the big S is a sum ("summa"). The notations are now vestiges, and in particular, differentials no longer directly represent quantities that students can manipulate and reason with.

The guiding principle of the informal infinitesimals approach is to restore this direct referential meaning to calculus notation. The approach is supported by the work in nonstandard analysis in the 1960s showing that calculus can be founded upon infinitesimals with equal rigor and power, but it uses the informality of Leibniz' reasoning rather than the formal development of the hyperreal numbers (e.g. Keisler, 1986). For instance, the derivative at a point dy/dx really *is* a ratio of two infinitesimal quantities, not code language for $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. The chain rule is canceling fractions. And an integral really is a sum of infinitesimal bits, each of which in MBS is given by the product $f(x)\cdot$dx. By taking differentials seriously, students can develop formulas for volumes of rotation, arclength, work, and many other ideas, formulas, and applications in first-year calculus (Dray & Manogue, 2010), and in vector calculus and physics (Dray & Manogue, 2003).

Registers and Signs

For Duval (2006), a representation *register* is a collection of signs and a set of transformations by which some of these signs can be substituted for others. Transformations within the same register are *treatments*; transformations of signs from one register to another are *conversions*. Barthes defines a *sign* as a combination of a signifier and a signified (1957/1972). For instance, a bunch of roses (signifier), together with the concept of passion it is representing (signified), comprise a sign. In our case, when a mathematical representation (e.g., "dx") signifies a concept (e.g., an infinitesimal increment), the combination of the representation "dx" (signifier) and infinitesimal increment (signified) is a sign. An interpretation is thus a signified concept. So if a representation stays the same but its interpretation changes, it becomes a different sign, since it signifies a different thing or concept. This points to a conversion to a different register, because within a given register interpretation should remain relatively stable. Such a conversion is often accompanied by a new lexicon of signs, interpretations, and treatments that which might support the new purpose or apply to the new context.

Consider the following example: We can use infinitesimals to develop the formula for the arclength of a curve in the plane between $x = 0$ and 1. We imagine a curve to be comprised of infinitesimal segments, each of which is the hypotenuse of a right triangle with legs dx and dy. Then the arclength of the curve would be the sum of these hypotenuse lengths: $\int_{x=0}^{1} \sqrt{dx^2 + dy^2}$. So far we have performed a modeling step, treating the dx and dy as quantities representing magnitudes, and have used the adding up pieces (AUP) register (which I detail soon). But this is no form to be evaluated for any particular curve, however. To be evaluated, the integral must first be *converted* by imagining all the dx's as uniform in size and then being factored from the integrand, to get $\int_{x=0}^{1} \sqrt{1 + (\frac{dy}{dx})^2}\, dx$ (if y is a function of x, say $g(x)$). Now the integral is of the form $\int_{x=0}^{1} f(x)\, dx$ (where $f(x)$ is $\sqrt{1 + g'(x)^2}$). So the integral can be evaluated by $F(1) - F(0)$, for some F as an antiderivative of f. Thus using algebraic manipulations we have converted to a new register that is suited for evaluating integrals. The $\sqrt{[\,]^2 + [\,]^2}$ structure lost significance and the $\int_a^b f(x)\, dx$ structure gained salience, and it became important for the original curve to be seen with y as a function of x.

The Adding Up Pieces (AUP) and Multiplicatively-Based Summation (MBS) Registers

The elements of the AUP and MBS registers are based on the work of Jones (2013, 2015a, 2015b). The interpretations involved in both registers are summarized in Figures 1 and 2, and the strange numbering on these (I2, etc.) draws from the learning progression in my class through which they emerge, which I reference elsewhere in detail (Ely, in review). The interpretations in the AUP register (Figure 1) entail the idea that a definite integral measures how much of some quantity A is accumulated over an interval of a domain, say from $t = a$ to b. This domain is partitioned into infinitely many infinitesimal increments of uniform size dt. For each infinitesimal increment dt there corresponds an infinitesimal increment of A, dA. The integral $\int_a^b$ adds all of these up to give the total accumulation of A over the interval from $t = a$ to b. In order for this to make sense, one must appeal to conception C5: The sum of infinitely many infinitesimal bits is a finite accumulation of A.

The AUP register allows one to transparently represent a general bit of the sought quantity and of the whole quantity as an accumulation of these bits. The treatments in the register include (a) writing a symbolic expression for a generic bit d*A* and (b) rewriting this expression for d*A* in terms of other infinitesimal quantities that specify the expression for the domain at hand, usually in terms of its corresponding domain increment d*t*. These treatments require viewing infinitesimals as legitimate quantities that behave normally under algebraic operations (including possibly some additional Leibnizian rules for operating with infinitesimal quantities). The treatments also rely on two other grounding conceptions: (1) equivalent expressions can be substituted for the same quantity (conception C6), and (2) a foundational understanding of covariation, which in turn relies on the basic understanding that variables vary (Thompson & Carlson, in press). The student must be able to coordinate changes of *A* with changes of *t* in order to reason that for each increment d*t* there is a corresponding increment d*A*.

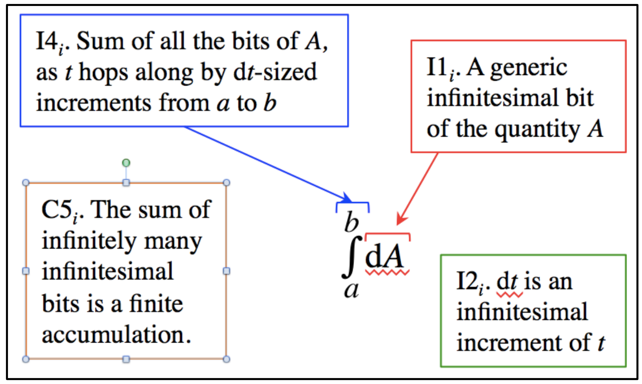


Figure 1 – Interpretive elements of integral notation in the adding up pieces (AUP) register

The MBS register includes many of the same notational interpretations as AUP, but it also adds to these the expression of each piece d*A* as a product $r(t)\cdot \mathrm{d}t$. This introduces the integrand, which is necessary for the Fundamental Theorem of Calculus (FTC) to apply. We follow Thompson, Byerley, & Hatfield's (2013) approach to treat the integrand $r(t)$ as a rate at which *A* accumulates over the increment d*t*. This ultimately allows us to recognize and use that an accumulation function *f* for *A* will have $r(t)$ as its rate-of-change function. Thus, one supporting conception in this interpretation of the product $r(t)\cdot \mathrm{d}t$ is the idea that *A* accumulates at a constant rate $r(t)$ over the d*t* increment, and that this constant rate is determined by the value of *t* closest to that d*t* increment. This relies on conception C4: A concept image for rate of change at a moment (Thompson, Ashbrook, & Musgrave, 2015). This momentary rate of change can vary constantly as *t* varies. The second supporting conception is the multiplicative structure associated with rate, expressed in conception $C1_i$ (see Figure 2). The

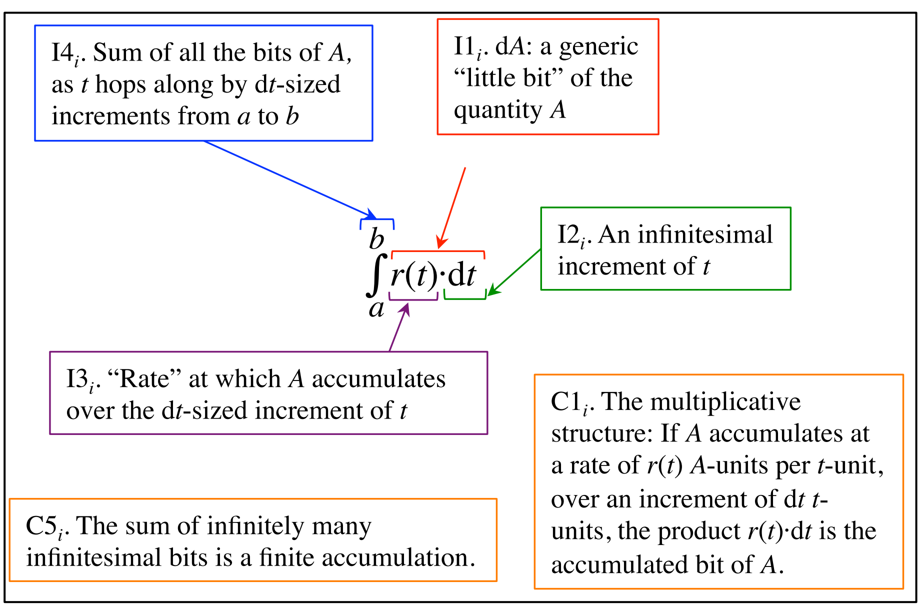


Figure 2 – Interpretive elements of integral notation in the multiplicatively-based summation (MBS) register

word “rate” is used here broadly. It does not necessarily mean that $r(t)$ is measured in a compound unit like miles-per-hour. Rather it means that as t changes, A changes by a proportional amount, and $r(t)$ provides that rate of proportionality (Lobato & Ellis, 2010). Most broadly, $r(t)$ serves as a factor allowing conversion from an increment of t to an increment of A, by $dA = r(t) \cdot dt$. Reasoning with quantities, rather than bare numbers or symbols, is crucial to this interpretation.

The MBS register promotes reasoning with the FTC: if we can find any accumulation function $f(t)$ whose rate-of-change function (i.e. derivative) is this $r(t)$, we can use it to recover the accumulated amount A, by determining $f(b) - f(a)$. The treatments within the MBS register include the same kinds of algebraic operations with summand and integrand as in the AUP register, and also include evaluating the integral $f(b) - f(a)$ by means of an antiderivative f.

Other Modes of Student Reasoning with Definite Integrals

Although we focus on AUP and MBS, these modes of reasoning are relatively rare among students in traditional calculus courses. For instance, Jones (2016) surveyed 150 undergraduate students who had completed first-semester calculus, using Prompts 1 and 2 on the next page. Only 22% of students made even a passing reference to summation of any kind on either prompt, and on each prompt less than 7% appealed to reasoning consistent with AUP or MBS. On the other hand, 87.3% of students appealed to an “area” interpretation on Prompt 1, and 76% used an “anti-derivative” interpretation on Prompt 2.

These two interpretations have also been described, and found prevalent, by other researchers. The *area* interpretation is that the definite integral represents an area “under” a curve in the coordinate plane, with the “d[]” denoting the variable on the horizontal axis, which forms the bottom of the shape. “The shape is taken as a fixed, undivided whole that is not partitioned into smaller pieces” (Jones, 2015b, p. 156). The *anti-derivative* interpretation is that the integrand came from some other other “original function” through differentiation; now the integral symbol represents an instruction to find this original function. The d[] dictates the independent variable “with respect to” which the derivative had been taken, and the limits of integration are the values that one must plug into the original function to get the numerical answer (Jones, 2015). Fisher et al. (2016) found that the majority of students in a standard calculus class used only the area interpretation when describing the meaning of a definite integral, and Grundmeier, Hansen, & Sousa (2006) found that only 10% of students mentioned an infinite sum when asked to define a definite integral.

Various studies claim that sum-based interpretations of the definite integral are much more productive in general for supporting student reasoning than are area and anti-derivative interpretations (e.g., Sealey, 2006, 2014; Sealey & Oehrtman, 2005, 2007; Thompson & Silverman, 2008, Jones 2013, 2015a, 2015b; Jones & Dorko 2015; Wagner 2016). For modeling in particular, the area and anti-derivative interpretations have serious limitations. The area interpretation is problematic when modeling in the myriad situations when the sought quantity is difficult to imagine as the area of a region (e.g., work, velocity, force, volume, arclength) (Thompson et al, 2013; Jones 2015a). The anti-derivative interpretation provides even less support for modeling, since it gives only a technique for evaluating a definite integral, not for creating one (Jones 2015a). These interpretations produce significant obstacles for students modeling with integrals in physics applications (e.g., Nguyen & Rebello, 2011).

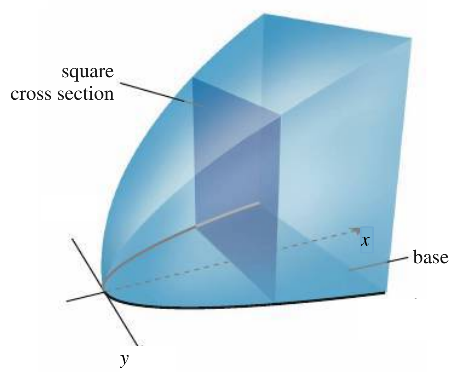
Along with AUP and MBS, there are other sum-based interpretations of integral notation, notably the Riemann sum (limit of sums). Nearly all calculus books define the definite integral using Riemann sums, but this fact seems to contribute little to building sum-based reasoning for the students who use these books. When investigating this apparent pedagogical disconnect, Jones, Lim, and Chandler (2016) found that instructors' teaching moves lead students to perceive the limit of Riemann sums not as a conceptual basis for understanding the definite integral, but merely as a calculational procedure that allows an integral to be estimated accurately. Another way that the limit process involved in the Riemann sum interpretation can form a conceptual obstacle for students is through the problematic collapse metaphor, through which students imagine the pieces losing a dimension in the limit, so the d[] loses its quantitative meaning (Oehrtman, 2009).

Data Collection

I taught an experimental Calculus I class using the informal infinitesimals approach, for science, engineering, and math majors at a large public university in the northwestern U.S. I conducted semi-structured interviews and analyzed student written work. I focus here on two prompts I used in the interviews: Prompt 1, which is verbatim from Jones (2013, 2016), and Prompt 3, a novel modeling context of a kind very different from what the students seen before (although they had done a few volume-of-rotation problems in class and on homework).

Prompt 1: Explain in detail what $\int_a^b f(x)dx$ means. If you think of more than one way to describe it, please describe it in multiple ways. Please use words, or draw pictures, or write formulas, or anything else you want to explain what it means.

Prompt 3: Set up an integral that represents the volume of this solid, whose base is the region bounded by the curves $y=\sqrt{x}$ and $y=-\sqrt{x}$, and whose cross sections perpendicular to the base and perpendicular to the x-axis are squares.



Results

I analyze here the reasoning displayed in the responses of two students, Dmitri and Galena.

Reasoning in the AUP Register

In response to Prompt 3, Dmitri's initial answer of ($\int_{x=0}^{1}(2y \cdot dx)^3$) was incorrect, but after reflecting for a minute he corrected it to $\int_{x=0}^{1}(2y)^2 \cdot dx$. He notes that if the slice was $(2y\mathrm{d}x)^3$, it would make a perfect cube, which can't be right. It should instead look like what is in Figure 4. He narrates as he draws and labels the slice: "This [indicates the width] is going to be dx. This will be the same thing as the other one: this one [indicates the slice's height dimension] is still 2y. This one's still 2y [indicates the slice's depth dimension]. But the width is still dx. So to find the volume of that, we'd have 2y squared times dx. And that is all. And that solves my problem." He then describes how the slices are aggregated, each time "you'd go up an infinitely small amount and then you'd do the same thing for that one, and you'd do that for all numbers between 0 and 1." Then he notes that the collection of all these pieces is the volume of the whole figure.

In this sequence of reasoning, Dmitri appeals to all the elements of AUP. He has imagined a domain partitioned into increments of infinitesimal size dx (I2), described a representative slice of the figure's total volume as the thing being summed (I1), and described the integral as the sum of all such pieces across the appropriate domain (I4). He notes that these have infinitesimal volume but that when you sum them all you get the whole region, which indicates he is using C5.

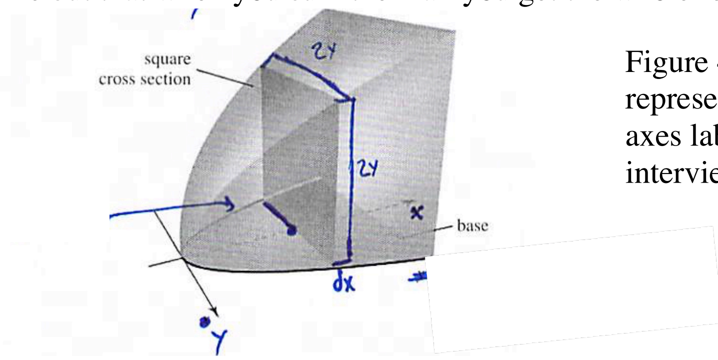


Figure 4 -- Dmitri's modeling of a representative piece of volume. The axes labels were written by the interviewer.

Dmitri checks his answer by appealing to dimensional quantities and units: even if dx is a "really small amount of meters, it's still meters – so meters squared times meters equals meters cubed, and that's the unit of volume." Dmitri uses AUP, not MBS; he never seems to need the summand to be in the form "$f(x)\cdot$dx." His initial answer is not at all in that form, and his final answer still does not have the integrand written as a function of x. Additionally, he appeals to multiplicative structure when he talks about the summand, he does not describe or treat the area part, the integrand, as a "rate" at which the figure's volume grows with each infinitesimal increment of the domain.

Reasoning in the MBS Register

In response to Prompt 1, to explain what $\int_a^b f(x)dx$ means, both Galena and Dmitri express notational interpretations I1-I4 and conceptions $C1_i$ and $C5_i$. Galena's succinct response is shown here, and I indicate how it displays I1-I4. There is no textbook with MBS in it yet, but if there was, Galena's account would be the "textbook" description of MBS:

: Okay, so, I'll just separate this into chunks:

So the dx is gonna be a small increment in time, like ideally it would be infinitely small. Um, so this [gestures to the "dx"] is a chunk of time, or whatever is on your x-axis. It doesn't necessarily have to be time; it could be meters if you were doing it in length. But it's a small increment of whatever x is. $I2_i$

Then $f(x)$ is the rate at which that grows over this [points to the dx] chunk of time, per se. And then, so this is a rate [points to the $f(x)$]. So it would be like meters per second, or whatever this x value is per whatever is the y value [she says these reversed but writes them correctly]. $I3_i$

Uh, and this [points to the entire integral] is making this a summation of these chunks. So this [points to $f(x)$dx] is going to be a chunk. ~~And this is~~ $I1_i$ the summation from a to b of those tiny chunks that you're adding up along the way. $I4_i$

Dmitri's responses to Prompt 1 also illustrate his reasoning using I1-I4, C1 and C5, and he explicitly refers to $f(x)$ as a "rate of change function; … let's say it's meters per second, then dx could be a really small increment, an infinitesimally small increment, of seconds."

Converting From the AUP Register to the MBS Register

An example of Galena's written work illustrates the process of converting between the two registers. The problem asks her to first set up, then evaluate, an integral representing the volume of the figure created by rotating around the x-axis the region enclosed by the curves $y = x + 6$ and $y = x^2$. Galena makes a couple of small mistakes in her work, but the switch between registers is clear. Her modeling work, to set up the integral, is shown in Figure 5.

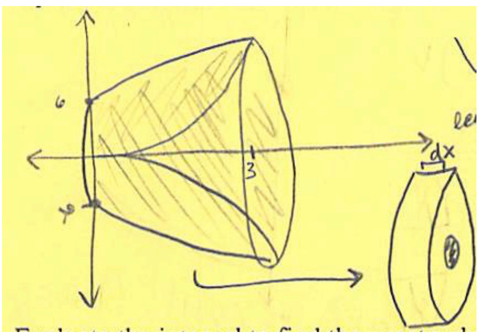
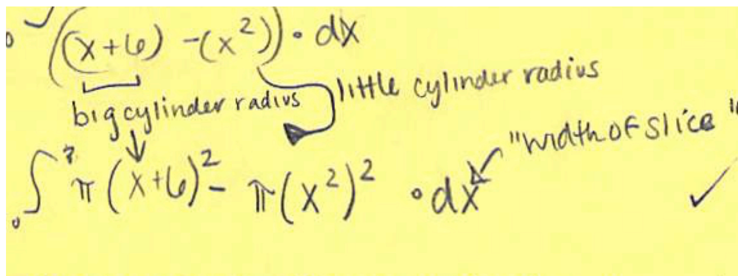


Figure 5—Galena's modeling in the AUP register



To evaluate the integral, Galena converts to interpreting the integrand $[\pi(x+6)^2 - \pi(x^2)^2]$ not as two dimensions of a slice but as a "rate of change," (she writes this). She then seeks to evaluate by finding an "accumulation function" evaluated at the starting and ending values of 0 and 3. The interpretation of the integrand has changed with the register shift, so she no longer refers to elements of the figure.

Discussion

These are a few illustrative examples of how students in an informal infinitesimals calculus course used the AUP register to model with definite integrals and the MBS register to reason with and evaluate definite integrals, and how there is an explicit change of interpretation marking the conversion between the registers. Since the two registers support these distinct purposes, it may support student learning for the instructor to teach the registers independently and to be explicit about the interpretations and affordances in each register. By explicitly teaching students the signs (notations and interpretations), treatments, and purposes of the two registers, we may help them develop meta-level awareness of the significance and affordance of their actions in the registers. An informal infinitesimals approach to calculus can help students develop these two registers, which are more powerful tools for reasoning with definite integral notation than the prevalent antiderivative and area interpretations.

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Students' Attitudes Toward Listing and Subsequent Behavior Solving Counting Problems

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Counting problems provide rich mathematical content and a variety of applications for students, motivating investigation into the difficulties students face while counting. In particular, an important result supported by previous quantitative and qualitative evidence is that listing may be an effective strategy for combating some student struggles in counting, particularly since it draws explicit attention to outcome structure. However, anecdotal experience has shown that students can resist listing and feel that it is tedious and not worth the effort. To investigate whether these negative mindsets exist outside these anecdotes, task-based interviews were conducted targeting student attitudes toward listing and their success in using listing to solve counting problems. Contrary to the anecdotal evidence, the students in the study expressed that they felt listing is a worthwhile activity, but their work on counting problems suggest that they would benefit from more explicit support relating to listing in their discrete mathematics classes.

Key Words: Combinatorics, Discrete Mathematics, Counting Problems, Mathematics Education

Introduction

Counting, or combinatorial enumeration, is an important part of students' mathematical curricula for a variety of reasons. Counting problems admit relevant applications in multiple fields, provide opportunities for students to engage in meaningful mathematical practices, and give contexts to help students understand difficult mathematical topics (Kapur, 1970). However, it has been shown that students at a variety of levels struggle to solve counting problems (Melusova & Vidermanova, 2015; Eizenberg & Zaslavsky, 2004; Batanero, et. al., 1997). While student difficulties persist, recent studies have found promise in a strategy known as listing: explicitly writing down one or more of the things you are trying to count while solving a counting problem (Lockwood & Gibson, 2016).

However, despite the quantitative and qualitative results providing evidence for the usefulness of listing, anecdotally it has been found that some students feel that listing is tedious and not worth the effort, preferring instead to look for key words in problem statements and determine an appropriate formula. Students in discrete mathematics classes where they were explicitly encouraged to list have been overheard saying things such as, "But there are so many!" and "I feel like I'm not learning when I list all of the outcomes. I learn better when I'm given the theorems to use, not doing something tedious." Seeing student resistance to listing despite evidence for its utility prompted investigation into the following research questions:

1. *What are undergraduate students' attitudes towards listing as a strategy for solving counting problems?*
2. *How willing are undergraduate students to use listing as a strategy to solve counting problems, and how successful are they are?*

Literature Review and Theoretical Perspectives

Previous research has shown substantial need to better understand how to help students overcome difficulties they face solving counting problems. Specific struggles that earlier studies have found include that student errors can persist even when given combinatorial instruction (Batanero et. al., 1997); intuition on how to approach problems can be misleading (Fischbein & Grossman, 1997, p. 35); students lack efficient verification strategies (Eisenberg & Zaslavsky, 2004); and students get confused on the types of objects being counted (Batanero et. al., 1997).

To help students overcome difficulties while counting, Lockwood (2013, 2014) argues that attention to outcomes—the objects which are enumerated when solving a counting problem—should play a significant role in teaching students to count. Listing therefore can be a useful tool, as it provides students with a concrete way of attending to the outcomes of a counting problem and tying them explicitly to the counting process used to solve the problem. Indeed, Lockwood and Gibson (2016) found quantitative evidence for listing as a potentially useful strategy for students. They also discovered the following features of student lists that seemed most helpful while counting: useful notation and appropriate modeling of outcomes, organized strategy, and evident structure. Useful notation and appropriate modeling of outcomes refer to the way the student encodes each outcome, and whether the encoding correctly articulates what constitutes a desirable outcome. Organized strategy pertains to the order in which these outcomes are written down by the student, which could include the Odometer Strategy (English, p. 458-61, 1991) in contrast to random outcome generation. Finally a student list is said to contain evident structure if its organization reflects the counting process that the student uses. Lists with evident structure make it easier for a student to justify that a particular process counts every outcome once and only once.

In this study, I follow Lockwood (2013, 2014) in viewing outcomes as an intrinsic part of combinatorial enumeration. Since attention to outcomes has been supported by theoretical perspectives (Lockwood, 2013; Lockwood, 2014) and shown to be useful from quantitative and qualitative evidence (Lockwood & Gibson, 2016), finding out how students might feel about listing is an important addition to existing combinatorics education literature. I frame the data analysis and results using Lockwood and Gibson's (2016) features of productive listing.

Methods

To study student attitudes toward listing and success in using it to solve counting problems, individual task-based interviews were conducted and video-recorded with 9 discrete mathematics students at a large university in the western United States. Each student solved 6 counting problems, which were aimed at eliciting listing since their outcome sets have small cardinality or contain outcomes that are easily encoded:

1. *Chocolate Truffles. A chocolate store sells five varieties of chocolate truffles: raspberry, sea-salt caramel, mocha fudge, dark chocolate, and white chocolate. How many ways are there to select two truffles (of possibly the same variety) to buy?*
2. *Committee. A department committee consists of four people. If there are seven senior faculty members, how many committees are there if Bob (one of the senior faculty members) must be on the committee?*
3. *Letter Arrangements. How many ways are there to arrange the letters A, B, C, D, Z, Z, if the two Z's must be together at the beginning or end of the arrangement?*
4. *Binary Strings. How many strings of 1's and 0's are there of length 8?*
5. *Door. How many ways are there to arrange the letters in the word DOOR?*
6. *Olympic Swimming. There are 10 swimmers in the women's 200-m fly final race at the Rio Olympics. How many ways are there to award the gold, silver, and bronze medals?*

While the sample size of this study is small, the interview design allowed for more insight into student attitudes about listing, since I had the opportunity to follow up and question students about their solutions. After the students solved the counting problems, they were given 15 Likert-response statements (responses could range from 1-strongly disagree to 5-strongly agree) which measured constructs related to their attitudes toward listing as a strategy for counting.

After data collection, each solution was coded as being correct or incorrect and whether a listing strategy was used or not. In addition, each list was coded as being *productive* or *unproductive*, where a productive list is one where the student arrived at the correct answer and was able to articulate a relationship between their answer and their list. This is a modification of Lockwood and Gibson's (2016) definition of productive listing. In addition, lists were examined to see if they contained the three features of productive listing found by Lockwood and Gibson (2016). Finally, descriptive statistics for each Likert statement were calculated.

Results

Responses to Likert statements

The Likert statements were aimed at answering several related questions about student attitudes toward listing. For brevity, I will focus on the following questions since they are most relevant to the research questions above: 1) do students think listing is a useful strategy in general, 2) do they find listing to be a worthwhile activity, and 3) what did they report about their teachers' listing activity. Questions 1) and 2) express slightly different constructs in that 1) looks into how students think about listing in general, while 2) gets at whether students report that they themselves find listing to be valuable. This allowed for the possibility that some students might think listing is a useful strategy for other or less-experienced counters, but not for themselves personally.

To investigate question 1), I looked at student responses to the following Likert items: item 1, "When solving counting problems, it is often useful to write down some outcomes;" item 6, "Before solving a counting problem, it's important to articulate explicitly what you're trying to count;" and item 14, "Creating a list of outcomes is not necessary for solving counting problems." In contrast to the anecdote previously mentioned, the students in the study actually expressed broad agreement with statements 1 and 6, which had means of 4.11 and 4.66, respectively. The mean response to statement 14 was lower at 3.44, but most of those responding clarified that they felt listing is useful; they just did not feel it was always necessary for solving every counting problem.

Similarly, student responses indicated that they also viewed listing as a strategy they personally thought was worthwhile. The statements which measured this attitude were Likert item 7, "I dislike writing outcomes when solving a counting problem, because I usually know the right formula to solve the problem right away;" item 10, "Creating a list of what I'm trying to count is tedious and not worth the effort;" and item 11, "Writing outcomes is a useful activity for me when solving counting problems." Items 7 and 10 expressed a negative attitude about listing being personally worthwhile, and so I expected students would agree with them. However, the means of these responses were 2.44 and 2.67, respectively, indicating that students largely disagreed with these particular statements. Likewise, statement 11 had a mean response of 3.67, indicating that students at least did not disagree with the notion that writing down outcomes is useful for them personally.

Overall, the answers to all six of the Likert items above went contrary to expectations. Despite the resistance to listing that I had observed anecdotally from students in the classroom, the results from the Likert-response items indicated that there may be more to their resistance than simply finding listing to be useless and unnecessary. Thus, I needed to look further to see what else besides negative attitudes might contribute to resistance to listing.

A partial answer to this was found in the responses that students gave to Likert item 5, which reads, "My teacher mentioned outcomes but never encouraged us to list." The mean of the responses to this question was 2.78, indicating that students largely neither agreed nor disagreed with this statement. A limitation of this statement as a stand-alone Likert item was that it is a compound statement, meaning that by only looking at the mean response it is difficult to determine which part of the item they agreed or disagreed with. However, students could clarify their answers in the interview as they went through the Likert items, and some of the student utterances indicated that they felt more support could be given to them in terms of productive listing.

For example, Student 1 gave this statement a 2, which meant that he disagreed, but his explanation showed that he disagreed in particular with the idea that his teacher mentioned outcomes while teaching him to count. He said, "[My teachers] encourage formulas, rather than listing, I guess." Student 8 expressed a similar sentiment when he gave the statement a 4. He said,

> *Student 8: "Yep, [my teacher] really doesn't. Like, maybe for the digit problem [my teacher will] list it out, but for all the others, [my teacher will] just use a formula, which is bad, I think, because most people if they don't--if they haven't touched like possibilities and that kind of stuff they don't know which one to use. They'll often get confused."*

In this context when he said, "possibilities," he was referring to outcomes, meaning that he felt it could be more helpful to him and his peers to see his teacher write down outcomes more often so that they would know which formula to use while solving counting problems.

Noting the small sample size of this study, I acknowledge that I cannot make any broad claims or generalizations from these students' responses. However, it was surprising and is worthwhile to know that these students overall seemed to have positive attitudes about listing, and some of their responses to Likert item 5 support the idea that some students might resist listing because they lack instructional support to know how to list productively. The results from the Likert items thus provide some insight into the first research question, which targets students' attitudes toward listing.

Work on Counting Problems

The students in each interview also answered a sequence of 6 counting problems, all of which were written to hopefully elicit listing as a strategy. Indeed, out of the 53 total solutions obtained, 28 of them included listing as a strategy. Since this is over half of the solutions, their work provides further evidence that the students in the study felt that listing is a worthwhile strategy. Interestingly though, when I looked at these students' listing behavior, I saw that their ability to list did not necessarily reflect the positive outlook they had on listing. Out of those 28 lists that the students used, only 17 of them were productive (which, again, meant the students were able to reach the correct answer and relate the list to the solution obtained). In addition, the percentage of correct solutions found using any strategy (69.81%) was slightly higher than the percentage of correct solutions when the domain (of all student solutions) is restricted only to those found using a listing strategy (67.74%). As noted in the literature above, Lockwood and Gibson (2016) had previously found a positive correlation between listing and solving counting problems correctly, so this prompted me to investigate why the students in my study seemed not to be as successful on problems in which they listed.

Looking at instances where listing was productive for students, a characteristic example of their work is Student 2's solution to the Committee Problem. Her list can be seen in Figure 1. Using her exhaustive list, she was able to obtain a correct solution of 10+6+3+1=20. Examining her list, I see that it exhibited all three characteristics of productive listing found by Lockwood and Gibson (2016). She had useful notation and appropriate modeling of outcomes, encoding the outcomes using the items 1, 2, 3, 4, 5, and 6 and writing unordered groups of three of these numbers. Her strategy for writing down outcomes was organized, making use of the Odometer

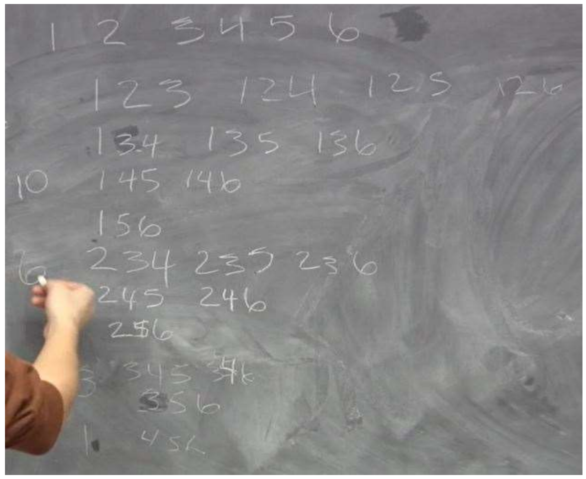

Figure 1. Student 2’s work on the Committee Problem.

Strategy (English, 1991), and her list also had evident structure, in that she was able to draw a clear connection between the list she obtained and the solution 10+6+3+1. In addition, while her productive list was exhaustive, I note that there were several productive lists in the study that were only partial lists of outcomes as well.

However, 11 of the 28 lists that students made in this study were unproductive, either leading students to incorrect solutions or, in one case, failing to contribute to a correct solution. For example, four out of the nine students made an incomplete list while solving the Chocolate Truffles Problem, which led them to the incorrect solution of 5^2=25. A characteristic example is Student 1’s list, which can be seen in Figure 2. In making this list, I see that he listed all of the outcomes in which a raspberry truffle is selected, and then extrapolated an observed pattern he saw in the outcomes: that there are 5 outcomes when one truffle type is fixed. He then reasoned that there are 5 total truffle types, so the solution must be 5×5=25. In this example, I see that he inappropriately modeled the outcomes of this problem using ordered pairs rather than unordered selections of truffles. While it is certainly possible that he would have reached this incorrect solution without the list he made, having a partial list of ordered pairs reflecting the multiplication 5×5 certainly didn’t help as he solved the problem.

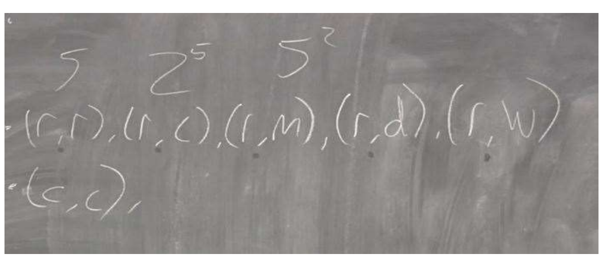

Figure 2. Student 1’s work on the Chocolate Truffles Problem.

While inappropriate modeling of outcomes was present in student work, other features of productive listing identified by Lockwood and Gibson (2016) were missing in the students’ unproductive lists as well. For instance, Student 4 used an unproductive list while solving the

Letter Arrangements problem. While making his list, he tried to write down all of the outcomes beginning with two Z’s and an A, appearing to be trying to use the Odometer strategy (English, 1991). However, he arranged the letters B, C, and D at random and was only able to come up with four outcomes beginning with ZZA, concluding from his partial list that the solution was 2×4×4=32 total arrangements. While, again, it is possible he would have obtained this solution without making a list, the disorganized strategy stands in contrast with the kind of listing shown to be productive in Lockwood and Gibson (2016).

Since the students’ unproductive lists in this study were often missing one or more of Lockwood and Gibson’s (2016) features of productive listing, this suggests that perhaps instructors could provide more support for students learning to solve counting problems by discussing how to create lists with the three features. It appears from these results that even if students are willing and able to list, they will not always be successful in generating productive lists.

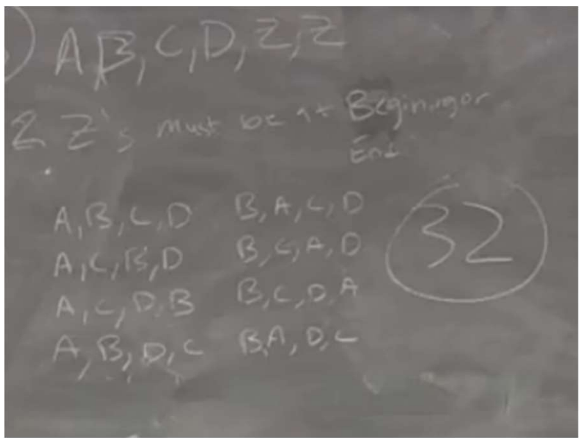

Figure 3. Student 4’s work on the Letter Arrangements Problem.

A second interesting observation about the students’ unproductive listing behavior occurred when the lists were examined by problem. Three of the problems in this study (the Letter Arrangements Problem, Binary Strings Problem, and Olympic Swimming Problem) can be solved by simply applying the Multiplication Principle, while the other three problems are likely to be

Table 1
Descriptive Statistics on Solution Accuracy with and without Listing for Each Problem.

Problem	Percentage accuracy of solutions using any strategy	Percentage accuracy of solutions using a listing strategy
Chocolate Truffles	33.33	42.86
Committee	37.50	40.00
Letter Arrangements	88.89	75.00
Binary Strings	88.89	80.00
Door	66.67	85.71
Olympic Swimming	100.00	100.00

more challenging or novel to students. The Chocolate Truffles Problem requires a multi-choose or a sum to solve it; the Committee Problem can be solved using an unordered selection; and the Door Problem can be solved using division. I found that while overall listing had no correlation with obtaining a correct solution, a closer look showed that listing appeared to have no effect or was a hindrance mostly on the problems requiring just the Multiplication Principle. For the three more challenging problems, the percentage of correct solutions when a listing strategy was used was higher than the percentage of correct solutions where any strategy was used. While there still were several unproductive lists for these three problems, and the percentage of accurate solutions for these more challenging problems was still low, these results support the conclusion that perhaps listing could be more beneficial for students solving more novel or challenging problems. These results are summarized in Table 1. While I cannot read too much into these numbers because of the small sample size, the findings at least suggest that there may be some effect of problem type on students' listing behavior and the extent to which listing is useful for students.

Overall for the students in this study, listing appeared to be most helpful on problems that were more novel or challenging, rather than simply requiring use of the Multiplication Principle. This may be in alignment with findings from Lockwood and Gibson (2016), in which the students were novices and were faced with some more challenging problems than were the students in my study. However, even on these more challenging problems, listing was not always helpful, and the students' willingness to list was not reflected in a need they may have had for more support in knowing how to list productively.

Conclusion

The aim of the study was to investigate students' attitudes toward listing, and to see if my findings would corroborate anecdotal evidence of student resistance to listing. Contrary to anecdotal experience, the evidence from this small-scale study suggests that students may agree that listing is useful and worthwhile, and that some instances of student resistance to listing actually may stem from not knowing how to list productively. Data from the Likert responses suggest that some students wish their teachers modeled productive listing and taught them explicitly how to list. I saw also from their activity while solving counting problems that they personally were willing to use listing as a strategy, but their positive attitude about listing was not reflected in the high percentage of unproductive lists that were created. I additionally observed that effectiveness of listing as a strategy may depend to some extent on the problem type and its degree of novelty.

Since student resistance to listing may result in part from a lack of instructional support rather than simply negative attitudes, a natural direction moving forward would be investigating how instructors can help model and teach productive listing for students in their discrete mathematics classes. In addition, a larger-scale study should be conducted to determine if the positive attitudes about listing I encountered persist when examining a larger sample. Finally, future studies should seek a more sophisticated understanding of the relationship between the utility of listing and problem type.

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Tinker Bell's Pixie Dust: Exploring the Differentiations Necessary to Engage in Emergent Shape Thinking

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Researchers have described the importance of seeing a graph as an emergent trace of how two quantities' values vary simultaneously. Researchers have also identified the many difficulties students face when constructing this conceptualization of graphs. In this paper I explore the role of two didactic objects on a student's conceptualization of graphs. In particular, I examine how a student's interactions with these didactic objects supported her in making key differentiations that enabled her to conceptualize a graph as emerging from simultaneously tracking two quantities' varying values. My findings revealed that a student must differentiate a place on a function's graph from the value of the function's output. Also, the student must distinguish tracking a point in the plane from creating the point by simultaneously attending to the variation of two quantities.

Keywords: Covariational Reasoning, Emergent Shape Thinking, Graphing

Students' conceptualizations of graphs remains a prominent area of study in both mathematics and science education. Researchers have documented how students' impoverished conceptualizations of graphs inhibit them from constructing productive meanings of function and rate of change (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2015; McDermott, Rosenquist, & van Zee, 1987; Oehrtman, Carlson, & Thompson, 2008). While researchers have highlighted ways of thinking that either support or inhibit students from developing propitious conceptualizations of graphs (e.g., Moore, Paoletti, Stevens, & Hobson, 2016; Whitmire, 2014), they have not documented how to support students who conceptualize graphs as static shapes in shifting to see them as emergent representations of the simultaneous variation of two quantities' values. In this paper I characterize the constructions and differentiations necessary to see a graph as representing the simultaneous variation of two quantities' values.

Background

Moore and Thompson (2015) describe two ways of thinking students hold for graphs: *static shape thinking* and *emergent shape thinking*. A student engages in *static shape thinking* when he conceptualizes graphs as objects in and of themselves. This student is likely to reason about a graph based on his perception of the shape. For example, a student who engages in static shape thinking might understand slope as the property of the line that determines whether the line falls or rises as it goes from left to right. This way of thinking is consistent with Bell and Janvier (1981) and Carlson's (1998) findings that students often reason about graphs based on their perception of the shape of the curve and confound pictorial attributes of a situation with the shape of the graph.

An alternative way of thinking about graphs is what Moore and Thompson (2015) called *emergent shape thinking*. They explained,

> "Emergent shape thinking involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation). As opposed to assimilating a graph as a static object,

emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities" (p. 4).

Moore and Thompson explained that students who think about graphs emergently are positioned to reflect on their reasoning and thus have the opportunity to form abstractions and generalizations from their reasoning. As a result, relationships students construct from these generalizations are not tied to particular shapes, labels, and orientations.

As Saldanha and Thompson (1998) explained, "understanding graphs as representing a continuum of states of covarying quantities is nontrivial and should not be taken for granted" (p. 303). To engage in emergent shape thinking the student must first conceptualize two attributes, call them x and y, and imagine representing the varying measures of these attributes along the horizontal and vertical axes, respectively. Then she needs to imagine projecting these measures into the plane and conceptualize the intersection of these projections, the correspondence point, as a way to unite the measures of x and y (Figure 1). When one constructs an object that unites two attributes and the relation between these attributes she has constructed what Saldanha and Thompson (1998) call a multiplicative object. Conceptualizing a point in the Cartesian plane as a multiplicative object that unites values of x and y is essential when engaging in emergent shape thinking because, as Saldanha and Thompson explained, by constructing a multiplicative object, when one tracks the value of x (or y) she is constantly aware that the other quantity also has a value. As a result, as the student tracks the correspondence point, he is simultaneously attending to both the value of x and the value of y.

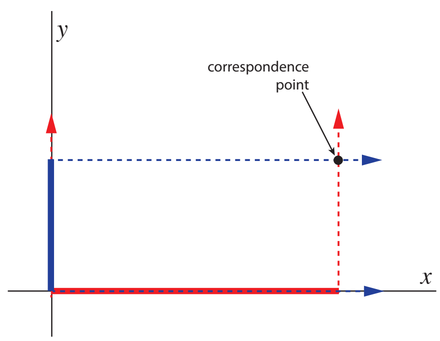

Figure 1: Conceptualizing a point in the Cartesian plane as a projection of two quantities' values, which are represented on the axes.

Methodology

I conducted one-on-one teaching experiments (Steffe & Thompson, 2000) with three university precalculus students (1 male STEM major, 1 female STEM major, 1 female liberal arts major). These students were selected to participate in the teaching experiments because they demonstrated different ways of engaging in covariational reasoning in a recruitment interview. My primary teaching goal was to support students in engaging in emergent shape thinking when constructing and making meaning of graphs. After I completed the interview process I engaged in retrospective analysis by identifying instances that provided insights into the students' conceptualizations of quantities, points in the Cartesian coordinate system, and covariation of quantities' values. I used these instances to generate tentative models of each student's schemes for graphing and covariational reasoning. I tested these models by searching for instances that confirmed or contradicted my model and repeatedly refined my model until it accounted for the student's mathematical activity.

Of the three students who participated in my study, only one student, Ali, came to consistently engage in emergent shape thinking. At the time of the teaching experiment Ali had just completed her freshman year at a large public university and had earned a B in her precalculus course. As a liberal arts major, this precalculus course satisfied Ali's math requirement and she did not intend to take calculus. In the follow sections I describe my initial model of Ali's schemes for graphing and covariational reasoning. Then I describe two didactic objects the research team used to support Ali in engaging in emergent shape thinking. I conclude by hypothesizing why these interventions supported Ali in constructing new meanings for points and graphs in the Cartesian coordinate system.

Results

In the second of four one-on-one teaching sessions it was apparent that Ali was not conceptualizing a point in the Cartesian coordinate system as a multiplicative object and thus was not engaging in emergent shape thinking. She was working on a task adapted from an instrument used to assess secondary teachers' mathematical meanings (Thompson, 2016). This animated item was originally designed to support researchers in better understanding in-service secondary mathematics teachers' schemes for covariational reasoning (Thompson & Carlson, in press). For the purpose of my study, I used this task to help me understand the nature of the multiplicative object Ali constructed when engaged in covariational reasoning.

I showed Ali a video that depicted a red bar along the horizontal axis and a blue bar along the vertical axis. As the video played, the lengths of the bars varied simultaneously in such a way that each bar had one end fixed at the origin. (See Figure 2 for selected screenshots from the video). The horizontal (red) bar's unfixed end varied at a steady pace from left to right while the vertical (blue) bar's unfixed end varied unsystematically. I explained to Ali that the length of the red bar represented the varying value of *u* and the length of the blue bar represented the varying value of *v*. I presented Ali with a printout that included the animation's initial screen depicting a set of axes and the initial placement and lengths of the red and blue bars. I asked Ali to sketch a graph of the value of *v* relative to the value of *u.* I anticipated Ali would be successful if she could imagine placing a point in the plane as a way to simultaneously represent the lengths of the red and blue bars. The video played repeatedly until Ali completed the task.

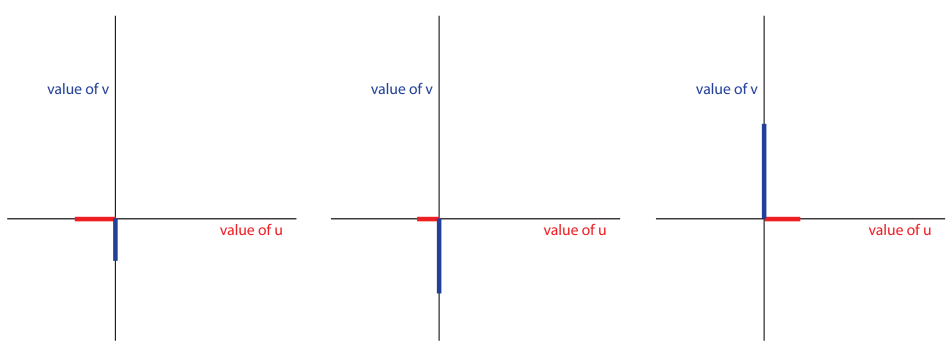

Figure 2: Three screenshots from the animated task.

After watching the video play through two times Ali began making dots within the plane and then connected those dots with a curved line (see Figure 3a). She explained that she figured out the graph by “looking at the motion of how the blue line is increasing and decreasing” and made a dot each time “the blue line kinda stopped and the line kinda dipped down.” When I asked her about the red line she said, “since the whole time this red line is increasing it (the graph) is going to the right.” While Ali was able to produce a correctly shaped graph by tracking the motion of the blue bar with respect to experiential time, her focus on the blue bar suggested that she was not coordinating the varying lengths of the red and blue bars and thus had a non-multiplicative conception of her graph. In the following paragraphs I provide evidence to support this claim.

First, consider Ali’s initial point (see Figure 3a). Since Ali sketched her graph on axes that displayed the initial lengths of the red and blue bars, Ali’s initial point should have been placed at the intersection of the projections of these bars (see Figure 3b). However, her initial point is only aligned with the projection of the blue bar. When I asked Ali to explain how she decided where to place her initial point she said, “I based it off of this blue line (*points to blue line on vertical axis*).” Notice that Ali does not mention the red line when discussing how she placed her initial point. Since Ali only attended to one of the bars when constructing the starting place for her graph, she was not conceptualizing her mark in the plane as a multiplicative object – as a way to unite two attributes simultaneously.

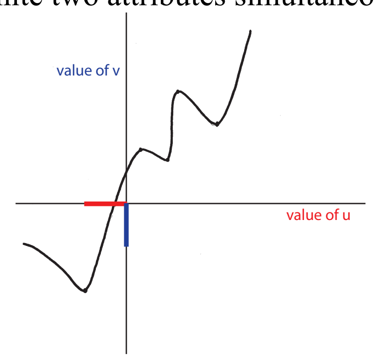

Figure 3a: Ali’s graph generally correct shape but incorrect initial point.

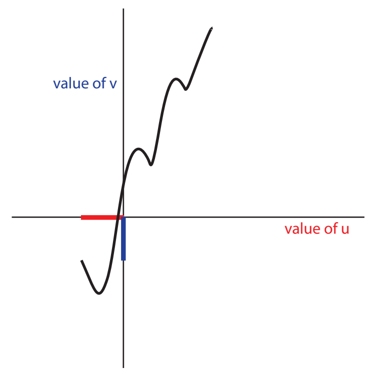

Figure 3b: Accurate graph for animated item in Figure 2.

Ali’s focus on the blue line extended beyond her placement of the initial point. When Ali described what her graph represented she said, “As the red is increasing (*traces hand left to right on horizontal axis*) it (the graph) is showing the motion or like the path of the blue line (*traces along curve from left to right*).” Ali’s conception of the curve was based in the motion of the blue line, not the coordination of the red and blue line. As a result, when Ali described the motion of the blue line she gestured along the curve. This suggests Ali confounded the variation of the length of the blue bar with the path of the curve and thus confounded the length of the blue bar with a place in the plane. Since the motion of the blue line and the curve were synonymous in Ali’s thinking, she had no need to attend to the red line when describing her graph.

This task was not designed to support Ali in making new constructions. Instead, it was designed to assess the role of multiplicative thinking in her covariational reasoning. Since her activity provided evidence that Ali was not thinking multiplicatively the research team used two didactic objects in the third teaching session to support Ali in engaging in emergent shape thinking. As Thompson (2002) explained, a didactic object is “‘a thing to talk about’ that is designed with the intention of supporting reflective mathematical discourse” (p. 198). The first didactic object was designed to support Ali in conceptualizing a correspondence point that simultaneously represented *two* attributes. The second didactic object was intended to support Ali in conceptualizing a curve as a locus of points.

Didactic Object I: Conceptualizing Correspondence Points

To support Ali in conceptualizing a point as a multiplicative object, I introduced the notion of a correspondence point as a way to represent the value of u and the value of v simultaneously. I modified the animation described in Figure 2 so that at any moment I could pause the animation and display the correspondence point as depicted in Figure 1. Following the recommendation of Thompson et al. (under review), I engaged Ali in an activity where I let the animation play, paused the animation, and asked Ali to use the pointer to show where the correspondence point would be. Each time I asked Ali to justify why the correspondence point would be in that specific location. Finally, I displayed the correspondence point to confirm Ali's conceptualization. I repeated this four times to ensure that Ali could construct the location of the correspondence point given the lengths of the red and blue lines and to ensure that she coordinated her conception of the correspondence point with both the red *and* blue lines.

After Ali repeatedly described the correspondence point with the animation paused, I asked her to imagine tracking the correspondence point as the animation played and to try to remember everywhere it had been. Ali watched the animation play through once, tracked the correspondence point with the mouse pointer, and then sketched a graph from her memory of where the correspondence point had been. While Ali's graph now had the correct shape and correct initial point, her image of her graph did not include points. She conveyed that her graph was made of imaginary points that had no coordinates.

This description suggests that although Ali could visually unite attributes to create her graph, she was not viewing her constructed graph in terms of the attributes she used to create the graph. One possible explanation is that to create the graph Ali needed to engage in two actions: first she needed to imagine uniting the length of the red bar and the length of the red bar through a correspondence point and then imagine tracking this point. After constructing the graph she was able to track the motion of a point, but did not retrospectively conceptualize the point as a representation of the simultaneous the lengths of the red and blue bars they varied together. As a result, her explanation of her graph was no different than before; she explained, "As v is increasing and decreasing u is just going to the right." Her conception of the completed graph was still non-multiplicative.

Didactic Object II: Tinker Bell's Pixie Dust

The researchers hypothesized that Ali was not conceptualizing the curve as a collection of points that emerged from simultaneously tracking two quantities' measures. The witness to the teaching experiment intervened to support Ali in conceptualizing her graph as being made by Tinker Bell, the fairy from Peter Pan's Neverland, flying along the path of the curve so that she left a trail of pixie dust marking everywhere she had been. As Thompson (2002) explained, supporting students in conceptualizing graphs being made of pixie dust supports them in coming to imagine lines and curves as being composed of points - pieces of pixie dust - where each piece of pixie dust simultaneously represents the measures of two quantities.

To ensure that Ali was familiar with Tinker Bell and her pixie dust, the witness asked Ali to describe what she knew about Tinker Bell. Ali explained that Tinker Bell is special because she can fly and has pixie dust so as she flies you "see where she has been in the pixie dust". The witness then asked Ali to think about her pen as Tinker Bell and everything she drew (the curve) as pixie dust. When he asked if there was any pixie dust on her graph Ali explained that each particle of pixie dust looked like an imaginary point. Additionally she said the pixie dust represented where Tinker Bell had been and Tinker Bell knew where to fly by "noticing where the value of *u* and the value of *v* were". This is significant because Ali included both the value of *u* and the value of *v* in her conceptualization of placing a piece of pixie dust. I hypothesize that thinking about how Tinker Bell knew where to fly gave Ali a way to differentiate between the two actions she used to construct the graph, coordinating two measures through a point (how Tinker Bell knew where to fly), and then tracking that point (where Tinker Bell flew).

After I introduced the idea of Tinker Bell's pixie dust there were two noticeable differences in Ali's actions. First, when Ali described the value of *v* she would point to the vertical axis as opposed to the curve. This suggests that introducing the notion of a piece of pixie dust enabled Ali to differentiate between the magnitude of an attribute and a place on the curve. As a result, Ali no longer explained a place on the curve, a piece of pixie dust, by attending to just *v*, the length of the blue bar. Instead, she thought about both the *u* and *v* when describing the pixie dust. The second difference in Ali's actions were in her description of how *u* and *v* changed together. Up until this point Ali always described how *u* changed, it increased, and how *v* changed, it increased, then decreased, then increased again. After introducing the notion of pixie dust Ali gave her first explanation of how *u* and *v* changed together that, from my perspective, revealed that she was coordinating two attributes.

> Ali: So as the value of *u* keeps on going towards the right the value of *v* um dips down. So *v* gets a bit closer to the value of *u* and then it dips down. Then as the value of *u* keeps going towards the right the value of *v* increases significantly (*moves pen up vertical axis*) then at a certain point where the value of *u* is about here (*points on horizontal axis*), the value of *v* decreases and then when the value of *u* is about here (*points on horizontal axis*), up until the value of *u* is around here the value of *v* increases and then dips down again (*moves finger up vertical axis*). Then again when the value of *u* is around here (*points on horizontal axis*) then the value of *v* increases again.

This reveals that when Ali attended to the meaning conveyed by her graph she maintained a multiplicative conception of a point and imagined the line coming from simultaneously tracking both the length of the red bar and the length of the blue bar.

Introducing the notion of pixie dust was critical because it gave Ali something real to think about on the graph. As a result, it enabled Ali to differentiate between the value of *v* and the point on the graph. Thus, she came to a reflective conceptualization of a point in the plane where she could imagine constructing a point by attending to the measure of *u* and *v* but also, she could conceptualize a placed point as a representation of both the value of *u* and the value of *v*. Additionally, thinking about where Tinker Bell flew versus how she knew where to fly enabled Ali to differentiate between the actions of placing a point and then tracking that point.

Discussion

At the beginning of the teaching experiment Ali was limited to static shape thinking. She constructed graphs by tracking one quantity with respect to experiential time. This led her to confound the measure of the output quantity with the location of a point in the plane. As a result she conceptualized the variation of the output quantity as the trace of the curve. While this suggests a dynamic conceptualization of graphs, with this way of thinking she did not conceptualize - and had no need to conceptualize - uniting two attributes to make her graph.

Introducing the notion of Tinker Bell's pixie dust supported Ali in making two key differentiations. First, conceptualizing the pixie dust gave Ali a new cognitive object to operate on. Additionally, Ali imagined this pixie dust to be different than the blue bar. Conceptualizing these two objects enabled Ali to differentiate between the value of the output, the length of the blue bar, and the point in the plane, the piece of pixie dust. More importantly, since Ali saw the pixie dust as different from the blue bar, Ali necessitated a way of thinking about placing the piece of pixie dust that involved more than blue bar. She responded to this intellectual need by coordinating her conception of the red and blue bar in order to think about the location of the pixie dust; Ali constructed the pixie dust as a multiplicative object.

The most robust form of emergent shape thinking involves more than constructing graphs by imagining tracking a correspondence point. One must also be able to reverse this way of thinking so that she imagines a curve having been produced by tracking a correspondence point. Constructing this reversible way of thinking is non-trivial as it involves reflecting upon the actions one engaged in when constructing the graph. As Moore and Thompson (2015) explained, "Emergent shape thinking involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation)" (p. 4). This implies that engaging in emergent shape thinking requires two actions: tracing the correspondence point and uniting two varying attributes through a correspondence point. Thinking about where Tinker Bell flew and how Tinker Bell knew where to fly enabled Ali to differentiate these actions so that each was available for reflection. As a result Ali was able to explain her constructed graph in terms of the actions she used to produce it: simultaneously attending to the red and blue bar as they varied.

This study provides insights into the understandings necessary to engage in emergent shape thinking and didactic objects instructors can use to support students in constructing these understandings. Perhaps most importantly, these results provide evidence that educators can support students in shifting from static to emergent shape thinking. This is essential as we call upon pre-service teachers, who often engage in static shape thinking (Moore et al., 2016), to support our students in conceptualizing and representing simultaneous variation in two quantities' measures. Future studies should explore the way in which these shifts might occur in classroom instruction.

I want to address a limitation of this work: the task described is not situated in a context. As a result, I caution interpreting Ali utterances of values of u and v as evidence of having quantified a situation and enacted a measurement process. Instead, it is likely that Ali's meaning for value of u and value of v were tied to the pictorial features of the representation in the Cartesian coordinate system. This claim is supported by Ali's ability to engage in emergent shape thinking in other novel graphing contexts, such as the City A and City B task (see Saldanha & Thompson, 1998), but her inability to construct multiplicative objects in an algebraic contexts. This suggests a limitation to studies situated entirely in the graphing world and motivates future studies that explore the ways in which students construct multiplicative objects in non-graphing contexts. Such studies have the potential to understand how students generalize their schemes for covariational reasoning.

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Gender and Discipline Specific Differences in Mathematical Self-Efficacy of Incoming Students at a Large Public University

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*This study investigates differences in mathematical self-efficacy and outcome expectations of 3107 incoming students enrolled in introductory level mathematics or statistics courses at a land grant university in the Midwest. Students were grouped by discipline (STEM (**S**cience, **T**echnology, **E**ngineering and **M**athematics), Social Sciences and Arts & Humanities) and by gender within each discipline. All students enrolled in an introductory mathematics or statistics course during their first semester at the institution were surveyed about their perceived mathematical self-efficacy and outcome expectations at the beginning of that semester. Our results suggest that discipline specific differences are dependent on the definition of STEM majors, namely distinguishing between math intensive and non-math intensive STEM majors. After accounting for this distinction gender differences in mathematical self-efficacy and outcome expectations disappear.*

Key words: Mathematics, Self-Efficacy, Gender, Social Sciences, and STEM

Introduction

The gender gap in the STEM sciences continues to afflict the U.S.'s economic and social prosperity. For the U.S., as a nation, to remain innovative, competitive and at the forefront economically and technologically, the country needs to continue growing the number of baccalaureates in the STEM sciences and provide diverse opportunities for individuals to pursue STEM related career paths. The National Science Board (NSB-2015-10) published in a recent, up to date, report that "policymakers, scholars, and employers have come to recognize that science, technology, engineering, and mathematics (STEM) knowledge and skills are critical to an extensive portion of the entire U.S. workforce and that a broad range of STEM-capable workers contribute to economic competitiveness and innovation."

In addition to unrecognized pathways, one of the recognized reasons for the existing shortage is the systematic exclusion of parts of the population due to the underrepresentation of women and minorities pursuing STEM or STEM-related degrees, a trend that has been challenging to reverse or to even slow down. Although much research has been devoted to understanding reasons for the shortage of women and other minorities in the STEM sciences (e.g., Betz & Hacket, 1983; Blickenstaff, 2005; Eccles et al., 1983; Else-Quest, Hyde, & Linn, 2010; Gainor & Lent, 1998; Hacket et al., 1992; Lent, Lopez & Bieschke, 1991, 1993; Lent, Brown, & Larkin, 1984, 1986; Wigfield & Eccles, 1992;) it is still not fully understood why these groups choose STEM careers at significantly lower rates than non-minority men. Contributing factors include gender differences in STEM science self-efficacy (Eccles et al., 1993; Jacobs, Davis-Kean, Bleeker, Eccles, & Malanchuk, 2005), lack of role models (Betz & Fitzgerald, 1987; Blickenstaff, 2005) or societal norms and expectations Eccles, 1987, Raty et al., 2002). Additionally, the complexity of the associations among these factors adds another layer of complicatedness to the phenomenon. For a more recent and complete review of the literature we refer to Wang & Degol (2013).

The primary focus of this research is on introductory mathematics and statistics classes taken at the beginning of college motivated by the fact that the mathematical and statistical sciences are seen as gate keepers (Gainen, 1995) to the STEM sciences. Without successful completion of calculus or differential equations continuing in many STEM science majors is difficult if not impossible. While research has explored gender differences in mathematical self-efficacy at the middle and high school level or at the college level within specific STEM disciplines, less is known about the critical transition period from high school to college. For students enrolling into STEM majors or interested in pursuing a STEM major introductory mathematics and statistics classes are commonly among the classes taken during their first semester at college. This motivates the question whether students' experiences in these classes contribute or are related to the gender gap. To provide at least partial insight into this question we investigate the levels of mathematical (and statistical) self-efficacy of incoming students at a large public state university.

Theoretical Framework

The theoretical framework builds on the construct of social cognitive career theory (SCCT; Lent, Brown & Hackett, 1994, 2000). SCCT is based on Bandura's social cognitive theory (Bandura, 1986, 1989, Bussey & Bandura, 1999). At the center of SCCT lies an individual's self-efficacy. Self-efficacy is defined according to Bandura (1986) as 'people's judgments of their capabilities to organize and execute courses of action required to attain designated types of performance' (p. 391). Thus when an individual contemplates a particular career path, the conviction the individual has about his or her ability as well as past, successful experiences plays an important role. With specific regard to a STEM field, if a person perceives that ultimately, he or she is not likely to be successful and has had what can be thought of as less than successful experiences, the likelihood that this person will choose that particular career path decreases. Self-efficacy is only one of three components of SCCT. The other two components are outcome expectations and personal goals. Outcome expectations refer to the belief about what outcomes are possible based upon specific courses of actions or experiences. An individual considers the question about what will happen if a course of action is completed. Environmental factors are often perceived as controllable and influential on the outcome, more than the individual's own behavior (Gro, 2008). Self-Efficacy and Outcome Expectations as the two primary pillars of SCCT are important when considering success of underrepresented groups, in particular females, in STEM career choices.

Research Questions and Research Hypotheses

The main research question of interest focuses on the levels of mathematical (and statistical) self-efficacy and outcome expectations of incoming students at a large public university such as Iowa State University. In order to obtain information related to gender gap issues in the STEM sciences, we group incoming students according to gender and discipline of declared major. We categorized declared major into STEM majors (distinguishing between math-intensive STEM and non math-intensive STEM majors), social sciences and arts and humanities. A student was considered as pursuing a STEM degree if the first or second declared major was in a STEM field.

Research Question

What is the level of mathematical or statistical self-efficacy and of outcome expectations of students entering a large public university, such as Iowa State University?

In order to obtain information related to the gender gap in the STEM sciences, incoming students will be grouped according to

(1) gender

(2) declared discipline, i.e. STEM, social sciences, arts and humanities.

Thus, our research question is aimed at baseline information, summarizing academic self-efficacy in mathematics and statistics courses.

Hypotheses

H1.1 *Self-efficacy, mathematical self-confidence and outcome expectations of incoming students are, on average, significantly higher in students with STEM majors compared to students with a social science or an arts and humanities major.*

H1.2 *Within each of the three groups, self-efficacy, and mathematical self-confidence and outcome expectations is comparable for women and men.*

Data and Statistical Analyses

Data for this study stem from a larger parent study, which collected data on 15,960 students enrolled into an introductory mathematics or statistics course during the semesters of Spring and Fall 2012 and Spring 2013. Of these students, 3107 students were described as *an incoming student*, for which we also had complete background data on ACT scores, high school credits in mathematics and natural sciences, as well as high school rank and GPA. Incoming students are defined, as students who entered the university directly from high school, were actively degree seeking, had U.S. residence status and enrolled into an introductory level mathematics or statistics course during their first college semester. Of the 3107 students 1868 completed a pre-survey on mathematical or statistical self-efficacy during week three of the semester. The survey instrument is based on an existing 36-item instrument called the "Survey on Attitudes Toward Statistics" (SATS-36) (Schau et al., 1995 and Schau, 2003). Schau's SATS-36 survey consists of six subscales measuring *Affect*, *Interest*, *Difficulty*, *Cognitive Competence*, *Effort* and *Value*, respectively. Although the survey is well known among statistics educators and has been utilized extensively since its introduction in the literature, the authors were not able to confirm the six dimensional structure but rather that items measuring Affect, Cognitive Competence and Difficulty loaded onto a single factor. This single factor reflects what is generally defined as perceived mathematical/statistical self-efficacy. Our finding is supported by VanHoof, S., et al. (2011). Some items not loading onto any factor motivated additional changes to individual items. These changes, most of them minor, and adapting the survey to mathematics students were made in consultation with a survey expert from the Center of Survey Statistics and Methodology (CSSM) at Iowa State University. In accordance with Lent's Social Cognitive Career Theory (Lent, Brown & Hackett, 1994, 2000) we further added six questions to measure outcome expectations, as outcome expectations are the second pillar in the SCCT framework in addition to self-efficacy.

We used exploratory factor analysis to estimate the latent factor structure (self-efficacy, value, effort, interest and outcome expectations) and evaluated each factor calculating Cronbach's alpha. The levels of Cronbach's alpha (standardized) are 0.87 (self-efficacy), 0.90 (value), 0.76 (effort), 0.61 (interest) and 0.87 (outcome expectations). Subsequently, we obtained the corresponding factor scores for each student and calculated numerical summaries of the factor scores and conducted corresponding two-sample t-tests according to the groupings defined in the Research Question.

Results

We will summarize the results in tabular form for each hypothesis. We begin with H1.1: Although the primary focus is on mathematical self-efficacy and outcome expectations we also include the results for students' perceived value and interest in mathematics. H1.1 hypothesizes that mathematical self-efficacy and outcome expectations of incoming students are, on average, significantly higher in students with STEM majors compared to students with a social science or an arts and humanities major.

Contrary to hypothesis H1.1 we were not able to identify significant discipline specific differences among the incoming student that participated in our study (see Tables 1 and 2).

Table 1
STEM Sciences versus Social Sciences (Soc. Science)

Dimension	Mean STEM	Mean Soc. Science	t-statistic	df	p-value	95% Confidence Interval for difference in means
Self-Efficacy	0.154	-0.084	2.049	72.3	0.0441	(0.006, 0.472)
Value	**0.352**	**-0.371**	**5.80**	**70.5**	**<0.0001**	**(0.475, 0.972)**
Interest	0.028	0.044	-0.139	72	0.8898	(-0.240, 0.209)
Outcome Expectations	0.075	-0.135	2.070	76	0.0419	(0.008, 0.412)

Table 2
STEM Sciences versus Arts & Humanities (Arts & Hum)

Dimension	Mean STEM	Mean Arts & Hum	t-statistic	df	p-value	95% Confidence Interval for difference in means
Self-Efficacy	0.154	-0.126	2.103	56.8	0.0399	(0.013, 0.547)
Value	**0.352**	**-0.550**	**6.402**	**55.8**	**<0.0001**	**(0.620, 1.185)**
Interest	0.028	-0.052	0.713	57.8	0.4789	(-0.145, 0.306)
Outcome Expectations	0.075	-0.117	0.858	60	0.3944	(-0.124, 0.309)

Both, mathematical self-efficacy and outcome expectations are comparable although based on the observed p-values less than 0.05 one may suggest a week tendency[1]. The same result holds for interest suggestion no significant differences between the disciplines. Highly significant, however, is the difference in how students in different disciplines value mathematics. A possible explanation based on the items belonging to the interest dimension is that students in STEM majors directly experience and appreciate the need for foundational mathematics knowledge in order to succeed in their downstream courses while for many social science and arts and humanities majors degree requirements often consist of no more than one class. Regarding the somewhat unexpected results for self-efficacy and outcome expectations we theorize that a possible explanation for this result lies in the definition itself of what majors are considered STEM majors. In recent years several majors such as economics, psychology or food sciences,

[1] Because of the large number of hypotheses tests, we do not use the usual cut-off value of 0.05 for the level of significance but in an effort to adjust for the multiple testing procedures consider p-values of 0.001 or less as statistically significant.

for example, have been grouped more frequently with the STEM sciences in an effort to provide a more inclusive definition of STEM and because many of these majors frequently include STEM related tasks or knowledge. Different majors within STEM, nevertheless have substantially different mathematical prerequisites, which prompted us follow up our analysis by distinguishing between so-called math intensive STEM majors and those that are not math intensive. For a major to be considered math intensive the major's degree requirement had to include science/engineering Calculus I, or equivalent. Table 6 provides the updated results displaying differences in mathematical self-efficacy and outcome expectations for students when distinguishing between math intensive and non-math intensive STEM majors. All four factors show a significant difference between both groups with students in math intensive STEM majors exhibiting higher levels of self-efficacy, value, interest, and outcome expectations. The same distinction further explains gender differences in mathematical self-efficacy and value originally observed in Table 3 for students in the STEM sciences. When accounting for the type of STEM major (math intensive or not) Tables 7 and 8 show that gender differences within each group disappear and can likely be attributed to random variation. Tables 4 and 5 support our second hypothesis H1.2 showing no gender differences in the social sciences and the arts and humanities.

Table 3
Gender Differences in the STEM Sciences

Dimension	Mean Female	Mean Male	t-statistic	df	p-value	95% Confidence Interval for difference in means
Self-Efficacy	**0.032**	**0.201**	**-2.921**	**572.1**	**0.0036**	**(-0.282, 0.055)**
Value	**0.140**	**0.433**	**-5.485**	**551.9**	**<0.0001**	**(-0.398, -0.188)**
Interest	-0.042	0.055	-1.838	620	0.0666	(-0.200, 0.007)
Outcome Expectations	0.071	0.076	-0.085	631.5	0.9323	(-0.121, 0.111)

Table 4
Gender Differences in the Social Sciences

Dimension	Mean Female	Mean Males	t-statistic	df	p-value	95% Confidence Interval for difference in means
Self-Efficacy	-0.093	-0.047	-0.150	16	0.8829	(-0.689, 0.598)
Value	-0.347	-0.484	0.348	13.6	0.7333	(-0.713, 0.988)
Interest	0.094	-0.186	1.006	16.7	0.3290	(-0.309, 0.870)
Outcome Expectations	-0.111	-0.244	0.424	13.7	0.6779	(-0.367, 0.488)

Table 5
Gender Differences in the Arts & Humanities

Dimension	Mean Females	Mean Males	t-statistic	df	p-value	95% Confidence Interval for difference in means
Self-Efficacy	0.004	-0.277	1.078	51.5	0.2859	(-0.119, 0.740)
Value	-0.549	-0.552	0.012	49.1	0.9903	(-0.567, 0.574)
Interest	-0.013	-0.098	0.387	52	0.7007	(-0.357, 0.527)
Outcome Expectations	0.003	-0.041	0.204	45	0.8393	(-0.280, 0.486)

Table 6
Math intensive (MI) versus non-math intensive (NMI) STEM Sciences

Dimension	Mean MI	Mean NMI	t-statistic	df	p-value	95% Confidence Interval for difference in means
Self-Efficacy	**0.192**	**-0.019**	**-2.847**	**287**	**0.0047**	**(-0.357, -0.065)**
Value	**0.523**	**-0.443**	**-15.441**	**280.4**	**<0.0001**	**(-1.089, -0.843)**

Interest	**0.070**	**-0.164**	**-3.527**	**300**	**0.0005**	**(-0.407, -0.104)**
Outcome Expectations	**0.120**	**-0.136**	**-3.326**	**295.4**	**0.0010**	**(-0.356, -0.092)**

Table 7
Gender Differences in the math intensive STEM Sciences

Dimension	Mean Female	Mean Male	t-statistic	df	p-value	95% Confidence Interval for difference in means
Self-Efficacy	0.110	0.212	-1.553	312.8	0.1215	(-0.214, 0.027)
Value	0.546	0.517	0.581	331.7	0.5617	(-0.070, 0.128)
Interest	0.086	0.066	0.333	333	0.7393	(-0.098, 0.138)
Outcome Expectations	0.193	0.102	1.447	372.6	0.1487	(-0.033, 0.214)

Table 8
Gender Differences in the Non-math intensive STEM Sciences

Dimension	Mean Female	Mean Male	t-statistic	df	p-value	95% Confidence Interval for difference in means
Self-Efficacy	-0.082	0.089	-1.210	179.5	0.2279	(-0.387, 0.101)
Value	-0.455	-0.423	-0.253	158	0.8007	(-0.280, 0.216)
Interest	-0.228	-0.054	-1.355	166.1	0.1722	(-0.427, 0.079)
Outcome Expectations	-0.107	-0.185	0.549	188.7	0.5839	(-0.204, 0.362)

Discussion and Conclusions

Although we did not test for significant differences between math intensive STEM majors and the social sciences and arts and humanities, respectively it is possible to conclude that the statistically significant differences found between math intensive and non-math intensive STEM majors will extend to statistically significant differences between math intensive STEM majors and the social sciences and arts and humanities as the sample means for math intensive STEM majors only increased from those of all STEM majors combined. Under the theoretical framework of Social Cognitive Career Theory with self-efficacy and outcome expectations as its two cornerstones the existing differences imply that the shortage of a strong STEM workforce as well as limitations on the number of pathways leading to a workforce will continue rather than beginning to embrace the STEM sciences. Our results were encouraging, however, in the sense that within each of the disciplines gender differences have subsided or continue to no longer present.

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Exploring a Pre-Service Teacher's Conceptions of Area and Area Units

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This paper highlights the data from a one-semester course with pre-service teachers in an ongoing study of their conceptions of area at a public university in the western United States. Their meanings of area and area units, both standard and non-standard, were explored throughout the semester. Analysis of our interviews with these pre-service teachers about their responses to area tasks allowed us to uncover three cognitive conflicts in conceptualizing area, namely: 1) how can non-square units be square(d)?, 2) how can we find the area of a shape when the area unit is neither square nor polygonal?, 3) how can we use square or polygonal area units to measure the area of a shape with curved boundaries? The work of one representative case study is reported here. This work will help educators develop tasks to initiate these cognitive conflicts for improved conceptualizations of area.

Keywords: area, area units, geometry, pre-service teachers

Area is a fundamental construct of geometry and is usually introduced in 2nd or 3rd grade. In the Common Core State Standards (CCSSM, 2010), *area* appears in 3rd grade under the Measurement and Data standards. In this grade, area measurement is discussed as covering a planar figure, or fitting squares (the area unit) into a planar figure, without any gaps or overlaps, and then counting the squares (CCSSM, 2010). In general, a 1-unit by 1-unit square is the standard unit that is used to measure area.

Even though the definition of area is straightforward, researchers demonstrate that current and future elementary teachers struggle with this concept of area (Enochs and Gabel, 1984; Simon and Blume 1994; Baturo and Nason, 1996; Reinke, 1997; and Menon, 1998). We, the current authors, were mathematics teacher educators at public universities at the time of the study. Over our collective years of working with pre-service teachers (PTs), we have noticed independently that our PTs struggle with defining and using area units, especially non-standard units. In conversation with each other, we found that our observations were very similar irrespective of our classes being at two different universities in two different states. This led us to our first study (Ghosh Hajra, McNeal, & Bowers, 2016), where we documented many of our PTs have difficulty defining a "square unit" and often reported answers in "square units" when measurements were actually taken using non-standard units. We found a wide gap between PTs' definitions of area and their use of area units. We also found that PTs particularly struggled when they had to find areas of irregular 2D shapes using non-standard units.

Our instructional goals for our PTs' learning include: 1) deepening their understanding of each geometric concept that they will teach and 2) preparing them to articulate fundamental features of each topic by pushing them to think beyond the level of mathematics that they will teach. In our instruction, we have begun to broaden the meaning of area to be the amount of 2D space taken up by a planar figure and measured using *any* two-dimensional object as the area unit. We describe *measurement* as a comparison of the given area unit with the 2D shape, as accomplished by covering the shape with an iteration of the area units and counting how many area units can fit in the shape. For example, consider the hexagon in Figure 1 as an area unit. We can iterate this unit to cover the blue region. The area of the blue region is then the number of hexagons that are needed to cover it, or 9 hexagons. Our area instruction thus aims to prepare our

PTs to respond to the following questions based on our course goals: What makes an appropriate unit of area? How does one use this unit to measure area? How does one report the resulting measurement, whether using a standard or non-standard unit?

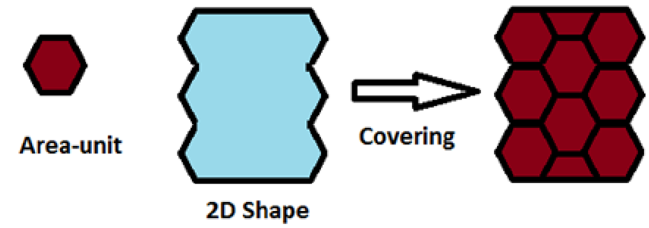


Figure 1: Measuring area using a non-standard area unit.

Like all classroom teachers, we aim to improve our instruction through close study of our students' work. We asked ourselves what effect continued work in class with a variety of non-standard units would have on PTs' understanding of area and its measurement. This led us to develop new area tasks for the students in our classes in Spring 2016. One of the limitations of the first study (Ghosh Hajra et al., 2016) was that we analyzed only students' written responses. The current study is an extension of the Ghosh Hajra et al. (2016) investigation, using data from the new group of students. The ideas expressed in the current paper emerged as the first author engaged in conversation with her students during the spring 2016 class and then in out-of-class interviews about their responses to the newly developed area tasks. Although our informal observations were the basis from which we articulated the points of cognitive conflict in PTs conceptualization of area-units, we undertook this latest study to enrich our understanding of these conflicts by looking more closely at students' thinking through the interviews. We also hoped this would move us toward validating the points of cognitive conflicts PTs have with area and area units. In this paper, we discuss the progress one of the 30 PTs in the study made in the process of measuring area. This particular student was selected because her thinking and points of confusion around the meaning and use of area units reflected the thinking of many other PTs in the class.

Theoretical Framework

The constructivist theory of learning provides the theoretical framework for this study in that it drives both our classroom instruction and our interpretation of the participants' thinking, whether derived from written work or from interviews. From the point of view of constructivism, new ideas are generated by individuals when new situations and experiences, which conflict with existing ideas, cause the individuals to modify their existing ideas (von Glasersfeld, 1995). Based on this theory of learning and in pursuit of our course goals, we pose non-traditional tasks as a regular feature of our instruction. These tasks (initially conceived as instructional materials rather than data tasks) are selected and sequenced to help PTs examine their own thinking and to provide a source of cognitive conflict. We hypothesized that tasks involving non-standard area units would be a significant source of productive conflicts and might cause reorganization of PTs' area concepts when needed.

Methodology

The first author was the PTs' instructor for a geometry course, the second part of a two-course mathematics sequence for elementary teachers. Thirty PTs enrolled in two sections of this course participated in a semester-long research project examining PTs' understanding of area units. The instructor used the Beckmann (2013) mathematics textbook that is aligned with the Common Core State Standards Initiative (2010).

In our first study (Ghosh Hajra et al., 2016), tasks were designed in the middle of the semester and PTs' written responses to those tasks were the focus of the report. PTs were not interviewed for additional clarification of their work. In the current study, we used *Units Task 1* (Figure 2) from Ghosh Hajra et al. (2016) as well as new tasks aligned with our instructional goals. The instructor posed several non-traditional tasks on area and, in class, listened to PTs' real-time responses for informal feedback on their understanding. This feedback was used to design follow-up class activities in a manner reminiscent of Simon and Blume (1994). In-class writing assignments, quizzes, tests, and the final exam were collected from all thirty PTs. A few clinical interviews (Clement, 2000) were conducted with eight of the PTs about their in-class work and other assignments individually. Each of the interviews was videotaped and transcribed, and the written work was digitized.

We used qualitative research methods to analyze our data. Our qualitative analysis drew on the techniques of grounded theory, constant comparison, and retrospective analysis. Our developing theories were grounded in our initial observations as classroom teachers and constantly compared to new classroom observations (Strauss & Corbin, 1990). We provisionally tested our theories in classroom interactions and by conducting interviews while instruction was ongoing. We developed, modified and generated new tasks through engaging in conversations with each other and with the students (PTs). We used the retrospective analysis procedure (Steffe & Thompson, 2000) to analyze videotaped interviews and students' written work to understand students' ways of conceptualizing area and area units after the conclusion of the period of instruction. Each author analyzed the interviews independently and later discussed them together. We followed all PTs closely throughout the semester to examine their conceptions of area and area units. This study reports only the work of one PT, Martha (pseudonym), who performed well in the first course in the sequence with the same instructor, and whose thinking was representative of the rest of the class as demonstrated in the entire data collection. This PT was selected as a case study on the basis of our retrospective analysis and after the conclusion of the instructional period.

Results

How can non-square units be square(d)?

In our first study, we observed PTs' unusual written responses to non-square units. This led us, in the current study, to watch closely how PTs engaged in the same tasks (and some new tasks) during class. Based on their work in class and written responses, individuals were selected for interviews to clarify our understanding of how they were thinking.

Early in the semester, the instructor presented the class with Task 1, shown in Figure 2. The goal of this task is to use the 2D shapes marked *a*, *b*, *c*, and *d* as area units to measure the area of the larger shapes *A*, *B*, *C*, and *D*, respectively. Martha used "unit^2" and "units^2" throughout even though her calculations were based on the given area units. While a "square unit" is defined as a 1-unit by 1-unit square that is used to measure the area of a shape, Martha's use of "units^2" in all tasks seemed to occur without regard to the actual unit given. For example, she wrote "11 unit^2" as her answer to Task 1 b) which asked her to find the area of shape *B* using a triangle.

Task 1: Find the areas of the shapes (A), (B), (C), and (D) using the following 2D shapes (a), (b), (c), and (d).

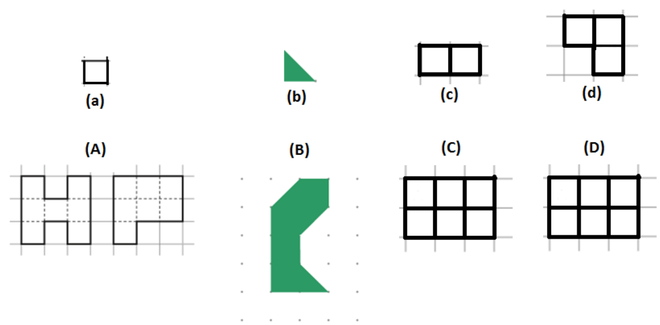


Figure 2: Area task involving standard and non-standard area units.

After several weeks of instruction, Martha reported the area of a shape in terms of the given unit, but still used the "squared" terminology. An example of this came up during the follow-up interview in which she was asked how she would interpret covering a rectangular piece of paper with a specific number of triangles and hexagons. Her responses are cited in Figure 3.

Interviewer: Suppose you have 20 triangles filling the rectangular shape [showing a rectangular piece of paper], what will be the area of the rectangle?
Martha: It will be 20 triangular units.

Interviewer: Say you try to cover this piece [a rectangular sheet of paper] with a hexagonal region [showing a hexagon drawn on a paper], and you need 36 such [hexagonal] pieces, so what will be the area of this [rectangular sheet of] paper?
Martha: It would be 36 hexagonal regions squared.

Figure 3: Martha's responses on task 2 (shown in Fig 4).

Martha used the number of triangular units when stating the area of the sheet of paper, but when she referred to the area of the same region in terms of hexagons, she added the word "*squared*". Martha's reporting of the answer was correct for the triangular units, but her response to the hexagonal unit was technically incorrect. Martha appeared to be uncomfortable with simply reporting the number and type of unit as an area, i.e., saying "36 hexagons". This is indicated by the extraneous (to our ears) use of the word "square". She understood that the unit of measurement should be specified, but clearly did not yet completely understand the meaning of a "square unit". Hence, the use of a non-square unit causes students to experience cognitive conflict and compels them to reconsider the meaning of "1 square unit". Martha and PTs like her do not seem to view a "square unit" as a 1-unit by 1-unit square. That is, saying that something has an area of "6 square inches" does not seem to mean that it has been covered with "six 1-inch by 1-inch squares". Simon and Blume (1994), who studied the multiplicative understandings of pre-service teachers in the context of area, found similar responses and observed, "It is likely that for some of these [PTs], square units do not conjure up an image of a square" (p. 485). In fact, one of our PTs went so far as to write on a quiz that "a square inch is not a square".

How can we find the area of a shape when the area unit is neither square nor polygonal?

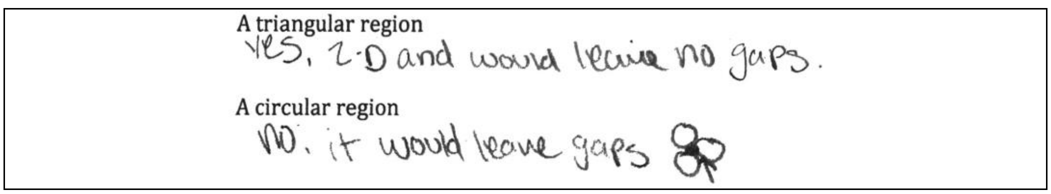
After a couple of weeks of instruction on area, the following task (Figure 4) was presented on a midterm exam.

Task 2: **Which of the following can be an area unit?** Provide reasoning for your choice.

A 1inch-by-1 inch square, a triangular region, a circular region, length of your finger, width of your finger, a finger nail, length and width of your finger, a hexagonal region, a toothpick, and a ribbon.

Figure 4: Area unit task.

Martha responded that all of the listed items could be area units except the circular region, length of your finger, and width of your finger. She correctly described why the triangular region can be an area unit explaining that it is a 2D shape and multiple copies of it can fit together to cover a shape without leaving any gaps (see Figure 5). However, Martha reasoned that a circular region could not be used as an area unit because the units would not fit nicely together without leaving gaps. She makes this conclusion regardless of the particular shape to be measured. Martha's thinking reflects the thinking of many other PTs in this study and others we have encountered in our teaching careers. It seems that most PTs believe area units need to have nice straight edges, i.e. only a polygonal 2D shape can be an area unit.


A triangular region

yes, 2-D and would leave no gaps.

A circular region

No. it would leave gaps

Figure 5. Written response of Martha on the area unit tasks.

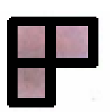
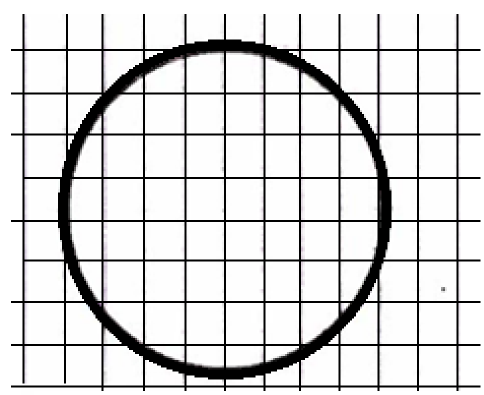
When the area unit under consideration is polygonal and the shape to be measured is also polygonal, it is reasonable to Martha that an area unit be decomposed because the area units fit nicely inside the shape without any gaps. The same holds true along the boundary. For example, in Figure 1, we saw that the hexagonal area unit needed to be decomposed into halves to cover the region. This is an application of the moving and additivity principles of area in the measurement of area. The moving principle states that moving an area unit rigidly without stretching maintains its area. The additivity principle states that if we combine a finite number of area units (whole or part units), the area of the resulting shape is the sum of the areas of the individual area units. When faced with a circular unit, the question of whether we can cover our shape with this unit became an issue even before considering the particular shape to be measured.

How can we use square or polygonal area units to measure the area of a shape with curved boundaries?

In the next task (Figure 6), Martha was asked to find the area of the circle using the non-standard polygonal area unit. To answer part (a), Martha wrote that the "exact" area cannot be found using the given area unit. For part (b), she tentatively mentioned that the moving and additivity principles of area could be used, but was not sure how to account for the parts of the circle that cannot be covered by a whole area unit. Martha wrote, "*No it is not possible because the Area unit is using straight edges which would leave parts of the circle unmeasured. If we used the moving and additivity principles to manipulate the area shape to fill in those missing parts we couldn't count the number of 'area units' it created to be exact."* In an interview about this task, she explained that, since the area unit is not the usual square grid, it was not possible to "*know exactly how many will fit in even if you do break it apart*." Note that with the polygonal area unit, she immediately showed understanding that she could break the unit for purposes of covering the circle, but the curved boundary became now the source of cognitive conflict. This part of her response also suggested that, even if the unit could be broken, it could not be used to fill curved spaces.

When the interviewer asked her if she could estimate the area, Martha was able to do so by counting the full area units first, finding that 10 complete area units would fit in the larger shape, and then decomposing the area unit further to obtain a total of approximately 17 area units. When asked if she would like to change her answer to part (b) of task 3, Martha said, "*Yes it's possible but I guess it wouldn't be, like it is possible but it's not possible to give you an exact—like a perfect measurement. Unless you know the exact, how like big that is [the circular region] compared to that [given area unit], like how many of those [given area units] go in here [within the circle] and how many of these [given area units] cause these [leftover parts within the circle] aren't the same.*" Martha seemed to feel that the units could be decomposed a few times, but that the process could not be completed—no straight-edged units could ever completely fill a region with a curved boundary.

Task 3: a) Find the area of the following region bounded by the circle using the given area unit.

Area unit:  The 2D shape: 

b) Is it possible to use the above area unit to measure the area of the above circular region?

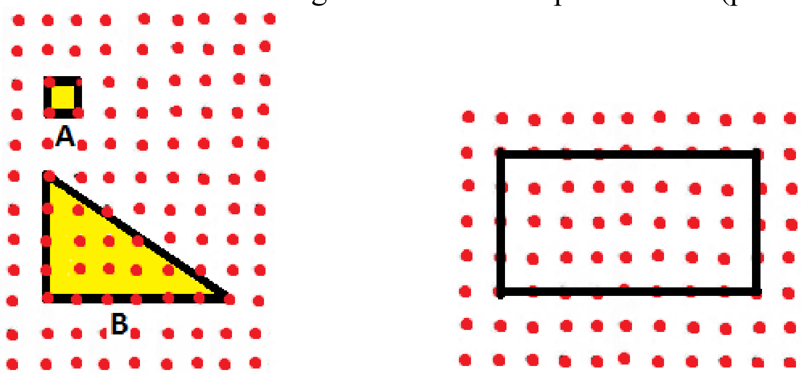
Figure 6. Task 3: Measuring the area of the circular region using a non-standard unit.

When using squares and triangles as area units, Martha was able to articulate the idea clearly. However, with non-polygonal non-standard area units and 2D shapes with curved edges, she had difficulties. Martha assumed her area calculation needed to be exact. Even though she was able to calculate an estimate of the area of the circular region using the non-standard unit, she did not think she could find the exact area.

Discussion and Conclusions

Our repeated questioning of Martha in and out of class over the semester generated cognitive conflicts that ultimately helped her recognize an area measurement as meaningful when obtained by covering a shape with a non-standard unit. Martha's understanding of area unit had thus grown in two ways: 1) identification of any 2D shape as a possible area unit (without regard to efficiency or ease of use) and 2) awareness that an area unit need not have straight edges.

Task 4: Consider the units of area given below as shapes A and B (picture to the left).



For each unit, indicate whether or not the 2D rectangular shape (picture to the right above) has an area with the given unit. If the shape DOES have an area with the unit, explain how you would find it using the unit. If it does not, explain why not.

Figure 7. Final exam question.

Martha continued, however, to have trouble making sense of her work when asked to interpret her "area answer" in terms of that non-standard area unit. On the final exam, when we repeated Task 1b, Martha gave her answer as "*11 triangle units squared*". On the same exam (Task 4 in Figure 7 above), she correctly identified the area unit and mentioned the decomposability of that unit. She wrote that both A and B could be used as area units. She wrote that 32 of unit A would be needed to fill up the space, but using unit B would require that we "*break it [the unit] apart filling in the gaps, which would be a little over 2 times*".

Our analysis suggests that engaging Martha in various tasks generated cognitive conflicts which influenced Martha's understanding of area. Our data also illustrates that we have not generated sufficient conflicts to disrupt her habitual use of square units.

Overall, students' understandings of the process of area measurement (separate from the formula "length times width") and of area units themselves (other than squares) were deeply enriched through tasks involving non-standard area units and through discussion of their responses. We propose two new types of tasks for work with future students. First, we would like to pose tasks that involve covering a shape with non-standard units (such as different-shaped post-it notes) with the idea of engaging PTs in thinking about fitting curved units into polygonal boundaries and polygonal shapes into curved boundaries. Finally, we propose to have PTs use rectangular units to cover larger rectangles as a way to coordinate their meaning of area with the definition of multiplication in the formula for the area of a rectangle.

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Contextualizing Symbols in Word Problems

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The Common Core State Standards recommend students to decontextualize word problems using symbols and contextualize symbols by defining the meaning of values. We observed pre-service teachers' difficulties with contextualizing symbols in word problems; hence, we incorporated supplementary word problems' modeling instructions for an arithmetic course with pre-service teachers. After six weeks of instruction, on a midterm exam, our pre-service teachers started using symbols to present arithmetic word problems. However, many of them still could not clearly define the meaning of the symbols they used. After completing the program, students demonstrated improvement in their reasoning with symbols. We believe difficulties with defining symbols are connected to weaknesses with active scientific vocabulary in terms of measurable attributes. Therefore, we propose mathematics courses for prospective teachers to accentuate scientific vocabulary regarding measurable attributes.

Key words: Algebraic reasoning, symbols, pre-service teachers, word problems

One of the main goals of the Common Core State Standards Initiative (CCSSI, 2010) is to promote quantitative reasoning. Thompson (2011) stated quantification as the "process of conceptualizing an object and an attribute of it so the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bilinear, or multi-linear) with its unit" (p. 37). For example, in the statement, *Ravi bought 3 kilograms of flour*, a qualitative attribute is *weight* and a unit of measure is *kilogram*. Here, the measurable attribute (*weight*) is not specified but can be deducted from the text.

Another aspect of children's education greatly emphasized by CCSSI is early development of algebraic reasoning. According to Kaput and Blanton (2005), algebraic reasoning requires students to "generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (p. 99). Lins and Kaput (2004) recommend developing algebraic reasoning in elementary grades. The National Mathematics Advisory Panel of the U.S. Department of Education (2008) acknowledged this, since research has shown most students struggle in algebra in secondary grades (Kieran, 1992). An early introduction of algebraic reasoning might help in transition to algebra in later mathematics classes. Low algebraic skills are believed to be a gatekeeper to progress in mathematics and science (Greenes et al., 2001).

One of the features of algebraic reasoning includes an ability to decontextualize word problems, i.e., "to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents" and contextualize symbols by attending to the meaning of quantities, i.e., "to pause as needed during the manipulation process in order to probe into the referents for the symbols involved" (CCSSI, 2010, p. 6). As a symbol, some choose basic letters (A, B, C, X, Y, Z, etc.) (Davydov, 1975; Dougherty & Slovin, 2004), while others use *parameter pointers*, letters or words, that point to key words connected with qualitative characteristics of values (Kofman, 2016). The latter are routinely used in physics (Serway & Faughn, 2006) and chemistry (Dingrando et al., 2006). Parameters, such as *V_1 is the volume of a gas before expansion* or *m is the mass of an obje*ct, are used to remind a problem-solver about the symbols' meaning in challenging word problems, which involve multiple symbols. Shifting gears toward problem-solving (CCCSI, 2010) increased the role of teaching students to use symbols, which help with solving word problems.

According to CCSSI (2010), children should start developing knowledge regarding measurable attributes as early as kindergarten and be able to “describe measurable attributes of objects, such as length or weight” (CCSS.Math.Content.K.MD.A.1, p.12). A well-developed understanding of measurable attributes helps students to develop an active scientific vocabulary—a basis for communicating math ideas needed to successfully solve mathematics word problems. CCSSI (2010) expects 6^{th} graders “to use variables to represent numbers and write expressions when solving a real-word or mathematical problem” and “understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set” (CCSS.Math.Content.6.EE.6, p. 44).

Since we want our elementary and middle school students to start developing algebraic reasoning by presenting word problems using symbols, we need to ensure teacher candidates (pre-service teachers) are fluent with this material because, according to Patton et.al. (2008), pre-service teachers “possess naive conceptions believing that teaching mathematics is only about delivering facts and memorizing mathematic procedures” (p. 494). Patton’s finding corroborate with early studies, which demonstrated undergraduates (non-physics majors) display difficulties with algebraic reasoning, particularly with interpreting variables when solving math problems (Rosnick, 1981). Also, MacGregor and Stacey (1997) have shown novice algebra students do not understand the meaning of symbols and commonly misinterpret symbols as representing objects or words.

We hypothesized pre-service teachers might exhibit similar difficulties. Although, the research regarding prospective teachers’ difficulties with generalizing patterns (Brown & Bergman, 2013; Hallagan, Rule, & Carlson, 2009) was undertaken recently, future educators’ active vocabulary with implicitly stated attributes has not been studied, yet.

Based on this, we closely monitored pre-service teachers (our students) performance on contextualizing symbols for simple arithmetic word problems. Observing their difficulties with defining the meaning of symbols in word problems, we undertook this research to analyze how prospective teachers contextualize symbols in word problems. To meet students’ needs, we strongly emphasized developing scientifically valid active vocabulary, which ensured fluent communication in terms of symbols, measurable attributes, and units of measure. Since high-level communication skills are a necessary condition for developing thinking (Vygotsky, 1978), we expected by increasing pre-service teachers’ abilities to communicate using correct terminologies, we could deepen their understanding of word problems; hence positively affect developing communication skills of their future students. Thus, the second stage of the research was to provide specific instructions to improve pre-service teachers’ abilities to identify measurable attributes and clearly define symbols used for presenting arithmetic word problems and acquire preliminary estimation of its impact.

Theoretical Framework

Our theoretical framework is based on the schemata approach. A *schemata* approach subdivides simple addition word problems into logical categories (Combine, Compare, Change, etc.) (Jitendra et al., 2007; 2015) and proposes schema for solving each type of word problem. We believe the schemata approach works because it helps students to concentrate on one learning dimension at a time, while allowing some variations. According to Marton and Pang (2006), namely ‘dimension variation’ elevates quality of the learning process. In the schemata approach, students learn to solve and present one type of problem (one learning dimension), while choosing between arithmetic operations (variation). This approach allowed us to concentrate on teaching students one dimension at a time—students solve one type of problem, while creating algebraic and visual word problems’ models specific for each type.

Contrasting variations were applied in the following aspects:

a) Problems were assigned that can be solved using addition or subtraction in each type of problem (Combine, Compare, and Change problems);
b) Students were taught to continuously differentiate between symbols (capital letters) and units of measure (lower case letters) by denoting them in different ways;
c) Students were taught to define parameter-pointers using precise definitions based on naming attributes (e.g., M is the distance Marta walked), which created a contrast with parameters' associative meaning (e.g., M is Marta).

Using the schemata approach combined with modeling and defining parameter-pointers allowed us to apply the dimension variation method when creating materials designed to improve pre-service teachers' knowledge how to interpret symbols they use.

Methodology

Participants and the word problem-solving (WPS) Treatment

Seventeen pre-service teachers from a western research university enrolled in an arithmetic course for elementary teachers participated in this study. The lead author taught the arithmetic course. This course was the first content course in a two-course sequence for elementary pre-service teachers. This course provided pre-service teachers with a deeper understanding of the real number system and arithmetic operations for whole numbers, fractions, and decimals for Grades K-6. The course textbook was written by Beckmann (2014). Mainly, chapters 1 through 9 were covered during the semester.

Along with standard course materials, supplementary materials on word problem-solving (WPS) were used throughout the semester. These supplementary materials (available upon request) were independently developed by the second author and were chosen because no books concentrating on developing teachers' vocabulary regarding measurable attributes and defining symbols for arithmetic word problems exist to the best of these authors' knowledge. These supplementary materials were self-explanatory workbooks, consisting of 13 chapters. Pre-service teachers read each of the chapters, each chapter comprised of multiple sections, worked on the examples, and completed the assigned problems for each section. The exercises involved presenting arithmetic word problems using multiple models and then solving the problems. The WPS workbooks focused on the following list of measurable attributes: number of objects, amount of money, length-type characteristics (depth, width, distance, height, and length), volume, weight (in terms of *mass*), temperature, and time.

The instructor provided feedback after pre-service teachers handed their assignments to the instructor during each class period. The instructor spent five-ten minutes in class discussing pre-service teachers' work and ideas from each section of the WPS materials. Pre-service teachers were given chances to correct their assignments until 100% fluency was shown with the assignments. When teaching the WPS supplementary program, the instructor constantly monitored pre-service teachers' progress, mistakes, and misconceptions, and focused on extending and sharpening students' mathematical vocabulary. The dimension variation approach was used to teach problem-solving topics. It includes materials teaching identification of the meaning of symbols (Table 1).

After the mid-term exam, pre-service teachers were asked to define each of the symbols they used when presenting word problems. Also, pre-service teachers were divided into groups of two or three and given word problems to work in class—they presented their solutions to the entire class. Pre-service teachers modeled arithmetic problems using visual and algebraic representations. When presenting problems in the form of algebraic equations, pre-service teachers were prompted to use parameter-pointers. For example, T_1 *is the number of flowers Linda had* <u>*at first*</u> or *R is the length of* <u>*Rob's*</u> *wire.*

Table 1
Objectives connected with contextualizing symbols in word problems

Dimensions of learning	Variations
Create models presenting word problems.	After presented separately, different types of problems were mixed together: combine, compare, and change.
Present problems using symbolic equations.	Denote symbols and measurement units for each problem, while using capital letters for parameter-pointers and lower case for measure of units.
Define parameter-pointers in symbolic equations.	Types of attributes (number of objects, amount of money, the distance between, …, etc.) were varied.

In the beginning, the WPS workbook contained word problems with easy to define attributes: "Linda has 5 erasers," and attributes were stated explicitly: "The weight of Linda's chair is 7 lbs." Later in the semester and during the final exam, more difficult, *implicit (not stated directly)* attributes were used in word problems, "Agnes collected 84 liters of rainwater." In such problems, pre-service teachers had to deduce the name of the qualitative attribute (*volume*) connected with the given value, 5 liters.

Data Collection and Analysis

The research design was a mixed-methods study. Data were collected from the pre-service teachers' writing assignments—two pre-tests, WPS workbook, quizzes, a midterm exam, and a final exam and instructors' classroom observations. A strong correlation in students' written work (correct usage of scientific vocabulary) and their usage of the vocabulary in oral presentations in the classroom were observed. For qualitative analysis, the pre-tests, mid-term exam, quizzes, and final exam written data were analyzed following an open, axial coding method (Strauss & Corbin, 1998). Each of the two authors read the written works and created a rubric. Then, we met to discuss our rubrics and created a common rubric (Table 2), based on pre-service teachers' presentations and explanations of the symbols used. For quantitative analysis, we created an excel spreadsheet of the responses for each task in the pre-tests, mid-term and final exam, which allowed us to follow an individual's progress throughout the semester. We recorded the responses using the rubric in Table 2 and counted the number of pre-service teachers in each category.

Table 2
Categories of symbols' descriptions

Coding categories	**Examples**	
Students demonstrate difficulties	Not defining symbols	Writing *Anna = 9*, A= 9, or X=9 and not saying what *A, Anna,* or *X* mean.
	Definition reflecting misconception	Defining symbols using key words from the text. *A means Anna*
Students present conceptually correct definition of symbols	Unclear naming of quality of quantity	Explaining symbols while inaccurately defining the qualitative characteristic of the symbols. *A is the grams of apples.*
	Clear naming of quality of quantity	Clearly associating symbols with their meaning. *A is the weight of apples.*

Results

The pre-tests run on the first two days of the semester demonstrated our pre-service teachers had difficulties to explain the meanings of the symbols they used in the word problems. After observing pre-service teachers' difficulties with CCSSI K-6th grade materials, we used the supplementary WPS materials for six weeks. The WPS sections regarding measurable attributes were thoroughly discussed in the classroom.

After six weeks into the semester working on the WPS materials, we found pre-service teachers continued to struggle with the concept of contextualizing symbols. More than half (9 out of 17) did not contextualize the symbols in the mid-term exam. For a *Combine* addition task, one pre-service teacher wrote "*P = Pearls*" (Figure 1a) instead of writing "*P is the weight of the box with pearls.*" Another wrote "*F = First box*" instead of writing "*F is the weight of the first box*." It is interesting to note, the same students appeared under the same categories for the two tasks (Table 3). It is noted, the materials were also reviewed a day prior to the midterm exam.

The final exam contained word problems with *implicitly* stated quantities (i.e., attributes are implied in the word problem, not directly mentioned) (see problem in Figure 2). In spite of this, we saw an increase in the *conceptually* correct definitions of symbols. The definitions reflected students' understanding of symbols, but were still somewhat unclear with precise defining measurable attributes (e.g., Figure 1b). We classified such answers as unclear, but in terms of symbol definition, conceptually correct. Across four different tasks in the final exam, 16 (15 in tasks involving length) students defined the symbols correctly or at least *conceptually* correctly (see Table 3).

Table 3
Comparison of pre-tests, mid-term, and final exam. The problems' difficulty in terms of contextualizing symbols was higher on the final exam than on the pre-test and mid-term.

<table>
<tr><th colspan="5">Pre-treatment:
Pre-test (beginning of instruction, 16 students; one student enrolled late)</th></tr>
<tr><th rowspan="2">Measurable attributes</th><th rowspan="2">Not defining symbols</th><th rowspan="2">Definition reflecting misconceptions</th><th colspan="2">Conceptually correct definition of symbols</th></tr>
<tr><th>Unclear naming of quantity</th><th>Clear naming of quantity</th></tr>
<tr><td>All categories</td><td>16</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td colspan="5">Mid-treatment:
Mid-term (after 6 weeks of instruction, 17 students using the WPS workbook)</td></tr>
<tr><td>Number of fish</td><td>3</td><td>6</td><td>0</td><td>8</td></tr>
<tr><td>Weight</td><td>3</td><td>5</td><td>0</td><td>9</td></tr>
<tr><td colspan="5">After-treatment:
Final Exam (after 12 weeks of instruction, 17 students using the WPS workbook)</td></tr>
<tr><td>Number of apples</td><td>1</td><td>0</td><td>4</td><td>12</td></tr>
<tr><td>Weight</td><td>1</td><td>0</td><td>4</td><td>12</td></tr>
<tr><td>Length</td><td>1</td><td>1</td><td>1</td><td>14</td></tr>
<tr><td>Volume</td><td>1</td><td>0</td><td>4</td><td>12</td></tr>
<tr><td colspan="5">Note: The numbers above show the number of pre-service teachers under each category for each problem type.</td></tr>
</table>

Students were taught to use any letter but "O" when creating symbols to prevent confusing "O" with *zero*. However, 15 students used "O" as a symbol to represent the weight of onions and only 2 students used other letters. Figure 3 demonstrates how the same symbol, P, has different meanings in a word problem involving volume. Since the measurable attribute was not explicitly mentioned in the problem, four pre-service teachers used the same words used in the word problem, but failed to use the correct attribute name, *volume*, when defining the symbols.

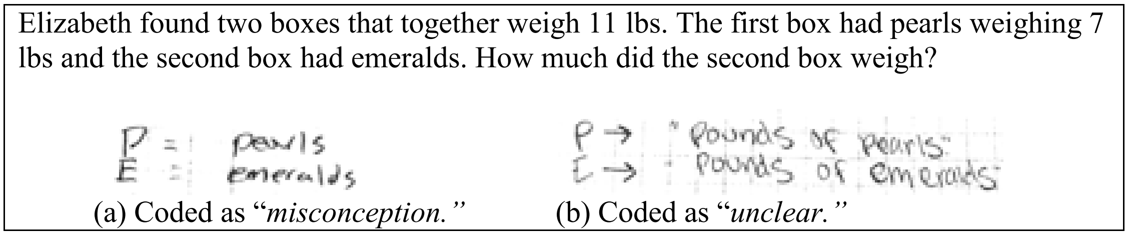

Figure 1. Symbol definition: a) misconceptions regarding parameters, b) missing attributes.

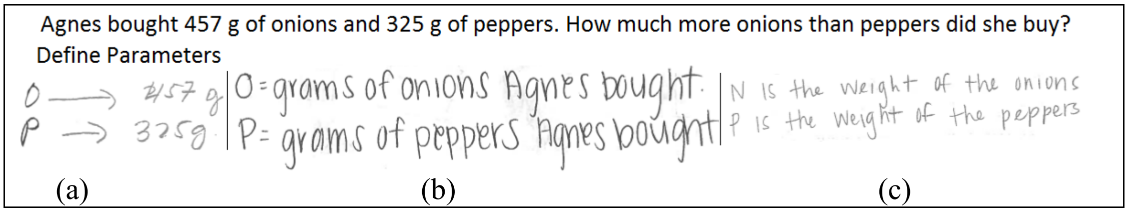

Figure 2. Defining symbols: (a) definition is unavailable; (b) definition is conceptually correct—there is a misconception regarding attribute; (c) correctly defined symbols.

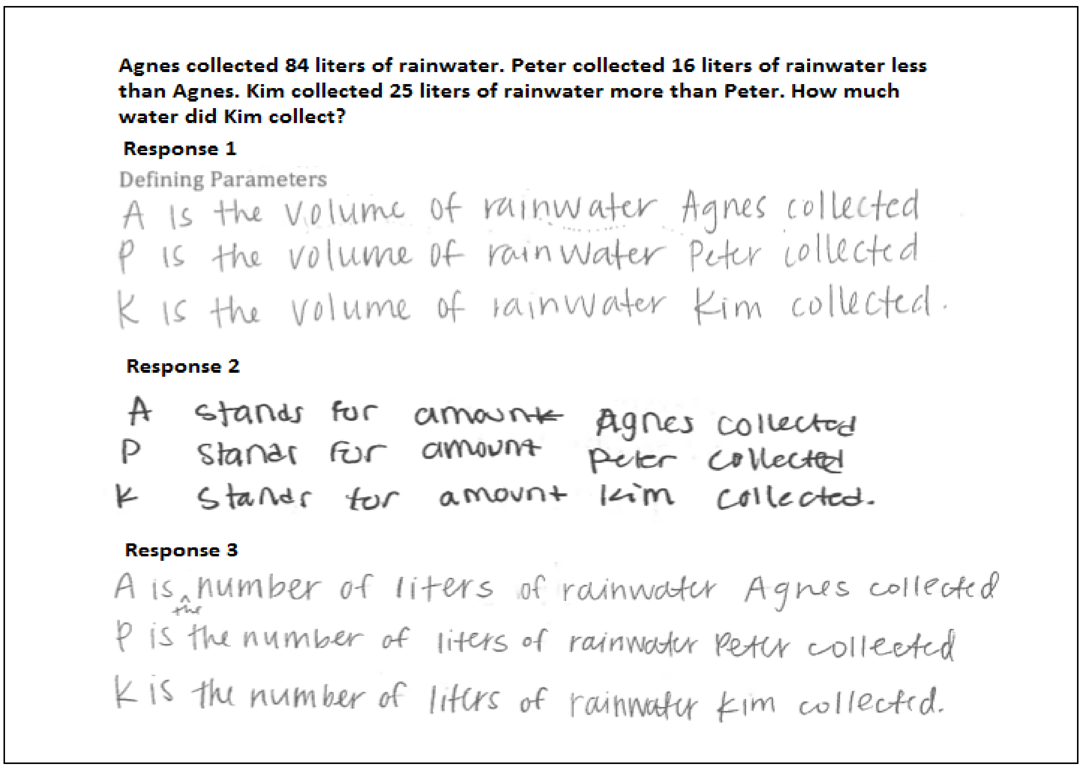

Figure 3. Responses of three pre-service teachers on the same task for the final exam.

Additionally, one student failed to define symbols for all the tasks in the final exam. This student did not complete the supplementary WPS workbook on her own and did not meet with the instructor to receive additional help.

Discussion and Conclusions

Our findings regarding students’ misinterpretation of letters as objects or words are consistent with other studies (Booth, 1988; MacGregor & Stacey, 1997). When presenting 1–2-step arithmetic word problems in symbolic form, our pre-service teachers demonstrated difficulties with defining symbols they used. Their work reflected both—misconceptions in regards to the symbols' meaning (e.g., *G is the green paint*) and misconceptions regarding defining measurable attributes represented by the values (e.g., *P = grams of pepper Agnes bought*). Supplementary instruction in word problem solving allowed us to improve pre-service teachers' precision in defining symbols.

One can argue that the pre-service teachers did not perform well, since they did not understand the task. This can be true for the pre-test. However, this is not true for all later tests. The pre-service teachers were taught throughout the semester using the WPS materials similar to the examples used in this paper. Hence, during the mid-term exam, pre-service teachers were well-aware of the type of questions and expectations for the symbol-meaning tasks.

The data reveal the difficulties demonstrated by pre-service teachers are real in regards to active scientific vocabulary connected with properties of matter measured and/or calculated. About 50% of the pre-service teachers, who already had prolonged instruction on defining symbols by naming the attributes of the values they represent, on the midterm exam, still demonstrated difficulties with the materials and only additional studies helped them improve their performance. The final exam demonstrated the improvements.

We corroborate the progress in pre-service teachers' abilities to define symbols is much greater than presented in Table 3 because (1) the word problems on the final exam were more difficult in terms of symbols' definition than on the mid-term exam and (2) because we compared performance of the pre-service teachers on the mid-term (discarding pre-test results) and final exams. Meanwhile, the performance on the mid-term exam was measured after the students had some WPS instruction regarding defining symbols used to present word problems.

In the future, it would be interesting to examine pre-service teachers' performances using a redesigned pre-test and compare treatment group results with a control group that does not have supplementary instruction. Based on our results, it can be concluded, mathematics courses for elementary teachers must put greater emphasis on teaching pre-service teachers scientifically-correct vocabulary regarding measurable attributes.

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Using Oral Presentations and Cooperative Discussions to Facilitate Learning Statistics

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In this paper we report on a study of assessment-based oral presentation tasks in a statistics course at a public university in the United States. We examine student attitudes towards using oral presentation tasks in learning statistics and their disposition towards statistics as well as their knowledge of the statistical concepts. Our results suggest that use of oral presentation improves students' mastery of statistical concepts and their disposition towards statistics. Moreover, responses to the anonymous course evaluation questionnaire provide insights on the benefits of using oral presentation tasks in statistics courses for students.

Keywords: Attitudes, beliefs, oral presentation, statistical significance

Introduction

There is a long-standing scholarly dialogue on how to deal with recurring concerns about student motivation and achievement, and how to improve student understanding of the role of statistics in providing answers to real-world problems (Meng, 2009). Ograjenšek and Gal (2016) propose re-examining the role of qualitative thinking in the early stages of learning statistics and integrating selected elements of qualitative research methods into statistics curricula. Cobb (2007) calls for further use of technology in teaching and more focus on simulation-based instruction. The use of technology to introduce and reinforce these essential concepts has also been promoted in the Guidelines for Assessment and Instruction in Statistics Education (GAISE) for introductory statistics at the college level (ASA, 2005). Using technology in teaching statistics has been discussed in a series of publications (e.g., Chance, Ben-Zvi, Garfield, and Medina, 2007). Gordon (2004) discusses the unique challenges faced by students and instructors of statistics in a service course. Issues with background deficiencies and lack of motivation are among the most critical factors when teaching occasional users of statistics.

One of the authors teaches statistics as a service course offered for a wide range of science majors. She designed the course with a focus on conceptual understanding and an emphasis on real-world applications and automated computations. These changes enriched students' learning experience and enhanced the overall course instruction. Technology can support conceptual understanding and data visualization. However, we believe that a change in the culture of teaching statistics and the way we interact with our students is needed. In this study, we consider pedagogical strategies that could improve students' conceptual understanding of statistics, thereby also improving their statistical awareness.

One of the strategies we discuss here is use of oral presentations in class for active learning. Fan and Yeo (2007) describe oral presentation as a classroom practice where students share their ideas verbally, and check their own doubts. They discuss that oral presentation provides students an opportunity to share their understanding about a concept in their own words. These verbal communication skills are believed to promote conceptual understanding (Berry and Houston, 1995) and are believed to increase students' confidence (Butler and Stevens, 1997). Recently, in Ghosh Hajra and Hasan (2016), we discussed using assessment-based oral presentation tasks in a mathematics content course for pre-service teachers. We found that our pre-service teachers gained confidence in their ability to teach mathematics and also improved their disposition towards mathematics.

Another strategy we discuss here is the use of cooperative discussions in class for active and meaningful learning. Cooperative learning environments encourage students to interact, critique one another, and learn from each other socially (Slavin, 1996). In this setting, students work in groups, share common goals, and are each responsible for the success of their group (Johnson and Johnson, 1994). Leikin and Zaslavsky (1999) propose four necessary components, which form a cooperative learning environment—learning in small groups, group interdependence, individual accountability within groups, and equal opportunity for interaction and communication for all participants.

Various studies (Meeuwsen, King and Pederson, 2005; Sutton, 1992; Webb, 1991) suggest multiple interpersonal and intrapersonal benefits of cooperative learning. These benefits include improved communication among students and enhanced creative thinking in students. Other studies have shown connection between disposition and learning (Maas and Schlöglmann, 2009; McLeod and Adams, 1989, Philipp, 2007). Disposition is one's attitudes, beliefs, and aptness to act in positive ways (NCTM, 1989). Attitude is a mental concept representing favorable or unfavorable feelings for identifiable entities, and a belief is known or perceived information about an object (Koballa, 1998). For example, statements about likes or dislikes reflect one's feelings toward an object; a statement such as, "*Statistics is hard*" represents one's beliefs. One is most likely to perform an action if one has a favorable attitude towards it (Fishbein and Ajzen, 1975). In particular, if students have favorable feelings and beliefs about oral presentations, they are more likely to use them in their own learning.

In this study, we investigate students' general beliefs and attitudes toward the use of oral presentation tasks in learning statistics. We examine if using oral presentation activities changes student dispositions towards statistics.

Theoretical Framework

Our theoretical framework is based on three theories: theory of constructivism, social constructivism, and multiple intelligence. Theory of constructivism hypothesizes that an individual constructs meanings through his own experiences (von Glasersfeld, 1995). Theory of social constructivism hypothesizes individuals' meanings are constructed through experiences in social interaction (Brooks and Brooks, 1993). From the viewpoint of multiple intelligence theory, every individual has different learning styles. Therefore, each individual needs different forms of learning opportunities (Fan and Yeo, 2007). Hence, we use oral presentations with cooperative discussions as a learning tool in class, in addition to other formal writing assignments, quizzes, and exams. We ask our students to discuss problems in groups and present their solutions to the class.

Methodology

Study participants and course description

This study took place at a public university in the United States of America. Two sections of a statistical modeling class were involved, one designated as control and the other as the treatment group. The two sections had the same instructor, same resources, and same written assessments. The participants were science majors who took an introductory statistics course as a prerequisite to this class. This course was their second course in statistics. The majority of the students were biology majors who took the course either as an elective or as a part of the requirement for their statistics minor. This course was designed as an undergraduate course, but some graduate students took it to supplement their statistical background and to prepare for the quantitative research component for their master's theses.

Study design

We applied an experimental design in this course. One of the two sections was assigned to the treatment group (19 students) and the second section (26 students) was assigned to the control group. This assignment was made before the start of the semester, and the students were unaware of such arrangements. The lecture notes, homework assignments, and tests were identical in both sections. The only difference was that oral presentation with cooperative discussion activities were used for the treatment group and not for the control group during the lecture and statistics lab sessions.

Base groups were formed in the first week of the semester with 3–4 students in each group in the treatment group. Base groups are groups that stay together for the entire course period (Barkley, Cross and Major, 2005). Members in a group were selected randomly according to the class roster after sorting them alphabetically by last name.

The instructor implemented two assessment-based oral presentation tasks in the treatment group: pre-structured oral presentation and impromptu presentation (Fan and Yeo, 2007). Fan and Yeo (2007) described pre-structured oral presentations as tasks that are prepared by the students in advance and impromptu presentations as tasks that are performed without rehearsals. Below, we describe the two tasks in the context of this study.

Task 1: Pre-structured oral presentation: Presentations were used in the treatment group during the lab sessions. In these sessions students were using the software R to perform data analysis. The class was given a handout with a list of solved data analysis problems. Each group was assigned one example to discuss. Twenty minutes were allotted for cooperative discussion. The problems were usually a data set with a research question. Students were given some programming code, description of the dataset, and research question. Students were required to understand, discuss, and interpret the code, along with interpreting the output and answering the research question. Each group had to elect a representative to explain their assigned problem to the class. Each student presented at least twice during the semester. The instructor was available to help during the cooperative discussion, but students were instructed to ask the instructor for help only after the entire group discussed the problem.

Task 2: Impromptu oral presentation: We implemented two forms of spontaneous presentations in the treatment group: a) Student presenters had to answer instant questions related to the presentation from the audience and the instructor after the pre-structured oral presentation. b) Each student had to summarize the day's lesson twice throughout the semester in the form of a 2-minute presentation. The software R was used to randomly choose a presenter at the end of each class. Students had to take notes and be prepared for presenting their 2-minute summary each day because they did not know whose name would be chosen at the end of the class.

We used weighted assessment categories for grading both classes. For the treatment group, oral presentation activities counted 5% and written homework counted 10% of their weighted total grade. For the control group, written homework counted 15% towards the total grade and there was no oral presentation component. Cooperative lab discussions were used only in the treatment group. However, students in both groups had the option to collaborate on some written homework assignments and on a major group projects outside class.

Data collection

We collected data in the form of pre-and post-surveys (modified survey questionnaire from Fan and Yeo (2007) (see Table 1), pre- and post-tests, and end-of-term online course evaluations. The pre- and post-survey consisted of questions on students' general beliefs and attitudes toward the use of oral presentation tasks, their beliefs in their own ability to use statistics in their own research and in their own dispositions towards statistics. The pre-and post-tests/surveys were conducted on the first day and last day of class respectively. Pre-and post-tests had a question with 5 parts on analyzing some R output from a simple linear regression model. We used the anonymous online course evaluation surveys at the end of the semester as our third data source. This survey included 10 questions where students ranked the instructor on a scale of 1–5 and answered two open-ended questions reflecting their thoughts about the most engaging and exciting aspects of the class and any recommendation for improvement.

Table1
Survey questions.

1.	Oral presentations improves my understanding of statistical concepts
2.	Oral presentation skill is important in learning statistics
3.	Oral presentation skill is important in scientific research
4.	Oral presentation makes me feel inadequate
5.	Listening to other classmates' presentations help me understand other's perspectives
6.	Oral presentation is a waste of time
7.	I am very worried about presenting statistical concepts in front of my classmates
8.	I see statistics as practical and useful
9.	I would like to have more oral presentations for my statistics classes
10.	Preparing for oral presentations helps me gain deeper understanding of statistical concepts
11.	I am confident in my ability to communicate statistical concepts
12.	I feel confident in my ability to use statistics in my research or course projects

Results

Quantitative analysis

Pre-Post Surveys and Tests

To analyze the data from the two surveys, we use a numerical scale to code the responses on a scale of 1–5, where 1=Strongly Disagree and 5=Strongly Agree. Our study is twofold—we aim to investigate changes in student attitudes toward oral presentations and changes in their disposition toward statistics. Hence, we divide the survey into two categories. Questions 1–7 and 9–10 addresses general attitudes and beliefs toward oral presentations. Questions 8, 11, and 12 represent student disposition toward statistics. We create the following score functions:

Oral presentation score = (Sum of responses to questions 1–7 and 9–10)/45*100

Disposition toward statistics score = (Sum of questions 8, 11, and 12)/15*100

Here we note that 45 and 15 are the maximum possible scores for each of the above scores respectively and can be obtained by multiplying the number of questions by 5 which is the score of a *Strongly Agree* response. The numerical scores of questions 4, 6, and 7 were inverted to account for the negative wording of the statement in these questions.

We consider the responses of each student as paired data. We used the averages to replace missing values for students who filled out one survey but missed the other. The paired differences in the three scores passed Anderson-Darling's normality test at the LOS=0.01, thus we apply a paired t-test to see if the observed sample differences are significant. Table 2 summarizes the results of our statistical inference. We have a strong evidence of improvement in the disposition towards statistics in both groups. However, the treatment group showed a 13% mean increase in their disposition towards statistics score compared to a 7% mean increase in the control group. Both the treatment and the control groups did not show a significant improvement in their oral presentation scores.

Table 2

Summary of the statistical tests on the three score function. The data is considered normally distributed if the p-value of Anderson-Darling's test exceeds 0.01. The observed sample difference is significant if the p-value of the paired t-test is less than 0.01.

Group Assignment	Score functions	p-value for the normality test	Mean sample of differences (After-Before)	p-value for the paired t-test
Treatment $n = 19$	Oral presentation score	0.753	-0.662	0.5526
Treatment $n = 19$	Disposition towards statistics score	0.0472	12.918	0.0014
Control $n =26$	Oral presentation score	0.471	-1.345	0.4303
Control $n =26$	Disposition towards statistics score	0.131	7.703	0.0007

Figures 1 and 2 display side-by-side boxplots for each of the two scores we introduced. The graphs show improvement in the disposition towards statistics scores at the end of the semester in both groups. It is interesting that neither of the two groups showed improvement in their attitudes towards oral presentation. On the contrary, some of them had slightly more negative opinion towards oral presentation at the end of the semester. Few students found public speaking stressful despite the fact that they learned a lot from presenting their ideas. It is also worth noting that students in both sections were either seniors or graduate students and at this stage of their career they had fixed ideas on oral presentations. Their overall responses were favorable of the practice and the treatment group had slightly more positive attitude towards oral presentation at the beginning of the semester.

To assess learning in the two groups, we administered a test before and after teaching a unit in the class. Both groups showed significant improvement in the post-test scores compared to their pre-test scores. This can be attributed to the collective learning activities that were used in class. Figure 3 shows a comparison of the pre-test scores and the post-test for the two groups. The treatment group had lower pre-test scores compared to the control group. The post test scores highlight the discrepancy between the two groups as the treatment group outperformed the control group.

To compare the test scores of the two groups, we apply Wilcoxon's rank-sum test with continuity correction. The rank-sum test on the pre-test scores resulted in a p-value = 0.9645. Thus, there was no significant difference in the pre-test scores between the treatment and control groups. Applying the same test to the post-tests resulted in p-value = 4.497×10^{-5}. This provides strong evidence that the post-test scores for the treatment group were significantly higher than those of the control group. The median of the post test scores for the treatment group is 95% compared to 73.75% for the control group.

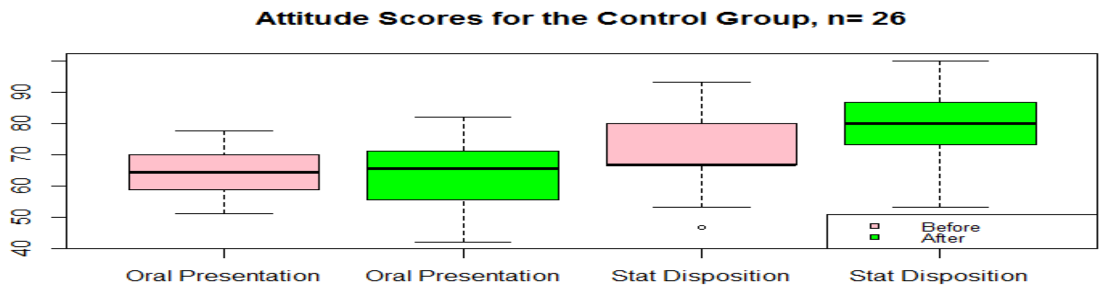


Figure 1: Boxplots for the survey responses on attitude scores for the control group before and after the oral presentation tasks.

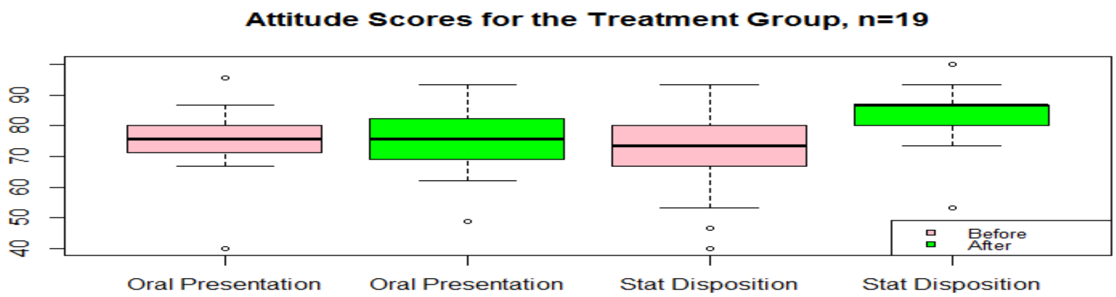
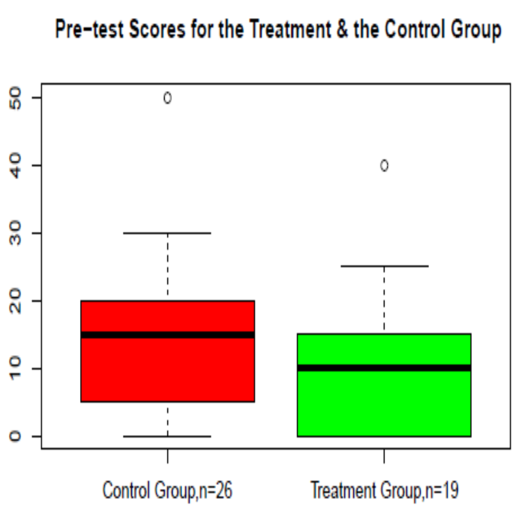


Figure 2: Boxplots for the survey responses on the attitude scores for the treatment group before and after the oral presentation tasks.



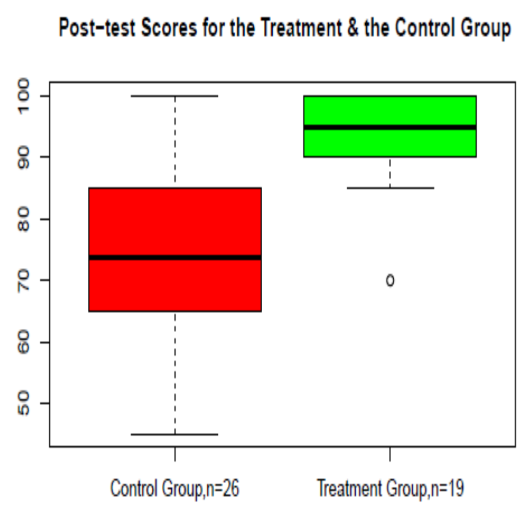


Figure 3: (Left) Comparison of the pre-test scores between the control and treatment groups. (Right) Comparison of the post-test scores between the control and treatment groups.

End-of-Term Online Course Evaluation Surveys

We used University's anonymous voluntary online course evaluation to solicit student feedback on their classes. It was interesting to compare and contrast the survey responses for the treatment and the control groups. The average instructor rating for the treatment group (n = 11) was 4.3 on a 5-point scale with a standard deviation of 0.8. The average for the control group (n = 14) was 3.9 with a standard deviation of 1.2. Overall, the treatment group had more positive comments about their experience in the class compared to the control group. The sample sizes are too small for this difference to be statistically significant. However, a difference of 8% in ranking the same instructor who used the same teaching material is probably due to the only difference in the instructional design, namely, the use of oral presentations and cooperative group discussions. Here are some written comments from the treatment group:

- *"In class it was really nice to consistently experience an active, excited discussion about the limitations and possibilities of statistical analyses."*
- *"Overall, the environment of this course made one feel that there was a strong support system to understand and apply statistics."*
- *"I think forcing all the public speaking was unnecessary and less helpful than other aspects of the class but I recognize that others may have benefited from it more."*

This course was taught again by the same instructor after completing this study. This time only one section was offered, and oral presentations and cooperative discussions were used in the class. Students' resistance to oral presentation activities seemed to diminish when they did not have another section to compare themselves to and they were not informed that they were a part of an experiment.

Conclusions

Our quantitative analyses provide evidence that implementing oral presentation with cooperative discussion tasks in teaching statistics resulted in significant improvement in the student disposition towards statistics as a field as well as their basic knowledge of the course content. Students came to class with a prior opinion on the use of oral presentations which did not change much during the class. It was an interesting coincidence possibly that the treatment group had an already more favorable attitude on the use of oral presentation tasks compared to the control group, and that the overall attitude scores did not change significantly at the end of the semester. This discrepancy could be due to the fact that the treatment group knew that they will be required to present their work before they filled the survey and their responses were influenced by the desire to please their instructor who required this additional activity in class. The students who took the class were mostly senior or graduate students, and they probably had fixed opinions on oral presentations based on their experience with other classes.

Limitations of the study

Due to the nature of the study, it was not feasible to randomly assign the students to either group. However, it is assuring that the pre-test scores indicated no significant difference between the groups. Lurking factors might include the class time (the treatment group met at 9 a.m. while the control group met at 11 a.m.). The course evaluation survey was a voluntary response survey and both sections had response rates below 60% of the total number of enrolled students. Generally, voluntary response sampling schemes tend to over represent people with strong opinions. Based on the course evaluation survey, 92.9% of the respondents in the control group were required to take the course compared to 63.3% in the treatment group. It is possible that students who elected to take the class are more motivated than those who are required to take it.

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Exploration of the Factors that Support Learning: Web-based Activity and Testing Systems in Community College Algebra

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A variety of computerized interactive learning platforms exist. Most include instructional supports in the form of problem sets. Feedback to users ranges from a single word like "Correct!" to offers of hints and partially- to fully-worked examples. Behind-the-scenes design of systems varies as well – from static dictionaries of problems to "intelligent" and responsive programming that adapts assignments to users' demonstrated skills within the computerized environment. This report presents background on digital learning contexts and early results of a cluster-randomized controlled trial study in community college elementary algebra classes where the intervention was a particular type of web-based activity and testing system.

Key words: Adaptive Tutoring System, College Algebra, Multi-site Cluster Randomized Controlled Trial

Many students arrive in college underprepared for college level algebra, despite its importance for future success in mathematics (Long, Iatarola, & Conger, 2009; Porter & Polikoff, 2012). Web-based Activity and Testing Systems (WATS) are one approach to supporting equity and excellence in mathematics learning in colleges. When it comes to technology and algebra learning in college, what works? For whom? Under what conditions? These ubiquitous questions plague educational researchers who are assessing the whats, whys, and hows of a technology intervention or addition to a course. Did the instructors have enough support to adequately implement the technology tool? Were the materials adequate to provide enough practice hours for students? Was instruction sufficient to prepare students to pass the final exam?

This preliminary report offers early results from a large project investigating relationships among student achievement and varying conditions of implementation for a web-based activity and testing system (WATS) used in community college algebra. Implementing a particular WATS constitutes the "treatment" condition in this cluster randomized controlled trial study. As described below, there are several ways to distinguish WATS tools. Some systems, like the one at the heart of our study, include adaptive problem sets, instructional videos, and data-driven tools for instructors to use to monitor and scaffold student learning.

Research Questions

Funded by the U.S. Department of Education, we are conducting a large-scale mixed methods study in over 30 community colleges. The study is driven by two research questions:

Research Question 1: What is the impact of a particular digital learning platform on students' algebraic knowledge after instructors have implemented the platform for two semesters?

Research Question 2: What challenges to use-as-intended (by developers) are faculty encountering and how are they responding to the challenges as they implement the learning tool?

Background and Conceptual Framing

First, there are distinctions among cognitive, dynamic, and static learning environments (see Table 1). Web-based Activity and Testing System (WATS) learning environments can vary along at least two dimensions: (1) the extent to which they adaptively respond to student behavior and (2) the extent to which they are based on a careful cognitive model.

Table 1. *Conceptual Framework of the Types of Instruction Based on Adaptability and Basis in a Theory of Learning*

		Static	Dynamic
Is a particular model of learning explicit in design and implementation (structure and processes)?	No	Text and tasks with instructional adaptation external to the materials	Adaptive tutoring systems (Khan Academy, ALEKS, ActiveMath)
	Yes	Textbook design and use driven by fidelity to an explicit theory of learning	“Intelligent” tutoring systems (Cognitive Tutor)

Static learning environments are those that are non-adaptive without reliance on an underlying cognitive model – they deliver content in a fixed order and contain scaffolds or feedback that are identical for all users. The design may be based on intuition, convenience, or aesthetic appeal. An example of this type of environment might be online problem sets from a textbook that give immediate feedback on accuracy to students (e.g., “Correct” or “Incorrect”).

Dynamic learning environments keep track of student behavior (e.g., errors, error rates, or time-on-problem) and use this information in a programmed decision tree that selects problem sets and/or feedback based on students’ estimated mastery of specific skills. An example of a dynamic environment might be a system such as ALEKS or the “mastery challenge” approach now used at the online Khan Academy. For example, at khanacadmy.org a behind-the-scenes data analyzer captures student performance on a “mastery challenge” set of items. Once a student gets six items in a row correct, the next level set of items in a programmed target learning trajectory is offered. Depending on the number and type of items the particular user answers incorrectly (e.g., on the path to six items in a row done correctly), the analyzer program identifies target content and assembles the next “mastery challenge” set of items.

Above and beyond such responsive assignment generation, programming in a “cognitively-based” dynamic environment is informed by a theoretical model that asserts the cognitive processing necessary for acquiring skills (Anderson et al. 1995; Koedinger & Corbett, 2006). For example, instead of specifying only that graphing is important and should be practiced, a cognitively-based environment also will specify the student thinking and skills needed to comprehend graphing (e.g., connecting spatial and verbal information), and provide feedback and scaffolds that support these cognitive processes (e.g., visuo-spatial feedback and graphics that are integrated with text). In cognitively-based environments, scaffolds themselves can also be adaptive (e.g., more scaffolding through examples can be provided early in learning and scaffolding can be faded as a student acquires expertise; Ritter et al., 2007). Like other dynamic systems, cognitively-based systems can also provide summaries of student progress, which better enable teachers to support struggling students.

No fully tested cognitively-based system currently exists for college students learning algebra. As mentioned, several dynamic systems do exist (e.g., ALEKS, Khan Academy “Missions”). The particular WATS investigated in our study is accessed on the internet and is designed primarily for use as replacement for some in-class individual seatwork and some homework. ***Note***: We report here on data collected from the first of two years. The second year of the study – which repeats the design of the first – is currently underway. Hence, we purposefully under-report some details.

Method

The study we report here is a multi-site cluster randomized trial. Half of instructors at each community college site are assigned to use a particular WATS in their instruction (treatment condition), the other half teach as they usually would, barring the use of the Treatment WATS tool (control condition). In addition, faculty participate for two semesters in order to allow instructors to familiarize themselves with implementing the WATS with their local algebra curriculum. Specifically, the Fall semester is a "field" semester to field-test the intervention and the Spring semester of the same academic year is the full "efficacy" study.

Using a stratified sampling approach to recruitment, we first conducted a cluster analysis on all 113 community college sites eligible to participate in the study (e.g., those offering semester-long courses in elementary algebra that met at least some of the time in a physical classroom or learning/computer lab). The cluster analysis was based on college-level characteristics that may be related to student learning (e.g., average age of students at the college, the proportion of adjunct faculty, etc.). This analysis led to five clusters of colleges. Our recruitment efforts then aimed to include a proportionate number of colleges within each cluster. The primary value of this approach is that it allows more appropriate generalization of study findings to the target population (Tipton, 2014). The first cohort of participants was a sample of 38 colleges similar to the overall distribution across clusters that was the target for the sample (see Figure 1).

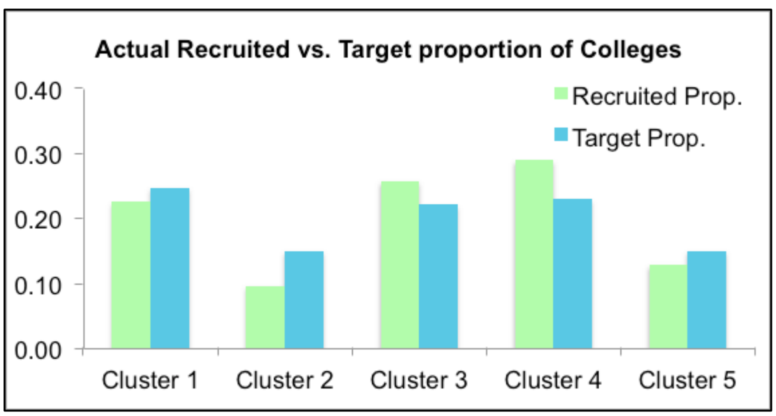

Figure 1. Recruited sample proportions and target sample proportions across clusters.

Sample for this Report

Initial enrollment in the study included 89 teachers across the 38 college sites. For this report on early results, we have used the data from 510 students of 29 instructors across 18 colleges. Student and teacher numbers related to the data reported on here are shown in Table 2.

Table 2. *Counts of Teachers, Students, and Colleges in the Study*

Condition	*Teachers*	*Students*	*Colleges*
Control	17	328	13
Treatment	12	182	11
Total	*29*	*510*	*18*

Results

The primary outcome measure for students' performance is an assessment from the Mathematics Diagnostic Testing Program (MDTP), which is a valid and reliable assessment of students' algebraic knowledge (Gerachis & Manaster, 1995). The primary aim of the quantitative analysis was to address Research Question 1, what is the impact of WATS use on students' outcomes? To this end, we employed Hierarchical Linear Modeling (HLM) (Raudenbush & Bryk, 1998) to predict students' end of semester MDTP scores. The HLM model includes a random effect of teacher to account for the nesting of students within instructors, and covariates that account for students' pretest MDTP scores at both student and teacher levels (i.e., student scores are aggregated at the teacher level; covariates were grand mean centered to achieve the intended covariate-adjustment). Importantly, in the model below, $WATS_j$ represents a dichotomous variable (dummy coded) indicating treatment assignment, and the main effect of the intervention is captured by β_{01}.

Model

$$\gamma_{ij} = \beta_{00} + \beta_{01}\boldsymbol{WATS}_j + \beta_{10}\boldsymbol{StuPre}_{ij} + \beta_{02}\boldsymbol{TeaPre}_j + \xi_{0J}\boldsymbol{Tea}_j + \varepsilon_{I0}\boldsymbol{Stu}_{ij}$$

The random and fixed effects for the model presented above are displayed in Tables 3 and 4, respectively.

Table 3. *Random Effects of the Model*

	Variance	*Standard Deviation*
Teacher ξ_{0J}	6.95	2.64
Level-1 Error ε_{I0}	37.69	6.14

Table 4. *Fixed Effect Results of the Model*

Variable	*B*	*Standard Error*	*p-value*
Intercept β_{00}	21.98	0.74	< .001
WATS β_{01}	2.59	1.17	.04
StuPre β_{10}	0.54	0.04	< .001
TeaPre β_{02}	0.35	0.17	.05

Controlling for students' pretest scores, we found that using WATS corresponded to a 2.59 point increase in students' post-test scores, a statistically significant positive effect ($p < .05$). Since the post-test is out of a 50 point total, the estimate corresponds to about 5 percentage points greater post-test score, on average, for treatment group students (2.59/50). The Hedges *g* value for this effect is 0.32, which is considered a small but noteworthy effect in educational research for studies of this size (Cheung & Slavin, 2015; Hill et al. 2008). The 95% confidence interval of the Hedges *g* value is .14 - .50.

We note this study suffered from high instructor attrition, which may bias the outcome of results. To investigate the robustness of the findings above, we are in the process of repeating this study with a second cohort of participants during the current (2016-17) academic year. Pooling the results of these two studies will help to determine the extent to which study results replicate with different populations. In this same vein, we plan to reanalyze the results using post-test scores that are estimated using item response theory (IRT). IRT is a measurement approach that takes into consideration potential differences in item characteristics when scoring individuals and places scores on a continuous metric. The use of IRT will allow us to take into consideration the difficulty and discriminability of items and represent these in the calculation of post-test scores, which can then be analyzed using the model presented above.

To address Research Question 2, a great deal of textual, observational, and interview data were gathered last year (and will be gathered again for the second iteration of the study). These data allow careful analysis of the intended and actual use of the learning environment and the classroom contexts in which it is enacted – an examination of implementation structures and processes. Indices of specific and generic fidelity derived from this work also will play a role in HLM generation and interpretation in the coming year.

As in many curricular projects, developers of the WATS in our study paid attention to learning theory in determining the content in the web-based system, but the same was not true for determining implementation processes and structures. The pragmatic details of large-scale classroom use were under-specified. Developers articulated their assumptions about what students learned as they completed activities, but the roles of specific components, including the instructor role in the mediation of learning, were not clearly defined. Thus, there was an under-determined "it" to which developers expected implementers (instructors and students) to be faithful.

Fidelity of implementation is the degree to which an intervention or program is delivered as intended (Dusenbury, Brannigan, Falco, & Hansen, 2003). Do implementers understand the trade-offs in the daily decisions they must make "in the wild" and the short and long-term consequences on student learning as a result of compromises in fidelity? As Munter and colleagues (2014) have pointed out, there is no agreement on how to assess fidelity of implementation. However, there is a growing consensus on a component-based approach to measuring its structure and processes (Century & Cassata, 2014). Century and Cassata's summary of research offers five components to consider in fidelity of implementation: Diagnostic, Procedural, Educative, Pedagogical, and Student Engagement (see Table 5).

Table 5. *Components and Focus in a Fidelity of Implementation Study*

Components	*Focus*
Diagnostic	These factors say what the "it" is that is being implemented (e.g., what makes this particular WATS distinct from other activities).
Structural-Procedural	These components tell the user (in this case, the instructor) what to do (e.g., assign intervention x times/week, y minutes/use). These are aspects of the *expected* curriculum.
Structural-Educative	These state the developers' expectations for what the user needs to know relative to the intervention (e.g., types of technological, content, and pedagogical knowledge needed by an instructor).
Interaction-Pedagogical	These capture the actions, behaviors, and interactions users are expected to engage in when using the intervention (e.g., intervention is at least x % of assignments, counts for at least y % of student grade). These are aspects of the *intended* curriculum.
Interaction-Engagement	These components delineate the actions, behaviors, and interactions that students are expected to engage in for successful implementation. These are aspects of the *achieved* curriculum.

The components in Table 5 are operationalized through a rubric, the guide for collecting and reporting data in our implementation study. A rubric articulates the expectations for a category by listing the criteria, or what counts, and describes the levels of quality from low to high. Each component has several factors that define the component. The research team has developed a rubric for fidelity of implementation that identify measurable attributes for each component (for example, see Table 6 for some detail on the "educative" component).

Table 6. *Example Rubric Descriptors for Levels of Fidelity, Structural-Educative Component.*

Educative: These components state the developers' expectations for what the user (instructor) needs to know relative to the intervention.			
	High Level of Fidelity	*Moderate Fidelity*	*Low Level of Fidelity*
Users' proficiency in math content	Instructor is proficient to highly proficient in the subject matter.	Instructor has some gaps in proficiency in the subject matter.	Instructor does not have basic knowledge and/or skills in the subject area.
Users' proficiency in TPCK	Instructor regularly integrates content, pedagogical, and technological knowledge in classroom instruction. Communicates with students through WATS.	Instructor struggles to integrate CK, PK, and TK in instruction. Occasionally sends digital messages to students using WATS tools.	Instructor CK, PK, and/or TK sparse or applied in a haphazard manner in classroom instruction. Rarely uses WATS tools to communicate with students.
Users' knowledge of requirements of the intervention	Instructor understands philosophy of WATS resources (practice items, "mastery mechanics," analytics, and coaching tools),	Instructor understanding of the philosophy of WATS tool has some gaps. NOTE: Disagreeing is okay, this is about instructor knowledge of it.	Instructor does not understand philosophy of WATS resources. NOTE: Disagreeing is okay, this is about instructor knowledge of it.
Users' knowledge of requirements of the intervention	Instructor understands the purpose, procedures, and/or the desired outcomes of the project (i.e., "mastery")	Instructor understanding of project has some gaps (e.g., may know purpose, but not all procedures, or desired outcomes).	Instructor does not understand the purpose, procedures, and/or desired outcomes. Problems are typical.

Defining and Refining Measures for the Fidelity of Implementation Rubric

The ultimate purpose of a fidelity of implementation rubric is to articulate how to determine what works, for whom, under what conditions. In addition to allowing identification of alignment between developer expectations and classroom enactment, it provides the opportunity to discover where productive adaptations may be made by instructors, adaptations that boost student achievement beyond that associated with an implementation faithful to the developers' view.

In using the rubric, we assign a number to each level of fidelity. This can be as simple as a 3 for a high level of fidelity, 2 for a moderate level of fidelity, or a 1 for a low level; or the items can be weighted. The general score for the intervention will be the total number of points assigned in completing the rubric as a ratio of the total possible, across all instructors. It will also be possible to create a fidelity of implementation score on each row for each instructor – these data will be used in statistical modeling of the impact of the intervention as part of a "specific fidelity index" (Hulleman & Cordray, 2009). We first total points for the item, then the component, and finally all components for a single score as an index of implementation.

Next Steps

In upcoming work, we will analyze a host of data on students, teachers, and colleges that may influence learning with WATS, including issues of feasibility of use in differing contexts, and measures associated with the nature of alignment or "fidelity" of implementation to WATS developers' expectations. Such analysis will help to inform important questions such as how and for whom WATS are most effective.

As indicated above, we will continue this study with a second cohort of new participants who will repeat the year-long study in the 2016-2017 academic year. Also, between now and the conference we will do more complex modeling of the data, with the introduction of IRT-informed scores and specific fidelity indices. Our specific objectives in the coming six months are to (1) continue analyses from the Spring 2016 efficacy study, and (2) conduct the field-test semester of the study with second cohort of participants.

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Supporting Instructional Change in Mathematics: The Role of Online and In-person Communities

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While studies continually show benefits of active learning strategies like inquiry-based learning (IBL), it is difficult to get faculty to adopt these methods. Particularly challenging is the third and final stage in Paulsen and Feldman's (1995) model, 'refreezing,' when instructors use feedback and support to decide whether to continue with the instructional changes they have made or return to their previous methods. In this paper, we show how a workshop to teach college mathematics instructors to implement IBL used both online and in-person communities to help provide the ongoing feedback and support necessary for 'refreezing.' We offer lessons for how to increase the relevance of and participation in online support communities. We also use an innovative analytical approach, Social Network Analysis, to understand the ongoing processes of how e-mail exchanges provide feedback and both intellectual and emotional support to workshop participants.

Keywords: Inquiry-Based Learning, Pedagogy, Instructional Change, Professional Development

Numerous studies have found benefits for the use of active learning methods in science, technology, engineering and mathematics (STEM) fields (Freeman et al., 2014). Freeman et al. (2014) stated that the benefits are so strong that, "If the experiments analyzed here had been conducted as randomized controlled trials of medical interventions, they may have been stopped for benefit—meaning that enrolling patients in the control condition might be discontinued because the treatment being tested was clearly more beneficial" (p. 4). While the evidence in support of the use of active learning strategies is strong, getting large numbers of faculty to adopt new methods is difficult (Fairweather, 2008; Henderson & Dancy, 2007; 2008; 2011). Professional development workshops are one strategy for helping instructors to adopt research-supported teaching methods. Workshops are the preferred method of National Science Foundation (NSF) program directors, particularly when they are multi-day, immersive workshops and include follow-up interaction between participants and organizers (Khatri, Henderson, Cole, & Froyd, 2013). Professional developers often include follow-up interactions in their designs through group email lists, reunion meetings, and mentoring (Council of Scientific Society Presidents, 2012), but there is almost no evidence that these plans are acted upon or whether they matter. In one study of a series of three workshops, only one of them successfully implemented the plan to engage participants in an email group (Hayward, Kogan, & Laursen, 2016).

In this report, we present findings from a weeklong, intensive workshop that engaged almost all participants in follow-up activities. This workshop was designed to help college mathematics faculty implement Inquiry-Based Learning (IBL) in their classes. IBL has its roots in the teaching methods of mathematician R.L. Moore (1882-1974) (Mahavier, 1999), but the term IBL is now used more broadly to include various practices that share the spirit of student inquiry through the core features of (1) deep engagement with rich mathematics and (2) collaboration with peers (Yoshinobu & Jones, 2013).

Conceptual Framework

Rather than impose a conceptual framework from the start, we let one emerge from the analysis and found a good fit to a three-stage model of instructor change developed by Paulsen and Feldman (1995), based on Lewin's (1947) theory of change. These authors described three stages: (1) *unfreezing*, (2) *changing*, and (3) *refreezing*. During *unfreezing* instructors gain motivation to change through experiencing incongruence between their goals and the outcomes of their teaching practices. In *changing*, instructors learn, apply, and reflect on new teaching strategies to help align their behaviors with desired outcomes. While teaching strategies may be fluid during this stage, in the final stage, *refreezing*, either the new strategies are 'reconfirmed' through positive feedback and support, or the instructors return to their original strategies.

In this paper, we present data to show that participants have applied their learning from the workshop by implementing IBL techniques and have therefore completed a crucial step of stage two, *changing*. However, our main focus in this paper is on how this workshop supported instructors through the *refreezing* stage by providing ongoing support and positive feedback, since little is known about this third stage (Connolly & Millar, 2006). Follow-up support can be challenging when workshop participants are from geographically diverse institutions and cannot easily reconnect face to face, but here organizers of this workshop leveraged other IBL-related resources and events and a group email list to help workshop participants through the *refreezing* stage. Some authors have argued that these "external networks of like-minded colleagues… can be important forces in promoting instructional reform" (Fairweather, 2008, p. 27), as they "provide a trusted community for faculty who may [have] no other support" (Stains, Pilarz, & Chakraverty, 2015, p. 1474). This is especially important given evidence that collegial interactions influence changes in instructional practices, even more than direct participation in professional development for some teachers (Penuel, Sun, Frank, & Gallagher, 2012). We explore the following research questions about *refreezing*:

(1) What resources can workshops use to support participants through *refreezing*?
(2) How can participation in online support resources be maximized?
(3) What are the processes of providing 'reconfirmation' through positive feedback and emotional and intellectual support?

Methods

This paper comes from a larger study of four workshops on IBL for college mathematics instructors held between 2013-2015. Each of the workshops was four days long and featured a mix of sessions designed to help participants learn about IBL and prepare to implement it in their classrooms. The workshop included four types of sessions that all featuring active participation and self-reflection, which are characteristics of effective professional development (Cormas & Barufaldi, 2011; Garet, Porter, Desimone, Birman, & Yoon, 2001). The sessions were: (1) Reading sessions to discuss research, (2) Video sessions to watch and analyze IBL classes, (3) Nuts & Bolts sessions to develop plans for how to structure and run an IBL class, and (4) Course Content sessions to work in small groups to develop IBL materials for their own courses. Data in this study come from the second of these four workshops, though all were very similar and post-workshop listserv cohorts functioned similarly as well.

We use a mixed-methods approach to understand how the workshop supported participants through *refreezing.* Through our quantitative analysis of survey data and tracking of email group activity, we share results about participants' engagement with support resources. We are currently conducting an innovative analysis of activity in the post-workshop email listserv to reveal the processes of *how* the email listserv helped to provide positive feedback and support participants both emotionally and intellectually. We are applying techniques of social network analysis (SNA) to examine the interactions between participants and staff via e-mail. SNA is appropriate for this context since emails are inherently social connections and these methods allow us to examine support processes as they develop over time.

For all four workshops, we collected surveys pre-workshop, post-workshop, and one academic year later. The surveys included all of the same items as workshop surveys published and presented previously (Hayward, Kogan, & Laursen, 2016), as well as some new items to assess participants' engagement with support resources. Survey data were analyzed using SPSS v. 21. Open-ended items were coded in Microsoft Excel. We tracked and coded listserv messages using Microsoft Excel, and we are performing a social network analysis on this message data in the statistical program *R* using the packages, *Social Network Analysis* and *iGraph.*

Results

The workshops served 35 participants. Of these, 54% identified as women and 43% as men (3% did not respond). They represented a variety of career stages and institution types, though most were untenured faculty (54%) and from four-year colleges (69%). Most were early in their careers with 2-5 years of teaching experience (51%) or less than 2 years (17%). All 35 participants (100%) completed the pre- and post-workshop surveys and 28 (80%) completed the one-year follow-up survey. Surveys were matched by respondent using anonymous identifiers.

Implementation of IBL

In order to assess whether participants had gone through the stage of *changing* their instructional methods, we assessed implementation of IBL in a number of different ways. On one-year follow-up surveys, 26 (93%) of 28 respondents reported using IBL methods in their classes, ranging from "some IBL methods" (25%) to "more than one full course" (39%). Seven did not answer the survey. In messages sent on the listserv, 30 of the 35 participants (86%) made comments indicating that they were using IBL methods. Combining these measures reveals that at least 31 participants (89%) were using at least some IBL methods. Changes in teaching practices from pre-workshop to one-year follow-up were consistent with IBL practice as well: decreases in instructor-led activities and increases in some student-centered activities, as detailed in Figure 1. Overall, these multiple data sources indicate very high usage of IBL methods.

Follow-up Activities, Resources, and Supports

Making a change (i.e. implementation of IBL) indicates completion of stage two of Paulsen and Feldman's instructional change model. Stage three, *refreezing,* is characterized by sustained use of new instructional methods reinforced by positive, ongoing support and feedback. To assess support and feedback, we asked participants on follow-up surveys to report their engagement with various IBL-related events and resources.

IBL events. Of the 28 respondents to the follow-up survey, 68% had participated in another IBL-related event in the year following the workshop. Events included IBL sessions at JMM, MAA meetings, MathFest, and the annual IBL conference. Some also presented at these events (7, 25%). Together, this means 22 participants (79%) attended or presented at another event.

While some events were more popular than others, it appears that the variety of related events made it possible for many participants to engage in at least one.

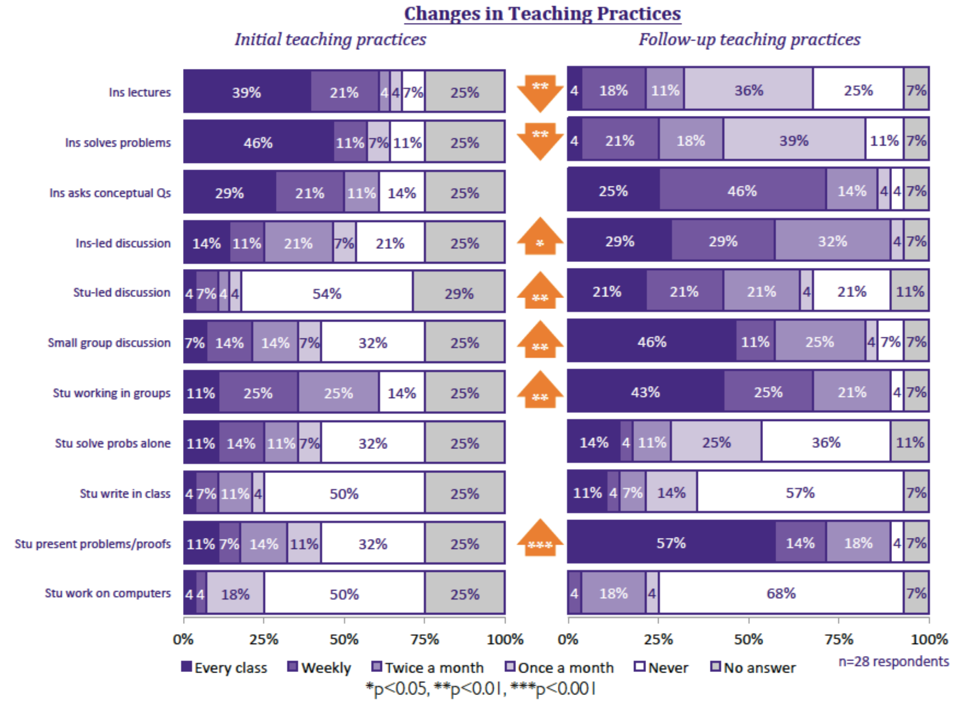

Figure 1. Comparison of teaching practices from pre-workshop to one-year follow-up.

IBL resources. Most respondents to the follow-up survey (25, 89%) reported using IBL resources. The resources used most frequently were the workshop listserv (21, 75%) and the *Journal of Inquiry Based Learning in Mathematics* (12, 43%). A few participants used the mentor program and mini-grants, which were both provided by the Academy of Inquiry Based Learning (AIBL). Again, participants engaged with a number of different resources, and the variety available may help to meet the various needs of different participants.

IBL support. On the follow-up survey, we asked participants to report the level of support they received from their institutions. Overall, the majority of participants felt supported at their institutions and rated key individuals as 'mostly supportive,' including department heads/chairs (64%) and departmental colleagues (50%). In describing how they were supported, some participants reported active support (12, 43%) through either encouragement or financial support, while others reported the absence of obstacles: they were 'free to do what they wanted' (5, 18%). Only a few reported feeling doubted or discouraged by colleagues (3, 11%). While these numbers suggest that collegial resistance is not a barrier for most, only 43% of respondents received the active support crucial to *refreezing*.

Participation in Online Follow-Up Activities

Past efforts by professional developers to engage workshop participants in online follow-up communications have failed (e.g. Hayward, Kogan, & Laursen, 2016). Social networking platforms like Facebook or Google+ seem like a good way to keep in touch, but those efforts too were unsuccessful for this workshop. In a semi-official capacity, one participant of the workshop created a private group on a social networking platform, and 24 people (55%) from the workshop joined it. Only 4 people (9%) exchanged 8 messages in the 2 days following the workshop, and in the two years since then, the group has had no activity. In contrast, 32 (91%) of workshop participants were active on the official group listserv at some point in the year after the workshop. In total, they exchanged 282 messages over the year. These observations suggest that 'push' technologies that deliver messages directly are more successful than requiring participants to log in and seek them out. Below, we share some other findings about why this group listserv was able to successfully engage participants in follow-up when other approaches have failed.

Promoting participation. Organizers sometimes prompted the list to generate discussions, but all members were free to send messages to the entire group at any time. The list was closed but un-moderated. Frequencies of individuals' participation are shown in Figure 2 below.

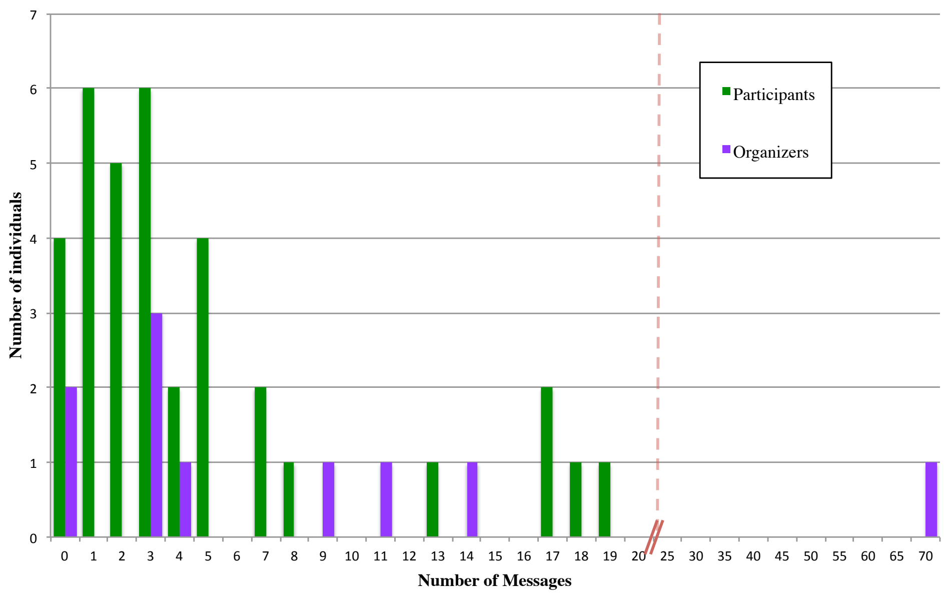

Figure 2. Frequencies of individuals' participation.

Most individuals sent fewer than 10 messages to the list in the year following the workshop. In fact, 27 participants (77%) sent 5 or fewer messages. Among organizers, most also sent fewer than 10 messages to the list. However, one organizer sent 66 messages, or about ¼ of all messages on the list. Many of these were announcements of upcoming events or resources, while others were prompts inviting check-ins from participants and raising topics for discussion. Such organizer prompts were often followed by flourishes of activity, including from participants who had not yet been active in the listserv, as shown in Figure 3.

In February, organizers also prompted individuals who had not yet been active in the group listserv. They checked in individually with these quiet participants in order to see how things were going and to offer assistance. Most of these participants responded individually, and two also later participated in the group list. In total, 94% of all 35 workshop participants were active in online 'e-mentoring,' either individually or as a group.

Relevance and helpfulness of discussions. As shown in Figure 3, organizers frequently prompted the list. The timing and content of these prompts are important. The biggest concentration of activity was in the fall (August through mid-October) soon after the summer workshop. This is an important time as many instructors prepare and then begin teaching their first IBL class. Organizers prompted the list with relevant topics, such as asking participants to share their 'first day plans,' offering advice, and checking in about how 'first-day' activities went. The listserv was relatively quiet again until another small burst of activity in January. Again, organizers provided relevant prompts by asking participants to reflect back on what did and did not go well during their first term using IBL, and querying participants' plans for the upcoming term. For the rest of the academic year, activity came in short bursts for a day or two as participants responded to a specific question or announcement. Over time, participants became more likely to initiate new exchanges, though organizers continued to pose occasional prompts.

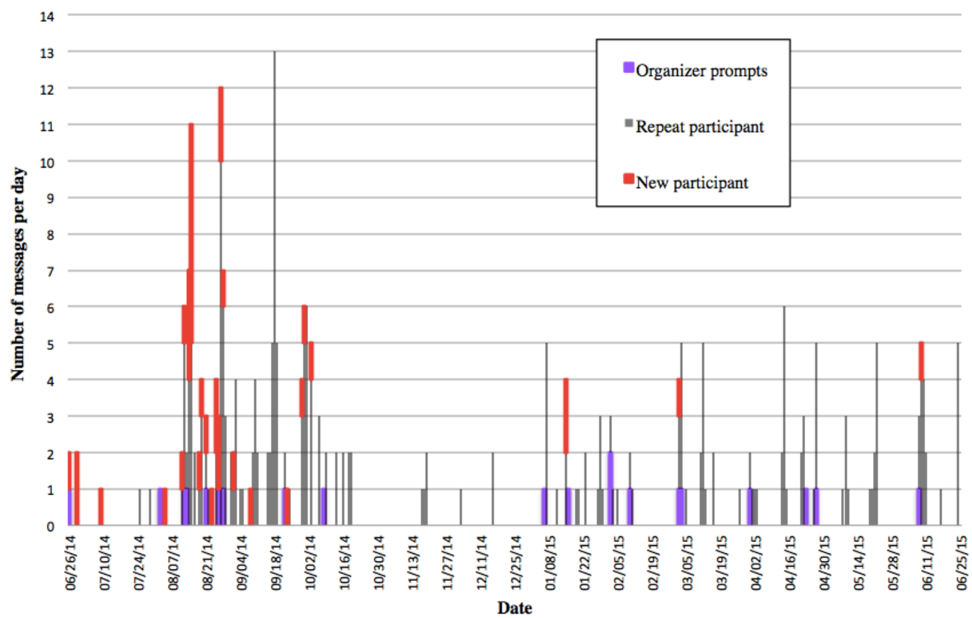

Figure 3. Chronological activity on the group email listserv.

On follow-up surveys, 39% said the list was a "great help," and 22% said it was "much help." None reported that it was "no help." Individually, most participants only sent a handful of messages and some did not engage with the list for a long time. However, evidence suggests that the collective discussions still helped even when they were not active participants: a few of the latest 'new participants' offered long, thoughtful reflections that included references to earlier discussions, suggesting they had been actively tracking and mentally engaging in the online cohort even though they had not posted. Moreover, some new participants stated that they had been 'lurking' and found ongoing discussions helpful.

Networks formed through email exchanges. While we have provided evidence of use of follow-up support resources, the ongoing process of reconfirmation through positive feedback is harder to assess with a one-time follow-up survey. However, while observing listserv traffic throughout the course of the year, we noticed that feedback loops sometimes developed: participants might raise a concern to the group, others would provide suggestions, and then the original participant would report back after testing out the suggestions. Such social exchanges prompted us to analyze not only the frequency but the content of messages sent on the listserv, using SNA methods to reveal the *process* of refreezing. We are coding each message to reveal its function and content, and to document how it relates to previous messages. For example, one message's content may be to celebrate (e.g. "My IBL class went so well today…") while another message may function as a connection by celebrating somebody else (e.g. "That sounds like an awesome class! Good job!"). Our coding scheme attends to the emotional function of messages—such as celebrating or commiserating—as well as intellectual content, such as trouble-shooting and idea-sharing, because we observe the list traffic to include a perhaps surprising degree of emotion-related messages. We suspect this interpersonal support may be particularly helpful in normalizing challenges of new implementers. The tone set by organizers also appears crucial to establishing the trust needed to share and seek help about teaching challenges.

We are exploring the relationships between content of messages and the type of connections that follow it in order to help understand *how* the participants and staff used the listserv during the *refreezing* stage. For example, we hypothesize that messages that seek information or help will tend to be followed by many messages in which others share opinions, advice, and resources, whereas a broadcast announcement will be followed up by fewer messages, largely to say thanks or expand on the initial announcement. This coding is painstaking but is well underway and will be finished by the RUME meeting in February, where we will present additional findings on the nature of listserv communications and how they support *refreezing*.

Discussion

Evidence from both the follow-up surveys and activity on the group listserv indicates that the workshop was largely successful in helping instructors through *changing* by beginning to use IBL practices in their classes in the year following the workshop. The third and final stage of Paulsen and Feldman's (1995) theory of instructor change, *refreezing*, often presents the biggest challenges, so support and positive feedback during this time are crucial so that participants do not return to their original strategies (Connolly & Millar, 2006). Feedback and support are most readily available at one's home institution, but only about 43% of participants described active support from colleagues. While support is essential during *refreezing*, more than half of survey respondents reported no active, local support.

Workshop participants and organizers can serve as another source of support for instructional change, but geographic distances make this challenging. The workshop organizers incorporated various methods to overcome these distances to help support participants through the *refreezing* stage, and participants reported high rates of IBL implementation. Participants took advantage of various IBL-related events and IBL-related resources. Some participants took part in in-person events, but the group e-mail listserv was the most widely used support. Both survey responses and participants' comments in emails indicate that the listserv was helpful.

The findings here show why this method of online support was successful when others have failed, and offer some lessons for those looking to support instructors involved in changing their teaching practices: First, online communication can be an effective way to help instructors interact with like-minded, supportive colleagues, especially if they do not have any at their home institutions. Second, listservs and other 'push' technologies that deliver relevant messages directly to participants seem to make it more likely that participants will engage, in comparison to methods that require participants to actively seek out information and help, such as online communities or forums. These findings are encouraging because listservs are relatively easy and inexpensive to set up and manage. Third, listservs function better when somebody is providing prompts that both encourage participation and keep discussions relevant to participants' experiences, in effect enabling list participants to provide each other with "just-in-time" implementation support as they try the new teaching strategies.

Ongoing analysis of listserv messages will reveal how participants and staff engaged in two important processes of the *refreezing* stage: (1) supporting each other emotionally and intellectually and (2) providing positive feedback by improving their implementation of IBL practices through trouble-shooting and idea-sharing.

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Student Proficiency with Transformational Geometry After a College Proof-Based Geometry Class

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Temple University

This report explores pre-service teachers' proficiency with concepts of transformational geometry at the end of a semester-long advanced geometry course. During the course, the instructor presented transformational geometry content, including congruence proofs, in an attempt to align with the Common Core State Standards for Mathematics. At the end of the course, the students, all pre-service teachers, appeared to mix ideas from the traditional approach involving triangle congruence criteria (SAS, ASA, SSS, AAS) and transformational approaches, and struggled with conceiving of transformation functions as objects. These difficulties appear to compound in their proof-writing attempts such that after citing appropriate transformational geometry ideas, such as the angle- or distance-preservation property, they would then supplement with congruence-based approaches in order to finish the proof. This has implications, especially for professional development, as this is the final mathematics class that these preservice teachers will take concerning transformational geometry prior to beginning their classroom instruction.

Key words: transformational geometry, proof

Identifying what prospective teachers know about geometric transformations gains new significance when we consider the prominence these functions are given in the Common Core State Standards in Mathematics (CCSSM). The draft of the Progressions for CCSSM in Geometry, 7-8, High School (Common Core Standards Writing Team, 2016) states that congruence should be defined in terms of rigid transformations, and that the traditional triangle congruence conditions (SAS, ASA, SSS, AAS) should then be proven using this definition. That is, students are expected to:

> *Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.*

The Progressions borrow heavily from Wu (2013), who points out that this approach allows one to define congruence independently of polygons and who also argues that the transformational approach connects the formal definition taught in high school geometry to the intuitive understanding of congruence that students develop in middle school. While there are important mathematical reasons to make such a change, it represents a substantial break from past practice and has significant implications for curriculum, inservice teacher education, and preservice teacher education. This contributed report approaches this complex issue of curricular change at a very granular level; exploring how preservice teachers in a university-based geometry course make sense of and write proofs about transformational geometry, as well as coping strategies they exhibit.

Harel (2014) cautions against heavy reliance on plane transformations due to the difficulties students and teachers have in understanding the transformations themselves. Some of these challenges and misconceptions include identifying the domain of a transformation as some subset of the plane rather than the plane itself (Hollebrands, 2003; Harel, 2014; Yanik, 2011), having an action or process conception of function rather than an object conception (Harel, 2014;

Hollebrands, 2003; Portnoy, et al., 2006), having a motion rather than mapping conception of transformations (Yanik, 2011), struggling to understand parameters such as the vector that defines a translation (Hollebrands, 2003; Yanik, 2011), and viewing geometric objects as perceived rather than conceived (Portnoy, et al., 2006) or drawings rather than figures (Hollebrands, 2003). In particular, Portnoy, et al. interviewed prospective high school teachers in a college geometry course and found that the students' conceptions of geometric objects as perceived objects and their process conceptions of transformations hindered their abilities to prove congruence using transformations rather than corresponding congruent parts.

We also note that this work is situated in the context of proof-based advanced mathematics classes and draws on student's proof-writing proficiency. For the sake of brevity, we do not report all of this research here, rather we note that there are multiple handbook chapters (e.g., Harel & Sowder 2007; Stylanides, Stylanides, & Weber, in press) describing the literature, and a comprehensive summary of undergraduate mathematics majors' difficulties with proof is given in Selden and Selden (2008) as well as Weber (2003). Student's difficulty with proof is well-documented (Moore, 1994, Weber, 2001). In both reading (e.g., Hodds, Alcock, & Inglis, 2014) and writing proofs (e.g., Raman, 2003; Weber & Alcock, 2004), they are more likely to focus on symbol-manipulation rather than the logical connections or connecting proofs to informal ideas (e.g., Raman, 2003). These certainly limit their ability to write proofs, but also to gain understanding from proofs that they produce (e.g., Weber, 2005; Weber & Alcock, 2009). Another important source of difficulty for students in writing proofs is that they do not appear to appreciate the fact that proofs are based on definitions of concepts (Alcock & Simpson, 2002; Bills & Tall, 1988). Finally, there is more recent research examining what students learn from lectures in advanced mathematics and why they appear to learn different things than the professors intend (e.g., Lew, et al., 2016). Students misunderstood terms that the professor used to convey content, focused on the symbolic manipulations that the professor wrote on the board, while ignoring or forgetting the conceptual explanations and ways of thinking about content and proofs that the professor said aloud (and described as the salient points of the lecture). That is, there appear to be several mismatches in communication styles between students and professors that inhibit student's comprehension of lectures about proofs. Thus, this study investigates the following research questions:

1) What resources (knowledge, skill) have students demonstrated, in terms of content, from their undergraduate geometry course to write proofs about transformational geometry, and are those sufficient to write proofs about congruence via transformations?
2) When writing proofs about congruence via a transformational approach, what proficiencies do students display? What difficulties do they display?
3) What are coping strategies that they exhibit when they encounter difficulties?

Theoretical Commitments

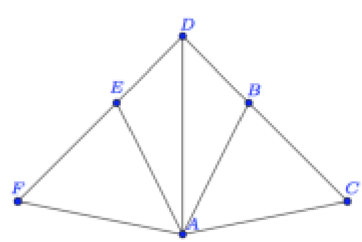
We hold three important theoretical commitments; the first is that students, and others, make decisions to act via practical rationality (c.f., Herbst & Chazan, 2003), making decisions about how to act in circumstances in order to achieve the best possible outcome in a given situation within the constraints that they perceive. In the case of students in this study, the context that most shapes their behavior is the fact that this is a proof-based collegiate mathematics class. As a result, we also draw on the notion of the *didactical contract* (Brousseau, 1986/1997), which can be understood as structuring the economy of the classroom in which students exchange work for grades (especially on exams). This student work addresses prompts that professors set, where research suggests (e.g., Herbst, 2003; Voigt, 1994) that there is frequent negotiation, often tacit, between professors and students about how to engage in a particular task. We claim that the use of partial credit when grading is one such form of negotiation--professors can, bluntly, give students a measure of how close they have come to fulfilling the expected role. In terms of what work, on the student's part, fulfills the expectations for proof within a particular domain, this negotiation typically begins with a professor's classroom presentation of proofs about and drawing on particular ideas and continues with the student's completion of homework about the topic (Thurston, 1994), indicating the needed elements and appropriate detail for a proof. The professor then grades the homework, using scores to indicate whether, and how close, the work comes to fulfilling the professor's understanding of the student responsibilities. The students can then use these marks to further calibrate their understanding of the didactical contract. This process is repeated on exams--although in the case of final exams, the marking will no longer serve as formative feedback to the students, only a summative evaluation of the professor's evaluation of how close the student came to fulfilling their responsibilities.

Methods

The Institution, Class, and Participants

Data from this study was collected in an undergraduate geometry class at a mid-sized urban doctoral-granting institution along the east coast. The course was an advanced, proof-based geometry course with an introduction to proof (or equivalent) prerequisite. The researchers solicited consent from the students to use data from the final exam after the end of the course and after grades had been submitted, and 17 of the students agreed to participate. The course description indicated that the course served as an introduction to Euclidean and non-Euclidean geometries, emphasizing proofs. The catalog also describes the course as targeted at pre-service secondary mathematics teachers. Because of this, the instructor (the first author) purposefully introduced transformational geometry, defining congruence and similarity via transformations. The instructor also stated and proved a variety of theorems about and with transformations. She also did examples of the types proofs that students would be expected to produce using a transformational approach. The data from this study is drawn from one three-part problem on the final exam. (Three out of nine problems on the exam focused on transformational geometry.)

Assume in the figure below that $\angle FDA \cong \angle CDA$ and $\angle DAE \cong \angle DAB$. Let R_1 be an isometry that maps $\triangle DEA$ to $\triangle DBA$.



(a) What could R_1 be? Define R_1 in terms of the given figure. (*You do not need to justify your definition at this point.)*

(b) Is R_1 unique? Why or why not?
(c) Using your definition of R_1, justify that $R_1(E) = B$. (*You should NOT refer to triangle congruence theorems!*)

In line with the expectations of the CCSSM, this task was designed to assess whether students could produce a proof based solely on the properties of transformations. In part (a), students had to identify the appropriate transformation as a reflection across line DA. Part (b) was intended as an independent question addressing the idea that the images of three non-collinear points uniquely define an isometry. Finally, part (c) required that students first cite the angle-preservation property of transformations to state that ray DF maps to ray DC and that ray AE maps to ray AB. The proof would then be completed by observing that while E represents the intersection of ray DF with ray AE, B represents the intersection of ray DC with ray AB, and therefore, the uniqueness of the intersection requires that the image of E is B. There is no way to ask this kind of question without giving students sufficient information to do it; that is, students cannot be asked to prove that triangles are congruent without giving them sufficient information to prove that the triangles are congruent. However, by giving them enough information and then focusing entirely on the justification, it may seem strange in part (c) to justify that R(E) = B when, in the preamble to question (c), R maps E to B by definition. The goal of (a) and (c) was to have the students create a definition in part (a) that they would draw on in developing their proof for part (c).

Methodology for Analysis

To analyze the definitions of R_1 that the students offered, we first examined whether they correctly identified the appropriate transformation as a reflection. Second, we analyzed whether the student specified a line (or line segment) to serve as the axis of reflection, which we then described as mathematically appropriate or not. Finally, we noted whether the students identified the images of particular points under the isometry. To analyze student responses to part (b), we first parsed whether the students correctly stated whether the transformation is unique, marking their responses as correct or incorrect. Subsequently, we evaluated the offered justifications; positing ways of thinking about the figure and transformations that would be consistent with the combination of the given justification and uniqueness response. For this work, we engaged in thematic analysis (Braun & Clarke, 2006), looking for conceptions such as having a motion rather than mapping conception of transformations (Yanik, 2011), and viewing geometric objects as perceived rather than conceived (Portnoy, et al., 2006) or drawings rather than figures. We used our analysis of parts (a) and (b) as an indicator of their understanding of transformations. Finally, to analyze students' proofs in part (c), we first indicated whether each student used a transformational approach, particularly the transformation that the student defined in (a). If the student did not, we described what approach the student took, specifically looking at the rate at which the inappropriate approaches drew on triangle congruence criteria (even though specifically warned not to). We noted whether the student carried out a direct or indirect proof, which properties the student drew on, such as distance preservation, and any claims that were needed but omitted (for example, about the intersection of rays). Finally, we gave a holistic evaluation of the proof-attempt; indicating what about it was productive and what was unproductive--where, by productive, we mean whether we could hypothesize a direct way for the work to lead to a correct proof. When students performed unproductive actions, we noted what was problematic. For each student, we read across their responses (and our analysis) to the three prompts and developed a summative description of their work, and hypothesized a conceptual model of transformations that the student might hold that would lead them to produce the exhibited work.

Data and Results

We first present summaries of student performance on items (a) and (b), then the analysis of their work on item (b), followed by a description and analysis of their proof-productions. Thus, we first note that 16 of the 17 students correctly identified that R_1 could be a reflection across the line DA. The one student who did not state that R_1 was a reflection stated:

R(D) = D, R(E) = B, R(A) = A

In this case, the student determined the images of three points under the transformation. Since the class had discussed that the images of three non-collinear points define a transformation, the evidence suggests that the student was claiming that the transformation was a reflection. The other 16 students made three different types of claims:

- 9 students made a claim that "R could be a reflection across the line DA" (or AD). One student added, "This would reflect everything on the left of AD to the right of AD but AD is fixed."

The first clause gives little insight into student thinking. The phrasing raises the question of whether the student imagines part, or all, of the figure moving (e.g., half at a time).

- 4 students made a claim similar to "Reflection with line of symmetry DA such that R(D) = D and R(A) = A" where they both named the line of symmetry and stated images of particular points.
- 1 student stated: "R would be a reflection across line DA. R(triangle DEA) = triangle DBA" (which implicitly draws on triangle congruence)

On Item (b), 9 students stated that the transformation was not unique, while 8 stated that it was, but only 4 students gave a correct justification. An example of such a justification:

No! This same isometry could have been done as a glide reflection and translation, etc. Three reflections could have worked.

In this case, the student gave a correct statement, and indicated knowledge of the three-reflections theorem that the professor presented in class, which stated that any isometry is equivalent to one, two, or three reflections. The students who gave incorrect justifications gave responses that indicated a variety of ways of understanding reflections and transformations generally.

Seven students gave a response that indicated that their understanding of unique implies that only one 'move' can be used (ignoring the idea that a composition of transformations is a transformation). These students wrote claims such as, "Yes because any other isometry would require more than one isometry." This type of claim suggests that the student has a conception that an isometry must be only one 'move,' rather than a composition of possible transformations. This way of thinking suggests that students have a definition of isometry as a 'move' rather than as a map. Similarly, one student wrote, "Yes, by definition isometries are onto and one-to-one which imply uniqueness." This claim that "onto and one-to-one" implies uniqueness is interesting, but not mathematically correct. There were two instances where students claimed that a rotation would suffice; one claiming a rotation of 180 degrees and another student claimed a rotation of 360 degrees. These students' responses do not give much insight into their thinking, but one hypothesis is that the first believes a rotation of 180 and reflection to be equivalent. The second may have been attempting to 'land' the figure back in the original image, but have not attended to the position of the points—essentially attending to the appropriate relationship between figure and ground, while ignoring others. Finally, there were two responses that suggest the students were writing a justification because one was asked for, but with no relationship that the researchers could hypothesize to the claims, for example, "it bisects $\angle D$ and $\angle A$." That is, apart from the students who gave correct responses and justifications and the two students who seemed to be attempting an answer, it appears that the students' justifications show that the concept of uniqueness of transformations is difficult for students. It appears that they are focused on the way of 'moving' the points rather than the outcome of the transformations. To relate this to the literature, they may be making the figure/plane distinction that Yanik (2011) identified, or, may (also) have a process-oriented understanding such as Portnoy, Grundmeier, and Graham (2006) described. A process-oriented conception would suggest that the individual focuses on the means of the transformation, imagining a particular output, without physically manipulating points, while an object-conception would allow the students to focus on the output of multiple processes simultaneously and then be able to compare those outputs for uniqueness. We note that this explanation, that students concurrently hold these two problematic conceptions, is speculative, based on their written production. We treat it as a hypothesis that would require further testing, and only one possible explanation for the work that they exhibited.

Looking to part (c), a correct proof uses the angle preservation property on both pairs of angles to show that the image of E must be at the intersection of the rays DC and AB and therefore must be B. First, one of the students, whose work on (b) appeared to be aimed at earning credit, wrote a collection of statements that were all related to the prompt, but which we could not interpret in any coherent manner, again suggesting that the work was aimed at earning partial credit. Second, four students drew on notions of triangle congruence; two of whom did so explicitly. For example, arguing that "Since triangle DAE is congruent to triangle DAB, we know DE = DB, and AE = AB." These students typically used triangle congruence to claim segment congruence.

All of the students used the distance preservation property or angle preservation property in some way; some only used these properties implicitly, while others correctly stated them and yet did not productively use them. Multiple students made false claims based on the distance preservation property, such as the claim that unequal points cannot be equidistant from the line of reflection. Many correctly used the angle preservation property, but omitted the needed claims about the intersection of rays; the following is a representative such proof:

> Since $\angle FDA$ is congruent to $\angle CDA$ and $\angle DAE$ is congruent to $\angle DAB$, then R(ray DE) will match to ray DB and R(ray AE) will match to ray AB respectively. Since this is an isometry length is invariant, so DE = DB and AE = AB. Thus R(E) = B.

Note that the first line begins by identifying two pairs of congruent angles and making a claim about where a pair of rays is mapped, indicating that the student has used the angle preservation property twice, although without stating it. However, the student omits any claims about the intersection of rays. Instead, the student then draws on the distance preservation property to state that length is invariant under an isometry and names line segments that are equal. Though accurate, this statement has not been justified and, in fact, relies implicitly on the fact that the triangles are congruent. The proof is complete without this statement, but the student possibly believes that citing length invariance is important. Four students only used the angle preservation property on one pair of angles, then attempted to 'repair' the result using other notions, such as that the preimage and image of a point are the same for reflections, or that a pair of segments would be parallel, while two of these students then used triangle congruence. Finally, four students gave correct proofs, though two of them used, without stating, both the angle preservation property and the intersection of rays.

Discussion

We argue that one important finding is that the majority of the students (future teachers) do not appear to be fluent with transformational geometry at the end of their one-semester course. This is an important finding from both the perspective of teacher education and undergraduate mathematics. From the perspective of undergraduate mathematics it reinforces the claim that students find advanced courses difficult and struggle with proof-writing (Selden & Selden, 2008), and supports an argument that they need significantly more time with the material to develop appropriate fluency. From the perspective of teacher education, this finding is important because it suggests that even teachers whose geometry class includes a transformational approach are not adequately prepared to teach it. It raises significant questions about their future instruction, their future students' learning, and policy questions about what additional levers for change need to be attempted.

A second important finding that we highlight is the fact that many students *implicitly* drew on triangle congruence ideas and sometimes did so explicitly, and, especially for part (c). We suggest that the notions of the classroom economy and students' practical rationality may have made this likely. In particular, we hypothesize that the students were attempting to maximize the points that they earned, and that, while the directions might indicate not to use congruence theorems, the students may believe submitting this proof is better than the alternative of possibly not submitting a proof. That is, they are trying to earn partial credit as a means to maximize their grade. We further claim that drawing on congruence theorems may be a form of coping--reverting to a more familiar and practiced set of content for proving, and away from the newer and less familiar content.

We also note two curious correlations that may be interesting to further explore. Students often use ideas implicitly, i.e., without stating them or warranting their use, and it is not clear whether they are omitting the name and warrant because they are hurried in an exam setting, because they find it obvious, or because they do not realize they need to warrant the use. There were only three cases where students transparently made an unsupported statement. Secondly, the students who did best on part c did not do especially well on part b. Most importantly, we note that this was a first study of students' understanding of transformational geometry and that we have drawn some preliminary hypotheses that can help explain the types of performances that they exhibited on their exam. We note that we did not interview the students; we did not, at the time, anticipate that this question would produce interesting data. Thus, one follow-up study will be to explore which of the proposed hypotheses for student difficulty are the most plausible by engaging in task-based interviews, as well as interviewing students at the completion of their final exam in a similar class. Moreover, because these are future teachers, asking for the kinds of explanations of ideas that they would give their students has value in both giving researchers insight into their understanding of the content and giving policy-makers a better understanding of what additional supports will be needed going forward.

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Exploring Experts' Covariational Reasoning

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In this paper, we discuss two experts' reasoning abilities when tasked with drawing a graph that relates two varying quantities. We present evidence that in some cases, these experts had constructed and coordinated the amounts of change of the quantities (while interpreting and constructing graphs). By comparing each experts' activities and corroborating previous researchers' findings, we argue that constructing a multiplicative object is critical to conceiving a graph and situation as constituted by covarying quantities. We identify particular complexities involved in the development of covariational reasoning including the conceptualization, coordination, and referent accumulation of the amounts of change of two quantities.

Key words: Covariational Reasoning, Multiplicative Object, Graphing

Many researchers have illustrated the importance of covariational reasoning in order for students and teachers to construct and represent relationships graphically, understand families of relationships (e.g., linear, quadratic, trigonometric, exponential), and develop mathematical ideas in calculus (e.g., limits, rates of change, differential equations) (Ellis, Özgür, Kulow, Williams, & Amidon, 2015; Moore, 2010; Moore & Thompson, 2015; Strom, 2008). In recent decades, researchers have developed constructs in order to characterize students' and teachers' quantitative and covariational reasoning. The results of such work has included frameworks of mental actions (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson & Carlson, 2016), refined characterizations of observed behavior (Castillo-Garsow, 2010, 2012), and examples of students' and children's reasoning about dynamic quantities (Ellis et al., 2015; Lobato, Rhodehamel, & Hohensee, 2012; Moore, Paoletti, Stevens, & Hobson, 2016). Researchers have also shown the importance and necessity of certain cognitive objects that an individual must construct in reasoning covariationally. In particular, Thompson and colleagues described that constructing a multiplicative object is critical to an individual constructing and representing covariational relationships (Thompson & Carlson, 2016; Thompson, Hatfield, Joshua, Yoon, & Byerley, 2016).

In this study, we extend the aforementioned body of research to include examples of experts reasoning in a dynamic situation. We use the covariational framework developed by Carlson et al., 2002 with particular attention to *amounts of change* to characterize the mental actions we inferred from the work of these experts. By comparing their reasoning and activity, we provide evidence of the importance of joining two quantities' amounts of change into a multiplicative object for interpreting and graphing relationships involving varying quantities.

Background

Our understanding of covariation stems from Saldanha and Thompson's (1998) description of "someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously… coupling the two quantities, so that in one's understanding, a multiplicative object is formed of the two" (p. 299). To further understand the notion of multiplicative object, we leverage Thompson's (1994) characterization of accumulations and accruals. Thompson described that a sophisticated conception of covariation and rate involves the construction and coordination between two quantities' accumulations and the accruals that make up these accumulations. We note that an accrual is understood in terms of a smooth accumulation itself and as a quantity whose magnitude is a difference between two states of another quantity's magnitude. We interpret Thompson's (1994) articulation of this relationship between covarying quantities' accumulations and accruals to involve forming interrelated multiplicative objects both between corresponding accumulated amounts of quantities and changes between these quantities.

Carlson and colleagues proposed a framework articulating levels of images of covariation and mental operations that support each level (Carlson et al., 2002). These mental actions can be described with regard to the elements that one is coordinating and the aspects of these elements one is attaining to. For example, amounts of change (Mental Action 3) are one such element that we focus on here. In regards to amounts of change, Carlson et al. (2002) observed students' lack of propensity to reason with these elements, indicating the difficulty of coordinating amounts of change into a multiplicative object. Corroborating the findings of Carlson et al., Johnson (2012a, 2012b) illustrated that maintaining an image of simultaneous accruals is non-trivial for students. For instance, Johnson described children's tendencies to reason about associated changes in magnitudes as happening independently in each quantity, which may be problematic for their ability to make sense of variation in rates of change. Furthermore, other studies have investigated students' construction of a multiplicative object uniting quantities' values (Thompson et al., 2016; Stalvey & Vidakovic, 2015; Frank, 2016). Thompson and colleagues (2016) investigated teachers' graphs representing animated magnitudes that were displayed on orthogonal axes. These researchers illustrated that a teacher's ability to unite the axes magnitudes into a multiplicative object was essential to her or his graphing the varying quantities.

Data Collection and Methods

The two participants of this study, Jake and Dan, were first year students in a mathematics education PhD program in a southeastern US university. Each participant had previously completed a Bachelor's degree in mathematics and taught secondary school mathematics for at least two years. These participants' interviews were part of a larger study aimed at investigating teachers' and college students' quantitative and covariational reasoning (see goo.gl/dAA7Re). The research team of this larger study (which includes the authors of this paper) conducted 60-minute semi-structured clinical interviews (Ginsburg, 1997) with 10 experts from the same university. By expert, we mean an individual who is enrolled in a doctoral program in mathematics or mathematics education with teaching experience in secondary or undergraduate mathematics. Thus, we mean expert as someone who one might expect to have sufficient experience engaging in tasks involving representing and reasoning about quantities.

One or two members of the research team led each interview. Each interview was video- and audio-recorded and each participant's work was digitized. We analyzed the data using grounded approach (Strauss & Corbin, 1998) combined with Thompson's (2000, 2008) notion of conceptual analysis. The data was first transcribed and instances offering insights into the experts' thinking were identified. After discussions with the full research team, Jake's and Dan's activity seemed to provide the most enlightening comparison for us, as both participants' behavior focused on quantities and did not resort to thematic associations or memorized formulas. Focusing on these two experts, we performed a conceptual analysis (Thompson, 2000) of their thinking in order to generate and test models of their thinking so that these models provided viable explanations of their behaviors.

Tasks

Each interview task began with having the participant watch a short (about 20 second) animated clip. The first interview was focused on *Going Around Gainesville (GAG)* (see Figure 1) and the animation played a "car" moving back-and-forth along the path indicated in the figure. The second interview was focused on *Taking a Ride* (see Figure 2). The first part of this task played a circle wheel spinning around the axle and the second part of the task involved an animation of a fixed square "wheel" in which the carts traversed along the frame of the "wheel." All three of these animations played so that the motion was at a constant speed.

Going Around Gainesville Part I

Some Georgia students have decided to road trip to Tampa Bay for Spring Break. Of course, this means traveling around Gainesville on their way down and back. The animation represents a simplification of their trip there and back. Create a graph that relates their total distance traveled and their distance from Gainesville during the trip.

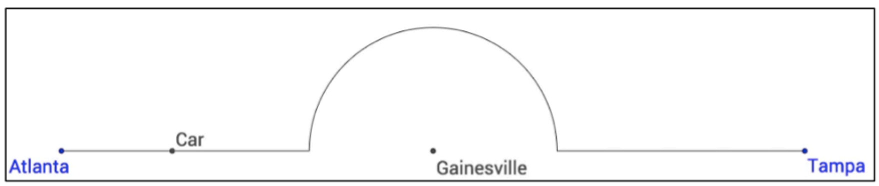
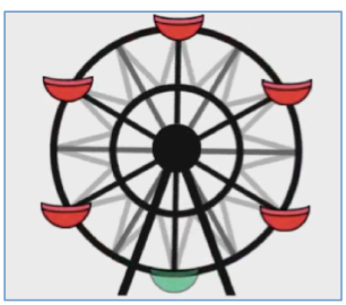


Figure 1. The *Going Around Gainesville* (*GAG*) task.[1]



Taking a Circle Ride
Graph the relationship between the total distance the rider has traveled *around* the Ferris wheel and the rider's distance from the ground.

Taking a Square Ride
Graph the relationship between the total distance the rider has traveled *around* the Square Wheel and the rider's distance from the ground.

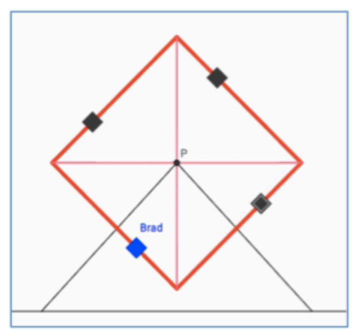


Figure 2. The *Taking a Circle* and *Square Ride* task.

We designed the situations in these animations to illustrate multiple measurable attributes and the tasks to prompt students to graph various relationships between two quantities without explicit reference to numerical values. Since we were interested in students' *images* of covariation, we did not want them to rely on learned facts or formulas to relate the quantities' magnitudes (Saldanha & Thompson, 1998). The focus on novel attributes and multiple orientations of graphs were intended to perturb students' ways of thinking about graphs (Moore, Silverman, Paoletti, & LaForest, 2014). We avoided time as one of the quantities for students to relate as they can often reason uni-variationally when time is one of the quantities under consideration (Leinhardt, Zaslavsky, & Stein, 1990).

[1] This task is a modification of the task provided by Saldanha and Thompson (1998).

Results

In this section, we compare and contrast Jake's and Dan's actions in responding to the interview tasks. Due to space constraints, we do not report on the entirety of their actions, but instead focus on particular aspects that were critical to their responses or instances in which we inferred marked differences in their actions.

Going Around Gainesville (GAG)

Both Jake's and Dan's activities in this task began in a similar way. After reading the task out loud, they each created and labeled axes and traced, what we perceived to be, an accurate graph (see Figure 3 and Figure 4). In each case, the interviewer then proceeded by asking the participant why he drew a graph composed of straight lines (or segments).

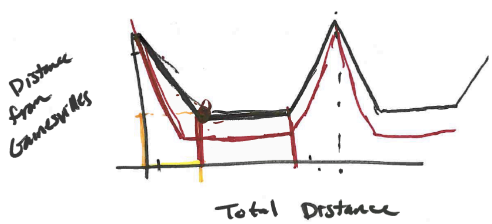


Figure 3. Jake's graph in GAG.

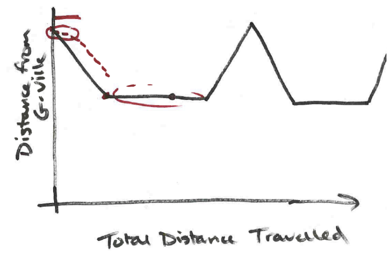


Figure 4. Dan's graph in GAG.

Each participant initially contended that the straight lines in his graphs were a consequence of the constant speed in the situation. Their initial verbal responses did not indicate their conceptions of two covarying distances, and we infer their constructing a thematic association (Thompson, 2016)—constant speed necessarily implies line—between their graphs and the situation. Upon further questioning from the interviewers, each participant concluded that the shape of his graph was actually not influenced by time or constant speed. Each explained,

> **Jake:** It's still like a, I guess, one to one relationship… for every one mile you move towards the total distance towards Tampa you're still getting that one mile closer to Gainesville at least for this part of the graph [*referring to the first linear segment of the graph*].
>
> **Dan:** There's a relationship between as the car's moving, it's consistently getting closer to Gainesville and it's also consistently traveling a distance. So, whatever it is doing distance-wise is also happening to its distance to Gainesville… 10 miles total distance traveled and it's also 10 miles closer to Gainesville.

In both responses, the participants identified a *relationship* between the quantities' *total distance* and *distance from Gainesville* by reasoning with an explicit amount of change in *each* quantity respectively. From their language ("for every" and "consistently"), we infer that this relationship resulted from participants' awareness that such an explicit amount of change could be constructed along any section of the first part of the road or first segment in the graph.

Taking a Ride

In the second interview, Jake's and Dan's activities unfolded in distinguishable ways, and thus we focus the majority of the results section here. Despite the participants' initially drawn graphs being noticeably similar, their ways of reasoning over the course of this task unfolded quite differently.

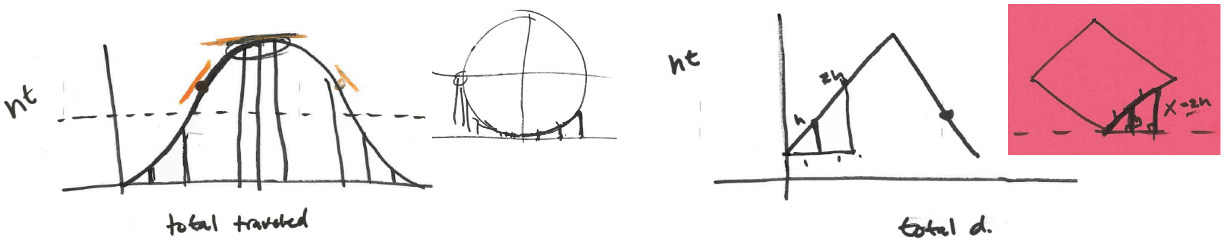


Figure 5. Jake's graph and diagram for *Taking a Circle* (a and b) and *Square* (c and d) *Ride.*

Jake—Taking a Circle Ride. Jake justified the curvature of his graph by claiming, "it's not a constant rate of change." The interview continued with Jake justifying this claim, concluding the situation instead involved "increasing rates of change," and then reasoning that this phenomenon was reflected in his graph. To explain why the situation was not a constant rate of change, Jake partitioned an arc around the circle wheel into equal parts (see Figure 5b). He then drew in segments to the "ground" from the points he marked on the wheel to indicate the corresponding "height of the rider" at each location on the wheel. He continued,

> "…so all of those are congruent, I guess, arcs [*referring to the arcs in Figure 5b*]. You see that the height is a lot different in comparison to these arcs being the same, um, so it's not linear. If it was linear we would expect these heights to all be the same for each arc along the circle that you move."

Due to Jake's construction and emphasis on the "congruent" arcs, we interpret Jake's activity to indicate that he was coordinating *changes* in arc length with *total* height at the terminal point of each partitioned arc. After further questioning, Jake continued to claim, "Because these heights aren't the same, we had kind of changing rates of change." We subsequently asked Jake to represent this phenomenon (of changing rates of change) on the *graph*. He responded, "Oh, well, I guess I mean, you can just look at tangents to the curve if you want." At this point in the interview, Jake had not engaged in behaviors we interpret to indicate his coordinating amounts of change in the context of the graph or situation, instead referring to "rates of change" and slopes of tangent lines. Thus, we were curious about Jake's mental actions associated with these phrases. We asked Jake if he could say more about tangents and slope. Jake responded,

> "So if you take the tangent of the curve at this point to be that [*drawing first orange segment on graph in Figure 5a*], the rate of change is increasing, whereas maybe as you take it through the maximum point [*drawing second orange segment on graph*] your rate of change is zero at that instant…. [Increasing rate of change] means … as you're moving along the wheel, as your total distance is changing, your height is also changing as well in the same direction but positive."

Jake claimed (1) a positively sloped tangent line at a point indicates the rate of change is increasing at that point and (2) the rate of change increasing means both quantities are increasing. As the interview progressed, Jake persistently repeated these two claims and he did not engage in activity that suggested his unpacking the tangent line as the result of "smaller and smaller refinements of the average rate of change" (Carlson et al., 2002, p. 358). Jake did not associate tangent lines or "rate of change" with coordinating what we perceived to be associated amounts of change of two quantities. Jake's coordination of *congruent arcs* around the wheel and *total height* where problematic in his constructing rate of change from coordinating *changes* in *both* quantities.

Jake— Taking a Square Ride. After watching the animation, and like the previous task, Jake immediately drew what we perceived to be an accurate graph (see Figure 5c). Jake explained the shape of his graph by claiming the rider was “increasing at a constant rate.” Jake, however, was perturbed by how to illustrate this relationship on the situation. After various attempts to do so, he redrew the square wheel (see Figure 5d), identified similar triangles, and explained that the quantities’ magnitudes maintained a fixed ratio. Jake explained, *“*For each unit that I move up, along the square, whatever this height is [*motioning to the total amount of height corresponding to the first “unit up”*] is going to be twice the original height, or if you want to call it a unit height.” Jake seemed confidant that this justified his linear graph, but subsequently expressed skepticism in using this argument to justify the statement: “height is changing by a constant rate.” We do not include this example in order to discredit Jake’s thinking, but instead to further illustrate that across the tasks in this interview, Jake did not persistently coordinate *amounts of change* of the quantities in ways that indicated his constructing a multiplicative object composed of amounts of change.

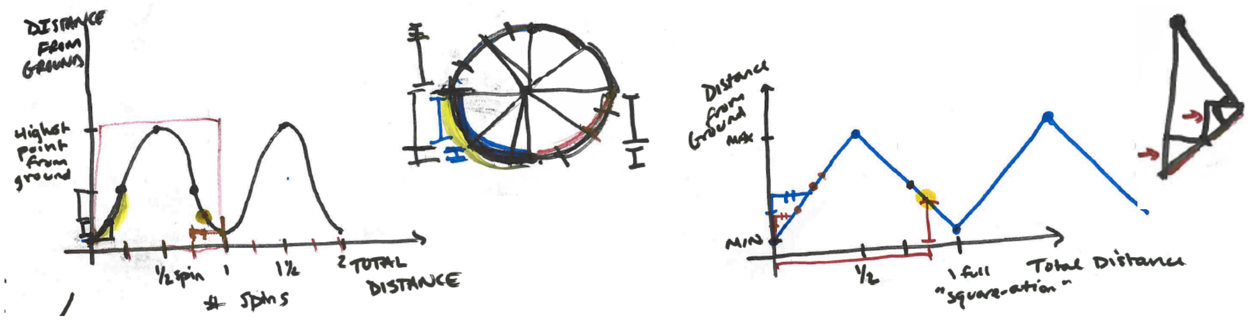


Figure 6. Dan’s graph and diagram for *Taking a Circle* (a and b) and *Square* (c and d) *Ride.*

Dan— Taking a Circle Ride. After watching the video Dan drew a set of axes, marked the minimum and maximum height along the vertical (height) axis and then partitioned the horizontal axis of his graph into half rotations of the wheel (see Figure 6a). He then plotted points on his graph corresponding to the rider’s height at each half rotation. He next questioned how to trace the graph between these points, “The question I'm having is, will this be a straight line [*motioning a straight line between the first two points he marked on his graph*] or will there be any curve in it.” This behavior is in contrast to Jake’s in that Dan focused on pairing two quantities’ values and considering how these quantities’ magnitudes vary between two pairs of magnitudes *before* drawing a graph.

Continuing, Dan drew a diagram of the wheel (see Figure 6b), marked a point at a quarter spin along the wheel and drew a vertical segment corresponding to the rider’s height at this location; this segment is “halfway as high as he can get.” He then referred to a “quarter spin” being “halfway” along both axes and plotted this point on his axes. Dan then partitioned the first quarter arc of the wheel into two equal arcs and drew vertical segments corresponding to the change of height of the rider along each arc (Figure 6b). He explicitly identified that these two arcs were equal and then compared the associated vertical segments; he stated that the first segment was “less change” than the second segment. He subsequently drew the first part of his graph corresponding to the rider’s trip along the first quarter of the wheel. Dan finished his graph by using this same strategy for each remaining quarter of the wheel.

We asked Dan how his graph related to his reasoning with the diagram. Dan explained by indicating a coordination of the amounts of change of total distance and the amounts of change of height from ground (i.e., reasoning with accruals). Dan explained,

> “These [*tracing along the arcs in Figure 6b*]... are all the same amounts on the same axis. So those are uniform tick marks … [*marking over the points on the horizontal axis*

corresponding to the arcs]. But … then you notice in the first *x* interval [*holding fingers around first interval on horizontal*], there's very little change in *y* [*points to vertical segment in Figure 6b corresponding to first change of height*] ... But then if you look at the next same amount of *x* [*holding fingers around the second interval on horizontal axis*], you get a much bigger change in *y* [*points to vertical segment corresponding to second change of height*]."

In this dialog, we infer that Dan was constructing the quantities on his graph by coordinating, along the axes, the accruals by which accumulations occurred (Thompson, 1994).

Dan—Taking a Square Ride. In this task, Dan acknowledged that he could construct his graph using the "same logic" as in the previous task. He partitioned each quarter of the square wheel into two equal parts and identified the corresponding change of height of the rider along each segment. Next, Dan identified a relationship between these changes in height by asking himself out loud, "… is this right here [*drawing arrow in Figure 6d pointing to a change in lowest height segment*] the same as this right here [*drawing second arrow in Figure 6d*]? And the answer is yes. So, and this is, I said for an eighth but you could do this for any arbitrary piece of the square." We infer that Dan held in mind that two arbitrary, but equal, distances traveled along the square wheel had corresponding changes in height that were also equal.

Conclusion

In this study, the participant's ability to construct and coordinate amounts of change was essential to his ability to reason covariationally about the shape of his graph. In the first interview, both Jake and Dan constructed amounts of change of each quantity in the situation and were able to use this construction to explain the relationship between the quantities. However, in the second interview, Jake did not construct associated amounts of change of the quantities. In these tasks, Jake instead relied on reasoning with tangent lines (*Taking a Circle Ride*) or the geometry of his drawn diagram of the situation (*Taking a Square Ride*). Jake did not unpack his constructed tangent line in terms of amounts of change or construct amounts of change from the geometry he described in his diagram. In both tasks Jake did not reach what he considered a satisfactory justification for his drawn graph.

In contrast, Dan's activity in the second interview was to construct equal changes of one quantity and compare corresponding changes in the other quantity. Dan represented the coupled changes of quantities on a graph by conceiving simultaneously accumulated amounts in terms of simultaneous accruals. The collection of these activities is an indication that Dan had constructed *amounts of changes* into a multiplicative object (Thompson, 1994).

This report extends previous observations from Carlson et al. (2002) and Johnson (2012a, 2012b) on the difficulties college-level students and children have in constructing and coordinating amounts of change of quantities. Indeed, we illustrate that maintaining an image of simultaneous accruals is also not trivial for experts. Furthermore, the evidence suggests that it was the construction of corresponding amounts of change that afforded Jake and Dan the ability to represent and interpret the animations covariationally.

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How Students Interpret Line and Vector Integrals Expressions: Domains, Integrands, Differentials, and Outputs

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This study expands on research happening in multivariate calculus education to an exploration of student understanding of line and vector integrals. We describe how students interpreted these types of integrals, including the various symbols in their expressions and their relationships to each other. We found that while students were able to associate mathematical objects with the individual symbols in the integral expressions, the main issues came in trying to coordinate these objects into a comprehensive whole for the entire integral expression. While the literature has discussed connections between the integrand, differential, and integral symbol, we also found that connecting the domain to the other parts of the integral expression was difficult as well. Thus, to give more attention to the domain in addition to these other parts, we have incorporated it into a general "domain-chop-evaluate-add" framework for reading integrals.

Key words: multivariate calculus, line integral, vector integral, domain, framework

Undergraduate education research has recently taken an interest in multivariate calculus, which is an important required course for many mathematics, science, and engineering majors (e.g., Harvard University, 2016; Massachusetts Institute of Technology, 2014). This research has begun to examine student understanding of function (Dorko & Weber, 2014; Martinez-Planell & Trigueros-Gaisman, 2012), graphing (Martinez-Planell & Trigueros-Gaisman, 2013; Weber & Thompson, 2014), partial derivatives (Martinez-Planell, Trigueros-Gaisman, & McGee, 2015; McGee & Moore-Russo, 2015), and integration (Jones & Dorko, 2015; McGee & Martinez-Planell, 2014). However, the studies dealing with multivariate integration have focused only on how students understand real-valued multiple integrals (Jones & Dorko, 2015), or on teaching strategies that may help students learn this specific type of integral (McGee & Martinez-Planell, 2014). There is no research we could find on other important types of integrals from the multivariate calculus curriculum, such as line integrals and vector integrals. By line integrals we mean integrals of a real-valued function over a curved line, *C*, as in $\int_C f(x,y)\,ds$. By vector integrals we mean integrals with a vector field for the integrand, such as $\int_C \vec{V} \cdot d\vec{r}$. Line and vector integrals are important both for later mathematical study (e.g., contour integrals) and later science and engineering study (e.g., work, flux, or circulation). Due to the lack in the research literature on student understanding of these important subtopics within multivariate integration, in this paper we seek to shed light on the following question: How do students interpret line and vector integral expressions, including the various symbols within those expressions?

Semiotics

Since this study deals with students interpreting symbols, we employed a semiotics framing and analysis (Peirce, 1998; Presmeg, Radford, Roth, & Kadunz, 2016; Radford, Schubring, & Seeger, 2008). Semiotics deals with how symbols or signifiers are used to refer to objects, in this case *mathematical* objects. Peirce (1998) defined a "sign" as containing three components: an *object*, or the actual thing represented by a symbol or signifier; a *representamen*, or the symbol or signifier used to refer to the object; and an *interpretant*, or one's understanding of the relationship between representamen and object. Note that an *interpretant* is not the *interpreter*. That is, an interpreter of the symbol or image creates an *interpretant* through the connections they see between the signifier and the signified. Peirce explained a difference between the *intentional* intepretant created by the one who generates the symbol or image, and the *effectual* interpretant created by one who "reads" the symbol or image. For the purposes of this paper, we focus mainly on *effectual* interpretants, or those created by students as they interpret the symbols in the integral expressions. Peirce also described that signs can potentially be *iconic* (physical resemblance), *indexical* (physical connection), *or symbolic* (conventional symbol) in nature. For the purposes of this paper, since we are using conventional symbolic notations of integral expressions, we are only examining signs that are *symbolic* in nature.

For successful communication using signs, the object and representamen must be known to go together by the one who interprets the representamen, such as an exclamation mark inside a triangle indicating danger, or that a stick figure is meant to represent a person. However, in mathematics it may be the case that a symbol has been invented to capture a particular idea – such as Leibniz introducing the integral symbol, $\int$, to refer to a summation, in which there is an implication that integrals are inherently sums (Katz, 2008) – but that a student is not aware of that intention. In that case, the student may associate that symbol with a different mathematical object, say, geometric area, implying that integrals denote areas within shapes. In this case Leibniz's *intentional* interpretant and the student's *effectual* interpretant do not match up. In this study, we were interested both in what objects students might perceive the line and integral symbols as referring to, and the interpretants created by the students in that interpretation process. That is, we are not necessarily assuming "successful communication" regarding the signs involved in the integral expression in terms of their normative, intentional meanings. Rather, we are interested in open-endedly seeing the objects that students determine a particular symbol to refer to and what the implications of that association are. We use the term "interpret a symbol" to mean that a student has linked the symbol to a mathematical object, through which an interpretant, or implication of that interpretation, might also exist.

Methods

The data used for this report were generated from semi-structured interviews on different types of integrals with 10 university students of varied backgrounds. Six students (A–F) had just completed multivariate calculus, coming from three different instructors. Two students (G and H) were further in their undergraduate mathematics studies, and two students (I and J) were graduate students in mathematics education. This diversity in background was deliberate in order to broaden the potential interpretations of symbols we might observe in the results. For this conference report, we focus on two of the questions from the interview protocol, which asked the students to describe a generic line integral expression and a generic vector integral expression, as follows: "What does $\int_C f(x,y)\,ds$ mean, where C is a curve (i.e. curved line) in the x-y plane?" and "What does $\int_C \overline{V} \cdot d\vec{r}$ mean, where $\overline{V}$ represents a vector field and C represents a curve (i.e. curved line)?" Standard follow-up questions were used to probe student thinking.

For our analysis, we considered each symbol of each integral expression, *C*, *f(x,y), V, ds, dr,* and $\int$, to constitute a separate *representamen* that could potentially refer to some mathematical object. The object was not specified for the students, so they could make their own interpretation of each symbol. After the interviews, we independently went through the data, one student at a time, to identify the mathematical objects they associated with each individual symbol. We then brought our separate analyses together to compare our results, which we found to be fairly consistent with each other, and to resolve any discrepancies. Once we had done this for each student regarding each symbol in both types of integrals, we went through the entire data set one more time to check if we had missed any objects in light of our joint discussions. For each symbol, we then identified which students shared the same interpretation. Lastly, after identifying the mathematical objects each student associated with each symbol, we then went back through the data, one student at a time, to see how the objects associated with each symbol, and the implications of those associated objects, fit together to give rise to an *overall* object for the entire integral expression.

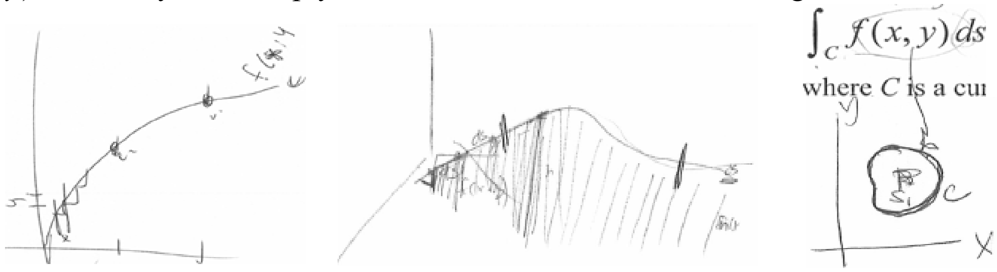
Results

We organize our results by first describing the students' interpretations of each symbol, in terms of the objects associated with the symbols, and the implications of those associations. We do so first for the line integral, and second for the vector integral. Also, within each integral expression, we describe how the individual objects associated with the symbols gave rise to an overall object associated with the entire integral expression, and the consequent implications.

Line integrals

Student interpretations of the integrand, f(x,y), and the domain, C. Since the interview prompt for the line integrals explicitly stated that *C* was a "curved line," it is not surprising that most students interpreted the symbol *C* as referring to a curved line. What is interesting, though, is that nine of the students (all except J) commonly interpreted the symbol $f(x,y)$ as *also* referring a curved line, and not to a curved plane. Only Student J consistently interpreted $f(x,y)$ as referring to a curved plane, though Students E and I also did so at certain points in the interview.

The students varied in how they interpreted the relationship between the objects associated with the symbols *C* and $f(x,y)$. The majority of the students (A, B, F, G, I, and J) interpreted both *C* and $f(x,y)$ as essentially representing the same mathematical object. For example, three of the students (B, G, and I) believed $f(x,y)$ represented a long (possibly infinite) curved line, and that *C* represented a certain finite segment of that same curved line, as seen in Figure 1a and 1b. Note that 1a shows $f(x,y)$ in the *x-y* plane (Student G), while 1b shows $f(x,y)$ in 3-space (Students B and I). Two other students (A and F) made an even stronger connection between $f(x,y)$ and *C* by representing them as the exact same closed loop in the *x-y* plane (Figure 1c). These students initially drew *C*, alone, as a closed loop in the *x-y* plane, but as the interviewer asked about where $f(x,y)$ fit in, they both simply indicated it would be the same thing as *C*.



(a) (b) (c)

Figure 1: Interpretations of *f*(*x,y*) and *C* as representing the same mathematical object

Student J and Student I (at a different time) both interpreted *f*(*x,y*) as referring to a curved plane, with *C* as a curved line *coinciding* with the plane *f*. While Student I thought *C* might refer to a closed loop traced out *inside* the curved plane itself (Figure 2a), Student J drew *C running through f* (Figure 2b). Interestingly, Student J actually described *f* as a finite object that bounded *C*, essentially flipping the roles of *f*(*x,y*) and *C*, as though the integral were $\int_{f(x,y)} C\, ds$.

Student J: I don't know… I would imagine that, probably, where your function is [*f*(*x,y*)], this plane right here tells you what part of the curve [*C*] you're integrating, or something.

(a) (b)

Figure 2: *C* as a curved line (a) *inside f*(*x,y*) and (b) *running through f*(*x,y*)

Student interpretations of the differential, ds. Eight of the students in this study (A, B, E, F, G, H, I, and J) interpreted the differential, *ds*, as referring to a small arc length, or a small portion of *C*, which agrees with the Δs in the Riemann sum that often forms the basis of the mathematical definition of the line integral. Three of these students (E, G, and H) further specified that the *ds* could be thought of as the *infinitesimal hypotenuse* of a right triangle with *dx* and *dy* as the legs. However, one of the students (C) interpreted *ds* as referring to small pieces of *area* instead of *length*, based on his idea of the integral as calculating the area underneath a portion of an *f*(*x,y*) graph. The tenth student (D) made no meaningful interpretation of *ds*, even when prompted by the interviewer.

Interestingly, half of the students (A, B, C, E, and F) attempted to alter the *ds* symbol in the line integral expression into other types of differentials, such as *dt*, *dx* and *dy*, *dA*, or *dr* and *dθ*. This seems to suggest that the symbol *ds* was deemed insufficient by these students in representing the mathematical object they believed *ds* referred to. By analogy, this may be like a stick figure that, while referring to a person, is inadequate for specifying the person's age. Presumably this may be due to the fact that computing a line integral by hand using anti-derivatives typically requires either a parameterization or a change to other types of coordinate systems. Thus, the *procedure* for computing line integrals seems to have influenced how several of the students fundamentally interpreted the differential, *ds*, of a line integral expression.

Student interpretations of the overall integral. Six of the students (B, C, D, E, H, and I), at one point or another, pieced the individual objects together to come up with an overall interpretation of the line integral as representing the surface area of a "curved sheet" stretching between *C* and *f*(*x,y*), matching the normative geometric interpretation of line integrals. However, only two of the students (E and H) had what could be considered a *stable* interpretation of the overall line integral (see Thompson, Carlson, Byerley, & Hatfield, 2014), as exemplified by Student H:

Student H: [Draws a parabolic curved line in the *x-y* plane, see Figure 3]. There's my curved line [*C*]… Then whatever my *f*(*x,y*) is [draws a curved line in 3-space above *C*]… I'm finding this region right here [shades in between *C* and *f*]… If my curve under here [*C*] is just a single line, I'm not going to have a volume, I'm just going to have an area. That could be, I mean, an area that ends up being curved and all this kind of stuff, but it's an area.

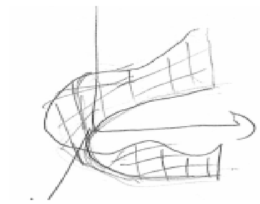


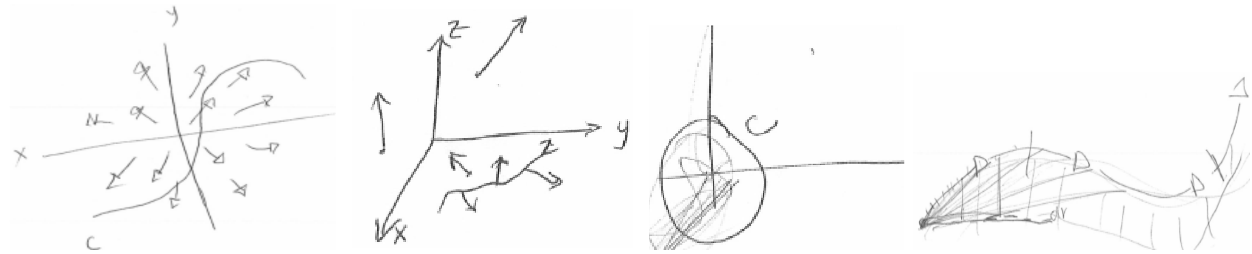
Figure 3: A line integral representing the surface area of a curved sheet between *C* and *f*

The other students who came up with this interpretation (B, C, D, and I) were clearly hesitant about it, and often began the prompt by making other interpretations instead. Student B initially described the line integral as a more "succinct" way of writing the multivariate integral $\iint_R f(x,y)\,dA$. Student C initially attempted to interpret this integral as the manifestation of some theorem, naming Green's theorem, Stoke's theorem, and the divergence theorem as possibilities. Student D frankly admitted, "I don't know, I'm guessing on this one." Student I was initially convinced that the line integral should represent the volume contained between the *x-y* plane and the region created by a closed loop, *C*, traced out inside of *f*(*x,y*) (see Figure 2a). Almost half of the students (A, G, J, and I) stated the line integral might represent the arc length of either *C* or *f*(*x,y*). Finally, two students (A and F) believed the line integral represented the area of the region inside of the closed loop they had drawn in the *x-y* plane, as seen in Figure 1c.

Vector integrals

Student interpretations of the integrand, $\vec{V}$, and the domain, C. For the vector integral expression, the students were split into three overlapping groups in terms of how they interpreted the integrand, $\vec{V}$. Five of the students (C, D, E, H, and J) interpreted $\vec{V}$ as a collection of literal "arrows" in that they drew several arrows pointing in different directions. Four of the students (E, F, G, and I) interpreted $\vec{V}$ as representing a *physical* phenomenon, such as the flow of water, wind currents, or the force of gravity. Three of the students (A, B, and C) represented the integrand as an algebraic function, including as $\langle -\cos t, \sin t, t\rangle$, *f*(*x,y*), and *F*(*r*(*t*)), respectively. This result is not to say that students in one group would be unaware of or unable to make interpretations from another group, but this result rather demonstrates the various approaches taken by the students to interpret $\vec{V}$.

All of the students interpreted *C* as a curved line, though in somewhat different ways, as seen in Figure 4. Three students (E, H, and J) drew a 2-dimensional vector field and represented *C* as a curved line passing through it (Figure 4a). Three students (C, D, and E) actually drew a 3-dimensional vector field, but then only drew *C* as a curved line on the 2-dimensional "*x-y*" plane (Figure 4b). Two students (A and F) thought of *C* as a closed loop in the *x-y* plane, without much reference to a vector field at all (Figure 4c). Another student (I) interpreted *C* as a collection of vectors from $\vec{V}$ lined up from tail to tip (Figure 4d). Finally, two students (B and G), while stating that *C* was a curved line, as specified in the prompt, never articulated exactly what it was or how it fit in with the other pieces of the integral expression.



(a) (b) (c) (d)

Figure 4: Various types of interpretations of *C* in relation to the integrand, *V*

Student interpretations of the differential, $d\vec{r}$. The students were most diverse in their various interpretations of the differential, $d\vec{r}$. There were a total of fifteen different types of interpretations the students gave to $d\vec{r}$, with some students coming up with more than one interpretation at different times as they struggled to give meaning to the vector integral. In fact, one student (I) came up with seven different interpretations for $d\vec{r}$ at different times! There were only three interpretations, however, that were shared by at least three separate students. First, Students F, I, and J stated that $d\vec{r}$ represented an "infinitesimal vector," agreeing with the $\Delta\vec{r}$ in the Riemann sum that often forms the basis of the mathematical definition of the vector integral. Students C, D, and I, however, wondered at times whether $d\vec{r}$ might actually represent the distance from the origin to points on the curve, *C*, as seen in Figure 5 and Figure 4d. Yet, the most common interpretation, made by five students (A, B, C, E, and F), was that $d\vec{r}$ needed to be converted to a different type of differential, including a conversion to *dt*, *dx-dy-dz*, *ds*, or *dr-dθ*. As with the line integral, the computation of vector integrals by hand using anti-derivatives may have influenced these students' fundamental interpretations of what a differential represents.

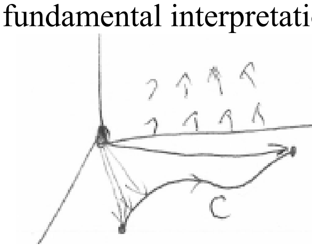


Figure 5: The differential, $d\vec{r}$, as representing distances from the origin to *C*

Student interpretations of the overall integral. The students were generally unsure about what overall object might possibly link all of the objects associated with the individual symbols. While five students (A, C, D, F, and G) stated that a vector integral might mean "work," they only seemed to state it from memory, explaining that they remembered having seen "work" examples in class. They confessed not knowing *why* the integral might calculate work, nor what work even meant in this context. Most of the students were explicit with the interviewer that they had no idea what the vector integral might represent, making statements such as, "Integrating over a vector field, um, I still don't really understand that part" (Student C) and "I don't know—I'm just trying to think of, like, values that might have some meaning in a vector field" (Student J). At the urging of the interviewer, the students attempted to imagine on overall object, which ranged ranged from a possible "surface area" (I), to an "average magnitude" of the vector field (G and J), to some kind of area underneath an unknown curve (B, H, and J), to an arc length (G and I), to an "overall direction" of the vector field (J).

Discussion

The students who participated in this study varied considerably in terms of how they interpreted the symbols in line and vector integrals. In many cases, the students associated the symbols with mathematical objects that align with typical *intended* objects, such as *C* representing a curved line, *f* representing a curve of some kind, or *ds* representing a small segment of a curve. However, the students were not always able to grasp the implications of these objects in relation to each other, and often struggled to amalgamate the objects defined by their own interpretations into an overall object referred to by the entire integral expression. For example, students struggled with the implications of *C* and *f* both being "curves," and sometimes reduced them to being the same mathematical object. Similarly, the students often struggled with the relationship between *C* and *V* in the vector integral, not knowing how to combine such distinct objects. Students also typically seemed unsure of how to relate the object represented by the integrand, *f* or *V*, with the object represented by the differential, *ds*.

Thus, the results of this study seem to imply that issues in understanding line and vector integrals do not necessarily come in the form of identifying the specific mathematical objects that the individual symbols refer to, but rather in combining these objects together to form a coherent object for the entire integral symbolic expression. This is similar to the idea of semiotic chains (Presmeg, 2006), though semiotic chains are used to describe the increasing abstraction or generality of a single mathematical object. In our case, the "chaining" has more to do with linking separate objects together into a new object that is referenced by the entire expression involving all of the individual symbols. Past research has looked at how students understand the connections between the symbols such as the integrand and the differential (Hu & Rebello, 2013; Jones, 2015; Sealey, 2014; Von Korff & Rebello, 2014), or the summation or accumulation of the quantities obtained from their product (Jones, 2015; Kouropatov & Dreyfus, 2013; Thompson & Silverman, 2008). From this arises a "chop, evaluate, and add" interpretation, in which little pieces represented by the differential are multiplied by objects represented by the integrand, which are then added up (a cue from the integral symbol). Yet, this interpretation does not involve the *domain* over which the integral happens. In our study, students also had issues incorporating the object represented by "*C*" with the differential and integrand. Thus, in order to help students see how *all* of the pieces of the integral relate to one another, *including* the domain, and to successfully chain the mathematical objects into a comprehensive whole represented by the entire integral expression, we recommend an interpretive framework for *reading* the symbols in an integral expression through a four-part process: (1) identifying the domain, (2) chopping the domain according to the differential, (3) evaluating the product of the integrand and differential within each piece, and (4) summing/accumulating this product across all of the pieces. This framework, depicted in Figure 6, shows how a student might need to visually break down the compact integral symbolic structure in order to make sense of the various pieces of the expression, how they fit together, and how they ultimately give rise to a meaning for the overall integral's value.

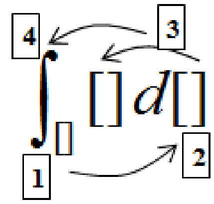


Figure 6: Domain-chop-evaluate-add framework for reading integral expressions

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Definite Integrals Versus Indefinite Integrals: How do Students See Them as the Same or as Different?

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Much of the calculus education research on student understanding of integrals has been separated into definite-integral-focused studies and indefinite-integral-focused studies. This means that research may not be capturing how students might see these two types of integrals in relation to one another, as opposed to in isolation of each other. This study examines whether students see these two types of integrals as representing the same basic concept, distinct concepts, or as sharing some concepts while diverging in others. The results show that a large majority of students ascribe the exact same underlying conceptions to both types of integrals, even describing the indefinite integral as representing the area under a curve also. We relate what aspects of each type of conception students saw as common to both types of integrals, and what features of the conception the students saw as different between them.

Key words: calculus, definite integral, indefinite integral, concept images

There has been growing interest in recent years on first-year calculus students' understanding of the definite integral. In particular, researchers have examined students' general understanding (Jones, 2013; Sealey, 2014), students' understanding of differentials and infinitesimals (Ely, 2010; Hu & Rebello, 2013a), students' understanding of accumulation and the fundamental theorem of calculus (Thompson, 1994; Thompson & Silverman, 2008), how students apply integrals to science and engineering (Hu & Rebello, 2013b; Jones, 2015a; Nguyen & Rebello, 2011), and ways to teach definite integrals to promote deeper understanding (Jones, Lim, & Chandler, in press; Kouropatov & Dreyfus, 2014; Thompson, Byerley, & Hatfield, 2013). In addition to this research focused on *definite* integrals, there has also been some attention given to student understanding of *indefinite* integrals as well. This smaller subset of the literature has examined student understanding of accumulation in connection with indefinite integrals, as well as student understanding of features such as zeroes, extrema, and inflection points (Swidan, 2011; Swidan & Yerushalmy, 2014; Yerushalmy & Swidan, 2012).

Yet, so far, the research literature on first-year integration has tended to be separated into *definite*-integral-focused studies and *indefinite*-integral-focused studies. This separation, unfortunately, does not speak to the way in which definite and indefinite integrals are often taught in close association with one another, with instruction on each happening within the same set of lessons. Thus, there remain important questions about how students themselves see these two types of integrals in relation to one another, as opposed to in isolation of each other. As a result, this report is meant to shed light on the following two questions: (1) Do first-year calculus students tend to see definite and indefinite integrals as representing the same basic concept, distinct basic concepts, or as sharing some concepts while diverging in others? (2) Given whether students see them as the same or different, in what ways do the students see them as representing the same or different ideas? These questions are important, because otherwise research may end up only speaking to compartmentalized portions of students' overall understanding of "integration" from first-year calculus.

Concept Images

This study is framed by Tall and Vinner's (1981) construct of *concept images*. A concept image is defined as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). For this paper, the "concept" under consideration is that which is termed "integral" in first-year calculus. We consider both definite and indefinite integrals to fall under the umbrella concept "integral" for the following reasons: (a) both are literally called "integrals," (b) both make use of the exact same symbols, "∫," and "*dx*" (or another differential), (c) students are taught to calculate both using similar methods, and (d) because they are often taught in close proximity to each other (Stewart, 2014; Thomas, Weir, & Hass, 2009).

Using this theoretical lens means that what students understand of *definite* integrals and what students understand of *indefinite* integrals together form a student's complete concept image for "integral" at the first-semester calculus level. Note that within a student's concept image for integral, any of the following conditions may hold. First, a student might have merged the two types of integral into a single idea, with no distinction between them. In this case, the student's concept image would have only one set of pictures, properties, and processes that encompass both types. Second, a student might have compartmentalized these two types of integrals. In this case, the student's overall concept image may have two distinct sets of pictures, properties, and processes, one for each type. Third, a student might have some kind of blending between the two, in which some pictures, properties, or processes might be shared by both types, and where other pictures, properties, or processes might be unique to each type. This study is meant to examine how students' concept images might fit under these different cases.

Data and Analysis

The body of data used for this study consists of surveys (described to the students as an introductory quiz) administered to 132 students on the first day of second-semester calculus at a large university in the United States. The purpose of surveying second-semester calculus students, as opposed to first-semester students, was that (a) these students were all guaranteed to have successfully completed a first-semester calculus course, and (b) these students came from a range of different instructors for first-semester calculus, making the sample more randomized than would occur by surveying students in first-semester calculus. In fact, the majority of students at the university come from locations outside of the school's home state. Interviews were also conducted with select individuals, but for the purpose of this abbreviated conference report, only the survey data is reported on.

The survey was divided into two parts, with a multiple-option section (Part 1) and an open-response section (Part 2). The purpose of having two parts was to create two separate opportunities to code a student into the categories "same," "different," or "mixed," according to what they perceived as the relationship between definite and indefinite integrals. Furthermore, Part 2 allowed for an analysis of which specific conceptions the students ascribed to both a definite and an indefinite integral.

In Part 1, the students were presented with four statements regarding the generic expressions $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$, and were asked to mark any they agreed with. Two statements (A and C) were intended to suggest that they had similar meanings, with the only differences being superficial (as in statement A). The other statements (B and D) were meant to suggest that they were different in conceptual meaning. The four statements were given as:

> (A) The fundamental difference between the meanings of the two integrals is that you need to plug *a* and *b* into your answer for the second one, but you don't have any numbers to plug in for the first one.
> (B) These two integrals are quite different from each other and represent different ideas.
> (C) The final answers you get for the two integrals might look different from each other, but the integrals essentially represent the same idea.
> (D) Even though you can use anti-derivatives to find answers for both of these kinds of integrals, the final answers you get mean very different things.

The multiple option Part 1 is obviously insufficient, alone, for constructing a picture of the relationship a student believes to exist between definite and indefinite integrals, or for knowing what conceptions are actually ascribed to each kind of integrals. Thus, a second, open-response section was included on the survey. In this second part, the students were given the prompt,

> Explain as much as you can what $\int f(x)\,dx$ means. Similarly, explain what $\int_a^b f(x)\,dx$ means. Use words, pictures, formulas, or whatever you want to help explain what they mean. How are they different? How are they similar? Please explain in as much detail as possible.

Part 1 and Part 2 of the survey were analyzed independently and then compared. For Part 1, students were coded as "same" if they marked only A, only C, or both A and C. Students were coded as "different" if they marked only B, only D, or both B and D. Students who selected at least one statement from A or C and one from B or D were coded as "mixed." Since Part 1 already dealt with *explicit* statements as to whether the two integrals are the same or different, Part 2 was not analyzed by whether the student *explicitly* stated they were the same or different. Rather, we coded a student's description of each kind of integral according to the conceptualization the statement seemed to assign to that integral. Based on previous research (Jones, 2013, 2015b), a student's description of each type of integral was categorized into (a) area, (b) anti-derivative, (c) summation/MBS, or (d) other. For a description of the MBS conception, see Jones (2015a). Each response coded into "other" was examined for the type of conception it represented, creating new categories (see Results). If the conception(s) contained in a student's explanation of the definite integral exactly matched the conception(s) contained in their explanation of the indefinite integral, the student's response was coded as "same." If there was no overlap in the conceptions contained in the explanations of the two integrals, the response was coded as "different." If a student ascribed at least one conception to both, but then also explained one of the integrals using a conception *not* ascribed to the other, the response was coded as "mixed."

Results

Part 2 of the survey

We organize this section by first describing the results from open-response Part 2 of the survey and then comparing them with the results from Part 1. Of the 124 completed responses to Part 2—meaning the students made statements for both types of integrals—most of the students (93 students, 75.0%) had descriptions of the definite and indefinite integrals using the exact same base conception(s). Table 1 shows a breakdown of the shared conceptions these 93 students applied to both definite and indefinite integrals. The “unspecified function” category was used for responses in which the student simply indicated that the integrals both represented functions, without specifying what that function was, including whether it was an anti-derivative. The “derivative” category was for students who claimed that the integrals represented *derivatives*, rather than an anti-derivative, though there is a possibility that some of those students could have made a mistake in what they wrote down. The “other” category contained unique meanings from the students, like a kinematics interpretations, or uninterpretable responses, like the “path on a graph.” Note that 11 of these 93 students gave more than one meaning to *both* integrals and are hence double-counted in Table 1. Nine of these 11 students claimed that both the definite and indefinite integrals represented both the area under a curve and an anti-derivative.

Table 1
Number of students in “same” that applied a given conception to both integrals

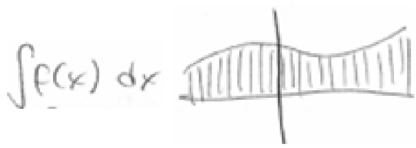
Area	Anti-derivative	Summation/ MBS	Unspecified function	Derivative	Other
65	22	2	4	5	6

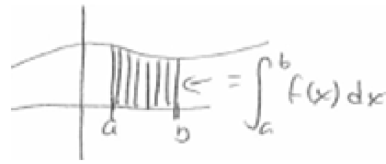
Given the numbers of students ascribing the “area” conception and the “anti-derivative” conception to both types of integrals, we present representative examples from the surveys to illustrate how these conceptions were simultaneously applied to both integrals. First, students often used “area under a curve” to describe both, wherein the main conceptual difference was the interval over which the area was taken. For indefinite integrals, the area was taken under the *entire* curve. While this may seem similar to what Hall (2010) called “potential area,” many students in this study were clearly thinking of *literal* area, and, in fact, seemed to equate $\int f(x)\,dx$ with $\int_{-\infty}^{\infty} f(x)\,dx$. The following two quotes and two sets of images (Figures 1a and 1b) are representative descriptions and drawings taken from among the surveys. Note that to save space in this paper, in Figure 1 some of the written integral expressions have been cut-and-pasted closer to the corresponding graphical images than they were in the original student written work.

Example survey response 1: $\int f(x)\,dx$ means find the area under the entire graph whereas $\int_a^b f(x)\,dx$ means find the area only between points *a* and *b* on a graph.

Example survey response 2: $\int f(x)\,dx$ means the area under the whole curve from $-\infty$ to $+\infty$… Meanwhile $\int_a^b f(x)\,dx$ is the area of a part of that curve.

(a)





(b)

Figure 1: Representative graphical depictions of the indefinite and definite integrals

A smaller subset of the students (22), interpreted both types of integrals as meaning an anti-derivative. The following two quotes are representative descriptions taken from the survey responses fitting into this category.

> *Example survey response 3*: They are similar in that you find the anti-derivative of *f*(*x*). They are different in that for the indefinite integral you are done after finding the anti-derivative while for the definite integral you plug in the values of *a* & *b*.
>
> *Example survey response 4*: $f(x)=3x^2$. $\int f(x)=x^3$. $\int_0^3 f(x)=(3)^3-(0)^3=27$. The integral of a formula provides you with a new formula capable of showing several properties about the original function. $\int_a^b f(x)\,dx$ is an equation that helps us learn more.

For responses in this category, both definite and indefinite integrals seemed to be represented as an anti-derivative function, and the essential difference seemed to be whether inputs needed to be inserted into the algebraic function or not.

We now briefly turn our attention to the 24 students coded as "different" for Part 2 (19.4%) and the seven students coded as "mixed" (5.6%). Of the 24 "different" students, the vast majority (21 students) described the definite integral as meaning the area under a curve and the indefinite integral as meaning an anti-derivative. These interpretations match with curricular meanings often given to the two types of integrals (e.g., Stewart, 2014)—which makes it interesting to note that only about a fifth of the students in this study made this particular interpretation for the two integrals. Among the "different" students, there was only one single student who described the definite integral as a Riemann sum (and the indefinite integral as an anti-derivative).

Of the seven "mixed" students, four said that both integrals shared the *anti-derivative* meaning, but that only the definite integral represented area under a curve. Two of the students described both integrals as sharing the *area* meaning, and that only the indefinite integral represented an anti-derivative. The final student indicated they both represented unspecified functions, with the definite integral representing area also.

Comparison with Part 1 of the survey

Part 2 of the survey demonstrated that a large majority of students' concept images hold the definite and indefinite integrals as representing the exact same basic concept, even as they note possible minor differences within that particular concept regarding each integral type. This result resonates with the options selected by the students on the multiple option Part 1 of the survey. Of the 132 students, 82 students (62.2%) were coded as "same," 20 students (15.2%) were coded as "different," and 30 students (22.7%) were coded as "mixed." To compare the options the students selected for Part 1 with whether they used the same or different conception(s) to describe the integrals in Part 2, Table 2 shows the breakdown of the students coded into "same," “different,” or “mixed” for both Parts 1 and 2. Note that “blank” in Part 2 indicates that the student did not write anything for that part of the survey.

Table 2
Students coded into each category for both Parts 1 and 2 of the survey

	Part 2 “same”	Part 2 “different”	Part 2 “mixed”	Part 2 “blank”
Part 1 “same”	69	4	4	5
Part 1 “different”	4	13	1	2
Part 1 “mixed”	20	7	2	1

Table 2 shows that most of the students who described the two integrals as representing the same concept in Part 2 also selected only statements in Part 1 that suggested they meant the same basic concept. Similarly, most students coded as “different” in Part 2 had also selected only statements in Part 2 portraying the integrals as different. The main discrepancy between Part 1 and Part 2 is the portion of students who were coded as “mixed” for Part 1, but that ended up invoking the exact same conception(s) for both types of integrals in Part 2. This result says something interesting about what some students may see as being *same* or *different*.

Discussion

Indefinite and definite integrals representing the same concept

Most students in this study seemed to conceive of both types of integrals as representing the same underlying conception. Most students described both integrals as representing the area under a curve, or both as representing an anti-derivative. In other words, it appears that many students’ concept images for “integral” may have merged both types of integrals into essentially the same mathematical object—as a geometric area, or as an algebraic function, or both. This suggests that we, as calculus educators, need to decide if we are comfortable with this conceptual merging of these two types of integrals, as implied by this study, or whether we would want to emphasize different meanings or conceptual objects for each type of integral. For example, many textbooks (e.g., Stewart, 2014; Thomas et al., 2009) describe them differently in that definite integrals arise from areas under curves and are defined as the limit of Riemann sums. On the other hand, indefinite integrals are typically presented as a family of functions with the related property of having all their derivatives equaling the integrand. Further, past research has suggested that Riemann-sum-based understandings of definite integrals may be important for applying integration to science and engineering (Jones, 2015a), and that anti-derivatives may relate well to continuous accumulation (Swidan, 2011; Yerushalmy & Swidan, 2012). With these various possible meanings, it may be worth our effort to decide carefully and explicitly what meanings we might want our students to closely associate with *definite* integrals and what meanings we might want them to closely associate with *indefinite* integrals. It is also important to decide what meanings should be given to *both* and what meanings should be *exclusive* to one or the other. As it stands, it could be problematic for the bulk of students to have fundamentally combined the two into, essentially, a single construct. This may limit the applicability of these two types of integrals to only situations that match the exact conception given to both.

Why are they believed to be the same?

One might initially believe that the reason many students ascribe very similar meanings to both types of integrals is because of the calculational similarity between them, through anti-derivatives. That is, because of the fact that students work out lots of exercises computing both definite and indefinite integrals through anti-differentiation, they may come to see them as the same basic thing. Yet, interestingly, this study's data does not support this possible conclusion. Rather, the majority of students interpreted both indefinite and definite integrals through the concept of *area under a curve*, even though this meaning is generally not given explicitly to indefinite integrals. What might account for this? We wonder if it is the case that students simply do not have much of a conceptual meaning for indefinite integrals, and that students consequently simply rely on their conception of definite integrals to invent a meaning for indefinite integrals. Past research has found that the "area under the curve" tends to be the most common conception students associate with definite integrals (Jones, 2015b), which matches the conception the students in this study gave to indefinite integrals as well. Thus, we conjecture that many students may be trying to find ways to simply extend the familiar "area under a curve" concept to the indefinite integral. They may have done so by extending the "bounded" definite integral to a "boundless" indefinite integral, which they then conceived of as representing the entire area under the whole curve, possibly from $-\infty$ to ∞. If our hypothesis is right, then much work needs to be done in promoting better possible conceptual understandings of the indefinite integral itself (e.g., see the work of Swidan & Yerushalmy, 2014; Yerushalmy & Swidan, 2012).

What some students may focus on for "same" or "different"

Finally, it is interesting to note that a subset of the students had selected statements in Part 1 of the survey indicating that definite and indefinite integrals were fundamentally different, but then described them both using the exact same conception in Part 2. While on the surface this may seem contradictory, it is actually quite reasonable. It appears that it matters what the *grain size* of difference is. For example, two equilateral triangles of different size could either be said to be the same since they are both the same basic shape, or different since they are not exactly congruent. Therefore, some students seemed to focus on certain aspects *within* a conceptualization as determining whether the two types of integrals were exactly identical or not. It certainly is a difference for one type of integral to be over the interval $(-\infty,\infty)$ and the other type of be over the interval $[a,b]$! Furthermore, it is different for one type of integral to require the insertion of inputs into a function, when the other does not. Thus, many of the students who were coded as "mixed" in Part 1 may really see integrals through the same basic conceptual lens, but then were more attentive to whether they felt those features *within* the conception were identical or not. It may be, then, that some of the "mixed" students on Part 1 were actually more closely associated with the "same" students than Table 2 depicts.

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Using Video in Online Work Groups to Support Faculty Collaboration

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Faculty in undergraduate mathematics departments are currently involved in making changes to their instruction, particularly by introducing different modes of student-centered type instruction. In this paper, we analyze a situation where faculty are involved in online collaboration using video of their own classrooms. We found that showing the video during the online work groups promotes more discussion of pedagogy rooted in instructional components than instructors watching the videos alone before. Pedagogy and students' mathematics also become important discussion points that encouraged and supported the instructors. Providing instructional components as a frame proved to be successful in supporting the video discussions as they stay centered on instruction.

Key words: Inquiry-oriented instruction, faculty collaboration, video, differential equations

Introduction

Recently, practitioners and researchers have turned their focus to the study of online resources meant to improve collaboration among faculty as a means to improve STEM education at the university level. For example, at a recent conference sponsored by AAAS and NSF, the Envisioning the Future of Undergraduate STEM Education Symposium, there were several sessions about faculty collaboration. Collaboration among faculty in different STEM disciplines at one university, among faculty in the same disciplines at different universities, and using technology of all types to facilitate and enhance collaboration were different foci considered at the conference. Online collaboration is particularly of interest as the technology to support this has made huge improvement in the last 10 years.

In this paper, we have chosen to consider one specific collaboration category. We will describe the use of videos from faculty classrooms during the implementation of faculty online work groups. This research is one component of a larger NSF funded project that is intended to build and conduct research on supports for instructors seeking to change their instruction. In this NSF project, the three supports (online work groups, curricular materials, and summer workshops) are studied together in order to support faculty as they implement one kind of active learning, inquiry-oriented mathematics (Rasmussen & Kwon, 2007). Later publications will report on the other supports and the effect on student outcomes.

Literature Review

When instruction that supports students' conceptual understanding is enacted, access to expertise and ongoing collegial support are important for sustaining that kind of instruction (Coburn, Russell, Kaufman, & Stein, 2012). At the K-12 level, a common support for teachers as they move to improve their teaching practice is professional development (PD). K-12 research points to a number of characteristics of effective PD programs. Impactful programs and supports need to be ongoing and sustained (Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009; Gallucci, 2008; Garet, Porter, Desimone, Birman, & Yoon, 2001; Hill, 2007; Kazemi &

Hubbard, 2008), be focused on subject matter, in terms of the mathematics, the students' mathematical thinking, and the instructional goals (Hill, 2007; Kazemi & Franke, 2004; Kazemi & Hubbard, 2008), be integrated into teacher's daily work (Darling-Hammond & McLaughlin, 1995; Franke, Kazemi, & Battey, 2007; Putnam & Borko, 2000) and provide teachers with feedback (Elmore, 2002).

Research on instructional change strategies at the undergraduate STEM level provides further insights. In a review of literature on instructional change strategies, Henderson, Beach, and Finkelstein (2011) identified two common but ineffective approaches: (1) top-down policies meant to influence instructional practices and (2) merely making "best practice" curricular materials available to faculty. In contrast, Henderson et al. found that effective change strategies which align with or seek to change beliefs of individuals involved, involve long-term interventions and are compatible with the institutional context are the most effective.

As far as research on using video, according to Borko, Koellner, Jacobs, and Seago (2011), "... by choosing video clips, posing substantive questions, and, facilitating productive conversations, professional developers can guide teachers to examine central aspects of learning and instruction" (p. 176). Borko and her colleagues describe using video of teachers in two ways: the teachers' own videos and videos from other classrooms. They found that the selection of goals for the video use was very important as was the selection of the video clips. The facilitator selected the videos which is different than our work, where the instructors selected their videos.

As one analysis framework, we used work by Sherin and van Es (2009). Their research focused on using videos to study teacher professional vision. Sherin and van Es analyzed the meetings focused around videos for the following categories: Management, Climate, Student Thinking, and Pedagogy. Their analysis using these and other codes showed that participating teachers did grow in their knowledge based reasoning, change their ways of paying attention to students and student thinking, and changed their instruction in the classroom.

Research Goal and Framework

Our research goal was to characterize the conversations that occur during an inquiry-oriented differential equations online work group when the participants are sharing videos of their instruction. As the project where the research was done has evolved, the team has developed a framework (Kuster, Johnson, Andrews-Larson, & Keene, under review) to analyze instructors implementing inquiry-oriented instruction. Four components of instruction have emerged which served as an additional framework for our analysis (table 1).

Table 1
Focal Instructional Components

Component	Definition
Generating Student Reasoning	Facilitating student engagement in meaningful tasks and mathematical activity related to an important mathematical point, eliciting student reasoning and contributions, actively inquiring into student thinking
Building on Student Thinking	Being responsive to student contributions and use student contributions to inform the lesson, guiding and managing the development of the mathematical agenda
Developing a **Shared** Understanding	Engaging students in one another's thinking
Connecting to Standard Mathematical Language and Notation	Teachers introducing language and notation when appropriate, teachers supporting formalizing of student ideas/contributions

Methods

Research Context and Participants

For one semester, instructors participated in a weekly online work group (OWG). The main goals of the OWG were to: 1) aid instructors in making sense of the materials (i.e., thinking through the sequences of tasks, how students might approach the tasks, how to structure instruction around the tasks to support student learning), and 2) assist instructors in developing and enhancing their instructional practices. In particular, attention was focused on the focal instructional components mentioned above.

The OWG included two multi-week lesson studies. Lesson studies are composed of approximately four week-long segments that have instructors doing the math that they will have their students do in class (week 1); discussing the possible student thinking (week 2); and sharing videos of themselves teaching the unit with attention to the focal instructional components (weeks 3 and 4). In those last two weeks, instructors seek feedback from the group to improve their teaching methods; they use the video clips of their classroom teaching to facilitate that discussion. Due to logistics of filming video in classes, video was shared during weeks 4, 5, 8, 9, and 10 of the OWG; with 4 and 5 as part of the first lesson study on one unit (autonomous differential equations) and for weeks 8, 9, and 10 (systems of differential equations).

Participants were interested in exploring instructional change and came from smaller universities across the US. In the inquiry-oriented differential equations (IODE) OWG there were four participant instructors (instructors teaching differential equations at the time of the OWG), one previous participant instructor (instructor who partook in the pilot OWG but was not currently teaching differential equations), two graduate research assistants, and one facilitator.

Data Collection

Each of the 12 OWG sessions took place via Google Hangouts as instructors were at their universities across the country throughout the semester. All OWGs were screen recorded by one of the researchers to be able to capture both audio and video of the OWG. As part of the OWG, instructors were to create videos of themselves teaching the same unit at two different times during the semester. The instructors then self-selected clips (cf. Borko et al., 2011) they wished to bring to the OWG, knowing that clips should be focused on the focal instructional components if possible. Clips ranged from 2-6 minutes in length and usually 2-3 were watched for each participant at a time.

Data Analysis

This study focuses only on the conversations in the OWG when the participants were sharing or discussing the shared videos (5 of the 12 total meetings). Two researchers watched the entirety of the OWGs and marked all the timestamps when video was discussed. To give perspective, Table 2 shows the percentage of the total time of the OWG sessions (approximately one hour each) that discussion of the videos specifically occurred.

Table 2

Percent of Coded Time of Discussion

Week	4	5	8	9	10
% of Coded Time	45.86%	53.78%	40.51%	40.16%	26.18%

We created content logs to summarize in detail the conversations of the OWGs. The content logs had multiple columns for us to be able to characterize the conversations (i.e., timestamps, topic of conversation, focal instructional components, speaker, and detailed notes). We began by coding two full OWGs together. To code the topic of conversation, we used a priori codes that were influenced by previous research on video clubs (Sherin & van Es, 2009). Using a constant comparative method (Strauss & Corbin, 1998) we open coded and also created new emerging codes to fully characterize the conversations in the OWGs. The focal instructional components were also used as an analytical framework for analysis of the conversation.

We watched the video until one researcher felt that the conversation had shifted directions and changed topics significantly. That chunk of time then became a row (or time block) in the content log, where we coded the topic of conversation and focal instructional components (if applicable). Whenever possible, video was divided so that each time block had as few topic of conversations as possible. However, multiple topics of conversation and focal instructional components often happened in one utterance so those could not be parsed out and the time block was coded as all applicable topics. For perspective, the average time block was 47 seconds across all five coded segments of the OWGs. We developed and refined a codebook with detailed definitions based on the a priori codes, as well as the new codes that emerged during analysis (see table 3 for the topic of conversation a priori codes). Collectively all disagreements were worked out. Finally, two researchers coded the three remaining OWGs and left any disagreements between them to be resolved by the third researcher.

Table 3
Topic of Conversation A Priori Codes

Codes	Definition
Pedagogy	Teacher instructional moves or ideas about teacher instructional moves
Student Mathematical Thinking	Evaluating and/or discussing students' thinking about mathematical concepts
Mathematical Content	Discussion of mathematics, IODE material, difficulty of mathematics
Social and Sociomathematical Norms	Discussion about getting students used to participating in group work; includes quality of group work and interaction
Environment	Classroom behavior, classroom setup, access to tools and technology

It is important to note that all time durations reported in results will be over estimates based on the way we divided up the video into the time blocks. Multiple codes could occur at the same time but may not have been used for the entire time block. However, as noted, we kept the time block as small as possible while still maintaining the integrity of the analysis; this aided in alleviating the limitation of the overestimates.

Results

First, we present a characterization of the conversation through the two perspectives (topics of conversations and focal instructional components). These results will be broken down by week and also shown as a synthesis across the five weeks of the OWG. Second, we discuss the role of when video was watched and how that influenced the conversation that occurred and the richness of said conversation.

Topic of Conversation and Focal Instructional Components

Table 4 includes the four new topic of conversation codes which emerged.

Table 4
Topic of Conversation Emergent Codes

Codes	Definition
OWG Logistics	How to record videos; what clips to share/select; what to watch; remembering video
Orienting	Setting the stage for the video that will be shown
Appraisal	Participants give value judgments about events from the videos, instructional moves

As noted, when codes were discussed they oftentimes were not alone. Here is an excerpt from the week 5 OWG where pedagogy, appraisal, norms, environment, orienting, OWG logistics, and student mathematical thinking were all coded. Note, this excerpt spanned four time blocks.

> *Instructor 1:* … So if we go to the video that is Instructor1_Owl_2. We'll start at 6:15. So this is, we had just spent some time talking about whether if we started with the owl population below 5 if they were going to go extinct or not. My students seemed to have that idea that because the derivative is negative the population is going to decrease and it is going to decrease down until it hits 0. Right - they kind of had a handle on that. So starting at 6:15, we want to talk about what happens for P between 5 and 8 … What I want you to look for, is that I bring up the Uniqueness Theorem myself. I didn't really give the students an opportunity to see it. And that is maybe based on past experience where they just aren't used to arguing with it yet. So I want you to look at how I brought that up and maybe if you think I brought it up too soon or what you have done if you're thinking, 'oh maybe, they're not going to get the Uniqueness Theorem.' So I'm going to stop talking so you can watch the video now.
>
> *Video watching occurred and additional conversation for approximately 8 minutes.*
>
> *Facilitator:* It would have been interesting, just as a suggestion to think about, you could have students come up to the front and present their ideas. It would have encouraged them to be more open about what they were thinking about.
>
> *Instructor 1:* Yeah, I don't have the students come up to the board too often in this class. The room is a little bit, I don't know, it doesn't quite fit for me somehow. It is also maybe that I have created a culture where that doesn't happen very often. Although, whenever I do have them come up to the board, they come quite willingly.
>
> *Instructor 2:* They were participating, they were bringing up things and you were writing them down. And you did a nice job of hearing what one person was starting to say and finishing up the other.

In the first lesson study, when video was watched during the OWG, there was a higher amount of time spent on OWG logistics. Particularly, much of the logistics were issues that participants had in finding the clips to watch. Videos of instruction were not screen-shared (a common feature of online collaboration tools) because audio was typically quiet and participants were not able to hear the video. Thus a shared video folder on an online server was made, as well as an online agenda that contained all relevant links. However, participants still had technology issues. For example,

> *Facilitator:* Let's at least start it. Let's watch the first couple. The 2:45-4:00, the 9:30-12:40 and then we might have to come back.
>
> *GRA:* Don't forget to mute yourself when you watch the video.
>
> *Instructor 3:* GRA, I'm clicking on the Drive link you sent but it just sends me to a verify your email address page and it is not going to the Drive at all.

To be able to resolve these issues the GRA would assist the instructor, which took time away from the OWG.

Because of space limitations, we are only reporting percentages on pedagogy and OWG logistics. Pedagogy was the largest topic of conversation that occurred across all OWGs (by week 4, 5, 8, 9, 10, respectively: 41.33%, 41.08%, 85.74%, 85.85%, and 59.50% and across all weeks, 59.15%). OWG logistics had the following breakdown: by week 4, 5, 8, 9, 10, respectively - 53.72%, 55.94%, 10.29%, 19.12%, and 22.50%, with a total of 36.98% across all weeks. Noteworthy is the difference in time spent discussing pedagogy and OWG logistics.

The objective of the OWG is to prepare instructors to teach inquiry-oriented instruction and thus the components were crucial for that discussion. Rarely were the components ever referenced by name, however, the practices that make up the components or a general sense of the components were discussed by the instructors. Generating was discussed 24.40% of the time, building 22.47%, shared 20.75%, and connecting 5.33%. The component Connecting was the least referenced component. While the week by week discussion varied, overall the remaining three components were roughly discussed the same amount of time.

Role of When Video is Watched

We note that during weeks 4 and 5 video clips were watched *during* the OWG and weeks 8, 9, and 10 they were watched *prior to* the OWG. Recall the inverse relationship between how often pedagogy and OWG logistics were discussed. When videos of instructors' classrooms were watched in the OWG (weeks 4 and 5), more OWG logistics were discussed than pedagogy. Yet, when video was watched prior to the OWG, logistics were discussed less and pedagogy more. Interestingly, when breaking down the pedagogy discussion into times when it was cross-coded with a focal instructional component, highlights something noteworthy (see figure 1). A large amount of the pedagogy discussion in weeks 8, 9, and 10, was not actually rooted in the focal instructional components. Participants noted in exit interviews they believe this was because the video was not fresh in their mind so the conversion could not be as deep.

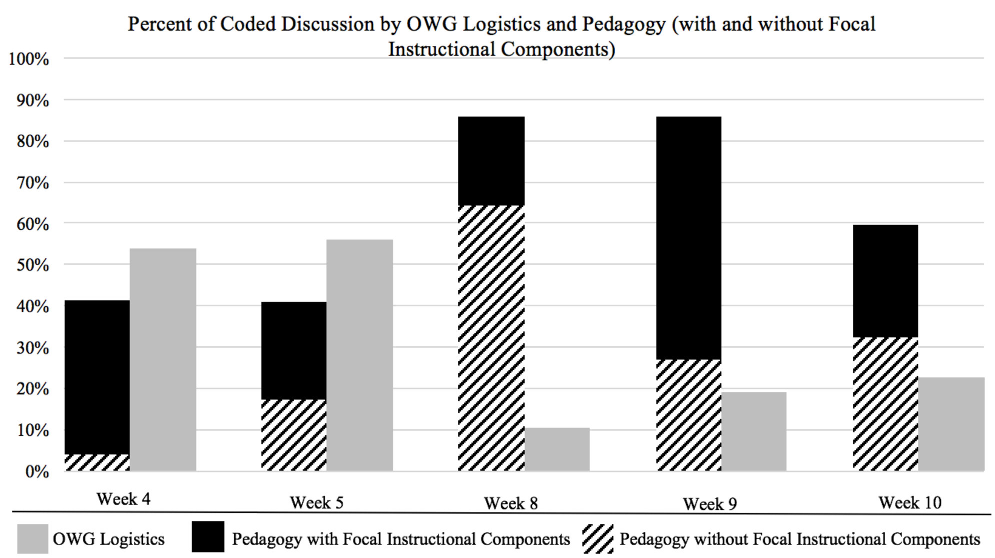

Figure 1. Percent of coded discussion by OWG logistics and pedagogy.

Conclusion

Overall, we have described the conversation around the use of video in faculty OWGs. As mentioned earlier, our research is part of a larger project, but the use of self-selected video and the discussion around it have provided us interesting new knowledge about what these conversations look like. We conclude this proposal with a brief interpretation of the results of our analysis and implications for future online faculty collaboration whose purpose is to support instructor change. The topics of conversation when the OWGs focused on the videos were heavily focused on the pedagogy during all the weeks of the OWGs. This is an important implication as faculty collaboration across universities is going to be increasingly more common. Traditionally, faculty collaboration has revolved around research and the development of new mathematics, but currently, faculty collaboration to discuss instruction is growing. Using the videos provided an object for pedagogy to revolve around.

Other conversation focused (in an interweaving way) on mathematical content, student thinking, and environment/norms. But we found that logistics were also common parts of the discussion. This category, along with orienting, had not been in our original code lists, but because of the online environment, they were important to document and interpret. We found that the orienting of other faculty participants to the classroom video was imperative to allow productive discussion to occur. Orienting needs to be encouraged, and different or more refined ways of orienting will be part of future work.

We particularly note the fine-grained analysis of the two methods of watching video, either during the actual OWG, or beforehand. Analysis showed that watching video during the OWG meant that technical support is needed to aid instructors in synchronously watching the videos, which in turn detracts from the amount of time that can be spent discussing the intended content (the focal instructional components). However, we also found that watching videos beforehand meant the discussion was not as rooted in the focal instructional components either because the video was not fresh in instructors' minds or they did not actually watch the videos. Further analysis is required to ascertain which method is concretely better, as our analysis revealed pros and cons for both approaches. Nevertheless, instructors noted they preferred to watch videos during the OWG in exit interviews.

As far as the focal instructional components that our project provided as a framework for thinking about inquiry-oriented instruction, we found that three of the components (generating ways of student reasoning, building on student thinking, and developing a shared understanding) were part of the conversation about an equal amount of time. The fourth component, connecting to formal notation, was present significantly less of the time. This is not surprising, as the content in differential equations involve much less formal structural development than other upper level undergraduate mathematics courses where our work is also focused.

The participants in the OWG were all from small universities where they were usually the only faculty member teaching Differential Equations and the only ones implementing this new kind of instruction. There is significant work going on across many universities in many projects that are attempting to improve STEM undergraduate instruction. We found that videos were a particularly powerful way to provide avenues for instructors to share their own classrooms in ways that were comfortable and affirming, reflect on their own authentic teaching and discuss pedagogy and students' mathematics.

Although the facilitator's role was not part of this proposal, it also has proven to be very important. Facilitators can learn from studying the research about OWGs that we are working on, to help better understand how to make the conversations most productive. Ultimately, the goal is to improve instruction in order to improve student outcomes. But taking this intermediate step of describing and analyzing one new and powerful tool, using video in OWGs, may help the mathematics education research community with the final goal.

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Undergraduate Abstract Algebra: Is Teaching Different at 'Teaching' Colleges?

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Reforming the way undergraduate math is taught has been the target of significant research efforts for decades; however, lecture remains the predominant form of instruction. While interest has been primarily focused on entry-level courses in order to recruit and retain STEM-intending students, quality instruction in upper division courses is also important. In a national survey of abstract algebra instructors, we investigated typical teaching practices, beliefs, and constraints that influence pedagogical decisions, and similarities/differences between those who do/not lecture. Of particular interest was exploring whether instructors at Bachelor's-granting institutions have markedly different circumstances than their counterparts at Master's- and Doctoral-granting institutions and the effect (if any) this has on their pedagogical decisions.

Key words: Collegiate teaching practices, beliefs, abstract algebra

In STEM higher education, but specifically in mathematics, a well-established tradition of lecture still persists despite the growing volume of mathematics education research that urges teachers to move beyond this practice. The ineffectiveness of lecture has been consistently reported (Freeman, et al., 2014) and students have been pretty clear about their dislike of lecture-based courses. Eric Mazur, Harvard professor and active-learning proponent, suggests that change is coming: "Though it remains the dominant form of instruction in higher education…the lecture may be on its last legs" (Lambert, 2012). Data from the HERI report (Eagan, et al., 2014) confirm this trend, reporting an increase in student-centered teaching practices and a decrease in lecture over the past 25 years; listing the most recent figure (2014) hovering around 51% of faculty reporting that they lecture in "all" or "most" of their courses; however, when dis-aggregated by discipline, "the data continue to show that nearly two-thirds of faculty across STEM sub-fields utilize extensive lecturing in all or most of their courses" (Eagan, 2016). This is particularly problematic given that the President's Council of Advisors for Science and Technology found that fewer than 40% of students entering college in pursuit of a STEM degree complete that degree (PCAST, 2012, p. i). It appears that students are frustrated by "faculty modes of teaching that suggest [they] take little responsibility for student learning" (Seymour, 2006, p. 4), as students cite ineffective teaching methods and uninspiring atmospheres in introductory-level STEM courses as the primary reason for attrition (PCAST, 2012, p. i, p. 5).

The mathematical professional societies (AMATYC, AMS, ASA, MAA, and SIAM) have taken notice and issued a joint statement indicating that while they would refrain from specifying pedagogical practices, they "feel that active student engagement is necessary for a mastery of algebraic ideas" and that "problem-based, inquiry-based, and collaborative learning activities are appropriate means of maintaining student engagement" (CUPM, 2015). Despite this recommendation, lecture is still the norm – even in upper division mathematics courses, which tend to have fewer constraints and more flexibility in terms of content (e.g., these courses are typically not coordinated across multiple sections and have few subsequent courses). Keith Weber reports that "the advanced proof-oriented courses for mathematics majors are typically taught in a lecture format" (Lockwood, 2015) and in a recent study by (Fukawa-Connelly, Johnson & Keller, 2016), it was found that when considering abstract algebra specifically, lecture is overwhelmingly the dominant pedagogical technique both in terms of percentage of instructors using it and percentage of class time devoted to its use.

In our previous study, the target population was mathematics instructors teaching at Master's- and Doctoral-granting institutions; i.e., 'research' universities. The purpose of the present study is to extend this work by investigating the teaching habits, beliefs, and constraints of instructors at Bachelors-granting institutions, traditionally 'teaching' colleges, to see if any differences emerge. To that end, we investigate the following research question: Does the way that abstract algebra is taught vary by institution type and can those differences be explained?

Literature Review

The literature has cited many reasons why instructors choose to lecture, not least of these is the belief that lecture is the best method and/or necessary for content coverage (as discussed by Roth McDuffie, & Graeber, 2003; Wagner, Speer, & Rosa, 2007; Yoshinobu, 2014). While we do not wish to discount the enormous influence personal beliefs have on instructional decision-making, we must acknowledge that a bevy of other external circumstances can factor considerably when instructors plan their courses. Research has indicated that instructors are reluctant to change practice when serving an administration that provides neither pressure nor incentive to do so and, furthermore, that concerns about promotion and tenure inhibit change when the administration is not perceived to value experimentation in teaching (Roth McDuffie, & Graeber, 2003). Another important factor is departmental and university culture. According to Henderson & Dancy (2007), working with colleagues who either lack knowledge about or withhold support of pedagogical reform inhibits an instructor's willingness to modify current practice. Finally, departmental resources must be considered as a factor. Instructors who are not offered classroom assistance, release time to plan and prepare tasks and revise syllabi, or financial support for professional development are unlikely to abandon the traditional lecture style in light of the high "start-up costs" demanded by a change in pedagogy (Henderson, Beach, & Finkelstein, 2011; Wagner, Speer, & Rosa, 2007; Roth McDuffie, & Graeber, 2003).

In higher education, a distinction is drawn between what are colloquially referred to as 'research institutions' and 'teaching colleges'. The motivation for this study was rooted in the notion that the culture at teaching institutions might be better suited to adopt non-lecture practices based on the widely-held belief that lack of research demands at these universities affords the instructors time to hone their teaching and thoroughly develop their courses. Additionally, instructors surrounded by faculty and administrators who similarly value teaching might be more likely to have the necessary support for experimenting with new pedagogy. Our literature review showed support for this broad characterization. Differences between teaching and research institutions are real and cannot be attributed to merely academic folklore.

At both types of institutions there are expectations for quality teaching and scholarly research; however, the emphasis can and does vary widely. Professors at research universities are judged and rewarded based on their research; conversely, teaching (or comprehensive) universities "do not effectively accrue status through research" (Henderson, 2009, p. 11). Not surprisingly then, the vast majority of research is produced by faculty at research institutions. A 2003 study of research productivity across higher education found staggering differences in the publication rate of faculty at R1 institutions as compared with Bachelors-granting schools: a 2.04 publication/faculty ratio at the former as compared with a paltry .06 (weighted average across categories) for the latter (Toutkoushian, et al., p. 139). Karen Webber explains this discrepancy:

"especially at research universities, publications and extramurally-funded grants are central to the institutional image and pocketbook, and thus strongly affect individual promotion and tenure" (2011, p. 110).When the most important consideration in tenure decisions is research productivity, it stands to reason that, "teaching is, at best, a secondary obligation" (Vest, 1994). In a 1984 study, Katherine Kasten put forth hypothetical candidates for tenure review with vitaes reflecting varying degrees of research and teaching quality. The statistical analysis, confirmed by interview data, demonstrated that research is the most important consideration indicating that "excellent scholarship … counterbalances virtually anything else except dereliction of duty" and there exists a "threshold for research below which no degree of teaching…can provide compensation" (p. 506).

In comparison, at teaching institutions, the 'publish or perish' ethos has yet to establish a strong foothold. Henderson (2009) explains that while traditional research holds the highest status, "other forms of publication have always been given credit in the faculty reward system…department heads, deans, and provosts have always been happy when faculty members have published, no matter where or what they published" (p. 18). Scholarship is appreciated, but students and teaching are the central focus. At comprehensive universities, where the faculty is likely to be unionized (Henderson, 2009), the contract clearly defines faculty teaching loads and criteria for promotion. In one such example, the most recent collective bargaining agreement of the PA State System of Higher Education (composed of 14 teaching institutions) outlines the teaching load to be 24 credit hours in an academic year with a minimum requirement of 5 office hours per week, and allows for the assignment of up to three unique academic preps per term (APSCUF, p. 74). This document further lists the categories for performance review and evaluation as effective teaching, continuing scholarly growth, and service – in that order (p. 24). While the NCES (Cataldi, Bardburn, & Fahimi, 2004) has shown the average faculty workload is 53.4 hours/week – a figure that does not differ significantly when considered by institution type – but the sort of work activities that occupy that time do vary significantly. Instructors at Doctoral-granting institutions spend on average 25.3% of their time in research as compared with less than 1% (on average) across all other types of institutions.

We believe that these institutional differences may lead to differences in the students' educational experience. On the one hand, diminished research expectations coupled with heavy teaching assignments would tend to suggest that these instructors are more likely to reflect upon and experiment with their pedagogical practice as compared with their research university counterparts. On the other hand, comprehensive universities are more likely to have scarce financial resources in part due to their disproportionate receipt of grant monies, which could limit participation in the types of professional development opportunities that introduce new pedagogical techniques or disseminate curricular materials and are less likely to find the time needed to make instructional changes due to the increased course loads common at teaching institutions. Webber (2011) estimates that the time needed to prepare well for class may equate to three to five times the number of hours for each hour spent in the classroom (p. 113), representing a non-trivial increase in time commitment for each additional course an instructor is assigned. Furthermore, faculty who are somewhat distanced from the research community, whether due to interest or circumstance, are perhaps less likely to be up to date on the literature advocating for pedagogical change when this work is being done by instructors who are neither their colleagues nor, in some circumstances, their peers. The purpose of the present study is to investigate whether there is evidence to support these suppositions regarding inclination for pedagogical change in the specific context of abstract algebra.

Data and Methods

Survey Design

In the previous study, we developed a survey designed to solicit information about the teaching practices, beliefs, and situational context of abstract algebra instructors. This survey was informed in part by both Henderson and Dancy's physics-education survey (Henderson & Dancy, 2009) and the Characteristics of Successful Programs in College Calculus surveys[1]. Our survey had sections to assess each of the following types of information: basic demographics and course context, teaching practices, beliefs and influences (including perceived supports and constraints), and knowledge of and openness to non-lecture practices. In this study, we wished to obtain the same information from a different population for the purposes of comparison. For that reason, it was methodologically important that the items under investigation remain largely unmodified. We chose the subset of items we wanted to ask and kept the formatting the same as the previous survey. Due to space constraints, sample items are not here, but a version of the survey can be found at the following link: pcrg.gse.rutgers.edu/algebrasurvey.

Participants

In the previous research, the initial sample consisted of 200 institutions from which 126 completed surveys were received. In this follow-up, a random sample of 400 institutions was drawn from the IPEDS list, targeting specifically Bachelor's-granting schools. From this, we received 112 responses, 91 of which were completed. For the purposes of this paper, all data has been combined into one data set and then disaggregated by terminal degree for all future analysis; 117 Type B = Bachelor's, 59 Type M = Master's, and 108 Type P = PhD – these designations will be used hereafter.

Methods

We first calculated basic descriptive statistics appropriate for each item and compiled demographic information. Where indicated, percentages were tabulated on the aggregate and for each identified sub-group. Group measures were compared using inference testing procedures such as ANOVA, Chi-Square test, or the Kruskal-Wallis Test as determined by the type of data under investigation. When appropriate, the Holm-Bonferroni correction was applied to control for the family-wise error rate affiliated with multiple comparisons. Details as applicable to particular tests can be found in the Results section. In general, the objective was first to provide a characterization of the abstract algebra course (as reported by our participants), specifically determining who teaches it, what is being taught, and how it is being presented. The secondary analysis was designed to explain those findings based on instructors' beliefs, resources, and constraints.

Results

In response to our first research question, *Does the way that abstract algebra is taught vary by institution type?*, the results indicate that in many regards it does, but there are some key characteristics that appear to be universal. Instructors teaching abstract algebra are generally not new faculty; roughly 78% have been teaching for more than six years (i.e. post-tenure) and this does not vary significantly by institution type; however, the experience with teaching AA is quite varied. On the aggregate, there is nearly an even distribution of experience across the three levels

[1] See www.maa.org/cspcc for more information about the CSPCC project and a copy of the surveys.

(1-4 times, 5-8 times, 8+ times), though this does not persist (Kruskal-Wallis test: $\chi 2 = 6.667$, df $= 2$, $p = .036$) when the data are dis-aggregated as the modal class rises proportionately with the type of institution. At Type P institutions, the majority of faculty have taught the course upwards of 8 times, whereas at Type B institutions the majority of faculty have taught the course only 1-4 times. Across institutions the predominant course structure was an undergraduate (61%) course for math majors; other responses included mixed audience courses and graduate courses. Approximately 9% of respondents reported teaching a "groups only" or "rings only" course, but the majority covered both topics in a "groups first" format and this was consistent across institution type. For nearly everyone surveyed, their algebra course is not taught as a capstone course and has a proof-writing prerequisite, although there are distributional differences by type with Type P being least likely to require this prerequisite course (57% as compared with 79% for Type B and 82% for Type M). The most significant, although unsurprising, differences were found in terms of follow-on courses. Type B schools were much less likely to require a second course (6.9%) than Type M (14%) or Type P schools (18.3%).

Moving away from demographic characteristics, the analysis looked at more subjective and personal measures such as pedagogical style and teaching practices. Using the prompt, *Have you ever taught abstract algebra in a non-lecture format?*, we coded respondents as either self-identified lecturers (hereafter referred to as "Lecturers") for responding *No* or *I have in the past, but I currently lecture*; or as self-identified non-lecturers (hereafter referred to as "Non-lecturers") for responding *I currently do*. Our findings support the notion that lecture is still the predominant mode of instruction in upper-division mathematics courses with 83% of our participants identifying as Lecturers; this represented 78% of Type B, 79% of Type M, and 91% of Type P instructors. On the one hand, it appears that our suggestion that instructors at teaching institutions would be less likely to lecture is supported ($Z = 2.23$, $p = .026$); however, a lecture rate of 78% does not indicate that sweeping pedagogical reform has occurred.

In terms of teaching practices, again there were a few noticeable differences, but for the most part institution types were more alike than they were different. When asked to report frequency per term for the following activities: *having students present proofs or counterexamples to the class, having students develop their own definitions, having student develop their own conjectures, having students develop their own proofs,* and *leading discussions in which students discuss why the material is useful and/or interesting*, no significant differences in mean frequency were observed between institution types. Similarly, when asked to report frequency per class meeting for the following activities: *I pause and ask students if they have questions, I use visual and/or physical representations of groups and group elements, I use diagrams to illustrate ideas,* and *I include informal explanations of formal statements*, there were no significant differences in mean frequency observed between institution types. Only for two activities: *I have students engage in small-group discussions or problem-solving* ($F = 7.984$, $p < .001$) and *I have students ask each other questions* ($F = 4.119$, $p = .018$) were statistically significant differences found: Type B schools engage in these activities more often than Type P schools. (It is important to note that the average increase in mean frequency was approximately .3 which, given the scale used, translates into roughly one additional occurrence per class, or a difference of engaging in the activity once per class versus every other class.)

In addition to reporting on frequency of certain activities, instructors were also asked to comment on the average percentage of class time devoted to certain pedagogical practices on a scale ranging from 0 (never) to 4 (75-100%) of class time. These included: *showing students how to write specific proofs, having students work with one another in small groups, having students* *give presentations of completed work, having students work individually on problems/tasks, lecturing, holding a whole-class discussion,* and *having students explain their thinking*. On average, the approximate amount of class time devoted to these techniques was under 25%, across practices and institution type; the notable exception was *lecturing* which occupied, on average, more than 50% of class time. In support of our hypothesis, Type B/M schools tended to devote more time to student-centered pedagogical techniques with significant differences observed for *having students work in small groups* ($F = 12.075$, $p < .001$) and *lecturing* ($F = 12.493$, $p < .001$). Each time, the mean difference between Type B/M and Type P schools was .5 on average, roughly translating to about 12.5% of class time.

Having found evidence suggesting that instructors at Type B schools self-identify as Lecturers less often and that the percentage of class time engaged in lecture for those who do is less than their Type P counterparts, our secondary analysis was focused on trying to explain these differences in terms of beliefs and resources/constraints. The instructors were asked to indicate degree of belief on a 4-point scale (-2 = disagree, -1 = slightly disagree, 1 = slightly agree, 2 = agree) in the following statements regarding teaching: *I think lecture is the best way to teach, I think lecture is the only way to teach that allows me to cover the necessary content, I think there's enough time for all the content I need or want to teach, When I last taught algebra, I had enough time during class to help students understand difficult ideas,* and *When I last taught algebra, I felt pressured to go through material quickly to cover all the required topics*. The instructors were consistent in terms of pressure and time concerns and really only differed in terms of evaluating the appropriateness of lecture. Type P instructors were significantly more likely on average to indicate that lecture was the only way to cover the necessary content ($F = 4.688$, $p = .01$) reporting moderate agreement where the others reported disagreement. Belief that *lecture is the best way to teach* failed to be statistically significant[2], but instructors at Type P schools did hold this belief more strongly than those at Type M and Type B schools as was expected.

The instructors were also asked to indicate degree of belief on the same 4-point scale in statements regarding students. These included: *I think students learn better when they do mathematical work in class, I think students learn better when they struggle with the ideas prior to me explaining the material to them, I think that all students can learn advanced mathematics, I think all students can learn abstract algebra,* and *I think students learn better if I first explain the material to them and then they work to make sense of the ideas for themselves*. Interesting here was that Type P instructors were the most pessimistic about the ability of the students to learn either advanced mathematics in general or abstract algebra specifically (although these differences did not achieve statistical significance). Consistent with the reported lecture practices, Type B/M instructors held significantly stronger belief in the students' need to do mathematical work in class ($F = 5.568$, $p = .004$).

Looking at differences in professional activities, there was little that was unanticipated. When ranking interest on a 4-point scale ranging from 1 (very weak) to 4 (very strong), the following activities were considered: *Discussing/reading about how students learn key ideas in abstract algebra, teaching abstract algebra, teaching other advanced classes, doing research in abstract algebra,* and *doing/reading research that could be considered the scholarship of teaching and learning*. Type P instructors had greater interest in algebra research and less

[2] Based on reported means of 0.5 for Type P, 0.3 for Type M, and 0.0 for Type B schools. This resulted in an F value of 3.73 with $p = .026$ which failed to meet the more conservative $p < .0125$ threshold for the Holm-Bonferroni correction for multiple comparisons.

interest in education research (reading or doing) and teaching. Significant differences were only observed for the research items ($F = 7.295$, $p = .001$ for algebra research; $F = 4.407$, $p = .013$ for educational research). Probably the most surprising finding here is that all faculty rated interest in teaching nearly a full point higher on average than either type of research activity

Type P instructors were lecturing at significantly higher rates than Type B/M; however, the lecture rates for all three institution types was quite high. To investigate the propensity for pedagogical modification, the respondents were asked if they would consider teaching in a non-lecture format and were directed to specific follow-up items investigating their reasons why they would not or why they have not. Type P instructors were significantly less likely (48%) to consider not lecturing as compared with Type B/M (65%) instructors ($z = -2.196$, $p = .028$). Those unwilling to consider a change were most likely to cite concerns over content coverage or a belief that it would go poorly (in that order), and surprisingly, this was consistent across all institution types. A very small number of individuals (5) indicated that they would not have departmental support for such a change. For those willing to consider a change, the number one impediment was lack of time to redesign their course, independent of institution type. Secondary concerns were lack of materials and not knowing where to start. Interesting here is that there were 12 individuals who would like to switch but feel that they lack departmental support (6 of whom were Type P instructors).

Discussion

Overall, abstract algebra instruction looks fairly similar across institution types, there being many commonalities regarding course structure, class activities, and time use. Virtually all instructors who hold Ph.D.s in mathematics receive degrees from research universities where lecture is the dominant paradigm. It is perhaps to be expected, then, that their instructional decisions – influenced in part by their own experiences as students – might reflect the teaching culture in which they were raised. A difference appears among Type B faculty, though, who self-report as Lecturers at lower rates than Type P (78% versus 91%, respectively) and who spend less (12.5%) class time lecturing than do Type P instructors. Moreover, the fact that using small group discussions and asking students to inquire into one another's thinking occurs approximately twice as often in Type B classes than in Type P further indicates a distinction in the day-to-day experience of students in these classes. That a clear majority of Type B instructors still report conducting class primarily through lecture, however, indicates that reformers still have a long way to go in spreading student-centered practices in mathematics (at least in abstract algebra), specifically with regard to existing curricular materials.

The current numbers should not detract from what may be a promising outlook: 65% of Type B/M Lecturers reported that they would consider switching to non-lecture pedagogies (significantly more than the 48% of Lecturers at Type P schools). The remaining 35% of Lecturers were most likely to cite concerns over content coverage or their belief that it would go poorly (in that order). It is interesting therefore, that 81% of these same Lecturers also report feeling no departmental pressure to cover a fixed set of topics, making the issue of content coverage largely one of internal orientation. Future directions for this research include examination of the actual content coverage by topic and its variance by institution type, the possible influence possession of a growth/fixed mindset plays in pedagogical decision-making, and researching the potential effect of the interaction between institution type and pedagogical style on the distribution of class time

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Mathematicians' Evaluations of the Language of Mathematical Proof Writing at the Undergraduate Level in Three Different Pedagogical Contexts

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This proposal discusses the extent to which mathematicians agree amongst themselves with regard to what are some of the linguistic conventions of mathematical proof writing. Data from a survey of 128 mathematicians are used to address this question. Participants were asked whether various excerpts highlighted in four partial proofs were unconventional in each of three different contexts: how proofs appear in undergraduate mathematics textbooks, what instructors write on the blackboard in undergraduate mathematics courses, and how students write proofs in these courses. These data point to a lack of agreement among mathematicians on the linguistic expectations of the proofs written by their students.

Key words: Mathematical language, Proof, Mathematics Textbooks, Mathematics Lectures

Research has shown that undergraduate mathematics students have difficulties when constructing (Weber, 2001), reading (Conradie & Frith, 2000), and validating (Selden & Selden, 2003) mathematical proofs. Among several different reasons for why undergraduates struggle with constructing mathematical proofs, Moore (1994) included students' unfamiliarity with the language of mathematical proof writing. However, there is a dearth of empirical and systematic research in the field of mathematics education on the language of mathematical proof writing at the advanced undergraduate level.

In particular, how advanced undergraduate mathematics students and mathematicians understand and use the technical language of mathematical proof writing is largely unknown. Author (Year) showed that the mathematicians and undergraduate students who participated in their study did not agree on the extent to which one should attend to English grammar, the introduction of new objects in a proof, and the context in which a proof was constructed when considering the exposition of said proof. While the interviews provided a clearer picture of how some mathematicians and students perceived the language of mathematical proof writing, the present study investigated how a larger sample of participants evaluated parts of the same proofs via an online survey.

This approach lends a quantitative perspective on how mathematicians and undergraduate mathematics students understand technical mathematical language. This work also further informs researchers' and instructors' understanding of both mathematicians' and students' expectations regarding the presentation of mathematical proofs at the undergraduate level. For the sake of brevity, this paper focuses solely on the results of the mathematicians' survey.

Related Literature and Theoretical Perspective

Prior Work on the Language of Mathematical Proof Writing

There is little systematic, empirical work on the language of mathematical proof writing. Konior (1993) studied over 700 mathematical proofs written in academic textbooks and mathematical monographs investigating the construction of mathematical proofs. He identified a common structure that framed the arguments of a proof by highlighting the plan of procedure and using cues to direct the reader through the proof. Burton and Morgan (2000) found that the norms suggested in professional writing guides (e.g. Gillman, 1987; Krantz, 1998) are sometimes broken by mathematicians, especially by those who were highly regarded in the field. Selden and Selden (2014) also described seven features of the style in which mathematicians write proofs (e.g. not including the statements of entire definitions within written proofs).While these studies begin to further the understanding of mathematical proof writing at the professional level, research on the language of proof writing at the undergraduate level is lacking.

As referenced above, a number of mathematicians (AMS, 1962; Halmos, 1970; Gillman, 1987; Krantz, 1997; Higham, 1998) have written texts describing how to properly and effectively use the language of mathematics for professional purposes such as published journal articles, dissertations, and books. Meanwhile, since the suggestions provided in these guides for mathematicians were written based on the authors' own assumptions and personal experiences, further work is necessary to investigate the extent to which these expectations of advanced mathematical proof writing are shared by the general population of mathematicians and how these conventions may apply to different contexts.

Linguistic Conventions of Proof Writing in Different Contexts

As a particular type of mathematical writing, we see mathematical proof as a particular genre of the language of mathematics. Mathematician Armand Borel (1983) equated mathematical proofs to the genre of poetry in natural language, saying, "our poems are written in a highly specialized language, the mathematical language […], unfortunately, these poems can only be understood in the original" (p .15). In this quote, Borel emphasized not only that the language of mathematics is distinct from the vernacular, but also that one *must* be knowledgeable in the language of mathematics in order to understand mathematical proofs. In this work, we assume that the genre of proof is a way of using mathematical language defined by both the formal properties and structures of language, as well as the communicative purposes of texts in particular contexts. This view of genre is consistent with the genre theory literature (Hyland, 2002). Our consideration of proofs in this light is in the pursuit of helping students to understand and follow the linguistic conventions of the genre, as work has done in other genres and discourses (Hyon, 1996).

In order to study the genre of mathematical proof writing, we sought to identify and validate the existence of linguistic conventions of mathematical proofs. We assume that conventions are rationally justifiable customs of practice to which members of that practice are expected to conform in the manner of Jackman (1998). Thus, we take linguistic conventions to be rationally justifiable customs of linguistic communication. Existing literature (AMS, 1962; Halmos, 1970; Gillman, 1987; Higham, 1998) has suggested possible conventions of writing mathematical proofs for professional contexts, such as correctly situating notation within a sentence according to proper grammar, and structuring the proof to guide a reader through the argument.

Meanwhile, it is important to consider how the context of the proof might affect how these conventions are followed as suggested by the mathematicians in Author's (2016) study. In particular, we investigate how mathematicians believe conventions of mathematical proof writing apply in the contexts of undergraduate textbooks, and in two classroom contexts: the way proofs are written on the board in class, and the ways in which proofs are written undergraduate students. The consideration of this variation of context within the genre of proof writing allows this work to highlight important similarities and differences in the contexts created by mathematical discourse, as Bondi (1999) had in her study of research papers, textbooks, and newspaper articles in the discourse of economics.

Researchers in higher education (Becher, 1987), linguistics (Hyland, 2004), and composition (Bizzell, 1982; Batholomae, 1985) have highlighted that different disciplines have characteristic discourse practices. Berkenkotter, Huckin, and Ackerman (1988) summarized the work of composition scholars Bizzell and Batholomae stating, "students entering academic disciplines must learn the genres and conventions that members of the disciplinary community employ. Without this knowledge, they contend, students remain locked outside of the community's discourse" (p. 10). We extend this necessity to acquire specialized literacy to undergraduate students of advanced mathematics, who—we argue—must understand the genres and conventions of mathematical discourse which includes the genre of mathematical proof in the different contexts that pervade their undergraduate study. Given the fundamental role of proof in mathematical practice (Thurston, 1994; Wu, 1996; Rav, 1999) and in light of Borel's (1983) quote above, understanding the language of mathematics in which proofs are written is of utmost importance for undergraduate students studying advanced mathematics.

In the present study, we investigate the conventions of mathematical proof writing from the perspective of mathematicians – the most prevalent instructors and examiners of undergraduate students' proof writing. As such, the present study investigates the following questions:

- To what extent do mathematicians agree among themselves on what the linguistic conventions of mathematical proof writing are in the three contexts of textbook proofs, blackboard proofs, and student-produced proofs?
  - Do conventions exist for the language of undergraduate mathematical proofs?
  - Does the context of said proofs affect what conventions are upheld in mathematical proof writing?

Methods

In order to evaluate how mathematicians perceive linguistic conventions in mathematical proofs, the survey adopted the methodology of breaching experiments in the style of Herbst (2010) and Herbst and Chazan (2003).The survey asked participants to make evaluations regarding the language used in several partial proofs, which were based on student work, but truncated to discourage participants from focusing on the logical validity of the purported proof being evaluated. Four of the seven partial proofs used in Author's (Year) study were included in the survey, each of the four proofs included in the survey presented three or four types of potential breaches of mathematical language.

These breaches were identified by Author (Year) as common, potentially unconventional uses of mathematical language found in student-produced proofs from 149 exams at the introduction to proof level. The breaches were categorized based on suggestions from the mathematical writing guides discussed above and their personal experiences with proof writing at the undergraduate level. One of the partial proofs and potential breaches included in the survey is illustrated below. Figure 1 shows the marked partial proof exhibiting the use of the unspecified variable, z, and the explanation for why someone might think it's unconventional, as presented in the survey. The explanations used in the survey are based on the mathematicians' discussions of the same potential breaches and proofs in Author (Year).

Each potential breach was presented on a separate page of the survey. On these pages, participants were provided a marked partial proof and an explanation of why a colleague might believe the corresponding proof excerpt had been written in an unconventional manner, as shown in Figure 1. Participants were then asked if they agreed that this proof excerpt was indeed unconventional for the stated reason and to what extent it affected the quality of the proof. These questions were asked for each of the contexts of a textbook proof, a blackboard proof, and a student-produced proof.

Marked Partial Proof Exhibiting the Potential Breach: Uses an Unspecified Variable	**Explanation of the Potential Breach**
Suppose $f : A \to B$, $g : B \to C$, $h : B \to C$, for sets A, B, and C. Prove: If f is onto B and $g \circ f = h \circ f$, then $g = h$. Suppose f is onto B and $g \circ f$. NTS: $g=h$ If f is onto B, $Rng(f)=B$ $\forall y \in B$, $\exists x \in A$ such that $f(x)=y$ let $(x, z) \in g \circ f$ such that $\exists y \in B$ $g(y)=z$	A mathematician suggested that this is unconventional mathematical writing because the variable *z* should be introduced prior to its use in the proof.

Figure 1. Example potential breach and explanation presented in the survey.

Participants were recruited from 25 of the top mathematics departments in the United States through email solicitation through their department secretaries. Mathematicians were sent a link that directed those who chose to participate in the study to the survey website. In total, 128 mathematicians (75 PhD students, 16 Postdoctoral fellows, and 37 faculty members) participated in the survey.

Analysis

The analysis for this study included investigating if the mathematicians answered the various aspects of the survey differently – in particular, whether they agreed or disagreed on the extent to which the potential breaches were unconventional in each of the three contexts (textbook proofs, blackboard proofs, and student-produced proofs).

Descriptive statistics were first considered to provide a holistic view of the data sets before conducting a number of statistical tests. Table 1 presents some of the findings from this study, indicating the proportion of the sample that agreed that the proof excerpt was unconventional for the reason provided, for each of the three contexts. To evaluate if the proportions of agreement indicated a high agreement within the samples, 75% and 25% of the sample were set to be the thresholds of a high agreement that a potential breach was unconventional and was not unconventional within the samples, respectively. Chi-squared tests for equality of proportions were conducted to check for proportions $p=0.75$ and $p=0.25$ with a level of significance of $\alpha=0.05/42$ (fourteen potential breaches in each of three categories). The results of these Chi-squared tests are indicated with ++ and - -, respectively. When considering this binomial data throughout this paper, the proportions of agreement were categorized in the following ways: high agreement that the use is unconventional (significantly different from and greater than 75%), high agreement that the use is not unconventional (significantly different from and less than 25%), or not shown to have high agreement within the sample.

Results

To what extent do mathematicians agree among themselves in these contexts?

Survey results suggest a lack of agreement amongst participants in whether the potential breaches are unconventional of mathematical language for the reasons provided. As can be seen in Table 1, the agreement percentages for fewer than half of all judgments made were significantly different from and above 75% or significantly different from and below 25%. However, both samples' responses also showed more internal agreement in the context of textbook proofs. Figure 2 shows the percentage of mathematicians who agreed that the potential breaches were unconventional in each of the three contexts. Lines connect the agreement percentages for evaluations in the same context and the shaded sections indicate the percentages significantly different and greater than 75% or significantly different and less than 25%. This section of the proposal discusses the types of potential breaches for which participants' responses showed high agreement and provides a post hoc analysis of the potential breaches for which the samples' responses did not show high agreement.

	Potential Breach of Mathematical Language	**Do you agree that this is an unconventional use of mathematical language for the reason provided? *(% Agree)***		
		Textbook Context	**Blackboard Context**	**Student Context**
Partial Proof 1	**Uses non-statement**	100% ++	100% ++	98% ++
	Uses an unspecified variable	59%	34%	37%
	Includes statements of definitions	41%	18%	12% --
	Lacks punctuation and capitalization	95% ++	34%	50%
Partial Proof 2	**Uses formal propositional language**	88% ++	74%	66%
	Uses unclear referent	93% ++	67%	70%
	Overuses variable names	98% ++	95% ++	93% ++
	Mixes mathematical notation and text	88% ++	28%	45%
Partial Proof 3	**Fails to make the proof structure explicit**	70%	29%	28%
	Uses mathematical symbols or notation as an incorrect part of speech	72%	19%	24%
	Uses informal language	77%	45%	48%
Partial Proof 4	**Fails to state assumptions of hypotheses**	64%	34%	40%
	Uses an unspecified variable with an existential quantifier	85% ++	55%	54%
	Lacks verbal connectives	97% ++	52%	72%

++ Significantly different from and greater than 75% of the sample, **--** Significantly different from and less than 25% of the sample (These tests were all evaluated with a level of significance α=0.05/42.)

Table 1. Mathematicians' responses indicating if they agree that the proof excerpt was unconventional for the reason provided in each context.

Types of potential breaches for which the participants' responses showed high agreement. Based on Figure 2, the mathematicians' responses showed that they found eight of the fourteen types of potential breaches to be unconventional in the context of a textbook proof for the reasons presented in the survey with high agreement. Moreover, there is high agreement among mathematicians that the proof excerpts exhibiting the use of non-statements or overuse of variable names are unconventional in each one of the three contexts. These findings provide further evidence that these eight potential breaches of the conventions of mathematical language are indeed unconventional in the context of textbook proofs for the reasons provided. The partial proofs that overused variable names or used non-statements were also seen by the mathematicians as unconventional in the two classroom contexts.

Finally, Figure 2 also shows the percent of mathematicians that agreed the inclusion of statements of definitions in a student-produced proof was unconventional is significantly different from and less than 25%. That is, there is a high agreement among the mathematicians that a proof excerpt including the statement of definitions is not unconventional in the context of a student-produced proof. Moreover, fewer than 42% of mathematicians agreed that the inclusion of statements of definitions was unconventional in any of the three contexts considered here. We note that this is in contrast to claims by Selden and Selden (2014) that mathematicians do not include the statements of entire definitions within written proofs. While the scope of the present study focuses on proofs at the undergraduate level, we note that two of the contexts (textbook proofs and blackboard proofs) are written by mathematicians. Thus, it may not be the case that the features of proof writing described by Selden and Selden (2014) extend to different contexts of proofs written by mathematicians or to the context of student-produced proofs.

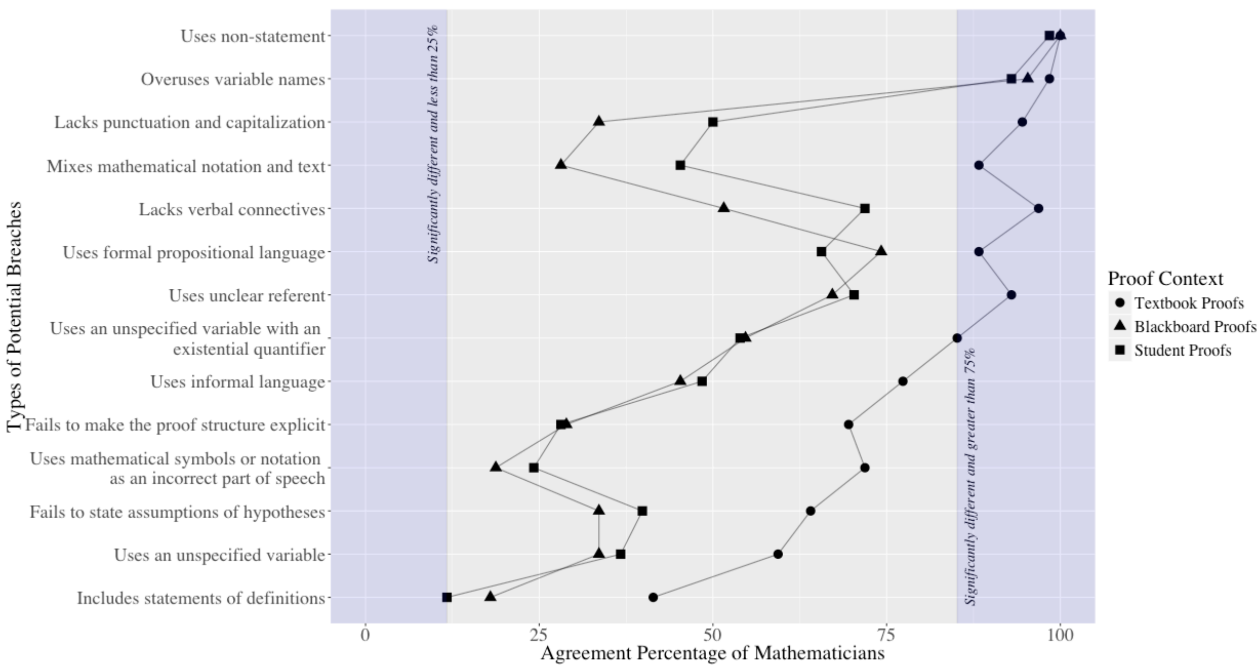

Figure 2. The mathematicians' agreement percentage for each potential breach in each context.

When the samples' responses did not show high agreement. For 29 of the 42 judgments made by mathematicians (fourteen potential breaches in each of three categories), percentages of agreement did not cross the thresholds for high agreement, i.e. percentages were not significantly different from and higher than 75% or significantly different from and lower than 25%. Figure 2 further shows that for five of the potential breaches, there was no high agreement among mathematicians *in any of the three contexts*, and that when we restrict the analysis to only the two classroom contexts (blackboard proofs and student-produced proofs), up to eleven of the fourteen types of potential breaches the results did not show high agreement. Finally, Figure 2 highlights that a number of these agreement percentages are close to 50%. In particular, eight of the 42 judgments had percentage agreements between 40% and 60%, including two judgments in the context of textbook proofs. These findings suggest that beyond failing to give confirmation that many of these potential breaches were indeed breaches of linguistic conventions in mathematical proof writing, that the disagreement among mathematicians may be higher when it comes to the classroom contexts, and that for some specific types of potential breaches the disagreement amongst mathematicians may be particularly extreme, even in the context of textbook proofs.

Moreover, it is clear that a larger percent of the mathematicians agreed that a potential breach was unconventional in the textbook context than when the same potential breach was assessed in either of the other contexts. In fact, Figure 2 suggests that for some of the types of potential breaches, the fewer mathematicians that agreed a proof excerpt was unconventional mathematical proof writing in the context of textbook proofs, the fewer that perceived that the same excerpt was unconventional in the classroom contexts.

Conclusion

The findings of this report highlight the existence of some potential breaches of mathematical language that mathematicians widely agree are unconventional in the context of textbook proofs. Specifically, mathematicians in our study widely agreed that including incomplete statements, overusing variable names for different mathematical objects, lacking proper punctuation and capitalization, carelessly mixing mathematical notation and text, failing to use connectives to bridge steps, using formal propositional language, using pronouns with unclear referents, and using an unspecified variable are all unconventional usage of mathematical language in textbook proofs. Moreover, mathematicians widely agreed on the specific rational justifications for why the proof excerpts breached linguistic conventions or mathematical proof writing on that context. On the other hand, mathematicians also widely agreed that one of the potential breaches studied (including full statements of definitions within proofs) was not unconventional in the context of student-produced proofs for the reasons provided, which suggests that Selden and Selden's (2014) claim that mathematicians do not include definitions in their proofs may not extend to other contexts and to mathematicians' expectation of how students write proofs.

Meanwhile this report also gives insight on how these mathematicians differed in their evaluation of the language of mathematical proof writing in the classroom contexts at the introduction to proof level. In particular, the results suggest that it is unclear what mathematicians expect a student-produced proof to look like. The mathematicians' responses did not indicate high agreement for twelve of the fourteen types of potential breaches in the student context, which may indicate the possibility that there is no shared understanding or expectation among mathematicians of how students should write proofs.

If it is indeed the case that there is not a consensus among mathematicians of how their students in introduction to proof courses should write their proofs, then how are instructors of these courses presenting proof writing to their students? Discussions amongst mathematicians, especially those who teach introduction to proof courses, concerning their expectations for language usage in the writing of proofs by their students would be a useful step towards a shared understanding of linguistic conventions of proof writing in the context of student-produced proofs. Further research is necessary to understand these varied expectations amongst mathematicians and how to address students' confusion when it comes to their professors' expectations of their proof writing. In turn, better understanding of mathematicians' expectations of their students' writing could enable the creation of interventions and curriculum to help undergraduate students in their transition to abstract and advanced mathematics courses.

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An Unexpected Outcome: Students' Focus on Order in the Multiplication Principle

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In an effort to better understand students' understanding of the multiplication principle, which is a fundamental aspect of combinatorial enumeration, we had two undergraduate students engage in reinvention of a statement of the principle during an eight-session teaching experiment. In this presentation, we report on the students' unexpected attention to the order in which they complete stages of counting process in a counting problem. We suggest that an early experience with a particular problem prompted them to think about order, and this way of thinking persisted throughout the experiment. The students' reasoning about order sheds light on ways in which students may think about order and about the nature of multiplication in counting. We conclude with potential implications and directions for further research.

Key Words: Combinatorics, Reinvention, Counting problems, Teaching experiment

Introduction and Motivation

The multiplication principle (MP), also known as "The Fundamental Principle of Counting" (e.g., Richmond & Richmond, 2009), is a foundational component of understanding of counting problems. The MP is the idea that for independent stages in a counting process, the number of options at each stage can be multiplied together to yield the total number of outcomes of the entire process (see Lockwood, Reed, & Caughman (in press) for a more in depth discussion of the MP; the statement by Tucker (2002) in Figure 1 is our preferred statement of the MP). This principle is often taught to students in discrete math classes and provides a basis for many counting formulas they are eventually taught.

In order to better understand student thinking about the MP, we had a pair of undergraduate students reinvent a statement in their own words over the course of eight hour-long interview sessions. While we have previously reported on this overall reinvention (Lockwood & Schaub, 2016), in this paper we focus on an unexpected feature of the MP to which the students attended. Specifically, the students' regularly and repeatedly expressed a desire to include in their statement a clause that the order of stages in a counting process does not matter. This was an unexpected focus that emerged during the teaching experiment. In this paper, we seek to address the following research question: *Why did the students in our study (unexpectedly) focus on order in their reasoning about and reinvention of the MP, and what does this suggest about the role of order in the MP?*

Background Review and Theoretical Perspective

Research about the MP in Combinatorics Education Literature

Previous work has demonstrated the importance the MP, and the lack of a well-developed understanding of the MP appears to be a significant problem and hurdle for students, particularly in terms of their ability to justify or explain formulas (e.g., Lockwood, Swinyard, & Caughman, 2015). We have found anecdotally that students can easily assume that they completely understand the MP in counting because multiplication is a familiar operation for them. As a result, they use the operation frequently but without careful analysis, and they tend not to realize when simple applications of the operation are problematic. Lockwood et al. (2015) had students reinvent basic counting formulas, and the students in that study did not appear to have a solid understanding of the MP. They worked with outcomes empirically but lacked the understanding of how those outcomes related to the underlying counting process involved with the MP. This work suggested the need for additional research that targets students' understanding of the MP as a fundamental counting process. In addition, Lockwood and Caughman (2016) wrote about students who struggled on a particular kind of problem (set partition problems), and the authors attributed the struggles in part to confusion about clauses about ordered stages in the MP.

Motivated by those two studies, Lockwood, et al. (in press) recently conducted a textbook analysis that examined statements of the MP in university combinatorics, discrete mathematics, and finite mathematics textbooks. This revealed a wide variety of statements of the MP (Figures 1 and 2 reveal two very different formulations of the MP). From this textbook analysis, Lockwood, et al. identified three key mathematical aspects to which many statements attended: *independence of options*, *dependence of option sets*, and *distinct composite outcomes*. We note that order is an implicit aspect of the MP – so implicit that it was not one of the features emphasized in the textbook analysis, in part because we assumed that students would already understand that they completed a counting process in some pre-determined order. Order is certainly an important component of the MP, but it varies in how explicitly it is addressed. For example, in Figure 1, Tucker (2002) explicitly describes "*m* successive (ordered) stages," while the order is implicit in the structure of k-tuples in Bona's (2007) statement in Figure 2.

> ***The Multiplication Principle****: Suppose a procedure can be broken down into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, ..., and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \ldots \times r_m$ different composite outcomes.*

Figure 1 – Tucker's (2002) statement of the MP

> ***Generalized Product Principle:*** $|X_1| \times |X_2| \times \ldots \times |X_k|$ *Let $X_1, X_2, \ldots, X_k$ be finite sets. Then the number of k-tuples (x_1, x_2,..., x_k) satisfying $x_i \in X_i$ is .*

Figure 2 – Bona's (2007) statement of the MP

We also draw on Lockwood's (2013) model of combinatorial thinking, which emphasizes the relationships between counting processes, sets of outcomes, and formulas/expressions. We emphasize that a particular counting process can generate a set of outcomes with a certain structure, and different ways of structuring the set of outcomes may reflect different respective counting processes. In the following section we discuss the key issue of order in this paper.

Order in the Multiplication Principle

To appreciate this issue of order in the MP, we consider two problems. First, *"How many ways are there to flip a coin, roll a 6-sided die, and pick a card from a standard 52-card deck?"* In this problem, the outcome is simply a set of results of these three activities. If the problem said "first pick a card, then roll a die, then flip a coin" then the order is determined, but the problem does not specify this. However, when one goes to solve this problem, a natural approach might be to specify a process with a particular sequence of stages with a particular order. For instance, we might solve the problem by specifying we will first of pick a card, then roll a die, then flip a coin, but this is a different counting process than rolling a die, then flipping a coin, then picking a card. In the first case, the outcomes would be {card, number, coin} triples, where the first entry is a card, the second entry is from the set {1, 2, 3, 4, 5, 6} and the third entry is from the set {H, T}. The formula/expression to represent that counting process would be 52*6*2. On the other hand, a different process of first flipping a coin, then rolling a die, then picking a card still counts the same outcomes (and answers the counting problem), but the tuples are now slightly different – {H/T, number, card} triples with the final expression of 2*6*52. The answers are numerically the same, but in each case we specified a differently ordered process in order to solve the problem. Notably, though, either ordering represents a similar set of operations, and ultimately it would not matter much, in terms of effort or efficiency, which order of the stages we used.

As a point of contrast, consider a slightly more complicated problem, *"How many even five digit PIN numbers are there with no repeated digits (leading zeros allowed)?"* Here, again, we can specify a counting process. If we specify our stages as considering the options for each position from left to right (first to last), we can run into a problem at the last position, which needs to be an even number. The number of options we have for that last position depends on how many of the previous numbers were even. It seems like we will need a complex case breakdown to address this, until we realize that we have the flexibility to specify the order of the stages in our process. Let us, as the first stage in the process, consider options for the last position (there are 5 even numbers, 0, 2, 4, 6, 8). Then, once that is specified we can return to the first position (or any other position) and there will be 9 options (everything except the digit used last), then 8, then 7, then 6. The answer is thus 5*9*8*7*6. Now, mathematically we could re-write this as 9*8*7*6*5 as a numerically equivalent expression, but we lose the valuable information of the counting process with the specified order. In this problem, the ordered counting process is helpful. The point is that a person counting has flexibility with regard to what process is being implemented. A judicious ordering of stages facilitates an efficient solution.

In terms of how order plays into statements of the MP, the MP specifies how to use multiplication to count, once a particular course of action (or a particular order of stages in a counting process) has been determined. As a result, the statement need not explicitly include a clause that allows for any order of stages, even though it is true that often multiple orders of stages is allowable (such as on the 2*6*52 problem). Rather, the point of the statement is to speak to when multiplication applies once a counting process is specified. Interestingly, because multiplication, as an operation, is commutative, the issue of order in the MP raises questions about how multiplication as an operation differs from multiplication in the context of counting. We briefly explore the commutativity of multiplication and how it relates to multiplication in combinatorial enumeration in the Discussion and Conclusion section.

Methods

For data collection, we conducted a teaching experiment (Steffe & Thompson, 2000) in which a pair of undergraduate students solved counting problems over eight hour-long sessions. The students were enrolled in vector calculus in a large university in the western United States, and we selected them because they had not been explicitly taught about the MP in their university coursework. The interviews took place outside of class time over a period of four weeks. Broadly, the students solved a series of counting problems, and they were asked to write down and characterize when they were using multiplication as they solved these problems. They wrote down several iterations of statements of the MP, and throughout the study the interviewer selected tasks to highlight various aspects of the MP and regularly asked clarifying questions.

The students engaged in three kinds of activities during the teaching experiment: solving counting problems that involve multiplication, articulating and refining a statement of the MP and evaluating given textbook statements of the MP. Although there was some overlap of activities in each session, Table 1 gives the overall structure of the teaching experiment by outlining the session number (and total number of tasks in each session), a sample task given in that session, and the predominant activity that occurred in each session. When designing tasks for the teaching experiment, we drew on the three key mathematical ideas presented in the textbook analysis (Lockwood, et al., in press). That is, when preparing for and implementing the reinvention, we designed tasks with the aim of addressing those three key mathematical issues.

Session	Sample Tasks for Each Session	Goal of Session
1 – 2 (11 total tasks)	How many ways are there to place two different-colored rooks in a common row or column of an 8x8 chessboard?	Solving counting problems
3 – 5 (9 total tasks)	How many ways are there to flip a coin, roll a die, and select a card from a standard deck?	Articulating a statement of the MP
6 – 7 (9 total tasks)	How many 6-character license plates consisting of letters or numbers have no repeated character?	Refining their statement of the MP
8 (7 tasks)	Please read the following statement [such as Tucker's (2002) in Figure 1]. How is it similar to or different from your own statement?	Evaluating given textbook statements

Table 1 – Overall structure of the teaching experiment

For data analysis, we first videotaped and transcribed the interviews. For this paper we especially examined episodes in which the students talked about order in their statement of the MP. We developed and used a conceptual analysis of order in the MP, and we used that to better understand what the students might have been thinking about and emphasizing.

Results

In this Results section, we attempt to provide evidence for Pat and Caleb's (pseudonyms, both students are male) emphasis on order. In particular, we first present the students' work on a problem in which they first seemed to grapple with order. Then, we present three brief episodes that emphasize their repeated interest in order and highlight certain ways of thinking about order.

Students initially reason about order

During Sessions 1-2, the students were not yet articulating or refining statements of the MP but were solving initial counting problems involving multiplication. In Session 2 we gave them the problem, "*How many ways are there to place two different-colored rooks in a common row or column of an 8x8 chessboard?*" A possible solution to this problem involves thinking of 8 options for the first rook row, 8 options for the first rook column, 2 options for whether the second rook is in the adjoining row or column, and lastly 7 options for the second rook's specific square. Thus we have 8*8*2*7 = 896. The students gave the following answer, which explains a solution of 2*2*8*8*7:

Pat: *You'll have two selections for color… So then you'll choose color, you'll choose if you're doing rows, putting them in the same rows or columns, then you'll have eight selections* *for which row or column, and then you'll have eight selections for first placement, and then you'll have seven remaining options for the second placement.*

Here the students are inadvertently multiplying by an extra 2 for the first choice of rook color. Accounting for color is unnecessary, because placing a white rook in Square 1 and a black rook in Square 2 is the same as placing a black rook in Square 2 and then a white rook in Square 1. If we also multiply by 2 for the rook color, we count every possible placement of rooks twice.

We then discussed this issue for some time with the students. They realized that they were overcounting, but they also seemed attuned to order and the language of "first" and "second." We see Pat's exchange with the interviewer in the following conversation, and the underlined portion especially suggests that he was emphasizing that the order in which the rooks were placed should not be taken into consideration in the solution:

Pat: *You could put white here first or white here second, black here first or black here second, white here first or white here second, as long as it ends up there with the other one ending up somewhere, the first place.*

Int.: *Okay. So you're talking about, kind of, this first and second language.*

Pat: *Yeah, yeah. <u>So it doesn't matter which order they're placed, as long as they end up in the same position.</u>*

They went on to say that there was no "temporal" element to the process, all that matters is the final outcome of two rooks on a chessboard. We feel that as a result of this experience, the students were later reluctant about making any commitments about the specific order in which stages in the counting process must occur. We contend that was an important episode for the students, as this seemed to the impetus for their focus on order in the teaching experiment.

Students incorporate of clauses about order in their statements of the MP

We now offer data examples that exhibit how the students' interest in order arose in the context of their reinvention of a statement of the MP. These examples are meant to demonstrate the nature of the students' thinking about order as a component of the MP. As noted above, because this was an unexpected aspect of their work, by examining it we can better understand how students perceive order, hopefully ultimately leveraging insights about such thinking to help students develop productive ways of thinking about the MP.

The students had provided an initial statement (*Use multiplication in counting problems when…there is a certain statement shown to exist and what follows has to be true as well*), which we perceived as a first attempt that still needed significant refinement. We gave them more problems to investigate multiplication, and in Session 4, we gave the students the problem described above, *How many ways are there to flip a coin, roll a die, and select a card from a standard deck?* The students quickly gave the correct answer of 2*6*52, and we prompted them to try to explain their reasoning and potentially to connect their answer to their statement of the MP. The discussion that ensued suggested the students' view that time and order no longer mattered. We hypothesize that their ways of thinking were affected by the experience of overcounting in the rook problem.

Pat: *So I feel like, if there's no order in which the selections have to be made, I feel like there's multiplication.*

Int.: *Okay.*

Pat: *At least that's part of it. I feel like it's more complicated than that. <u>But I feel like as long as we're just selecting a coin toss, a dice roll, and a card, it's the same number of outcomes no matter how you do that</u>. And so if there's no order in which you have to* *make your selection, I feel like that makes all the movements possible, if you put it in order it will still be the same number of outcomes but you can do it however you want, and you'll have the same number. So regardless of how you say where to start, you'll get the same number of possible outcomes. <u>So it doesn't actually matter what order you do the selection that's, that would be mult... I don't know if that leads to multiplication or if it's just multiplication reflects that fact</u>.*

Caleb: Yeah. I think it might be more of the second one.

Pat: Yeah.

At the beginning of the statement (the first underlined portion), Pat articulates the idea that he will get the same number of outcomes no matter the order in which he does the three tasks. This is true, and for this problem in particular the order in which the three tasks are completed does not affect the total number of problems. However, note that in order actually to complete the multiplication and determine the solution, one must specify (and commit to) a particular order. That is, we could yield 2*6*52 or 6*52*2, or any other arrangement of the tasks, but in actually multiplying a specific sequence must be determined. In this particular problem, each ordering of the tasks yields essentially the same amount of work, and there is no clear benefit to doing tasks in any particular order. However, as we demonstrated with the PIN number problem above, sometimes it behooves a student to specify a particular order in which to complete stages of the counting process. So, Pat's observation about order on this problem is correct, but order does not always behave in this same way on other counting problems.

At the end of his statement (the second underlined portion), Pat reflects on whether the idea of order not mattering implies multiplication or if it is simply a reflection of multiplication. This is a noteworthy observation that highlights questions about the nature of multiplication and also the nature of a statement of the MP.

Questions about order came out again in Session 4 when the students try to articulate multiple pathways (options) into their statement of the MP. Here we see that they did not want to incorporate into their statement how they order the stages in the counting process.

Caleb: But it has, it's more than just that because that really only talks about one outcome and one pathway. It has to do with all these, like how many times you multiply too, and when the whole, <u>but I'm trying not to incorporate like how you order it</u>.

Int: Ok. Great. Hang on, so you're, real quick, so you're saying, so you're trying not to incorporate how you order it, and what do you mean by that?

Caleb: <u>So I'm not saying like one comes first and then you multiply another and then another. I'm trying to just say there's a total number of pathways and a total number of options</u>...

As a final example of their reasoning about order in the MP, we share an episode when we gave them the relatively straightforward problem of *How many ways are there to put four distinct people in a line?* The students were easily able to identify a counting process and solve the problem correctly (they said that they had four option for which person goes in the first spot, then three options for who goes in the second spot, and so on), but they were reluctant to acknowledge a specific order in which they solved the problem. We asked them to list out all 24 outcomes in a way that reflected their counting process, and they wrote an alphabetical list of the letters A, B, C, and D. Then, we asked them to list differently, in which the letter A was in the third position. To us, the two lists would reflect two different orders in which to complete the counting process, and we hoped it would help them realize that they could specify a certain order. As the exchange below shows, the alternative way of listing did not register for them, and in fact they saw no difference in the two orders that would produce two different lists.

Int.: *So then presumably you could also list out where C and D are in the third position right?*
Pat: *Yeah, you could.*
Int.: *I mean, you could get all 24. Does that seem like that matters at all to you, or is it just like –*
Caleb: *No, it means absolutely nothing.*

From these three examples, we suggest that the first problem involving rooks had a deep impact on the students' thinking about order in the MP. They were hesitant to enforce any sort of ordering on the problems, and they wanted to include a clause about order in their reinvented statement of the MP. Although in the end the students did create a cohesive and refined statement of the MP, this foray into order is to show an unexpected outcome of working on a reinvention of the MP, and it serves as a reminder that students often have robust and consistent ways of thinking that may differ from our (as researchers) initial expectations.

Discussion, Conclusions, and Implications

Perhaps because order is often treated implicitly in many textbook statements of the MP, it was not a feature that we expected students to discuss explicitly (as opposed to independence and overcounting, which we suspected would be key mathematical issues for students). However, order was one of the main issues to which Pat and Caleb students repeatedly returned, and this allows for a couple of important points of discussion.

First, our findings indicate that the notion that multiplication in counting involves ordered stages is not always natural for students to consider. And yet, our results suggest that it is important for students to think carefully about order and the role it plays in the MP. More specifically, the fact that counting processes specify an order emphasizes that we as counters can have flexibility, and we can choose to complete stages in a counting process in a way that will be effective or efficient (as the PIN number problem suggests). This is an issue that we feel should be emphasized among students as they learn to count

The results also highlight a potential distinction between multiplication as an operation (which is commutative over the nonnegative integers) and multiplication in a combinatorial context. The fact that different orders of stages might result in a better solution to a counting problem suggests that commutativity of multiplication may not be a key feature of multiplication we want to emphasize in combinatorial contexts.

We acknowledge that since this is just one set of students, we cannot say for certain how other students will treat the same issue. And, we believe that this is likely a remnant of their work on one particular problem (rooks on a chessboard), and that the experience of solving that problem had a lasting impression on them. However, the fact that order was such an important feature of Pat and Caleb's work suggests to us that it represents a strongly held mathematical idea, and we conjecture that other students might also think similarly about order in the MP.

In further research we would like to do more teaching experiments that acknowledge student thinking about order in a more purposeful way. Due to the unexpected nature of this issue, we would like to create more problems and questions to help students become attuned to order in future teaching sessions. We strongly believe that having a well-defined understanding of order within the MP is a key mathematical aspect of understanding the when and why one can use multiplication when solving counting problems.

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Computational Thinking in and for Undergraduate Mathematics: Perspectives of a Mathematician

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We report on a mathematician's perceptions and awarenesses related to incorporating problem-based activities requiring computational thinking into an upper level undergraduate mathematics course. Computational thinking is understood as the thinking, strategies, and approaches for problem solving that parallel the design of computational algorithms which can be followed and executed by a computer. Data from this case study is qualitative in nature, and seeks to present an in-depth account of one professor's experiences developing and teaching computational thinking in and for mathematics. Analyses highlight the similarities and differences amongst the values and opportunities perceived for computational thinking versus other more ubiquitous mathematical approaches, as well as the perceived tensions and challenges in trying to foster such values and opportunities.

Key words: Computational Thinking; Undergraduate Mathematics; Awarenesses; Disciplinary Goals; Mathematics Professor

This study reports on the case of a mathematician's aims and objectives when incorporating a focus on computational thinking in an upper level undergraduate course focused on problem solving, simulations, and mathematical investigation. Computational thinking can be loosely defined as the thinking involved in, and related to, computer programming. This can include screen-based programming, paper-based or embodied pseudo-coding, as well as other approaches to problem solving that parallel the design of computational algorithms which could be followed by a computer (Aho, 2012; Wing 2006). While the idea of using computer programming in mathematics is not new (e.g., Feurzeig, et al., 1969; Howson & Kahane, 1986; King et al., 2001; Marshall et al., 2014a; Papert, 1980), research has been primarily focused on elementary or secondary school learning (e.g., Floyd et al., 2015; Gadanidis, 2014, 2015; Sneider et al., 2014) with recent attention turning towards ways computer programming can be used in undergraduate mathematics (Muller, et al., 2009; Marshall & Buteau, 2014). However, purposes and best practices for computational thinking in undergraduate mathematics learning are far from well understood.

This research is part of a broader research program that aims to address the what, why, and how of computational thinking in and for undergraduate mathematics learning. In this paper, we focus specifically on the experiences and perceptions of a mathematician in his design and implementation of a mathematics course that sought to incorporate computational thinking as a key practice. In particular, we address the following research questions:

1. What does a mathematician teaching an undergraduate level problem solving course describe as the values and opportunities for student learning when incorporating computational thinking in and for mathematics?
2. What challenges and tensions were experienced from a teaching perspective?

We use a case study approach and qualitative analysis, which are suitable for collecting and interpreting in-depth stories of teaching and learning (e.g., Stake, 2000; Yin, 1994). We use Mason's (1998) framework of levels of awareness to analyze both the disciplinary and pedagogical values associated with computational thinking in and for mathematics, as well as our participant's perceptions of opportunities, challenges, and tensions.

Background

Research has highlighted an extensive set of skills and competencies that may be developed by incorporating computational thinking in mathematics learning. These include, but are not limited to: self-motivation to do and explore mathematics; experimentation; development of mathematical intuitions and approaches; critical reflection; working with abstraction and different representations (Howson & Kahane, 1986; King et al., 2001; Marshall & Buteau, 2014; Marshall et al., 2014). Addressing mathematics problems through the use of computer programming has also been described as transformative to learning (Papert, 1980), fostering learner-driven active engagement, as well as resilience and creativity in the face of new and challenging problems.

Parallels between learning computer programming in general, and developing computational thinking for mathematics in particular, have been noted. Wing (2008) observes that: "In computing, we abstract notions beyond the physical dimensions of time and space. Our abstractions are extremely general because they are symbolic, where numeric abstractions are just a special case" (p.3717). In considering higher levels of mathematics, abstractions become more complex than simply numeric abstractions and research has suggested that the more condensed an abstract object is, the more challenging it is to reason with (e.g., Hazzan, 1999). For example, operating on a set of numbers is seen as a more abstract, and more complex, endeavor than operating on a single number. As such, computer programming may offer a sort of scaffold in fostering abstract reasoning by increasing students' experiences with abstraction while providing tangible and immediate feedback on how such abstractions may be operated upon.

Weintrop et al. (2016) developed their taxonomy for computational thinking in mathematics and science by analyzing characteristic practices that were seen as the most important in meeting both the needs of students and the disciplinary practices of mathematics and science professionals. They cite their work as contributing a set of actionable guidelines for bringing computational thinking into mathematics and science classrooms quickly and effectively that can "serve as resource to address "what" and "how" questions that accompany the creation of new educational materials" (p.129). Their taxonomy is depicted in Figure 1, with some elaboration of practices below.

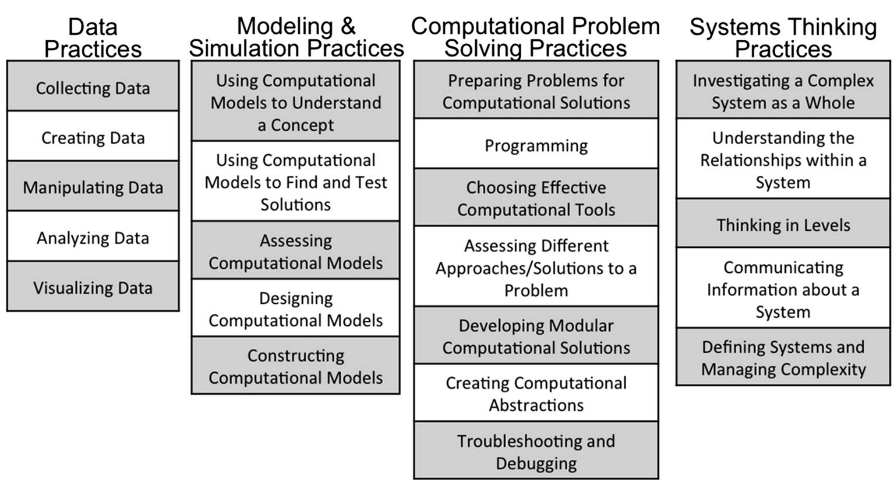

Figure 1: Computational thinking in mathematics and science taxonomy (Weintrop et al., 2016, p. 135)

As the professor participating in our study focused his course on problem solving and simulations, we take a moment to explicate some of the facets in the two corresponding dimensions of the taxonomy. *Modeling and Simulation Practices* focus on skills in using, analyzing, designing, and constructing computational models. Models may be used "to test many different solutions quickly, easily, and inexpensively" (p.137), and understanding how the model relates to the phenomenon being represented can help students articulate the similarities and differences between the two. Noticing similarities and differences is part and parcel to rich mathematical thinking (Mason, et al., 1982), and in the context of computational thinking can be fostered by questions such as "what assumptions have the creators of the model made about the world and how do those assumptions affect its behavior?" and "what layers of abstraction have been built into the model itself and how do these abstractions shape the fidelity of the model?" (Weintrop, et al, 2016, p.137). *Computational Problem Solving Practices* include interpreting and preparing problems for modeling, choosing between and assessing different tools and approaches, developing solutions, and debugging or revising. Students must be aware of which problems can be effectively addressed through computational solutions as well as how to reframe a problem so that "existing computational tools – be they physical devices or software packages – can be utilized" (p.138). Choosing amongst possible strategies, tools, or solutions, and developing modular approaches can foster critical mathematical thinking and logical deduction, while equipping students with a set of approaches that can be applied to larger, more complex problems. Of interest in this research study is whether and which practices emphasized in the taxonomy are recognized, valued, and incorporated by the professor teaching a problem-based mathematics course, and which practices may be seen as more challenging to incorporate.

Theoretical Framework

Mason's (1998) article discusses the necessary levels of awareness which distinguish between a novice, an expert, and a teacher in the discipline of mathematics. These levels of awareness are linked to the structure of attention, which "encompasses the locus, focus, and form of attention moment by moment" (p.250). Through shifts of attention, awareness is broadened and an individual may move from novice, to expert, to teacher through his or her attention and sensitivity toward different aspects of the discipline. Mason (1998, p.256) identifies and develops three forms of awareness:

- *Awareness-in-action*: this is the awareness which focuses attention on what to do in the moment. It is "highly personal and context specific" (p.256) and enables us to act, to know what to do in response to some stimulus. Awareness-in-action includes the "power to select, distinguish, demarcate, discern, detect differences; … to see (construct) something as an example of something else… to abstract… to connect… to express… [and] to decide" (p.257).
- *Awareness-in-discipline*: this awareness is awakened when one becomes aware of his or her awareness-in-action. In other words, it is an awareness "which enables articulation and formalisation of awarenesses-in-action" (p.256). Awareness-in-discipline is broadened by the ability to formalize algebra and geometry, to appreciate how and why we act in the moment, to enact the "habits of thought, forms of fruitful questions, and methods of resolution of those questions" (p.259) applicable to the discipline of mathematics.
- *Awareness-in-counsel*: this is the awareness required of teachers. It is an awareness of awareness-in-discipline; the "self-awareness required in order to be sensitive to

> what others require in order to build their own awareness-in-action and –in-discipline" (p.256). Awareness-in-counsel "provides access to sensitivities which enable us to be distanced from the act of directing the actions of others, in order to provoke them into becoming aware of their own awarenesses" (p.261).

In Mason's perspective (and we agree), teaching is not about telling students what to do, but rather it involves eliciting and fostering their ability to decide for themselves what to do. In the context of undergraduate mathematics education, an instructor's awareness-in-counsel is evidenced in his or her choices (and the expressed reasons behind these choices) of what content, practices, habits of minds to emphasize, as well as in the features of the problems or tasks chosen to elicit and support the shifts of attention that will help novice become expert.

As an analytic lens, we use the construct of awareness to interpret and analyse a mathematician's decisions and reflections regarding the values, opportunities, challenges, and tensions of teaching a problem-solving mathematics course with a focus on computational thinking. We focus in particular on our participant's awarenesses related to the taxonomy of computational thinking and practices developed by Weintrop et al. (2016) as they are associated with, or different from, mathematical thinking and practices more generally. In particular, we examine what our participant identified as valuable ways of knowing and practicing mathematics with computational applications, what opportunities were envisioned or provided, what challenges emerged, and what tensions were felt in terms of realizing those values and opportunities in the face of the emergent teaching and learning challenges.

Methodology

Our participant was an experienced and well-received mathematics instructor, whom we shall refer to as Dr. Y. Dr. Y has taught at the undergraduate level for over 25 years, and this was his third time teaching a course which emphasized computational thinking for mathematics problem solving. Since this research focuses on an in-depth qualitative analysis of an individual's experiences and decisions regarding teaching and learning, a case study approach (e.g., Stake, 2000; Yin, 1994) is appropriate. Data include field notes taken during the course by one of the researchers, personal reflections and instructional materials provided by Dr. Y, as well as unstructured interviews conducted at the completion of the course.

The course taught by Dr. Y was an undergraduate third year problem solving course for mathematics majors. The main objectives of the course included enriching students' existing content-based knowledge and enhancing their problem solving strategies and thinking. Classes were conducted in 50-minute blocks, and students spent the majority of class time working on problems, which they could choose to do individually or in small groups. The problems were chosen to elicit or provide different perspectives on material that was covered in other courses, as well as to stimulate reflection on various new and familiar ways of engaging in solving problems. Students were expected to discuss and justify their solutions with their peers and their instructor, and were required to present solutions to the class at various times throughout the semester. Weekly assignments were assessed, as well as two in-class tests and a final exam. As part of his description of the course, Dr. Y emphasized (and this was confirmed via field notes) that very little of class time was spent on lecturing. Dr. Y reported that lecturing was "done only to introduce a problem or a set of related problems, and provide context and motivation. Students realize that the problems are not in any way 'randomly' selected, but instead are tied to their previous knowledge and experiences."

Students in the course were described as "mathematically mature" – each had taken several second year courses, with experience working with theorems and proofs, and had "developed a critical attitude toward the level of rigour required of a proof". However, students' computer programming background varied widely: some had no programming experience beyond using one-line commands in Maple of Matlab, while others had taken computer science courses and / or had written programs in several languages. Dr. Y indicated that as of Fall 2016 this would change, as students will be required to complete a new programming requirement by the end of their second year. In this problem solving course, student learning objectives included "practicing math the way math researchers do" and "learning to write programs in Python and use them to investigate problems in math". At the time of this research, Dr. Y indicated that approximately 40% of class time and assessment was devoted to computational thinking, with an emphasis on the "basic building blocks" of coding such as conditional statements and loops. Detailed course objectives were co-constructed throughout the course as students discovered and developed an extensive list of strategies for problem solving.

Results and Analyses

We organize this section around our two research questions, addressing respectively:

1. Values and opportunities: we examine Dr. Y's perceptions via his choices in pedagogical structures, such as features and affordances of exemplar tasks, as well as his expressed motivations and intentions for course activities and objectives;
2. Challenges and tensions: we rely on field notes, reflections, and interview data to analyse Dr. Y's experiences enacting his teaching agenda within institutional constraints and in response to student feedback.

Values and opportunities

As previously mentioned, Dr. Y spent minimal class time lecturing students and the value he placed on providing opportunities for students to "discover" and "develop" were made explicit in course objectives. Exemplifying his *awareness-in-counsel*, he sought to foster students' powers in selecting and distinguishing problem-solving strategies (*awareness-in-action*) and to compel them to formalize this awareness via their articulation of (e.g.) which kinds of problems could be reformulated in such a way as to be solvable with programming (*awareness-in-discipline*). To this end, students encountered a variety of problems and tasks which required reformulation, and in the cases where the problems could be fruitfully solved through programming, students were required to use, assess, and design computational models for solving – features of the *modeling & simulation practices* described by Weintrop et al. (2016). One of the early examples of a programming activity to which students were exposed introduced the concept of a two-dimensional random walk and required students to construct a program which simulated N steps of the walk (start at the origin, and with equal chance move up, down, left, or right).

Dr. Y described some of his reasons for using this problem:

> "The problem itself is not so difficult. Most students have little or no problems writing and running the code. However, testing the code is a challenge, because there is a new element in it – randomness. Students are familiar with testing a 'deterministic' program, where they could predict the answer, run the program and see whether or not their answer agrees with the computer's answer. But how does one test a program whose output is based on randomness? This is more difficult. Visualization helps, trial and error, tinkering with the code… Eventually students have to figure out the distance between the initial and terminal points of a random walk with N steps, and then compute the average distance as the random walk is repeated many times. Now they can compare the average distance they obtained to a theoretical result, thus having a way of checking that their code is correct."

In this excerpt, we identify several points of interest. First we note Dr. Y's use of a "not so difficult" problem that increased in complexity through the practice of testing and verifying the code. Following Papert's (1980) lead, such a "low-floor / high-ceiling" approach requires minimal prerequisite knowledge yet offers opportunities to investigate more complex ideas, practices, and relationships. Referring to Weintrop et al.'s (2016) taxonomy, we note opportunities for students to develop *computational problem solving practices*, such as preparing problems for computational solutions ("writing and running the code"), assessing different approaches / solutions ("testing the code"), and troubleshooting and debugging ("tinkering with the code"). We interpret the selection of this problem as an instantiation of Dr. Y's *awareness-in-counsel*, namely, the sensitivities required to direct novice students' attention toward complex computational practices required for disciplinary expertise. A second point of interest is Dr. Y's acknowledgement of the complexities associated with random systems, and the role of visualization in helping to attain a theoretical result. Randomness can be challenging for mathematics learners to address, and novice strategies for dealing with randomness have included various ways of reducing its level of abstraction (e.g., Chernoff & Mamolo, 2015). By contextualizing questions related to randomness within computational thinking, students may be provided with scaffolding for developing abstract reasoning – the tangible feedback given by the program they construct, test and adapt offers insight that may not be readily perceived in other contexts. Similarly, the insights gained through the practice of visualization have been linked to conceptual development in higher mathematics (e.g., Tall, 2007), as well as achievement and understanding (e.g., Koehdinger, 1992). A final point we note is the connection, within this task, of mathematical disciplinary practices and content and computational practices. The computational practices support mathematical ones and facilitate the development of mathematical content, while the act of verifying that students' code is correct highlights an important difference between the two disciplines and their practices – specifically, as Dr. Y put it, that "providing evidence that a computer code is correct is different than a formal math proof!"

Challenges and Tensions

With respect to institutional constraints, some challenges were already mentioned – the diversity in students' prior programming experience, the scheduling of three short classes per week, as well as the size of the course. The course runs with 30-40 students in a regular classroom (rather than a computer lab), and students are required to bring their own portable devices in order to engage with class activities – while most students brought notebook computers, some tried to work on their phones and others had no devices. A few of the observed challenges included syntactical differences in Python depending on what operating system was in use – for instance, the tab function will work for indenting in iOS but yields an error in Windows. While not a major concern, such discrepancies needed to be negotiated during students' peer-peer interactions and work. A more significant issue that emerged occurred when students either did not own, or did not bring, their own portable computer. This led to group work that positioned some students at the periphery, while others were central to the activity. Dr. Y expressed frustration at this imbalance, and noted that "students who watched someone else type and work on screen seemed to learn less, and were less confident in their coding." This was a particular challenge during problem-solving activities wherein students would go straight to on-screen programming solutions, without preliminary discussion or paper-work, making it difficult for other students (and the instructor) to follow their thinking. Dr. Y also noted that the 50-minute classes were not ideal for his pedagogical approach. The time limit constrained the types of problems selected for the course, stemmed peer-peer discussions, and inconvenienced students – Dr. Y remarked: "just as students get in the ‘zone’ [problem-solving zone] tackling some rich mathematics, the class ends and they’re off to biology or physics or something completely different.” Watson (2008) has touted the importance of extended experiences with mathematics, noting that it is more difficult to learn the subject when attention is constantly shifting from one task to another. We see this as particularly relevant for computational thinking in mathematics problem solving, as the added need to reformulate a question in programming terms and to coordinate theoretical and computational solutions may require additional time, especially for inexperienced learners. In reflecting on these issues, Dr. Y’s attention seemed to shift from acknowledging the constraints to posing questions around how to better navigate them, suggesting a broadening of his *awareness-in-counsel*. He raised questions around how to construct appropriate activities that include programming and which could be meaningfully tackled given the aforementioned constraints of time, class-size, and student access to computing devices.

A notable tension that emerged for Dr. Y concerned the use of group work and how this impacted ways in which he was able to fairly assess students. He noted that previous offerings of the course had relied primarily on take-home assignments that could be completed in groups, but were to be submitted individually. Dr. Y noted “in a problem-solving course, it makes sense to assess problem-solving and it’s difficult to do that in a midterm or exam.” However, concerns emerged: “most students were handing in perfect, or near-perfect, solutions, often very similar to their peers’ solutions, and yet different from the kinds of work they were producing and presenting in class. It was obvious that they did not take the course requirements seriously.” In addition, Dr. Y identified challenges in assessing programming-based problem solving as “when students work on-screen, usually overwriting the code, it’s hard to follow their thinking.” As a result, Dr. Y introduced a significant in-class assessment component comprised of tests and exam, and accounting for 80% of the course grade. Dr. Y lamented, “I’m not sure what else to do.”

Concluding Remarks

Despite the tensions and challenges which emerged, Dr. Y’s perceptions of the role of computational thinking in and for undergraduate mathematics were positive. Dr. Y reflected:

> “Very often, when we teach math, we ‘forget’ how results, theorems, or definitions have been arrived at. We present a proof, but rarely talk about how it was constructed, and in particular, we don’t talk about failed attempts! By attempting to solve a problem, students experience some of that essential process of creating mathematics. By writing a computer program, they can engage in rich mathematics, develop important habits of mind, and produce something tangible in the end”

Dr. Y plans to increase the emphasis on computational thinking in future instances of the course, and we plan to extend this research following him through changes inspired by the shifts of attention and broadening of awarenesses that occurred during the course of this study. With a growing emphasis on computational applications of mathematics, both in industry and education, there is a need to better understand the role that mathematics courses can play in fostering computational thinking. Attention toward how the broad array of computational practices for mathematics may be developed is needed, particularly for those practices which closely relate to the mathematical habits of mind necessary for disciplinary expertise. Yadav et al. (2014) noted that fewer than 15% of the pre-service teachers surveyed recognized critical thinking as part of computational thinking, and fewer than 10% viewed computational thinking as helpful in understanding the “why” behind problems, suggesting work is needed to raise awareness of what is computational thinking (in and of itself, as well as in connection to mathematics) and what are fruitful learning experiences for students.

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"Explanatory" Talk in Mathematics Research Papers

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In this paper we explore the ways in which mathematicians talk about explanation in their research papers. We analyze the use of the words explain/explanation (and various related words) in a large corpus of text containing research papers in both mathematics and physical sciences. We found that mathematicians do not frequently use this family of words and that their use is considerably more prevalent in physics papers than in mathematics papers. In particular, we found that physicists talk about explaining why disproportionately more often than mathematicians. We discuss some possible accounts for these differences.

Key words: corpus linguistics, mathematical language, mathematical explanation.

The notion of explanation in mathematics has received a lot of attention in both mathematics education and the philosophy of mathematics. In mathematics education, scholars have been particularly interested in proofs that explain mathematical theorems (i.e. proofs that provide an insight into why a mathematical claim is true) and their role in the mathematics classroom (e.g. Hanna, 1990). Philosophers of mathematics have discussed at length possible equivalents for mathematics of existing philosophical theories of scientific explanation (e.g. Steiner, 1978). Some of these discussions bring to bear the extent to which explanation is relevant to the actual practice of mathematicians and often cite individual mathematicians' views on mathematical explanation (more often than not that mathematician seems to be Henri Poincaré, Paul Halmos, or William Thurston). In this report we explore the extent to which mathematicians talk about explanation in their research papers, and the ways in which they do so.

Literature review

In an influential paper in mathematics education, de Villiers (1990) argued that proof serves several different roles in mathematics, that proof is not only used in mathematics as a way to verify results, to provide conviction of the truth of those results (see also Bell, 1976). One of those other functions of proof was to *explain* mathematical results, to provide an insight or understanding into why these results were true, as opposed to just evidence in support of that result. Hanna (1990) made a similar distinction in the context of the teaching and learning of mathematics, discussing the idea that certain proofs fulfilled this explanatory function better than others, to the point that among the set of all proofs one could identify proofs that explain why a theorem is true, while others simply demonstrate that a theorem is true. Mathematics educators have generally suggested that in the mathematics classroom, mathematical explanation should be an important, if not the primary role of proof (de Villiers, 1990; Hanna, 1990; Hersh, 1993).

This distinction between proofs that explain and proofs that demonstrate has a longer history in the philosophy of mathematics. Steiner (1978) put forward a model of mathematical explanation, arguing that a mathematical proof could be better defined in terms of what he called *a characterizing property* of a concept in the theorem, as opposed to other alternative defining characteristics such as the abstractness or the generality of the proof. Steiner's top-down approach to modeling mathematical explanation by providing a general definition of explanatory proof (and thus creating an absolute distinction between explanatory and non-explanatory proofs) has been criticized by other philosophers of mathematics. In particular, Hafner and Mancosu (2005) argued that ascribing explanatoriness to specific proofs should be done based on practicing mathematicians' evaluations, not philosophers' own intuitions (such as Steiner's). The extent to which practicing mathematicians not only agree with philosophers' characterization of mathematical explanation, but simply talk about explanation in their practice plays an important role in the general argument for the *existence* of explanation in mathematics (which not all philosophers believe). As such, it is not uncommon for a discussion of mathematical explanation to mention how much mathematicians talk about it. For example, Steiner claimed that "mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain" (p.135), and Hafner and Mancosu (2005) supported their claim that mathematicians seek and value explanation in mathematics by presenting several examples of what they called *"explanatory" talk* in mathematical practice: passages of research mathematics papers in which the authors explicitly discuss the role of explanation in their own work. However, we do not currently have empirical evidence, other than these small selections of introspective accounts, about the extent to which talk about mathematical explanation is part of mathematical discourse. We believe one of the reasons this has not been studied at a larger scale may be methodological: a researcher would have to be able to process and analyze a large number of mathematical research papers or conversations among mathematicians.

One method of studying mathematical discourse at such a scale is to use the techniques of corpus linguistics, a branch of linguistics that statistically investigates large collections of naturally occurring text, known as corpora. Methods developed by corpus linguists can be used to investigate many different types of linguistic questions. Here, we report a study that employs some of these techniques to address the following questions: to what extent do mathematicians discuss explanation in their research papers, how does it compare to the extent to which they discuss other important related notions (such as *showing* or *proving* given mathematical results), and how does it compare to discussions about explanation in other types of scientific discourse?

Theoretical perspective

Discussions about mathematical explanation tend to differentiate between explanations of other mathematics (i.e. mathematics X explains mathematics Y, or X is an explanatory proof of theorem Y), and explanations of physical phenomena (i.e. mathematics X explains physical phenomenon Y). Colyvan (2011) refers to these two types of explanation as *intra-mathematical* and *extra-mathematical*, respectively. Here we focus on intra-mathematical explanations.

Hafner and Mancosu (2005) further differentiated between two uses of intra-mathematical explanations: those that are "instructions" on how to master the tools of the trade (as in explaining how to employ a certain mathematical technique), and those that "call for an account of the mathematical facts themselves, the reason why" (p. 217). While Hafner and Mancosu considered the latter to be a "deeper" use of mathematical explanation, which is also the focus of the larger philosophical discussion around explanatory proofs, others have emphasized the importance of the former type of explanation in mathematical practice. For instance, Rav (1999) insisted that one of the main reasons mathematicians read proofs is because of all the mathematical know-how embedded in them, emphasizing the mathematical methodologies and problem solving strategies/techniques contained in proofs. According to Rav, "proofs are for the mathematician what experimental procedures are for the experimental scientist: in studying them one learns of new ideas, new concepts, new strategies—devices which can be assimilated for one's own research and be further developed." (p. 20) Indeed, there is empirical evidence (from both small scale interview studies and large scale surveys) that mathematicians maintain that one of the main reasons they read proofs is to gain insights into how they can solve problems that they are working on (Weber & Mejía-Ramos, 2011, Mejía-Ramos & Weber, 2014).

An interesting question related to the specific ways in which mathematicians talk about explanation in their papers, relates to these two types of *"explanatory" talk*: to what extent do mathematicians discuss explanations of why a certain mathematical statement is true, compared to their talk about explanations of how to do something in mathematics?

Methods

One of the main ways in which mathematicians around the world communicate about mathematics is through research papers stored in the ArXiv. The ArXiv is an online repository of electronic preprints of scientific papers in the fields of mathematics, physics, astronomy, computer science, quantitative biology, quantitative finance, and statistics. These papers constitute a large corpus of scientific text that can be used to analyze mathematical discourse.

We downloaded the bulk source files (mostly TeX/LaTeX) and converted the source code to plain text, which we could then analyze using standard software packages for corpus analysis. We then sorted these articles based on their primary and secondary subject classification (Alcock et al., 2017, discussed the details about the processing of these source files). All analyses reported here are based on a proper subset of this corpus, containing all mathematics and physics articles (based on their primary subject classification) uploaded in the first four months of 2009. This left us with 6988 mathematics papers (30,892,695 words) and 14861 physics papers (58,859,660 words).

Results

Frequency of explicit *"explanatory" talk* in mathematics papers

Table 1 shows the frequencies of all words linguistically related to the word *explain* (henceforth *explain-words*) in our corpus of 6988 mathematics papers. Explain-words showed up 4871 times in this set of papers, or approximately once every 1.4 papers. While this certainly provides an existence proof of explicit *"explanatory" talk* in this corpus, it is not very surprising (it would very rare if no word based on the word explain showed up in these many mathematics papers). In order to get a sense of the extent to which these frequencies were high or low in this type of mathematical discourse, we compared them against the frequencies of words related to other important mathematical activities.

Tables 2 presents the frequencies of words linguistically related to the notions of showing, solving, and proving, which were chosen based on their relevance in mathematical explanation. Measured against these other frequencies, mathematicians used explain-words rather infrequently. Indeed, mathematicians used explain-words in their papers approximately 11 times less frequently than show-words or solve-words and nearly 23 times less often than prove-words.

One possibility is that explain-words are simply not used much in this kind of scientific discourse in general. Thus, even though the importance of scientific explanation is so obvious that it does not need to be justified by looking at *"explanatory" talk* in science, it could be the case that this type of talk is not that common in scientific research papers either. In order to test this hypothesis we studied the use of explain-words in the 14861 physics papers in our corpus (Table 3). Explain-words showed up 21305 times in this set of papers, approximately once every 0.7 papers, or twice as often as they showed up in the mathematics papers. Thus, based on the comparison of the use of explain-words in mathematics and physics papers, it seems that mathematicians discussed explanations more infrequently than physicists.

Explain-words	Frequency
explain	1827
explained	1690
explanation	498
explains	484
explaining	175
explanations	119
explanatory	51
unexplained	22
unexplainable	4
explainable	1
Total	4871

Table 1. Frequency of words related to *explanation* appearing in the mathematics papers

Show-words	Frequency
show	31691
shows	12890
shown	10235
showed	2414
showing	2129
Total	59359

Solve-words	Frequency
solution	25845
solutions	15956
solve	2204
solving	1717
solvable	1618
solved	1342
solves	1071
solvability	429
solver	145
unsolved	95
solvers	56
nonsolvable	39
unsolvable	32
cosolvable	29
equisolvable	18
unsolvability	12
Total	50608

Prove-words	Frequency
proof	56452
prove	29481
proved	12842
proves	4160
proofs	3892
proving	2661
proven	1902
provable	159
reprove	58
disprove	43
provability	29
reproved	29
disproved	17
unprovable	13
unproven	12
reproving	11
disproving	10
reproves	10
prover	7
unproved	7
subproof	5
disproof	4
Total	111804

Table 2. Frequencies of words related to *showing, solving*, and *proving* appearing in the mathematics papers

Explain-words	Frequency
explain	7768
explained	6513
explanation	3564
explains	1601
explaining	914
explanations	675
unexplained	177
explanatory	62
explainable	23
unexplainable	8
Total	21305

Table 3. Frequency of words related to *explanation* appearing in the physics papers

Finally, the search for explain-words may be thought of as requiring an extremely explicit discussion of explanation, one that would leave unnoticed a significant amount of the *"explanatory" talk* in these papers. Hafner and Mancosu (2005) offered a list of eight expressions that they had found to be commonly used in the mathematics and philosophy of mathematics literature to describe the search for explanations. Table 4 presents these expressions along with the specific concordance search we made to investigate their prevalence in both the mathematics and physics papers, and the frequencies with which these alternative expressions appeared. We note that the total number of occurrences of these expressions is only about 10% of the total amount of explain-words in each set of papers (with disproportionately more occurrences of these expressions in the physics papers than the mathematics ones) and thus this analysis does not affect the finding made by only investigating the use of explain-words.

Alternative expression	Concordance search	Mathematics	Physics
"the deep reasons"	deep* reason*	5	16
"an understanding of the essence"	understand* the essence	0	5
"a better understanding"	better understand*	161	767
"a satisfying reason"	satisfy* reason	0	0
"the reason why"	reason* why	312	924
"the true reason"	true reason	3	1
"an account of the fact"	an account of the fact	0	0
"the causes of"	cause* of	16	609
	Total	497	2322

Table 4. Frequencies of alternative expressions of related to *"explanatory" talk*

Explaining why vs. explaining how

In order to investigate mathematicians' discussion of *explanations of why* a certain mathematical statement is true (Hafner and Mancosu's "deep" explanation), in comparison to their talk about *explanations of how* to do something in mathematics (related to Rav's notion of mathematical know-how), we created a concordance of the corpus of papers and identified every instance an explain-word had been immediately followed by the words why or how (e.g.

unexplained why, explanation how). We did this by searching the concordance for *expla* why and *expla* how, and checking that all results were indeed uses of explain-words. We then repeated the process with the corpus of physics papers. As, shown in Table 5, there is a clear difference between the ways that explain-words show up in the mathematics and the physics research papers.

	Mathematics	Physics
expla why	247	952
expla how	458	353
Total	705	1305

Table 5. Frequencies of explain-words immediately followed by the words *why* or *how* in the mathematics and physics research papers

We note that when taken together the total of *expla*-why and *expla*-how expressions were roughly as common in math papers as they were in physics papers, with approximately one of these expressions showing up every 10-11 papers in the corresponding set, and also a relatively small subset of the wider use of explain-words (roughly 14% and 6% of explain-word usage in mathematics and physics, respectively). However, the distribution of these two different types of expressions in the two sets of papers was significantly different (Fisher's exact test, $p < .001$), with mathematicians using nearly twice as many *expla*-how expressions than *expla*-why expressions, and physicists on the other hand using a little under three times as many *expla*-why expressions than *expla*-how expressions.

Discussion

Our analysis of *"explanatory" talk* in a large sample of mathematics papers does not offer support for a claim often made in the philosophy of mathematics: that this type of talk is prevalent in mathematical discourse. When compared to explicit discussion of other related mathematical practices (showing results, solving problems, and proving theorems), mathematicians do not seem to discuss explanation nearly as much. Furthermore, when compared to another scientific discourse, we found that mathematical discourse contains only a fraction of *"explanatory" talk* as research papers in physics. Indeed, we believe these findings suggest that the prevalence of *"explanatory" talk* in mathematical discourse has been widely exaggerated.

Furthermore, by analyzing the frequency with which variations of the expressions *explain why* and *explain how* occur in mathematics and physics research papers, we found that, to the extent to which they engage in *"explanatory" talk*, mathematicians seem to be much more interested in discussing explanations of how to do something in mathematics, than in explanations of why things are the way they are in mathematics. In physics we found the situation to be the opposite. This is particularly interesting given mathematics educators' and philosophers' of mathematics preoccupation with the type of intra-mathematical explanations of the form X explains why Y (where X and Y are mathematical assertions), and particularly with the notion of explanatory proofs (in which proof X explains why theorem Y is true). This focus may have been inherited from the more traditional study of the notion of scientific explanation, which is not only naturally concerned with this type of explanations (the desire to explain the real world is full of why-questions), but according to our findings may also be more commonly discussed in scientific discourse in terms of answers to why-questions. However, our findings suggest that this focus may also be misguided for those interested in studying the notion of mathematical explanation as it more commonly occurs in the discourse of professional mathematicians. Indeed, as suggested by Rav (1999), it seems that when it comes to proofs and explanations, mathematicians are primarily interested in learning how to solve other problems, possibly over learning the reasons why some mathematical results hold true.

Now, one must be careful about several inferential jumps made in this kind of analysis. First, while the ArXiv may well be the largest, most widely used repository of this type of preprints and postprints in the world, we have analyzed a very specific type of mathematical discourse, leaving open the possibility that studies of mathematical discourse in others settings (conversational or other digital communications) could lead to contrasting findings. Second, we have analyzed these research papers for a limited type of *"explanatory" talk*, one required to contain explain-words or a limited number of alternative, related expressions. While this was an obvious place to start to investigate *"explanatory" talk* in mathematical discourse, it is certainly possible that the analysis of other expressions related to mathematical explanation may skew our results. These limitations of the present study indicate clear avenues for future empirical research on mathematical explanation.

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Managing Tensions Within a Coordinated Inquiry-Based Learning Linear Algebra Course: The Role of Worksheets

Vilma Mesa, Mollee Shultz, Ashley Jackson

How do nine instructors teaching a linear algebra course at a research university manage tensions that emerge because of the requirement of teaching the course with an Inquiry-Based Learning approach within a coordinated system? Using Herbst's practical rationality framework (Chazan, Herbst, & Clark, 2016; Herbst & Chazan, 2011) we identify features of the course organization that contributed to tensions between professional obligations that were resolved via the production of worksheets that teachers gave to the students. We noted differences in how these tensions were handled, and provide some evidence that such differences might be related to the research orientation the instructors brought and to their status in the institution. We formulate some hypotheses that can shed light on how to assist in changing post-secondary instructional practices.

Key words: Linear algebra, Inquiry-Based Learning, Professional Obligations

Objectives

Promoting change in instructional practices in undergraduate mathematics education has been an important concern for over three decades. The calculus reform from the 80s and 90s resulted in key changes to the calculus curriculum, more prominently by bringing more contextualized and representation-rich problems that could capitalize on new hand held technologies (Douglas, 1986; Ganter, 1999; Harver, 1998). Federal support for calculus innovation through grants by the National Science Foundation increased as these were seen as a vehicle for making science, technology, engineering, and mathematics (STEM) fields more appealing to students. The need for instructional change was fueled by reports that mathematics teaching was one of the main reasons why students left STEM fields (Seymour, 1995, 2002; Seymour & Hewitt, 1997). Women in particular indicated feeling unwelcome in science and mathematics courses.

Changing instructional practices is a difficult enterprise, however, as documented by decades of research in teacher training in the K-12 system and as noted by Henderson, Beach, and Finkelstein's (2011) seminal literature review on change strategies in undergraduate STEM education. Henderson and colleagues noted that part of the problem relies on the change strategy that institutions use. They note that "developing and testing 'best practice' curricular materials and then making these materials available to other faculty" (p. 952) is ineffective in generating the anticipated change that would happen because of the availability of new materials and documents. They also noted that the second typical approach, "'top-down' policy-making meant to influence instructional practices" (p. 952) is equally ineffective as it generates arguments that usually threaten and affect collegiality so necessary for departments to function well. They note instead that the most effective strategies are aligned with or seek to change the beliefs of the individuals by involving them over a long-term process that promotes an understanding a college or university as a complex system, thus making the changes to be fully compatible with the given environment.

What happens when such approach to change is implemented in a department? This study contributes to answering that question by documenting the tensions that emerged as a department chose to implement inquiry-based learning methods (IBL) of instruction in a linear algebra course that involved about 300 students and 11 instructors. IBL is an approach that "invites students to work out ill-structured but meaningful problems… [and] construct, analyze, and critique arguments… present and discuss solutions alone at the board or via structured small-group work, while instructors guide and monitor this process" (Laursen, Hassi, Kogan, & Weston, 2014, p. 407). We followed the implementation with two goals in mind: first, to understand how the faculty operationalized teaching with inquiry-based learning methods and second, to document and understand how they managed the need to cover the prescribed content allowing at the same time free exploration of the ideas to comply with the spirit of IBL. Having information about what happens when a department seeks to institute teaching changes in a key mathematics course such as linear algebra is informative for departments interested in pursuing similar moves. This study primarily contributes to the literature on teaching change in undergraduate settings.

Theoretical framework

We assume that teaching and learning are phenomena that occur among people enacting different roles—those of teacher or students—aided by particular resources, and constrained by specific institutional requirements. We are neither concerned with the knowledge, beliefs, or attitudes of the individual teacher who enacts a particular instructional approach nor with the knowledge, beliefs, or attitudes of the students of those teachers. Rather, we seek to understand how the people in these roles navigate their obligations as teachers and students to ensure that the purposes for which they are gathered together are fulfilled.

Herbst and colleages (Chazan, Herbst, & Clark, 2016; Herbst & Chazan, 2011), using the notion of *practical rationality*, have proposed that teachers respond to four distinct professional obligations while teaching and that some of these create tensions that put teachers in a double-bind as they make decisions in the classroom. In this study we attended to the disciplinary and the institutional obligations. The disciplinary obligation refers to the expectation that teachers will represent the knowledge and practice of mathematics appropriately (e.g., precise ideas, correct language). This may include the responsibility of checking the mathematical quality of tasks, textbooks, or other resources provided to students. The institutional obligation refers to the expectation that teachers will fulfill their role as a part of larger organizations (e.g. the department, university, etc.). It involves following regimes such as official pedagogies, policies, and assessment that exist independently of teachers' individual preferences. Thus the obligation to a given course's curriculum is a manifestation of the institutional obligation, while the disciplinary obligation can "oversee[s] and question[s] the quality of the representation of the discipline offered by the curriculum" (p. 1067). We describe key tensions within the IBL linear algebra course as a clash between these two obligations precisely because of the need to teach the course using IBL.

Methods

This qualitative study took place in a coordinated linear algebra course of 11 sections taught at a research university. Two major goals for the course are stated in the syllabus as: "to learn linear algebra and to learn how to write a rigorous mathematical proof. Students should leave this course prepared to use linear algebra as well as to succeed in further theoretical courses in mathematics." The syllabus also states that the course is difficult and names two alternative courses offered to those interested in the "computational side of linear algebra."

Participants in the study were all the nine faculty members who taught the course in Fall 2015 at this university. The participants included four post-doctoral fellows, four tenured faculty, and one tier-three lecturer. Instructors were further classified as applied or non-applied mathematicians based on their background and current research interests. There were four applied (Bethany, Ed, Henry, and Miles[1]), four non-applied (Laura, Lewis, Thomas, and Ulrich), and one self-described non-applied mathematician with research interests in applied fields. Three of the instructors (Bethany, Ed, and Henry) had taught the course in the year immediately before the semester in which we collected the data and had developed enough material for the course; three instructors (Ed, Laura, and Lewis) taught the course in the semester following the data collection term (See Table 1).

Table 1: Characteristics of study participants.

Participant	Status	Research	Terms teaching Linear Algebra with IBL
Bethany	Post-doc	Applied	Winter 2015, Fall 2015
Ed	Post-doc	Applied	Winter 2015, Fall 2015, Winter 2016
Henry	Clinical faculty	Applied	Winter 2015, Fall 2015
Laura	Tenured faculty	Non applied	Fall 2015, Winter 2016
Lewis	Tenured faculty	Non applied	Fall 2015
Miles	Post-doc	Applied	Fall 2015, Winter 2016
Monica	Tenured faculty	Both	Fall 2015
Thomas	Tenured faculty	Non applied	Fall 2015
Ulrich	Post-doc	Non applied	Fall 2015

We collected several types of data: interviews with instructors, field notes from about half of the course planning meetings, observations and focus groups with students, instructor and student surveys on instructional practices, bi-weekly logs about the course from some of the faculty, and various documents: the textbook, the pacing chart, worksheets, quizzes, syllabus, and exams. We analyzed the interviews thematically by question, seeking to identify threads that were relevant to the institutional and disciplinary obligations that instructors responded to. We performed an analysis of the textbook to identify the elements that were more prominently discussed during planning meetings contrasting its content with the content present in a non-IBL textbook. We created a matrix with initial assertions, identifying the faculty to which the assertion applied to and the different sources that corroborated the assertion, in this way engaging in a cross-case analysis and triangulation of the data sources.

In order to corroborate our interpretations, we sent a summary of our findings to our participants so that they could verify or rectify what we have found. Three of eight participants (Bethany, Henry, and Laura) responded making specific suggestions about the textbook analysis (part of a longer version of this paper) and corrected some factual information regarding the context of the course. Given their comments and who responded we have reason to believe that our interpretations accurately reflect the situation that we are about to describe.

There are two limitations to our study. First, we are still in the process of collecting student performance data. As of now, we have data from focus groups on student perceptions of how IBL supported their learning, but we do not have access to student final grades in the course or in subsequent courses. Thus, although we could talk about impact of the change we are in no position to do so. Additionally, this study seeks only to document processes of change not the impact of the change on students. The students were quite satisfied with the learning they were experiencing, but self-report data is hardly sufficient as a tool to convince administration of the appropriateness of a programmatic instructional change. In a subsequent phase we plan to perform a historical analysis of data to study impact of the implementation

[1] Names are pseudonyms.

of IBL. Second, the first and third authors were involved as evaluators in a training grant that involved the participating faculty. The goal of the grant was to increase the number of undergraduate students in the lower division courses experiencing IBL. As evaluators, their role was to document how the IBL training strategies were deployed and to assess which ones were perceived by the faculty as beneficial in helping them understand how to teach with IBL. As such, some of the data collected sought to providing feedback to faculty rather than provide data for our research questions. For that reason, data related to faculty feedback (observations and focus groups) play a secondary, albeit important role, that of triangulation, in this analysis.

Results

The department had put in place a number of mechanisms to ensure that all the students in the 11 sections of the course were exposed to similar content and experiences. These mechanisms came in the form of the pacing chart, that indicated the sections of the textbook to cover each day; the textbook itself that students had to use to do their daily reading quiz and that instructors should use to design worksheets; the homework, which was the same for all students in the course; and the exams (two midterms and a final) which were also common and graded with a common rubric. Two tensions emerged from the need to fulfill the obligations while adhering to those mechanisms._The first tension expressed itself through complaints regarding the presentation of content in the textbook required by the department that differed from what some faculty thought should be the definitions that should be used, an evidence of their allegiance, or obligation towards upholding the knowledge in the discipline. This tension is especially interesting to observe in an IBL setting because instructors distribute supplementary material in the form of worksheets where they can choose to support or deviate from the textbook.

The second tension emerged from the two explicitly stated goals of the course, "to learn linear algebra and to learn how to write a rigorous mathematical proof" and what instructors saw was the best way to approach the linear algebra content, either by exemplifying notions through concrete examples that then get formalized or by starting with definitions that are then exemplified with concrete examples. We saw the faculty resolving this tension in their worksheet production by either emphasizing applications of concepts or by increasing the level of abstraction and development of mathematical theory. We believe that their choices were related to their research interests. We expand on the first tension here.

According to Henry, the selection of Brestcher as the textbook for the course was predicated on the desire to choose a book that helped develop intuition. This was the book he felt students could best learn from. Henry chose the textbook after reviewing various options and, after having given it "a try," found it sufficiently readable and usable by the students. His perception was that the problems were "decent" and sufficient for helping students develop the needed intuitive understanding of linear algebra ideas, despite its shortcomings regarding the definitions used for linear independence and linear transformations, and the particular ordering of the chapters. During the first semester in which all the sections used IBL with Bretscher the faculty heatedly discussed these shortcomings during the weekly course planning meetings.

Confirming those discussions, the interviews with faculty suggested that nearly all instructors (7 out of 9) recognized problems with the mathematics in the required textbook (e.g., "This book is almost like not having a book. It has good problems and historical treatment, but doesn't have the definitions that mathematicians use," Thomas). Instructors with applied backgrounds handled this problem differently than instructors with non-applied background. The former group sought to maintain consistency of content and pacing across the different sections of the course, making an effort to align the content with the required readings. For example, Bethany, a professor with an applied background, said,

> A lot of it came down to how much we should rely on the book versus how much you should rely on what you think is convention. From my perspective, this is the book that students read, and we should try our best to provide them a consistent picture. I'll sacrifice nonstandard definitions for the sake of that consistency.

Similarly, Ed said that once the textbook is chosen, he sticks to it even if it is not ideal. He figured he could be creative with the worksheets and be more rigorous if he wanted to. He also said that Bretscher's definitions (e.g., linear transformations) were good for conceptual understanding but not for theory. Henry, who choose the textbook for the course, was also an applied mathematician. In contrast, faculty with a non-applied background prioritized mathematical development. Lewis did not require students to read the book because he was unhappy with the sequence of topics: "The book was awful on many counts, even their definitions were suspect." Monica, while unhappy with the textbook, did not deviate from it. She said, "Certain things were in the wrong order," for example, the way the book introduced vector spaces. She liked the use of examples to develop students' intuition, "...but then the book stays with the example of $\mathbf{R}^n$ and if that's the first time you hear about a vector space, which is really an abstract concept, …it does harm [if] that's all we know." Like Ed, she found the worksheets to be a place that she could supplement the book. She tried to build an abstract conception of vector spaces and general structures in class and in the worksheets and also through lectures in parallel to the textbook's examples in $\mathbf{R}^n$.

Henry and Lewis wrote the original worksheets that were used by most of the instructors: Bethany, Ed, and Miles used and revised Henry's worksheets while Thomas and Laura used or referred to Lewis'. We noted that the first group includes all instructors with applied background, while the second group includes faculty with non-applied background. Henry's worksheet (Figure 1) from the section on subspaces of $\mathbf{R}^n$, bases, and linear independence (section 3.2) contains exercises that use of the term *redundant*, a term introduced by Bretscher (2013, p.125) to create an intuitive build-up to linear independence. However, the worksheet also contains a problem that walks students through a proof using the standard definition of linear independence.[2] Lewis's worksheet (Figure 2) on linear transformations (sections 2.1 and 2.2) begins, "Our definition will be a bit different than the one in the book but you will see that it is equivalent to the definition in the book." His worksheet on linear independence, bases, spanning sets, and basis (section 3.2) begins with the standard definition of *linear independence*, *span*, and a *basis*, and briefly mentions *redundant* in his "Something to think about" section at the end of the worksheet. Thus, Henry started with the book's definition and related it to the standard definition, while Lewis started with the standard definitions and related them back to the textbook.

[2] A subset S of a vector space V is called **linearly dependent** if there exists a finite number of distinct vectors $u_1, \ldots, u_n$ in S and scalars $a_1, \ldots, a_n$, not all zero, such that $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$. In this case we also say that the vectors of S are linearly dependent. A subset S of a vector space V that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of S are linearly independent (Friedberg, Insel, & Spence, 2002, p. 36-37).

3. More fun with linear independence!

   (a) Are the vectors $\vec{v}_1 = \begin{bmatrix}1 & 0 & 0 & -2\end{bmatrix}^T$ and $\vec{v}_2 = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^T$ linearly independent? Why or why not?

   (b) Now, consider the five vectors $\vec{v}_1 = \begin{bmatrix}1 & 0 & 0 & -2\end{bmatrix}^T$ and $\vec{v}_2 = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^T$, $\vec{v}_3 = \begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}^T$, $\vec{v}_4 = \begin{bmatrix}-2 & 2 & 3 & 4\end{bmatrix}^T$ and $\vec{v}_5 = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T$. Which is the first redundant vector? Write it as a linear combination of the preceding vectors.

   (c) Write a nontrivial linear relation among $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$, and $\vec{v}_5$. *(Hint: your work in (b) has the outside chance of being useful.)*

4. Wasn't that fun? What fun thing should we do next? Let's suppose we have a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$.

   (a) If $m = 5$ and we know that we have the linear relation

   $$-\vec{v}_1 + 2\,\vec{v}_2 + 3\,\vec{v}_3 - \vec{v}_4 + 0\,\vec{v}_5 = 0,$$

   explain why $\{\vec{v}_1, \ldots, \vec{v}_5\}$ are linearly dependent (by identifying a redundant vector in the set).

   (b) Explain why the existence of a nontrivial linear relation $c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = 0$ (nontrivial means that at least one $c_i \neq 0$), shows that there is a redundant vector, and hence that $\{\vec{v}_1, \ldots, \vec{v}_m\}$ are linearly dependent.

   (c) Conversely, if $\vec{v}_i$ is a redundant vector in the set $\{\vec{v}_1, \ldots, \vec{v}_m\}$, show that there is a nontrivial relation among the vectors in this set.

Figure 1. *Sample of Henry's worksheet from the section on subspaces.*

A *basis* of V is a collection of elements that is both *linearly independent* and a *spanning set.*

1. Decide if the following collection of vectors are a basis for $\mathbf{R}^3$. If not, say what property or properties fail.

   (a)
   $$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

   (b)
   $$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}2\\1\\3\end{bmatrix}.$$

   (c)
   $$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}2\\1\\2\end{bmatrix}.$$

   (d)
   $$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}2\\1\\2\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

2. Let V be the set of vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ in $\mathbf{R}^3$ satisfying the relation

   $$x_1 + 2x_2 - 4x_3 = 0.$$

   (a) Prove that V is a subspace of $\mathbf{R}^3$.
   (b) Find a basis for V.

Something to think about

Vector $\vec{v}$ in Bretscher's book was overheard saying "I hate being a redundant vector. If I was the zero vector, that would be okay. But if I'm redundant just because others were ahead of me in the line, then that's not fair!" Do you sympathize with vector $\vec{v}$?

Figure 2. *Sample of Lewis' worksheet from the section on subspaces.*

These examples illustrate the tension between the obligation to comply with the institutional requirement of using a textbook that will be consistently used across all sections of the course, and the disciplinary obligation to represent accurate mathematics when the goal is to foster proving and illustrate how instructors managed them through the writing of the worksheets, suggesting a possible explanation rooted on the research background of the instructors. This is speculative, as other factors (e.g., institutional status) might have played a role as well.

Discussion and Implications

The instructors were able to use the worksheets as a place to manage the tensions that arose from their institutional and disciplinary obligations. The instructors who were unhappy with the content of the textbook were able to provide supplementary material to their students that, in their opinion, displayed the mathematics more appropriately. In particular, we found instances where instructors preferred different yet equivalent definitions, and used the worksheets to connect them to the curriculum.

All the instructors expressed deep commitments to their students, which indicates that these decisions reflected their interest in student success. Research on how courses are designed can help the research community build knowledge on what speaks to faculty teaching students in STEM environments. The preliminary evidence from this investigation suggests that the tensions that can emerge as mathematics departments seek to increase the number of students experiencing new ways of teaching can be navigated via a coordination system that allows faculty some space for exerting their professional commitments.

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Function Sameness to “Function” Meaning

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Exploring students’ conceptions of sameness is an avenue for exploring their understandings of the objects being compared. More specifically, finding what students think it means for functions to be identical can help us figure out what students think it means for something to be a function, since identity within a category (in this case the category of function) is inextricably tied to the defining aspect of that category. This paper has three primary aims: to illustrate the importance of using students’ assessments of sameness as a means to discover their concept images, to describe a particular student’s concept image of function and of function sameness, and to suggest that the math education research community develop a more refined understanding of a “process”(cf., Breidenbach, Dubinsky, Hawks, & Nichols, 1992) conception of function.

Key words: equivalence, function, concept image, sameness

I designed this study on function sameness with the following research question in mind:

(1) Which function representations do students view as representing identical functions?

However, my research motivations go beyond the stated research question at the service of the larger question:

(2) What do students think “function” means*?*

Questions 1 and 2 are inextricably linked. Finding what students think it means for functions to be the same can help us figure out what students think it means for something to be a function, since identity is inextricably tied to what something *is.* This paper uses a case study to illustrate the importance of using (1) to answer (2) and to suggest that the math education research community investigate what students with a process conception of function (cf., Breidenbach, Dubinsky, Hawks, & Nichols, 1992) view as representing the same process.

It was natural to adopt Tall & Vinner’s (1981) usage of *concept image* and *concept definition*, since these constructs capture the essence of my research question. Someone’s concept definition is the definition that he would give in words to that particular concept or string of words. A person’s concept image is anything he associates with a particular concept or string of words, including mental pictures, computational processes, and properties. I will use the term “concept image” interchangeably with “understanding,” “conception,” and “meaning.” We can reframe (1) as “what is a student’s concept image of sameness of function?” and (2) as “what is a student’s concept image of function?”.

As mathematicians, we view functions f and g to be *the same function* if and only if f and g have the same domain and $f(x)=g(x)$ for all values of x in the domain. A student, on the other hand, might have a different conception of function sameness. To a student, “f and g are the same function” might not mean the same thing as “$f=g$”; this would be unsurprising in light of the research that shows that students often view “=” as a command to calculate (Powell, 2012). I investigate what a student thinks it means for two representations of functions to be of the same function, disentangled from her assessment of whether or not “equals” means “the same as” in the case of function. In other words, I investigate a student's concept image of "function sameness," *not* her concept image of "function equality".

Research shows that many students have a weak conception of function (Bardini, Pierce, Vincent, & King, 2014; Breidenbach et al., 1992; Clement, 2001; DeMarois & Tall, 1996; Musgrave & Thompson, 2014; Oehrtman, Carlson & Thompson, 2008; Sfard, 1992; D. Tall & Bakar, 1992; Thompson, 1994; Vinner, 1983). For example, Clement (2001) shows that a high percentage of precalculus students fail at completing basic tasks in assessing whether particular relations are functions and impose incorrect conditions on what it means to be a function. In particular, several students classified a non-continuous relation (that happened to be a function) as a non-function on the grounds that it was non-continuous, even though when giving their concept definition of function, they did not mention continuity. The method that Clement (2001) uses to assess students' concept images is a common way of investigating students' concept image of "function"; several other researchers (e.g., Bardini et al., 2014; Breidenbach et al., 1992; Clement, 2001; Tall & Bakar, 1992; Sfard, 1992; Vinner, 1983; Vinner & Dreyfus, 1989) assess what students consider to be instances and non-instances of function. This method (of having students assess instances and non-instances of function) is useful because it helps researchers see beyond simply the student's stated definition of function.

Subject and Methods

I asked students to identify pairs of representations of functions as representing the same function or not. By finding what criteria students use to decide whether two functions are identical, we can begin to discern what students take functions to be, since the identity criteria for a class of things defines what those things are. Consider sets: if someone says that $\{2,3\}$ is not the same set as $\{3,2\}$, then we can infer that this person believes that a set is something more than simply a collection of objects. In the case of function, a student may hint that he thinks a function is simply an equation by indicating that different equations automatically indicate different functions. For example, Sfard (1988) describes how some students revealed their views of functions as computational processes when they refused to view the functions f and g defined on the natural numbers by $f(x)=x^2$ and $g(0)=0$, $g(x+1)=g(x)+2x+1$ as the same function, despite acknowledging that they output the same values.

The goal of my study, which takes the form of an hourlong semi-structured clinical interview, is to model the student's concept image of function sameness through a series of tasks and accompanying questions, with an eye toward assessing the student's concept image of function. The majority of the interview protocol consists of function comparison questions: the student is given two different descriptions or representations of possibly different functions and must determine if indeed these representations describe the same function. Four subjects were enrolled in the study, but here I discuss only one, Jane. At the time of interview, Jane was a senior microbiology major earning an A or a B in an upper-division transition-to-proof course.

Jane's Concept Images

Jane began the interview with a mathematically correct concept definition of function sameness, but she soon began to think that more conditions are necessary. Throughout the interview she talked about two possibilities for sameness: her original, mathematically correct notion, which she called "equivalence,"[1] and then something she considered an additional possible criterion for sameness that would create a narrower definition. Jane clarified that equivalence was necessary for sameness but perhaps not sufficient. She said that there is another possible criterion for sameness, which she described as sameness of mathematical processes. So, Jane considered two possible characterizations of sameness of functions: sameness as equivalence, and sameness of mathematical processes together with equivalence. Her notion of mathematical process sameness as a criterion for function sameness fits with her concept image of "function," which I will describe in detail later.

Jane's Concept Image of "Equivalence"

As early as Task 2, Jane coined her term "equivalence" and stated that it was necessary, but perhaps not sufficient, for sameness of functions:

Excerpt 1: Task 2: What does it mean for f and g to be the same function? I: Interviewer; J: Jane

I	What does it mean for f and g to be the same function?
J	I don't know if a math function being the same is the same as it being equivalent...what I mean by equivalent is that if every x I put in through this function gives me the same y as every x that I put in...like… if x-sub-1 corresponds to y-sub-1 here...if I can put x-sub 1 into either of these equations and always get the same y, I would say they are equivalent. I don't know if in math terms if they would be the same

As the interview progressed, it became clearer that Jane's notion of "equivalence" of functions was the mathematically correct conception of equality of functions. She continued to express that equivalence was necessary for sameness, articulating that the functions must have the same domain and must output the same value anytime they are given the same input. Whenever she could use equivalence to show that two functions were not the same, she did. For example, she argued that the functions defined by $f_1(x) = x^2/x$ $f_2(x) = x$ are not equivalent and hence not identical by virtue of having different domains, and she argued similarly for functions represented graphically (where one function was a subset of the other). Moreover, when asked to give examples of functions that are not the same and prove that they are not the same, she described pairs of functions that differ on particular values. Jane stressed that she knew that the functions were not the same due to differing on a particular input value. She used the language "input" and "output" and avoided "independent variable" and "dependent variable". For example, she described the function defined by $f(x+1) = (x+1)^2$ as at least equivalent to $g(x) = x^2$ because "the $g(5)$ here it's still like, if you are gonna graph it and you get to 5, either way you are still going to get (5,25), so ... if you could always put the same input in and get the same output. If there's the same relationship between the points, I would say they are equivalent."

Jane's Concept Image of Sameness of Mathematical Process

[1] I use "equivalence" to refer to what Jane calls "equivalence," which is (in the case of function) what mathematicians would call "sameness".

As mentioned earlier, Jane thought that it was possible that there was more to sameness than "equivalence". In several instances in the interview, she described why two equivalent functions would not be the same, if in fact equivalence is not sufficient for sameness. She referred to this extra criterion as "the same mathematical process" and designated examples and non-examples of this criterion. In the remainder of this paper I discuss Jane's notion of sameness under her hypothetical assumption that equivalence is not a sufficient criterion, and thus hone in on her notion of "same mathematical process".

At several points during the interview, Jane revealed her notion of "same mathematical process." She said that the functions in Task 6 ($h_1(x) = |x|$ and $h_2(x) = \sqrt{x^2}$) are equivalent but not the same, but that the functions in Task 8 ($f(x) = 5$ and $h(x) = 1(0x + 5) + 5 - 5$) *are* the same (and thus also equivalent). She argued that the functions h_1 and h_2 were different by virtue of doing different "mathematical things," despite being what she calls "equivalent".

Excerpt 2: Task 6: $h_1(x) = |x|$ and $h_2(x) = \sqrt{x^2}$. Are h_1 and h_2 the same function? Why or why not? I: Interviewer; J: Jane

I	What part of sameness is it lacking?
J	Um, like, it's not mathematically, like, I don't know if mathematically it's doing the same thing.
I	What do you mean?
J	Like, whereas this just takes the magnitude of it, this does a mathematical thing to it which comes up with the same answer.

Yet, she argued that the functions f and h in Task 8 were "doing the same thing," and further hinted that transformability may be a relevant property for determining sameness. She described a process of simplification and contrasted it with a situation in which standard algebraic transformation procedures would not apply.

Excerpt 3: Task 8: Are $f(x) = 5$ and $g(x) = 0x + 5$ the same function? Are $f(x) = 5$ and $h(x) = 1(0x + 5) + 5 - 5$ the same function? I: interviewer; J: Jane

J	You just have to simplify to like what does this mean or what's the simplest terms we can put this function in. (...) But I feel like there's some function where like, oh the sum from like k=1 to infinity of 2k+1 is equal to like, (n+1) squared or something... these are like mathematically different things you do to the numbers, it makes them equivalent, like at any k value ...or n value...those things are equivalent. So for me that's equivalence. But for me this [points at $g(x) = 0x + 5$] is about the same thing, because you just simplify it and you get $f(x) = 5$.
I	What about this one? [points at $h(x) = 1(0x + 5) + 5 - 5$]
J	Well for me, this now, it's like you have your 5 and your negative 5 and those just cancel, and you multiply the 1 across and you still get 0x+5 which is just 0 times whatever value this is so always 0 so always just 5. So like, I just feel like you can just simplify it, and if you can just simplify it, then it's the same.

I	Would you not say you could simplify this infinite sum to the formula $(n+1)^2$?
J	Yeah I guess you probably could...it just feels different to me….because like this (sum) is a different sort of structure...it's like hey if we are taking all of these values and adding them up...you know...we get some value, and this (formula) is like what's the value you put this one number in. **This (sum) is still an input-output deal but like, it's like a mathematically different process to get the output** but here [points to $f(x) = 5$ and $h(x) = 1(0x+5)+5-5$]...these feel like the same process.

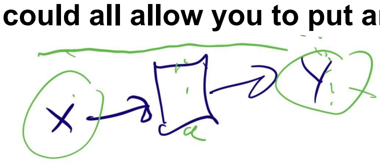
When Jane compared a piecewise function with a non-piecewise function, she yet again emphasized that, although the functions are equivalent, they differ because "you're using a different tool to get to a place".

All the examples that Jane described as equivalent but not the same were not transformable via typical algebraic manipulations. However, this isn't to say that transformability alone is sufficient for sameness in her eyes; she immediately identified the functions $f_1(x) = \frac{x^2}{x}$ and $f_2(x) = x$ as being non-equivalent, and thus not the same, due to having different domains. Moreover, Jane described $g(x) = x^2$ and $f(x+1) = (x+1)^2$ as the same function, acknowledging that not only are they equivalent, but also the same, with the explanation that "they are taking an input, and, um, making...mathematically doing the same thing to give an output. So in my mind those are the same function". In light of this example, it seems that Jane believes that for functions to be the same, they must have some degree of similarity in what it is that they do to their input. For her, transformability via familiar algebraic procedures ("simplification") indicates a sufficient degree of sameness of process.

Jane's Concept Image of Function

When we consider Jane's concept definition of function, her inclination to consider the sameness of process makes more sense:

Excerpt 4: Task 1: What is a function?

J	Well, the sort of grade school way we get is like, a little box. We have an x go in, and a y go out. You have an input, **do something to it** and get a y out. Um, I uh, I don't know uh...so uh...**you could have maybe different functions that could all allow you to put an x in and get that same y** 
I	And those would be different functions?
J	They could all be different functions.

She describes this function machine and that it does something. If a function does something, how it does it seems to matter. For her, it seems that doing something to its input is not simply a matter of mapping to a particular value, but mapping to a particular value via a particular process - this makes sense in light of her saying "different tool to get to a place" (mentioned earlier).

In Excerpt 4 she also seems to say that different functions can have the same inputs and outputs (second bolded portion), indicating that the process the function performs is an essential aspect of the function. Further, Jane's assessment of $f(x+1)=(x+1)^2$ and $g(x)=x^2$ as defining the same function illustrates her view that a function performs some process on its inputs. For her it is clear that "squaring" is a process regardless of what you name the input. However, taking the absolute value is a different process than squaring and then taking the positive square root (Excerpt 2).

In summary, if we view Jane's concept image of function as a process that accepts particular inputs and then gives outputs according to that process, then it makes sense that for her, sameness of functions involves having the same inputs and the same outputs for each input (equivalence) and *also* having the same process.

Classifying Jane's Function Concept in the Action-Process-Object Framework

Breidenbach et al. (1992) use the term "action conception" to refer to a conception about functions as methods to use to produce a result. Thompson (1994) describes the student as viewing the function as a "recipe" for calculation. Breidenbach et al. (1992) describes the student as having a process conception of function when the student views all of the steps of computation together as a process, where a process is "when the total action can take place entirely in the mind of the subject without necessarily running through all the specific steps" (p. 249). In other words, when a student is thinking of a process, she is not thinking of a step-by-step computation, but is instead thinking of the transformation as a whole. When a student sees a function as an object (e.g. a set of ordered pairs) and manipulates it as such, then she has an object or structural conception. Sfard (1992) additionally describes a *pseudostructural* conception of function as viewing a function as its representation, such as conflating a function with its equation or an image of its graph.

Jane definitely does *not* have an action conception of function. She does not seem to need to plug in specific values. She discusses the processes that functions perform without actually carrying out these processes herself (e.g., when she is comparing sameness of processes). When asked "What is a function?", she does not give any of the descriptions that Breidenbach et al. (1992) associate with an action conception. It is evident that Jane does not have a pseudostructural understanding of function either; this is especially evident when we consider that she allows the use of different variable names and inputs to define the same function.

Instead, it appears as if Jane has a process conception of function. Her description is similar to one of the descriptions Breidenbach et al. (1992) associated with a process conception: "a function is some sort of input being processed, a way to give some sort of output" (p. 252). Additionally, she has the ability to discuss the general notion of a function's output without having to compute it herself.

Discussion

Jane considered transformable equations as representing the same process. Why this is the case is not clear, but one hypothesis is that through schooling, she reflexively "simplifies" any expression that she sees. She may think of, say, multiplying by 2, adding 3, and then subtracting 3 to be the same thing as simply multiplying by 2, just more complicatedly expressed.

If (to a student) a function is a process, then sameness of functions reduces to sameness of processes. A different student also considered transformability to be sufficient for sameness of process, but gave a different answer to Task 6: he argued that taking the absolute value is just an abbreviation of the process of squaring and then taking the radical. His rationale was that taking the absolute value is not a "mathematical process" and just a shorthand way of telling a function to square and then take a radical. Further investigation is required to see the rationale that students give for transformability and sameness of processes, possibly leading to a more refined notion of "process conception".

There is potentially interesting research that can be done regarding the historical community's notion of sameness of processes. Sfard (1992) discusses how, like with functions, the mathematical community considered fractions as representing a division or measuring process rather than numbers. Did mathematicians have anything to say about the similarity between the processes 1/2 and 2/4? Was there a sense in which the historical community considered them to be representing the same process?

Using the notion of sameness of representations is useful in other mathematical education research settings. I am currently working on a project in which I ask students about the derivatives of the functions defined by $f(x) = x^3$ if $x \neq 2$, $f(x) = 8$ if $x = 2$, and $g(x) = x^3$. Some students claimed that f and g are the same function with identical graphs but different derivatives, leading me to wonder what these students think it is that they are differentiating (an equation?).[2] Another student (Jane, in fact) claimed that despite the functions f and g having the same graphs, they could not be the same function, since (according to her) she could use the power rule to find $g'(2)$ but could not for $f'(2)$.

One of the goals of another research project I am involved in is to see what students think a "graph" is (and in particular, if students conflate "graph" in the sense of "set of ordered pairs" with "displayed graph" or "image of graph"). We used the notion of sameness to further this investigation by asking students to assess whether certain pairs of displayed graphs are indeed the same graph. For example, we asked students if an image of $y = x^2$ with same-scale axes is a different graph than the image of $y = x^2$ with axes of different scale in order to learn if the student thought that a graph is defined by more than its ordered pairs.

The notion of sameness can potentially be useful in exploring students' understanding of structures. Consider asking students about the groups Z_2 and $Z/2Z$. These are different groups (Z_2's elements are integers, $Z/2Z$'s elements are equivalence classes of integers). However, if a student thinks that they are the *same* group, then we know that the student might *not* think of a group as a set together with an operation (if he did, then different sets would imply different groups!). Similarly, we could learn about a student's understanding of equivalence classes by asking him if [4], as a member of Z/2Z, is the same as [10].

[2] Harel and Kaput (1991) describe a similar phenomenon.

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Students' Quantitative Reasoning in a Revenue Maximization Context

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The purpose of this study was to investigate students' quantitative reasoning when solving a multivariable problem in a revenue maximization context. We conducted task-based interviews with 12 pairs of business calculus students. Analysis of verbal responses and work written by the students revealed that in reasoning about the relationships among the quantities (sales, discount, and total revenue) in the problem, nearly all the pairs of students created new quantities. The creation of these quantities helped the students to reason about the effect of the discount on sales and total revenue. An important finding of this study is that the students took different approaches to the meaning of the discount and only five pairs of the students interpreted the discount as intended in the design of the problem. Directions for future research are discussed.

Key words: quantitative reasoning, optimization problems, business calculus

This study used Thompson's (1993) definition of quantitative reasoning: analyzing a problem situation in terms of the quantities and relationships among the quantities involved in the situation. According to Thompson, what is important in quantitative reasoning is not assigning numeric measures to quantities but rather reasoning about the relationships between or among quantities. Quantitative reasoning, as used in this study, refers to how students described and represented relationships between or among quantities and how they created and used new quantities to solve the problem they were given. The term quantity has been defined and used in similar ways by several researchers (e.g., Ärlebäck, Doerr, & O'Neil, 2013; Moore & Carlson, 2012; Thompson, 2011). This study used the definition of quantity proposed by Thompson (1990): "a quantity is a quality of something that one has conceived as admitting some measurement process" (p. 5). Examples of quantities in this study include sales, discount, and total revenue.

Much research has investigated students' reasoning about quantities in physical contexts such as in kinematics (e.g., Beichner, 1994; Bingolbali & Monaghan, 2008; Monk, 1992), heat and energy (e.g., Prince, Vigeant, & Nottis, 2012), temperature (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), and volume (e.g., Monk & Nemirovsky, 1994). However, we know very little about how students make sense of quantities in an economic context and how they reason about those quantities, which is the motivation of this study. This study examined business calculus students' quantitative reasoning when solving an optimization problem that is situated in the context of revenue maximization. The following research question guided this study: What do business calculus students' responses to optimization problems involving multiple covariates that are situated in the economic context of revenue maximization reveal about their quantitative reasoning?

Literature Review

Research on students' quantitative reasoning at the undergraduate level is scarce: much of the existing research that has looked at students' quantitative reasoning is at the elementary and the secondary levels (e.g., Lobato & Siebert, 2002; Thompson, 1993; Yerushalmy, 1997). A few studies (e.g., Moore, 2014; Moore & Carlson, 2012) have looked at students' quantitative reasoning at the undergraduate level. Yerushalmy (1997) studied six Israeli secondary school students' reasoning while modeling a multivariable situation. The researcher investigated how the students related three quantities (cost, number of days, and number of kilometers driven) as well as how they described the various representations they used to relate the quantities. Over a period of four one-hour meetings, the students worked in two groups to determine the total cost of renting a car from a rental car company that charges its customers "1000 shekels for a day and an additional 5 shekels per kilometer" (p. 435). Yerushalmy reported that to represent the relationship among total cost, the number of days a car is driven, and the number of number of kilometers a car is driven, the students used real-valued algebraic functions of two variables, three-column tables, and a three dimensional graph. Yerushalmy argued that even though a majority of the students had difficulty using correct mathematical notation to represent the relationship algebraically, the use of three-column tables helped some of the students to correctly determine an algebraic equation relating the three quantities. Only one student attempted to represent the relationship in a three dimensional plane by combining a pair of two dimensional planes (total cost versus number of days a car is rented and total cost versus number of kilometers a car is driven). The findings of this study suggest that representing relationships among several quantities using algebraic equations and graphs in the context of renting a car is particularly difficult for students.

Moore and Carlson (2012) examined how nine students, drawn from three sections of a precalculus course at a large public university, engaged in quantitative reasoning while reasoning about the volume of a box. The box was "formed by cutting equal-sized squares from each corner" (p. 51) of an 11 inch by 13 inch sheet of paper and folding the sides up. Each of these students participated in a task-based interview where they were asked to "write a formula that predicts the volume of the box from the length of the side of the cutout" (p. 51). Moore and Carlson found that, at first, a majority of the students did not recognize that the length and width of the box co-varied with the length of each square that was cut out from the sheet of paper. Consequently, the students conceived of the box as having a static base with dimensions 13 inches (length) by 11 inches (width) instead of a dynamic base with dimensions (13-2x) inches by (11-2x) where x is the length of each square that is cut out from the sheet of paper. The students were eventually successful in creating a correct formula for the volume of the box. Moore and Carlson argued that "it was only after the students imagined the process of making the box and considered how the relevant quantities of the situation changed in tandem that they created a correct volume formula" (p. 57). Similar results were reported by Lobato and Siebert (2002) in the context of a wheelchair ramp. In general, the findings of Moore and Carlson (2012) and those of Lobato and Siebert (2002) show that covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) is an essential understanding that students need if they are to be successful in relating co-varying quantities using algebraic equations. In the study reported in this paper, we examined how undergraduate students reasoned about relationships among co-varying quantities in a revenue maximization context.

Theoretical framework

This study draws on the theory of quantitative reasoning, an evolving theory in mathematics education whose origin can be traced to the early work of Thompson (1990). Thompson (2011) described three tenets that are central to the theory of quantitative reasoning. These tenets are: a quantity, quantification, and a quantitative operation. According to Thompson (1990), "a quantity is a quality of something that one has conceived as admitting some measurement process" (p. 5). Thompson added that a quantity is a mental construction. Thompson (1993) distinguished between a quantity and a numerical value: a quantity has a unit of measurement and a numerical value does not. Thompson (1993) added that “quantities, when measured, have numerical values, but we need not measure them or know their measures to reason about them” (pp. 165-166). For example, we can think of Company A’s profit for a given trading period, Company B’s profit for the same trading period, and the amount by which Company A’s profit is bigger (or smaller) than Company B’s profit, without having to know the actual numerical values of the profit. According to Thompson (2011), quantification “is the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute’s measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit” (p. 37). In a sense, the process of quantification entails assigning numerical values to the attributes of an object. A quantitative operation is the process of forming a new quantity from other quantities (Thompson, 1994b). Thompson (1993) stated that “comparing two quantities with the intent to find the excess of one against the other” (p. 166) is a specific example of a quantitative operation formed by comparing two quantities additively. In economics, for example, comparing (by way of finding the difference) total revenue and total cost with the intent to find the excess (profit or loss) of total revenue against total cost is a quantitative operation known as a quantitative difference.

The design of our study was influenced by the theory of quantitative reasoning in three phases, namely task design, data collection, and data analysis. Drawing on the theory of quantitative reasoning as a theoretical framework for our study, we designed the mathematical task (shown in the methods section) that provided students with opportunities to reason about relationships between or among quantities. This task provided opportunities for students to reason about relationships among sales (number of computers sold), sales discount, and the total revenue generated when a business sells computers to a school. Our interview protocol allowed us to engage students in reasoning about relationships among quantities during the data collection process. Finally, a major part of the data analysis phase focused on looking for evidence for when students created new quantities (performing quantitative operations), how they used these quantities to reason about relationships between or among quantities, and whether or not these quantities helped the students to solve the problem posed in the task.

Methods

This qualitative study used task-based interviews (Goldin, 2000) with 12 pairs of business calculus students. There were four tasks in total (Mkhatshwa, 2016). In this paper, we report on how the students reasoned quantitatively about one of the tasks, adapted from Hughes-Hallet et al. (2002):

> The Smith family runs an electronics business in Southern California. The family is considering signing a contract to supply a small junior high school with laptops, the exact number to be determined by the principal of the school later. For any supply of up to 300 laptops, the price per laptop will be $900.
>
> For any supply of more than 300 laptops, the school will receive a $2.50 discount per computer (on the whole order) for every additional computer over 300 supplied.
>
> The Smith family would like you to advise them whether or not to sign the contract. They want to make sure that they make the most amount of revenue possible from this contract. What advice can you give to the Smith family on whether or not to sign the contract and why?

This task was designed to examine students' reasoning about the relationship among three quantities, namely sales (the number of computers ordered by a small junior high school), the discount offered on orders of over 300 computers, and the total revenue that is generated by the sales when the discount is taken into consideration. The discount was intended to be applied on the whole order for any order of more than 300 computers. For example, the selling price for each computer in an order of 301 computers would be $897.50, the selling price for each computer in an order of 302 computers would be $895.00, and so on. This means that the amount of the discount on every computer varies with the number of computers ordered. As we will show in our results below, half of the pairs of students did not interpret the discount as intended in the design of the task.

Setting and participants

This study was conducted at a research university in the north-eastern part of the United States. The study participants were 24 undergraduate students who had recently completed a business calculus course. Twenty-two of these students were business majors (e.g., management, marketing, accounting) while the other two students were considering majoring in business-related programs. Twelve students took business calculus in the spring semester of 2015 and the other twelve students took it in the fall semester of 2015. In addition to taking business calculus: (1) ten students had taken AP economics (AP microeconomics and AP macroeconomics) in high school, (2) twelve students had taken a high school course (AP or non-AP) in calculus, and (3) twenty-one students had taken at least one college-level economics or business class (e.g., intermediate microeconomics, managerial accounting) prior to participating in this study. At the time of conducting the study, fifteen of the study participants were sophomores and nine participants were freshmen.

Data Analysis

Data analysis was done in two stages. In the first stage, we used a priori codes. We carefully read through each interview transcript and coded instances where students reasoned about: (1) new quantities (e.g., diminishing marginal returns), (2) relationships between or among quantities (e.g., revenue decreases as more computers are sold at a discounted price), and (3) representations of relationships among quantities (e.g., using graphs and algebraic equations). In the second stage, we looked for patterns in students' responses to the task. These patterns include common understandings or difficulties in students' reasoning about the effect of the discount on sales and total revenue. The common understandings or difficulties in students' reasoning found in the second stage of our analysis provided answers to our research question.

Results

Our analysis of the data revealed that eleven pairs of students created new quantities (e.g., the rate at which the revenue is increasing, the accumulation of the discount, and the point of diminishing marginal returns) which they used to reason about relationships among sales, the discount, and the total revenue generated. Of these eleven pairs of students, six pairs of students reasoned with the context of the task but not as intended in the design of the task. Instead, these six pairs of students applied the discount only to the additional number of computers over 300 ordered and created a new quantity, the rate at which the revenue is increasing. This quantity was then used to reason about the relationship among sales (number of computers sold), the discount, and the total revenue generated. Another five pairs of students reasoned with the discount as intended in the context of the task: they applied the discount to the whole order. These students created a new quantity, the accumulation of the discount, which they used to reason about the relationship among sales, the discount, and the total revenue generated. Abby and Shawna are the only pair of students who did not create any new quantity in this task. These students indicated that they needed to have the demand and supply equations (which were not given in the task) in order to solve the problem posed in the task.

Applying the discount only to the additional computers over 300 ordered

Kierra and Isaac are representative of the six pairs of students who applied the discount only to the additional computers over 300 ordered. These students created a new quantity (the rate at which the revenue is increasing) which they used to reason about the relationship among the quantities: sales, the discount, and the total revenue generated. Kierra and Isaac are the only pair of students (out of the six pairs of students) who created a graph (Figure 1) to show the relationship among these quantities. By applying the discount only to the additional computers over 300 ordered, Kierra and Isaac came to the conclusion that the relationship between the revenue and sales is such that the revenue will continue to increase as more computers are sold beyond 300. In the following excerpt, which occurred at the beginning of working on the task, Kierra and Isaac reasoned about the relationship between the number of computers that are sold and the total revenue.

Isaac: Well I think it’s, I think, I kind of picture it as almost like a graph

Researcher: How would it look like as a graph?

Kierra: Like it would, for the first 300 [computers] it [revenue] would go up at the same rate like at one rate of $900 and then like once it hits that 300, like 301, it slightly changes the slope of the line of going up of each laptop by $897.50 so I guess like, there is never gonna be a time when like they will be losing money. They will just be like slightly gaining money, nine hundred minus $2.50

Researcher: You said something about the slope changing at like 300, right?

Isaac: Yeah, because like anything above 300 is when they give that discount of $2.50

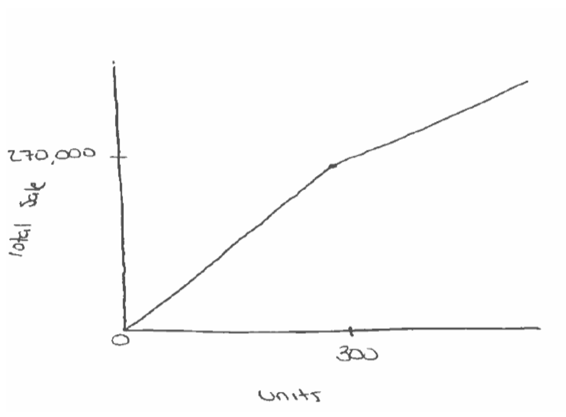

Figure 1. Isaac's graph of the relationship among the number of computers sold, the revenue, and the discount.

Kierra mentally created a new quantity (the rate at which the revenue is going up) when she stated that the revenue “would go up at the same rate like one rate of $900.” Her statement that “there is never gonna be a time when like they will be losing money” suggests that she viewed the application of the discount on orders of more than 300 computers as insignificant in that the revenue continues to increase no matter the size of the order. Kierra and Isaac recognized that graphically, the effect of applying the discount will be a slight change in the slope of the graph of the total revenue function after 300 computers as shown in Figure 1. When asked to elaborate on how the slope would change, Isaac created Figure 1. He also stated that the revenue "will go up like a steep rate" as more computers are sold up to 300, which suggests that he also created a new quantity, the rate at which the revenue will go up. Isaac continued to reason about this new quantity when he indicated that "the slope would be less steep" when he referred to how the revenue continues to increase as more computers are sold over 300 in Figure 1. At the end, Kierra and Isaac advised the Smith family to sign the contract for any number of computers ordered by the school, which was reasonable advice based on their understanding of how the discount is applied.

Applying the discount to the whole order

Yuri and Kyle are representative of the other five pairs of students who reasoned about the context of the task as intended, where the discount is applied to all of the computers ordered. This pair of students created a new quantity, the accumulation of the discount, when reasoning about the relationship among sales, the discount, and total revenue generated. In addition to verbalizing the relationship among sales, the discount, and the total revenue generated, Yuri used an algebraic equation shown in Figure 2 to represent this relationship and Kyle used a graphing calculator to graph Yuri's revenue equation. Kyle's graph was a concave down parabola.

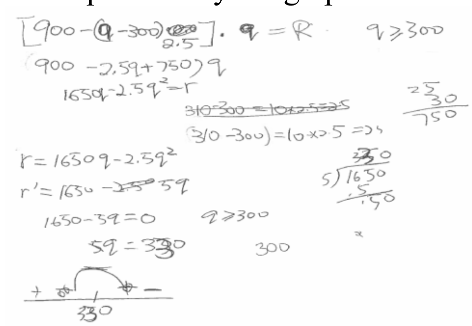

Figure 2. Yuri's algebraic representation of the relationship between the revenue and the number of computers sold.

Yuri and Kyle described the discount as "cumulative." They explained what a cumulative discount is by giving examples. Kyle indicated that "if you buy three hundred and two you get five dollars off each computer." Yuri added that "if you sold three hundred and ten computers, you get twenty-five dollars discount per computer you sell." Yuri and Kyle's understanding of the discount as cumulative suggests that these students mentally constructed the accumulation of the discount as a new quantity. With the understanding of the discount as cumulative, Yuri and Kyle calculated the total revenue from selling 310, 350, and 400 computers respectively. They found that the revenue from selling 310 computers is the same as the revenue from selling 350 computers and that the revenue from selling 400 computers is less than the revenue from selling 310 computers. Yuri and Kyle then advised the Smith family to consider the size of the order placed by the school prior to signing the contract. They indicated that an order of "315 computers" will generate more revenue "than if they sold 415 computers." This suggests that Yuri and Kyle recognized that, in the long run, the accumulation of the discount will cease to increase the Smith family's revenue and will, instead, result in loss of revenue.

After doing the work shown in Figure 2, Yuri concluded by saying that the Smith family business should sign the contract if the school orders at most 330 computers. He, however, did not talk much about the details of his work in Figure 13 such as when and why he took the derivative of the revenue function denoted by the letter r in Figure 2. Kyle also concluded by advising the Smith family to sign the contract when the school orders 330 computers "since that's the most money they can make on this deal [contract]." When asked how he arrived at this conclusion, he said [referring to the graph of the Yuri's revenue function which he graphed on his calculator], "I did second calc max to find the maximum and the max quantity is three hundred and thirty for a revenue of two hundred and seventy-two, two hundred and fifty." That is, Kyle used a graphing calculator to determine the revenue-maximizing quantity of 330 computers.

Discussion and Conclusions

This study examined undergraduate students' reasoning about the relationship among several quantities, namely sales (number of computers ordered), the discount offered, and the total revenue generated. An important finding of this study is that in reasoning about this relationship, the students took different approaches to the meaning of the discount and created new quantities. Six pairs of students did not interpret the discount as intended in the design of the task. These students applied the discount only to the additional computers sold over 300 and created a new quantity, the rate at which the revenue is increasing. Five other pairs of students interpreted the discount as intended in the design of the task. They applied the discount to the whole order and created a new quantity, the accumulation of the discount. The creation of these new quantities helped the students to make sense of the relationship among the quantities in the task especially the effect of the discount on number of computers sold and the total revenue generated in addition to helping them answer the problem posed in the task which is advising the Smith family business whether or not to sign the contract. Other research (e.g., Lobato &, 2002; Moore & Carlson, 2012) has shown that when engaged in reasoning about relationships among three quantities, students tend to talk about two quantities explicitly while treating the third quantity as an implicit quantity. Consistent with findings of this research, the students in our study tended to talk, explicitly, about the relationship between the number of computers sold and the total revenue generated while treating the discount as an implicit quantity. Previous research (Yerushalmy, 1997) has reported on students' use of graphs and algebraic equations when representing relationships among several variables. In our study, only one pair of students (Isaac and Kierra) used a graph and only one student (Yuri) used an algebraic equation to represent the relationship among the three quantities in the task they were given. It might be important for future research to examine the opportunities for students to learn to reason about economic quantities such as sales discount presented in business calculus textbooks and classroom instruction.

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Preservice Secondary Teachers' Abilities to Transfer from Graphical to Algebraic Representations of Functions

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In this study, I examined 14 preservice secondary teachers' abilities to transfer graphical to algebraic representations of functions. The analysis showed that the vast majority of the participants had problems in noticing critical behaviors of function graphs and in using them to construct algebraic forms. About half or fewer of the participants noticed qualities such as x-intercepts, vertical asymptotes, slant asymptotes, and concavity/extrema, with only a few of them successfully using such qualities in constructing algebraic forms. Only a few noticed and used qualities such as horizontal asymptotes, point discontinuities, domain, and end behaviors in constructing algebraic forms. It is advisable that the teaching of the function concept incorporate transformational activities beyond algebraic to graphical transformations and focus more on the critical characteristics of functions.

Keywords: Function, Concept image, Graph, Preservice secondary teacher

Introduction

Understanding concepts in multiple representations is critical in mathematics (Hiebert & Carpenter, 1992). Yet making connections among representations is not trivial. One should not only know how to transform within the same and across different forms of representations (Dufour-Janvier, Bednatz, & Belanger, 1987; Even, 1998), but also understand subconcepts and translate the meanings among multiple representations (Hitt, 1998; Lesh, Post, & Behr, 1987; Markovits et al., 1986).

There are many studies that deal with individuals' conceptions of functions related to graphical representations of functions. Several of these studies report that students and teachers tend to associate functions only with equations or with graphs of continuous functions (Carlson, 1998; Hitt, 1998). However less frequently reported is research that concerns whether individuals can indeed relate function graphs to function equations. As such, in this study I investigate individuals' abilities to transfer function representations that are related to equations. Such information would be beneficial not only to the understanding of individuals' cognitive structure on the function concept but also to the teaching of the function concept. This study answers the research question: "What do preservice teachers notice and use when transferring from graphical to algebraic representations of functions?"

Background

Theoretical Framework

The overarching framework of this study is the construct of concept image—"the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). The construct of concept image has been widely used to understand students' cognitive structures of the function concept in association with the formal concept definition of function—"a relation between two sets *A* and *B* in which each element of *A* is related to precisely one element in *B"* (p. 154). Due to the nature of this study, however, I focus on the characteristics of functions that are essential to the conversions between graphical and algebraic representations of functions, especially fractional functions $f(x)/g(x)$ with $f(x)$ and $g(x)$ polynomial or radical functions. Fractional functions embed many of the important characteristics of functions. As such, students' mathematical behaviors with fractional functions provide a good portrait of their understanding of function characteristics in general.

In typical college algebra or precalculus textbooks, where manual representations of rather elaborate graphs of functions are handled with no aid of the calculus concepts, a graph of a function equation $y = f(x)$ is sketched through the concepts such as domain/range, zero, and limit. The domain/range determine where the graph lies; zeros determine where the graph crosses the x axis, with $f(0)$ determining where the graph crosses the y axis; and limits determine to which lines the graph is tangent—that is, the vertical/horizontal/slant asymptotes—as well as whether the graph has point discontinuities (holes) and what the end behaviors of the graph are like. Yet understanding functions and their graphs in relation to the above concepts is not trivial for many individuals. Included below are some studies that concern individuals' understanding of characteristics of functions.

Literature Review

In general, many individuals have a limited concept image of functions. They associate functions to single equations or continuous functions and lack understanding of important characteristics of functions (Breidenbach et al., 1992; Even, 1998; Hitt, 1998).

For the concepts of domain and range, studies show that students do not hold the concepts in their cognitive structure nor do they use it appropriately in problem solving. A college student in Aspinwall, Shaw, and Presmeg and Nenduradu 's study (1997) showed a discrepancy between her image of second-degree polynomials and the domain of second-degree polynomial functions. The student thought that second-degree polynomials had vertical asymptotes, but at the same time she thought that the domain was all real numbers. Hitt (1998) also showed that many beginning secondary teachers could not articulate whether or not a curve represented by an algebraic form—circle or ellipse—was a function. The teachers also did not identify subconcepts of functions, such as domain and range, in graphical representations of functions. Williams' study (1998) also showed that for the vast majority of 28 calculus students, the concept of domain and range was not present in their concept map of functions.

For the concept of limit, most studies deal with students' understanding of limit in association with the formal concept definition of limit. Only a limited number of studies concern individuals' understandings of the limit concept related to graphs of functions—asymptotes, holes, and end behaviors. Even' study (1998) showed that a prospective teacher had difficulty telling whether a discontinuous graph with holes was a function or not. Lauten, Graham, and Ferrini-Mundy (1994) showed that constructing a function that satisfies some conditions might not be easy for college students. When a student, Amy, was asked to provide an example of a function that satisfied given conditions of f(1) = 2, f(3) = 6, and f(5) = 9, Amy only attempted to find a single equation that met the conditions. Failing the attempt, she then moved to a graphical representation and provided a continuous curve by connecting three points she plotted. She also failed to distinguish between the roles of x and y when describing limits, making it difficult for her to see a function value as a vertical distance. Amy also did not use a calculator as a toolkit. She had minimal use of a calculator, mainly using it to find values through the trace keys when problems were accompanied with graphs. She however did not use a calculator independently to explore ideas before answering questions.

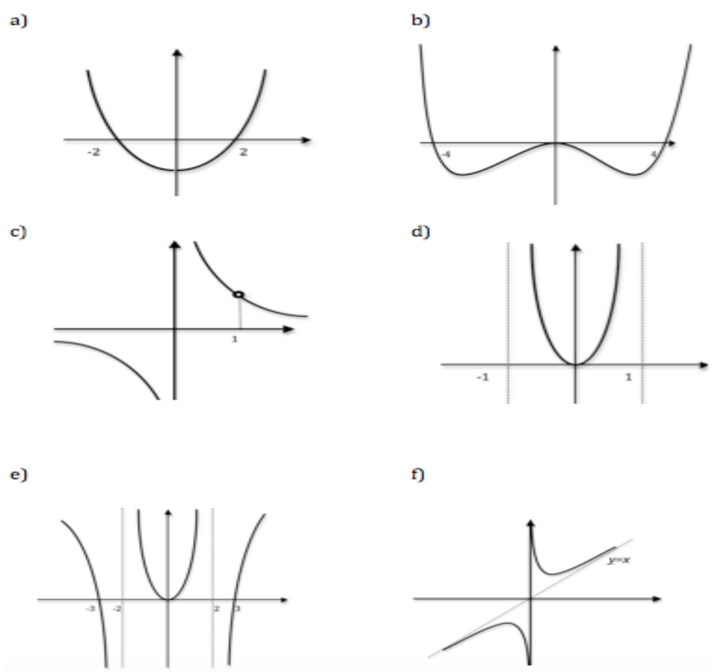
When the graphing technology is used properly, however, students could develop a sound understanding of the characteristics of functions related to the limit concept. Students in Yerusalemy (1997) graphed rational functions using computer software programs and constructed the definitions of asymptotes based on the graphs. They also created equations that have graphs lying between the rational function $f(x) = (x^2 - 1)/(x^2 + 1)$ and its horizontal asymptote and investigated how asymptotes could be evaluated from the symbolic manipulation of algebraic equations.

Methodology

The participants of this study were 14 undergraduate mathematics majors in the secondary teaching track at a medium-sized state university in the Southeast. The levels of participants' mathematics backgrounds varied—three taking Precalculus, one taking Calculus I, and the other ten taking Calculus II or above. The participants were individually interviewed twice, for about one-and-a-half hours each time, in the form of semi-structured clinical interviews. The interviews were recorded with a video camera and were transcribed. Their written responses were also collected.

The interview item relevant to this study is Q1 below from the second interview, which required graphical to algebraic representational transfers for 6 graphs that could be associated with fractional functions. Preservice teachers were asked to "think aloud" during the interviews.

Q1: For each of the graphs below, give an example of a function equation that can match the given graph:



For analysis, I used an open coding strategy (Strauss, 1987). I coded the characteristics and graphical behaviors of functions that participants noticed and used in the representational transfer as well as their understandings and difficulties shown in the transfers.

Results

Analysis showed that the preservice teachers had difficulty in constructing algebraic equations that could yield given graphs of functions. Most of them solely depended on their memory and randomly guessed the corresponding equations, or did not even attempt to guess, without careful inspections of graphs. Even in the cases in which they noticed certain characteristics or behaviors of graphs, they were incapable of constructing algebraic equations that could embed such qualities. Except for Graph (a) in which 9 out of 14 participants came up with a reasonable form of a parabolic equation, they struggled for all the other graphs. With Graph (b), which could be represented as a higher-order polynomial function, only 3 provided a correct algebraic form. For the other 4 graphs, which could not be represented as polynomial functions, only one teacher or none of the teachers was able to construct a correct equation (see Table 1).

Table 1 Correct algebraic representation of graph

	Graph (a)	Graph (b)	Graph (c)	Graph (d)	Graph (e)	Graph (f)
The number of participants who provided a correct equation	9	3	1	0	1	1

Note: Both $y = x^2 - 2$, which was based on the y-intercept, and $y = x^2 - 4$, which was based on the x-intercepts, were accepted as correct answers for Graph (a).

Preservice teachers' ability to notice critical characteristics or behaviors of graphs was also very limited. Although many of them mentioned x-intercept/zeros (7), vertical asymptote (10), and slant asymptote (8) at least once, only a few of them mentioned horizontal asymptote (3), hole/point discontinuity (2), concavity/maximum/minimum (4), and domain (2). None of them mentioned the end behaviors of the function graph, which involved the limit concept (see Table 2).

Their noticing of the characteristics or behaviors of graphs also did not lead to their using of the information in the graphical to algebraic transfer. They were able to use qualities such as x and y intercepts and concavity/extrema when constructing corresponding algebraic forms if they noticed the qualities. However, they could not embed graphical behaviors related to the concepts of limit and domain when constructing in algebraic forms even if they noticed them. The details follow.

For the concept of the y-intercepts, 5 teachers noticed it in Graph (a) and 3 of them used it to construct algebraic form with success. The other two mentioned that they needed to shift the graph of $y = x^2$ to have the y-intercept of -2 (which they thought was the value of the y coordinate of the y-intercept), but did not know how to modify the equation $y = x^2$ to have its graph translated by 2 to the negative direction of y. For the concept of the x-intercepts, 6 teachers successfully used it at least once to construct an algebraic form for Graphs (a), (b), (d), and (e).

Three used it only for the Graph (a), two used it for the Graphs (a) and (b), and one used it for Graphs (a), (b), and (e). Out of reach for most preservice teachers was the Factor theorem—for a polynomial $y = p(x)$, $p(c) = 0$ if, and only if, $x – c$ divides $p(x)$—in relation to the graphs. Although they knew that the x-intercept of $(c, 0)$ of the graph of $y = p(x)$ would satisfy $p(c) = 0$, they could not use the Factor theorem to write $p(x)$ as $(x – c)$ times something. Instead, they performed a trial and error method to construct a form that would yield $p(c) = 0$. For example, for the Graph (a), they acknowledged that (2, 0) and (-2, 0) were the x-intercepts. However instead of creating the quadratic form $(x - 2)(x + 2)$ by using the Factor theorem, they manipulated $x^2 + constant$ repeatedly until they came up with the form $y = x^2 – 4$, which yielded the two zeros, 2 and -2. Many of them showed the same trait for the Graphs (b) and (e), with some of them failing to provide a correct form by doing so.

Table 2 Characteristics or behaviors noticed or used at least once

	Noticed	Used With Success
The y-intercept	5	3
The x-intercept or zero	8	6
Vertical asymptotes	11	6
Horizontal asymptotes	3	0
Slant asymptotes	8	1
Hole (point discontinuity)	3	2
End behaviors, excluding horizontal and slant asymptotes	0	0
Concavity/maximum/minimum	5	5
Domain	2	1

Note: It is possible that an individual creates an incorrect equation even if she uses one or more characteristics in a correct way, if she fails to use other characteristics correctly.

Preservice teachers also struggled to notice the graphical behaviors related to the concept of limit—vertical/horizontal/slant asymptotes, holes, and end behaviors—and/or to use them to construct algebraic equations. For the vertical asymptotes, although 11 teachers noticed vertical asymptotes in at least one of the Graphs (c), (d), (e), and (f), only 6 of them were successfully able to use them at least once to construct an algebraic form. Their performance was also inconsistent, indicating that they did not have a firm understanding of the idea that a vertical asymptote of a graph was related to a zero of the denominator of $p(x)/q(x)$. Five of them successfully used the vertical asymptotes to construct an equation for the Graph (e), 3 for the Graph (d), and 1 for the Graph (f).

In the case of the horizontal asymptotes, only 3 of them noticed it in the Graph (c). However, even those three teachers did not use the information to construct an algebraic form. One teacher came up with the equation $y = 1/x$ by simply by using the vertical asymptote, without checking whether the equation had a horizontal asymptote of the x-axis; another teacher came up with the equation $y = \log(x)$; and the other teacher did not provide any algebraic form.

Three teachers noticed the characteristic of point discontinuities, with only one of them successfully using it to construct an algebraic form. This teacher did not mention vertical/horizontal asymptotes when he came up with his first equation $y = 1/x$ from his memory. He then noticed the hole and changed the equation to $y = (x - 1) / x(x - 1)$ by incorporating the hole at (1, 1). For the other two teachers, one came up with $y = 1/(x - 1)$ as the answer without noticing the hole and the other came up with none. As for the slant asymptote, although 8 teachers noticed it in Graph (f), only one teacher was able to use it to construct an equation. This one teacher came up with a correct equation $y = (x^2 + 1) / x$, but his understanding of asymptotes was somewhat shaky. He mentioned that a rational function, $p(x)/q(x)$, would have a horizontal asymptote of $x = 0$ if deg $(q(x))$ > deg $(p(x))$; a horizontal asymptote of x = the leading coefficient of $p(x)$/ the leading coefficient of $q(x)$, if deg $(q(x))$ = deg $(p(x))$; and a slant asymptote if deg $(p(x))$ = deg $(q(x)) + 1$. Yet he came up with the equation $y = x / (x^2 - 1)$ for the Graph (d) by focusing on the zero and the vertical asymptotes, without realizing that his equation, $y = x / (x^2 - 1)$, would yield a horizontal asymptote of $y = 0$, which the given graph did not have. He also said, "my problem is I don't know how to get a slant asymptote," after constructing a correct equation $y = (x^2 + 1) / x$ for the Graph (f). For the end behaviors of graphs, none of the teachers mentioned end behaviors of graphs that were not related to horizontal or slant asymptotes.

For the concavity/extrema/increase/decrease, of the 5 preservice teachers who noticed the characteristics, 4 used the information correctly to determine the degree of polynomial as 4. However, only 3 of them were able to use the information in conjunction with the zeros of the function to come up with a correct function equation.

For the concept of domain, preservice teachers in general did not pay attention to or use the concept of domain when constructing a function equation. For the Graph (d), for example, although many of them noticed that there were two asymptotes in the graph, only two of them specifically mentioned the graph was lying on the interval, (-1, 1). Further, those two teachers did not know how to use the domain information together with other characteristics of the function. One teacher came up with an algebraic equation of $y = x^2$ with $x \in (-1, 1)$ even after she mentioned the graph had two vertical asymptotes of $x = 1$ and $x = -1$, influenced by the look of the graph that was similar to a parabola. The other simply guessed that the equation $y = csc\ (x)$, would be an algebraic form for the graph by thinking that the interval of (-1, 1) was somehow related to trigonometric functions. Four preservice teachers, including one of the two above, came up with a variation of $y = x^2$ as an answer for the Graph (d). Three teachers came up with $y = x / (x^2 - 1)$ or $y = x^2 / (x^2 - 1)$ by reflecting the vertical asymptotes with or without considering the parabola figure in the graph.

Discussion and Conclusions

Prior research showed that many post-secondary students including secondary preservice teachers have misconceptions of functions and have a concept image of functions as continuous functions or associate them only with equations (Carlson, 1998; Even, 1998). However, there is little information about whether individuals indeed understand functions that are associated with equations. Here in this study, I attempted to answer part of this question and investigated preservice secondary teachers' abilities to transfer graphical to algebraic representations. In particular, I focused on what behaviors of function graphs they noticed and used when transferring graphs to equations.

The results showed that most teachers held only a few characteristics of functions in their concept image of function and had difficulty noticing and using them when transferring graphs to algebraic equations. Many preservice teachers noticed and used the concepts of x and y-intercepts and vertical asymptotes, but most of them could not make use of the Factor theorem efficiently to construct an equation. Although they knew that $f(a) = 0$ if $(a, 0)$ was an x-intercept of the graph of $y = f(x)$, they did not see $(x - c)$ as a divisor of $f(x)$ and write $f(x)$ as $(x - c)$ times something.

Preservice teachers also had problems in noticing and using other concepts. For the concept of domain, as with the beginning secondary teachers in Hitt's (1998) or Williams' study (1998) who did not hold the concept of domain in their concept map of functions, preservice teachers did not consider the domains of functions when constructing function equations. Even in the case in which they acknowledged the domain of a function, they did not know how to construct an equation with that domain. As with a college student in Aspinwall, Shaw, and Presmeg's study (1997) who had a conflicting understanding that second-degree polynomials had vertical asymptotes and at the same time the domain of all real numbers, one preservice teacher in this study also showed a discrepancy in dealing with a parabola-like graph. Although she noticed that the graph was defined on (-1, 1) with the vertical asymptotes of $x = 1$ and $x = -1$, she constructed the equation $y = x^2$ with $x \in (-1, 1)$.

For the concept of point discontinuities, Even's study (1998) showed an example of a teacher who could not tell whether a discontinuous graph with holes was a function or not. In this study, only three out of 14 teachers noticed a hole in a graph, and of those, only one of them was able to embed the information when constructing an equation. It was also noticeable that none of the teachers mentioned the end behaviors of function, which is part of the limit concept, and only one teacher who noticed a slant asymptote was able to construct an equation with the slant asymptote.

In light of the findings of this study, the teaching of critical concepts of functions might have to be handled with more care. In addition to the traditional, algebraic to graphical transfer activities that have been primarily used in classrooms, other activities that focus on the same concepts from different angles might have to be considered. The transformational activities such as those included in this study are an example. Instruction using graphic utilities such as Yerusalemy's (1997), which was effective for the understanding of vertical asymptotes of rational functions, is another. As shown in Lauten, Graham, and Ferrini-Mundy (1994), students show different understandings of the function and limit concepts in different representational contexts, and further, the use of technology does not guarantee students' understandings of the function characteristics. Only with well-designed activities that could offer students opportunities to explore critical ideas of the various function concepts is there a great possibility for students' better understanding of functions.

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Preservice Secondary Teachers' Abilities to Use Representations and Realistic Tasks

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This study reports three preservice secondary teachers' abilities and tendencies to use representations in problem solving as well as their abilities to use realistic tasks after taking mathematics content and methods courses that emphasized the roles of representations and realistic tasks. Qualitative analyses showed that the preservice teachers developed beliefs that representations and realistic tasks are important components of secondary education and used motivational tasks in their instruction. However, they used the tasks mainly as the application of learned facts rather than as the departure of students' construction of mathematical ideas. They also showed tendencies to use algebraic approaches in problem solving for grade 5-12 level tasks and had difficulties connecting algebraic and geometric representations when solving high school level algebra problems.

Keywords: Representations, Teacher Education, Mathematical Tasks, Problem Solving

Objectives

The importance of representations in mathematical teaching and learning has been emphasized in many studies (Brenner et al., 1997; Hiebert et al., 1997; Leinhardt, Zaslavsky & Stein, 1990; Thomson, 1994ab). Yet studies have suggested that many teachers have less than optimal understanding of the roles of representations and have difficulties using them in their instruction and in their own problem solving. Nevertheless, not much is known about how and in what ways teachers, especially secondary teachers, develop conceptions or knowledge of representations. This study addresses this under-documented area. It examines in what ways a series of mathematics and mathematics education courses that emphasized the roles of representations and tasks helped preservice secondary teachers to develop their mathematical knowledge for teaching. In particular, this study answers the following two research questions: (RQ 1) How were preservice secondary teachers' tendencies and abilities to solve problems using representations after taking courses that emphasized the roles of representations and tasks? (RQ 2) How were preservice secondary teachers' abilities to use tasks in instruction after taking courses that emphasized the roles of representations and tasks?

Framework and Literature Review

This study was conducted under the premises that knowledge and external representations are closely related and that mathematics is a human activity—"the activity of mathematicians that involves solving problems, looking for problems, and mathematizing subject matter" (Gravemeijer, 2008, p. 285). According to Hiebert and Carpenter (1992), "mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections" (p. 67). When internal representations are connected, the connections produce networks of knowledge, which are structured hierarchically or as a spider's web. These internal representations and their connections, however, cannot be observed. Under the assumption that there is a relationship between external and internal representations, learners' knowledge is often viewed through their abilities to use and connect external representations.

On the other hand, representations play a central role in the Realistic Mathematics Education (RME). In RME, mathematics is viewed as an organizing activity that makes sense of the world, and learners are viewed as mathematicians who act as the reinventors of mathematics. By constructing, connecting, and evolving representations while being actively engaged in "realistic" contexts, learners construct or reconstruct mathematical ideas (Gravemeijer, 2008). As such, realistic tasks play a critical role in RME. Realistic tasks in RME are different from traditional word problems in that tasks serve as the start of knowledge construction instead of as the application of learned facts at the end of instruction. Realistic tasks have representations built-in, are truly problematic so that students have motivations to solve them, and are open enough so students can take various paths to solving the problems (Fosnot & Jacob, 2010; Gravemeijer, 2008).

In this study, I use the lenses of representation and realistic task—the lens of representation for teachers' problem solving tendencies and abilities, and the lens of realistic task for task design and implementation—to examine teachers' knowledge. As the participants in this study took various courses, designed following some principles of RME, where representations and realistic tasks were particularly emphasized, these lenses served as appropriate analytic tools to look into their knowledge.

I here begin with a brief literature review that regards secondary teachers' tendencies and abilities to use representations in problem solving, as well as their conceptions about representations. The literature on teachers' conceptions of or tendencies and abilities to use realistic tasks that is consistent with the RME description is rare at the secondary level, and I include two studies that show the effects of RME with Dutch secondary teachers.

Teachers' Conception, Use, and Understanding of Representations

Studies have shown that many secondary teachers hold less than optimal conceptions or knowledge of representations in teaching and learning. They focus mostly on symbolic representations in teaching (Cunningham, 2005); they tend to use algebraic approaches in their own problem solving and favor students' algebraic approaches over other approaches (Nathan & Petrosino, 2003); and they view non-symbolic representations as informal objects, not necessarily as mathematics itself (Stylianou, 2010). In Moyer's study (2001), even after an intervention that emphasized the role of manipulatives and technology in instruction, many in-service middle grade teachers used manipulatives mainly in teacher-directed ways in their instruction and viewed the use of manipulatives as playing, exploring, or changing of pace, but not necessarily as making sense of mathematical concepts or ideas.

Many studies have also shown teachers' lack of abilities to connect representations in problem solving. Hitt (1998) showed that many beginning secondary teachers were unable to articulate whether a curve represented by an algebraic form—circle or ellipse—was a function or not. Even (1998) showed that secondary teachers have difficulties in determining the number of zeros of an algebraic equation, $y=ax^2+bx+c$, due to their inability to connect the equation to its graph. In Presmeg and Nenduradu (2005), a teacher used mostly tabular/numerical representations in solving problems. Even for a problem in which he used a graphical representation, he came up with a line graph in spite of his table showing a quadratic pattern.

Teachers' Conceptions of Realistic Tasks

In the Netherlands, the RME theory has been used in teacher education programs to prepare preservice teachers, and there have been some reports attesting to positive effects from the programs. According to Korthagen and Kessels (1999), a national evaluation study of all Dutch education programs showed that preservice secondary teachers who were trained within the RME framework had higher ratings than their counterparts. Wobbles, Korthagen, and Broekman (1997) also showed that a group of preservice teachers trained in RME not only was much more in favor of RME principles than was a random group of experienced inservice teachers, but also characterized their teacher education program favorably—as using mathematics in contexts, having an inquiry approach, using mathematics as an activity, and using different explanations. However, when asked about important characteristics of "high school" mathematics education, only a few of the 10 preservice teachers mentioned that using mathematics in "realistic" contexts or mathematics as an activity are important characteristics.

As evidenced by research above, there is much to be done about secondary teachers' conceptions of the roles of representations and realistic tasks and their abilities to use multiple representations in problem solving. Yet research documenting the kinds of interventions implemented and the effects of such interventions is very limited. This study concerns this under-documented area. The details of this study follow.

Methodology

Research Contexts

The subjects of this study are three preservice secondary mathematics teachers—Jennifer, Kristal, and Shane (pseudonyms)—who took four mathematics content and methods courses. As undergraduates, the teachers majored in mathematics with an emphasis in secondary teaching and minored in mathematics education. In their junior year, they participated in the mathematics content courses, Math A and B (pseudonyms). After completing their undergraduate degrees, they entered a teacher education program, offered at the same university, to obtain secondary teaching credentials in mathematics. During the teacher education program, they had two mathematics education courses, ED A and B (pseudonyms). The three participants were specifically chosen for this study because they were the only teachers in the secondary teaching cohort who had taken Math A and B and Ed A and B.

The Math A and B courses had two main goals: helping future teachers reconceptualize the key concepts of grade 5-12 mathematics through the analysis of student work and tasks, and providing them with opportunities to construct mathematical ideas themselves through the problem solving of college level mathematics. Throughout the courses, the roles of representations and tasks in teaching and learning were emphasized through reading, discussion, problem solving, and task analysis by using *Young Mathematicians at Work: Constructing Fractions Decimals and Percents* (Fosnot & Dolk, 2002), *Making Sense: Teaching and Learning Mathematics with Understanding* (Hiebert et al., 1997), and the *Contexts for Learning* series. College level tasks based on historical contexts, such as Euclid, Archimedes, and Omar Khayyam as well as the tasks based on literature, such as Thompson (1994a) and Carlson et. al (2004), were also used so that preservice teachers themselves would experience construction of mathematical ideas. Most of the tasks used in the courses were realistic and naturally embedded representations. It is noteworthy, however, that much of the materials used in the courses were either at the grade 5-8 level or at the college level, due to their focus on student mathematics thinking at the grade 5-12 level and a lack of professional materials at the high school level that followed principles similar to RME.

The two graduate mathematics educations courses, Ed A and B (offered in the fall and winter quarters, respectively) focused mainly on grade 5-12 mathematics. The main activities for the courses were problem solving and discussion of articles regarding many issues in mathematics teaching and leaning. The two major components of the courses were *Young mathematicians at work, Constructing algebra* (Fosnot & Jacob, 2010) and the *Cookie*s unit from a reform based high school curriculum, *Interactive Mathematics Program*. Much of the class hours were spent on solving algebra problems and discussing students' construction of ideas in algebra. As a final portfolio, they submitted 15-20 pages of writing on the *Cookies* unit in which they explained how the unit brought out mathematical ideas—such as equivalence, inequality, variation, the Cartesian connection, and optimization—in relation to representations—such as the number line model and the Cartesian coordinate plane.

Data and Analysis

Two different types of data were used for analysis: two individual interviews and the Performance Assessment for California Teachers (PACT) teaching events. The first interview lasted about an hour and was conducted after the participants finished the Ed A course. The second interview lasted about 2 hours and was conducted while they were taking the Ed B course. Interview questions included items that could measure participants' problem solving tendencies and abilities, their conceptions of representations, and their student teaching experience and PACT. Both interviews were recorded with a video camera and were transcribed. Participants' written responses on the interviews were also collected. The PACT teaching event was collected at the end of their teacher education program.

There were 7 interview questions that regarded the first research question—"How were preservice secondary teachers' tendencies and abilities to solve problems using representations after taking courses that emphasized the roles of representations and tasks?" The questions were similar or identical to items in the MKT survey (Hill et al., 2008) or tasks in Dufour-Janvier, Bednarz, and Belanger (1987), Even (1998), and Knuth (2000). For the grade 5-8 level, included were three problems related to a division of fractions, $2/3 \div 1/6$; a multiplication of decimals, $1.5 \times .7$; and an algebraic expression, $(x+2)(x+3)$. For the high school level, included were four problems related to the number of solutions of $x^4 = x + 2$; the number of solutions of $\sqrt{x} = x - 2$; the solutions of an equation, $?x + 3y = -2$, with a graph of the equation provided; and the signs of a, b, and c in $y = ax^2+bx+c$, with a graph of the function provided. For analysis, I coded the kinds of approaches the participants used in problem solving and the correctness of their work.

For the second research question—"How were preservice secondary teachers' abilities to use tasks in instruction after taking courses that emphasized the roles of representations and tasks?", PACT events as well as interview items that were relevant to task design and implementation were analyzed. Two interview items asked the participants to generate a task and to explain how they would implement the task for students who had little instruction on the mathematical topics of linear functions and of quadratic functions. Three interview items asked them to explain the roles of representations, the roles of realistic tasks, and their experiences in student teaching. The PACT event was a collection of instructional materials—video clips of teaching segments, students' work samples, daily reflections on instruction, and commentaries—from a week-long learning segment of instruction during their student teaching. As such, it showed participants' abilities to design and implement tasks in instruction. Among multiple tasks included in the PACT event, I selected major tasks for which the preservice teachers spent 30 minutes or longer of instructional time. Because an implementation of realistic tasks normally requires a considerable amount of time, those tasks had a better chance to show participants' abilities to use representations and realistic tasks. Jessica and Kristal had two tasks and Shane had one task that met the selection criteria.

For the analysis of data related to the second research question, I modified the Context Scale of the Assessment of Facilitation of Mathematizing (AFM; Fosnot, Dolk, Zolkower, & Seignoret, 2006). The AFM Context Scale was a measuring tool, examining teachers' abilities to use contexts in teaching on three levels based on teachers' use of representations and tasks. Rather than classifying the tasks into three levels as in the AFM Scale, I classified the participants' tasks into two levels. The discriminating factor between the two levels was how they used tasks in instruction: If a task was used as an application of learned facts, it was rated Level 1; if a task was used as a departure of students' construction of mathematical ideas, it was rated Level 2.

Results

Tendencies and Abilities to Use Representations in Problem Solving

Analysis showed that despite taking the courses that focused on multiple representations and the connections among representations, preservice teachers showed tendencies to use traditional, algebraic approaches in problem solving for both grade 5-8 and high school level problems when they were not prompted to use visual/geometric representations. In addition, when they were prompted to use visual/geometric representations, they were more successful on the grade 5-8 level problems than on the high school level problems.

As for their tendencies to use visual/graphical representations, for each of the three grade 5-8 level problems, only one participant used a visual representation to answer the question when not prompted to do so. Kristal used an area model to explain why $(x+2)(x+3)$ is equivalent to x^2+5x+6, and Jennifer and Shane used the FOIL method to explain it. The same trait was shown for the questions regarding the division of fractions, $2/3 \div 1/6$, and the multiplication of decimals, 1.5 x .7. Shane used number line models to explain both operations, but Jennifer and Kristal used the invert and multiply method for the fraction problem and the multiplication algorithm for the decimal problem. Their tendencies to use algebraic approaches on the three high school level algebra problems were higher than those on the grade 5-8 level problems. Except for the case where Shane used a graph to answer one of the three problems, all three participants used algebraic approaches to answer the high school level questions. All three used algebraic approaches and successfully found a solution of ?$x + 3y = -2$; Jessica and Shane used algebraic approaches and successfully determined the number of solutions of $\sqrt{x} = x - 2$; Jessica and Kristal used algebraic approaches, but failed to determine the number of solutions of the equation, $x^4=-x+2$.

Despite having tendencies to use traditional, algebraic approaches for the grade 5-8 level problems, when they were prompted to use visual/geometric representations, they came up with representations to explain the grade 5-8 level problems. All three participants used area models to explain 1.5 x .7 and $(x+2)(x+3) = x^2+5x+6$, and Kristal and Shane used a number line model and an area model to explain $2/3 \div 1/6$. However, they were less successful in solving high school level problems using geometric representations, even with prompts. Although all three participants provided a solution successfully by examining the graph of ?$x + 3y = -2$, when prompted, they could not determine the number of solutions of $\sqrt{x} = x - 2$ or of $x^4=-x+2$ by using graphs. A major obstacle for them was that they did not know where to start. Kristal and Jessica said that with no y present in the equations, they did not know what graphs had to be considered. Yet when they were told that graphs of $y=x^4$ and $y=-x+2$ could be considered, they sketched the graphs and explained using graphs that there were two real solutions of $x^4=-x+2$. In the question where they had to determine the signs of a, b, and c in $y = ax^2+bx+c$, using the given graph, none of them could determine the sign of b successfully. They could determine the sign of a from the concave down shape of the parabola graph, but they were unable to use the location of the vertex to determine the sign of b.

It was noteworthy that they were unable to associate graphs to the equations, $x^4=-x+2$ and $\sqrt{x} = x - 2$, after taking many courses that emphasized the connections among representations. It was also noteworthy that their positive conceptions of representations did not help them to solve the problems. They said in an interview: “Graphs are important since they are used in all sorts of things. But if you don’t understand what graphs mean, then the graph has no meanings and it can’t serve its purpose”; “a teacher’s knowing many ways of representing is important because students learn differently”; and “representing in many ways is important because there would be some students who might not understand in one representation.” However when they were given tasks in which they had to solve by connecting representations, they were unable to do so.

Knowledge of Tasks

Their abilities to design and use realistic tasks shown in the two interviews were quite satisfactory, with all of them rated Level 2. For the linear function topic, Jennifer used a money context that showed a constant rate of change of 5; Kristal used a paper-cutting context which led to the linear function $y=2x$; and Shane used a context involving two different constant rates of change of jumps. All of them also explained that they were using the contexts as the start of the construction of the ideas such as slope, rate of change, and the initial value, by connecting tables and graphs to the problem contexts. For the quadratic function topic, Jennifer used a square number sequence context of 1,4,9, … and emphasized that the differences in the steps formed a linear function; Kristal used $y=x^2$ by connecting it to the linear function $y=x$ and $y=x^2 + c$ graphs; and Shane used a tile context in which the nth step included n^2 number of tiles. All of them explained that they were using the tasks as a departure of a meaning-making process, focusing on the development of concepts such as the rate of change of a quadratic function as a linear function or its connection to linear function by connecting tables, graphs, and problem contexts.

The tasks in their PACT events were rated lower than those on the linear and quadratic function concepts. Jennifer had two word problem tasks on linear functions, rated Level 2, which were similar to the tasks she provided in an interview. Kristal had two hands-on activities on linear inequality, rated Level 1, which she used as a practice of learned rules. Shane had a hands-on task on parallelogram, rated Level 1, which was basically a parallelogram constructing activity using the conditions of parallelogram learned in previous classes. Jennifer was the only one who used tasks as the departure of meaning-making process, and Kristal and Shane used the tasks as applications of learned facts.

Some factors that contributed to the lower ratings on PACT tasks were their students’ academic levels (or their beliefs about how students with lower academic abilities should be taught) and their lack of understanding about realistic tasks. In Kristal’s case, for example, although she believed that students in general needed to be given opportunities to construct ideas themselves, she learned from her teaching that her low academic level students could not learn mathematics in that way. She thus adjusted her teaching style, following behaviorism principles, and used mostly skill-based or low-level tasks in her instruction. Another contributing factor was their lack of understanding of "realistic" tasks. All three described realistic tasks as "real-life," "hands on," "fun," "interesting," "something that made them engaged," "making them want to learn math," and "incorporate something hands-on." But none of them mentioned that the use of tasks as the starting point of constructing mathematical ideas was important in task implementation. As such, when they were asked in interviews to design tasks on the topics with which they were familiar through the intervention courses, such as linear function and quadratic function, they came up with tasks that were similar to the tasks used in the courses and explained in the ways that they would use the tasks as the departure of understanding. But when they were given a topic with which they were unfamiliar, such as parallelogram in Shane's case, they focused on the fun, hands-on, interesting, and real-life part of tasks rather than on the use of tasks as the departure of construction of ideas.

Discussion and Conclusions

This study examined three preservice secondary teachers' problem solving tendencies and abilities to use representations as well as their abilities to use tasks in instruction after taking a series of mathematics and mathematics education courses that focused on representations and realistic tasks. The results of this study are bi-fold. As for their abilities and tendencies to use representations, on the positive side, they were able to solve problems or to explain ideas using representations other than symbolic for the grade 5-8 level problems when they were prompted to do. On the negative side, however, they could not solve high school level problems by connecting algebraic and geometric representations as in the teachers in Gagatsis and Shiakalli (2004) and Presmeg and Nenduradu (2005), even when they were prompted to do so. Algebra problems requiring the connections between symbolic and graphical representations, such as those in Dufour-Janvier et al. (1987) and Even (1998), were very challenging to them. Furthermore, when they were not prompted to use visual/geometric representations or approaches, they still showed tendencies to use algebraic or traditional approaches as in teachers in Nathan and Petrosino (2003).

As for their use of tasks in instruction, on the positive side, they developed beliefs that realistic tasks or multiple representations are important parts of learning. Unlike the grade 5-8 teachers in Moyer (2002) and Stylianout (2010) or in Wobbles, Korthagen, and Broekman (1997), the preservice secondary teachers in this study believed that it was important to understand mathematical ideas in many representational contexts, that students' construction of ideas was an important characteristic of secondary mathematics, and that realistic tasks were important. They also designed many tasks that embedded multiple representations that could potentially help students construct critical ideas. Yet on the negative side, they understood realistic tasks as tasks that were related to real-life, fun, and interesting, but not as tasks that were used as the departure of construction of ideas.

This study suggests that preparing secondary teachers should be done with special care. Teachers need to be provided with opportunities to design or modify tasks using traditional mathematics curriculum or ill-designed tasks. As shown in this study, it is difficult for them to design and implement tasks that can lead to students' construction of ideas on unfamiliar topics, if they learn the importance of tasks only with well-designed tasks. By having opportunities to discuss deficits in ill-designed tasks and to modify the ill-designed tasks into well-designed realistic tasks, they might develop knowledge of tasks that can be transferred into their teaching in various situations.

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Preservice Elementary Teachers' Understandings of Greatest Common Factor Versus Least Common Multiple

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Little is known about preservice elementary teachers' understandings of greatest common factor (GCF) or how they relate to their understandings of least common multiple (LCM). As part of a larger case study in which an emergent perspective (Cobb & Yackel, 1996) was used to investigate preservice elementary teachers' understandings of topics in number theory, task-based interviews elicited participants' conceptions about modeling GCF and LCM using manipulatives, pictures, and story problems and the procedures for finding GCF and LCM using prime factorizations. Additional classroom data served to support findings. Participants held stronger understandings of modeling LCM than they did with modeling GCF. In contrast, participants' understandings of the procedure for finding GCF were far more robust than their understandings of how to find LCM.

Key Words: Number Theory, Preservice Elementary Teachers, Task-Based Interviews

Number theory content is integrated throughout elementary and middle school mathematics education in the United States, ranging from learning about evens and odds in early elementary school to greatest common factor (GCF), and least common multiple (LCM) in 6^{th} grade (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). However, research suggests that preservice elementary teachers may not have the understanding of number theory (e.g., Zazkis & Liljedahl, 2004) necessary to increase U.S. elementary and middle school student achievement, which is currently unsatisfactory (U.S. Department of Education, 2015). Zazkis and colleagues contributed the bulk of what is known about (Canadian) preservice elementary teachers' understandings of topics in number theory, such as even and odd numbers (Zazkis, 1998), multiplicative structure (Zazkis & Campbell, 1996), and prime numbers (Zazkis & Liljedahl, 2004). Many of Zazkis et al.'s participants exhibited procedural understandings of the content and difficulty working flexibly with various number concepts. For instance, Zazkis (1998) found that participants struggled to associate "evenness" with "divisibility by two" and the presence of a two in the prime factorization.

Brown, Thomas, and Tolias (2002) investigated preservice teachers' conceptions of LCM. They found that their participants used one or more of three approaches for finding the LCM of two numbers. Two of these approaches required a great deal of brute force. The third and most efficient approach made use of the prime factorization of the two numbers, identifying the highest power of each prime factor. Of the participants that attempted to use the prime factorization method, not all were successful. Successful participants did not adequately explain why this method works, with the exception of one student. For students to possess a conceptual understanding of LCM, the researchers suggested that they first need a flexible understanding of prime factorization and how it relates to factors, multiples, and divisibility. Brown, Thomas, and Tolias also suggested that students require a connected understanding of LCM across representations, including prime factorizations and story problems.

Greatest common factor and least common multiple are interrelated concepts, but little is known about preservice elementary teachers' understandings of GCF or how they relate to their understandings of LCM. This paper aims to illuminate this relationship using data from a qualitative case study that investigated the nature of preservice elementary teachers' understandings of topics in number theory. An emergent perspective (Cobb & Yackel, 1996) was used to analyze and interpret multiple forms of data, including classroom observational notes, student coursework, and responses from two sets of one-on-one task-based interviews.

Methods

The emergent perspective (Cobb & Yackel, 1996) served as the lens for collecting and analyzing data. Through their empirical investigations, Cobb and Yackel identified certain aspects of classroom microculture and their corresponding psychological constructs that all contribute to the construction of student understanding at the classroom level. The researchers did not mean for "the classroom level" to be a physical location; instead it refers to the types of activity in which students are engaged. Cobb and Yackel claimed that researchers can use this framework to explain the social influences on the individual's developmental understanding. The psychological perspective was the primary lens used in this study, because the bulk of the data represent individual conceptions. On the other hand, via the social lens, classroom norms, expectations, and experiences that framed participants' perspectives on mathematics teaching and learning were explored and used to inform individual conceptions.

This interpretive case study (Merriam, 1998) centered on preservice elementary teachers who were seeking a mathematics concentration and enrolled in a number theory course at a large doctoral-granting university in the western United States. This served as a bounded unit (Merriam, 1998). By seeking a mathematics concentration, participants could be certified to teach mathematics through 8^{th} grade. All participants were Caucasian and female, typical of the elementary education major. While data for this study came from multiple sources, one-on-one task-based interviews with six participants (Brit, Cara, Eden, Gwen, Isla, and Lucy) served as the focus of the data analysis. Open thematic and constant-comparative coding (Corbin & Strauss, 2008) were used as part of the coding process. Member and peer checking as well as data triangulation were used to ensure trustworthiness.

Results

The first three sections that follow outline specific tasks used to elicit participants' understandings of GCF and LCM, and they synthesize participants' responses. The fourth and final section informs participants' individual conceptions by describing the coursework experiences that may have influenced their responses.

Modeling Greatest Common Factor

During the one-on-one task-based interviews, the interviewer asked participants to create a GCF story problem that would require someone to compute the GCF of 28 and 32. All participants (with the exception of Eden) eventually attempted to create a GCF story problem. Aside from Brit, who began her response by attempting to create a GCF story problem, participants engaged in different activities to help them respond to this task. Eden and Isla verbally recalled the basic definition of GCF; others began by using numerical methods to find the GCF of 28 and 32. The interviewer prompted participants to model the GCF using pictures or manipulatives if they did already use these models to create a story problem.

Participants modeled GCF using two distinct types of representations, similar to the two types of meanings or models of division. Ball (1990) referred to these as the *measurement model* and the *partitive model* of division. However, participants were familiar with the phrasing used in Beckmann (2008) from previous courses, the "How many groups?" model and the "How many in each group?" model. Beckmann's language was adapted to account for the complexity of GCF structure. With "How many subgroups?" representations of the GCF, the groups of *A* and *B* objects are broken up into the same number of subgroups, and this number is maximized. In other words, to find the GCF of *A* and *B*, we find the largest number of subgroups that both *A* and *B* objects can be broken into. With "How many in each subgroup?" representations of the GCF, the groups of *A* and *B* objects are each broken into subgroups with an equal number of objects in each, and this number is maximized. In other words, to find the GCF of *A* and *B*, we find the largest number of objects in each subgroup of both *A* and *B*. Implicit in this language is that all of the objects are equally distributed amongst the subgroups without remainders.

Cara and Lucy modeled or described how to model the GCF to assist them in creating story problems. They both drew from the "How many subgroups?" representation of the GCF. After finding the GCF of 28 and 32, Cara described how to break up 28 and 32 objects to show the GCF is four.

Cara: So you would end up with four groups of a certain number in it. So for 28, you would have four groups of seven, and with 32 you would have four groups of eight. So the number in your groups would be different, but the amount of groups is the same, showing that that represents the [greatest] common divisor.

Here, Cara first found the GCF numerically and used that number to model the GCF, rather than using the model to *find* the GCF. Cara went on to create a "How many in each subgroup?" representation of the GCF, and she used it to assist in creating a second story problem. Cara compared her model using this representation to the one she described using the "How many subgroups?" meaning of GCF, again emphasizing how one of the factors of 28 and 32 was immaterial. Cara was the only participant to demonstrate flexibility with these meanings of GCF.

Cara: So if we wanted, we could do it the opposite way. Where the groups are even, or the amount in the groups are even, but then the amount of groups might not be even in this case. So you would put 4 in each group.

Gwen also used a "How many in each subgroup?" representation of the GCF to assist in creating a story problem. Brit and Isla attempted to create "How many in each subgroup?" story problems first, rather than start with a visual model. However, Isla felt her story problem needed further explanation, and she explained using a visual model.

In her attempts to model GCF, Eden demonstrated a conflated understanding of the two meanings of GCF. She wanted both the number of subgroups and the number of objects in each subgroup to be maximized and equal. She was the only participant to not produce a valid visual model of GCF. However, not all participants demonstrated a strong understanding of how to model GCF visually; only a few participants successfully explained how to use manipulatives or pictures to *find* the GCF of two numbers. Instead, participants appeared more comfortable calculating the GCF using numerical means and using it to model breaking up the two quantities.

In contrast to their relative success in modeling GCF using manipulatives and pictures, none of the participants created a valid GCF story problem. As the only participant to unsuccessfully model GCF visually, Eden did not attempt a story problem. The other participants' story problems had a variety of flaws. Cara and Gwen did not include a statement that required the reader to maximize the quantity in question, essentially asking for a common factor rather than the *greatest* common factor. Gwen, Isla, and Lucy did not successfully contextualize the GCF structure in their story problems; instead, they referred to "objects" and "things." Many of the participants also struggled to pose an appropriate question. For instance, Isla's question did not specify that each group of "things" needed to be subdivided equally.

Isla: You had 28 things and your friend had 32 things. How could you each group your things where you have the same number in your largest group, but you couldn't have remainders left over?

Brit's scenario was the most successful, but the wording of her story problem was not concise or clear; it required reorganization and clarification.

Brit: OK, so if I have 28 dinosaur stickers and 32 flower stickers and I want to group the dinosaur stickers and the flower stickers together… and I want to give them to individual students. So I want to know what is the greatest… how many, how many dinosaur stickers and flower stickers am I going to need in each group? I want to use all of them in an equal amount of groups. So I want to know how many stickers are going to be in each group.

Modeling Least Common Multiple

The interviewer also asked participants to create LCM story problems that would require someone to compute the LCM of six and eight. Unless participants also modeled LCM using manipulatives or pictures, the interviewer later prompted them to do so. Every participant successfully modeled LCM visually, using pictures, manipulatives, or both. And all six participants explained how to use their visual models to find the LCM of two numbers.

Participants were less successful at creating LCM story problems. The most common struggle with modeling LCM story problems was that most participants posed a question asking for *any* common multiple, rather than the *least* common multiple. However, participants were overall more successful in modeling LCM story problems than GCF story problems. Two participants began with visual models or the meaning of multiplication before suggesting an LCM story problem. For example, Brit started by modeling the LCM of six and eight using Cuisenaire rods. She determined that the dark green rod was six units long and the brown rod was eight units long. Then she created one-color trains using the rods. Brit presented her solution in terms of what she would have students do, even though the question was not phrased in that context.

Brit: What I would have kids do… is line up the blocks until they match, which turns out to be three brown ones and four green ones. Then they would do three times eight is 24 and six times eight is 24, would be the least common multiple.

Brit explained that it was a "really good way to visually see it… All you have to do is line up the blocks in a row and connect it back to the math." Brit used this representation to help her organize her story problem. While initially wordy, Brit produced a relatively successful LCM story problem.

Brit: So I have a nice society where my only money is \$6 coins and \$8 coins… I only have those two amounts. So I want to know what amount I can make using the \$6 coins, just the \$6 coins, can make the same amount as just the \$8 coins. And I want the least, because I don't want to carry around that many coins in my pocket at the same time. So I want the least amount that I can make with \$6 coins that is the same as what I can make with \$8 coins.

In contrast to Brit, Cara reminded herself that the LCM of six and eight was 24, which is four times six and three times eight, or four groups of six and three groups of eight, in order to create her story problem.

Cara: So if you have six chocolate chips in this cookie, how many cookies with eight chocolate chips would you need to have the same amount of chocolate chips… er, how many cookies of six do you need, or six chocolate chips, and how many cookies of eight to equal the same amount of chocolate chips?

Like Brit, Cara chose an appropriate context, and interpreting the problem in terms of the meaning of multiplication (i.e., $a \times b$ means a groups of b objects) appeared to help with her development of the problem. However, her question concerning the number of cookies was not asking for the LCM of six and eight. Rather, it asked the reader to find factors by which we would need to multiply six and eight in order to obtain a common multiple. Cara also did not qualify this multiple. She simply stated that the total number of chocolate chips needs to be the same, not necessarily the *least* number of chocolate chips.

Eden started the interview task by generating a story problem. "The microwave timer will go off every six minutes and the oven timer will go off every eight minutes. When will both timers go off at the same time?" While this was an appropriate context, it is necessary to have a statement comparing the two timelines; it is unclear when the microwave and the oven timers last went off at the same time. Additionally, Eden did not specify that we want the *next* time the two timers will go off, which would imply the *least* common multiple rather than any multiple. Isla and Lucy also posed story problems using the context of time, and they ran into similar issues with phrasing. Gwen generated a new type of story problem. She referred to a single quantity and described how to arrange the quantity into even groups of six and also rearrange it into even groups of eight. In other words, this quantity is both divisible by six and divisible by eight.

Using Prime Factorizations

The interviewer posed a series of tasks involving prime factorizations to each participant, including one modeled after a task from Zazkis and Campbell's (1996) study concerning preservice elementary teachers' understandings of multiplicative structure. Given the prime factorization of a number, $M = 3^3 \times 5^2 \times 7$, participants discussed whether M is divisible by two, seven, nine, 11, 14, 15, 26, and 63. Participants in this study accomplished this with ease, which differed from Zazkis and Campbell's participants, many of whom struggled. As a follow-up question, participants were presented with a second prime factorization, $N = 2 \times 3^2 \times 5^3 \times 13$, and asked to find the GCF and the LCM of M and N. All participants determined the GCF with ease and successfully reasoned through how to find the GCF (identify the common prime factors and the largest prime powers that they have in common, and multiply). Because participants gave well-reasoned accounts for why the procedure for finding the GCF using prime factorizations works, their explanations were coded as demonstrating conceptual reasoning.

Participants were less successful in identifying and reasoning about the LCM of M and N. Eden and Gwen immediately and accurately found the LCM of M and N, which was $2 \times 3^3 \times 5^3 \times 7 \times 13$. Eden was not sure why the procedure for finding the LCM works; "that's just what we do." Gwen recognized, however, that the LCM must have all of the prime factors of M and N in order for the LCM to be divisible by M and N. Gwen did not discuss the powers in the factorization.

Brit and Cara initially found incorrect values of the LCM, but could not reconcile their answers with what they knew to be true about the LCM. This led them to rethink how they found their LCMs and eventually find the correct value. Brit initially thought the LCM to be $2 \times 3 \times 5 \times 7 \times 13$, because it is "what's left over" once you divide *M* and *N* each by their GCF and multiply. However, Brit was convinced that the product of *M* and *N* was equal to the product of their LCM and GCF (a fact participants explored in their number theory class), which was not true for her LCM. "In this case, [*N*], [the LCM] has to include at least two 3s, at least three 5s, and a 13. And for here, [*M*], it includes the three 3s, at least two 5s, and a 7." This observation helped Brit find the correct LCM, but she was not confident in her answer. It was not until she multiplied *M* and *N*, then divided by the GCF, and found the same value for the LCM that she trusted her answer. Cara initially could not decide whether the LCM was $2 \times 7 \times 13$ or $2 \times 3 \times 5 \times 7 \times 13$. She eventually decided on the latter, reasoning that all of *M* and *N*'s prime factors need to be represented in the LCM of *M* and *N*. When she tried to explain her reasoning further, Cara brought up the example of 6 and 8, which caused a conflict because the product of their prime factors, 2×3, was not equal to their LCM, 24. By figuring out how to use the prime factorization of 6 and 8 to find their LCM, Cara was able to correctly find the LCM of *M* and *N*.

Neither Isla nor Lucy were successful in finding the correct LCM of *M* and *N*. Isla insisted that the LCM was $2 \times 3^2 \times 5^2 \times 7 \times 13$, because it accounts for all of the prime factors of *M* and *N*, and it takes the smallest of the exponents. Isla could not explain why that made sense. Lucy acknowledged that you multiply *M* by something to get the LCM of *M* and *N*, and you also multiply *N* by something to get the same number. When prompted to explain what that meant for the prime factorizations, she suggested that we take the prime factors that *M* and *N* have in common and multiply them, which would give her 15. This seemed completely unrelated to the valid conception she had only moments before. When Lucy was reminded of what she had said about having to multiply *M* by something, she immediately realized that 15 could not possibly be the LCM, but she could not think of a way to find it. In short, four participants eventually arrive at the correct LCM for *M* and *N*, but only Brit demonstrated a relatively complete understanding of why the procedure for finding LCM works.

Course Experiences

Observational notes and participant coursework revealed that participants had a number of experiences in their number theory course that may have contributed to their responses to the interview tasks. Data revealed that participants did not have an opportunity to create, or even answer, GCF story problems in their number theory class. They were, however, given the opportunity to briefly explore visual and story problem representations involving LCM. Participants were asked to create and analyze an LCM problem for elementary or middle school students to solve as part of a homework assignment. Some participants designed tasks that required students to use Cuisenaire rods to find the LCM of two numbers (a procedure introduced in their number theory course) while others wrote story problems involving LCM in other ways. Later in the course, participants also solved and created story problems related to the Chinese Remainder Theorem. Gwen's LCM story problem was phrased similarly to the Chinese Remainder Theorem problems that participants worked with in class.

Participants worked with prime factorizations quite a bit in their number theory class, and they consistently demonstrated a great deal of success with them. They found prime factorizations on homework assignments and tests in order to determine the GCF and the LCM of two numbers. The reasoning behind these procedures was discussed at length in class.

Participants also had to use the properties of exponents to fully factorize numbers such as $a = 10^8 \times 30^5$. Implicit in all of participants' work with prime factorizations was the idea that numbers are divisible by their prime factors and by the products formed by subsets their prime factors.

Discussion

Given the interrelated nature of GCF and LCM, one might expect preservice teachers' understandings of these topics to be similarly robust or weak, depending on the individual. However, data suggest this is not the case. Participants demonstrated significantly less difficulty when modeling LCM visually and with story problems than they did when modeling GCF. The opposite appeared to be true when participants attempted to find the LCM and the GCF of two numbers procedurally using prime factorizations. All six participants used and explained the GCF procedure with ease. Only two participants immediately and accurately found the LCM, but the other four participants had failed attempts at finding the LCM, with two eventually succeeding. Only one participant successfully explained the reasoning behind the procedure.

As Brown, Thomas, and Tolias (2002) suggested, a flexible understanding of prime factorization is necessary for a robust understanding of LCM. However, as the data in this report suggest, it is not sufficient. The reasoning that participants used for determining the GCF of two numbers given two prime factorizations was similar to the reasoning that they used to determine the divisibility of *M*, however. This may suggest why participants demonstrated a stronger understanding of GCF with prime factorizations than they did of LCM. Additionally, from a set theory perspective, the GCF of two numbers is obtained by finding the intersection of their sets of prime factors, whereas the LCM of two numbers is obtained by finding the union of their sets of prime factors. Finding the union of two sets is a more complex process than finding the intersection of two sets. Also, allowing for some duplicate prime factors but not others, especially in the case of finding the LCM of two numbers, further complicates the procedure. This could explain participants' struggle with the procedure for finding LCM.

In contrast, it is curious that participants demonstrated significantly more success when modeling the LCM of two numbers visually and with story problems than finding the LCM using prime factorizations. It is also interesting that the opposite was true of GCF. It is possible that participants' number theory course experiences enabled them to be more successful in modeling the LCM than modeling the GCF of two numbers, because they had done so in class and on assignments. However, participants also found the LCM and the GCF of two numbers using prime factorizations in class multiple times. And course experience did not necessarily lead to success in finding the LCM. The reasoning behind these procedures was also discussed in class, and participants only consistently recalled the reasoning for the procedure for finding the GCF.

It is important that preservice elementary teachers have robust understandings of the concepts that they could one day teach (e.g., Conference Board of Mathematical Sciences, 2012). Further investigation is warranted to determine the necessary interventions to ensure these future teachers have a flexible understanding of GCF and LCM, and their relationship. However, the additional success that participants demonstrated modeling LCM, as opposed to modeling GCF, does suggest that number theory courses designed for preservice elementary and middle school teachers provide more opportunities for students to model GCF using manipulatives, pictures, and story problems.

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Examining Lecturer's Questioning in Advanced Proof-Oriented Mathematics Classes

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There has been a substantial increase in mathematics education research in how proof-oriented university mathematics courses are traditionally taught. In this paper, we focus on the questions that lecturers pose to students. Specifically, we audio-recorded 11 proof-oriented mathematics lecturers and analyzed all of the questions they asked their students. We categorized each of the 1,031 questions according to a coding system we describe as well as identified wait time and subsequent speaker. We describe trends across all 11 lecturers, highlighting the limited opportunities students had to engage in important mathematical practices, and identify variances between how different lecturers used questions. We present qualitative data highlighting common and uncommon questioning techniques and conclude with a discussion of our results.

Key words: Questioning, Proof-oriented mathematics courses, mathematical responsibility

In recent years, there has been a substantial increase in mathematics education research in how proof-oriented university mathematics courses are traditionally taught (e.g., Fukawa-Connelly, 2012; Fukawa-Connelly & Newton, 2014; Lew et al., 2016; Mills, 2014; Pinto, 2013; Weber, 2004; Weinberg et al., 2016). Our work differs from the cited literature in two ways. First, previous research in this area has tended to focus on the information that mathematics professors convey to their students and how they choose to convey this information. In the current study, we focus on the questions that lecturers pose to their students. Second, previous investigations into mathematicians' pedagogy of advanced mathematics are largely comprised of case studies that describe a single mathematicians' pedagogy. In the current study, we examine the pedagogical practices of 11 mathematicians, from which we can analyze consistencies and differences across our sample. The goal of this paper is to make three points: (1) Although teacher questioning was common in our sample of lecturers, students' opportunities to participate in lectures were usually limited. (2) There was substantial variation in the types of questioning that mathematics lecturers used, which suggests that lecturing in advanced mathematics is not a uniform pedagogical practice. (3) Although it was not common in our data set, some lecturers used questioning to engage students in important mathematical practices.

Literature Review

Teacher questioning can serve many purposes. Teachers can use questions to elicit student thinking (e.g., Martino & Maher, 1999), set the stage for future investigation (e.g., Boaler & Brodie, 2004), generate discussion (e.g., Stein & Smith, 2011), focus students' attention (e.g., Lew et al., 2016), and promote mathematical discourse in general (e.g., White, 2016). We focus on one specific role -- inviting student participation (e.g., Martino & Maher, 1999; Mesa, 2010).

According to White (2016), mathematics education researchers examining questioning have aimed to characterize teacher questioning in three different ways: First, some researchers have investigated patterns of teacher questioning. A notable example of this type of research is

Wood's (1998) funneling and focusing patterns of questioning. Second, other researchers focused on the cognitive level of questions that were asked, such as how teachers' questions relate to Bloom's taxonomy (e.g., Emerson, 2011). In the third category of research, "question types have generally been provided in broad categories to elicit students' thoughts and participation" (White, 2016, p. 20). The current paper fits into this third category of research; we categorize teacher questions by what types of mathematical products they are designed to elicit.

From our perspective, one way that we can characterize the extent that instruction is student-centered is by identifying the extent to which the teacher and the student are responsible for generating the mathematical content in the course. Rasmussen and Marrongelle (2006) claim that one end of this continuum would be "pure discovery," in which students are responsible for generating all of the mathematical ideas. The other end of this continuum is "pure telling"—a lecture with no teacher-student interactions. A weakness of a classroom based on "pure telling" is that the instructor is responsible for supplying nearly all of the mathematics in the classroom, including how concepts are defined and represented, the procedures that are used, and whether a solution is correct or not. Consequently, students are denied the opportunity to engage in important mathematical practices such as defining, conjecturing, and representing (Weber et al., 2010). However, few lectures in advanced mathematics consist of "pure telling." Professors usually make *some* attempt to elicit ideas from students, and the most common means to do so is via teacher questioning. The goal of this paper is to understand what types of mathematics are solicited in the questions that professors ask and the extent to which students are given genuine opportunities to participate in the advanced mathematics lectures that they attend.

Methods

We first describe our participants, including how we recruited participants and the content of their courses. We then describe data collection methods. We conclude by describing how we coded the lessons.

Participants. We recruited participants by sending e-mails to every lecturer at three institutions teaching a proof-oriented course in advanced mathematics. We asked if we could observe, take notes on, and audio record one of their lectures. Lecturers were not told the purpose of the study. Eleven lecturers agreed with at least three instructors from each institution. The course content included Number Theory, Real Analysis, Linear Algebra, Abstract Algebra, Geometry, and Differential Geometry.

Data collection. For each lecturer, a member of the research team attended a class in which an exam was not given. The researcher audio-recorded the lecture while transcribing everything that the lecturer wrote on the blackboard into the researcher's notes using a LiveScribe pen (i.e., a pen that one can use to simultaneously audio record and take notes so that the timing of the notes is coordinated with the audio-recording).

Coding the lectures. Each lecture was transcribed. Then a member of the project team flagged every instance in which a lecturer asked a question or provided an invitation for students to participate. We first categorized the content of each question as either mathematical (What is a ring?) or non-mathematical (e.g., Do you have any questions about the syllabus? Do you know when your exam is?). Questions that were non-mathematical were excluded from subsequent analysis. In what follows, we discuss how we coded the remaining 1,031 questions. For each question, we listened to the audio-recording and recorded the number of seconds that passed before the lecturer or a student spoke. We also noted who spoke next, the lecturer or a student.

In the next phase of analysis, we sought to categorize questions by the types of mathematics elicited from the students. We formed categories of teacher questioning using a semi-open coding scheme, drawing on grounded theory in the style of Strauss and Corbin (1998). Our initial coding scheme was adapted from the framework that Fukawa-Connelly (2012) developed to account for lecturers modeling of mathematical behavior in advanced mathematics (which was informed specifically by the work of Selden and Selden (2009) and Alcock (2010)). The categories included the selection of proof frameworks (Selden & Selden, 2009), choosing a next step in an argument, justifying or warranting a step in a proof (e.g., Weber & Alcock, 2005), and exemplifying a concept or step in a proof (Alcock, 2010). However, not surprisingly, these initial categories did not capture many of the questions that we encountered. In many cases, these questions were easily captured by existing categories described in the literature, including checks for understanding (Mills, 2013), verbal tics (e.g., adding "right?" at the end of sentences) (Mesa, 2010), and eliciting facts (Viirman, 2015). When these occurred, we expanded our list of categories to include these categories. Some questions did not fit into our initial categories or the categories from the extant literature. When this occurred, we created new categories and descriptions to account for these questions.

With the final coding scheme, each lecture was assigned to one of the original four coders for final coding of the questions. The codes, descriptions of the codes, and examples of each code, are provided in the results section. Subsequently, a second coder re-coded a randomly chosen 20-minute segment of each 80-minute lecture (approximately 25% of the duration of the lecture), or 258 questions. The coders agreed on 236 of the codes (91.47% agreement, Cohen's Kappa = .90), representing a very high level of agreement.

Results

We first present our final codes, descriptions, and examples of each code. We then present data highlighting how these question types were used across the lectures and how there was variation in individual lecturers use of questions. We then present qualitative data highlighting how we coded the lectures and notable questioning techniques we observed.

Coding and Categories. Table 1 details the final coding scheme agreed upon by the team.

Table 1
Descriptions, and example questions by code.

Code	Description	Example Question(s) (Source)
Check for Understanding (C)	Questions or invitations that ask if students are okay, understand, or are with the lecturer.	"Alright, any questions about this calculation?" (L4)
Tic (T)	Statements formulated as a question that are habitual and frequently repeated throughout a lecture.	"No matter where I am on the cylinder I am going to fall on one of the [student: circles] circles and one of the lines, yeah?" (L11)
Fact, Computation, or Recall (F)	Questions or invitations that ask for specific pieces of information (e.g. definitions, computations, theorems, etc.).	"So what is *f* plus *g*?" (L5) "[…] what was the identity element in Z_n?" (L9)
Big Idea (B)	Questions or invitations that address broad concepts that are explored throughout or across lectures.	"[…] in fact, the question was […] how can we characterize what's coming here?" (L9)
Next Step (NS)	Questions or invitations that ask students to recommend a course of action that would continue the logical progression of a proof or example.	"(Student name)? What can we do with this information?" (L2) "Therefore, now what?" (L7)

Proof Framework (PF)	Questions or invitations that address higher level logical structures of a proof.	"What would I need to prove in order to make this relevant to the Euler Phi function?" (L6)
Warrant (W)	Questions or invitations that ask for a justification for a statement or claim.	"How do I know that (Student name)?" (L2)
Evaluate a Claim (E)	Questions or invitations that provide a truth-value for a statement.	"Does *x* minus two divide this function?" (L5)
Addressing a Convention or Notation (CV)	Questions or invitations that address a convention or notation	"How do you write the identity in cycle notation?" (L5)
Clarifying Student Responses (CS)	Questions or invitations that seek clarification of students' ideas or restate what students say	"You're asking […] once it stabilizes from one step to the next is it stable all the way?" (L4)
Other (O)	Questions or invitations that do not fit into the other categories	"Are ideals that contain 1 very interesting?" (L2)

Examining findings across all lectures. We coded 1031 questions providing an average of 93.7 questions per lecture. Collectively, Table 2 highlights the limited opportunities lecturers provided their students to engage in important mathematical practices. For example, lecturers most often provide students with opportunities to address factual questions (F, 26.4 per lecture) followed by next step questions (NS, 10.5 per lecture). Excluding checks for understanding and tics, lecturers asked all other question types an average of 22.6 times per lecture.

Because we do not consider Checks for Understanding or Tics to be genuine invitations, we omit these categories from further analysis. Without these two question categories there are a total of 654 questions or 59.5 questions per lecture. Table 2 also provides the number of questions per lecture within each wait time category by question type. This data highlights the limited time students had to hear, consider, and respond to a lecturer's question. For instance, students had less than 3 seconds to respond on over 80% of questions. Only 4.5 questions per lecture had a wait time of more than five seconds. Although in many cases a student responded to questions in less than 3 seconds, we note that less than 3 seconds is likely not enough time for all students to hear, consider, and respond to a question.

Table 2
Total number of questions (N) and wait time (0 s. - >5 s.) per lecture by Question Type.

Question Type	N	0 s.	1-2 s.	3-5 s.	>5 s.
Check for Understanding	10.5	--	--	--	--
Tic	23.8	--	--	--	--
Fact, Computation, or Recall	26.4	13.1	6.8	4.0	2.5
Big Idea	1.4	1.1	0.1	0.2	0.0
Next Step	10.5	4.8	3.8	1.4	0.5
Proof Framework	2.8	1.5	0.9	0.3	0.2
Warrant	5.6	2.8	1.7	0.5	0.6
Evaluate a Claim	3.1	2.0	0.6	0.4	0.1
Addressing a Convention or Notation	2.5	1.2	0.9	0.3	0.1
Clarifying Student Responses	3.2	2.6	0.5	0.0	0.1
Other	4.0	2.1	1.3	0.3	0.4
Total	93.8	31.2	16.6	7.4	4.5

Students responded to an average of 24.1 questions per lecture. Further, whereas Table 2 highlights that lecturers infrequently gave students opportunities to participate in important mathematical practices (e.g., providing a warrant), Table 3 shows that, when given the opportunity, students addressed most question types with about the same relative frequency. Hence, students were willing to engage in important mathematical practices during a lecture but were not given significant opportunities to do so.

We defined a question as a *genuine invitation to participate* if the question was not a Check for Understanding or Tic and either a student responded or the lecturer provided more than five seconds of wait time. On average there were 26.6 genuine invitations to participate per lecture. Of the questions that meet the genuine invitation criteria, factual questions and next step questions accounted for 16.3 questions per lecture. In contrast, there were relatively few genuine invitations to provide warrants (2.7 questions per lecture), give insight into a proof's framework (1.5 questions per lecture), or evaluate a claim (1.1 questions per lecture).

Table 3

Number of questions (N), student responses (S), teacher responses (T), and genuine invitations (GI) per lecture (and as a percent) by Question Type.

Question Type	N	S	(%)	T	(%)	GI
Fact, Computation, or Recall	26.4	10.5	(39.7)	15.9	(60.3)	12.0
Big Idea	1.4	0.00	(0.0)	1.4	(100.0)	0.0
Next Step	10.5	4.1	(38.8)	6.5	(61.2)	4.3
Proof Framework	2.8	1.4	(48.4)	1.5	(51.6)	1.5
Warrant	5.6	2.5	(43.5)	3.2	(56.5)	2.7
Evaluate a Claim	3.1	1.0	(32.4)	2.1	(67.6)	1.1
Addressing a Convention or Notation	2.5	1.0	(40.7)	1.5	(59.3)	1.0
Clarifying Student Responses	3.2	2.5	(77.1)	0.7	(22.9)	2.5
Other	4.0	1.3	(31.8)	2.7	(68.2)	1.5
Total	59.5	24.3	(40.5)	35.5	(59.5)	26.6

Identifying differences across lecturers. Although lecturers generally limited students' opportunities to engage in mathematical practices, there was variance in regards to how lecturers used questions. As a way to compare this variance, consider Table 4. We note that 25 of the 31 proof framework questions and 78 of the 117 next step questions came from two lecturers, L2 and L8, and that a majority of the lecturers provided students with little (3 or less questions) to no opportunities to engage in the practice of writing proofs. We also note the wide variance in the number of genuine invitations per lecture, from as few as 0 (L2) to as many as 66 (L8).

Table 4

Number of instances of each Question Type and Genuine Invitations (GI) by Lecturer

Lecturer	C	T	F	B	NS	PF	W	E	CV	CS	O	GI
L1	4	1	12	3	4	2	4	4	11	1	4	10
L2	1	28	21	1	23	12	10	1	1	3	12	54
L3	12	6	29	0	8	0	12	2	1	11	1	29
L4	7	0	41	0	8	0	1	1	3	1	1	43
L5	9	0	18	0	1	0	3	7	2	5	1	23
L6	3	0	7	0	3	3	5	0	0	2	0	12
L7	32	80	64	3	5	0	5	4	1	10	5	47
L8	28	56	81	4	55	13	19	9	7	2	14	66
L9	13	31	8	2	6	0	2	2	1	0	5	2
L10	1	35	0	2	0	1	1	0	0	0	0	0

L11	5	25	8	0	4	0	0	4	0	0	1	7

Examples of lecturers' questioning. In this section, we provide qualitative data for two purposes: (1) to highlight how we coded questions within a lecture and (2) exhibit examples of common and uncommon questioning techniques we observed. For each question, we provide the question code and wait time using the convention [Code; WT (Wait Time): Number of seconds]. We annotated the transcripts for readability without changing the overall flow of the interaction.

A common questioning phenomenon we observed involved a lecturer posing a question to students without providing a genuine invitation to participate. In these cases, we often observed that lecturers responded to the question on their own, asked a follow-up question, or moved on without addressing the question. As an example of this, we present an excerpt from L8 discussing that any ring element is in the ideal *I* when the element 1 is in the ideal.

[1] L8: All right, here's a first question, okay, if *R* is a ring with 1, say *R* is commutative with 1 and *I*, an ideal in *R*. Okay? [T; WT: 1]. Is 0 in the ideal? [NS; WT: 1]. Yeah, how do I know that [S1]? [W; WT: 0].
[2] S1: The zero set is an ideal.
[3] L8: Yeah, but how do I know it has to be in there to begin with? [W; WT:0]. The zero set is an ideal, zero absorbs things. In particular, an ideal is what? [F; WT: 1].
[4] S2: A subring.
[5] L8: A subring. It has to have zero, right? [T; WT: 0]. *I* has zero. Does 1 have to be in *I*? [F; WT: 2]. Not necessarily. What if 1 is in *I*? [F; WT: 5]. What can you say? [NS; WT: 1]. Well 1 is what? [F; WT: 0]. The multiplicative identity, so what does that mean [S3]? [F; WT: 1].
[6] S3: That *I* is a ring with identity?
[7] L8: Yeah, but what does it mean for 1 to be the identity? [F; WT: 0]. What's the property of the multiplicative identity? [F; WT: 1].
[8] S3: An element times that [the multiplicative identity] is itself.
[9] L8: An element times that is itself. So *r* times 1 is? [F; WT: 2]
[10] S3: *r*.
[11] L8: *r*, okay? [T; WT: 0] But if 1 is in *I*, where is *r* times 1? [F; WT: 1].
[12] S3: In *I* as well in *R*.
[13] L8: Yeah. So what can I say about *I*? [NS; WT: 1]. It's all of *R*. Do you guys see that? [C; WT: 0]. If 1 is in *I*, then *I* is actually equal to *R*.

L8 posed several questions in this excerpt that she then answered herself ([1], [3], [5], [13]) or followed with another question ([1], [5], [7], [11]). In each of these cases, L8 provided little or no wait time. Of the 18 questions present in the excerpt, only 6 questions were genuine invitations to participate. Additionally, this excerpt exemplifies the high number of factual questions lecturers ask; nine of the eighteen questions in this excerpt were coded as factual.

Although generally lecturers did not provide students with significant opportunities to participate, this was not true across all lecturers. For example, L2, a real analysis instructor, provided students many genuine opportunities to participate by calling on students in a particular sequence, without the students necessarily volunteering to answer questions. The following excerpt shows the lecturer using his questioning technique to engage students in a proof of the fact that multiplying a converging series by a constant generates a series that also converges.

[1] L2: [S1 name], you want to start, how do you prove this one [*writing "Proof of 1" on the board]*? [PF, WT: 1]
[2] S1: Start with what you know.
[3] L2: Always a good strategy. What do you know? [PF; WT: 1].
[4] S1: We know that uh, a_n converges to A.
[5] L2: [*Writes "Given that* $\{S_N\}$ *converges to A, where* $S_N = a_1 + \ldots + a_N$"] Okay, [S2 name]. That's what we know. [NS, WT: 1].
[6] S2: Uhh, cS_n equals the [inaudible].
[7] L2: [*L2 indicates there are too many "S" letters on the board and decides to change his notation from* S_N *to* T_N *and writes "Need to show that* $\{T_N\}$ *converges to cA, where* $T_N = (ca_1) + \ldots + (ca_N)$"] Okay, so. Write what you know. Write what you're going to show. Now [S3 name], I guess you're next. So this is what you know, this is what you're trying to show. What would you do next? [NS, WT: 1].
[8] S3: I'd grab a c from each of the terms.
[9] L2: [*Writes* "$T_N = cS_N$"] Like that? [CS, WT: 0].
[10] S3: Yeah.
[11] L2: So what is true then? [NS, WT: 1].
[12] S3: It converges.

Overall, we noted that L2's technique of sequentially calling on students provided students with many genuine invitations to participate (54 across the lecture). Whereas on average there were only 2.82 proof framework questions per lecture, L2 asked two such questions ([1], [3]) in this interaction alone. L2 also used a clarifying question [9] to verify that he accurately represented what the student intended to convey in his board work.

Discussion

In this study, we examined the questioning techniques of 11 lecturers in advanced mathematics courses at three institutions. Recall that we defined a *genuine invitation to participate* as questions that were not Tics or Checks for Understanding that either a student answered or the lecturer allowed greater than five seconds of wait time. This data highlights that generally invitations for students to participate in advanced mathematics classes are limited. Further, on average, the vast majority of invitations for students to participate require students to cite facts, perform computations, recall previously covered material, or provide the next step in proof or example. Lecturers infrequently used questions to prompt students to consider or discuss a proof's framework or provide warrants.

Although we draw some generalizations by looking across all lectures, we also highlight that there is variance in how individual lecturers use questions. This variance can be seen both with respect to the use of certain question types and to the number of genuine invitations lecturers provided. Hence, we must be careful when making generalizations about lecturers questioning activities as our data suggests that lecturing in advanced mathematics is not a uniform pedagogical practice.

We also note that in some instances, lecturers used questioning to engage students in important mathematical activities. As presented, L2 was unique in calling on students in some order to answer questions, which resulted in providing students many genuine invitations to participate. Such novel questioning techniques that elicited genuine invitations to participate also highlight the variance in lecturer questioning in advanced mathematics.

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Transitioning to Proof with Worked Examples

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In this study, I explored the use of a worked-examples-based proof-writing framework as a pedagogical tool to improve undergraduate students' ability to construct proofs. Over the course of three months, I ran a series of three workshops with five undergraduate students who had no prior experience with formal mathematical proof. In each workshop, participants worked through worksheets containing completed worked examples of mathematical proofs, followed by partially completed worked examples of proofs (to be completed by the participants), and, lastly, exercises. I collected and coded participants' written work and reflections and explored changes in student proof-writing across workshop sessions. In this paper, I describe themes across student work and provide qualitative data supporting the benefits of incorporating the use of such a worked-examples-based proof-writing framework when introducing students to mathematical proof.

Key words: proof, worked examples, proof-writing framework

This study is motivated by the ongoing difficulties that students have in learning mathematical proof. As Moore (1994) points out, the transition from computational/procedural courses, such as calculus and differential equations, to proof-based courses is particularly difficult for undergraduate students. These difficulties are myriad and complex, and based on the experiences of both students and faculty, the traditional lecture-based classroom falls short of helping students overcome these difficulties. Fortunately, there is a growing body of knowledge on alternative methods of instruction such as guided reinvention, flipped classrooms, and worked examples. The purpose of this study is to explore the use of a worked-examples-based proof-writing framework as a pedagogical tool, implemented in a series of introductory workshops on mathematical proof-writing, to improve students' ability to construct proofs.

Literature Review

The instructional practice at the center of this research is that of *worked examples*. In mathematics education, the word "example" has several interpretations. The word example can refer to a particular instance of a concept (e.g. 7 is an example of a prime number). Alternatively, the word example can refer to the demonstration of technique as the written solution to a particular problem/exercise, a worked-out example – or simply *worked example*; worked examples are typical of undergraduate mathematics textbooks (Weber, Porter, & Housman, 2008). Lithner (2003) found that in procedure-oriented courses students "almost always" make use of worked examples in completing their homework and that this approach was used by students of varying mathematical ability. Other recent studies suggest that undergraduate students in proof-based courses might use worked examples to inform their construction of proofs (Weber, 2004). While Lithner (2003) expressed concern that the use of worked examples leads to students completing homework assignments without developing conceptual understanding, cognitive psychologists have emphasized the merits of having students use worked examples in problem solving (e.g., Zhu & Simon, 1987; Atkinson, Derry, Rankl, & Worthman, 2000).

As pedagogy, the *worked examples* model has, in recent years, received considerable attention from researchers and educators, especially in mathematics, physics, and computer science. While the defining features of the *worked examples* pedagogical model are subject to debate, the common thread is the goal of providing the novice (student) with an expert's problem solving framework, which is to be emulated in the interest of gaining conceptual understanding (Atkinson, Derry, Renkl, & Wortham, 2000). Three factors moderate the effectiveness of teaching by worked examples (Atkinson et al., 2000):

- Intra-example features – e.g. use of multiple modalities, clarity of substructure goals, completeness/incompleteness
- Inter-example features – e.g. example's proximity to matched problems, multiple examples per problem type
- Students' individual differences in processing examples – e.g. social incentives, self-explanation

In this study, I made use of findings in the *worked examples* literature to develop a proof-writing framework as well as a methodology for using that framework in a series of introductory workshops on mathematical proof-writing. In doing so, I took into consideration the inter- and intra-example features of successful *worked-examples-based* pedagogical models to create worksheets that deconstructed the proof-writing process according to that framework. This framework - that is, the specific way in which proof-writing is broken down into a series of subgoals - will be referred to as a *worked-examples-based proof-writing framework*.

Methods

Participants. The participants for this study were five second-year undergraduate students majoring in engineering, all with the very similar mathematical backgrounds. All of them had completed the first-year calculus sequence and were enrolled in an ordinary differential equations course at the time of the workshops. None of these students reported having any experience with mathematical proof, beyond high school geometry.

Materials. The theorems and exercises used in the workshops were drawn from introductory texts on mathematical proof-writing to reflect the material one would typically encounter in a transition to proofs course. For each session, worksheets were prepared which contained introductory information, such as definitions and descriptions of proofs types, fully worked-out examples, partially worked-out examples, and exercises.

Structure of Worked Examples. The structure of the worked examples was intended to decrease the cognitive load of the proof-writing process by encouraging students to deal with the various aspects of proof, as defined by Selden and Selden (2009), separately. These aspects of proof form the "conceptually meaningful chunks" or subgoals of the worked examples. The structure outline is as follows:

1. Proof construction
    a. Breakdown of problem/theorem statement into hypothesis and conclusion
    b. Brainstorming
        i. Identifying relevant and related definitions, axioms, and theorems
        ii. Identifying the end goal by considering the next-to-last step of the proof
    c. Translating the problem/theorem statement into hypothesis and conclusion into appropriate mathematical terminology and notation
    d. Application of definitions, axioms, and theorems to hypothesis

   e. Experimentation (attempts to connect definitions, axioms, and theorems logically or algebraically)
2. Formal proof write-up/presentation
3. Reflection (Only for Sessions 2 & 3)

As Catrambone (1994, 1996) argues, explicitly labeling subgoals improves the effectiveness of the worked examples. Furthermore, the subgoals of part (1) – the proof construction - are expected to elicit directed self-explanation from students. Part (1) of the worked examples corresponds primarily to *Hierarchical Structure, Construction Path*, and the *Problem-Centered* part of the proof, while part (2) – the formal write-up of the proof - corresponds primarily to the *Formal-Rhetorical* and *Proof Framework* parts of the proof.

Hierarchical Structure refers to "knowing what the proof has to accomplish and coordinating any sub-proofs or constructions" (Fukawa-Connelly, 2012, p. 328). By rewriting the problem/theorem statement in if-then form (assuming the original statement has not been presented in this way), the worked example identifies what the proof has to accomplish (conclusion) and the conditions under which this must be accomplished (hypothesis). For example, in Worked Example 1, the statement "Prove that the sum of two odd integers is even" is rewritten as "*If* two integers are odd, *then* their sum is even."

The brainstorming part of the worked examples addressed the *Hierarchical Structure* and the *Problem-Centered* part of the proof. Here, the goal of the proof is further clarified, key ideas are determined, and "the 'right' resources" are brought to mind. In Worked Example 1, the definitions of an odd integer and an even integer are called upon. The next-to-last step (x + y = 2N) is also written, further clarifying the goal of the proof.

In translating the proof statement into appropriate mathematical notation and terminology, the worked example introduces the *Formal-Rhetorical* part of the proof. This requires "primarily *behavioral knowledge* to complete" (Fukawa-Connelly, 2012). In worked example 1, the statement "*If* two integers are odd, *then* their sum is even" is rewritten as

$$If\ x, y\ \in \mathbb{Z}, then\ x + y\ is\ even.$$

The Application of definitions, axioms, and theorems to the hypothesis addresses the *Hierarchical Structure* and *Formal-Rhetorical* parts by "coordinating any sub-proofs or constructions" and "coordinating aspects of the proof"(Fukawa-Connelly, 2005).

The experiment phase of the worked example addresses the *Construction Path* and *Problem-Centered* parts of the proof. Here, a viable construction path is sought by exploring logical and algebraic connections between all aforementioned definitions, axioms, and theorems. During the presentation of this portion of the worked examples, I encouraged students to practice self-explanation.

Finally, having established a viable construction path, the formal write-up/presentation of the proof is focused on *Formal-Rhetorical, Construction Path,* and *Proof Framework* parts of the proof.

Data Collection & Analysis. In an interpretive qualitative study such as this, data is typically collected through interviews, observations, and document analysis (Merriam, 2002). In place of conducting individual interviews with participants, in Sessions 2 and 3 students were prompted to write written reflections for each problem. I coded their work and reflections for emergent themes, first on a granular level, then grouping those codes according to the five aspects of proof. Data collection methods are summarized in the following table:

Data Collection Method	Description
Written Reflections	For the Partial Examples in Sessions 2 & 3 students were asked to provide written responses to the following questions: 1. Why did you choose to prove this statement via direct proof/contraposition/contradiction? 2. What parallels can you draw between this proof and the examples above? 3. Which parts of our proof framework were most useful in working on this proof? 4. If you were unable to complete this proof, what gave you difficulty?
Observations	Each seminar session was audio recorded. Students were observed as they worked during each seminar.
Document Analysis	All student written work and reflections were collected and scanned at the end of each session.

Results

There were several notable themes in the ways in which students' proof writing changed over the course of the three workshop sessions. The most notable changes took place during the first session. Student work varied significantly on the Pretest, but, with only a small number of exceptions, no valid formal mathematical proofs were submitted by any of the students.

However, following the introduction of the work examples, all of the students were able to write at least one complete and valid proof on the Session 1 worksheet, with most doing so for more than one example/exercise. In the following table we see that the number of correct and mostly correct proofs increased significantly between the Pretest (PT) and Session 1, and that the number of incorrect proofs remained relatively low over the next two sessions. Here, "Mostly Correct" refers to proofs with minor errors or omissions, such as leaving variables undeclared and minor algebraic or arithmetic errors. Furthermore, while most of the proofs submitted on the Pretest were empirical or colloquial in nature, all of the proofs (or proof attempts) submitted after the introduction of the worked examples were deductive.

Session	Correct	Mostly Correct	Incorrect	Incomplete	Empirical or Colloquial	Deductive Proof	Out of
PT	1	1	20	2	17	7	24
1	9	9	4	3	0	25	25
2	5	9	2	2	0	18	18
3	6	6	7	0	0	19	19

Below are Rasa's and Paul's solutions for the same exercises on the Pretest and the Session 1 worksheet. The task in this case was to prove that for integers $a,b,$ and $c,$ if $a|b$ and $b|c$, then $a|(b+c)$.

	Pretest	Session 1
Rasa	If b is divisible by a and c is divisible by a, then b+c is divisible by a, because it is the same as adding the sum of the two separate fractions, since when adding two fractions with the same denominator, the numerators are added and the denominator stays the same.	If $a \mid b$ and $a \mid c$, then $a \mid (b+c)$ Let $b = m \cdot a$ and $c = n \cdot a$ $b+c = (a \cdot m) + (a \cdot n)$ $b+c = a(m+n)$ Let $N = m+n$ By definition since $b+c = aN$, then a is a multiple of b+c
Paul	$bn = a$ $cn = a$ n is any integer $b+c = a$ $bn_1 = cn_2 = (b+c)n_3$ $b = c = b+c$ $\frac{b}{n_2 n_3} = \frac{c}{n_1 n_3} = \frac{b+c}{n_1 n_2}$ n is a different integer for a, b, and b+c due to its difference in values for when a is an integer and the values $a \mid x$ where x is different for any value a is given	Let $a, b, c \in \mathbb{Z}$ an integer Then $\exists\, n, m \in \mathbb{Z}$ such that $an = b$ & $am = c$, $an + am = b+c$ • $a(n+m) = b+c$ where $n+m$ is P $aP = b+c$ Let $P \in \mathbb{Z}$ an integer such that $P = b+a$ $\therefore a \mid b+c$

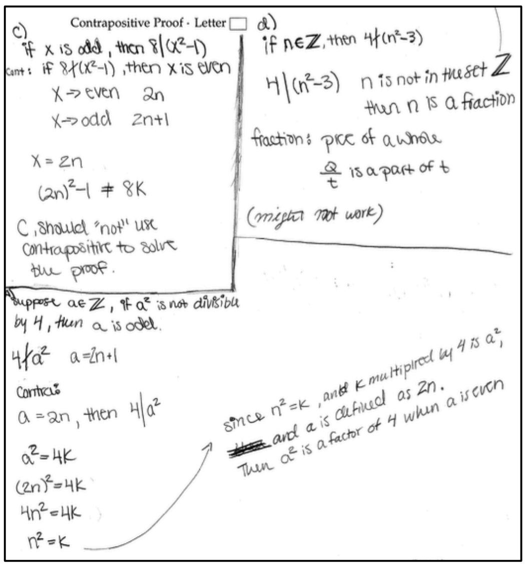
Rasa's proof on the pretest was primarily colloquial in nature, but her proof on the Session 1 packet was almost entirely correct, with the exception of undeclared variables, *m* and *n*. Modeling the formal write-ups in the worked examples, her proof followed mathematical convections such as use of notation, separating and labeling algebraic work, and writing in complete sentences. Paul, who attempted to operationalize the definition of divides on the Pretest, also wrote a correct proof on the Session 1 worksheet. Like Rasa, he wrote in complete sentences, indented his algebraic work, and clearly stated his conclusion.

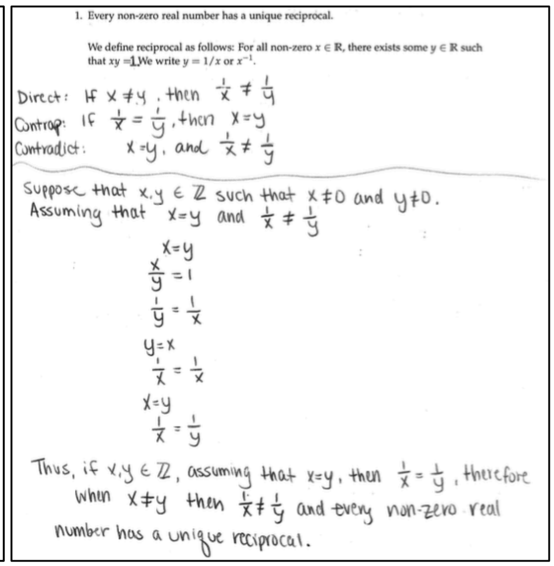
In addition to correctness, on a more granular level, the formal-rhetorical features of students' proofs continued to evolve over the course of the three sessions. In particular, all of the students included conventional notation and language, and declared all primary and secondary variables in the partial examples in Session 1. Here, primary variables refer to those which are used to represent quantities referenced in the hypothesis/assumptions of a given statement to be proven, and secondary variables refer to those used to operationalize a definition. In most cases students became less consistent with declaring secondary variables in the later sessions. The following table compares the total number of variables declared by all students in each session with the minimum number of variables required to construct correct proofs on those exercises. To establish these minima, I wrote out proofs for each example and exercise and had several mathematicians look over that work to determine if I had used the minimum number of variables possible to construct valid proof.

Example/Exercise	Primary Variables Total / Minimum	Secondary Variables Total / Minimum
Pretest Summary	27 / 40 = 67.5%	10 / 45 = 22.2%
Session 1 Summary	33 / 37 = 89.2%	33 / 45 = 73.3%
Session 2 Summary	17 / 20 = 85.0%	22 / 38 = 57.9%
Session 3 Summary	32 / 38 = 84.2%	25 / 48 = 52.1%

On average, students made and maintained significant progress after the Pretest in declaring primary variables, from 67.5% declared on the Pretest to 89.2% on the Session 1 worksheet and then maintaining roughly 85% for the last two sessions. However, in declaring secondary variables, there was a significant increase from the Pretest to Session 1, but then a decline in the final two sessions.

Finally, by Session 3, most students had begun to demonstrate a strategic approach to proof-writing, engaging in brainstorming and experimenting prior to attempting to write formal proofs. Consider, for example, Heather's work in Session 2 (left) and Rasa's work from Session 3 (right):





While Heather was unable to write an entirely correct proof for any of the exercises in this session, she did experiment with several statements to determine which might be most easily solved as either direct proof or proof by the contrapositive. In doing so, she started each proof and made notes regarding what "might work" and whether to use contraposition or not. For 6a, she did in fact write a mostly correct proof, although once again, variables were not properly declared. Rasa began her work on Exercise 1 in Session 3 by rewriting the given claim according to the three proof methods covered in these workshops, prior to selecting proof-by-contradiction and attempting to construct a proof. The work of both students here indicates planning as a separate activity from the formal writing-up of a proof.

In their written reflections, students most frequently identified brainstorming and experimenting as being the most valuable features of the worked-examples-based proof-writing framework. The following examples highlight some of their reasons for saying so:

Student	**Response**
Nadia	Definitely brainstorming is very important to understand what we are trying to proof but experimenting was important to fully understand the process.
Heather	There is NO way to know if the statements work w/ out experimentation.
Rasa	Experimenting and thinking which process would work best beforehand helped the most.

Paul	Experimenting and brainstorming are the easiest, without a plan to do the proof you cannot experiment to find answers and do trial and error methods.

Overall, these responses suggest that students found these steps useful because they allowed them to make sense of what the proof required prior to attempting a formal write-up.

Discussion

Most undergraduate mathematics students in the United States typically encounter formal mathematical proof-writing for the first time in a transition (or introduction) to proofs course. Such courses tend to following computation-centered courses, such as the typical freshman level calculus courses, and typically span only one semester. Because formal mathematical proof plays such a central role in advanced mathematics courses, it is important that we take full advantage of these transition courses to equip students with tools and experiences that will allow them to succeed in later proof-based courses. In this study, I aimed to explore the ways in which a worked-examples-based proof-writing framework could support students during this transitionary period.

This study was conducted independently of any transition to proofs course in order to focus on the ways in which this worked-examples-based proof-writing framework would affect novice students' proof-writing. However, the implications of this study for instruction lie primarily in the potential use of this type of framework as a tool to supplement instruction in an introductory course to proof. Worked examples are by their nature supplementary and of primary value to the novice.

Lithner (2003, 2004) expressed concern that in relying too heavily on worked examples to solve problems, students sacrifice opportunities to build on their conceptual knowledge and problem-solving strategies. For this reason, this study treats worked examples as a means to "provide an expert's problem-solving model for the learner to study and emulate" (Atkinson et al., 2000, pp. 181-182) and as a means to prompt guided exploration. As has been found in much of the worked examples literature, without examples to study, students may develop, and over time reinforce, novice strategies that ignore deeper structures in problem-solving activities (Weber et al., 2008).

The keys findings of this study suggest that a worked-examples based framework can both help novice students develop some of the basic proficiencies necessary for constructing and formally writing up mathematical proofs, and over time, facilitate productive habits, such as brainstorming and experimenting prior to attempting a proof write-up. These basic proficiencies include declaring variables, operationalizing definitions, using conventional mathematical language and notation, and structuring proofs in an organized, readable way. By encouraging brainstorming and experimenting, using this framework may help students develop some of the strategic knowledge required for mathematical proof-writing.

Regarding implementation, more research (see future research recommendations below) may be required to determine how one might best incorporate this framework into a standard transition to proofs course. However, given that the students who participated in this study received little instruction, and instead learned to construct proofs primarily by modeling worked examples, it may be that worksheets such as those used in this study could be used as supplementary materials in an introductory course on proof. Of course, they could be tailored to suit the specific content of the course and the needs of the instructor and students. The advantage to this type of purely supplemental use of the proof-writing framework is that little or no change is required of the instructor in terms of preparation or preferred teaching style.

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Approaches to the Derivative in Korean and the U.S. Calculus Classrooms

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This study explored how one Korean and one U.S. calculus class defined the word "derivative" as a point-specific object through the limit process on the difference quotient, and as a function on its domain. The analysis using Commognitive approach showed that both class used similar visual mediators for the limit process/object, but addressed different components of the definitions; Discussion of the derivative as a function before it was defined were frequently found in the U.S. class but rarely found in the Korean class; Words for the derivative at a point, and words for the derivative as a function explicitly differed in the Korean class compared to the U.S. class; and the derivative was first defined as a function through correspondence between x-value and the derivative value in Korean class, but through expansion of x values from a number to variable and corresponding changes in the U.S. class.

Keywords: commognition, calculus, derivative, language, teaching.

Introduction

Calculus is considered as a first college level mathematics course that students encounter in the United States (U.S.). However, they often learn Calculus in high school as a form of Advanced Placement (AP) or a regular course. Similarly, in South Korea, most high school students who are on the college track learn basic calculus concepts because they are included in the national curriculum. This study looks at calculus lessons about the derivative from South Korea and the U.S. to explore how language specific terms for "the derivative at a point," and "the derivative of a function" are discussed in their lessons. In contrast to the English terms, both of which include "derivative," the corresponding terms in Korean, "Mi-bun-Gye-Sum" (translated to "differential coefficient") and "Do-ham-su" (translated to "leading function"), do not include a common term. Specifically, this study addresses the following research question:

> How is the derivative realized as an object at a point and an object on an interval in one high school class in South Korea, and one AP calculus class in the United States?

This study adopted Sfard's (2008) commognitive approach as an analytical tool. The purpose for the analysis was not to compare the way the derivative is taught in two countries, but to apply the same analytical lens in mathematical discourse involving two languages, focusing on how the two terms were addressed as a number or a function in each language.

Theoretical Background

Various studies considered languages as a factor for students' mathematical thinking. Some studies explored the relationship between specific language features and students' performance in mathematics in Chinese (e.g., Wang & Lin, 2005), Korean (e.g., Kim, Ferrini-Mundy, & Sfard, 2012), and Irish (e.g., Ni Riordain, 2013). Others considered cultural differences including languages in teaching and learning such as indigenous non-native English speaking students' learning, whose teachers were native English speakers (e.g., Favilli, Maffei, & Peroni, 2013; Russell & Chernoff, 2013); they revealed that the language difference was a main factor of teachers' decision on the content and the level of difficulty for the class, and teachers' knowledge about their students' native language was a key component of the teacher's knowledge for teaching mathematics. The current study also explores teaching mathematics in different languages, but also takes other means of communication into consideration such as use of visuals while exploring discussions of "derivative" in Korean and English by adopting the *commognitive* approach (Sfard, 2008) as a discourse analysis framework. It combines cognition and communication, and explains mathematical thinking through one's discourse through the four characteristics: *word use*, *visual mediators*, *routines*, and *endorsed narratives* (Table 1).

Table 1.

Features of Mathematical Discourse in Commognitive Approach

Feature	Descriptions	Further Descriptions
Word use	Use of words signifying mathematical objects	Different speakers can use a word differently. It is an "all-important matter" because "it is responsible to a great extent for how the user sees the world" (Sfard, 2008, p. x).
Visual Mediators	Non-verbal means of communication	Because ways people attend to visuals depend on contexts, mediators need to be viewed as part of the thinking process, not auxiliary means of pre-existing thought.
Routines	Well-defined repetitive patterns	Patterns can be found in speakers' use of words and visuals, or in the process of creating and endorsing narratives.
Endorsed Narratives	Utterances that speakers endorse as true	Students' endorsed narratives are often different from what the professional mathematics community endorses as true (e.g., "Multiplication makes bigger" changes to "Multiplication can make smaller.").

Note. Adopted from Park (2015, p. 234)

The first two characteristics are also considered as realizations. Sfard (2008) defined realizations of a signifier (either a word or visual mediator) as perceptually tangible entities that share Endorsed narratives, and mathematical object as a collection of these realizations. For example, a word "function" can be realized with a word, "mapping" or "relation," graph, equation, or gesture for a curve or straight line. With these four discursive characteristics, the development of various mathematical objects has been examined (e.g., Kim et al., 2012; Sfard, 2008). The current study addresses the development of "the derivative" through its realizations – words and visual mediators – in discourse from a Korean and an American calculus class.

The derivative, which is commonly mediated with symbols, is realized as a number (e.g., $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$) and as a function (e.g., $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$). The realization of "the derivative" includes several process and object transitions. First, for "the derivative at a point," the difference quotient (DQ) is considered as an initial object, and then the process of the limit over smaller and smaller intervals is applied, and then the final object from this process is be the derivative at a point. The limit process is often mediated with graphs of multiple secant lines; symbols for the limit; words such as "as *h* approaches 0, the DQ approaches..."; numbers for DQ. Through this process, the "derivative" is objectified as a number. Then, for "the derivative function," the derivative at a point is considered as an initial object, and the derivative process of finding the derivative at every point is applied. This process can be mediated with several tangent lines; several dots on an x-y plane; symbols including different letters (e.g., *x*); multiple numbers for the derivative. The derivative of a function would be objectified from this process.

Method

The purpose of this study is to explore mathematical discourse about the derivative from one Korean and one American calculus classroom. To this end, lessons for the derivative were videotaped 7 times for the U.S. classroom and 10 times for the Korean classroom when the teachers started the derivative unit. The video camera was located in the back of the classroom to minimize the interruption, and field notes were taken. The two instructors also participated in a 30-minute interview about what they believe as important to teach in their class. The interviews were also videotaped and used as a complementary data for the classroom data.

Participants

One Korean high school teacher, Mrs. Kim, and one U.S. AP calculus teacher, Mr. William (pseudonyms) (Table 2) were recruited via email sent to the group of teachers recommended by mathematics education professors at the researcher's institution.

Table 2.

Teachers' backgrounds and classes

	Mrs. Kim	Mr. William
First language	Korean	English
Degrees	BS in Mathematics	BS and MS in Mathematics
Teaching experience	13 years	10 years
Calculus teaching	3 times	7 times
Number of Classes/week	Three 50-minute classes	Five 90 minute classes
Teaching method	Blackboard and chalk	SmartBoard, interactive graphs
Number of Students	34	32

Coding Scheme

Videos from each class were transcribed, and then the excerpts including realizations of the derivative were selected. Excerpts for "the derivative at a point" were coded as a) *Initial Object: Location* where the initial object was defined; b) *Initial Object* where the limit process was applied; c) *Limit Process:Location* where the limit process was applied; d) *Limit process:Change* in the object; e) *Limit object* from the limit process, and f) *Limit Object*:*Location*: where the limit object was defined. Table 2 shows an example for such realizations. The first column shows codes for the limit process/object. The first row shows Mrs Kim's words or visual mediators.

Table 3.

Words and visual mediators in episode about limit process/object

Codes	Mrs. Kim's Words	Graph [Fig. 2]
Initial Object	ARC [is] the slope of the line	Top secant line
Initial Object: location	Passing through two points apart.	(a, f(a)), (b, f(b))
Limit Process: location	How do we move the points? Closer, closer…	$\rightarrow a$
Limit Process: Change	What happens to the slope? It's getting smaller and smaller (drawing three secant lines and arrows)	Three secant lines, arrow
Limit Object	We see the slope, the differential coefficient, and the instantaneous rate of change.	Tangent line
Final Object: location	At one point	A point *a*

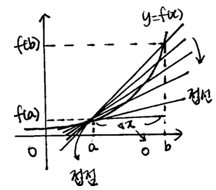

Figure 1. Realization of the differential coefficient with graphs

The excerpts about the derivative as a function were coded as a) *Limit Object*: the initial object where the derivative process was applied (e.g., the derivative at a point $x=a$); b) *Derivative Process-location: x* on the derivative process (e.g., as *a* vary); c) *Derivative Process-change*: change in the object (e.g., multiple numbers); d) *Final object*: the final object from the derivative process (e.g., graph or equation). The derivative process was categorized by five types: (a) *Expansion,* when excerpts expands from a number to a universal value on the location (e.g., "a certain point" to "any points") and/or the change (e.g., multiple tangent lines); (b) *Correspondence*, when excerpts map *x* values to the derivative function; (c) *Variation*, when excerpts include description of how the derivative varies on an interval (e.g., increasing); (d) *Universality*, when excerpts include explicit realization of the derivative defined at every point where the original function is differentiable; and (e) *Specification*, when excerpts include a transition from the derivative of a function to the derivative at a point (e.g., substitution).

Results

The results present realizations of "differential coefficient" and "leading function" in Mrs. Kim's class, and "the derivative at a point" and "the derivative of a function" in Mr. William's class. Only one of the visualizations for these cases was presented here due to the limited space.

Limit Process and Object in Mrs. Kim's Class

Among 44 episodes in Mrs. Kim's lessons about the differential coefficient, 25 of them included the limit process. The realizations for the limit process included words with symbols (11 of 25), symbols (7), words with graphs (5), and words (2) (Figure 2). These realizations highlighted the limited visual mediation of the limit process, but consistent use of words for the initial and final objects. First, most episodes only included the realization of the locations of the limit process with dynamic words or points on the curve without the change for the limit process. The change was addressed only once with graphs of secant lines. Second, uses of different visual mediators for the limit process were not consistent; two points on the curve were used for the location of the limit process, which was not consistent with the two *x* values in the symbol ($\lim_{x\to a}$). Third, in most cases involving symbols, both the location and change of the limit process were implicit; in the episodes about evaluating "differential coefficient" using the definition (e.g., Let $f(x+y)=f(x)+f(y)+3xy$, $f'(0)=3$. Find $f'(-4)$.), she transitioned from the difference quotient to the differential coefficient only with symbols ($\lim_{x\to a}$) without other explicit mediation of the limit process. It should be also noted that, Mrs. Kim consistently used the term "the differential coefficient" for the derivative as a point-specific object throughout the derivative lessons including the ones about the derivative process (e.g., the slope of a linear function, and the slope at any point on a graph). In most episodes, words mediating the initial and final objects were consistent (e.g., "slope" for both, or "average rate of change" and "the rate of change"). The term "differential coefficient" was defined synonymously with the terms for the final object.

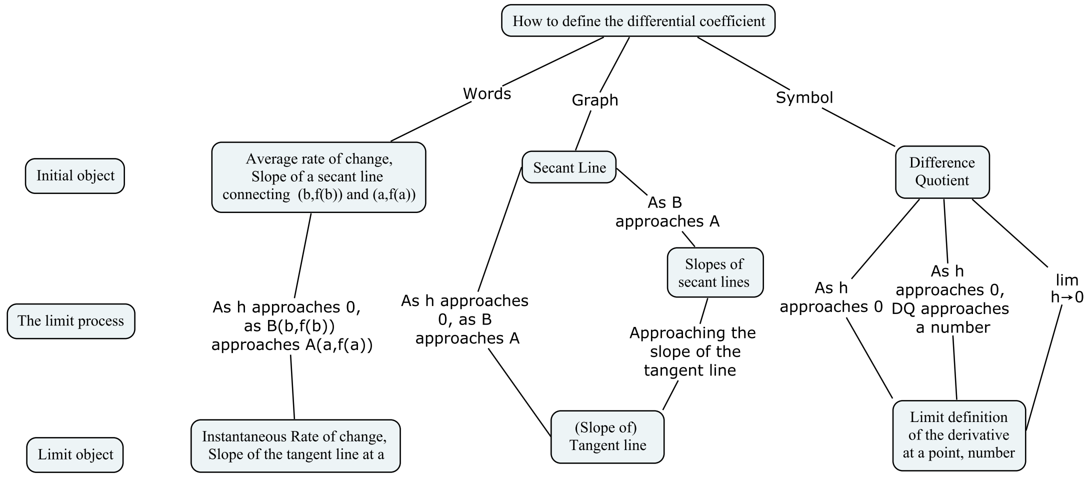

Figure 2. Realization tree for limit process/object in Mrs. Kim's class

Derivative Process and Object in Mrs. Kim's Classroom

Among 20 episodes about the leading function in Mrs. Kim's lesson, 17 addressed the derivative process, consisting of correspondence and expansion (6), computation (6), specification (4), and universality (1). Realizations of "the leading function" highlighted the limited use of visual mediators for the derivative process, and explicit word use of "differential coefficient" for the initial object, and "leading function" for the final object. First, the transition from the differential coefficient to the leading function was explained with symbols and words for the derivative process as correspondence; a diagram mapping a specific *x* value to "the differential coefficient," which was first realized as a "function" and then "the leading function." The two terms, "the differential coefficient" and "the leading function" were not directly related until "the differential coefficient " was computed as a value of "the leading function" with symbols (i.e., specification). "The differential coefficient" and "the leading function" were realized as the "same," only through symbols. Graphs were not used in any types of derivative process. Mrs. Kim mainly computed the leading function mainly following the limit process/object without mentioning any types of the derivative process. Her word use separating "the leading function" and "the differential coefficient" was also found. For example, she first described "the differential coefficient" of a linear function as "same" "always," and then used "the leading function" for the constant function. Also, she used "the leading function" for the equation, and "the differential coefficient" for its value at a point.

Limit Process and Object in Mr. William's class

Among 17 episodes in Mr. William's lessons about the derivative at a point, 11 of them included the limit process. The limit process was mediated with words (3), graphs, gestures and words (3), and symbols and words (5). Different components of the limit process/object was included in the discussion when the different types of realizations were used. First, when words and graphs were used, realizations included both the location and change for the limit process. However, when the symbols were used, the realization mainly included the location for the limit process besides one case where the words for the computation process mediated both the location (e.g., "substitute,") and the change (e.g., "zero over zero"). Second, the ways the limit process was discussed also differed across different mediators. The location for the limit process was mediated with words, gestures, and dynamic graphs for the values on the *x*-axis (e.g., " two *x* values," horizontal hand gestures, and "h" with numbers approaching 0), which corresponded to the limit notation in symbols (e.g., $\lim_{x \to a}$). However, on the stationary graph, his gestures for the location were mediating two points on the curve, not on the x-axis. Third, mismatches among realizations with different visual mediators were observed. Specifically, his gesture mediating the location for the limit process on a graph was not consistent with what is drawn on the graph. Also, the letter for a moving point included in a dynamical graph (e.g., (x+h) approaching x) differed from the letter included in symbols (e.g., x approaching a) that the graph mediated. Regarding word use, words for the initial object, the process for the limit process, and the final object were consistent in most cases (e.g., ARC and IRC; the slope for both the initial and final objects). However, there were several cases, the word "slope" was inconsistently used to mediate only either the initial object, the process for the limit process, or the final object. In some cases, secant lines and tangent lines mediated the limit process/object without word "slope."

Derivative Process and Object in Mr. William's Classroom

Among 16 episodes about the leading function in Mr. William's lesson, 15 addressed the derivative process, consisting of correspondence and expansion (1), variation (2), correspondence (2), universality (3), computation (4), and specification (3). He mainly used gestures and graphs to transition from the derivative as a value to a function. First, the location for the derivative process was mediated with his gesture of drawing and moving a tick mark on the board horizontally, and the change was mediated with multiple tangent lines. Second, the differentiability and velocity were addressed as specification after those words were realized as a function with graphs. "Differentiability" was addressed as a specification of the derivative at "every point" on its graph, and "velocity" at a point was realized as a specification of the velocity over time (e.g., gestures for the tangent lines on a curve, and then a hitting gesture for one point on the board). Similarly, the word "derivative" was used as a function after the graph of the derivative was drawn, and compared to the use of the same word "derivative" as a number, but the graphs for the "derivative" as a number (e.g., a tangent line, a point on the derivative graph) were not directly compared to the graphs of the derivative function. Regarding word use, transitions from the realization of the derivative as a point-specific object to a function were mainly made with the word "slope." The word "derivative" was not explicitly used in the derivative process. Although Mr. William specified two uses of "derivative" as a number and as a function several times while comparing the limit and derivative objects, he did not used the word in the derivative process through which he objectified the "derivative" as a function.

Discussion and Conclusion

The analysis of realizations of "derivative," in one Korean and one American calculus class led to 4 observations regarding word use and visual mediation. First, similar visual mediators were used in the realization of the limit process/object in both classes, but different components were included in the realizations in each class. While using words and graphs for the limit process, Mrs. Kim mainly mediated the location for the limit process, but Mr. William always mediated both location and change. Mrs. Kim's use for words and visuals for the initial object, limit process, and final object were consistent (e.g., "slope," "rate of change"), but Mr. William's words and visuals often varied (e.g., "the slope of a secant line" for the process and symbols for the final object without "slope"). They used the terms ("the derivative at a point" and "the differential coefficient") only for the final object mediated with symbols. Second, the realizations of the derivative as a function before it was defined were frequently found in Mr. William's class but rarely found in Mrs. Kim's class. Mr. William used "velocity" "acceleration" and "force," with the phrase "always changing," but Mrs. Kim never used "the leading function" before it was defined, or the synonyms for the "derivative" (slope, rate of change) as a function. Similarly, the word "differentiability" was realized as existence of "differential coefficient" at a point in Mrs. Kim's class, but was realized as a specific case of "the derivative function," in Mr. William's class. Similarly, the notation $f'(a)$ was used both classes, but only Mr. William used the word "any" for *a* before the derivative of a function was defined. Third, the words realizing the derivative at a point and the words realizing the derivative as a function explicitly differed in each class. The words differentiating the derivative as a point-specific object from the derivative as a function were nouns in Mrs. Kim's class. She explicitly used "the differential coefficient" for the derivative as a number, and "the leading function" for the derivative as a function, especially when discussing the relation between these two terms. In contrast, words realizing the derivative as a number or a function were often attached phrases to the "derivative," adjectives or adverbs, in Mr. William's class (e.g., "slope every point along *x,*" for the derivative function and "individual slope" for the derivative at a point). Fourth, the types of the derivative process through which the derivative was first defined as a function was different in each class: the *correspondence* between *x* values and the differential coefficients in Mrs. Kim's class, and the *expansion* of the derivative at multiple points in Mr. William's class. In Mrs. Kim's class, there was no direct transition from the term "the differential coefficient," to "the leading function." Instead, the word "function" was used between the terms "the differential coefficient," and "the leading function." In Mr. William's class, the main visual mediator for the expansion was graphs and the word "slope" throughout the initial object, derivative process, and the final object.

Although the differences in realizations between the two classes listed above seem related to their different use of words, the analysis does not imply such differences are caused by the two different words for the derivative as a point-specific object and the derivative as a function in Korean and English. However, the realization of the derivative as a function or the transition from the derivative at a point to the derivative function seems related to the key words mediating those objects. Considering the initial motivation of this study, the differences in language related terms, this study shows that Mr. William's use of words were more consistent while realizing the relation between the derivative at a point and the derivative of a function (e.g., use of "slope," or "velocity" throughout the initial object, derivative process, and the final object), than Mrs. Kim's. With the consistent use of words and corresponding visual mediators, the nouns realizing the derivative at a point and the derivative of a function were used in a more coherent way (e.g., one tangent line, multiple gestures for several tangent line, and then the graph of the derivative) in Mr. William's class. Their uses of words and visuals were also different when they transitioned from the derivative at a point to the derivative of a function, and when other related terms were realized (e.g., "differentiability" in each class), when and how the terms synonymous to "the derivative" was visually mediated (e.g., "rate of change" was realized as a function first, and then realized at a point in the flea example). Mr. William's consistent use of same words and visuals mediating both the derivative at a point and the derivative function, and Mrs. Kim's separate use of words for "the differential coefficient," and "the leading function" may not have been affected only by the different terminology. However, the differences in realizations of the derivative at a point and the derivative of a function seem consistent with the use of the word "derivative" for both objects in English, and use of two different terms, which does not show the relation between the two objects, in Korean.

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Learning to Notice and Use Student Thinking in Undergraduate Mathematics Courses

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This study evaluated the outcomes of an intervention focused on developing mathematics graduate teaching assistants' (GTAs') skills of noticing and effectively responding to instances of student mathematical thinking that have significant potential to further students' learning. Four GTAs participated in a semester-long intervention that included individual analysis and group discussion of video of undergraduate mathematics lessons. The MOST Analytic Framework (Stockero, Peterson, Leatham, & Van Zoest, 2014) was introduced to aid in these activities. The GTAs also completed a pre- and post-interview to document their real time noticing and an assessment of common content knowledge. Results indicate that the intervention was successful in improving the GTAs' noticing skills in a variety of ways and in their ability to propose student-centered responses.

Key words: Graduate Teaching Assistant Training, Teacher Noticing

Research has shown that student-centered instruction leads to more effective learning for people of all ages (National Research Council, 2005). Higher education has been slow or unsuccessful in implementing student-centered instruction (Barr & Tagg, 1995; Felder & Brent, 1996), however, with transmissive instruction (i.e., lecturing) still prominent (Ramsden, 2003; Svinicki & McKeachie, 2014). Challenges for adopting student-centered instruction include student resistance, instructor comfort level, and the time needed to see results (Felder & Brent, 1996; Seymour, 2002). Some researchers suggest promoting changes in higher education teaching methods through GTA training (e.g., Cano, Jones & Chism, 1991). Since effectiveness of GTAs typically affects undergraduate students in their early years of study, GTA training is also important in student retention (Cano et al., 1991; Speer, Gutmann, & Murphy, 2005). With a workforce shortage in science, technology, engineering, and mathematics (STEM) fields (President's Council of Advisors on Science and Technology, 2012), student retention is crucial in university STEM departments (Seymour & Hewitt, 1997; Suchman, 2014).

At the K-12 level, a teacher's ability to notice aspects of instruction as it unfolds has been recognized as important in the implementation of student-centered instruction. Many studies have found that professional noticing of [students'] mathematical thinking—defined to include attending to, interpreting, and deciding how to respond to students' strategies and understanding (Jacobs, Lamb, & Philipp, 2010)—can be learned and improved through teacher education (e.g., Jacobs et al., 2010; McDuffie et al., 2014; Sherin & van Es, 2009; Stockero, Rupnow, & Pascoe, 2015). Although teacher noticing interventions are not widely practiced in higher education, the gains made with K-12 mathematics teachers suggest that similar results may be possible.

Also foundational to effective mathematics teaching are the six domains of mathematical knowledge for teaching proposed by Ball, Thames, and Phelps (2008). Critical to this study is common content knowledge (CCK), the mathematical understanding and proficiency used in diverse contexts not exclusive to teaching. Without CCK, a teacher could not adequately guide students in building such knowledge. In addition, mathematics teachers would not likely be able to determine which instances of students' mathematical thinking are important to notice without a strong command of CCK. Because both noticing skills and CCK are important for effective mathematics teaching, it would be of interest to investigate if and how these factors are related.

This work examines the outcomes of a GTA training intervention focused on analyzing undergraduate mathematics lesson videos with a teacher noticing framework as a means to support GTAs' use of student thinking more effectively, and thus the enactment of student-centered instruction in their classrooms. Of particular interest is measuring the effectiveness of the intervention in improving GTAs' noticing of mathematically significant pedagogical opportunities to build on student thinking (MOSTs) (Leatham, Peterson, Stockero, & Van Zoest, 2015) and in supporting their ability to propose student-centered responses to such instances. This work seeks to answer the following research questions: (a) How effective is the intervention in improving GTAs' noticing of MOSTs?; (b) How effective is the intervention in supporting the GTAs' ability to propose in-the-moment student-centered responses to instances they identified in video?; and (c) What is the relationship between the GTAs' CCK and their noticing skills?

Theoretical Framework

With the goal of improving GTAs' use of student mathematical thinking in undergraduate mathematics classrooms, this study used the MOST Analytic Framework (Leatham et al., 2015) to characterize instances of student mathematical thinking that are not only important to notice, but also the most fruitful to discuss in a lesson to support students' mathematical learning. MOSTs—**M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking—are defined as "instances of student thinking that have considerable potential at a given moment to become the object of rich discussion about important mathematical ideas" (p. 90). To be a MOST, a moment must satisfy three characteristics: student mathematical thinking, mathematically significant, and pedagogical opportunity. To satisfy these characteristics, the student mathematics must be inferable and related to a mathematical point, the mathematical point must be appropriate to the learning level of the students and a central goal for student learning, the student mathematics must create an intellectual need for students to understand the mathematical point, and it must be the right time to address the intellectual need at that moment.

Methodology

Participants and Intervention

The participants were four mathematics GTAs from a Midwestern U.S. university. They had completed one to two years of graduate study and taught for one to six semesters. They had all completed training required by the mathematics department: one week of GTA orientation prior to their first semester of study, a course entitled Teaching College Mathematics, and a six-week seminar during their first semester of teaching. Participation in the study was voluntary.

The GTAs engaged in a ten-week intervention facilitated by the first author in fall 2015. The goal was to improve the GTAs' skills in attending to, interpreting, and responding to MOSTs in a student-centered manner. The intervention design was adapted from Stockero and colleagues' work with prospective secondary mathematics teachers (Stockero, 2014; Stockero et al., 2015). Pre- and post-intervention, each GTA completed a one-on-one, video-recorded interview in which they watched an undergraduate mathematics lesson video clip, stopped the video if they thought a *mathematically important moment that the instructor should notice (MIM)* occurred, and then described why they selected the moment and what they might do if it had happened in their own classroom. A *MIM* definition was not given to the GTAs to establish baseline data.

In each week of the intervention, the GTAs used video analysis software to individually analyze an undergraduate mathematics lesson video in preparation for a weekly group meeting. In the first three weeks, the GTAs tagged *MIMs* and described in text what they noticed and why they chose each instance. After three weeks, the GTAs read a paper that defined the MOST Analytic Framework (Stockero, Peterson, Leatham, & Van Zoest, 2014). In that week, they reexamined two videos they had already analyzed and chose instances that they believed were MOSTs. In the remaining six weeks of the intervention, the GTAs were prompted to tag and describe MOSTs in new classroom videos. They were provided a text prompt to specifically address each MOST criterion in their written responses in the last five weeks of the intervention.

The researcher analyzed the same videos as the GTAs for MOSTs—the types of instances that were the goal for noticing. The researcher examined the GTAs' video timelines, compared their instances and the researcher-identified MOSTs, and selected instances to discuss at the group meeting. A variety of instances were selected, including those marked by multiple GTAs—both MOSTs and non-MOSTs—as well as MOSTs that were not noticed by the GTAs.

In the group meetings, the researcher pushed the GTAs to articulate what an instructor would have to notice in each moment and why it was mathematically important. The GTAs first worked toward building a definition of *MIMs* in group discussion. Later, discussion focused on whether instances fit the MOST criteria. In the last three weeks, the GTAs proposed *building* moves in response to the MOSTs discussed—a teacher move that engages students in collaboratively discussing the significant student mathematical thinking that is present in the instance (Stockero et al., 2014). These moves would use student mathematical thinking to further the learning of all students, which aligns with effective student-centered mathematical instruction (NCTM, 2014). The GTAs then proposed building moves in their subsequent video analyses.

At the conclusion of the intervention the GTAs completed the Calculus Concept Inventory (CCI) (Epstein, 2013) to understand whether there was a relationship between GTA performance on the CCI (a measure of their mathematical CCK) and their noticing of MOSTs.

Data Collection and Analysis

The data for this study included the CCI results and the video timelines produced by the GTAs, both during the intervention and the interviews. The CCI results were scored for correctness. The score of each GTA was then compared to the rest of the group to see if there were any obvious differences in scores that could account for differences in noticing skills.

Each instance identified by a GTA was coded to examine changes in the GTAs' noticing. First, like in the work of Stockero and colleagues (2015), each was coded according to *agent* (who or what was noticed) and *mathematical specificity* (the way in which the mathematics was discussed). Instances that had a student agent were also coded for *focus* (what about the student(s) was noticed). See Figure 1 for descriptions of coding categories and codes. Second, like Stockero, Rupnow, and Pascoe (under review), each instance was coded according to whether it was a MOST and whether the reasoning provided by the GTA was consistent with the MOST criteria. A GTA instance was coded as a MOST if it occurred at the same time in a video as a MOST identified by the researcher. In a consistent MOST, the GTA also identified the characteristics of the instance that qualified the instance as a MOST according to the framework.

Because a goal of this intervention was that the GTAs propose more student-centered responses, the pre- and post-interview instances were also coded according to whether the GTA-proposed response to each instance was student- or teacher-centered. A student-centered response is one in which the teacher would involve one or more students in responding to the instance, whereas a teacher-centered response would only involve the teacher.

Coding Categories and Descriptions	Codes	
Agent: Who or what was noticed	Teacher Teacher/ Student Student/ Teacher	Student Group Student Individual Math
Mathematical Specificity: Whether and how the mathematics was discussed	Non-Math General Math Specific Math	
Focus: For instances with a student agent, what about the student(s) was attended to	Affective Interaction General Understanding Mathematical Interaction	Noting Student Mathematics Analyzing Student Mathematics

Figure 1. Noticing coding scheme. Adapted from Stockero et al. (2015; under review).

After the coding was complete, the data were summarized to look for changes in GTAs' noticing and responding throughout the intervention, focusing on the components of noticing (agent, mathematical specificity, and focus), MOSTs, and how the GTAs might respond to the moments they selected. To provide a common unit of measure, percentages were calculated for each code out of each GTA's and the group's total number of instances in each video. The data was then split into three stages—early, middle and late in the intervention—and summarized accordingly. In this analysis, *baseline* refers to the first three videos of the intervention before the introduction of the MOST framework. *Middle* stands for the three videos immediately following the introduction of the MOST framework, and *final* refers to the last three videos of the intervention. Data from the interviews were analyzed separately due to the difference in the nature of the interviews, in which the GTAs engaged in in-the-moment video analysis where repeated viewings and lengthy deliberation about instances were not possible.

Results

Components of Noticing

Agent. Because the goals of the intervention placed an emphasis on students and their mathematical thinking, changes were examined in the GTAs' noticing of instances in which students were the primary agent (i.e., Student/Teacher, Student Individual, and Student Group agents). Table 1 provides the percentages of such instances in each stage of the intervention and the pre- and post-interviews. It can be seen that both individually and as a group the general trend was an increase in the GTAs' noticing that had a primary student agent from stage to stage. Impressively, the majority of GTAs averaged 100% and the group averaged 94% of primary student noticing in the final data. The table also shows improvement in the GTAs' noticing of instances with a primary student agent from pre- to post-interview, with GTAs 1 and 2 showing the most growth in this type of noticing. Most notably, 100% of the GTAs' noticing in the post data was primarily on students, indicating that they developed the ability to focus their noticing on students over the teacher or the mathematics itself in their in-the-moment analysis of video.

Mathematical Specificity. With mathematical significance and student mathematical thinking being two of the characteristics of a MOST, a possible indicator of improvement in noticing of MOSTs is the ability to speak about the mathematics of an instance in a detailed manner, aligning with the Specific Math code. Table 2 shows that baseline data percentages for

Table 1
Noticing of Primary Student Agent by Stage and Interview

Participant	Baseline	Middle	Final	Pre	Post
GTA 1	41%	100%	100%	29%	100%
GTA 2	50%	89%	100%	41%	100%
GTA 3	63%	100%	100%	75%	100%
GTA 4	28%	67%	75%	80%	100%
Group	46%	89%	94%	56%	100%

Specific Math were rather high for all GTAs with the exception of GTA 4. The middle data showed an increase in mathematical specificity for all GTAs, with the most considerable increase of 77% being that of GTA 4. Perhaps most important is that all of the GTAs exhibited 100% Specific Math noticing in the final data. Table 2 also indicates improvement in Specific Math noticing from pre- to post-interview. GTA 1's Specific Math noticing was maximized and maintained from pre- to post-interview. GTA 4 showed the most improvement from 40% in the pre-interview to 100% in the post-interview data. Like the final data, the post-interview data demonstrated 100% Specific Math for all participants. Thus the GTAs not only discussed the mathematics with a high level of detail when they had time to reflect and write about each instance, but also in a real time video-based interview setting.

Table 2
Specific Math Noticing by Stage and Interview

Participant	Baseline	Middle	Final	Pre	Post
GTA 1	97%	100%	100%	100%	100%
GTA 2	85%	100%	100%	59%	100%
GTA 3	81%	100%	100%	75%	100%
GTA 4	6%	83%	100%	40%	100%
Group	67%	96%	100%	69%	100%

Focus. The focus code most aligned with the goals of the intervention was Analyzing Student Mathematics, since the MOST Analytic Framework requires that an inference be made about what the student is saying mathematically. While the focus code only applied to instances with a student agent, the reported percentages are out of all instances identified by the GTAs to reflect an overall sense of their noticing. Table 3 indicates that Analyzing Student Mathematics was absent or low in the baseline data, both individually and as a group. By the middle data, substantial increases were made by all GTAs, with GTAs 1, 2, and 3 improving by 72% to 84%. GTA 3 reached 100% in the final data. The pre- and post-interview data exhibited remarkable growth for in-the-moment noticing with this focus. GTA 4 demonstrated the largest growth at 100% from pre to post, while increases for the other GTAs ranged between 75% to 87%. Interestingly, GTA 4 exhibited Analyzing Student Mathematics much more in the post-interview, an in-the-moment context, than in the final data. It is possible that GTA 4 more completely communicated what they noticed with spoken word in the interview than in written word in the text that accompanied their video timelines. In general, these results suggest that the intervention was successful in developing the GTAs' ability to focus on and interpret student mathematical thinking both when analyzing video individually and in an in-the-moment context.

Table 3
Analyzing Student Mathematics by Stage and Interview

Participant	Baseline	Middle	Final	Pre	Post
GTA 1	16%	100%	94%	14%	100%
GTA 2	0%	72%	67%	0%	75%
GTA 3	24%	96%	100%	13%	100%
GTA 4	0%	33%	47%	0%	100%
Group	10%	75%	77%	7%	94%

MOST Analysis

Changes were examined in the GTAs' noticing of consistent MOSTs, the main goal of the intervention. Like the other analyses, the percentages presented were calculated out of the total set of instances marked by the GTAs. Table 4 shows that all GTAs improved in their noticing of consistent MOSTs in each stage of the intervention, with the group's average percentages increasing from a baseline of 19%, to 73% in the final data. The pre- and post-interview data showed considerable increases in the GTAs' in-the-moment noticing of consistent MOSTs, with increases ranging from 26% to 80%. However, the raw percentages in the post-interview data are not overwhelmingly high. It is worth recalling that the prompt for both the pre- and post-interview was to identify *MIMs*, which may explain why there was not a higher percentage of MOSTs identified in the post-interview. An idea underlying the intervention was that the MOST Analytic Framework would provide a way to characterize mathematically important moments that the instructor should notice, but perhaps the connection between *MIMs* and MOSTs was not seen by the GTAs. Still, the data suggests that the intervention was successful in improving the GTAs' ability to notice MOSTs and reason about them in accordance with the MOST Analytic Framework, both when analyzing video individually and in an in-the-moment context.

Table 4
Noticing of Consistent MOSTs by Stage and Interview

Participant	Baseline	Middle	Final	Pre	Post
GTA 1	24%	61%	87%	14%	40%
GTA 2	19%	36%	76%	11%	50%
GTA 3	25%	57%	78%	13%	67%
GTA 4	8%	33%	53%	20%	100%
Group	19%	47%	73%	14%	64%

Responses

Another goal of the intervention was to increase the amount of student-centered responses that were proposed to instances identified in video. Table 5 indicates the percentage of instances in which the GTAs proposed a student-centered response. Substantial increases in the percentage of such responses were made from pre- to post-intervention for all GTAs. In fact, with the exception of GTA 4, 100% of the responses provided by the GTAs were student-centered in the post data. These results suggest that the intervention was successful in improving the GTAs' skills in proposing student-centered responses in an in-the-moment context.

CCI Scores

The GTAs' performances on the CCI assessment, both as a raw score and as a percentage, were quite similar to one another. Raw scores ranged from 17-19 out of 22 possible points, with an average of 18; percentages ranged from 77-86%, with an average of 82%. Because all of the GTAs had about the same aptitude for calculus concepts, there was no evidence that differences in mathematical content knowledge accounted for differences observed in their noticing.

Table 5
Student-centered Responses by Interview

Participant	Pre	Post
GTA 1	0%	100%
GTA 2	57%	100%
GTA 3	83%	100%
GTA 4	33%	50%
Group	43%	88%

Discussion

Results showed that the intervention was successful in improving the GTAs' noticing in a number of ways and in two different video analysis contexts. The GTAs greatly increased in their noticing of instances primarily focused on students, the percentage of instances in which they discussed the mathematics of an instance in a specific manner, their focus on analyzing the student mathematics of an instance, and their noticing of consistent MOSTs. These results add support to the successes of interventions in K-12 mathematics education that use video and a defined framework to improve the noticing skills of mathematics teachers (e.g., McDuffie et al., 2014; Schack et al., 2013; Stockero et al., 2015, under review) and suggest that such interventions can be successful at the undergraduate level as well.

The intervention was also successful in improving the GTAs' skills in proposing student-centered responses. This finding builds upon those of existing studies (Jacobs, Lamb, Philipp, & Schappelle, 2011; Jacobs et al., 2010; Schack et al., 2013) that suggest that professional development structured around noticing students' mathematical thinking in video and classroom artifacts can develop teachers' abilities in not only *attending to and interpreting [students'] strategies and understandings*, but also the skill of *deciding how to respond* (Jacobs et al., 2010).

While the results of this study suggest that similar interventions could be successful in training GTAs in noticing student mathematical thinking, the small number of participants is a limitation. Future work could involve replicating this study with more GTAs, at other universities, or with another set of videos. Limitations aside, the findings suggest that the intervention was a step in the right direction for advancing student-centered instruction in undergraduate mathematics courses. The development of noticing skills, while achieved in a professional development setting, has the potential to improve classroom instruction and thus the retention of first- and second-year undergraduate students (Cano et al., 1991; Speer et al., 2005).

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Attention to Detail: Norms for Proof Evaluation in a Summer Mathematics Program

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In this study, we explore the norms by which students and undergraduate mentors in a summer mathematics program evaluate proofs of theorems in number theory. By utilizing cognitive interviews during which students and mentors evaluate number theory proofs written by a hypothetical student, we find that for students as well as mentors, "rigor" is a dimension of mathematical acceptability of proofs distinct from, though related to, proof validity. Additionally, we find that both students and mentors frequently adhere to strict unwritten norms that govern how they believe proofs should be constructed and presented, and that these norms may be more rigid than the intended proof-writing norms of the mathematicians who teach in the summer program. This study suggests some potential challenges associated with the growing practice of asking undergraduate student graders to evaluate proofs written by students in introduction-to-proof courses.

Key words: Proofs, Proof Validation, Cognitive Research, Informal Mathematics Education

One of the goals of a typical undergraduate program in mathematics is to instill in students an understanding of mathematical proof: the purposes for which mathematicians use proof, the process by which mathematicians prove results, and the ways in which we decide whether an argument is valid and should be accepted by the mathematical community. However, students learn about disciplinary norms governing proofs in varied and sometimes idiosyncratic ways. An instructor in an undergraduate mathematics course may rule a proof produced by a student acceptable or unacceptable according to rules that have not been made fully explicit for students. For example, an instructor in an introduction-to-proofs course may suggest that a proof of the formula for the sum $[1 + 2 + 3 + \ldots + n]$ using induction is more "rigorous" than the classic reordering argument that pairs this with the sum $[n + (n – 1) + (n – 2) + \ldots + 1]$. An instructor who says this may intend to signal to students that in formal mathematics, we are reluctant to accept reasoning that is hidden "behind" the ellipses in these expressions without careful investigation; however, students may infer from the instructor's choice that in formal mathematics, rigid proof schemata such as induction are more "rigorous" than arguments that use flexible reasoning. As a result of exchanges such as these, students may develop the belief that the acceptability of a proof is determined more by the manner in which an argument is written rather than by the logical coherence of the argument itself. In this study, we investigate ideas about "rigor" and proof acceptability co-constructed by faculty, mentors, and students in a summer mathematics program, and document some cases in which students demonstrate norms for proof evaluation that may be inconsistent with those that the faculty who teach in the program intend to convey.

Background

There is a growing body of literature on *proof validation*, the process of deciding whether a proof is valid, as performed by mathematics students and instructors. Validation of proofs is an intellectually complex process requiring many different types of reasoning, including the construction of formal and informal deductive arguments and example-based reasoning (Weber, 2008). Harel and Sowder (1998) suggest that many students, by the time they begin learning about formal mathematical proofs in college, have developed *external conviction proof schemes* in which validity derives from the "ritual of the argument presentation"; that is, degree to which a proof structurally resembles a formal argument scheme. Additionally, many students have *empirical proof schemes* and may be convinced of the validity of a conjecture by testing specific cases. These non-analytical proof schemes may interfere with students' ability to assess the validity of nonstandard arguments. Selden and Selden (2003) suggest that exclusive attention to surface features of a proof, or failure to attend to the global logical structure of a proof, may cause students to make incorrect judgments about the proof's validity.

Mathematicians use a variety of socially-determined criteria when evaluating a proof, such as whether they understand the concepts embedded in the proof, whether the argument is convincing, and whether the theorem being proven is consistent with the existing body of accepted mathematical results (Hanna, 1983). Courses that introduce students to proof-writing in mathematics often have the dual aims of helping students develop the ability to identify and produce valid arguments, and enculturating students into practices and disciplinary norms regarding the writing and evaluation of proofs that are typical among professional mathematicians. While several studies have addressed the mental processes involved in distinguishing valid proofs from invalid ones, relatively few have studied how students' judgments of proof validity interact with their (possibly separate) assessments of whether a proof is acceptably presented. While mathematicians do not share a single common standard for evaluating the validity and presentation of proofs (Inglis *et al.*, 2013), we view the instructor of a proof-writing course as an exemplar of a particular set of norms and practices that the course is intended to transmit, in part or in full, to its students, and are interested in the various pedagogical forces that influence how students assimilate these norms themselves.

In this study, we aim to address the following research questions:

1. What criteria, other than proof validity, do students who are learning to read and write formal mathematics use to judge proofs?
2. To what degree are novice proof writers able to distinguish flaws that make a proof less readable or less complete from those that render a proof invalid?
3. To what degree do novice students' and their mentors' norms for evaluating proofs conform to those of the community of professional mathematicians, as embodied by the instructors teaching their courses?

Setting of Study

Our study took place at a summer mathematics program for talented high school students in the United States. Students in the program are recruited from all geographic regions of the United States; while most enter the program with no prior formal experience in undergraduate mathematics, a few have learned to write proofs by participating in high school mathematics competitions.

During the program, students learn how to write mathematical proofs while taking a course in number theory; some students return to the program in subsequent summers and take other courses in undergraduate mathematics. Students are assigned to study groups, with each group supervised by a mentor who has attended the program for several years. Most mentors are undergraduate STEM majors at leading research universities, though some mentors are senior high school students.

Students in the program attend classes each morning and afternoon; during the evenings, they participate in extended study sessions with their groupmates, supervised by their mentors. They write proofs of theorems in number theory and submit these proofs for "grading" by their mentors. Although these proofs do not receive numerical grades, mentors provide feedback on the proofs' correctness and style, and sometimes recommend that students revise work that is incorrect or incomplete.

Method

We conducted a survey of the first-year students in the program, their mentors, and the mathematician faculty member who teaches the number theory course to determine whether participants in the program were familiar with the use of the word "rigor" as a criterion used to evaluate proofs, and to what extent participants view "rigor" as distinct from proof validity. When asked to agree or disagree with the statement, "When we talk about proofs, the word 'rigorous' has the same meaning as the word 'valid,'" 6 respondents strongly disagreed, 13 moderately disagreed, 13 moderately agreed, and 4 strongly agreed. The instructor of the number theory course was among those who strongly agreed. When asked to explain the distinction between validity and rigor, if they believed such a distinction exists, many respondents suggested that validity points to whether a proof is correct, while rigor deals with the level of detail in a proof and the extent to which the argument explicitly states and justifies each step.

Based on the survey results, we selected two study groups in the summer program for follow-up interviews. We invited students and mentors from these two study groups, along with the mathematician who taught the number theory course, to participate in cognitive interviews in which they evaluated several proofs written by a hypothetical student. Because we selected interview subjects from study groups in which multiple students shared the view that a proof is rigorous if and only if it is detailed, we make no claim that the interview results are representative of the proof-evaluation norms of the entire student body of the summer program. However, we conjecture based on the results of the preliminary survey that the views of rigor and proof validity demonstrated by the students and mentors we interviewed were shared by many other students and mentors in the program.

In the interviews, we asked six students, their two mentors, and the mathematician instructor to review the first theorem shown in Table 1 on the next page, either produce a proof of the theorem or write some notes on the main ideas of the proof, and then evaluate three hypothetical student proofs of the theorem on three dimensions – validity, rigor, and understandability – each on a scale of 0 to 3. We then repeated this routine with the second theorem below. In asking interview subjects to rate these proofs, we hoped to gain additional insight about whether students and mentors made distinctions between validity and rigor in their evaluations of specific arguments, and about whether students' and mentors' overall evaluations of proof quality were consistent with the norms the instructor intended to set.

Results

Students' and mentors' responses to the proofs given in the interview, and in particular their assessments of the rigor of these proofs, varied considerably. Table 2 on the next page shows the ratings the interview subjects gave the three proofs of Theorem 1 for Validity, Rigor, and Understandability, along with the proof each subject rated the best overall and the most rigorous.

Table 1: Descriptions of Hypothetical Student Proofs

Theorem	Proof	Description of Proof
If a, b, and n are integers such that a < b and n > 0, then an < bn.	1A	Detailed argument that omits essential step demonstrating that difference between *bn* and *an* is a natural number
	1B	Correct argument that omits names of algebraic properties that justify steps
	1C	Correct and detailed argument by induction on *n*
If a, b, q, and r are integers with b > 0 and a = bq + r, then GCD(a, b) = GCD(b, r).	2A	Correct argument carefully using definition of GCD
	2B	"The common divisors of *a* and *b* are the same as the common divisors of *b* and *r*" argument
	2C	Argument that makes incorrect inference

Table 2: Interview Subjects' Ratings of Proofs of Theorem 1

Subject	Proof 1A			Proof 1B			Proof 1C			Best	M Rig
	V	R	U	V	R	U	V	R	U		
Student X1	3	2	2	2	1	2	3	3	2	1A	1C
Student X2	3	2	2	2	1	1	3	2	2	1A	1A
Student X3	3	2	3	3	3	3	3	3	2	1B	1B
Student X4	3	3	2	2	1	2	2	2	3	1A	1A
Mentor X	2	3	1	2	1	2	2	1	2	1A	1A
Student Y1	3	3	3	3	1	2	3	2	3	1A	1A
Student Y2	3	3	2	3	2	2	3	3	2	1B	1C
Mentor Y	2	1	2	3	2	3	3	1	1	1B	1B
Instructor	2	2	3	3	3	3	3	3	3	1B	1B/1C

We asked each subject to explain his or her ratings after evaluating each proof. Several themes emerged during these explanations, suggesting discrepancies between proof-evaluation norms of students and those of their mentors, or discrepancies between norms of students and mentors and those of the mathematician teaching the number theory course.

Level of Detail Versus Essential Reasoning: The Case of Proofs 1A and 1B

We constructed Proofs 1A and 1B to differ in two essential ways. The first is that Proof 1A obfuscates its argument slightly by introducing a new variable t that is equivalent to n, while Proof 1B is more direct. The second is that while Proof 1A is more generous in including names of algebraic properties used in the argument, such as the substitution principle, Proof 1B is the only argument that uses the class's definition of "less than" (that $x < y$ if there exists a natural number k such that $x + k = y$) to establish that $an < bn$. (Proof 1A states that because $at + kt = bn$, we have $at < bn$, without stating or showing that kt is a natural number.) Thus while Proof 1A includes an overall greater level of detail, Proof 1B arguably gives more attention to the most crucial step of the argument.

While both mentors and the instructor noted the key omission in Proof 1A and rated the proof's validity accordingly, all six students rated the proof's validity a 3, and five of the six rated Proof 1A more rigorous than Proof 1B. Most of these five students stated that they rated

Proof 1A more rigorous because it more consistently stated reasons for the algebra steps in the proof and named the algebraic properties used. However, Student X3 mentioned specifically that while Proof 1B left out a greater number of details overall, Proof 1A left out reasoning that seemed more essential. This suggests that Student X3 had a way of thinking about proof quality that took into account the relative importance of various details.

Rigidity in Methods and Styles of Proof: The Case of Proof 1C

We constructed Proof 1C in order to see how interview subjects would respond to the hypothetical student's use of induction to prove a theorem that can be proven directly using the definition of "less than" and the distributive property. The ratings and interviews suggested that while some students found Proof 1C to be the most rigorous of the three (despite preferring one of the other proofs), others considered the proof less rigorous than others in part because of the specific induction language used in the argument. Student X4, who gave this proof a rating of 2 for rigor, said, "It's kind of like, why are you using induction for something that probably could save you some time?" She later clarified that the use of induction was not the reason for her rating, but rather that the author of the proof might have written a better argument had he or she chosen a different method. Mentor X also took issue with some of the language used in the induction proof; most of her concerns revolved around the fact that the argument, at various times, specified a value for n (*e.g.*, for the base case) and proceeded to make statements based on that assumption rather than writing these statements in conditional form. For example, the base case assumed that $n = 1$ and later said that $an < bn$; of this, Mentor X said, "That's not like… that's just not true… you would say, maybe, 'thus, when $n = 1$, $an < bn$,' right. That language sort of implies that they don't understand what a base case does, what the significance of a base case is." It seems that in this case, Mentor X read the claim that "$an < bn$" at the end of the base case as a general claim rather than as an instantiated claim about the case $n = 1$. These comments, along with her other markings on the proof, suggested that Mentor X had adopted a rigid way of reading induction arguments that had some difficulty accommodating differences in how students might represent the inductive logic of a proof.

In addition, Student X3 expressed concern that parts of Proof 1C had been written as indented chains of algebraic steps rather than in paragraph form. When pressed on this, subjects suggested that some mentors in the summer program encouraged students to write proofs in paragraph form, rather than block-indenting sequences of calculations. This is one of several stylistic choices about proof-writing that students in the program may infer to be norms of the mathematical community, but that in fact are artifacts of the program culture and of the way mentors grade proofs. We have collected scanned images of the graded proofs of students in the summer program and hope to perform a more in-depth investigation of the norms for mathematical writing that mentors in the program convey to students.

"Logic" Versus Chains of Axiomatic and Definitional Reasoning: The Case of Proof 2B

We constructed Proofs 2A and 2B to differ in only one significant way. While Proof 2A explicitly uses the formal definition of greatest common divisor given in class (that d is the GCD of a and b if d is a natural number; d divides both a and b; and if c divides both a and b, then c is less than or equal to d), Proof 2B proves Theorem 2 by showing that a and b have precisely the same common divisors as b and r, and thus concluding that these two pairs must have the same GCD. Under the surface lay a structural difference that, in our preliminary investigations, seemed to have some import for students and mentors in the program: while Proof 2A proceeds through a sequence of steps, most of which can be directly justified using axioms or definitions from the number theory course, Proof 2B ends with a verbal argument that appeals to the notion of GCD as the maximum of the set of common divisors of two integers, and attempts to persuade the reader that if two pairs of integers have the same set of common divisors, they must have the same GCD. The latter approach departs slightly from the norm of precise definitional reasoning to which students become accustomed early in the number theory course, though the number theory course itself includes a number of theorems (such as Fermat's Little Theorem) whose proofs contain steps that appeal to nonlinear, *ad hoc* reasoning rather than axioms.

Of the five students and one mentor who finished reviewing the proofs of Theorem 2, only two students rated Proof 2B as entirely valid. Two students rated the proof's validity a 2, and Student X4 and Mentor X rated the proof's validity a 1. Student X1 rated the proof a 3 for validity but only a 2 for rigor; when asked to explain his ratings, he said, "some of the steps aren't explained clearly, such as the end, where it says that… it's sort of just explaining logic instead of using any theorem or previous axiom." In the preliminary survey, two other respondents, including Student Y1, mentioned the use of "logic" as a possible source of disagreement among students and mentors over whether a proof is rigorous; one respondent stated that "sometimes our [mentors] tell us to use more algebra instead of just logic and assuming."

Student X3, whose interview responses suggested the least commitment to rigid, style-based norms for proof acceptability among those students interviewed, rated the proof a 3 for both validity and rigor, but rated it a 2 for understandability. She suggested that the author of the proof should have constructed the set of common divisors of a and b and the set of common divisors of b and r, and shown that these two sets are equal. Responses to Proof 2B suggest that the students interviewed were not entirely comfortable with arguments that depart from chains of reasoning based on axioms and previously proven theorems and that include steps for which the justification is fluid and unfamiliar.

Norms of a Professional Mathematician: The Case of the Course Instructor

After we completed interviews of students and mentors, we interviewed the instructor of the number theory course using the same protocol. Prior to rating any proofs, the instructor stated his position that "validity" and "rigor" are not separate constructs, and expressed concern that students might believe that rigor is a separate or stronger standard for proofs than validity. When rating each proof, he assigned the same rating for both validity and rigor.

The instructor initially rated each of the three proofs of Theorem 1 a 3 in all three categories; he changed his validity and rigor ratings for Proof 1A to 2 only after considering that the author of the proof made, in his view, two mistakes: failing to use the substitution principle correctly, and failing to observe that kt is a natural number. He expressed little hesitation in giving ratings of 3 for Proofs 1B and 1C, and showed little concern about the justification steps skipped in Proof 1B.

When the instructor was given the opportunity to review Theorem 2 prior to seeing the proofs, he noted that he is familiar with two different approaches; he then described the approaches used in Proofs 2A and 2B. When shown Proof 2B, he expressed amusement that he had foreshadowed the argument given, and rated the proof a 3 in all three categories, as he had with Proof 2A.

We can envision two different hypotheses to explain the fact that the instructor awarded higher ratings for most of the proofs in the interview than the students and mentors did. One possible explanation is that the instructor's extensive experience teaching number theory has familiarized him with the arguments provided and desensitized him to possible errors in logic that may reveal unclear thinking on the author's part. Another hypothesis is that the instructor, when rating a proof, was more interested in whether the author had a complete chain of reasoning linking the hypotheses to the desired conclusions than in issues of how the proof is written. At various points in the interview, the instructor pointed out instances of writing that he did not consider optimal, but openly dismissed these as inessential to the central issue of whether the proof under review was valid, in contrast with mentors and students who considered some of these issues (such as the explicit naming of axioms about the algebraic structure of **Z**) crucial in evaluating the validity and especially the rigor of proofs. This contrast suggests a disparity between the proof-evaluation norms of students and mentors in the summer program and those of professional mathematicians as embodied by the instructor of the course.

Discussion and Implications

The interview results suggest that some students in the summer mathematics program had criteria for "rigor" or completeness of proofs distinct from the criteria they used to evaluate the validity of proofs. In some responses to interview questions, students rated proofs as valid, but indicated that they believed that their mentors may not find these proofs entirely acceptable. In some cases, students indicated that they themselves did not consider certain proofs (such as Proofs 1A and 1B) totally acceptable despite finding them entirely valid.

We conjecture that in some introduction-to-proof courses, instructors pay explicit attention to issues of style and detail in proof-writing in the belief that strict attention to these issues supports the development of students' ability to read and write proofs. We make no claim that our study sheds light on the validity of this particular belief. However, we observe that in most of the interviews conducted in this study, subjects did not always make clear distinctions between flaws in a proof that render it more difficult to read or follow and flaws that render a proof invalid. Although the instructor of the number theory course gave little direct instruction on norms for writing and evaluating proofs in class, mentors provided regular feedback on students' proofs and, based on a preliminary analysis of their markings, sometimes advised students to rewrite proofs in ways that would have yielded at most negligible improvements in their readability. We hope to ascertain through follow-up interviews of the subjects of this study whether the mentors' feedback instilled in students norms for writing and reading proofs that are more rigid than those shared by most of the community of professional mathematicians. We also hope to conduct a more thorough analysis of the graded proofs we collected over the duration of the program to develop a more precise inventory of the norms that governed the mentors' evaluations of proofs.

We believe that our study has implications for the teaching of undergraduate mathematics in light of the growing practice of appointing relatively inexperienced students to grade papers in courses that involve proofs, borne of funding decreases that have forced mathematics departments to shift some of this intellectually challenging work from graduate teaching assistants to undergraduate graders. At minimum, we suspect that some undergraduate graders tasked with grading proofs may benefit from professional development that encourages flexibility in evaluating different approaches and styles of proof presentation. We acknowledge, however, that such professional development may not mitigate the intellectual demands associated with evaluating whether unfamiliar arguments are mathematically valid.

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Student Mathematical Connections in an Introductory Linear Algebra Course

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In an introductory linear algebra course, students are expected to learn a plethora of new concepts as well as how these concepts are connected to one another. Learning these connections can be quite challenging for students due to the vast number of connections and student inexperience with mathematical logic. The study reported here consisted of an investigation into how inquiry-oriented teaching methods could be employed in an attempt to create opportunities for students to develop mathematical connections in an introductory linear algebra course.

Key words: linear algebra, mathematical connections, inquiry-oriented teaching

Introductory linear algebra courses have traditionally been quite challenging for students. There are several reasons for this, including the fact that students are introduced to a plethora of brand new concepts and terminology. Further, many of these concepts are connected to one another in various ways, and students are expected to learn these connections as well. While many researchers and teachers would agree that students should be able to make mathematical connections, the phrase "mathematical connection" is often loosely defined. This study considers one particular type of mathematical connection in an introductory linear algebra course: *logical implication connections*. Relationships between various linear algebraic concepts are often summarized in theorems of logical equivalence such as the Invertible Matrix Theorem (IMT) (Lay, 2011). The statements in this theorem are all logically equivalent, meaning any statement in the theorem logically implies another (and vice versa). Thus, the logical implications present in the IMT could be described as logical implication connections.

While the IMT provides a convenient presentation of logical implications in introductory linear algebra, it is somewhat restrictive due to the fact that it only applies to square coefficient matrices (as only square matrices can be invertible). However, subsets of the logical implications inherent in the IMT could be applied to non-square matrices. The IMT could actually be divided into two "sub-theorems," which will hereby be known as the First and Second Theorems of Logical Equivalence; these theorems are presented in Figure 1.

Theorem 1: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

a) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
b) A has m pivot positions; that is, A has a pivot position in every row.
c) Every vector $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
d) The columns of A span $\mathbb{R}^m$.
e) The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

Theorem 2: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

a) The equation $A\mathbf{x} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
b) For each $\mathbf{b}$ in $\mathbb{R}^m$, the linear system corresponding to $A\mathbf{x} = \mathbf{b}$ does not have a free variable; that is, the linear system only has basic variables.
c) A has n pivot positions; that is, A has a pivot position in every column.
d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e) The columns of A form a linearly independent set.
f) No column of A is a linear combination of the other columns.
g) The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

Figure **1**: Unlike the Invertible Matrix Theorem, these theorems of logical equivalence are not restricted only to the case of square matrices.

As learning these mathematical connections can be challenging for students, it would be beneficial to improve the teaching of these connections. The study described in this report was part of a larger study that attempted to determine how inquiry-oriented teaching methods could be implemented in an introductory linear algebra course that, due to considerations such as large class size and limited amount of class time, would not lend itself to the traditional demands of inquiry-oriented teaching. Regarding the teaching of mathematical connections, one goal of this study was to answer the following research question: How do students take advantage of inquiry-oriented teaching to make connections in an introductory linear algebra class?

Literature Review

It is not uncommon for students to know that two linear algebraic concepts are connected but not understand *why* they are connected. This issue was well described by Harel:

> So if a student thinks of 'linear independence' to mean 'the echelon matrix which results from elimination has no rows of zeros,' without being able to mathematically justify this connection, then he or she does not understand the concept of linear independence. (Harel, 1997, p. 111)

This issue of the quality of student understanding has been previously discussed by Skemp (1987) in his description of instrumental and relational understanding. The understanding presented in Harel's example is *instrumental understanding*; that is, knowing what to do but not why. According to Skemp, true understanding of a concept involves *relational understanding*, which is "knowing both what to do and why" (Skemp, 1987, p. 153). This characterization of the quality of understanding could be applied to mathematical connections. Thus, a student has made an *instrumental connection* if the student has formed a connection but does not understand why that connection exists; similarly, a student has made a *relational connection* if the student has formed a connection and understands why that connection exists. For example, a student could present relational understanding of a logical implication connection if he or she can form a chain of logical implications beginning with one statement in a theorem of logical equivalence and ending with another. Unfortunately, students at this level often struggle with mathematical logic, and in particular, many students struggle to form these chains of reasoning (Dorier & Sierpinska, 2001).

Regarding inquiry-oriented teaching, there are several ways to define inquiry depending on the context or academic subject. Rasmussen and Kwon (2007) characterize student inquiry in a mathematics class through Richards' (1991) definition of mathematical inquiry, which is the mathematics of mathematically literate adults. Thus, mathematical inquiry involves participating in mathematical discussion, solving new problems, listening to mathematical arguments, and proposing conjectures. With this interpretation of mathematical inquiry, inquiry-oriented teaching involves creating opportunities for students to engage in mathematical inquiry.

Setting and Methods of Analysis

This study was conducted through an action research methodology that began with a pilot study in the summer of 2015 and continued into the fall of 2015 and spring of 2016. Each of these action research cycles consisted of research in an introductory linear algebra course that was taught by the researcher. While the results of the pilot study and fall 2015 research cycle informed the spring 2016 research cycle, this report will primarily focus on the spring 2016 cycle.

In the spring of 2016, the researcher taught an introductory linear algebra course at a large state university in the Pacific Northwest. The class consisted of sixty students; the majority of these students were engineering majors, while others were mainly math and computer science majors. The course was a two credit course, which placed considerable time constraints on the instructor. As a result of these constraints, inquiry-oriented teaching activities were largely reserved for concepts closely related to logical implication connections. In particular, students worked on several activities focusing on span and linear independence, as these concepts play parallel roles in the first two theorems of logical equivalence; two of these activities are presented in Figure 2.

Without solving a linear system or using any elementary row operations, determine whether the following sets of vectors span the given space. For each set of vectors, formulate a conjecture about span based on that set.	Without solving a linear system or using any elementary row operations, determine whether the following sets of vectors are linearly independent or dependent. For each set of vectors, formulate a conjecture about linear independence or dependence based on that set.
• Do the vectors $\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}$ span all of $\mathbb{R}^2$?	• $\left\{\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}\right\}$
• Do the vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}$ span all of $\mathbb{R}^3$?	• $\left\{\begin{bmatrix}1\\3\\-5\end{bmatrix}, \begin{bmatrix}3\\9\\-10\end{bmatrix}\right\}$
• Do the vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$ span all of $\mathbb{R}^3$?	• $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-3\\-4\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}\right\}$
• Do the vectors $\begin{bmatrix}1\\7\\0\end{bmatrix}, \begin{bmatrix}3\\8\\0\end{bmatrix}, \begin{bmatrix}-2\\9\\0\end{bmatrix}$ span all of $\mathbb{R}^3$?	• $\left\{\begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}\right\}$
• Do the vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}$ span all of $\mathbb{R}^2$?	

Figure **2**: Each of these activities were designed as opportunities for students for explore span and linear independence in ways that would allow them to develop logical implication connections involving span and linear independence.

Data on student mathematical connections was largely collected from interviews with nine students from the aforementioned class. These interviews were conducted shortly after the Invertible Matrix Theorem had been covered in class. Each interview was approximately an hour in length and consisted of students finding the solution set of a linear system, a vector equation, and a matrix equation. Each problem lent itself to a different theorem of logical equivalence. For example, the coefficient matrix corresponding to the linear system had a pivot position in every row, thus making every statement from Theorem 1 true for that coefficient matrix. Similarly, the vector equation lent itself to Theorem 2 and the matrix equation lent itself to the IMT. After an interviewee completed one of the problems, the interviewee was asked to describe his or her work. The researcher would then present the interviewee with a list of vocabulary terms that had been discussed in class. The interviewees were asked to discuss as many of the vocabulary terms as they could and how they relate to each problem. The interviewer would often ask for justification of particular claims that the interviewee had made and would sometimes directly ask the interviewee whether he or she could discuss a particular vocabulary term. This was all done in an attempt to determine what logical implication connections the interviewees could evoke that incorporated some of the familiar terms involved in the theorems of logical equivalence.

In analyzing the interviews, the researcher attempted to determine what mathematically correct logical implications corresponding to the three theorems of logical equivalence each interviewee evoked. Evidence of logical implication connections took several forms. Many logical implications involved words such as if, then, means, because, and so. For example, "The vectors, if a linear combination of those produce every single vector in that space, then they span that space" would be considered a logical implication connection. While many logical implications were evoked entirely by the interviewees, some logical implications were evoked as a result of an interviewee responding to a question asked by the interviewer. After determining what logical implications the interviewees evoked, the researcher then attempted to determine which of these connections were relational connections; this was largely accomplished by determining which logical implication connections a student was able to justify.

Results

In general, the interviewees tended to evoke more connections relevant to the second theorem of logical equivalence than they did the first. This is in itself not entirely surprising; the theorems presented in Figure 1 were the versions of the theorem discussed in class, and the second theorem contains more statements than the first. As the concept of invertibility and the IMT were still new to the interviewees, they tended to evoke relatively few connections exclusive to the IMT. Due to this, the results reported here will primarily focus on connections that are not exclusive to the IMT.

Logical Implication Connections Relevant to the First Theorem of Logical Equivalence

In evoking connections relevant to the first theorem of logical equivalence, the interviewees tended to reference span, pivot positions, and linear combinations. Interestingly, several interviewees presented interpretations of span that were likely consistent with the formal definition of span, but interviewees rarely explicitly referenced the formal definition. That is, several interviewees were able to provide geometric interpretations of span or were able to describe span via linear combinations without explicitly saying the phrase "linear combinations." For example, consider Will's explanation of why two particular vectors span $\mathbb{R}^2$:

Will: Because these two aren't scalars of each other, they're going in different directions. They each have their own x_1 and x_2 components. If they were the same, they'd just end up looking like that.

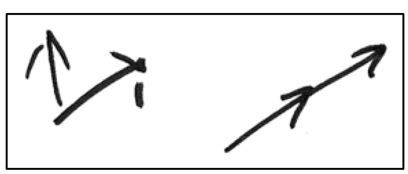

Figure **3**: Will provided a geometric description of what it means for two vectors to span $\mathbb{R}^2$. The illustration on the left represents an example Will provided of two vectors that span $\mathbb{R}^2$, while the illustration on the right represents an example Will provided of two vectors that do not span $\mathbb{R}^2$.

Seth provided an explanation that incorporated both matrix and geometric interpretations of span:

Seth: If you had a matrix, let's take this one [Seth draws the 2×2 identity matrix], then this one would span all of $\mathbb{R}^2$ because no matter how you rearrange this, you can create – uh, I'll expand it [Seth then changes his matrix to the 3×3 identity]. So uh, this one can create, because you can multiply this by infinitely many scalars outside for each row, you can create infinitely many planes, like, if you think about this geometrically, planes in any coordinate system.

Jimmy appeared to allude to linear combinations, but also referenced a geometric interpretation of span:

Jimmy: Well, for spanning, you want, uh. Every direction to be covered, every direction on the plane to be covered by some scale, er, some combination of those vectors, I think.

Jason explained that three particular vectors span $\mathbb{R}^3$ because "they go in different directions. They're not, uh, linear combinations of each other." While Jason referenced linear combinations, it was not in reference to the formal definition of span, but rather, as a description of what must be true of a set of vectors in order to span an entire space. Bill heavily alluded to linear combinations but did not explicitly reference linear combinations:

Bill: Span is having the ability to make any vector within a space. You can, like I said, manipulate any piece of the outcoming vector. You can change it by changing one of the more, one of the scalar multiples along there, not scalar multiple, scalar weights along the way you go. In this case we did at x_1, x_2, x_3. If you could change each of those to then manipulate one of the vectors in the overall value within the system, you could then change the outcome. That goes into the span. If you can do that then it does span $\mathbb{R}^3$, it does span $\mathbb{R}$ whatever. It has the ability to reach any vector, any point within that space.

Bill's description of scalar weights and manipulating vector may provide evidence that he is describing linear combinations, although he does not explicitly reference linear combinations. Thus, Bill's interpretation of span is likely consistent with the formal definition of span, even if he cannot provide the formal definition.

It should be noted that while several students provided geometric descriptions of span, geometric interpretations were not heavily emphasized in class. They were briefly referenced from time to time, but concepts were never defined from a geometric perspective. Further, prior to span being defined, the class had discussed the problem of determining whether any vector in an $\mathbb{R}^n$ space can be expressed as a linear combination of a particular set of vectors. However, when span was formally defined, it was defined more generally as the set of all linear combinations of a set of vectors. Despite this, several interviewees appeared capable of determining whether a particular set of vectors spans an $\mathbb{R}^n$ space by determining whether the vectors were linearly independent, linear combinations of each other, or go in different directions. Thus, it is likely that students developed these alternative, yet mathematically correct, interpretations of span as a result of the inquiry-oriented activity previously described.

Logical Implication Connections Relevant to the Second Theorem of Logical Equivalence

Many of the connections the interviewees evoked relevant to the second theorem of logical equivalence involved pivot positions, linear independence, and basic and free variables. The interviewees tended to refer to basic and free variables in their logical implications more than any other concept; this was particularly interesting, as the interviewees from the previous semester tended to refer to pivot positions more than any other concept. The interviewees also appeared to have serious misunderstandings of the homogeneous equation. For example, Fred appeared to believe that *any* homogeneous equation can only have the trivial solution:

Interviewer: If I had given you zeroes here instead of 3 and 2, would that still have a solution? That homogeneous linear system?

Fred: Yes, because homogeneous equation always have at least one solution, which is the trivial solution.

Interviewer: And what was the trivial solution again? Can you remind me one more time, what was that?

Fred: Trivial solution is $A\mathbf{x} = \mathbf{0}$, so zero is always the solution, for example $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Jason appeared to hold a similar view:

Interviewer: Can you define homogeneous equation for me? What does that mean?

Jason: It means there's only one solution. I can't remember what it was.

Seth appeared to confound the trivial solution with the homogeneous equation:

Interviewer: Do you remember what the trivial solution is?

Seth: Uh, it's when $A\mathbf{x} = \mathbf{0}$.

Cecily, who was incredibly close to relational understanding of the connection between pivot positions and linear independence, made a similar mistake:

Interviewer: Why is it that not having a pivot position in every column tells you that these columns cannot be linearly independent?

Cecily: Because if there's not a pivot position in every column, then it can have infinitely many solutions. And for it to be linearly independent, it can only have the trivial solution.

Interviewer: Okay. So what can only have the trivial solution?

Cecily: The matrix, the linear system.

Interviewer: Okay. So what is the trivial solution?

Cecily: That's $A\mathbf{x} = \mathbf{0}$, right?

I reminded Cecily that she was describing the homogeneous equation before asking her what the trivial solution is; she claimed she did not know. As these interviewees had misconceptions about the homogeneous equation, it is likely that any connections evoked that involved the homogeneous equation could only be instrumental; further, it suggests that these several interviewees have misunderstandings of the formal definition of linear independence. Indeed, similar to span, some students provided geometric descriptions of linear independence; Bill, for example, claimed that a particular set of vectors was linearly independent "because they can all point in different directions." Others essentially appeared to instead interpret linear independence through basic and free variables instead of the homogeneous equation. For example, consider Seth's explanation of linear independence:

Interviewer: How, how come if it has no free variables, that means it's linearly independent?

Seth: Well if it has no free variable, that means that there was a pivot in every column, which would mean that it would have no free variables, because there wouldn't be, like, say a 2 out here. And, uh. This vector would always have a solution.

Seth's response was not unique. Several other interviewees tended to refer to basic and free variables often in their descriptions of linear independence, as did many students on one of the class exams.

It should be noted that when we discussed the homogeneous equation in class, we did not do this through an inquiry-oriented activity; I believed that the concept did not warrant such an activity, as students in the pilot study and fall semester appeared to understand the homogeneous equation fairly well through a mixture of lecture and whole class discussion. Looking back at the day that we discussed the homogeneous equation in the spring semester, I noticed that we concluded our initial coverage of the homogeneous equation with the following discussion of a homogeneous matrix equation that only had the trivial solution:

Instructor: So, could I have free variables?

Student: No.

Instructor: Kay. I can't have any free variables. Why not? Why can't I have free variables?

Student: You'd have infinitely many solutions.

As this was how we concluded our initial coverage of the homogeneous equation, it is possible that some students essentially replaced the concepts of trivial and nontrivial solutions with basic and free variables. That is, students made an instrumental connection between the homogeneous equation and basic and free variables, and as they did not quite understand what the homogeneous equation is and when it has nontrivial solutions, they instead considered when the homogeneous equation would have free variables. Then, when the formal definition of linear independence was provided in terms of the homogeneous equation, they tended to view linear independence in terms of basic and free variables instead of the homogeneous equation. Thus, many students in this semester relied on their instrumental connection between the homogeneous equation and free variables in order to compensate for their lack of understanding of the connection between the homogeneous equation and linear independence, thus interpreting linear independence largely through basic and free variables. This was likely exacerbated by the aforementioned linear independence activity, in which students could refer to the familiar concept of basic and free variables to determine whether the sets were linearly independent or not. Students who had not come to rely as heavily on free variables likely developed more geometric interpretations of linear independence as a result of the linear independence activity, as the sets in the activity were in $\mathbb{R}^2$ and $\mathbb{R}^3$, which can be easily visualized. Once these students had developed a more geometric interpretation of linear independence, they may have felt that the formal definition was no longer necessary for an understanding of linear independence.

Conclusions

In light of the results from the interviews, it appears as though the inquiry-oriented activities that focused on span and linear independence were successful in creating opportunities for students to develop their own interpretations of span and linear independence. The role of geometric descriptions in the class was limited, yet several students developed interpretations of span that appeared to be more geometric in nature; further, these interpretations often heavily alluded to linear combinations while not explicitly referencing linear combinations in an algebraic sense. Regarding linear independence, the activity allowed students to reinforce interpretations of linear independence that heavily relied on basic and free variables; it also allowed students to develop geometric interpretations of linear independence that relied on the notion that linearly independent vectors are not linear combinations of each other.

While the inquiry-oriented activities were successful in creating opportunities for students to form their own interpretations of span and linear independence and how they relate to other concepts, the implementation of these activities could be improved. Students appeared to replace their understanding of the formal definition of span and linear independence with the understanding they developed as a result of the activities. This may have limited the students' ability to form logical implication connections involving these concepts and their formal definition. In retrospect, the instructor should have devoted time to exploring how these student developments relate to the formal definitions of span and liner independence. Investigating how the inquiry-oriented activities could be improved in this regard remains an avenue for future research.

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Mathematicians' Collaborative Silences

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In this paper I re-analyze the transcripts from Smith (2012) to investigate silence in mathematicians' collaborative work. I provide an existence proof that silence, at times, forms a significant aspect of mathematicians' embodied work. Based off a discussion of the nature of embodied interaction, the paper concludes that it is likely that silence forms a significant aspect of mathematicians' collaborative work, more generally, both in discovering new mathematics, and in ordering the mathematicians together toward the task of discovering new mathematics. Because this use of silence is different from that of everyday conversation, this raises important pedagogical questions regarding students' apprenticeship into the mathematics profession.

Key words: [Silence, Ethnography, Mathematical Practice, *Habitus*]

It may seem paradoxical to study the use of silence in mathematical activity—to study, as it were, the times when mathematicians are *not* doing something. However, as St. Ambrose notes, we are at least as likely to err due to inopportune speech as to silence—for instance, we all can remember blurting out a comment that derailed a productive and interesting conversation, when we should have listened carefully, and held our tongue. Indeed, we often "speak because [we] do not know how to be silent" and are "rarely silent, even when nothing is gained by speech." Therefore "it is more difficult to practice silence than speech," and "we ought to learn silence, so we can speak well" (*De Officiis*, I.5).

The appropriate uses for silence, like the appropriate uses for speech, however, are context specific. Not only must we learn how to be silent, we must learn how to be appropriately silent, in the various activities and interactions we undertake. For instance, in English, silence following an invitation can communicate a rejection of the invitation (Liddicoat, 2011); and this fact needs to be learned by second language learners. Or, both teachers and students learn to respond to the silence at the end of a designedly incomplete utterance (Koshik, 2002) which marks that a teacher's statement has been left incomplete and so transformed into a question. Or, to take an extreme example, in some Apache interactions, silence is the predominate form of communication, and speaking during the silence would be disruptive (Basso, 1970).

Mathematical collaboration, is, like all concerted activity, coordinated and accomplished through perceptible non-linguistic actions (e.g. eye-gaze, speech rhythm), including silences, through which people position themselves and their colleagues around the tasks they undertake, and direct themselves and their colleagues together toward achieving common goals (McDermott, Gospodinoff, & Aron, 1978; Liddicoat, 2011). Because actions have this social function the bodily actions mathematicians use in their collaboration, even when those actions do not signify mathematical realities, are important aspects of mathematical collaboration.

In learning to be a mathematician, then, it is important to learn not only the disciplinary norms regarding kinds of speech and reasoning, but to learn when speech would be *inopportune* and disruptive; and to learn to respond appropriately to a colleague's silence. One can investigate empirically whether mathematicians employ silence in their collaborative activity. Moreover, because silences longer than a second are unusual in everyday conversation, and communicate the need for someone to speak (Liddicoat, 2011), if silences are used more extensively than in everyday conversation, that fact is pedagogically important.

Theoretical Perspective

In an influential essay (1973 [1935]) Marcel Mauss, a leading French Anthropologist, developed the concept of technique or technology of the body. This concept continues to be employed in leading anthropological works (e.g. Asad, 2003; Mahmood, 2005; Strhan, 2015), literary theory (e.g. Allan, 2016, though he doesn't explicitly use the term; Saussy, 2016), and theories of embodiment (e.g. Saussy, 2016). Without an analysis of the techniques of the body, an anthropological investigation of technology is incomplete (Mauss, 2007).

Mauss defines a technique as "an action which is *effective* and *traditional*" (p. 75), that is, learned. A technique of the body, then, is an effective, learned action which employs the body as an instrument for achieving a goal. Because the French, like the English, terms that could be employed to describe a technique of the body do not quite convey the correct sense, Mauss used the Latin term *habitus* to refer to a technique of the body, a usage which has been carried over into the English writings that draw off his work.

Mauss used the concept of technology of the body to explore diverse activities like walking, swimming, sleep, sexual positions, child-birth, etc. and argued that in fact all our bodily techniques that, as adults, we employ in our day-to-day activities are technologies of the body; that is, in part psychological, in part physiological, and in part learned. For instance, the human body is well suited for walking, but the particular gait we employ is culturally specific, and so learned.

It is the learned aspect of *habitus* that makes his notion educationally relevant. In addition to learning to know particular ways of reasoning, and particular facts, there are particularities of *habitus* that we learn as we apprentice into various professions. Because mathematicians are, like all of us, embodied, the work of doing mathematics includes a particular *habitus* and in their course of studies, students are apprenticed into this *habitus*.

Though the concept is widely used and valid, one shortcoming of Mauss' description of techniques of the body is that he doesn't adequately attend to the spatial and interactional aspects of our *habitus* (Crossley, 1995). Techniques of the body not only achieve goals like walking across a room or proving a mathematical theorem, they accomplish what McDermott, et al. (1978) call positionings, the ongoing, in the moment, work of orienting a group of people together toward a common activity.

Two aspects of positionings are worth mentioning. First, the guestures involved in positionings are often not overt, macro gestures like "look over there", but the whole panoply of subtle cues like eye-gaze, body posture, and sort of speech (or lack of speech) particular members of a group use. Indeed, all the perceptible gestures potentially contribute to the positioning, and cannot be, over the long-term, opposed to the over-all configuration the group has taken up. The gestures are not contrary to the order because of the second aspect worth mentioning: When people depart from an order, the group may nevertheless maintain that order. They do so when the departure treated as a breach of the order, and the person who departed, directed (through overt or covert means) to return.

The perspective sketched here is similar to the one employed by Nemirovsky, Kelton and Rhodehamel (2013), which "propose[s] that mathematical knowing is constituted by—not dialectally related to—embodied tool use." However, it differs in two key ways. First, it recognizes that, as Muass states, our "first and most natural technical object, and at the same time technical means, is [the] body" (p. 75), so that the body *itself* is a tool. This addition facilitates an attention to the ways learned embodied actions constitute mathematical knowing, even in the absence of, or in abstraction from, other technologies. Second, my approach is more attentive to the ways we use our bodies not only as a tool for doing mathematics, but as a tool for positioning each other in a given mathematical task, and so using our bodies *in concert*. An emphasis on the *habitus* also has the potential to make an explicit link with examinations of the Politics of mathematics education and so and critically interact with Foucauldian, Bordieuan, and perhaps even Marxist examinations of mathematics education (Burkitt, 2002; Asad, 2003; Mahmood, 2005; Bang, 2014; for the Politics of mathematics education, see e.g. Pais & Valero, 2012; Kollosche, 2016).

Literature Review

There has been relatively little anthropological or sociological investigation of the materiality of doing mathematics, and the face-to-face communicational practices indigenous to the mathematics community. Indeed, sociological investigations of doing mathematics *at all* are rare (Greiffenhagen, 2006; Greiffenhagen & Sharrock, 2005).

Those studies that have investigated the doing of mathematics have tended to focus on the linguistic aspects of doing mathematics. For instance, Weber (2008) and Weber and Mejia-Ramos (2011) investigated the techniques mathematicians use to evaluate proofs. Their investigations, however, focused on the types of reasoning (e.g. formal or informal) and the goals of reading proofs; but not on the techniques of the body the mathematicians used to undertake those styles of reasoning and reading, and to achieve their goals.

On the other hand, sociological investigations of the embodied nature of abstract mathematical reasoning—that is, of mathematical *habitus*—though few in number, do establish that mathematics is not a purely mental phenomenon, but that the doing of mathematics employs various bodily and material techniques and technologies. For instance, Greiffenhagen (2006; 2014) demonstrates the importance of the blackboard, and its spatiality, in doing a mathematics proof, particularly through the facilitation of particular gestures which embody mathematical activity. For instance, when attempting to create "space" between two numbers in order to prove an inequality, a mathematician may orient him, or her, self to that metaphorical space by gesturing along the blank spaces created by blackboard inscriptions.

Smith (2012) also investigated the material practice of doing mathematics. He found that when mathematicians struggle with mathematics, their actions and perceptions make use of particular material and bodily aspects of their world. For instance, several mathematicians he studied exhibited what he called *proximal inhibition*, in which they are both drawn toward a physical representation of the mathematics, and yet, because of a lack of knowledge of *where* to approach the symbolized mathematics, are held back.

These three studies establish that mathematicians use various *habitus* in their doing mathematics—mathematics is not a purely mental operation, rather, like a harp player, or someone walking across the room, mathematicians employ their body as an instrument for accomplishing their work. However, the analysis of this *habitus* has never attended to silence.

Silence also tends to be a neglected, but important, aspect of communication (Saville-Troike, 1989, p. 148) and of *habitus* (Acheson, 2008). Furthermore, communities have specific communicational norms regarding the toleration of silence in interaction, the meaning of silences, and the function of silences in achieving various goals, and so an investigation of the functions and uses of silence in a particular community is pedagogically relevant: If silence

Table 1

	Episode name in Smith, (2012).	Total time (s)
Joseph & Bill	Tubes	70
	Chain Map	241
	Odd Scalars	158
	Indexical Map	120
	Finger Moves	140
Matt & Bart	Bart Pushed Back	67
	Inverted Pyramid	183
	Adjusting a Triangle	167
Fay & Martha	Graph Confining	158

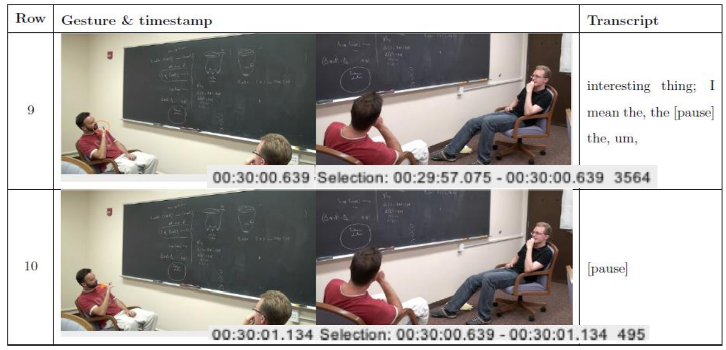

Figure 1: A page of Smith's (2012) transcript (p. 261), taken from episode 3 (Joseph & Bill). The top frame includes an unmeasurable silence, the bottom contains the first half second of a 5 second silence (at the beginning of the episode).

forms a part of the *habitus* mathematicians employ in their work, in positioning themselves and their colleagues around the (symbolized) mathematics, and so orienting themselves to the task at hand and achieving the goal of discovering new mathematics, that use of silence is learned.

This leads to the research question addressed in this paper: Is silence an aspect of the *habitus* cultivated by mathematicians, that is, an aspect of their body technique employed both in their collaborative work of doing mathematics, and in their positioning themselves and their colleagues in relation to their task of doing mathematics.

Methodology

For this study, I reanalyzed the transcripts included in Smith's doctoral dissertation (2012). Smith's research question was "What are some ways in which mathematicians structure their experiences of struggle while working in pairs in person on a current problem?" (p. 46) To answer this question, he recorded ten total hours of collaboration between practicing mathematicians (Matt & Bart), and between PhD students nearing their defense and their major professor (Joseph & Bill and Fay & Martha). (I use the pseudonyms Smith used.) He included the latter two pairs based on a judgment that their collaboration was sufficiently similar to the collaboration of mathematicians to answer his research question but also regular enough to schedule video-recording. After collecting the data, he narrowed his investigation to 9 episodes, between 1:09 and 4:01 long, 21:43 total, that were particularly salient for his research question. Joseph and Bill's activity was analyzed in 5 episodes, Matt and Bart's in 3, and Fay and Martha's in 1.

Smith (2012) includes a transcript of all the episodes in appendices. These are intended to allow the reader to verify the results of his in depth microethnography (Erickson, 1996). The transcript consists of 1,029 screen shots—0.79 frames per second—time stamps, arrows indicating motion, and transcriptions of speech (see Figure 1 for a sample). If the participants were silent for all or part of the time recorded in a screen-capture, the transcript says [pause]. 309 frames (30%) record pauses, though the length of many of these pauses is unmeasurable (I assume it is less than 1 s. for this analysis). The only text accompanying a number of frames, however, is [pause], and the length of these silences can be measured by recording the initial time-stamp of the frame the silence starts on, and the final time-stamp of the frame the silence ends on. When the length of a silence can be measured, I call it a measurable silence.

To determine how prevalent silence was in these mathematicians' cooperative work, I recorded the length of all the measurable silences by noting the initial and final time stamps for any pauses in their speech. In five frames there was only a very small amount of text (for instance, one letter), but the rest of the text was recorded as [pause]. I included these frames as parts of measurable silences, though I adjusted the total time of the slide slightly. For this analysis I did not attend to their gestures, but it is worth noting that any time was spent writing on or erasing the board in only three silences, all under five seconds.

Some very short pauses were measurable, and, because Smith was not attending to silences, we cannot conclude that pauses shorter than a particular time are measurable—Fay and Martha's session in particular includes a number of silences that would be unmeasurable in the transcripts of other groups' works. In order to have a relatively uniform grain-size, and because silences shorter than 1 second are common in American English (Liddicoat, 2011), only silences longer than one second are included in this analysis. Silences shorter than one second are assumed to be a part of the natural rhythms of speech, and not a separate phenomenon.

Because ten two second silences may be very different from one twenty second silence, to get more information about the length of the silences, I also collected data on silences longer than 5, and 10, seconds.

Table 1 lists the names Smith (2012) gave to the episodes he analyzed, and the total length of each episode. The episodes are ordered, first, based on participants, and then, within each category, on percentage of time spent in 1+ second pauses. Throughout this paper, episodes are listed in the same order as in Table 1. In Figure 4, where the episodes are arranged vertically, the same top-to-bottom order is preserved. In Figure 2 and 3, the episodes are arranged horizontally, and the top-to-bottom order is preserved left-to-right in those Figures.

Results

The results are summarized in Figures 2-4. Figure 2 presents the percentage of time in each episode spent in silences longer than one, five, and ten seconds, respectively. In five of the nine episodes, four of them involving Joseph & Bill, the mathematicians spent more than 20% of the episode silent; in two of the episodes, more than 50% was silence. In four of the nine episodes, more than 20% of the time was spent in 5+ second silences. In three of nine episodes, more than 15% of the time was spent in silences longer than ten seconds, and in a fourth, 13.4% of the time was spent in silences longer than 10 seconds. In two episodes, more than 35% of the time was spent in silences longer than 10 seconds.

Figures 3 and 4 help visualize that use of silence. Figure 3 displays all the silences longer than 1 second in each episode. Because the episodes are not all the same length, the total time of each episode is displayed as a secondary category. Figure 4 contains a timeline of each episode. Time in which both mathematicians were silent are symbolized with a line, time in which either spoke, or the silence was shorter than a second, are displayed by blank space. An inspection of Figure 4 suggests that the mathematicians' work may be divided into periods in which silence is common and periods in which silence is rare; however, there is not enough data here to address that conjecture.

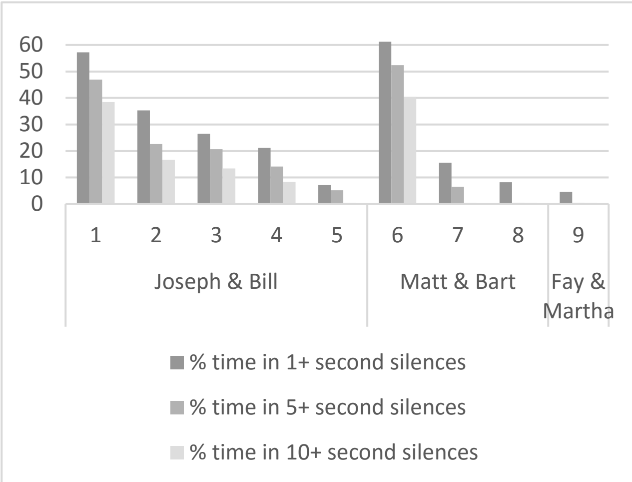

Figure 4: Percentage of time, per episode, in 1+, 5+, and 10+ second silences

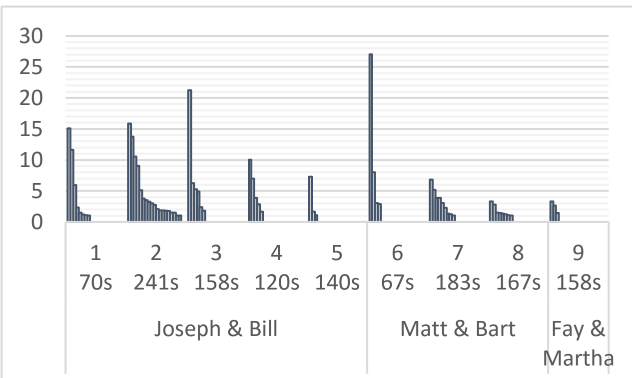

Figure 3: Length of each silence, in seconds, in each episode. The subcategories on the horizontal axis list the total time of each episode.

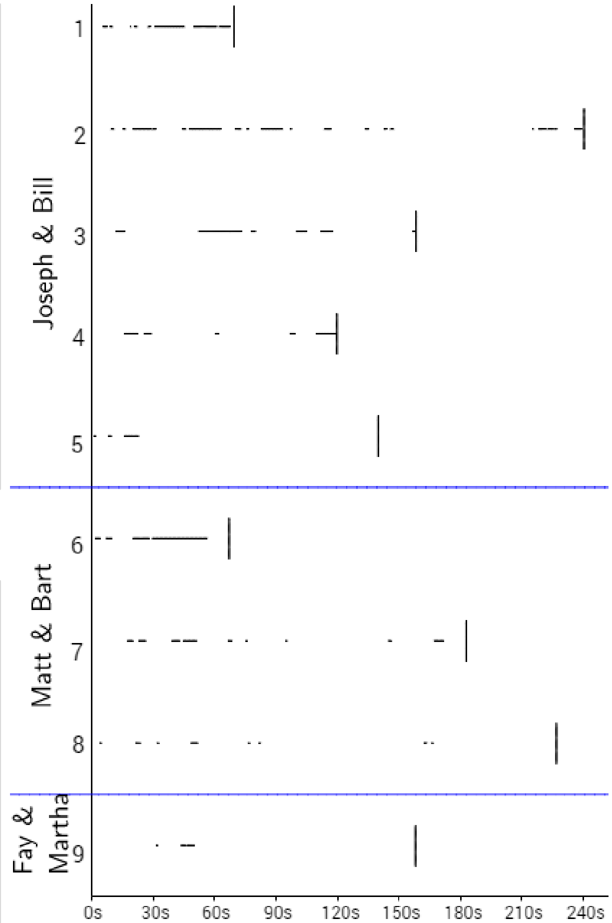

Figure 2: Timelines of the episodes analyzed in Smith (2012). Silences of over one second are represented with a line, times when the mathematicians were speaking are left blank.

Discussion

It would, obviously be invalid to try to generalize these findings statistically. The significance lies in the fact that, in episodes that a researcher selected as an interesting example struggle with mathematics, without any reference to silence, the researcher still selected episodes containing a very large amount of time in lengthy, mutual, silences. These are not times of inactivity, but of intense activity. Moreover, the fact that the mathematicians engaged in lengthy silences implies that these silences are a part of efficient, learned, bodily mathematics activity. That is, they form an aspect of the *habitus* of four of the mathematicians whose work was analyzed in Smith's dissertation. (Fay and Martha are not seen employing it much.)

Furthermore, in American English, silences of over a second are often treated as a lack of speech (Hepburn & Bolden, 2013), and people orient to the silences themselves. Contrary to the usual expectation, the mathematicians whose work is analyzed here are not only silent for long periods of time, they are silent *together*. That the silences are co-produced is important for two reasons.

First, this provides evidence for the generalizability of the episodes analyzed here. In not quickly breaking the silences, and attempting to redress the breach of the conversation, the mathematicians treated the lengthy silences as an *entirely normal* element of their work. That the silences are treated as normal indicates that while silence may not be a universal feature of the *habitus* of mathematicians, it is a relatively common aspect of the mathematical *habitus*.

Second, because in all our activities we are engaged in positionings of the people we are interacting with, and because silences are usually noticeable, it seems likely that silence was an element of their mutual positioning—that is, that the mathematicians oriented each other to the task at hand, in part, through their silence.

Conclusions

In these episodes, silence was an important aspect of the mathematical *habitus*, that is, both in its orientation to solving a mathematical conundrum, and critically, in positioning a peer toward the same goal. The mutual tolerance for silence is in contrast to findings regarding silence in student group-work. In student work, silence is sometimes an aspect of the effective embodied practices students employ to answer vexing mathematical problems; however, other students often read the silence as a problem, and attempt to rectify the problem by eliciting a response (Petersen, 2015; Petersen, under review).

It may, at first glance, be that mathematicians use silence more extensively in their attempts to resolve vexing questions than students do, or they may use it roughly the same amount. However, they have learned not to treat the silence as a breach of group work, and a lack of speech needing rectified, but to respond to it, at least sometimes, by silently engaging with the mathematics. This response to silence is pedagogically relevant, because it differs sharply from student responses, and from the use of silence in every-day conversation.

We can also, perhaps, conclude form this response that in fact, mathematicians incorporate silence into their work more extensively than students do—if they did not, they would treat the silences like students have tended to—though there may also have been specific gestures accompanying the silence that communicated the quality of the silence, and its use.

The difference between the mathematicians' use of silence and the students' use raises several questions for future research: Is there something specific to the practice of mathematics that mathematicians utilize silence in their work, and this use of silence needs to be learned as students are initiated into doing mathematics? Was there something specific about the gestures that accompanied the silence in these episodes that communicated that the silence was a potentially productive, active silence, not an awkward pause, or lapse in the work? How do various teaching styles help students to learn to use silence as mathematicians do, and how do they inhibit that learning? If these silences were accompanied by gestures that communicated the potential productivity of the silence, how are we teaching our students to subconsciously notice those gestures, and to be positioned toward engagement with the mathematics, or at least, to recognize that their peers are actively working, even if not speaking? And finally, are there technological aspects to these silences—do different external technologies provide affordances or hindrances to employing silence like these mathematicians did? Further fine-grained analyses of student and professional mathematical practices are needed to address these questions.

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Graduate Student Instructors learning from peer observations

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Graduate Student Instructor (GSI) professional development addresses an urgent need to improve STEM retention. This paper focuses on a semester-long professional learning community in which six mathematics GSIs engaged in regular cycles of peer observation, feedback, and reflection. In contrast to most GSI development work, this approach emphasized that GSIs give, not just receive, peer feedback. Analyses of post-semester interviews indicated that all GSIs enhanced their noticing of students. Moreover, insight into peer feedback was developed along three dimensions: (1) the importance of being an objective observer, (2) the impact of working with equal-status peers, and (3) the value of critical feedback.

Keywords: Graduate Student Instructors (GSIs); Graduate Teaching Assistants (GTAs); Professional Development; Noticing; Reflection

Introduction

Introductory college calculus is a major barrier for students pursuing STEM careers (Bressoud, Carlson, Mesa, & Rasmussen, 2013); low student success rates in calculus contribute to a lack of persistence, which has become an issue of national concern in the US (PCAST, 2012). Fortunately, a growing body of evidence highlights the positive impact of student-centered teaching practices (Freeman et al., 2014), particularly in improving student persistence (Kogan & Laursen, 2014). Despite this evidence, college mathematics classrooms are still dominated by instructor-centered teaching (Lutzer, Rodi, Kirkman, & Maxwell, 2005). Thus, there is an urgent need to improve instruction in introductory undergraduate mathematics courses in the US.

Graduate Student Instructors (GSIs) play a crucial role in teaching these introductory mathematics courses. Yet, GSIs typically receive little professional development (Austin, 2002). To implement student-centered teaching practices, GSIs need to learn to attend to and respond to student thinking (Franke, Carpenter, Levi, & Fennema, 2001; Sherin, Jacobs, & Philipp, 2011). Accordingly, this paper explores how peer observations help GSIs enhance their noticing of student thinking. In contrast to observations by faculty or more experienced GSIs (Miller, Brickman, & Oliver, 2014), peer observation supports noticing through giving, not just receiving, feedback. It also helps alleviate the costs of scaling and sustaining traditional methods of observations, which may create an undue burden on faculty members and more experienced graduate students. In the present study, six mathematics GSIs met regularly in a professional learning community (PLC; Stoll, Bolam, McMahon, Wallace, & Thomas, 2006) and engaged in cycles of peer observation, feedback, and reflection through the PLC.

This paper addresses two research questions: (1) how was GSI noticing impacted by peer observation? and (2) which features of peer observation supported or inhibited noticing? Analyses of post-semester interviews indicated that all six GSIs felt more reflective about their teaching. Moreover, they described: the importance of being an objective observer, the impact of working with equal-status peers, and the challenges of providing critical feedback. Based on these results, this paper argues that peer observations provide a number of additional learning benefits that extend beyond traditional observations of GSIs.

Theoretical Framing

Enacting student-centered pedagogies requires GSIs to build on the resources that students bring to the classroom. To build on these resources, GSIs need to engage in three related processes: attending to, making sense of, and responding to student thinking (Jacobs, Lamb, & Philipp, 2010). The study of such decision making comprises the field of *teacher noticing* (Sherin et al., 2011). The goal of the present work was to help GSIs enhance their noticing of students, rather than focusing primarily on themselves.

PLCs can enhance noticing, as instructors reflect on their practice with peer support (van Es & Sherin, 2008). PLCs are *communities of continuous inquiry and improvement,* with five key features: (A) shared values and vision, (B) collective responsibility, (C) reflective professional inquiry, (D) collaboration, and (E) group, as well as individual, learning (Stoll et al., 2006). In this study, the PLC gave GSIs opportunities to provide feedback, not just receive it. Given the benefits of peer assessment (Reinholz, 2015c), it was hypothesized that this would enhance noticing more than simply receiving feedback from others. Recognizing that not all feedback is equal (Hattie & Timperley, 2007), GSIs were helped to provide critical, supportive feedback to their peers. When feedback focuses on *processes*, it is more likely to draw attention to student thinking, in contrast to feedback focused on *people*, which will draw attention to the GSIs themselves (Reinholz, 2015b). Person-focused feedback, such as praise, actually inhibits learning (Hattie & Timperley, 2007; Mueller & Dweck, 1998).

Providing feedback position GSIs as competent (Engle & Conant, 2002), and there is evidence that individuals may learn as much from providing feedback as receiving feedback (Reinholz, 2015c). Thus, conducting observations rather than just being observed provided GSIs with opportunities for enhanced noticing. In particular, it allowed GSIs to enter the classroom as a third party without the cognitive load of teaching. This paper adds to the study of noticing and GSI professional development by elaborating these opportunities for improved noticing.

Method

Participants

Six GSIs teaching either calculus 1 or 2 at a large research-intensive university participated in the study. The calculus classes were comprised of (each week): (a) three 50-minute lectures, (b) one 50-minute recitation, and (c) one optional 100-minute workgroup. The GSIs each taught a combination of 3-4 recitations or workgroups. The recitations consisted of GSIs: answering homework questions, completing examples, providing short worksheets, and administering quizzes. The workgroup sessions were collaborative problem solving sessions, modeled on the Emerging Scholars Program (ESP; Treisman, 1992). A key insight from the ESP was that providing students with additional challenge, rather than remediation, was a more effective way to support their success in calculus. The collaborative groupwork sessions were also designed to promote community and collaboration amongst the students.

GSIs received no incentives for participation in the PLC; all four calculus 2 workgroup instructors participated as a part of the department's efforts to improve instruction, and two calculus 1 instructors were chosen by the department to participate. There were four female and two male GSIs, five domestic GSIs and one international GSI, and the GSIs had a variety of teaching backgrounds; the four female GSIs were in their first year of teaching in the department, and the male GSIs had been teaching for a number of years.

Design

The GSIs in the study met as a PLC and conducted regular peer observations during a single semester. The PLC was facilitated by a mathematics educator, who shared videos, articles, and feedback on teaching with the GSIs. The facilitator also assigned short "homework assignments," which required GSIs to implement active learning strategies in their recitations or workgroups. The PLC typically met every other week, for a total of seven one-hour sessions.

To help GSIs develop a shared vision (PLC principle A), GSIs reflected on and discussed their prior experiences as learners during the first PLC meeting. To support collective responsibility (principle B), reflective professional inquiry (principle C), and collaboration (principle D), the facilitator refrained from providing "answers" to the GSIs, instead creating opportunities for collective reflection and discussions of teaching. To support individual and group learning, GSIs had one-on-one conversations with their peers after observation, and the observations were later discussed collectively in the PLC (principle E). To create a safe space for these public conversations, the facilitator promoted a culture of sharing: each meeting began with a debrief on GSI experiences during the past two weeks. Moreover, the PLC discussed norms of giving feedback and normalized struggle as a part of learning.

The GSIs each completed 5-6 peer observations total, with three of their peers (two observations per peer). These observations were based on Peer-Assisted Reflection (Reinholz, 2015a). Each observation involved: (1) the GSI setting goals for the observation, (2) a peer observing and video recording the session, (3) a debrief conversation between the two GSIs after they both observed each other, and (4) a whole-group debrief during the next meeting.

To support feedback and reflection, the GSIs each completed peer feedback forms. The observed GSI began by listing their goals for what they wanted a peer to pay attention to. Then the peer provided specific examples to answer three questions: (1) What opportunities did students have to talk about mathematics?; (2) What opportunities did students have to work with other students?; and (3) What else did you notice, both related to the instructor's goals and otherwise?

Data Sources and Analysis

Pre and post interviews were conducted with the GSIs. In addition, all group meetings were audio recorded, and peer observation forms were copied. The pre-interviews provided context and background on the GSIs; the post-interviews were used as the basis for the analyses that follow. The post-interviews focused on the following areas: teaching philosophy, Peer-Assisted Reflection, experiences exchanging feedback, and beliefs about feedback. The goal of the interviews was to holistically understand how the GSIs experienced exchanging peer feedback, including: how they felt, what they learned, and what challenges they encountered.

All interviews were transcribed and coded by the researcher. The goal of coding was to understand how GSI noticing was impacted by peer observation. Drawing from techniques in grounded theory (Glaser & Strauss, 1967), a first pass of coding was conducted to identify emergent themes. These themes were: (1) objective observers, (2) equal-status peers, and (3) critical friends. Once these themes were identified, the researcher completed a second pass of coding to look for the prevalence of themes across the six post-interviews. The presentation of results that follows is illustrative, intended to highlight important areas for future research. All names below are pseudonyms.

Results

Objective Observers

All six GSIs discussed how they became more reflective about their teaching and improved their noticing of student thinking as a result of observing their peers. For example, Leo contrasted his years of prior experience with his engagement in the PLC,

> I didn't really think that much about teaching. I would sort of hope my students did well on the tests and give me good [ratings], but thinking about the process is something that I've really gotten out of this, and to really try to empathize a little and put yourself in the students' shoes and ask what is this teacher doing, or what should this teacher be doing.

Leo describes that teaching was something he did for many years, but "didn't really think that much about." In contrast, the PLC provided Leo with time and space to reflect on his teaching, learning to put himself in "students' shoes." Leo described the importance of observations, which allowed him to be in a classroom unburdened with the responsibilities of teaching,

> Well when you're not constantly running around helping people with math, it's really easy to tell when groups have sort of lost focus. You also get a better feeling for, I think, the dynamic between people, seeing how certain groups view their teacher…

In other words, peer observations supported Leo to improve his noticing of students, because they provided him with an opportunity to focus *only* on students, rather than all of the other responsibilities associated with teaching. Similarly, Tina described enhanced noticing resulting from being an observer,

> I was able to pay more attention to students' interactions in other workgroups. I guess I learned something about how the students interacted...I feel like there were the different groups. There was the group that had a ringleader that would get everyone going and would lead everything, and then there were some groups that would just not be working, and then there were groups that would be working pretty well together.

Broadening from the specifics of student-student interactions, peer observations allowed the GSIs to compare the different types of classroom environments that their peers created. For instance, Celeste reported on insights developed by comparing three different peer classrooms,

> I knew that I have some problems with my recitations, I knew that I'm not as good as I should be. And observing Tina and Tara and Elayne I saw, OK, this one's not working so probably I should not do it, and this one is working.

Celeste describes noticing what was "working" and "not working" in her peers' classrooms, which informed what she herself would do as a teacher. In this way, being present in a variety of peer classrooms allowed Celeste to see various gradations in teaching practices, which is a key aspect of identifying a high-quality performance (Sadler, 1989).

The observations also provided GSIs with concrete instances of student-centered teaching. For example, Elayne emphasized the value of watching Edgar teach, who focused on "guiding students" rather than just "giving them the answer,"

> Well I learned a lot about just the whole guiding students to the answer instead of giving them the answer, just watching other people- like I keep bringing up Edgar, because I think he was one of my favorite people to observe because he would literally just ask questions the whole time and not give any answers.

Elayne further described how such observations changed her views on teaching,

> A big role that I found this semester was just learning to ask the right questions and having patience… if the student is able to get to the answer on their own instead of you just giving them the answer, it builds their confidence and they retain it longer. Even though it might take three times as long for the student to get there instead of you just showing it to them, in the end they're going to do better in the class and be able to learn the math better if you allow them to get to it eventually.

As the above interview excerpts highlight, observing their peers provided opportunities for the GSIs to notice new things in the classroom. Although changes in GSI teaching practices were not analyzed, prior research showed that working with GSIs in the same department in a similar setting resulted in measurable changes in practices (Reinholz et al., 2015).

Equal-Status Peers

An important feature of the PLC was that the GSIs observed peers of relatively equal status. This contrasts approaches that focus on "experts" (experienced GSIs or faculty members) observing or being observed by "novices" (new GSIs). This allowed the GSIs to form community with their peers. As Leo noted, the PLC helped him shift from competition to collaboration,

> It was kind of a nice supportive environment. I really liked our group meetings where we sort of realized we're all in the same fight. Sometimes there's a little bit of competition, at least in my mind, between [GSIs], because you really want to have good [student ratings] and that's sort of only measured relative to a baseline. So you're like I want to be the best, I want my students to love me the most. But really more interesting are these questions of how do we prepare our students, all of our students, the best, and how do we teach the best. It was good to have actual regular meetings with other teachers in a way that... I don't know. It was a good emphasis on pedagogy, reminding myself why I'm actually there. It's not to get high scores, it's to teach kids math.

The GSIs also discussed the culture of mathematics and the pressure to understand all of the mathematics that they were teaching at a deep level. When the GSIs observed their peers and realized that their peers also found aspects of the mathematics challenging, it was reassuring for them. Even Edgar, who was a relatively experienced GSI, noted that the peer observations helped him overcome aspects of his imposter syndrome,

> [T]hey're also not crazy experts with the material. In learning that I felt more comfortable…There were instances where I was like I know how sequences and series work, and then I'd try and teach somebody how sequences and series work and I'd be like ah, fair enough, I don't know how sequences and series work…just seeing that [other

> GSIs] were also struggling with that is reassuring, that I shouldn't feel the imposter syndrome or anything like that.

Edgar's comments speak to broader cultural issues around mathematics, in which mathematics is often equated with intelligence (Nasir & Shah, 2011) and there is great pressure for the GSIs to act as authorities in the discipline. In observing Tina, Edgar noticed that she would often look at the solutions to problems during in the middle of workgroup sessions, and he realized that it was all right for him to do the same thing,

> So I was like, OK. I've always kept the solutions in my back pocket, so then it feels weird to, like, here are the solutions right in front of the group. Leaving and saying work on this and then refreshing privately, so to speak, so you maintain the aura of knowledge.

Here Edgar describes a concrete strategy, leaving and looking at answers away from the group, that allowed him to maintain what he perceived as his necessary authority as an instructor, while "refreshing" his understanding of the mathematics.

The idea of an "aura of knowledge" speaks strongly to narratives tying mathematics and intelligence (Nasir & Shah, 2011) and the perception of authority that GSIs felt that they had to maintain. Related to these narratives, Tara expressed anxiety in being observed,

> I mean sometimes the students would ask really hard questions and I wasn't completely sure of the answer, so I was worried that I'd be judged for being stupid by the other [GSI] basically.

As Tara expressed, the GSIs felt pressure to be experts. Addressing this "anxiety" has potential to support GSIs through peer observation and in GSI development more generally.

Critical Friends

All six GSIs stated that they found critical feedback to be more helpful than praise. For instance, Celeste discussed how overly positive feedback did not support her learning,

> Tina and Tara…they were always happy with the things that I wanted them to look at and I don't think that's very accurate…I think they wanted to be encouraging, like keep doing that, it's good. But I kind of liked Elayne's [feedback] the best because she actually provided actual things that I have to improve.

Upon receiving this not-so-helpful feedback, Celeste recognized that when she provided the same types of feedback to her peers it must also not be so helpful for them. As such, she altered the feedback she provided to peers to be more critical,

> I know that at the beginning I was like everything's great, nice, you're doing good. So I did that, and I know I did it. I didn't know them or what they would think, how they would react, would they get angry, so I wanted to be positive. But after Elayne I understood that's not the point. I knew when we talked that that's not the point, but it's different when you actually experience it. After that I tried to be more critical.

Celeste describes the initial barrier to providing critical feedback; she did not want to hurt the feelings of her peers or be judged by them. Yet, as she received critical feedback from Elayne, she realized that this was an important part of supporting her peers to grow, and changed her feedback accordingly. Edgar similarly described critical feedback as supportive,

> It's kind of like if I have to write a cover letter for my next job application and I hand it to my good friend Joe, and Joe says this is awesome, well done, I think you're going to get the job, you're a cool person, I would hire you. I'm like thanks Joe, you're nice. And then I give it to my good friend Stephanie – and I don't have any friends named Joe or Stephanie, these are made up names – and she says well, you know, it's passable. I've seen cover letters like this, I've written cover letters like this. It's good, but you could do better. There's this and this. I write like this, so when I read your handwriting doesn't make any sense to me. Take it or leave it, because when people read my handwriting they say the same thing to me. Tonal choices. This whole paragraph, what does it mean? It doesn't mean anything, I didn't get anything from it. What were you saying with that paragraph? It's like thanks Stephanie, I feel like I'm going to get the job now because I'm going to get rid of that paragraph and write something useful.

Here Edgar contrasts being "nice" with being "supportive." Edgar describes two imaginary friends, Joe and Stephanie giving him feedback on a cover letter. Joe is *nice* because he provides encouragement, but Stephanie is *supportive* because she provides critical feedback that can be used as fodder for improvement. In this professional context, Edgar emphasizes that support is more useful than niceness, and will actually help him get a job.

Discussion

The present paper provides evidence that peer observation can enhance noticing. In particular, when GSIs are positioned as competent to provide meaningful feedback to their peers, they can learn through observing others and form meaningful community with equal status peers. As such, equal-status peer observation can improve GSI professional development. For instance, it offers a low-cost alternative to observations conducted by faculty or experienced peers, because the very process of observing GSIs becomes a learning experience for the observer rather than a "cost" for the observer in service of another GSI's learning. Moreover, it gives GSIs an opportunity to interact with students in a different capacity, increasing their understanding of their students. Despite the benefits, peer observations can be inhibited by the inter-personal challenges of GSIs criticizing other GSIs who they view as friends or colleagues. Addressing this issue requires building an environment that supports supportive, critical exchange.

This paper also suggests new directions for research in teacher noticing. While peer observation as a tool for noticing appears promising, further research is required. In particular, the mechanisms through which peer observation can support individual reflection need to be further elaborated. Moreover, further research is required to understand how this type of reflective community practice impacts the actual teaching of GSIs in the classroom. These are promising avenues to continue this work.

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McNuggets, Bunnies, and Remainders, Oh My!

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Given its applications in computing, coding, and cryptography, the Chinese Remainder Theorem is a worthwhile, accessible, and unexplored area of number theory. The purpose of this qualitative case study was to investigate strategies and reasoning that students exhibited while solving problems chosen to elicit thinking in elementary number theory topics related to the Chinese Remainder Theorem. We interviewed pairs of students from three different courses in order to investigate the similarities and differences that may occur as a result of varying mathematical backgrounds and partner dynamics. We identified a range of strategies including manipulating final digits, listing multiples while accounting for remainders, and implementing divisibility rules. This paper presents a portion of our findings discussing strategies for two of our three cases on several tasks from our interviews.

Key words: Chinese Remainder Theorem, Frobenius Problem, Pre-Service Teachers

Objectives and Background

Although many mathematicians would agree that knowledge of elementary number theory helps students to build a firmer foundation upon which other mathematical knowledge is based, the area remains relatively unexplored in mathematics education research. For instance, several studies have revealed the difficulties that undergraduates have with divisibility, prime factorization and remainders (Campbell, 2002; Zazkis, 1998). However, authors have yet to investigate students' understanding of subjects including the Chinese Remainder Theorem and Frobenius Problem. We believe these areas deserve further study, given students may be able to develop an intuitive understanding of the Chinese Remainder Theorem by exploring concrete problems that elicit the concept of modular arithmetic.

The purpose of our research was to investigate the strategies and reasoning that students exhibit while solving problems chosen to elicit thinking in elementary number theory topics. We interviewed pairs of students from three different courses—Mathematics and Liberal Arts, Fundamental Mathematics II, and Calculus 1—in order to investigate the similarities and differences that may occur as a result of partner dynamics and varying mathematical backgrounds. More specifically, our aim was to investigate the following: What strategies do students in the different undergraduate mathematics courses use when attempting to solve the Frobenius problem and Chinese Remainder Theorem tasks? Additionally, we explored the extent to which partner dynamics influenced the strategies that were used.

Existing literature on elementary number theory has centered around students' understanding of divisibility, parity, and primality. We identified two areas within elementary number theory in which the research has shown students struggle: preference for procedural demonstration over conceptual demonstration of knowledge, and general difficulty with rational and whole number division and remainders (Brown, Thomas & Tolias, 2002; Campbell, 2002; Kaasila, Pehkonen, & Hellinen, 2010; Zazkis & Campbell, 1996). Students at all levels have demonstrated difficulty in these areas, and these findings had bearing on the strategies that our participants used.

A recurring issue from the literature revealed a preference for performing algorithms and processes over utilizing conceptual knowledge as a source of certainty with division. Zazkis and

Campbell (1996) found that rather than investigate the prime factorization of a number, preservice elementary teachers preferred to multiply all of its factors and then divide that result by the possible composite factor. The students lacked both understanding of prime factorization and the confidence that comes with a more conceptual understanding. Brown, Thomas and Tolias (2002) note that participants concentrated on "the role of division and a consistent tendency to choose to perform procedures rather than making inferences" (p. 51). For example, students referenced the specific action of dividing and "seeing if it would go evenly" to justify identifying the factors of a given integer (p.52). Participants in Campbell's (2002) study were determined to perform long division in order to verify the remainder of a prime decomposed number, instead of identifying the remainder in terms of the remaining factors.

Undergraduate mathematics majors showed similar difficulty in a number theory course, understanding the concept of congruence operationally rather than relationally (Smith, 2006). Specifically, they viewed a congruence statement in terms of a completed operation, instead of as a relationship between two equivalent quantities. Additionally, students would often solve linear congruences using complicated memorized procedures rather than the properties of the congruence relation. While our students did not work directly with congruence, we found that they encountered similar difficulties and very much relied on procedural manipulation over conceptual.

Students at all levels also demonstrate issues with whole number and rational division, specifically in that students attempt to apply ideas from whole number division to rational number division, and are thus unable to correctly identify remainders. Campbell's (2002) study exposed this issue by allowing participants the use of a calculator during tasks. Not a single participant was able to correctly identify the whole number remainder from the rational quotient displayed on a calculator, though many attempted a fractional approach. For example, students would calculate 68/3, which would display as 22.666..., and claim the remainder was 6. Given that whole number division provides an answer with two parts, the quotient and the remainder, participants experienced cognitive conflict with interpreting the responses from their calculators. These preservice teachers struggled to incorporate their intuitive knowledge of remainders with their decimal representation as part of a rational number.

In a wider-reaching study, Kaasila, Pehkonen, and Hellinen (2010) found that roughly 55% of preservice teachers and 63% of upper secondary students were unable to correctly represent the remainder of an abstract division problem. The subjects either recognized that the result of the division problem should not be an integer but could not determine the remainder, or completely ignored the remainder and responded with an integer answer.

The Chinese Remainder Theorem is built upon these aforementioned concepts of divisibility and remainders. Prior to our research, we anticipated that our participants would encounter difficulties similar to these discussed, but we were hopeful that they would be able to develop strategies to construct solutions to the problems. Our study yielded both of these results.

Theoretical Perspective

Merriam and Tisdell (2016) describe the goal of qualitative research as determining how people make sense of their lives and construct meaning. Thus, the role of the researcher is to both interpret and narrate these meanings and experiences. Given our stance that the individual learner is inseparable from "the realm of the social," this research is framed by the theory of social constructivism (Ernest, 2010, p. 43). A major tenet of social constructivism is the notion that "human beings are formed through their interactions with each other as well as by their individual processes" (Ernest, 2010, p. 43). With this perspective in mind, we sought to explore the strategies of our individual participants, as well as how the dynamic of working with a partner influenced or altered their approaches to our tasks. The structure of our project lent itself to collective problem-solving as well as solo work. However, the assumptions of social constructivism were still applicable in the instances of individual problem solving, as individual thought originates through internalized conversation. Implementing this framework allowed us to address the social influence that the presence of another participant had on the construction of knowledge.

Methods

As mentioned above, participants were undergraduate students recruited from three undergraduate mathematics courses: Mathematics and Liberal Arts, Fundamental Mathematics II, and Calculus I. We chose these three courses because we feel that each has a student population that differs from the others, which we thought might result in the use of different strategies and allow us to perform a comparative case study analysis of the data.

Three 60-minute video-recorded interviews were conducted, each with a pair of students from one of the target classes. The semi-structured clinical interviews began with general information regarding the students' majors, previous math classes, and mathematical experiences. The main focus of the interview was to have participants solve a set of problems that we chose in order to reveal cognitive resources for understanding elementary number theory concepts, including the Frobenius problem and the Chinese Remainder Theorem (see Findings for descriptions of tasks). Participants were encouraged to work together, discussing problem solutions and sharing their reasoning aloud, and follow up and clarifying questions were asked as needed. Each participant had access to a graphing calculator and was instructed to think aloud when performing different procedures or checking various solutions. Lastly, artifacts recording student work were collected at the end of the interviews. Students were not asked to delineate their strategy entirely on paper, though several participants opted to do so.

We chose to analyze the data as three cases: the three partner groups defined from the three different mathematics courses. Due to the nature of the experience that students in each of these courses have, we believe that the strategies that students used when attempting our tasks were somewhat dependent upon the mathematical backgrounds from which they came, including which mathematics course they were taking. In addition, each pair exhibited different partner dynamics which likely contributed to the choices of strategies used.

We used a grounded theory approach to analyze the data, including interview video transcripts, artifacts, and interview notes. The literature informed the open coding process that we used for the first pass individually (Miles & Huberman, 1994). We then used a constant-comparative method, individually and in collaboration with each other, to identify the strategies that participants used in solving the tasks (Strauss & Corbin, 1990). Following the individual passes through the transcripts, we met to address any differences in coded strategies until we reached a consensus. Two distinct themes were of interest from the video-recorded data: student strategies and partner dynamics. Our final list of codes included 9 distinct student strategies, as well as a code for identifying student issues with remainders. Field notes and transcripts were simultaneously coded and discussed for the distinct working relationships of the partners in each interview, including isolated individual work and pairs working in unison. Finally, we compared the student strategies across our three cases to identify similarities and differences discussed below.

Findings

The remainder of this paper will focus on two of the three cases from our study. Yolanda[1], a freshman theater and history major, and Rose, a freshman secondary history education major, were both enrolled in Mathematics and Liberal Arts, although they came from separate sections of the course. This appeared to affect the manner in which the two approached working together. Yolanda was typically the leader in her classroom group, taking charge and delegating work. Rose's classroom group, however, generally approached problems as a whole unit, talking through strategies and steps together. These differences caused some tension throughout the interview as the students negotiated the activity of working together. Their collaboration did increase as the interview progressed, with Yolanda assuming the role of the leader, but this was not without its faults. This was particularly evident during Task 3, when Rose mentioned that she did not want to solve the problem using guess and check, and Yolanda replied "Well, that's what we're doing. We don't know how to do your equations, we're gonna have to do something else." Rose shrugged and proceeded to follow Yolanda's lead.

Our second case was Marie and Fiona, who are both elementary education majors and enrolled in the same Fundamental Mathematics II course, the second in a three-course sequence of mathematics content for preservice education majors. All three of the courses in this sequence are student-centered courses with a heavy emphasis on collaboration and problem solving within small groups. This training was evident while the two were working on the interview tasks, as they consistently shared ideas and fed off of each other's results. The two took turns taking the lead on tasks. In addition, Marie and Fiona were the only group of the three to persevere and find a solution to Task 3, which may be the result of their prior experience in problem solving and group work.

We present our analysis of three of the five tasks, giving first an overview of the task and then the students' solution strategies.

Task 1: McNuggets (Contextual Frobenius Problem)

Task 1 required students to determine the number of boxes of size 6, 9, & 20 needed to buy 53 McNuggets. Rose and Yolanda began the task separately. Yolanda used an additive strategy that transitioned into a multiplicative strategy, starting by subtracting six from 53 twice to get 41. Then she subtracted nine, and noticed that the result, 32, was not divisible by 6, but that if she subtracted twenty, that result would be divisible by 6. Her strategy was based on guessing, and she started by subtracting six because it was listed first in the problem.

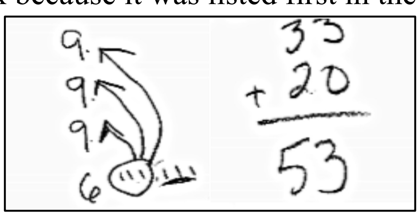

Figure 1: Rose's work showing the groups of 10s created from 9s and 6

Rose immediately decided to work with ensuring a 3 was in the ones place, because "getting to the 50, [she felt], would be easier after getting to the 3." She decided to work with the nines: "well, I was thinking, like, a 9, a 9, and a 9, 'cuz if you do a 6, then it breaks up into like 3 and

[1] All student names are pseudonyms.

those 3 go into those three 9's, and then you have 3 left over for the 53." By breaking a box of 6 into a 3 and three ones, Rose noted she was making the nines into groups of ten and getting the three leftover (See Figure 1). At this point, she knew she had thirty-three McNuggets and needed only to add a box of twenty to finish. We argue that Rose was more strategic in her process, and her use of the 'groups of nine' shows a progression toward a multiplicative strategy rather than an additive one.

Fiona and Marie began Task 1 by dividing 53 by each of the possible box sizes and determining if either that box size or a combination of that box and a single box of another size would work. They then used a strategic guess and check method, starting with the largest number: "subtracting 20. And then I got 33. Then I did 33 divided by 9 and got 3 even boxes of 9 and then you'd have 27, so then there was 6 left, 'cuz 33 minus 27 is 6." Their strategy had elements of both multiplicative reasoning ("33 divided by 9 and got 3 even boxes") and additive reasoning.

Tasks 3 and 5: Bunnies and Construction with Remainders (Chinese Remainder Theorem)

The Bunnies task proved to be the most challenging for all of the pairs. The intent of the problem was to introduce the concept of division with remainders through the context of grouping bunnies for various activities on a ranch that would allow students to construct a number satisfying specific constraints. Participants were asked to find a number of bunnies that satisfied divisibility by 5 with remainder 4, by 8 with remainder 6, and by 9 with remainder 8.

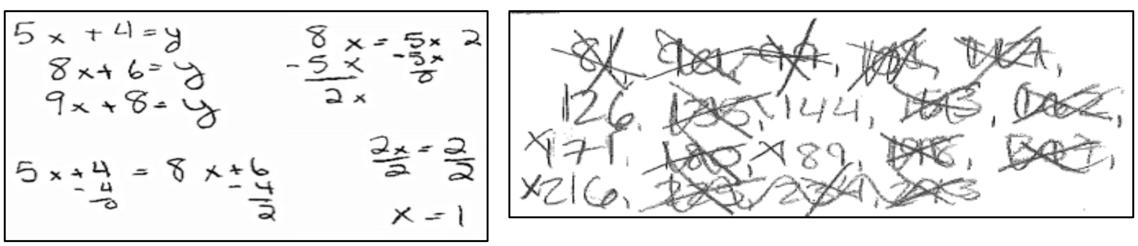

Figure 2: Rose's attempt at solving a system of equations (left) and Marie & Fiona's first attempt at multiples of 9 (right).

Yolanda and Rose utilized mainly guess and check. At first, Rose wrote a system of equations, but when the two were unable to solve the equations (Figure 2), Yolanda took the lead suggesting that they try 249, since it was "less than 250." The two then checked whether 249 yielded the correct remainders when divided by 5, 8, and 9. When 249 did not work, they examined 248 in the same manner. Here Yolanda recognized that in order for the number to have a remainder of 4 when divided by 5, it would have to end in a 4 or a 9, limiting the possible numbers that they would need to check. This was a good example of a guessing strategy that became more strategic and incorporated an understanding of divisibility rules. Unfortunately, the problem proved too challenging and the pair decided to move on.

Marie and Fiona were the only pair to persist to a solution for Task 3. They immediately recognized that the number cannot be evenly divisible by 5, 8, or 9, and they used divisibility rules to note that it would not end in 0 or 5. They started with 249, which helped them recognize that the number would "have to be 4 away from a multiple of 5." This particular understanding of the remainder proved problematic, because the students treated numbers that ended in a 1, which would be *four less than* a multiple of five, as a number that would satisfy the condition of having a remainder of 4 when divided by 5. Later, the students used a similar reasoning to justify the inclusion of numbers ending in 6 (again, four less than a multiple of 5). Their primary strategy on the task was to create a list of multiples of 9 and systematically check those multiples based on their rules for division by 5. They were left with a much smaller list on which to verify the remainder requirement when dividing by 8 (Figure 2).

It is interesting to note that both students were attentive to remainders when considering division by 5 and by 8, but neither recognized an initial mistake of *not attending to* remainders when divided by 9. Once they recognized this error, the students created a new list of the multiples of 9, adjusting for the remainder of 8. They then listed multiples of 8 adjusted for the remainder of 6, and multiples of 5 adjusted for the remainder of 4.

Task 5 was a decontextualized Chinese Remainder Theorem task similar to Task 3. All pairs identified the similarity of this task with “the bunny problem.” Pairs were asked to find a number that satisfied divisibility by 3 with remainder 2, by 4 with remainder 1, and by 5 with remainder 3. Marie and Fiona used the same strategy of listing multiples of the divisors and adding the remainder to each multiple in order to identify a number common to all three lists.

Similarly, Yolanda and Rose chose to explore the final digits of each of the divisors and adjusted the digits for the remainder each time. Yolanda first noted that the desired number could not end in a 0 or a 5 as that would make it divisible by 5. Rather, she claimed that the number should end in a 3 or an 8 by adding the remainder of 3 to each final digit. The pair listed the ones digits of multiples of 3 and then added 2 to each digit to account for the remainder (Figure 3).

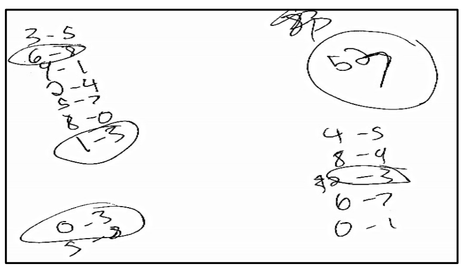

Figure 3: Yolanda and Rose’s solution for Task 5 identifying the ones digit.

They then repeated this process for the ones digit of multiples of 4 until they saw that 3 appeared in all three lists. Yolanda concluded that she and Rose only needed to check the “decades that end in 3” until they found a number that satisfied all of the requirements. Once again we found a preference from our participants to begin with the divisibility rule for 5 and build a strategy from this starting point, because this rule depends only on the ones digit.

Conclusion

We found nine different strategies, six which were exhibited by these two groups while they completed the number theory tasks: guess and check, divisibility rules, use of equations, consideration of final digit, additive reasoning, and multiplicative reasoning. We note that participants often incorporated multiple strategies on each problem, making it difficult to attach a single label to certain tasks or groups. For example, we saw that guess and check was often a starting point, but the strategy evolved into a more sophisticated use of additive or multiplicative reasoning.

We were also interested in comparing the strategies used across our cases in order to investigate the possible effects of participants’ background and partner dynamics on choice of strategy. While our data set is small, we did see some differences. Marie and Fiona were perhaps the most flexible in strategy use. When they ran into roadblocks, they pushed through the frustration and tried something new. They were unafraid to be incorrect, despite the fact that their teacher was in the room with them. We believe that this was largely due to the problem-solving mathematics environment that they were used to. They were accustomed to collaborating equally with others and encountering difficult problems.

Although Yolanda and Rose were somewhat accustomed to working in groups, we saw that they had different conceptions of what group work looks like, which caused difficulties for them. Working together did serve them well, however, as they were able to push each other past areas where they struggled on the tasks and they exhibited different strategies as a result. We believe that the evident success of working together on Task 5 would have likely led to success on Task 3 had the participants chosen to revisit it.

The results of this study gave us some insight into what cognitive resources students rely upon in order to understand the Chinese Remainder Theorem and Frobenius Problem, which can be applied to their understanding of modular arithmetic. This is important to those who teach number theory courses, as these basic strategies can provide a foundation upon which to build more formal topics. We also found that these topics are accessible to a number of different populations, independent of mathematical background. This provides an avenue through which to investigate preservice teachers' understanding of division and remainders in general, which can inform classroom instruction on divisibility.

Limitations and Areas for Future Consideration

We chose to present contextualized problems before decontextualized problems in order to provide students with problems that appeared more applicable to their own lives. However, we noticed that participants frequently fixated on minute details of the wording of the problems, making comments about animal hoarding and the absurdity of wanting to purchase exactly 53 McNuggets, derailing their solution process. It may have been beneficial to present decontextualized problems first for this reason.

We note that the students in both partner groups were enrolled in classes taught by the authors. In order to minimize reactivity, we structured the interviews so that students were not interviewed directly by their instructor, although she did remain present in the room and responsible for operating the video camera. In both interviews there were moments when the participants would look toward their instructor for a sign of approval or disapproval regarding their work on the tasks. Although each researcher attempted to remain expressionless and limit comments regarding clarification to any of the strategies, we recognize that this still likely had an impact on the participants' responses.

Furthermore, we acknowledge the choice in soliciting individual students rather than pairs. Marie and Fiona had been working together in their math class, but Yolanda and Rose were paired together from different classes. Having never collaborated before, the partnership was somewhat inauthentic as the two had different conceptions of what working together would look like. For future extensions, it would be worth attempting to solicit pairs of students that would sign up together rather than individual students.

We are disappointed that we did not have enough time to fully investigate the effect of classroom culture on our interviews, as was initially intended. We had also hoped to see some models of student strategies that resembled the Chinese Remainder Theorem in our interviews. Although this was not a result in this study, we believe that constructing a design experiment would be a worthwhile endeavor. One could also explore participants' previous mathematical knowledge and how they transfer these experiences into a number theory class, adopting actor-oriented transfer as a theoretical framework (Lobato, 2003).

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Students' Conceptions of Mappings in Abstract Algebra

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In an effort to understand ways students approach constructing homomorphisms and isomorphisms between groups, six undergraduate math and engineering students in a lecture-based introductory abstract algebra course were interviewed. These students experienced varied success in creating isomorphisms and homomorphisms, which allowed both successful techniques for map creation and stumbling blocks to map creation to emerge from the data. Some successful techniques for determining if groups were isomorphic included checking the orders of the groups, looking for invertible maps between groups, and determining the identity element and orders of elements of each group. Successful strategies for approaching the creation of homomorphisms included checking if the groups were isomorphic, seeing if a proposed map would preserve closure, and using strategic trial and error. Stumbling blocks included the inappropriate use of definitions, an inability to interpret definitions, and misunderstanding the distinction between the names and roles of elements in different groups.

Key words: Abstract Algebra, Homomorphism, Isomorphism, Qualitative

Purpose and Background

Experts have identified isomorphism and homomorphism as two of the most central topics to abstract algebra (Melhuish, 2015). Yet though some research has been done on how students approach isomorphism, including designing an inquiry-oriented curriculum that addresses isomorphism (Larsen, Johnson, & Bartlo, 2013), research explicitly on students' understanding of homomorphism has been scarce. Thus, the purpose of this study is to examine students' approaches to finding both isomorphisms and homomorphisms and what prevents students from finding appropriate mappings between groups.

A homomorphism between groups is defined as follows: "Let $(G,\star)$ and $(H,\boxdot)$ be groups. A map $\varphi\colon G \to H$ such that $\varphi\colon (x \star y) = \varphi(x) \boxdot \varphi(y)$ for all $x, y \in G$ is called a homomorphism" (Dummit & Foote, 2004). Thus a homomorphism is a map that preserves the structure of the original group in the second group. It does not require the groups to have the same cardinality; group G may be larger or smaller than group H. There is always at least one homomorphism between groups; namely, the trivial homomorphism, in which every element of G is mapped to the identity in H. Further, an isomorphism between groups is defined as follows: "The map $\varphi\colon G \to H$ is called an isomorphism and G and H are said to be isomorphic or of the same isomorphism type, written $G \cong H$, if φ is a homomorphism, and φ is a bijection" (Dummit & Foote, 2004). Thus isomorphisms are a specific type of homomorphism in which the cardinalities of both groups are the same. For example, V, the Klein four-group, is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. However, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$ because the homomorphism property is not satisfied.

Previous studies have examined isomorphism in different ways. Early studies mostly provided students with two Cayley tables or stated two groups and asked if they were isomorphic or how they could tell they were isomorphic. The Dubinsky, Dautermann, Leron, and Zazkis (1994) study indicated that when students considered isomorphisms between groups, they considered the cardinality of each group, but not whether the homomorphism property was satisfied. Leron, Hazzan, and Zazkis (1995) also noted students' tendency to check the cardinality of a group, but their students continued to test multiple other properties by finding the identity element (in Cayley tables), the orders of individual elements (the smallest positive integer m such that a^m is the identity for each element a), the number of elements of given orders in each group, whether a group is generated by a single element, and if it is commutative. Despite the many factors to check, students would still struggle if more than one way to construct an isomorphism existed, demonstrating a "craving for canonical procedures" (p. 168).

Other studies have considered isomorphism in the context of proof. In related studies, Weber and Alcock (2004) and Weber (2002) asked undergraduate and doctoral students to prove a number of theorems related to isomorphism and to prove or disprove specific groups were isomorphic. While both doctoral and undergraduate students were able to prove the initial simple propositions, the doctoral students continued to be successful in proving the remaining five propositions, while collectively only two (of twenty) proofs of the remaining propositions given by undergraduates were successful. Much like in Dubinsky et al. (1994), these difficulties largely related to undergraduates' tendency to form arbitrary mappings once they had ascertained that a bijection between groups could be formed. They would not apply other properties of the groups when trying to find or disprove the existence of an isomorphism.

Recent studies on isomorphism have shifted the focus to how to develop local instructional theories that could be transformed into an inquiry-oriented curriculum that included topics such as isomorphism. In the process of examining how students used their existing ways of reasoning to engage with mathematically rich tasks, other student views of isomorphism have come to light. In 2009, Larsen recorded a teaching experiment in which participants were expected to generate a definition of isomorphism. In that study, participant Jessica noted that the definition of isomorphism should include bijection because "…it has to go both ways" (p. 133). Her statement brought to light another approach to isomorphism: invertible mapping. Later, Larsen et al. (2013) noted that the homomorphism property was more challenging for students to unpack than the bijection property. Additionally, Larsen (2013) noted, "students' use of the homomorphism property is usually largely or completely implicit" (p. 722). Thus a number of tasks in his curriculum engage students in forming an explicit homomorphism in order to help students formulate the definition of homomorphism and, later, isomorphism.

In these isomorphism studies, some research has been conducted on homomorphism in the process of researching isomorphism. However, a few studies have examined homomorphism more closely in the context of proof. Nardi (2000) noted students' struggles in proving the First Isomorphism Theorem for Groups stemmed from three major sources: an inability to recall definitions or a lack of understanding of definitions, poor conceptions of mapping (such as thinking a homomorphism was an element of a group), and not realizing specifically what each part of the proof was proving. Weber (2001) observed that despite undergraduates' ability to recall relevant theorems, they struggled to move past syntactic, "definition unpacking" techniques when trying to prove theorems related to isomorphism and homomorphism, such as proving a group was abelian given the map was a surjective (onto) homomorphism. He also noted doctoral students had a tendency to use conceptual knowledge to formulate proofs more strategically and experienced more success in proving.

Methods

The participants for this study were six sophomore or junior university students in a lecture-based introductory abstract algebra course. Four were mathematics majors and two were engineering majors considering double majoring in math or transferring into the math program. Students' college math backgrounds other than introductory calculus and a proof course varied but included courses in combinatorics, discrete math, vector geometry, linear algebra, multivariate calculus, differential equations, operational methods, and real analysis.

Students were recruited from two instructors' courses with one student coming from Instructor A's section (denoted participant A1) and five from Instructor B's section (denoted participants B1, B2, B3, B4, and B5). Instructor A had taught both group and ring homomorphisms earlier in the semester and his students, including A1, had been tested on that material. Instructor B had just begun teaching about homomorphisms and isomorphisms when his students were interviewed. Four of his students were interviewed after learning about isomorphisms but before learning about homomorphisms in class (B1-B4), and one student was interviewed after learning about both isomorphisms and homomorphisms (B5).

Participants were recruited in two ways. The author asked for an announcement to be sent to Instructor A's students with interested students sending the author a message. In Instructor B's section, the author visited the class and asked interested students to provide their email address in order to be contacted. Students were given their preference of $10 or an hour of tutoring to avoid biasing the sample towards strong or weak students.

Each participant engaged in a semi-structured interview (Fylan, 2005) lasting approximately one hour. The interview questions were drawn from those in Figure 1, but time and students' backgrounds prevented some students from seeing certain questions. However, all students answered questions 4, 10, 11, and 12. The interviews were all audio-recorded and five of the six were video-recorded. Participants' written work was also collected. To analyze the data, participants' interviews were transcribed and open coded for students' problem solving strategies regarding isomorphism and homomorphism. This coding generated themes, which were verified by utilizing multiple iterations of coding (Anfara, Brown, & Mangione, 2002).

Results

Participants' approaches to mappings fell into three major categories: successful methods of determining if a map was an isomorphism or if such a map could be generated between groups; successful methods of determining if a map was a homomorphism or if such a map could be generated between groups; and stumbling blocks to success.

Isomorphisms

Students exhibited a variety of successful strategies when trying to determine if two groups were isomorphic or if a given map could be an isomorphism. Because a one-to-one and onto mapping must exist for a bijection to exist, groups must have the same cardinality to be isomorphic. Thus, one strategy students used was to check if the groups were the same size. Four of the participants (A1, B2, B3, and B5) utilized this strategy at some point with three of them checking this characteristic first when faced with any of problems 11-17. For example, in response to the isomorphism part of question 11 ($\mathbb{Z}_5 \rightarrow 5\mathbb{Z}$), B2 answered, "I don't think, um, I can form isomorphism between them because, um, I don't think it is bijective, um, because I think this group [points to $5\mathbb{Z}$] is larger than this one [points to $\mathbb{Z}_5$]."

Another successful strategy was to look for an invertible map between given groups or to note that a previously stated map was invertible. A1, B1, and B2 all utilized this strategy, with B1 using this as her main approach. For example, consider the following exchange when asked the isomorphism part of question 16.

B1: I believe so. Like, to get from $\mathbb{Z}$ to $2\mathbb{Z}$, you just multiply by 2. To get from $2\mathbb{Z}$ to $\mathbb{Z}$, you just divide by 2.

I: You said $\mathbb{Z}$ to $2\mathbb{Z}$, multiply by 2. Is that an isomorphism? Homomorphism?

B1: Well that in itself is, that's only a one way mapping. Its inverse also exists so paired together, isomorphism, I'd say.

When pressed on details of the inverse map, namely explaining why it was acceptable even though division was not the operation defined for $\mathbb{Z}$ and $\frac{1}{2}$ was not an element of $\mathbb{Z}$, she said it did not matter because "1 isn't in $2\mathbb{Z}$. There is nothing that will get you to a non-integer. If it's written as 2 times something, dividing by 2 just gets rid of that." She realized that the maps she had defined were one-to-one and onto. In reference to the same problem, A1 noted $2\mathbb{Z}$ to $\mathbb{Z}$ was "the same one. Since they were isomorphic before, it works the other way." He realized that when an isomorphism exists, the mapping between the groups can be inverted so each group is the domain of an isomorphic mapping to the other.

1. What is a homomorphism?
2. How do you determine if [a map or whatever used above] is a homomorphism?
3. What is an example of a homomorphism?
4. What is an isomorphism?
5. How do you determine if [a map or whatever used above] is an isomorphism?
6. What is an example of an isomorphism?
7. Is the homomorphism example you gave also an isomorphism?
8. Is the isomorphism example you gave also a homomorphism?
9. Interpret the following definitions of homomorphism and isomorphism.
   Definition: Let $(G,\star)$ and $(H,\boxdot)$ be groups. A map $\varphi: G \to H$ such that $\varphi: (x \star y) = \varphi(x) \boxdot \varphi(y)$ for all $x, y \in G$ is called a homomorphism.
   Definition: The map $\varphi: G \to H$ is called an isomorphism and G and H are said to be isomorphic or of the same isomorphism type, written $G \cong H$, if
   φ is a homomorphism, and φ is a bijection.

For each of the following pairs of groups in 10-17, is it possible to form an isomorphism between them? Why or why not? Is it possible to form a homomorphism between them? Why or why not?

10.

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

+	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	d	c	a	b
d	c	d	b	a

11. $\mathbb{Z}_5 \to 5\mathbb{Z}$
12. $\mathbb{Z}_5 \to \mathbb{Z}/5\mathbb{Z}$
13. $\mathbb{Z}_5 \to \mathbb{Z}_6$
14. $\mathbb{Z}_3 \to \mathbb{Z}_6$
15. $\mathbb{Z}_6 \to \mathbb{Z}_3$
16. $\mathbb{Z} \to 2\mathbb{Z}$
17. $2\mathbb{Z} \to \mathbb{Z}$

Figure 1. Interview protocol for semi-structured interviews.

B2 and B4 considered another potential strategy for determining if groups were isomorphic: creating a Cayley table for each group. Additionally, B2 recognized this was only a feasible option for small groups. In question 5, when asked how to determine if a group was an isomorphism, he said he would like to make a Cayley table if possible, but if the group was too big, it would be challenging: "a Cayley table will be 16x16 and I cannot make it, but, you know, 4x4, that would be fine, so kind of [a] small example." In the reverse situation, when presented with the Cayley tables in question 10, A1 and B1 also compared the orders of elements or mapped the identities of each group to each other without being prompted to do so.

Homomorphisms

The most straightforward argument given for determining if a mapping was a homomorphism stemmed from determining it was an isomorphism. Because all isomorphisms are homomorphisms, if a student determined an isomorphism could be formed between groups, a homomorphism could automatically be formed too. A1, B1, and B5 clearly recognized this property, making statements similar to what A1 said: “It’s part of the definition to be isomorphic: you have to have homomorphism in there.” The other three participants may have also recognized this concept based on their explanations of the definitions of isomorphism and homomorphism, but they never directly used this concept to answer other questions.

Closure (associated with cardinality arguments) was a property students used to check if a map they proposed could be a homomorphism. This idea was especially used to rule out maps they had created in response to questions 11 and 13. Five of the students successfully used this strategy, with B4 making a typical response as shown in Figure 2. She used her work with Cayley tables that would not close (on the mid-right and bottom right of Figure 2) to conclude she could not map all of the elements in $\mathbb{Z}_5$ (pictured on the mid-left) to distinct elements in $5\mathbb{Z}$ and have a closed group result. From this, she concluded that she could not create a homomorphism or isomorphism. Note that although her conclusion that she could not form an isomorphism was valid, her conclusion that there was no possible homomorphism was flawed because the trivial homomorphism, $\theta\colon \mathbb{Z}_5 \to 5\mathbb{Z}$ such that $\theta(x) = 0 \;\forall\; x \in \mathbb{Z}_5$, would be possible. In fact, only A1 recognized the trivial homomorphism would be a homomorphism without being prompted to consider it. He and other participants who successfully generated homomorphisms appeared to notice patterns in the orders of the groups, which allowed maps to be generated quickly. For example, when looking for a map from $\mathbb{Z}_3$ to $\mathbb{Z}_6$, B1 noticed the 0’s should

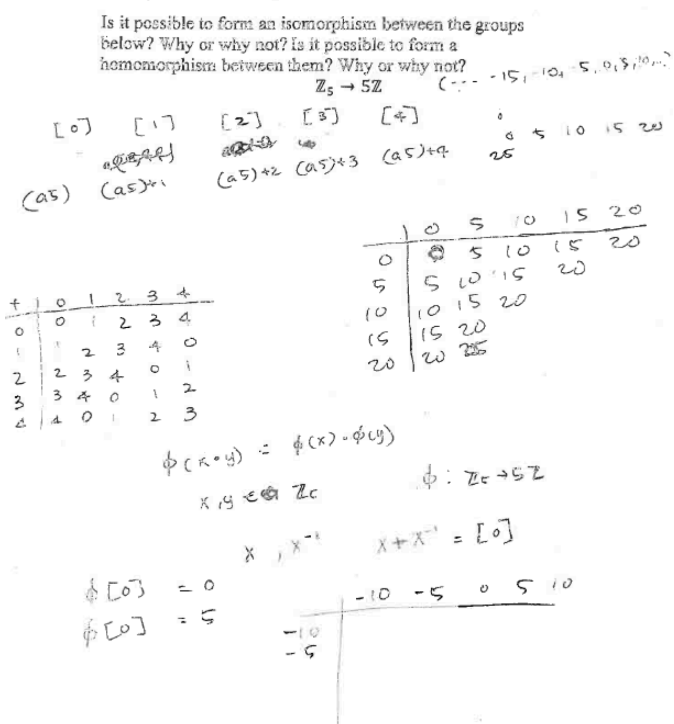

Figure 2. Work sample from B4 examining possible maps between $\mathbb{Z}_5$ and $5\mathbb{Z}$.

correspond and that $\{0,2,4\}$ acted just like $\mathbb{Z}_3$. Similarly, B2, who had successfully answered question 15 before being asked question 13, noted a non-trivial homomorphism could not be created between $\mathbb{Z}_5$ and $\mathbb{Z}_6$ because "3 divides 6 right? So, like, it should have something important about mapping…."

Although most students attempted to use one of the strategies above first, most students eventually resorted to trial and error to look for homomorphisms. Some students used pure trial and error techniques, such as B5, who attempted to map elements of $\mathbb{Z}_6$ to $\mathbb{Z}_3$ in seemingly random order and then check if the homomorphism definition was satisfied (i.e. mapping $\{0,3,4\} \to 0, \{1,2\} \to 1, \{5\} \to 2$). Both he and B3, who used this seemingly random approach, stated that they thought these problems were "trick questions" because there was no obvious technique to apply. Participants who had more success in finding suitable maps and in ruling out incorrect maps tended to use their knowledge of the orders of the groups to narrow possible choices for mappings, such as in B1's comparison of $\{0,2,4\}$ to $\mathbb{Z}_3$.

Stumbling Blocks

A number of participants struggled to recall and utilize definitions effectively. Although most students defined an isomorphism as a bijective mapping or one-to-one and onto mapping that had the property $\phi(a * b) = \phi(a)\phi(b)$ (or similar statements), some students struggled to unpack what "one-to-one" or "onto" meant. For example when B3 was asked what one-to-one meant, she first attempted to give the formal definition but could not recall it. When asked just to state what she thought about one-to-one and onto, she replied that in a one-to-one mapping "one [element] maps to one [element]" as she drew a set diagram illustrating a mapping. However, she still could not explain what onto meant, even in the context of her diagram. When asked to interpret the definition of homomorphism given in question 9, B2 claimed, "It should be abelian….Because you can switch these two, x and y, and then it just, isn't it automatically saying if you say yx, it'll just be y and x?" Additionally, students B2 and B5 attempted to map elements to multiple locations, thus violating the function relationship implicit in the definitions.

Although all participants attempted to use the formal definitions of homomorphism and isomorphism at times, some struggled to move past that point to interpret what they meant. Consider the following exchange after B4 was asked to give the definition of isomorphism.

B4: I don't even know if this is right. [Writes $(a \circ b) = a \circ b$.] I don't know what the circles are, but it's when it's preserved under the same something—I know it but I can't tell you.

I: Ok, well I feel like you're trying to give me the formal definition….Do you have just an intuitive sense of what it is?

B4: It's when two things are multiplied or something under a, say the mapping is α, if you do map of $a \cdot b$ is equal to map of $\alpha(a) \cdot \alpha(b)$ under, yeah, and it's onto and one-to-one, I think.

I: Ok. So you've given me a nice definition. Do you have any sense of, like in practical terms, what that might look like?

B4: No.

I: Ok, so if I asked you for an example you would say…

B4: Would probably give it to you in math terms. I don't know a real example. Yeah.

Some students also struggled to distinguish between the roles and the names of elements. This issue especially arose with the 4x4 Cayley tables in question 10, when students frequently assumed they had to map element a to element a because it was the same letter rather than determining the identity of each group or utilizing other techniques. This tendency to map similarly named elements to each other appeared in later problems as well. For example, in question 13, every participant first tried to map $0 \to 0, \ldots, 4 \to 4$, and all but A1 and B5 concluded that there could not be any homomorphism between the groups because the map matching similarly named elements did not work. Even when presented with the trivial homomorphism as a possible solution to question 13, B3 struggled to accept the mapping because she did not know what to do with the other elements in $\mathbb{Z}_6$ (i.e., 3+4=2 is mapped to 0, but 3+4 = 1 in $\mathbb{Z}_6$ "isn't going anywhere").

Even when students identified the roles of elements in different groups, they did not always know how to create a mapping from this information. For instance, B2, who in response to question 5 had stated that creating Cayley tables could be useful in determining if an isomorphism existed between groups, struggled when presented with the tables of question 10. He successfully located the identity element of each group, but he was unsure what to do with this information until being prompted to consider mapping the identity elements to each other.

Conclusion

In this study, like Larsen et al. (2013), students were more comfortable working with the bijective property than the homomorphism property. This seems plausible given that because the cardinality of a group can be determined rapidly, determining if a bijection exists between two groups is most likely not as difficult as determining if the homomorphism property holds for all pairs of elements of a group. Although the homomorphism property allows specific maps to be tested quickly, students had to rely on trial and error (albeit strategic trial and error) to create reasonable maps to be tested. Additionally, students with limited concept images (Tall & Vinner, 1981) of one-to-one, onto, function, and homomorphism were at a disadvantage when trying to create homomorphisms and isomorphisms. Students A1 and B1, who both attempted and solved the most problems, both utilized the concept of a mapping being invertible if it is an isomorphism and demonstrated the ability to use strategic trial and error to find homomorphisms quickly, indicating a robust concept image of maps, much as the successful doctoral students utilized their knowledge of theorems strategically to write proofs (Weber, 2001).

Like other studies, students considered the cardinalities of the groups to determine if they could be isomorphic (Dubinsky, et al., 1994; Leron, et al., 1995; Weber & Alcock, 2004; Weber, 2002). Despite literature indicating that comparing orders of elements of groups when trying to determine if a map is an isomorphism was a common technique (Leron, et al., 1995), only two participants utilized the technique in this study; however, this may be due to the limited number of questions in which both groups were finite and had the same cardinality.

Future studies should use a larger sample of students who have all been exposed to homomorphism and isomorphism before being interviewed. Additionally, more questions with isomorphic groups should be explored to see if students utilize techniques like considering the orders of elements and creating invertible maps when faced with more groups of the same cardinality. Because of the dearth of studies on homomorphism, it would be illuminating to conduct teaching experiments examining how to enhance students' concept images of these mappings. This might help students consider mapping elements that look different to one another despite elements of similar appearance being present in the other group (e.g., map $1 \to 2$ instead of $1 \to 1$ when generating a homomorphism from $\mathbb{Z}_3 \to \mathbb{Z}_6$). Finally, comparing how other content areas, such as linear algebra and graph theory, create and use homomorphisms and isomorphisms could provide insights into the role of mappings throughout mathematics.

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A Success Factor Model for Calculus: The Relative Impact of and Connections between Factors Affecting Student Success in College Calculus

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What factors (in terms of the student) contribute to success in college calculus, and what are the relationships between and relative importance of these factors? This study addresses these questions by building on the Academic Performance Determinants Model (Credé and Kuncel, 2008). A new model called the Success Factor Model for Calculus was developed using semi-structured and task-based interviews with fourteen first-semester college calculus students. The data suggests that creative mathematical reasoning and knowing-why are not required for success on college calculus tests. Alternatively, motivation is a determining factor in success in that students can perform well on exams by being motivated to know how to solve specific types of problems. Motivation is decreased by some course-specific factors, such as lack of structure and accountability, and its effect on success is decreased sometimes by a lack of study skills and habits.

Key words: Calculus, Creative Mathematical Reasoning, Success Factor Model, College Success

Approximately 300,000 students enroll in college calculus in the United States each year (Bressoud, 2015). With calculus being a gateway course to science and engineering disciplines, it is concerning that an estimated 25-30% of these students are not successful in the course and that both student confidence and enjoyment of mathematics decreases significantly between the beginning and end of the college calculus course (Bressoud, Carlson, Mesa, & Rasmussen, 2013). To address the challenges of teaching calculus and determine directions for future research, a comprehensive picture of the factors that lead to or prevent student success in the course is needed. As part of a larger study, this research used a comparative analysis of data from fourteen student interviews to address the following research questions:

1. *What factors contribute to or hinder student success in college calculus?*
2. *What are the relationships between and relative importance of those factors?*

Framework and Review of Literature

The Academic Performance Determinants Model

Student difficulty in calculus may arise from design aspects of the course, such as the types of questions asked on tests and ultimately what type of reasoning is required to be demonstrated on exams. It also may also be impacted by student-specific characteristics such as background knowledge, level of effort or motivation, or knowledge of how to study appropriately. Credé and Kuncel (2008) suggest that some of these factors mediate the effects of others on student performance. Their model of Academic Performance Determinants (see Figure 1) shows relationships between these determinants and illustrates how some affect performance more directly than others. Their model is not specific to mathematics or any other discipline. It was used in this study as a starting point for developing a similar model that is specific to the learning of calculus.

Credé and Kuncel distinguish between study motivation and study skills and habits. Study motivation refers to a student's willingness to study and the "sustained and deliberate effort" they exert in studying (p. 428). In Anthony's (2000) survey of college instructors and students, both groups indicated that a student's level of motivation was the most important factor that contributes to student success. However, there is a discrepancy in the amount of practice that is expected from instructors and the amount that students believe is necessary (Cerrito & Levi, 1999, Bressoud, 2015).

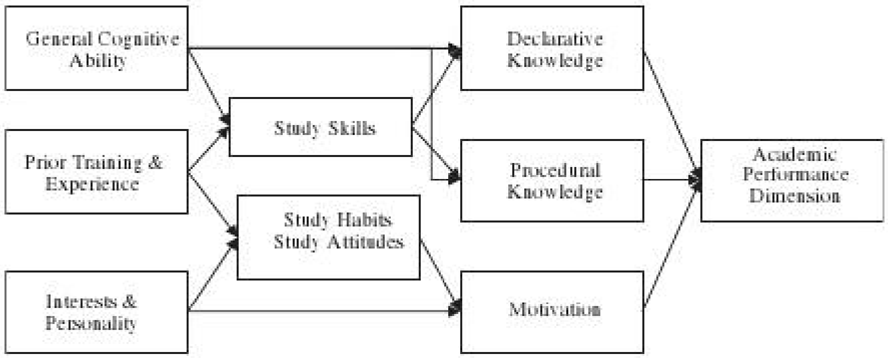

Figure 1. Academic Performance Determinants Model from Credé and Kuncel, 2008.

The Academic Performance Determinants Model suggests that study skills, habits and attitudes impact performance through the acquisition of certain kinds of knowledge. Note that since this model was not developed specifically for mathematical problems, certain types of knowledge or reasoning may not be accounted for in this model. For example, procedural knowledge means something different in mathematics. To aid in the need to consider mathematical knowledge and reasoning, we utilized two additional sets of definitions described next.

Types of Knowing and Reasoning

Building on the work of Ryle (1949) and Skemp (1979), Mason and Spence (1999) distinguish between four types of knowing – knowing-that something is true, knowing-how to do something, knowing-why you do something or why something is true, and knowing-to do something in a particular situation. They explain that while there are certainly connections between the types of knowing, and often one can facilitate another, they are distinct, in that one does not guarantee or precede another. Mason and Spence claim that classroom education focuses on teaching knowing-that, how, and why which amounts to knowing *about* a subject, but that this does not equate to knowing-to. Knowing-to is often a significant barrier in solving non-routine problems.

Lithner and his colleagues (Lithner, 2006; Palm, Boesen, & Lithner, 2006) have developed a conceptual framing that addresses the issue of solving non-routine type problems. They distinguish between creative and imitative mathematical reasoning. Creative mathematical reasoning (CMR) requires learners to produce a solution method they have never seen or have forgotten. Imitative reasoning, in contrast, is devoid of "attempts at originality" (Palm, Boesen, & Lithner, 2006, p. 6). Imitative reasoning can be further categorized as either memorized or algorithmic reasoning. Memorized reasoning (MR) requires simply recalling information to produce a complete solution, such as a facts, definitions, or even proofs. That is, it requires knowing-that. Algorithmic reasoning (AR) requires knowing-to use a particular procedure and knowing-how to use that procedure. Knowing-to in problems that can be solved using algorithmic reasoning is simplified by triggers or surface features from previously studied similar problems. In a problem that requires creative mathematical reasoning, knowing-to is significantly more challenging because of the absence of these surface similarities.

Methods

Participants and Data Collection

This report outlines one piece of a larger study that examined the impact of AP Calculus on students who repeat Calculus 1 in college. Ninety-minute interviews were conducted in the fall semester of 2012 between mid-October and early December. Fourteen interview participants were chosen from two small private and four large public universities in two states in the southeast. These six universities were on the semester system and students were recruited anywhere from 8-14 weeks into the 16-week semester. Participants were first-year students enrolled in a Calculus I course who had taken AP Calculus and taken the AP Calculus exam. Participants were required to have made a C- or lower on a recent exam.

A semi-structured, conversational interview process (Drever, 1995, Smith, 1995) was used for the first 45 minutes. The interview protocol consisted of 17 open-ended questions and was influenced by the Academic Performance Determinants Model. The second portion of the interview was task-based (Kelly & Lesh, 2000) and was diagnostic. Its purpose was to determine whether imitative reasoning (see above for description) could have been used to solve each missed test problem. To make this determination, test problems were compared to problems from other course resources, such as examples from class notes or the textbook or assigned homework problems. Participants were asked to identify whether they were able to solve the similar problems prior to the test and also to attempt the test problem in the interview after reviewing how to solve similar problems from the resources.

Analysis

Analysis of Part I of the interviews led to the emergence of 17 themes relating to students' experiences in calculus. We used open coding to identify experiences among participants and axial coding to group codes into themes (Strauss & Corbin, 1997). A theme was eliminated if it was not mentioned by at least three participants.

Analysis of the second portion of the interviews involved writing summaries for each student's attempts at solving missed test problems and determining what type of reasoning could have been used to solve these problems. After reading each interview transcript, we wrote a 1-2 paragraph description of the student's problem solving process for the attempted test problems and similar problems. These descriptions included the following three items: 1) whether the participant was able to solve the similar problem without assistance, 2) whether the participant was then able to solve the test problem after learning to solve the similar problem, and 3) if not, what barrier(s) the student encountered in solving the test problems (such as insufficient pre-requisite mathematical knowledge). Item 1 spoke to the student's preparation for the test, while item 2 revealed whether imitative reasoning could have been used to solve the problem. Item 3 for each problem was compared across participants and commonalities were noted.

The summaries from the Interview Analysis Part II were combined with some of the results from Interview Analysis Part I to develop a Success Factor Model for College Calculus. The model captured all observed factors affecting success on Calculus exams for the fourteen participants and the relationships between these factors. Themes from Part I were included only if we saw a consistent relationship between the factor and student success. For example, while participants consistently claimed that increased course structure led to greater motivation and subsequently greater performance, smaller class size (while desirable) did not present such a consistent impact.

We initially used certain success factors and terminology while leaving out others that we had no data to support or evaluate. We developed the initial draft of the Success Factor Model after examining individually and comparing the summaries of four participants. As we examined additional transcripts and summaries, we began to change the proximity of some factors to success, add additional factors and remove others, as well as add and remove connections between factors. This was an iterative process that involved going back and forth between the overall model and the individual participants' experiences.

Once the model was complete with its factors and pathways, we set out to determine the relative importance or impact of each factor using a meta matrix and subsequently used the meta matrix to create case-ordered matrices (Miles & Huberman, 1994). That is, we sorted the participants by certain variables and then looked across the other categories for similarities amongst cases with the same characteristic, to determine which variables might be associated with others. We grouped students by their relative success in college calculus. Doing this allowed us to identify which factors were consistent for successful students and which factors varied, or were not essential. This process illuminated the different pathways students could take to success in Calculus (on exams) and the common factors that hindered other students' progress.

Results

The Success Factor Model below depicts the factors of success and the relationships between the factors. Results also include the relative impact of the factors on student success in calculus.

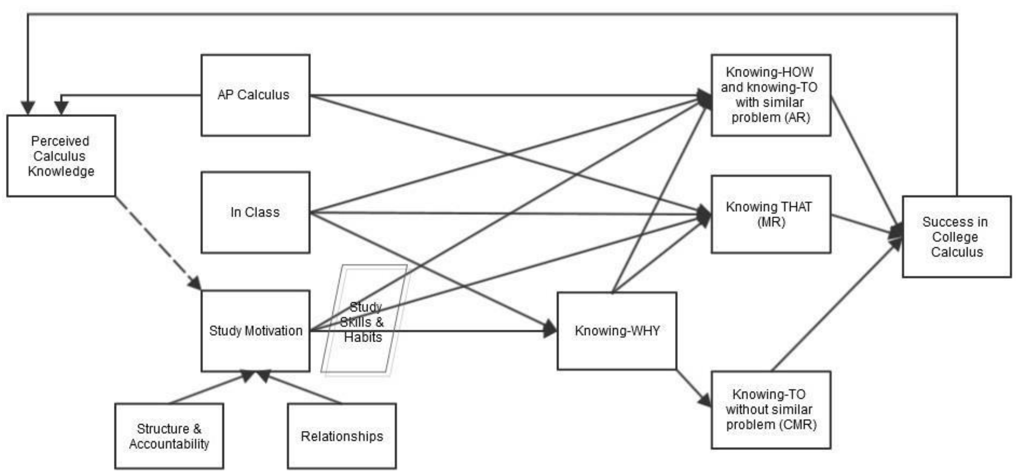

Figure 2. Success Factor Model for College Calculus.

Arrows connecting factors A and B indicate an impact of factor A on factor B. Bold arrows indicate a direct relationship; an increase in the factor A positively impacts factor B. A dashed arrow indicates an inverse relationship; that is, an increase in factor A results in poorer results for factor B. Distance of a factor from success on the diagram does not determine relative impact of that factor; larger distance merely shows that there is a mediating factor that determines how much of an impact that factor will have on success. Due to space restrictions, only selected factors, relationships, and their relative importance will be discussed.

The Relative Impact of the Direct Factors of Success

We first discuss the direct factors of success, displayed on the right-hand side of the Success Factor Model, and their relative importance for students' success on their exams. While any one of these three factors may be required to solve a particular problem, some were more frequently required and/or posed more frequent barriers for the participants than others.

One of the direct factors of success is knowing-to using CMR. Ten of the participants needed to apply some degree of CMR to successfully complete at least one of their discussed test problems, but no student had enough CMR required on his or her test so as to prevent passing the test without it. This result is consistent with that of Bergqvist (2007) who found that 15 of 16 Swedish calculus tests could be passed using only imitative reasoning.

Additionally, very few of the problems required CMR for calculus material; rather, most elements of problems requiring CMR dealt with pre-requisite knowledge or skills. For example, when Blake (all student names are pseudonyms) was asked to find the absolute minimum and maximum of a function, he was able to take a derivative and knew to set it equal to zero, but was then unable to solve the equation. In a limited number of cases, the element requiring CMR came at the beginning of the problem and did prevent the student from being able to continue, thereby costing the students all or most of the points for the problem. These cases were limited, however.

As with knowing-to using CMR, difficulties involving knowing-that were almost all limited to pre-requisite knowledge. While some participants did lose points for not knowing-that on one or two of the discussed problems, this was not a primary issue for any of the participants.

Twelve of the 14 participants were able to make more progress on at least one of their two discussed test problems during the interview than they had made during the test after first learning to solve a similar problem from their resources. Very little variation was found in the directions for the problems from the tests and class resources; therefore, recognizing the problem type did not prove to be problematic for the majority of participants. The implication is that students who missed test problems that could be solved by knowing-to and knowing-how with AR could have improved their grade on the test by having worked similar problems prior to taking their test. For these students, lack of success stemmed from either inadequate study motivation or study habits, both of which will be discussed.

Knowing-why Is Not a Direct Factor of Success

Knowing-why was not essential for solving any of the problems students had missed that were discussed during the interviews and was certainly not essential for passing the test. Knowing-why did prove to be helpful to some students, while other students opted to bypass understanding by using AR. Two students, Frank and Michael, were unable to solve a test problem that asked for the location of a function's horizontal tangent line. When initially reviewing this problem in the interviews, Frank and Michael studied the same similar example from the textbook. Frank focused on the textbook's discussion rather than the solution. After approximately thirty seconds, he quickly attended to the explanation that the derivative is equal to zero when the tangent line is horizontal. He did not proceed to study the provided solution and therefore did not need to use AR. Frank demonstrated an understanding of the connection between the concept of a horizontal tangent line having a slope of zero and the process of setting a derivative equal to zero. He no longer needed an association with a particular type of problem or set of instructions to be able to solve his test question; he now knew-to because he knew-why. In contrast, when Michael looked at the textbook, he focused on the solution rather than the discussion. He explained that in the provided solution, "they would set it equal to zero and get their x values. They then plugged those back into the original equation to get the y's." When asked why they did that, he replied, "I'm not exactly sure why." However, this did not prevent him from being able to then solve the corresponding test question. This finding supports the claim of Mason and Spence (1999); understanding is related to, but not dependent on or necessary for knowing-to. "You can 'understand' but not know-to act,…you can know-to act and yet not fully understand" (p. 140). This finding raises questions about the type of knowing instructors most value and test.

One important distinction mentioned by several participants was that their college calculus course emphasized knowing-why but their AP Calculus course had not. Katelynn described her college course as significantly more challenging because of this knowing-how component. She stated, "I'd definitely say the big difference I've noticed then to now – [in high school] we learned more the how to do things as opposed to why you do them. Like I remember learning like sin x over x equals one, but I had to learn how to derive it this year. That was a lot harder." Other participants made reference to the material in college going more "in depth."

The Impact of Study Motivation is Mediated by Study Skills and Habits

The data suggests that the need for good study motivation and habits may in many cases be circumvented by other factors, but also that having strong study motivation and habits can be the deciding factor in whether some students succeed. Students at four of the six institutions could have been making A's on tests by learning to solve specific problems from their resources prior to the exam and then using AR. Students at the other two institutions were limited somewhat by the presence of multiple problems requiring CMR. However, even these students could have improved their exam scores to at least a B- by making improvements only with AR.

The impact of study motivation on knowledge acquisition for the participants was mediated by their study habits. That is, the effect of a student's willingness or effort to study was either amplified or tempered by the student's ability to study appropriately. Maggie, for example, had very high study motivation, but poor study habits. When asked if she would be able to make at least a C in the course, she declared, "I'll MAKE it happen!" But when she was asked to describe *how* she would do this, she referred only to doing more of the same activities she had already been doing that had not led to success. For example, she said "I will probably read my whole textbook, if I have to." For Maggie, high motivation did not automatically translate into greater knowledge acquisition that produced results.

Structure, Accountability, and Relationships Impact Motivation

The participants' study motivation was impacted by the amount of structure of their course and the amount of accountability provided by their instructors, as well as the relationships they had with their instructors. The most commonly discussed element of course structure was the grading of homework, or lack thereof. Most participants had no regularly graded assignments in high school, but did have this in college, and most claimed it being graded was the only factor that would motivate them to complete it. This was an exception to an important pattern - in most cases, participants described their high school course as much more structured and indicated this was a positive aspect of the high school course. College instructors were not thought to have many expectations of students. Albert explained, "her job is to teach us, not make sure we care about class." Maggie's comments echoed this sentiment. She said, "I think it's like, we're supposed to do it, be responsible for ourselves. It's college, so…" Students seemed to understand

and validate the responsibility put on them in the college environment, but they offered no evidence that it made them more successful. In contrast, Katelynn discussed her AP teacher "staying on top of [the students]" and claimed she would not have passed the course if he hadn't. AP courses tended to provide more regular examination, frequent reminders from teachers, and accessibility of help from teachers.

The relationship between teachers and the participants was one of the most widely discussed differences between high school and college calculus courses. Some participants indicated that their relationship was not just academic but also personal – one student had attended the wedding of his teacher and another described his teacher as "involved and relatable, explaining "I actually went out to dinner with him after I graduated, like me and four friends." Several participants had taken previous courses with their AP Calculus teacher. One explained that "it was almost like we had become a family because everyone had had [this instructor] for so long." The impact of these relationships was significant because the students did not want to "let down" their teachers. Multiple participants increased their amount of studying simply because of their relationship with their teacher. Jeffrey and Frank went so far as to say they would have been content with a B in their AP courses, but they did not want their teachers to think they weren't trying, or that he or she was a bad teacher, so they put it the extra effort to get an A.

There was a glaring void of these kinds of stories from the college courses. Just as the relationship motivated the students in high school, the lack of relationship in the college course was demotivating. Haley explained, "I don't know him as well [as my high school teacher] so I'm not motivated to do as well because I know him. It's terrible to get bad grades, but not because I know him and I'll be embarrassed." When students had a relationship with their teacher, they were motivated to work harder because of how their teacher would perceive them; when the relationship was absent, so was the extra motivation.

Conclusions and Implications

The Success Factor Model presented in this study highlights the need for larger scale studies that assess the relative impact of the success factors and their connections. It also challenges us to ask which of the factors of success are currently being taught or impacted by instructors, and moreover, which *could* be. For example, some have suggested that the best place to teach study skills and habits is in the classroom, rather than divorcing it from content (Taylor & Mander, 2003, Wingate, 2006). This will lead to a discussion of not only what is effective in increasing student success, but how certain recommendations for aspects of course design, such as increased structure or accountability, are perceived by college instructors and how attitudes and resource limitations may challenge such recommendations. Similarly, how do instructors' beliefs about what is important align with how they test students? For example, our findings suggest that students can be successful in college calculus without knowing-why, but this may not be the instructors' intent. The ability or inability to work non-routine problems may have little to do with a strong conceptual understanding, and instructors may be testing for one when desiring the other (Selden & Selden, 2013, Tallman, Carlson, Bressoud, & Pearson, 2016). Finally, we need to know the potential impact of emphasizing and developing each particular factor on students' success in subsequent courses. For example, if calculus courses become more structured, will students find later, less structured courses more challenging, or could the development of CMR in Calculus I lead to greater success in Calculus II?

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C×N: Investigating the Creative Proving Process Using Neuroscience Methods

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With this research, we seek to find theoretical constructs that correlate with participants' neural activity that occurs as they are presented slides of mathematical proofs. We first asked three graduate student participants to complete two graduate level proofs (one each of abstract algebra and real analysis) using a LiveScribe pen. We then generated slides of their written work and researcher-generated proofs that we used during electroencephalography (EEG) trials. Having coded the slides along 22 theoretical categories, we used step-wise model selection to determine suitable models for variance in neural activity. Preliminary results indicate that the best code-based models at a given instant can account for between 25 and 50 percent of the variance in electrical activity near the EEG electrode for that model when participants observe their own proofs and between 33 and 75 percent during researchers' proofs.

Keywords: Proof, Electroencephalography, Insight, Mathematical Creativity

Throughout the mathematics education literature, researchers and mathematicians reference moments of insight during problem solving (e.g., Burton, 1999; Hadamard, 1945), which are also described as AHA moments (e.g., Liljedahl, 2004). And, although this notion is a common colloquialism among mathematicians and mathematics educators, little empirical research has focused on evidence beyond self-reported accounts of individuals' moments of insight as they develop formal proofs (Savic, 2015). To investigate and move beyond the self-reported moments of insights, we draw on neuroscience methodologies to provide hard data for exploring insight during proof. Through an experimental methodology using electroencephalography (EEG), coupled with an extended taxonomy created for local proof comprehension (Savic, 2011), we aim to identify and explore neural activity related to proof. Specifically, we seek to identify which theoretical constructs coded for a written proof might be used to model variance in participants' neural activities as they read their own and others' proofs. Ultimately, we are motivated by a broader investigation exploring the possibility of identifying neurological evidence of theoretical constructs related to proving, specifically whether we can identify neural correlates to moments of insight, an aspect of the creative process (Wallas, 1926).

Background Literature/Theoretical Framework

Wallas (1926) outlined four stages of creativity: preparation, incubation, illumination, and verification. Subsequent literature has drawn on these categories and used them to explore the creative process, beginning with mathematicians' reflecting on their own creativity (e.g., Hadamard, 1945; Poincare, 1946; Borwein, Liljedahl, & Zhai, 2014) and moving towards qualitative research relying on participants' subjective recollection of moments of insight (e.g., Burton, 1999; Liljedahl, 2004; 2013). However, Liljedahl (2004) conceded that there might be more that can be done to investigate insight, admitting, "Upon reflection, I now see that the clinical interview is not at all conducive to the fostering of such phenomena [insight]..." (p. 49). With this research, we aim to provide an additional approach exploring moments of insight.

Mathematicians and mathematics educators alike discuss moments during proof production in which a prover gains a clearer sense of why the conjecture they have set out to prove is true (e.g., Burton, 1999; Raman, 2003). Although researchers draw on varying perspectives and language to describe such moments and the psychological processes involved, these moments seem to occur during which the prover develops a sense of a "key idea" of the proof. We use Raman's (2003) definition of key idea as something that "gives a sense of understanding and conviction. Key ideas show why a particular claim is true" (p. 323). Liljedahl (2004) stated that "At the moment of insight, in the flash of understanding when everything seems to make sense and the answer is laid bare before you, you know it, and you call out – AHA!, I GOT IT!" (p. 1).

There have been many neuroscience studies dedicated to insight during problem solving (e.g., Bowden & Jung-Beeman, 2003). Kounios et al. (2006) found that "greater neural activity was observed for insight than for non-insight preparation in bilateral temporal cortex (left more than right, in both experiments)" (p. 887). Finally, insight has been correlated with phenomena such as P300 "positive deflection occurring 300 ms after stimulus presentation," and N200 - "negative deflection 200 ms post stimulus onset" (Dietrich and Kanso, 2010, p. 824). However, the prompts used in these psychology experiments are often remote association tasks, in which three words are given (e.g., electric, high, and wheel) and the participant is supposed to come up with a fourth word (e.g., chair) that can form an association among those three words (i.e., by being able to modify each word). Such tasks, while perhaps generating a moment of insight, may not afford the incubation time that people might need while proving theorems (Savic, 2015).

Researchers from several neuroscience domains – including memory, vision, and motor control – have demonstrated strong evidence supporting the notion that by either remembering or imagining an event or action individuals tend to generate neural activity that is similar to the activity generated during the actual event or action (e.g., Farah, 1988; Stavrinou et al., 2007). Our central hypothesis is that when subjects re-experience their proofs (i.e., see their proofs during the experiment), they might experience similar thinking to that which occurred during the moment of insight that produced a critical step for completing the proof. So, it stands to reason that the neural activity generated during this "re-living" experience will be similar to the brain activity generated during the production of the proof.

Methods

Our experimental methodology combines both theoretical and neuroscience components. We used a modified taxonomy for local proof comprehension (Savic, 2011) to generate codes for each slide of a slideshow that each participant watched during the EEG trial. In the first phase (days 1-2), three graduate mathematics majors (Marshall, Francis, and Buzz) were provided with a Livescribe pen to record his or her proving process, which we call "original" proofs. This pen is able to record the participant's written work and in real-time, allowing each pen stroke to be timestamped (to the second) and synchronized with the participant's verbal utterances during their proof. The pen also allows the participants' work to be saved as digital images. They were provided with two theorem statements (Figure 1), one in abstract algebra and the other in real analysis. We selected theorems that a senior-level abstract algebra and real analysis should reasonably be expected to prove.

Task 1: Prove that no group is the union of two of its proper subgroups.
Task 2: Prove that, if $a \in \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are functions continuous at a, then $fg: \mathbb{R} \to \mathbb{R}$ is continuous at a. [Here $(fg)(x) = f(x)g(x)$. Note that $fg \neq f \circ g$.]

Figure 1: Theorem statements to be proven by participants

In the second phase (days 3-4), each participant's digitized proofs were chunked and converted into uniform-sized digital images (700x700 pixels). In addition to the two proofs created by the participants, we generated two pre-written proofs of related, but different, conjectures which we call "canned" proofs. We generated these proofs to be concise, mathematically correct (not misleading) proofs of proximal difficulty as the assigned proofs for the participants to allow us to compare brain activity for a proof generated by the participant with one that was not.

In the third phase (day 5), participants came to the laboratory and were fitted with a 128-channel net, where the EEG data was obtained. Participants were instructed to watch the slideshows of the "canned" proofs and try to follow along with the argument as they would in a lecture or reading a text. For the "original" proofs, we instructed them to try and remember their thought processes as they generated each proof, focusing on their reasons for writing the content of each slide. The canned abstract algebra proof was shown first in its entirety, followed by their own abstract algebra proof, and then the canned real analysis proof, and finally their own real analysis proof. Each slide was shown for three seconds with a subsequent one-second break to blink, and all four groups of slides were shown in one sitting. Following the presentation of the last proof, participants were interviewed, with the entire interview taking between 15 and 45 minutes. This semi-structured interview inquired about academic history (e.g., mathematics courses completed) and demographic data. We also asked each participant to explain what s/he thought it meant to be the key idea of a proof. We then provided each participant with paper copies of their own proofs and the researcher-generated canned proofs, and asked him or her to identify the most important part(s) of the proof.

Variables and analyses

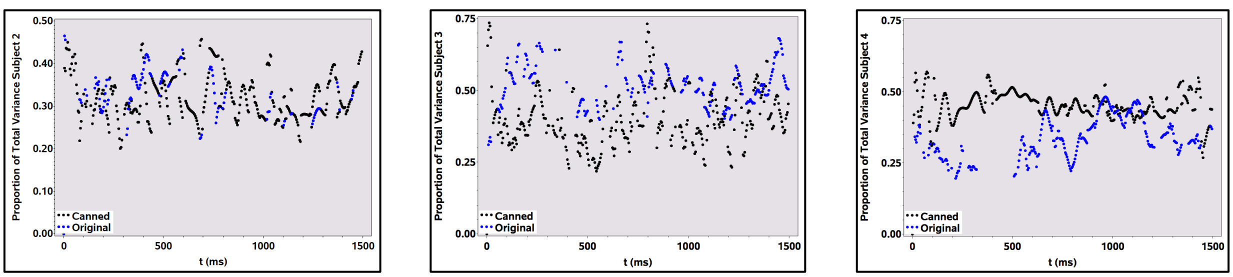
After data collection, at least two researchers from the team separately coded the collections of slides according to the modified proof comprehension taxonomy (Figure 2) and met to compare codes. Discrepancies between codes were resolved through discussion between the coders. We also generated an additional code of "student-reported potential for insight" (SRPFI) based on the slides corresponding to parts of the proof that the participant identified as important during the post-EEG interview. For the EEG data, for each time point (every 4 milliseconds between 0 ms and 1500 ms) in every slide across the four trials, we regressed the measured amplitude of the EEG electrode with the theoretically coded variables, choosing the final model through stepwise regression. In this process, we chose the electrode with greatest predicted total proportion of the variance (R^2) explained using the fewest coded variables. Here, variance in a given electrode is identified as the difference across all slides between the voltage of measured electrical activity by that electrode for a fixed time elapsed after first seeing each slide. We selected these models at each time point to generate a spatial and temporal description of the evolution of the modeled neural activity induced by the slides of the proofs.

Assumption (A), Contradiction statement (CONT), Delimiter (D), Exterior reference (ER), Interior reference (IR), Relabelling (REL), Statement of intent (SI), Similarity in a proof (SIM), Algebra (ALG), Conclusion statement (C), Definition of (DEF), Formal logic (FL), Use of exterior reference (UE), Use of interior reference (UI).

Figure 2: List of Codes from modified local proof comprehension taxonomy (Savic, 2011)

Results

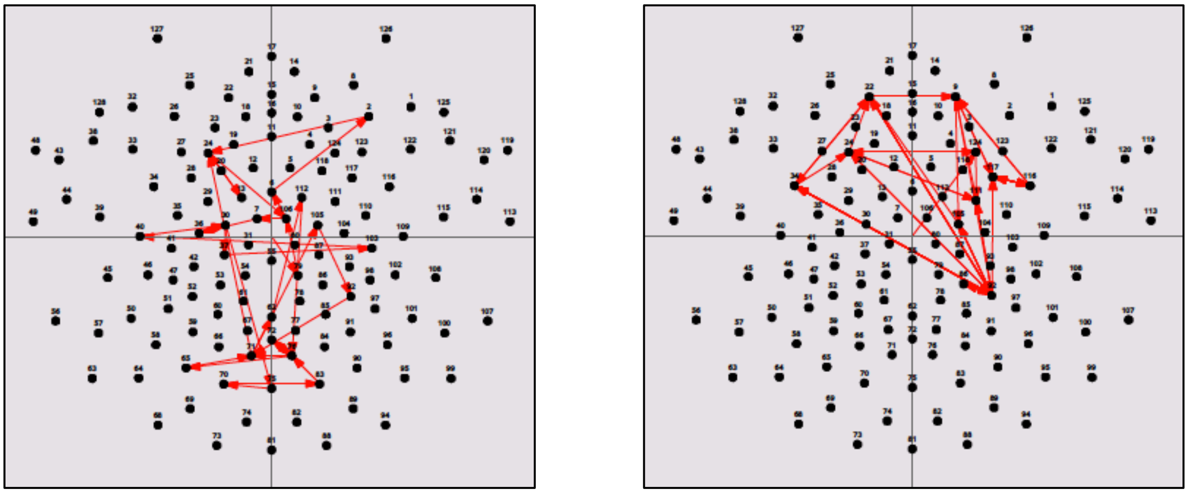
Our preliminary analysis indicates that the theoretical codes are able to account for between 20% and 75% of variance (y-axis in Figure 3) in modeled neural activity, though the codes accounted for around 35% to 50% of variance in neural activity for the majority of the time (x-axis). As one might infer from Figure 3, the current data do not show a conclusive difference between the potential for the theoretical codes to explain (or predict) variance between the canned and original proofs.



(a) Marshall (b) Francis (c) Buzz

Figure 3: Graphs of the proportion of total variance predicted by the model of best fit along time

We have also generated maps of the progression over time of the EEG electrodes with most variance explained (Figure 4). The figures below display the placement of the 128 nodes from the neural net. The highest nodes in quadrants 1 and 2 are located under the eyes. The origin corresponds with the top-most, center part of the participant's head, quadrant I corresponds with the right frontal lobe; quadrant II corresponds with the left frontal lobe, quadrant III corresponds with the left occipital lobe, and quadrant IV corresponds with the right occipital lobe. Initial comparisons between the maps of the participants' responses to the researcher-generated "canned" proofs and their responses to their own proofs indicate that the participants' react to the two types of proofs differently. For instance, the modeled electrodes during Marshall's reaction to the "canned" proof (Figure 4a) initially concentrated in and around the parietal lobe and shift to the temporal and frontal lobes. This implies that Marshall's initial reaction to a slide from the "canned" proofs tended to rely initially on visual processing before shifting to critical processing areas in the brain.

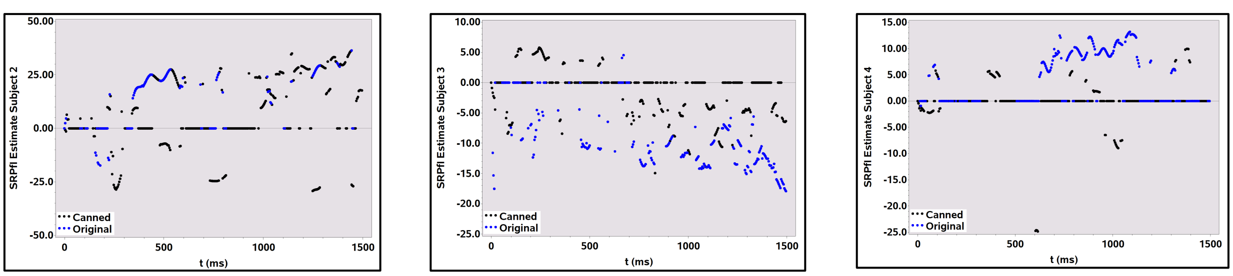


(a) (b)

Figure 4: Progression of neural activity with best model fit for Marshall's canned proofs (a) and original proofs (b).

In contrast, when Marshall was presented with "original" slides, the neural activity modeled in our regression takes place almost entirely in the temporal and frontal lobes (Figure 4b). Since his dominant electrodes were not in the visual processing area (occipital lobe), this supports the notion that Marshall is relying on his prior experiences as he observes the "original" slides. In other words, Marshall is likely engaged in some level of "re-living" his own proof. Finally, we see bilateral movement between left and right hemispheres on both 4a and 4b, which is "essential for complex mathematical reasoning" (Dehaene et al., 1999; cited in Desco et al., 2011, p. 282).

We found that all three participants had the code SRPFI (student-reported potential for insight) included in many of the regression models, seen in Figure 5, where the y-axis is microvolts squared and the x-axis is time. This indicates a potential link between student-reported insight and neural activity. However, due to the preliminary nature of the findings and sample size, we only hope to expand our research on that link.



(a) Marshall (b) Francis (c) Buzz

Figure 5: Graphs of the microvolts^2 of variance influenced by SRPFI

Future Research

We hope to extend our current findings by (1) exploring interpretations of our current data and the theoretical implications this might have on the codes we use to model neural activity, (2) collecting data with more participants, and (3) exploring different neural data collection techniques (e.g., alternative brain-computer interfaces such as fNIRS – functional near-infared spectroscopy). We expect that further analysis of the data collected to this point will help inform future work by raising questions of which theoretical constructs are able to better predict variance in neural activity, in turn narrowing our focus with respect to the existing theoretical codes and also informing our selection of potential codes to use in the future. Can we, with only coding, predict when a person has an insight in his or her proving process? Finally, since the codes considered local proof comprehension, we plan on using other coding schemes for holistic proof comprehension.

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Emerging Insights from the Evolving Framework of Structural Abstraction in Knowing and Learning Advanced Mathematics

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Only recently 'abstraction on objects' has attracted attention in the literature as a form of abstraction that has the potential to take account of the complexity of students' knowing and learning processes compatible with their strategy of giving meaning. This paper draws attention to several emerging insights from the evolving framework of structural abstraction in students' knowing and learning of the limit concept of a sequence. Particular ideas are accentuated that we need to understand from a theoretical point of view since they reveal a new way of understanding knowing and learning advanced mathematical concepts and have significant implications for educational practice.

Keywords: Limit Concept; Mathematical Cognition; Sense-Making; Structural Abstraction

Introduction

Theoretical and empirical research shows the existence of differences in knowing and learning concerning different kinds of knowledge (diSessa, 2002). A general framework on abstraction cannot encompass the whole complexity of knowing and learning processes in mathematics. Rather, in investigating the nature, form, and emergence of knowledge pieces, various micro-genetic learning theories may be developed, which will be quite specific to particular mathematical concepts, individuals, and their underlying sense-making strategies. As a consequence, the complexity of knowing and learning processes in mathematics cannot be described or explained by only one framework. Instead, we acknowledge that comprehensive understanding of cognition and learning in mathematics draws on a variety of theoretical frameworks on abstraction.

The literature demonstrates significant theoretical and empirical advancement in understanding 'abstraction-from-actions' approaches, particularly the cognitive processes of forming a (structural) concept from an (operational) process (Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991). Abstraction-from-actions approaches take account of a certain sense-making strategy, namely what Pinto (1998) described as 'extracting meaning'. However, only recently 'abstraction on objects' has attracted attention as a form of abstraction that provides a new way of seeing the complex knowing and learning processes compatible with students' strategy of what Pinto (1998) described as 'giving meaning'.

The purpose of this paper is to provide deeper meaning to a recently evolving framework of a particular kind of 'abstraction from objects': structural abstraction. The structural abstraction framework is evolving in the sense that the framework functions both as a tool for research and as an object of research (Scheiner & Pinto, submitted). In more detail, we use the structural abstraction framework retrospectively as a lens through which we reinterpret a set of findings on students' knowing and learning of the limit concept of a sequence. This reinterpretation is an active one in the sense that the framework serves as a tool to analyze a set of data, while the framework is also refined and extended since the reinterpretation produces deeper insights about the framework itself. Especially, these more profound insights are what we need to understand from a theoretical point of view since they have relevance for significant issues in knowing and learning advanced mathematical concepts.

We begin by providing an upshot of our synthesis of the literature on abstraction in knowing and learning mathematics. Our synthesis is to suggest an orientation toward the evolving framework of structural abstraction as an avenue to take account of an important area for consideration – that is, drawing attention to the complex knowing and learning processes compatible with students' 'giving meaning' strategy. The structural abstraction framework constitutes the foundation of the second part of the paper providing emerging insights in knowing and learning the limit concept of a sequence. These insights offer theoretical advancement of the framework and deepen our understanding of knowing and learning advanced mathematics.

Mapping the Terrain of Research on Abstraction in Mathematics Education

Abstraction seems to have gained a bad reputation because of the criticism articulated by the situated cognition (or situated learning) paradigm, and, as a consequence, has almost disappeared. This criticism rests primarily on traditional approaches considering abstraction as decontextualization and often confusing abstraction with generalization. The recent contribution by Fuchs et al. (2003) shows that such classical approaches to abstraction still exist:

> "To abstract a principle is to identify a generic quality or pattern across instances of the principle. In formulating an abstraction, an individual deletes details across exemplars, which are irrelevant to the abstract category […]. These abstractions […] avoid contextual specificity so they can be applied to other instances or across situations." (p. 294)

However, scholars in mathematics education argued against the decontextualization view of abstraction. Van Oers (1998, 2001), for instance, argued that removing context must impoverish a concept rather than enrich it. Several other scholars have reconsidered and advanced our understanding of abstraction in ways that account for the situated nature of knowing and learning in mathematics. Noss and Hoyles (1996) introduced the notion of *situated abstraction* to describe "how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed" (p. 122). Webbing in this sense means "the presence of a structure that learners can draw up and reconstruct for support – in ways that they can choose as appropriate for their struggle to construct meaning for some mathematics (Noss & Hoyles, 1996, p. 108). Hershkowitz, Schwarz, and Dreyfus (2001) introduced the notion of *abstraction in context* that they presented as "an activity of vertically reorganizing previously constructed mathematics into new mathematical structure" (p. 202). They identify abstraction in context with what Treffers (1987) described as 'vertical mathematization' and propose entire mathematical activity as the unity of analysis. These contributions demonstrate that research on abstraction in knowing and learning mathematics has made significant progress in taking account of the context-sensitivity of knowledge.

Several other contributions shape the territory in mathematics education research on abstraction. Scheiner (2016) proposed a distinction between two forms of abstraction, namely *abstraction on actions* and *abstraction on objects*. This distinction has been further refined in Scheiner and Pinto (2014) arguing that the focus of attention of each form of abstraction takes place on physical objects (referring to the real world) or mental objects (referring to the thought world) (see Fig. 1).

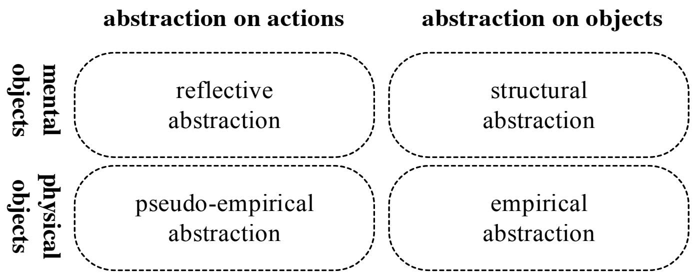

Fig. 1: A frame to capture various kinds of abstraction (reproduced from Scheiner & Pinto, 2014)

We consider this distinction as being productive in trying to capture some of the variety of images of abstraction in mathematics education (for details see Scheiner & Pinto, 2016). It acknowledges Piaget's (1977/2001) three kinds of abstraction, including pseudo-empirical abstraction, empirical abstraction, and reflective abstraction, that served as critical points of departure in thinking about abstraction in learning mathematics. Research on abstraction in mathematics has long moved beyond classifying and categorizing approaches in cognition and learning. For instance, Mitchelmore and White (2007), in going beyond Piaget's empirical abstraction and in drawing on Skemp's (1986) conception of abstraction, described abstraction in learning elementary mathematics concerning seeing the underlying structure rather than the superficial characteristics. Abstraction in learning advanced mathematics, however, is almost always defined in terms of *encapsulation* (or *reification*) of processes into objects, originating in Piaget's (1977/2001) idea of *reflective abstraction*. Reflective abstraction is an abstraction from the subject's actions on objects, particularly from the coordination between these actions. The particular function of reflective abstraction is abstracting properties of an individual's action coordination. That is, reflective abstraction is a mechanism for the isolation of specific properties of a mathematical structure that allows the individual to construct new pieces of knowledge. Taking Piaget's reflective abstraction as a point of departure, Dubinsky and his colleagues (Dubinsky, 1991; Cottrill et al., 1996) developed the APOS theory describing the construction of concepts through the encapsulation of processes. Similar to encapsulation is reification – the central tenet of Sfard's (1991) framework emphasizing the cognitive process of forming a (structural) concept from an (operational) process. In the same way, Gray and Tall (1994) described this issue as an overall progression from procedural thinking to proceptual thinking, whereas proceptual thinking means the ability to flexibly manipulate a mathematical symbol as both a process and a concept. Gray and Tall (1994) termed symbols that may be regarded as being a pivot between a process to compute or manipulate and a concept that may be thought of as a manipulable entity as procepts.

Scheiner (2016) revealed that the literature shows an unyielding bias toward *abstraction on actions* as the driving form of abstraction in knowing and learning advanced mathematics. This almost always exclusive view arises directly from the trajectory of our field's history; originating in Piaget's assumption that only reflective abstraction can be the source of any genuinely new construction of knowledge. While abstraction-on-actions approaches have served many purposes quite well, they cannot track detail of students' knowing and learning processes compatible with the strategy of giving meaning. The recently evolving framework of structural abstraction has attracted attention as a promising tool to shed light into the complexity of students' knowing and learning processes compatible with their strategy of 'giving meaning'.

The Evolving Framework of Structural Abstraction

The evolving framework of structural abstraction (Scheiner, 2016) further elaborates Tall's (2013) conception of this particular kind of abstraction. The crucial aspect lies in the argument that structural abstraction takes account of two processes: (1) complementizing meaningful aspects and structure underlying specific objects falling under a particular concept, and (2) promoting the growth of a complex knowledge system through restructuring various knowledge pieces. Several assumptions guide the evolving framework of structural abstraction assumptions:

Concretizing through Contextualizing

Structural abstraction takes place on mental objects that, in Frege's (1892a) sense, fall under a particular concept. These objects may be either concrete or abstract. Concreteness and abstractness, however, are not considered as properties of an object but rather as properties of an individual's relatedness to an object in the sense of the richness of a person's representations, interactions, and connections with the object (Wilensky, 1991). From this point of view, rather than moving from the concrete to the abstract, individuals, in fact, begin their understanding of (advanced) mathematical concepts with the abstract (Davydov, 1972/1990). The ascending from the abstract to the concrete requires a *concretizing* process where the mathematical structure is particularized by looking at the object in relation with itself or with other objects that fall under the particular concept. The crucial aspect for concretizing is *contextualizing*, that is, setting the object(s) in different specific contexts. Different contexts may provide various senses (Frege, 1892b) of the concept of observation.

Complementizing through Recontextualizing

The central characteristic of the structural abstraction framework is that while, within the empiricist view, conceptual unity relies on the commonality of elements, it is the interrelatedness of diverse elements that creates unity within the approach of structural abstraction. The process of placing objects into different specific contexts allows particularizing essential components. Structural abstraction, then, means attributing the particularized meaningful components of objects to the mathematical concept. Thus, the core of structural abstraction is *complementarity* rather than similarity. The meaning of advanced mathematical concepts is developed by complementizing diverse meaningful components of a variety of specific objects that have been contextualized and recontextualized in multiple situations. This perspective agrees with van Oers' (1998) view on abstraction as related to *re*contextualization instead of *de*contextualization.

Complexifying through Complementizing

The structural abstraction framework takes the view that knowledge is a complex system of many kinds of knowledge elements and structures. Complementizing implies a process of restructuring the system of knowledge pieces. These knowledge pieces have been constructed through the above-mentioned process or are already constructed elements coming from other concept images, which are essential for the new concept construction. The cognitive function of structural abstraction is to facilitate the assembly of more complex and compressed knowledge structures. Taking this perspective, we construe structural abstraction as moving *from simple to complex* knowledge structures, a movement with the aim to build coherence among the various knowledge pieces through restructuring them.

Emerging Insights from the Structural Abstraction Framework

In this section, we summarize emerging insights we gained, and still gain, by using the evolving framework of structural abstraction retrospectively as a lens through which findings on students' (re-)construction of the limit concept of a sequence were reinterpreted. The study by Pinto (1998) provides the context in which she identified mathematics undergraduates' sense-making strategies of formal mathematics. From a cross-sectional analysis of three pairs of students, two prototypical strategies of making sense could be identified, namely 'extracting meaning' and 'giving meaning':

> "*Extracting meaning* involves working within the content, routinizing it, using it, and building its meaning as a formal construct. *Giving meaning* means taking one's personal concept imagery as a starting point to build new knowledge." (Pinto, 1998, pp. 298-299)

The literature on abstraction-from-actions provides several accounts of how students construct a mathematical concept compatible with their strategy of 'extracting meaning'; however, there are almost no accounts of how students construct a concept compatible with their strategy of 'giving meaning'. It is important to note that the evolving framework of structural abstraction is problem driven, that is, addressing the need of bringing light into the complexity of students' knowing and learning processes compatible with their strategy of 'giving meaning', rather than filling a theoretical gap just because it exists. The reinterpretation of empirical data on students' strategies of giving meaning in the light of the theoretical framework of structural abstraction proved to be particular fruitful - not only to provide deeper insights into the strategy of giving meaning but also as a way to deepen our understanding of the phenomenon of structural abstraction that revealed to new theoretical development (Pinto & Scheiner, 2016; Scheiner & Pinto, 2014). In the following pages, we highlight the main theoretical advancements.

The idea of complementizing meaningful components underlying the structural abstraction framework reflects the idea that whether an individual has 'grasped' the meaning of a mathematical concept depends on the specific context where the objects falling under the particular mathematical concept have been placed in. This supports Skemp's (1986) viewpoint that "the subjective nature of understanding […] is not […] an all-or-nothing state" (p. 43). The reanalysis of the data indicates that the object of researchers' observation should be directed to students' *partial constructions* of the limit concept. These partial constructions may be specific and productive to particular contexts but may remain not fully connected and may be unproductive in other contexts (for instance, in making sense of the formal definition). The empirical data shows that, in the case of the students who give meaning, several meaningful elements and relations in understanding the limit concept of a sequence are involved, although a few elements are missing (or distorted). However, some students are able to (re-)construct some meaningful components at need by making use of their partial constructions, while others are not able to do so.

The reanalysis indicates that some students have developed *resources* that enable them to (re-)construct the limit concept of a sequence at need. Scheiner and Pinto (2014) presented a case where a student developed a generic representation of the limit concept of a sequence that operates well in several, although not all, contexts and situations. This particular representation, however, allows the student to (re-)construct the limit concept in other contexts and situations. The reinterpretation of the data sheds light on the phenomenon that individuals may do not 'have' all relevant meaningful components, but, rather, they may have resources to generate some meaningful components and make sense of the context at need. In that sense, the 'completeness' of the complementizing process cannot ever be taken as absolute.

Several scholars suggested exposing learners to multiple contexts and situations. An important insight from using the structural abstraction framework retrospectively is that exposure to multiple contexts is at least important for particularizing meaningful components: various objects falling under a particular mathematical concept have to be set into different specific contexts in order to make visible the meaningful components or mathematical structure of these objects. In so doing, the objects may be 'exemplified' through a variety of representations, in which each representation has the same reference (the mathematical object); however, different representations may express different senses depending on the selected representation system (see Fig. 2). The distinction between *sense* and *reference* has been specified by Frege (1892b) in his work *Über Sinn und Bedeutung*, indicating both the sense and the reference as semantic functions of an expression (a name, sign, or description). In short, the former is the *way* that an expression refers to an object, whereas the latter is the *object* to which the expression refers. According to Frege (1892b), to each representation correspondents a sense; the latter may be connected with an *idea* that can differ within individuals since people might associate different senses with a given representation. Though multiple contexts and situations are needed, a new context that does not provide a new sense will unlikely to be productive for the concept construction. The framework indicates that the *nature of the contexts* the objects are set is determinative of their value toward the complementizing process.

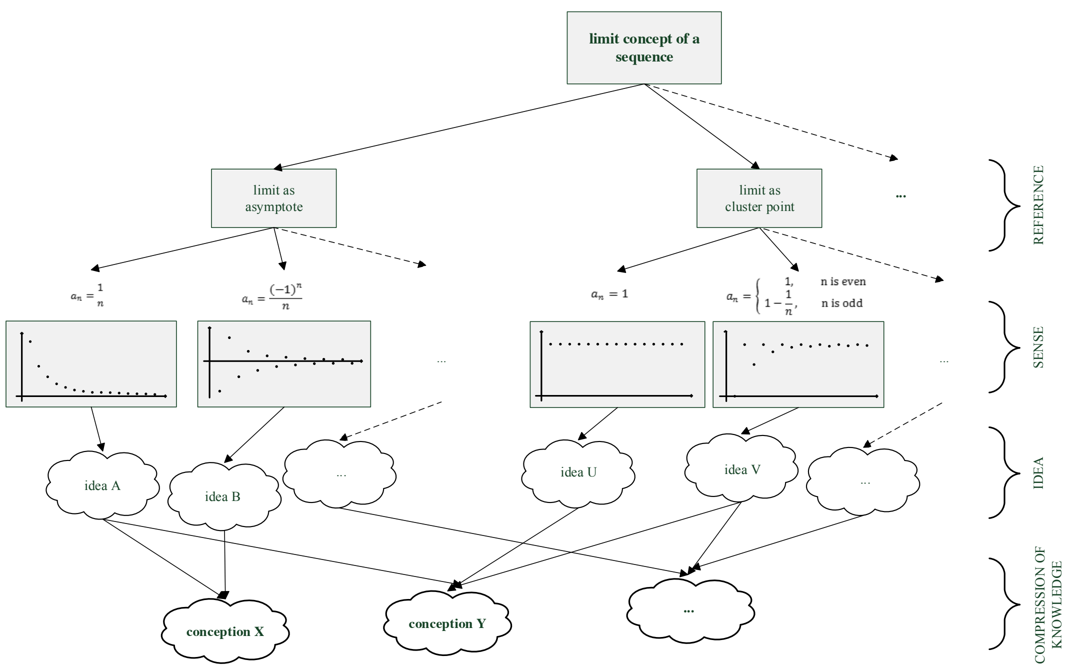

Fig. 2: Reference, sense, and idea

Research also indicates that students may have difficulties with the *relationships* between the sense and the reference as well as difficulties in *maintaining* the reference as the sense changes (Duval, 1995, 2006). Thus, based on the insights we have gained from reanalyzing the data (Pinto & Scheiner, 2016; Scheiner & Pinto, 2014), we can assume that these difficulties may (at least partly) be overcome by providing students a particular resource (such as a generic representation of the mathematical concept) that serves as a guiding tool in complementizing the meaningful components indicated in the different senses. From this perspective, a 'representation for' is a tool that provides theoretical structure in constructing the meaning of the concept of observation. It necessarily reflects essential aspects of a mathematical concept but can have different manifestations (Van den Heuvel-Panhuizen, 2003). Concerning the learning of the limit concept of a sequence, the reinterpretation of the data indicates that a slightly modified version of a student's representation (see Fig. 3) may support the complementizing process when the limit concept is recontextualized.

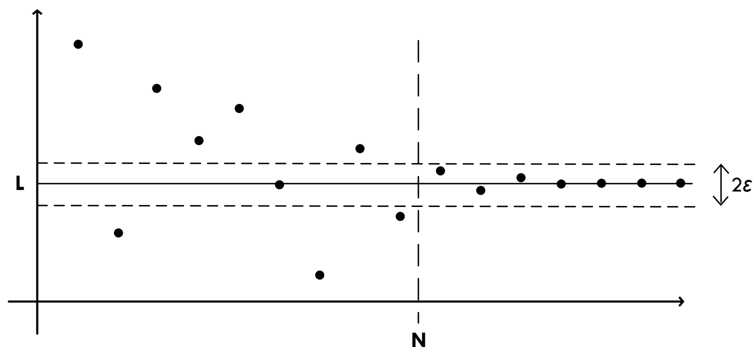

Fig. 3: A generic representation for learning the limit concept of a sequence

Notice that this generic representation for learning the limit concept of a sequence takes account of several students' common conceptions identified in the research literature, including those as (1) the limit is unreachable, (2) the limit has to be approached monotonically, and (3) the limit is a bound that cannot be crossed (see Cornu, 1991; Davis & Vinner, 1986; Przenioslo, 2004; Tall & Vinner, 1981; Williams, 1991).

The reanalysis of the empirical data gained from Pinto's (1998) study has shown that students giving meaning built a representation of the concept and, at the same time, used it as a representation for recognizing and building up knowledge – the reconstruction of the formal concept definition, for instance. The analysis shows that these students consistently used *representations of* mathematical objects to create pieces of knowledge; or, in other words, the representations were actively taken as *representations for* emerging new knowledge and making sense of the context and situation. This shift from constituting a representation of the limit concept to using this representation as a representation for (re-)constructing the limit concept in other contexts can be described in terms of shifting from a *model of* to a *model for* (Streefland, 1985) – a shift from an *after*-image of a piece of given reality to a *pre*-image for a piece of to be created reality. This mental shift from 'after-image' to 'pre-image' indicates a degree of awareness of the meaningful components and the complexity of knowledge structure that allows the transition from a 'representation of' as a result of various representations expressing specific objects set in different contexts to a 'representation for' constructing and reconstructing the limit concept, inter alia, in formal mathematical reasoning. We suggest that a generic representation, as presented in Fig. 3, may provide an instructional tool that supports raising the awareness of meaningful components in learning the limit concept of a sequence. In other words, such a generic representation may direct students' perception of meaningful components.

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Physics Students' Use Of Symbolic Forms When Constructing Differential Elements In Multivariable Coordinate Systems

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Analysis of properties of physical quantities represented by vector fields often involves symmetries and spatial relationships that are best expressed in three dimensional non-Cartesian coordinate systems. Many important quantities, both scalar and vector in nature, are determined by paths, areas, or volume integrals of multivariable functions. The differential quantities in these systems are not trivial for students to understand and implement correctly. As part of an effort to investigate physics students' understanding of the structure of non-Cartesian coordinate systems and the associated differential elements when using vector calculus in Electricity and Magnetism (E&M), we interviewed four pairs of students in the junior-level E&M course. In one particular task, students were asked to construct differential length elements for an unconventional spherical coordinate system. A symbolic forms analysis (Sherin, 2001) of student reasoning revealed both known and novel forms, and found that student difficulties with vector differential quantities were primarily conceptual rather than symbolic.

Key words: Coordinate Systems, Symbolic Forms, Physics, Differential Elements, Vector Calculus

Introduction

The equation is a fundamental construct that appears ubiquitously in physics as a means to provide a mathematical description of physical situations, from the analysis of opposing forces on an object at the introductory level to the use of non-Cartesian vector elements in electricity and magnetism (E&M). The equation becomes a translation across the mathematics-physics interface that students are expected to routinely navigate. The ability to work with mathematical systems, equations, and expressions becomes especially relevant in upper-division coursework, in which the mathematics and physics increase in sophistication. In many cases the use of mathematics includes accounting for underlying physical symmetry, extracting information from physical situations for calculation, or interpreting the results of calculation physically, all of which are cited as areas of student difficulty in upper-division E&M (Pepper *et al.,* 2012).

Our work seeks to address students' understanding of differential vector elements used to solve various vector calculus problems in E&M, specifically answering the following research question: How do students construct differential vector elements in a given coordinate system?

Sherin's (2001) *symbolic* forms is one method of analysis developed to understand how students interpret and construct equations. We use the symbolic forms framework to provide insight into how upper-division students develop the structure of differential vector elements and determine how each component is represented in the final equation.

Background

In upper-division E&M, students reason about physical symmetries of electric and magnetic fields and interpret physical scenarios to construct and implement differential elements into calculation. In the majority of cases, these differential elements begin as vector quantities, having a particular direction, where students are summing up the dot products of various vector fields over specific lines or surfaces. To add complication, field symmetries in E&M often dictate the use of a non-Cartesian coordinate system, meaning the differential elements take on a different form than the traditional dx or $dydz$, a line element and area element both in the $\hat{\imath}$ direction.

In cylindrical and spherical coordinates, motion along curved surfaces results in differential length elements that are arc lengths and include scaling factors ($i.e.,\ dl_\theta = rd\theta\hat{\theta}$ for spherical coordinates in physics). Area elements are constructed by determining which variables are changing and which are constant, and then multiplying the differential length elements associated with the change(s). Therefore an understanding of the differential length vector across each coordinate system emerges as a fundamental mathematical construct in the application of vector calculus in our upper-division electricity and magnetism courses.

Previous studies have assessed student understanding of integration and differentials (Doughty *et al*., 2014; Hu & Rebello, 2013; Nguyen & Rebello, 2011) and have addressed calculation, understanding, and application of gradient, divergence, and curl in both mathematics and physics settings (Astolfi & Baily, 2014; Bollen, *et al*., 2015). Other studies have identified difficulties in applying Gauss's and Ampère's Laws, to common aspects of an E&M course that involve a surface integral and line integral, respectively (Guisasola, *et al*., 2008; Manogue, *et al*., 2006; Pepper *et al*., 2012). However, little work has explored student understanding of the differential vector element as it appears in the non-Cartesian systems used in associated physics problems.

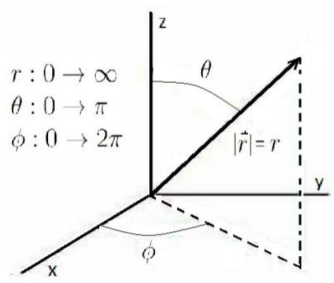
Theoretical Perspective

Based on the knowledge-in-pieces model (diSessa, 1993), a symbolic forms analysis affords the opportunity to investigate students' thinking about equations (Sherin, 2001). Symbolic forms began in a third-semester physics course, where students were provided with several word problems modeling physical situations common to introductory physics. The development of a symbolic forms analysis provided a critical lens for the investigation of students' construction and sense-making of equations at the introductory level.

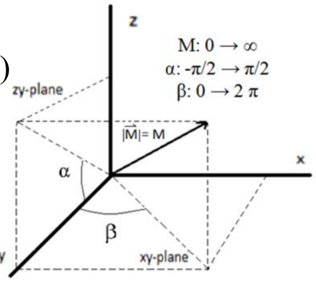
The specific nature of a symbolic form comes from the combination of a symbol template with a conceptual schema. A symbol template, such as $\square + \square + \square$, is an externalized structure of an equation. Each box contains variables and/or numbers depending on what a student deems fit. A student's conceptual schema is the sum of internalized ideas students use to determine and fill in the template. For example, $\square + \square + \square$ would appear in a students' equation if the student decided they needed to sum multiple quantities. The symbolic form associated with this particular template-schema pair is known as *parts-of-a-whole* (Sherin, 2001). Additionally a student would be able to read information out of an equation, such as relationships between quantities, by interpreting the templates used to construct an equation.

Work with symbolic forms has since expanded into the upper division and to explore integration in mathematics. Meredith and Marrongelle (2008) adapted the conceptual aspects of symbolic forms to describe what features of electrostatics problems cue students to integrate. They found students using the *dependence* form when eliciting the reliance on a particular variable and the *parts-of-a-whole* form when summing up multiple components. The symbolic forms framework has been used to analyze physical chemistry students' use of partial derivatives in thermodynamics (Becker & Towns, 2012). The ideas of symbolic forms were used to address calculus students' ideas when making sense of integrals (Jones, 2015); students' exposed conceptual understandings often included graphical representations of given functions.

(a)



(b)



(c)

$$\vec{dl} = dr\hat{r} + rd\theta\hat{\theta} + rsin\theta d\phi\hat{\phi}$$
$$d\tau = r^2 sin\theta dr d\theta d\phi$$

(d)

$$\vec{dl} = dM\hat{M} + Md\alpha\hat{\alpha} + Mcos\alpha d\beta\hat{\beta}$$
$$d\tau = M^2 cos\alpha dM d\alpha d\beta$$

Figure 1. (a) Conventional (physics) spherical coordinates; (b) an unconventional spherical coordinate system given to students, for which they were to construct differential length and volume elements. The correct elements for each system are in (c) and (d), respectively.

Methods

In order to investigate student construction of differential length vectors, a task was developed in which students were asked to reason about an unconventional spherical coordinate system, which we call “schmerical coordinates” (Figure 1). By developing an unconventional coordinate system, we could determine students’ abilities to work with the underlying conceptual ideas, rather than their ability to recall a memorized answer.

Clinical think-aloud interviews were conducted with four pairs of students (N=8) following the first semester of a two-semester, junior-level E&M sequence. Students were asked whether the given coordinate system was feasible and to determine and verify the differential line and volume elements. Pair interviews facilitated a more authentic interaction between students and minimized the input and influence of the interviewer. Groups are identified as AB, CD, etc., signifying pairings of students A and B, and so on.

Interview data were transcribed and analyzed using open coding to identify common actions and recurring ideas across interview groups. Initial analysis categorized these ideas as aspects of students’ *concept images* (Tall & Vinner, 1981) and building actions. Concept image aspects include *component and direction, dimensionality, differential,* and *projection*. Building actions involved *recall* of and *mapping* to other coordinate systems, as well as *grouping* of specific terms. Subsequently, we addressed the sequence these ideas and actions arose in the construction and checking of the differential vector elements (Schermerhorn & Thompson, 2016).

A secondary, finer-grained analysis investigated how students developed, implemented, and filled known and novel symbol templates as part of a symbolic forms analysis.

Results

Analysis of student work resulted in the identification of symbolic forms related to the construction of non-Cartesian differential length vectors (Table 1). Some invoked symbolic forms match those established by Sherin (2001): *parts-of-a-whole*, *no dependence*, and *coefficient*. In addition, other, novel template-schema combinations were common across all student groups, emerging due to the vector nature of this context and the increased mathematical sophistication of the upper-division content. We labeled these forms *magnitude-direction* and *differential*.

This report focuses on students’ initial attempt at construction before they were asked about the differential volume element. At this stage of construction no group was able to successfully determine the differential length element due to inattention to certain ideas, such as *projection*, or overreliance on recalling spherical coordinates (Schermerhorn & Thompson, 2016).

Table 1. Symbolic forms identified in students' construction of a differential length element.

Symbolic Form	Symbol Template	Conceptual Schema
Parts-of-a-whole	$\square + \square + \square$	Accounts for multiple components that contribute to a larger whole (Sherin, 2001).
No dependence	$[\ldots]$	Indicates an expression is independent of, or not a function of, a specific variable. The expression is therefore absent of the variable (Sherin, 2001).
Coefficient	$[\square\ldots]$	Quantities or variables put in front of an expression; Seen as just numbers or constants, possibly having units (Sherin, 2001).
Magnitude-direction	$\square\hat{\square}$	Used to denote a vector expression including the magnitude of a quantity (having units) paired with a unit vector to indicate a specific direction.
Differential	$d\square$	Represents taking a small amount of or infinitesimal change in a quantity.

Previously identified symbolic forms

Students' attention to the overall structure of the differential length vector led them to recognize the need for multiple components coupled with motion in each of the three spatial directions, consistent with Sherin's *parts-of-a-whole*. Student F demonstrated an appropriate conceptual schema when starting construction:

F: There are three different dl's. There is dl with respect to M, dl with respect to α and dl with respect to β.

Group EF worked on each component independently; student E then summed these components to express their full (incorrect) vector differential:

E: You sum them, so $\overrightarrow{dl}$ is those added together $dM\hat{M} + Md\beta\hat{\beta} + Md\alpha\hat{\alpha}$.

Group CD recognized the need for each part by leaving space to write the components into the expression (Fig. 2).

Students also used language consistent with the *no dependence* symbolic form:

A: [writes $Md\alpha$] For α? [sweeps arm up and down] Yeah, for α, it doesn't have any dependence on this other angle over here.

The arm motion was made by A to trace out how the angle α is changing. Here student A reasoned about the geometrical space to recognize that the differential length element resulting from a change in α is independent of β.

As students made determinations about the scaling factors (*e.g.*, the $M\cos\alpha$ associated with the $M\cos\alpha\, d\beta\hat{\beta}$) they commonly used language associated with Sherin's *coefficient* symbolic form. As the *coefficient* form is typically coupled with additional forms in equation building, the invocation of this form will be discussed later.

Figure 2. Beginning stages of construction for C and D showing the coupling of the *parts-of-a-whole* and *magnitude-direction* symbolic forms.

Newly identified symbolic forms

Given the relevance of mathematics associated with vectors and differentials in this task, this work has identified new template-schema pairings, the *magnitude-direction* and *differential* symbolic forms, emerging from students' constructions.

When constructing differential length vectors, students must attend not only to the magnitude of the length component but also the direction in which the length is taken. All groups recognized the need to include unit vectors in their full expressions. From this we identify the *magnitude-direction* symbolic form, which in a larger context accounts for students recognizing the need to multiply a quantity, representing a magnitude, by a unit vector denoting the direction associated with that particular quantity.

While the *magnitude-direction* symbolic form is similar to the *scaling* and *coefficient* symbolic forms (Sherin, 2001), these existing forms are insufficient to describe the constructed vector expressions. Consider a vector representing a distance of 5 meters north. In two dimensions we could represent this as $(5\text{m})\hat{\jmath}$. The *coefficient* form is not applicable in that the magnitude of a vector expression has specific meaning, especially in physics. The *scaling* form is used to describe unitless or dimensionless quantities that attenuate the size of what follows, and thus contradictory given the emphasis on the vector describing a length. Therefore, the *magnitude-direction* symbolic form is more applicable to describe the choices of our students.

As students determined the expressions for the magnitudes of each component, they discussed the inclusion of differential terms. Student C exemplifies common reasoning:

C: You have a change in your $\widehat{M}$ is going to be your dM, it's your change in your M.

From this and similar statements we identify the *differential* symbolic form, which is used to signify a small change in or small amount of a given quantity, a conceptual schema consistent with students' ideas of differentials as identified in the literature (Artigue *et al.,* 1990; Hu & Rebello, 2013; Roundy *et al.,* 2015; Von Korff & Rebello, 2012). The form itself is similar to what appears in graphically oriented symbolic forms for integration (Jones, 2013) but here, the differential is a standalone quantity with its own attached schemata. Distinction is also made from Sherin's (2001) *base±change* form, where the change is seen as being applied to a quantity or as a correction term dependent upon the base quantity ($i.e.,\ v_o \pm at$).

Student invocation and combination of symbolic forms

Overall, each group invoked and combined symbolic forms in a similar sequence (Fig. 3). At the beginning of construction, there was almost an immediate nesting of the *magnitude-direction* symbolic form into each box of the *parts-of-a-whole* form, specifically accounting for the vector components. Group CD explicitly develops this structure by writing out the three needed units vectors and leaving space to write in the magnitude of each component (Fig. 2).

Students often paired the coefficient and differential terms when the necessity of appropriate dimensions dictated it, as seen in the following two (independent) excerpts:

A: …This doesn't have any units of length, so it needs to have some *M* term.

C: …So, if it's going to be some trig thing but sine of something isn't a length so we're going to have to also have something else in there.

The determinations of the coefficients of each term occurred at multiple stages of construction for a variety of reasons, including recall and mapping to spherical coordinates to a geometric argument that recognized *M* as the radius of a differential arc length. We see the reasoning about dimensionality, in particular, as an abstracted use of the *dependence* symbolic form, where

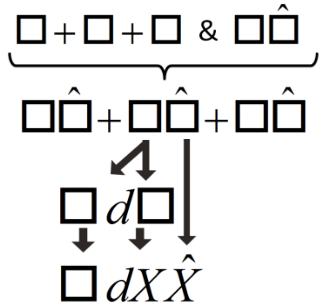


Figure 3. The general flow of students' use and combination of symbol templates during the interview task.

instead of the dependence of the expression being on a specific variable, it is driven by dimensional analysis. C and D paid particular attention to dimensionality and even returned to their expression later in construction with concern about whether the angles carried any units of length. The dependence on units of length sometimes appeared early in construction with the pairing of *magnitude-direction* and *parts-of-a-whole*:

E: So, we're going to have components right... each one has to have units of length.

This often resulted in most students recognizing they did not need to attach a coefficient to dM since it already possessed the necessary units.

As part of students' conceptual schema during construction, many groups paired the differential variable and the corresponding unit vector (*i.e.*, $dM\ \widehat{M}$, $d\alpha\ \hat{\alpha}$, and $d\beta\ \widehat{\beta}$). Groups AB and GH did this automatically during construction (Fig. 4). Other groups initially tried to forcefully insert a differential into their expression. After recognizing $M\cos\alpha$ was a projection into the *xy*-plane, CD wrote a "*d*" in front of the whole expression (Fig. 5). Attempting to insert a trigonometric function into the expression, student F tried to express an infinitesimal arc length as $r\sin d\theta$. After fixing this, group EF focused their construction on having a differential length component in a particular direction containing a differential with that variable:

E: So you're going to have a length component in the β-hat direction…so, basically we're going to need…an M…so it's M times some Δ, I think it's M times $\Delta\beta$, a small β, because it's like if you take r times its small θ then that is the arc length (Fig. 6).

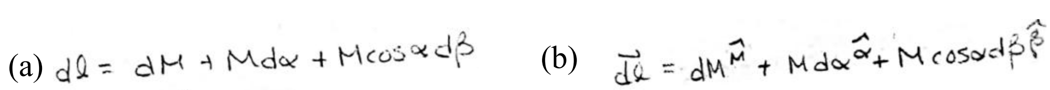


Figure 4. Students G and H before (a) and after (b) recognizing the need to include unit vectors.

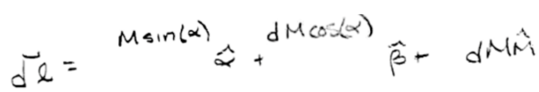


Figure 5. Students C and D incorrectly incorporating the idea of a differential.

Figure 6. Students E and F constructing the beta component of the differential length, initially leaving space to write the needed coefficient and unit vector (a). After discussion they include a coefficient which lacks the needed projection term $\cos\alpha$.

Here we see E and F including the differential element appropriately and using the idea of arc length to fill in the preceding space that was left intentionally blank, similar to C and D with the larger structure (Fig. 3). For E and F, this resolved into the length component, $\overrightarrow{dl}_\beta = Md\beta\hat{\beta}$, which still lacked the needed trigonometric term. As the groups appropriately connected the unit vector with the variable in the *differential* symbolic form, they accounted for the larger pieces of the differential length vector and focused efforts on determining the coefficient terms.

Conclusions

Applying a symbolic forms analysis to student construction of a differential length vector enabled us to see what students think the vector expression should look like as well as what concepts they use to construct the vector elements. The vector context and the increased mathematical sophistication of the upper-division content led to the recognition of existing symbolic forms as well as the identification of new forms. Further analysis identified a typical sequence of invocation of these forms during construction.

Sherin's symbolic forms framework was developed to frame some student difficulties with physics equations as related to the symbol template in addition to or rather than their conceptual understanding. However, our symbolic forms analysis found that students were able to recognize the general structure needed for the equation and invoke the correct template. Difficulties constructing the differential length element were primarily related to conceptual schemata, *e.g.*, students constructed an appropriate expression for the β-component in terms of dimensional and differential considerations, but neglected the projection that introduces the $\cos\alpha$ term.

Results also suggest that attention to dimensionality was significant to students' choices during construction, yet some students struggled to determine the units of certain terms. This is an especially important finding, as previous research on symbolic forms neglects how students' attention to units impacts their problem solving (Sherin, 2001).

Geometric reasoning was prominent in students' construction. In many cases students would attempt to visualize the lengths traced by the vector as small changes were made to individual variables. Students attended to the multiple components needed to express the differential length vector and properly connected the differentials to unit vectors of the same variable. In cases where this proved difficult for students, recall mediated expression construction, similar to upper-division physical chemistry findings dealing with partial derivatives (Becker & Towns, 2012). In our study, however, recall of spherical coordinates during construction led students to expressions that incorrectly included a $\sin(\alpha)$ term (Schermerhorn & Thompson, 2016).

These findings suggest that instructional changes should focus on the concepts associated with the building of the differential, specifically on the determination of the coefficients for the angle components. Subsequent efforts include the development of student-centered instructional materials to be used in E&M and/or mathematical methods of physics courses. In particular, we are developing and piloting a more scaffolded version of the interview task as an instructional activity that guides students through the construction of components of the differential length vector, construction and checking of the volume element, and construction and choice of differential area vectors in schmerical coordinates. Given the importance of the differential area element in E&M calculations (*e.g.*, to determine electric and magnetic flux), current investigations explore how students reason about and construct differential area elements. Additional efforts will investigate the effectiveness of the instructional activity on student understanding of differential length and area vectors.

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A Comparison of Calculus, Transition-to-Proof, and Advanced Calculus Student Quantifications in Complex Mathematical Statements

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Abstract: This study investigates Calculus, Transition-to-Proof, and Advanced Calculus students' meanings for quantifiers in conditional statements involving multiple quantifiers. Three students from each course participated in clinical interviews. Students were presented with the Intermediate Value Theorem (IVT) and three other statements whose sentence structure was similar to the IVT except for reordered quantifiers and their attached variables. The results reveal that Advanced Calculus and Transition-to-Proof students made distinctions between the different statements more often than Calculus students. Several student meanings for quantification were found to be necessary for making distinctions between each of the four statements. We also address student quantifications that emerged for the phrase "Suppose f is a function."

Key words: Student Quantification, Undergraduate Students, Comparative Analysis, Intermediate Value Theorem (IVT), Logic

The teaching and learning of Calculus is important for all STEM programs. Many studies have investigated students' meanings for important Calculus concepts such as limits, differentiation, and integration, and how these concepts are presented to students (Carlson et al., 2002; Martin, 2013; Oehrtman, 2009; Orton, 1983; Thompson, 1994; Vinner & Dreyfus, 1989; White & Mitchelmore, 1996; Zandieh, 2000). However, little attention has been paid to the language in which mathematicians express Calculus ideas in textbooks and discourse.

In this paper, we focus on students' meanings for quantifiers in complex mathematical statements. By complex mathematical statements, we mean statements that have both if-then structure and multiple quantifiers. The Intermediate Value Theorem (IVT) is one example of such a statement: "Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a real number c in (a, b), such that $f(c)=N$" (Stewart, 2003, p. 131). Similar to the IVT, many Calculus theorems can be classified as complex mathematical statements, even though the topic of quantification is not addressed in popular Calculus textbooks (Bittinger, 1996; Larson, 1998; Stewart, 2003). Quantifier words such as "for all," "there exists," and "unique" may have different student meanings than mathematical ones (Dawkins & Roh, 2016; Dubinsky & Yiparki, 2000; Epp, 1999; Epp, 2003). Students at different mathematical levels, including Calculus, may interpret these words differently than mathematicians. Although some studies have investigated students' and teachers' understandings for quantifiers, many of these studies have dealt with a single quantifier or statements outside of mathematical context (Dubinsky & Yiparaki, 2000; Epp, 2009; Piatek-Jiminez, 2010; Tabach et al., 2010; Tsamir et al., 2009). On the other hand, studies that have dealt with multiple quantifiers in mathematical contexts have placed more emphasis on Transition-to-Proof and Advanced Calculus students' understanding of quantification than Calculus students' understandings of quantification (Dawkins & Roh, 2011; Dawkins & Roh, 2016; Roh & Lee, 2011, 2015, Selden & Selden, 1995). This study explores the following question: *What meanings do students at various mathematical levels have for quantifiers in complex mathematical statements from Calculus?*

Theoretical Perspective

Previous studies show that students often do not problematize the distinction between "for all… there exists" ($\forall\exists$) and "there exists… for all" ($\exists\forall$) statements, and that $\exists\forall$ statements are more frequently misinterpreted than $\forall\exists$ statements (Dubinsky & Yiparaki, 2000; Piatek-Jiminez, 2010). Previous studies also note that students may reorder the variables attached to the quantifiers in their explanation of a complex mathematical statement because they do not necessitate the independence of the first variable and dependence of the second variable (Dawkins & Roh, 2016; Roh & Lee, 2011, 2015). For example, in the IVT, one must consider each value of N before finding the associated c that depends on each N.

Student quantifications[1] for various $\forall\exists$ and $\exists\forall$ statements and relationship between variables attached to the quantifiers may change if we consider different moments. Our goal is to describe our best perception of each student's own meanings for quantifiers at different moments. We thus utilize the phrase "student meaning" throughout this paper the same way in which Piaget views that each individual constructs his own meanings by assimilation to schemes (Thompson, 2013). Students' meanings for quantifiers *in the moment* are preferred because some meanings may be stable, but other meanings may be "meaning(s) in the moment" (Thompson et al., 2014). Thompson et al. (ibid) describe a meaning in the moment as "the space of implications existing at the moment of understanding" (p. 13), so students could be assimilating information in the moment and forming new meanings. A student's thoughts may begin to emerge or different meanings may be elicited in different moments. Thus, we consider several different moments of interaction for each student because different moments of interaction may result in different types of student quantification.

In this study, we identified two types of quantifiers in complex mathematical statements—*explicit* and *implicit quantifiers*, and we seek to understand student meanings for both of these types of quantifiers. *Explicit quantifiers* are directly worded in a mathematical statement. The IVT wording "for all" explicitly states a universal quantification, while "there exists" explicitly states an existential quantification. *Implicit quantifiers* (Durand-Guerrier, 2003) are intended in a mathematical statement but are not conveyed through direct phrasing such as "for all" and "there exists." In the the IVT statement, the hypotheses "Suppose that f is a continuous function" has a hidden universal quantifier because this statement is, in a mathematical convention, a generalized statement and thus f stands *for all* continuous functions.

Methods

Two-hour long clinical interviews (Clement, 2000) were conducted with nine undergraduate students during the spring and summer of 2016 at a large southwestern university in the United States. Students were placed in a category based on the highest course they had already completed. These students had various STEM majors and completed these courses with a variety of different instructors. Three students volunteered from each mathematical level: Calculus I, Transition-to-Proof, and Advanced Calculus. All four authors of this paper participated in data collection (as either interviewer, camera operator, or witness) and analysis.

Interview Tasks. Students were asked to explain their understandings of the four statements shown in Table 1 and to evaluate the truth-values of each of the statements. (Only the statements in the left-hand column of Table 1 were presented to students.) The four statements in Table 1

[1] In this paper, quantification refers to the process by which students quantify variables.

exhaust all combinations for ordering explicit quantifiers and their attached variables. The symbolic representations of the explicit quantifiers found in the conclusion of each statement are also shown beside each statement in Table 1. Three of the statements are false. Statement 2 is the IVT, and the only true statement. The variety of statements allowed us to analyze students' comparisons and contrasts amongst the explicit quantifiers.

Statements	Symbolic Representations
Statement 1: Suppose that f is a continuous function on the closed interval $[a, b]$, where $f(a) \neq f(b)$. Then, for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$.	$\forall c \exists N$ such that $f(c)=N$
Statement 2: Suppose that f is a continuous function on the closed interval $[a, b]$ where $f(a) \neq f(b)$. Then, for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b), such that $f(c)=N$.	$\forall N \exists c$ such that $f(c)=N$
Statement 3: Suppose that f is a continuous function on the closed interval $[a, b]$, where $f(a) \neq f(b)$. Then, there exists a real number N between $f(a)$ and $f(b)$, such that for all real numbers c in (a, b), $f(c)=N$.	$\exists N \forall c$ such that $f(c)=N$
Statement 4: Suppose that f is a continuous function on the closed interval $[a, b]$ where $f(a) \neq f(b)$. Then, there exists a real number c in (a, b), such that for all real numbers N between $f(a)$ and $f(b)$, $f(c)=N$.	$\exists c \forall N$ such that $f(c)=N$

Table 1. Statements presented in the clinical interviews.

Data Analysis. Our analysis was conducted in the spirit of grounded theory (Glaser & Strauss, 1967) using videos of the student interviews as well as the students' written work. Hence, the categories that describe students' meanings for quantifiers emerged from our data and not from previously created categories. Each interview was analyzed moment-by-moment to identify moments where distinctions could be made about a student's meanings for the quantifiers. A new moment began when a student was presented with a new question or task, changed their evaluation of a statement, or if the student provided a new description of the quantifiers in a given statement. Thus, we separated each student interview into moments where we found evidence for his or her quantifier meanings as our unit of analysis. Once the difference in explicit and implicit quantifications was noticed, students' meanings were re-organized and separated into one of these two categories. In the final stage of data analysis, each interview was re-coded moment-by-moment with different types of explicit and implicit quantification. We counted the number of moments that occurred in each category amongst all students. Moments were tagged by the interviewee's mathematical level so that we could make comparisons about student meanings of quantifiers for each group.

Results

The students in this study often provided various truth-values for the statements, and their meanings for the quantifiers also varied. Advanced Calculus students all concluded that the four statements had distinct meanings throughout their interviews, but there were moments with Transition-to-Proof and Calculus students where they said some or all of the statements were equivalent in meaning. Students often had moments where they explained Statements 2-4 in the same ways mathematicians would reason about Statement 1 ($\forall c \exists N$).

Student Meanings for Explicit Quantifiers

Our findings include several meanings for explicit quantification that are necessary for students to have in order for them to interpret the four complex mathematical statements in our task set. We noticed that students with correct truth-values and thorough explanations of the statements were able to distinguish between meanings for different types of quantifiers. These students articulated distinctions for single quantifier words "for all" and "there exists," as well as distinctions for the order of quantifiers "for all… there exists…" and "there exists… for all…" They also recognized and explained the importance of the order of c and N in the statements.

Distinguishing "For all" and "There exists." Some students exhibited distinct meanings that differed for each of the phrases "for all" and "there exists." Some moments involved student specification of the meanings of these phrases in their own words, such as describing "for all," as "for each one of these," or clarifying "there exists" as "I can find at least one." Other student moments were characterized by an interchange or alteration of the meanings for these phrases. Even though the phrases "for all" and "there exists" were used in the statements, students' explanations sometimes included using universal quantifier language such as "every," and "all" for the variable attached to the existential quantifier. Likewise, during some moments, students used phrases such as "I can find an" or "there is a" for the variable attached to the universal quantifier. Such student utterances indicate that in these moments, students did not have distinguishable meanings for the singular universal and existential quantifiers.

Hannah was a Calculus student who did not distinguish between "for all" and "there exists" in some moments. Hannah stated that Statement 4 ($\exists c \forall N$) had the same meaning as another previous statement. She also claimed in this moment, "anywhere you choose c to be, there is a value of N that's a real number because the function is continuous." Hannah said we could choose c-values *anywhere*, which indicates that she was treating the existential quantifier as if it were a universal quantifier. She also said that we could find *a* value of N, even though the statement says that we need to find a c that works for *all* values of N. This dialogue indicates that in this moment she was also treating the universal quantifier as if it were an existential quantifier.

Distinguishing "For all… there exists…" and "There Exists… for all…" Statement 1 ($\forall c \exists N$) and Statement 3 ($\exists N \forall c$) both have "there exists" with N and "for all" with c, but the ordering of these quantifiers is different. Statement 2 ($\forall N \exists c$) and Statement 4 ($\exists c \forall N$) also reorder in a similar fashion. Some student moments included explanations of why each statement in these pairs had a distinct meaning. Other student moments were classified by student views that these pairs were equivalent in meaning. Our findings are similar to those of Dubinsky and Yiparaki (2000), who showed that students may view reordering quantifiers, with variables attached, as inconsequential. Mike, a Transition-to-Proof student, explained why he thought Statement 1 and Statement 3 are similar:

> [Statement 3] and [Statement 1] are saying the same thing and [Statement 2] and [Statement 4] are saying the same thing. These two (*Statements 1 and 3*) are saying that there is only one output for all the inputs. But [for] Statement 1 and 3, I could assume this… for all real numbers c in the interval (a, b) I could assume that's true.

Mike acknowledged that he believed Statement 1 and Statement 3 are equivalent in meaning and Statement 2 and Statement 4 are equivalent in meaning. His classification insinuates that he did not distinguish between these quantifier distinctions. We have further evidence of his lack of distinction because he stated that both Statements 1 and 3 are about *one output* and *all* c's. Mike recognized that these pieces of the quantification are the same. However, since he emphasized the singular quantifiers and concluded that the statements are equivalent, we claim that he did not distinguish the difference in the order of the quantifiers in the statements.

Variable Ordering and Dependence. We also found a difference in how students treated the ordering of the variables c and N in the statements. All but one student mentioned that the ordering of the variables was different in some of the statements. However, during some moments, students claimed that the reordering the variables had no effect on the statement meanings. Zack, a Calculus student, explained why he thought Statement 2 ($\forall N \exists c$) was different, but equivalent, to Statement 1 ($\forall c \exists N$) in this moment:

> Um, so I mean obviously this [*N*] is now flipped with *c*, at least in [Statement 2]. I don't know how this [switch] necessarily affects [the statement]. So, when I explained it on the last one, I thought that *N* was a dependent value depending on what *c* is… Such that *f*(*c*) is equal to *N*… I don't think that, I'm sorry. So it's like that… I'm interpreting [Statement 2] the same, that it's just saying that really for any real value of *N*'s there exists a value of *c*, but *c* is still the value. Like… this (*points to c in f(c) in statement*) is the independent value versus this (*points to an N in the statement*) is the dependent value.

We claim that Zack recognized the variable order in Statement 2 in this moment, but he did not treat the first variable, *N*, as being chosen independently of the *c*-values. Zack mentioned that *c* is an independent variable and *N* is a dependent variable in the function in both statements. Zack appeared to choose *c*-values first because he thought that independent variables should be chosen first. However, he did not seem to connect the variable order in the quantification to his own choice for which variable should be held independent of the other.

Many student moments were different than Zack's moment; in these moments, students noticed the order of the variables and also displayed an understanding of the ramifications of this variable switch in different statements. Jay, an Advanced Calculus student, explained why Statement 1 ($\forall c \exists N$) and Statement 3 ($\exists N \forall c$) are different:

> For this one [Statement 1] all of the... like the values for individual *c*'s can be different. So like given *c,* I give you an *N*, but that doesn't have to be the same *N* as some other *c*. But for here [Statement 3] that's not true. There exists a real... so I give you *N* before you give me *c,* meaning I know what the answer is before I even know what *c* is.

Jay was not only aware of the variable order in this moment, but why the variable ordering affects the statements' meanings. Jay contrasted the order in which the first person gave a *c* or *N* and the other person gave a variable in return. His contrast suggests that he understood that the first variable was to be thought of independently of the second variable. He also used the first variable's information to give information about the second variable, which indicates that he considered that the second variable is dependent on the first.

Explicit Quantifier Distinctions Across Mathematical Levels. The four different meanings in Table 2 were more prevalent amongst the more advanced students interviewed. The results of this comparison of percentages from each of the three groups are shown in Table 2:

Explicit Quantifier Meanings (by % of Relevant Moments[2])	Calculus	Transition-to-Proof	Advanced Calculus
Distinguished "for all" and "there exists"	4/17 (23.53%)	21/22 (95.45%)	17/17 (100%)
Distinguished "for all… there exists…" and "there exists… for all…"	0/8 (0%)	7/12 (58.33%)	6/6 (100%)
Recognized Variable Order	10/13 (76.92%)	4/4 (100%)	5/5 (100%)
First Variable Independent & Second Variable Dependent	4/19 (21.05%)	14/15 (93.75%)	11/11 (100%)

Table 2. Comparison of students' explicit quantifier meanings.

[2] We define a *relevant moment* as any moment where a student explained his meaning for the specified construct.

Table 2 summarizes that higher-level mathematics undergraduates exhibit more moments with distinctive meanings for explicit quantifiers. The Advanced Calculus students had a greater percentage of moments where they exhibited each of these four meanings than Calculus students. Calculus student moments not only showed students' confounded meanings of "for all… there exists…" and "there exists… for all…," but they also revealed Calculus students' tendencies to confound the singular quantifiers "for all" and "there exists" as well. While the Transition-to-Proof students had a higher percentage of moments than Calculus students for distinctions between "for all" and "there exists," there were still several moments where they confounded meanings of "for all… there exists…" and "there exists… for all…" Calculus student moments were most similar to their advanced peers in their comprehension of recognizing variable order, but the variable dependency moments revealed that this recognition was not associated with students' understanding for why variable ordering is important to statement meanings.

Students' Implicit Quantifications

Calculus students had fewer moments where they distinguished between different types of explicit quantifiers and their attached variables than their advanced peers. However, meanings for implicit quantifications varied amongst the groups. The phrase "Suppose f is a continuous function" elicited ambiguity in some student moments. All four statements in Table 1 are written with the intent that readers will apply each statement to *all* continuous functions. We found three ways that these students implicitly quantified f: universally, existentially, or case-by-case.

Universal Implicit Quantification. Some students did quantify the phrase "Suppose f is a continuous function" as intended. Students used words such as "arbitrary" or the phrase "*any* continuous function" to describe their meanings of the statements.

Case-by-case Implicit Quantification. Three students considered that some of the statements that we were giving them were not firmly true or false for some moments. They preferred to evaluate some statements as "sometimes true" or "sometimes false" instead of strictly true or false. Their reasoning for this choice was made apparent when we offered them graphs and they considered that the statement was true for some functions, and false for some functions. Zack claimed in one moment, "No, I still agree that this statement [Statement 1] would be sometimes true… because in my mind this graph (*points to what we would consider the counterexample graph*) proves it… proves that I can't say the statement is true one hundred percent of the time." These students are classified as "Case-by-Case" because they were considering the if-then statement as having a variable truth-value instead of being firmly true or false. Our evidence supports Durand-Guerrier's (2003) finding that students tend to think of a conditional as an open statement which may be true or false, depending on the case at hand.

Existential Implicit Quantification. Ron, a Transition-to-Proof student, interpreted the phrase "Suppose f is a continuous function" with an existential quantification. He described his lack of certainty about the intent of the statement, and described his conclusions in this moment:

> I am not sure if f is limited to there being an existence of a continuous function or it's "suppose that any function." So because the wording is ambiguous in my mind I am not sure. I am just gonna keep it true for now because I am going to assume that "Suppose f is a continuous function" is going to be equivalent to the wording being "Suppose that *there is an existence* of a continuous function f on the closed interval a to b."

Ron's implicit quantification of "Suppose f is a continuous function" affected the rest of his arguments because he believed he only needed one function to make each statement true. He also thought that proving each statement false required exhausting all functions.

Implicit Quantification Across Mathematical Levels. The summary of all the students' meanings in the moment for implicit quantifiers is shown in Table 3:

Implicit Quantifications (by % of Relevant Moments)	Calculus	Transition-to-Proof	Advanced Calculus
Universal	6/8 (75%)	4/8 (50%)	7/8 (87.5%)
Case-by-Case	2/8 (25%)	3/8 (37.5%)	1/8 (12.5%)
Existential	0/8 (0%)	1/8 (12.5%)	0/8 (0%)

Table 3. Comparison of students' quantifications for "Suppose *f* is a continuous function"

As shown in Table 3, the majority of moments when students were implicitly quantifying *f* were distinguished by universal quantification in all three groups of students. However, there were some student moments from all three mathematical levels that exhibited a case-by-case quantification of "Suppose *f* is a continuous function." Universal quantification was more frequent amongst Calculus and Advanced Calculus students than Transition-to-Proof students. This phenomenon is addressed in the discussion. Our only existential implicit quantification moments originated from one Transition-to-Proof student, Ron.

Discussion

This study indicates that students had several different meanings for explicit and implicit quantifiers when reading the IVT. Based on our findings, students need to distinguish the singular quantifiers "for all" and "there exists," and they need to distinguish "there exists… for all…" and "for all… there exists…" They also need to recognize when variables are ordered differently across a set of statements before they focus on how this reordering alters the meanings of the statements. These explicit quantifier meanings are foundational to understanding the explicit quantifiers in complex mathematical statements, and the Calculus students in this study had fewer moments with each of the explicit quantifications necessary for statement coherence than the Transition-to-Proof and Advanced Calculus students. Yet, we teach Calculus theorems with quantifiers that need these meanings for theorem comprehension.

Even though there were more moments where Transition-to-Proof and Advanced Calculus students exhibited distinct meanings for different types of explicit quantifiers, truth-values still varied from student to student. There were moments from students of all three levels who implicitly quantified "Suppose *f* is a continuous function" in different ways than the authors of the statements intended for the statements to be quantified. The quantitative data may seem to indicate that students decrease their abilities in a Transition-to-Proof course. This is not meant to indicate that Transition-to-Proof is harmful for students. This may mean that students are now exposed to new mathematical quantifications, but have yet to reflect on when and how to apply all of these types of quantification.

These results should be considered when making curriculum and instructional decisions for all mathematical courses, but particularly for Calculus and Transition-to-Proof courses. Further research needs to investigate how students come to learn appropriate meanings for both explicit and implicit quantifiers for many different theorems. Our study was limited to the IVT, but was also limited to nine students. Studies that involve a larger number of students in each academic level are suggested to more accurately measure the prevalence of each meaning amongst each group of students.

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Beyond the Product Structure for Definite Integrals

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Over the past decade research has shown that a Riemann sum based interpretation of the definite integral supports a robust understanding of the underlying structure of the integrand/differential relationship and facilitates students' ability to make sense of contextual integral models. However, current studies center this understanding on the multiplicative structure $f(x) \cdot \Delta x$ which does not account for many practical uses of integration. In many situations, the Δx is most productively conceived as a component of another quantity which might then be incorporated in any of a variety of quantitative models, such as an inverse square law rather than a simple product. To fill this gap, this study utilized Dewey's theory of inquiry to identify three interpretations of the definite integral which proved productive for students when modeling definite integrals that extend beyond the traditionally studied product structure.

Key words: Definite Integral, Riemann Sum, Adding Up Pieces, Quantitative Reasoning, Physics and Engineering

Introduction and Literature Review

In 2001 Sherin lamented "it seems reasonable to suggest that, prior to physics instruction, students should learn to invent at least some simple types of mathematical models and to express the content of those models." This sentiment is indicative of the general shift in attitude towards the calculus curriculum in recent years proposed by education researchers; that calculus curriculum should focus on an introduction to the concepts of limit, differentiation, and integration in a way that supports students' acclimation of mathematical ideas into their applied STEM fields, simultaneously leveraging that contextualization to support students' interpretation, mathematization, and argumentation (Jones, 2013, 2014, 2015b; Meredith & Marrongelle, 2008; Oehrtman, 2009; Redish & Smith, 2008; Sealey, 2006, 2014; Sealey & Oehrtman, 2005; Thompson, 1994; Thompson & Silverman, 2008).

Although differentiation and limits are both important aspects of the calculus curriculum, special attention should be paid to the definite integral for both its utility within mathematics, as well as its application to modeling in real world contexts (Hibbeler, 2004; Sealey & Oehrtman, 2005; Thompson & Silverman, 2008). Sealey (2014) provided a framework for characterizing student understanding of Riemann sums and definite integrals, decomposing this understanding into five parts; orientation to the constituent quantities, a product layer in which one attends to the quantitative relationship between $f(x_i)$ and Δx, the summation of these products, a limiting process applied to this summation, and a function layer in which ongoing accumulation may be considered. Von Korff and Rebello (2012) further adapted this framework into a physics context promoting the concept of infinitesimal products. Jones (2013) applied Sherins' (2001) symbolic forms to students' interpretations of the definite integral, classifying their reasoning into categories such as area under a curve, function matching, and adding up pieces. Jones (2015) later refined his description of the adding up pieces symbolic form and described a multiplicatively-based summation (MBS) conception that was "highly productive" in engaging students in both the mathematical structure of the definite integral as well as in modeling for physics-based contexts. This Riemann sum interpretation of the definite integral $\int_a^b f(x)\,dx$ focuses on adding up many terms derived from a multiplicative relationship between a (possibly) varying integrand $f(x)$ and small or infinitesimal differential dx. The products in this conception must be abstracted from a parallel multiplicative relationship in what we call a “basic model” between a constant quantity F and a (possibly) larger value X, respectively. Such decompositions to the Riemann sum structure have been stressed by math, engineering, and physics education researchers as essential for STEM student success, including mathematics majors (Doughty, McLoughlin, & van Kampen, 2014; Jones, 2013, 2014; Meredith & Marrongelle, 2008; Redish & Smith, 2008; Sealey, 2014; Sherin, 2001).

Meredith and Marrongelle (2008) made the apt point that a product structure derived from a basic model $F \cdot X$ to generate terms of the form $f(x) \cdot \Delta x$ in a Riemann sum is only adequate for modeling accumulation of a quantity when the integrand is a rate of change or density, yet many contexts do not naturally decompose into this quantitative structure (pp. 575-576). Oehrtman (2015) showed that an MBS interpretation was insufficient for students constructing more complex definite integrals from basic models that are not a simple product. To clarify reasons for students’ success, Oehrtman also analyzed the symbolic forms students applied to the basic model, how they transferred that quantitative reasoning to terms of a Riemann sum or the differential form of a definite integral, and how those symbolic forms interacted with the students’ symbolic forms for the integrals. Thus, only investigating the MBS conception of the definite integral leaves a potentially significant gap in our understanding of the quantitative reasoning required for successful modeling with definite integrals and interpretations of differential forms apart from a generalized product of integrand and differential. Our study seeks to further explore the ways in which students conceptualize definite integrals that do not match the traditional $f(x) \cdot dx$ multiplicative structure and identify key aspects of their reasoning which lead to productive results. Specifically, we pose the research question:

> What interpretations of a definite integral and basic model are productive and unproductive for students as they progress through increasingly complex definite integral modeling tasks?

Theoretical Perspective & Methods

As this study seeks to identify how students engage in and overcome difficulties modeling definite integrals beyond the product structure they typically first learn, we chose to employ John Dewey’s theory of inquiry (Dewey, 1938; Hickman, 1990) to their problem-solving efforts. Dewey characterized knowledge as a product of the activities a student actively engages in when encountering non-routine problems. These activities involve strategically identifying mental tools with which to make sense of the problem, testing that tool in the context of the problem, and utilizing the tool to clarify the problematic situation in some way for the student. New knowledge emerges through the dialectic interplay between application of the tool against the problem and evaluation and refinement of the tool against the perceived progress. With this interpretation of learning in mind, we will classify an interpretation, i.e. tool, of the definite integral as productive if it is perceived by the student in advancing their progress towards resolving a problematic situation. We note that a productive interpretation need only clarify the situation for the student and does not have to be consistent with the researchers’ own views for resolving the problem.

Four pairs of students were selected to participate in an hour and a half long interview focusing on physics applications of the definite integral. The choice to work with pairs rather than individual interviews was made so students could discuss their ideas with each other, rather than an authority figure. The students were encouraged to discuss their thinking aloud while working through problems on a whiteboard until they were satisfied with their solution, after which the interviewers would ask follow-up questions to clarify any notes made during the encounter.

Because our study seeks to understand students' progress beyond an integrand×differential quantitative relationship, it was important to recruit students who were likely already familiar with the MBS conception described by Jones. As such, Group C consisted of a senior undergraduate and second-year masters student who had recently taken an Advanced Calculus course, while the remaining three groups, A, B, and D, were recruited from a second semester calculus course in which the instructor utilized a calculus curriculum emphasizing the Riemann sum approach for definite integrals in classroom labs (Oehrtman, 2016). After our initial round of interviews, Group B was asked back for a second one hour interview due to the strength of their collaboration and quality of data provided. These interviews were video-taped and later transcribed for coding.

Each group was first asked to discuss their general interpretation of a definite integral, followed by a series of tasks chosen to reveal their reasoning while modeling definite integrals as they progressed from simple rate×time contexts to situations that obscure the product structure of a differential form (Figure 1). The intent was to track the students' operative interpretations of the definite integral, the components that comprise the integral, the quantitative relationships involved in the basic physics formulas, and the students' perception of the interconnection between these concepts. We anticipated many students would approach some of the simpler problems heuristically, avoiding reasoning quantitatively, but that the progression of tasks would require an eventual shift to adding up pieces in progressively sophisticated ways in order to be successful.

1. When fully wound, a toy car will travel in a straight line for just over 17 seconds. Its speed t seconds after it is released is $v(t) = 3\sin\left(\frac{t^2}{100}\right) m/s$. Write an integral that gives the distance the fully wound car will travel during the first 10 seconds it travels.

2. Linear density is a measure of an objects mass per unit length. This means for example if an object has length 3.2 ft and **constant** linear density of $.5\ lb/ft$ it will have a total mass of $(3.2)(.5) = 1.6$ lbs.

   King Arthur's lance is 10 feet long and heavier at one end, decreasing from 1.2 lb/ft at the base to 0.2 lb/ft at the tip. That is, the linear density x feet from the base is $\delta(x) = 1.2 - 0.1x$. Write an integral that gives the mass of King Arthur's lance.

3. When pollen from a red cedar tree is released from their cones it travels through the air. Pollen from a mature tree settles on the ground with an estimated density of $\delta(r) = \frac{37}{10+r}\ g/m^2$ a distance r meters from the tree.

   Write an integral that gives the mass of pollen distributed within 100 meters of a mature tree.

4. Coulomb's law says that the force between two electrically charged particles is $F = k_e\frac{q_1q_2}{r^2}$, where k_e is Coulomb's constant ($k_2 = 8.99 \times 10^{21} N \cdot m^2/(\mu C)^2$), q_1 and q_2 are the charges of the particles in microcoulombs (μC), and r is the distance between the charges. Coulomb's law only directly applies to two point charges.

   Write an integral that gives the electrostatic force between a uniformly distributed charge of 29 microcoulombs on a rod of length 7 meters and a particle with charge of 2 microcoulombs 3 meters from the rod along its axis.

5. The energy measured in joules to lift an object straight up h meters is $E = M \cdot g \cdot h$, where M is the mass of the object measured in kg, and g is the gravitational force measured in m/s^2.

   Suppose a 10-meter chain with a total uniform mass of $15kg$ is freely hanging from the roof of a building. Write an integral that represents the total energy required to lift the chain to the top of the building.

Figure 1. Task Sequence.

Group A was only able to progress through task 2, while the remaining three groups were able to advance to the fourth task with varying levels of success. Task 5 was chosen as a supplemental question for Group B to work through during their second interview session and was not presented to any other group.

The video transcriptions were coded using open and axial coding methods of grounded theory (Strauss & Corbin, 1998). Because we were interpreting this data through the lens of Dewey's theory of inquiry, the initial coding pass sought to identify problematic situations for students, in the sense of non-routine engagement and reciprocal influences of their evolving conceptions for definite integrals and basic models with their understanding of the modeling tasks. After these instances were identified, each was broadly coded for all cognitive tools utilized, obvious relationships to previous research, and any specific tools which lead to productive results. Subsequent classifications looked to identify associations between these codes both throughout a specific interview, as well common themes amongst the different groups to develop a narrative regarding students' conceptualizations of these definite integrals. Common problematic situations and tools which led to productive results were identified and the information was synthesized into 3 defining characteristics of productive modeling of the definite integral, Adding up Pieces, Quantitative Reasoning, and Utilization of "nearly constant," which will be discussed in the following section.

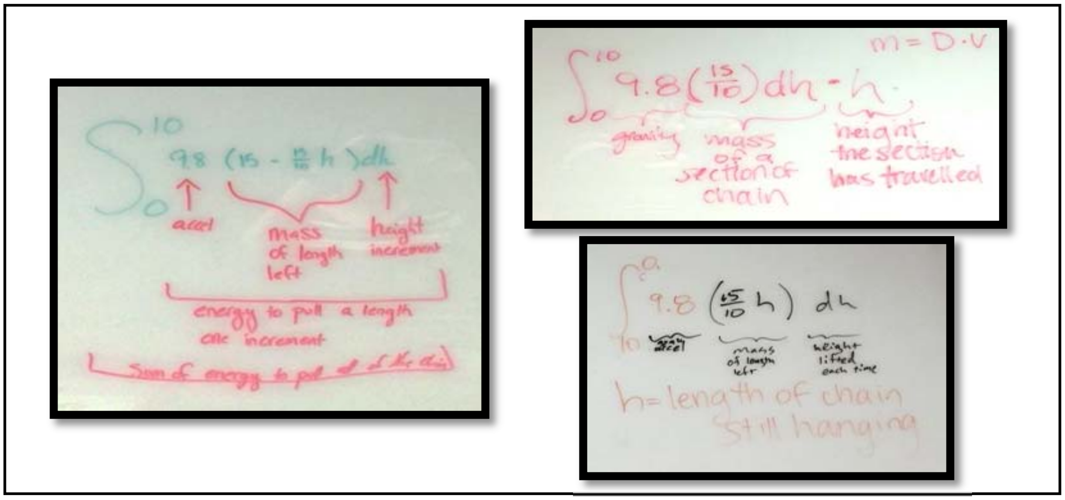
Results

Adding up Pieces

Echoing previous research, our findings suggest that conceptualizing a definite integral as a summation allows for students to model physics equations in a way consistent with both mathematical and physics/engineering constructs. However, our interpretation of adding up pieces differs slightly from that of previous research as we allow for integration of quantitative relationships more complex than the traditional $f(x) \cdot \mathrm{d}x$ structure. Due to the nature in which we recruited our subjects, all pairs demonstrated at least some underlying Riemann sum structure for the definite integral. Group A showed the weakest correlation between Riemann sums and definite integrals, often relying on area under the curve symbolic forms or graphical interpretations, although when pressed they could describe these interpretations in the context of representative rectangles (Jones, 2013, 2015a). This weak connection to adding up pieces played a role in Group A only completing two of the tasks, and only the first completely correctly.

Quantitative Reasoning

Aside from the adding up pieces conceptualization, quantitative reasoning is perhaps the most crucial aspect of modeling definite integrals (Thompson, 1994, 2011; Thompson & Silverman, 2008). Within the construct of a definite integral which is comprised of a non-routine quantity, there are multiple layers of quantitative reasoning a student must navigate. First, the student must attend to the quantitative relationship of the definite integral itself. Within that structure the student must also attend to the relationship between the integrand and differential. Certain contexts lend themselves to obvious relationships, such as Tasks 1 and 2, however, the more complicated structures have yet another quantitative relationship layer; how the differential is situated within a quantitative formula. In Task 3 this extra consideration did not prove overly difficult for most groups, however, in Task 4 it caused numerous problematic encounters for the students and required significant effort to resolve. We note that there are often multiple ways to reason about the basic model within a definite integral quantitatively. Figure 2 displays three



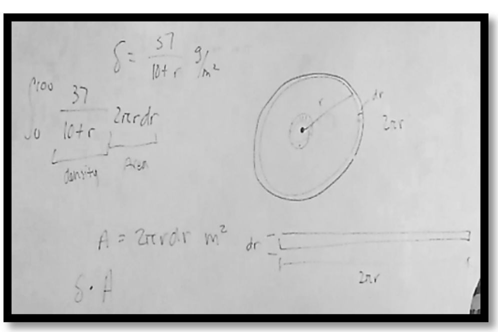
different ways in which group B conceptualized Task 5 during their second interview; the top two were simultaneously modeled by each student individually while discussing the problem relatively fluently with one another. In light of this encounter, we conjecture that students need not conceptualize the quantitative relationship within a basic model in a specific way to be productive in modeling a definite integral. Rather the importance lies in attending to the quantities that comprise a given formula and identifying their relationship to each other in a meaningful way, while simultaneously considering any underlying implications of the basic model, such as the multiplicative structure distance = speed × time is only valid when speed is constant.

Utilization of "Nearly Constant"

In each of the problems presented there is an underlying basic model which holds for

Figure 2. Three quantitative relationships demonstrated by group B in Task 5.

constant values. Every group used these formulas as a part of their final integral structures,



B: So, because the density of pollen is not uniformly distributed you have to find, you have to find what the density is pollen is at every different r. And it's really, it's similar where two r's are really similar. So your dr is your change in that r, and within this range the density is gonna be really similar but if you just say multi, put that plugged in to 100 you'd be assuming uniform density over the whole circle which would be incorrect.

A: Yeah.

I: Alright so earlier you said the $2\pi r$ is the circumference. So why circumference? It has area underneath it, so why circumference?

B: Uh, so you have the circumference because that's the distance around the circle, which when you cut the circle and spread it out and make it into a rectangle that would be the length of the rectangle. But in order to get an area you have to multiply a rectangle length times its width. And your width would be your dr which is your change in r which is a very small number where the density is nearly uniform.

Figure 3. Group C discusses "nearly uniform" density in Task 3.

asserting that the reason you must integrate is that at least one of the quantities in question is varying. Every pair articulated some form of justification for why they use the formula for a "nearly constant" segment, as shown in Figure 3.

For many pairs, inserting the given formula into the integral during the first few tasks was a routine procedure that only became problematic and subsequently justified when questioned by the interviewers. This was especially evident in Task 1, where every group made reference to the relationship that velocity is the derivative of position. However, when pressed, most groups were able to identify that over a small duration of time, speed was basically constant, which allowed for a confident explanation of the role of the differential within the integral. This justification of process, which was evident in early tasks, was not as prevalent in the data for Task 4. Instead there was a shift of focus to "what's changing," which led to many unproductive encounters and in one case led to dividing by the differential (Figure 4).

Group B demonstrated a strong adding up pieces conception and quantitative reasoning skills in all previous tasks, and were fluidly speaking about breaking up the rod in Task 4 into segments and adding forces together to obtain a total force. So in all other aspects they were conceptualizing this definite integral in ways compatible with current research. However, the desire to find the moving part of the equation permeated their reasoning, hindering advancement. Although very problematic for the students, the resolution of this interpretation was only possible through an interviewer's intervention. It is possible that an extra layer of abstraction provided additional difficulties for this pair, as their initial problem contained no specific quantities. Numerical quantities were provided halfway through their encounter with this problem and the task was updated to reflect specific values for future interviews.

J: And then we have 1 over s^2, that's what is changing, s^2?

M: Yeah.

J: So we need to change that to $(s + dl)^2$.

M: Yes, cause s is the distance and the formula just has distance squared. I think that's it, but... that doesn't look like an integral to me.

J: Yeah, dl's down there and it's squared. That's not normal.

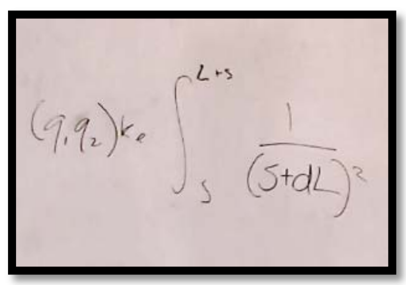


Figure 4. Group B dividing by the differential in Task 4.

Familiarity of context

Although not characterized as productive, and perhaps unsurprising, the unfamiliar context and quantities of electrostatic force proved problematic for the students in this study. Every group that attempted this task required at least some clarification of Coulomb's law. The more troublesome complication arose when the groups tried to conceptualize the adding up process that would need to take place if they were to integrate. Unfamiliar with the principle of superposition, two of the groups envisioned the integral needing to account for forces compounding upon one another as the integrand ranged over its domain. An example of this is shown in Figure 5.

Because students' lack of familiarity with superposition of forces was not intended to introduce complications, one of the interviewers stepped in to provide the supplementary explanation. This finding echoes Meredith and Marrongelle's 2008 assertion that "understanding of the physical situation is necessary, but not sufficient for students to use their mathematical resources".

J: From what I can understand, the function that they gave us, f, like this is just one member that we would get. This is the force between these two parts [points towards a diagram explaining Coulombs Law]. Over here we have a force between these two parts, and then a force between these two parts, and a force between these two parts, and we're adding all of those forces together.

M: Uhhh...

J: It's kind of like the other ones we were doing, it's like, the thing that's changing is the forces and that's what we're adding together.

M: Yeah...[laughs] but... I don't know. It doesn't make sense to me that it doesn't like compound, you know?

J: Compound?

M: Well cause wouldn't the charge of this be like, like right here have that plus all of this [points to the far end and then motions across the full length of the rod]. Wouldn't that have...

J: Ohhhh

M: That's why I was saying if it was moving it would be easier. Because then you're looking at this and then nothing but space. But you're looking at this [points to the third subsection] and then like that little part [motions to the two subsections preceding the third subsection].

J: I get what you're saying. So is it like a double integral?

M: [whispers] I don't know.

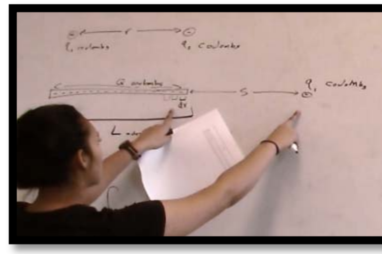


Figure 5. Group B discussing compounding force.

Discussion

The results of this study illustrate three key interpretations of the definite integral, adding up pieces, quantitative reasoning, and attention to the quantity which is nearly constant, which we have identified as crucial for successful modeling of not only multiplicative basic models but also more general basic models found in many physics and engineering contexts. While discussed separately, these different themes are often interrelated and require the student to coordinate subtle interplay between them. For example, the decision to break a situation into multiple pieces to create a Riemann sum interacts with their conception of variable and constant quantities in the basic model and with their conceptions of the definite integral. As in previous research, we found the adding up pieces conception of the definite integral highly productive for students, however, when paired with the more complex basic model there are additional layers of this construct for the student to work though. It may be clear that the integral will add up small chunks of the basic model to obtain the desired total, while the quantitative reasoning behind breaking up the physical (or abstract) situation to obtain these small chunks can remain problematic for students. To overcome this difficulty, students must be able to attend to how the concept of a small chuck being "nearly constant" allows for the use of integration of the basic model, which can support their decision of how and why they are breaking up the situation into smaller constructs.

Based on the data collected in this paper, and the call for modeling of definite integrals in the other STEM fields, it is our contention that Jones' (2015a) MBS conception of the definite integral can be productively extended to a Quantitatively Based Summation conception (QBS) for the definite integral, of which the MBS conception would be an often utilized subset. Similar to the MBS conception, this interpretation of the definite integral in a QBS is incorporated into a Riemann sum approach to the definite integral, but requires focus on the rich quantitative reasoning about the basic model and transferred in flexible ways to the differential form for the definite integral.

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Exploring Undergraduates' Experience of the Transition to Proof

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We report a qualitative analysis of 14 undergraduate students' experience in a semester long introduction to proof course. Half were mathematics majors. Our research aims to characterize, conceptually and empirically, students' transition from a focus on computation to proof in mathematics. Our analysis focused on how students saw the course as different from prior courses, how they described their work in it, and whether being successful in the course required new or different learning activity of them. This approach—targeting students' overall experience of the course—differs from prior research that has tracked students' challenges, focused on their work on specific proof problems, and explored how to support and improve their work (e.g., Selden & Selden, 2003). Our work has promise for informing the design of transition to proof courses and how those courses are organized and taught.

Key words: transition to proof, proof reasoning, students' experience, qualitative analysis

The Transition to Proof and Proving

Many undergraduates experience difficulty in learning to prove mathematical propositions, including those who major in mathematics (Baker & Campbell, 2004; Moore, 1994; Selden, 2012). For many, experience with proof prior to college is quite limited (Anderson, 1994; Jones, 2000). Students' prior competence (and interest) in mathematics is typically centered on producing accurate answers to easily recognized tasks—"exercises" in Schoenfeld's terms (1992). That work involves recognizing specific types of such "problems" quickly and applying well-practiced procedures to solve them. These abilities do not support, and may interfere with students' work to conceptualize, write, or evaluate proofs. In addition to differences in the didactical contract between computation-heavy courses like calculus and proof-intensive courses, Selden (2012) reports that students encounter many difficulties, including mastering logic and definition, generating and using examples and counterexamples, understanding concepts and theorems, and evaluating the arguments of others. In short, the transition to proof is complex and challenging, and much of students' prior ability appears inadequate, if not problematic for addressing these challenges.

Many mathematics departments have recognized this problem and experimented with different curricular and instructional approaches to support students' entry into proof, including dedicated "transition to proof" courses. These typically focus on precision of language and notation, reasoning, and proof and sample accessible statements from numerous content domains to prove or disprove. But designing these courses is not a trivial task, and they have often shown limited positive effects (Selden & Selden, 2003; 2014). A crucial limitation effecting course design is that research has not systematically focused on the experience of students. Instead, course design and instructional decisions have often been shaped by assumptions about students, the kinds of challenges they face, and structures that support their learning. To begin to address this limitation, we explored how students describe their experiences in one such “transition to proof” course.

Conceptual Framework

To explore students’ experience of the shift in mathematics work from producing answers to composing proofs, we drew on prior work in conceptualizing “experience” and “transition.” Smith and Star (2007) proposed that precollege students’ mathematical experience could be composed into four dimensions: Achievement, disposition, differences between prior and current mathematics work, and learning activity. Assessing change on these dimensions in turn supported grounded judgments of the significance/depth of students’ transitions. The present study built on this work in its focus on the following dimensions of students’ experience of the introduction to proof course: (1) The nature of the course and how it differed from prior courses, (2) the reasoning involved in proving, and (4) the learning activities that support success. The first and third dimensions correspond closely to Smith and Star’s (2007) dimensions of differences and learning activity. It also drew on that study’s use of student-constructed graphs to assess changes in students’ confidence.

In analyzing the data we collected, we also found the constructivist focus on how prior resources are recruited to cope with new challenges productive, whether those resources prove effective or not (e.g., Smith, diSessa, & Roschelle, 1994). This perspective provided a frame for understanding how components of students’ experience and skill developed in more computationally-oriented mathematics work may be applied and reworked or adapted in the efforts to succeed in the quite different mathematical work of proof and proving.

The Course & The Participants

The 14 participants in the study were all graduates of a multi-section semester course designed to introduce them—both mathematics majors and not—to proof and proving. The mathematics department hoped that the course would support the success of both groups of students in upper-division courses that emphasize proof. The course introduces appropriate syntax and notation, basic concepts, and proof techniques before proof of “entry-level” statements from various content domains (e.g., linear algebra, real analysis, and number theory). The course pedagogy is not lecture-based. Instead instructors give short presentations and present proofs (or parts of proofs) before students spend substantial time working on proof tasks themselves, often in small groups. The course also includes elements of “flipped classrooms;” students read and answer basic comprehension questions prior to working on problems in class that draw on that content. Evaluation was primarily based on homework and exam grades ($n = 3$ before the final). Students were required to use LaTeX to post their solutions to homework tasks, but their solutions on all exams were hand-written.

We interviewed students in the summer after they completed the course. We invited all 110 enrolled students from the prior semester to participate; 17 (~15%) responded positively, and 14 (~13%) completed an hour-long interview about their experience, which were audiotaped for analysis. Nine students were male and five were female; three were international students. The sample was diverse by major. Six were mathematics majors; two others were majoring in actuarial science. In addition, we interviewed mechanical engineering, chemistry, packaging, biology, and economics majors. Two of those were pursuing dual majors. Most of the other "non-math" majors were pursuing minors in mathematics where the course was a requirement. Table 1 provides an overview of the sample, including their final grades.

Table 1. *Overview of Participants*

Student	Gender	Home	Major(s)	Standing[a]	Minor[b]	Grade
1	F	US	Mathematics	1		3.0
2	M	US	Actuarial Science	2		3.5
3	M	US	Mechanical Engineering	4	Math	4.0
4	M	US	Chemistry	3	Math	3.0
5	M	US	Actuarial Science	1	Math	4.0
6	M	Intl	Mechanical Engineering	3	Math	4.0
7	M	Intl	Computational Mathematics	3		4.0
8	F	US	Mathematics	1		3.0
9	F	US	Packaging	2	Math	2.5
10	F	US	Mathematics	2	Act Sci	3.0
11	M	US	Human Biology	3	Math	3.5
12	M	Intl	Mathematics	3		4.0
13	F	US	Economics & Statistics	2		4.0
14	M	US	Economics & Mathematics	2		4.0

[a]"Standing" reflects the amount of university coursework; in some cases, standing exceeds the number of years at the university. [b]S5 and S9 were strongly considering a mathematics minor but had not decided.

We focused our interview questions on understanding students' experience of the course in their terms, relative to their prior work in high school and college mathematics. We first secured basic information about them (e.g., major, standing, section/instructor, minor, and prior and intended math courses). Then we asked about their sense of the nature of the course (in contrast to their prior courses), their sense of success (or not) in it, what they did to be/become successful, and specifically how their view of and work on proof tasks may have changed in the course. As complement to seeking their verbal responses, we also asked them to graph their confidence in the course across the semester. These Confidence Graphs (Figure 1) often surfaced new information about our focal issues, as students explained the shape and location of their graphs.

We also asked about their sense of their instructor's view of the course, whether the course experience influenced their understanding of calculus, and whether they expected to be successful in any subsequent math courses. Finally, we asked what they would tell incoming students about the course. We also regularly observed (about twice per week) the classroom teaching and interactions in one section of the course for most of the semester—the section taught by the course coordinator. Observers took field notes in an observation template. These observations proved essential in preparing for the interviews, interpreting and responding to student responses, and in the analysis of the resulting data.

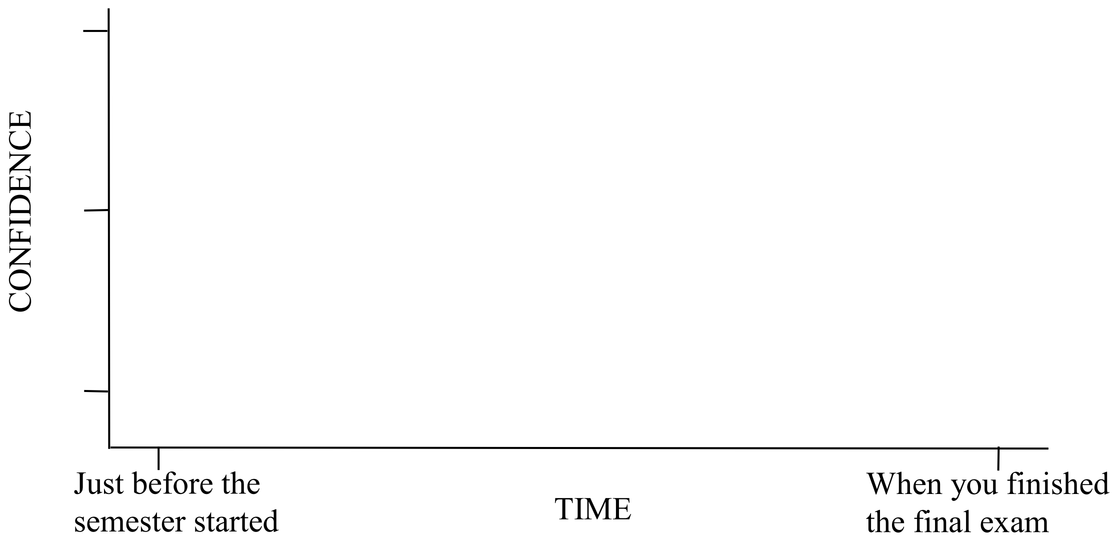

Figure 1. Confidence Graph (reduced in size). Each participant drew a graph of confidence over the course of the semester. The X-axis represents time, from just before the course began through the final exam. The Y-axis represents their confidence in doing well in the course.

Analysis of the Interviews

Our analysis was qualitative, “bottom-up” (though guided by our foci), and cyclical in nature. The results of initial analyses led to more detailed and focused rounds of analysis around specific issues. First, immediately after each interview, interviewers posted detailed summary descriptions of their interviewee’s responses. All members of the research team then read carefully and discussed the similarities and differences evident in those summaries. Next, the audio records were transcribed verbatim, and particular passages were examined to check our summary descriptions and provide exact wordings. We analyzed the Confidence Graphs for their general graphical properties (e.g., initial sense of confidence and where slopes of the graphs were greatest, positively and negatively) and particularly for points in the semester when students’ confidence dropped. Many students described the reasons for these drops and actions they took in response. In response to our questions about activities in their classrooms, students described some differences in instructors’ teaching methods. But because we observed in only one section, we could not assess the accuracy of their claims or the magnitude of instructional differences across sections. This is an important issue that we have to address in our on-going work.

Results

We report results in four main components. First, we present our summary of how the students described the course in comparison to prior mathematics courses. Second, we discuss one aspect of that comparison—how students used the terms “problem” and “answer” in describing their course work. Third, we characterize how students described their reasoning in solving proof tasks. Finally, we describe how the students viewed and evaluated their success in the course and the learning activities that they engaged in.

The Different Nature of the Course

Students were quite articulate about how the course differed from their prior mathematics course experiences, but their focus and emphasis varied. We found three main characterizations of difference: (1) Work in the transition to proof course explained why mathematics worked the way it did, where prior work had focused mainly on producing answers, (2) the course valued process over answers, and (3) the course changed students' conceptions about proof and writing in mathematics. Very few students reported substantial prior experience with proof, in either high school or college. Those who cited two-column proof in high school geometry were largely dismissive of that work as irrelevant.

Most frequently, the students stated that the course *explained* mathematics, "why things are the way they are," much more than prior courses. More than half of the students said it explained "the rules" or "where things come from." S14 and S6 cited derivatives and operations with matrices, respectively, as examples of learning why procedures from prior courses actually worked. This result indicates that the course served an explanatory purpose for students, providing conceptual understanding over and above their learning about language, notation, and proof techniques. This is a central function of proof in mathematics (e.g., Thurston, 1994), but it was pleasing to hear students articulate it.

Second, students contrasted the computational, answer-focused nature of their work in prior courses (e.g., calculus) to the process- or argument-focused work in the transition to proof course. As we note below, numerous students described the course in terms of what it was not: Writing proofs was not straightforward or step-by-step (S2, S11) and did not involve applying formulas or solving equations (S6, S12, S14). About a quarter of the sample stated that the course valued the process of reasoning more than the result.

Third, about half of the students talked explicitly about the new focus on proof and the dramatically increased focus on writing. They commented on changes in how they perceived mathematical writing and noted how much more writing was required in the course. Some realized that proofs can include words, not just symbols (S12, S13); others came to new understandings about the importance of accurate grammar in writing compelling proofs (S2, S3). They saw that mathematical terms have much more specific meanings than do words in everyday language and communication and that that specificity was important.

New Meanings for "Problem" and "Answer"

In characterizing their work in the course, students often used terms that typically describe work in more computationally-oriented courses, particularly "problem" and "answer." Most participants used the term "problem" to describe the proof tasks they were assigned, in class and for homework. More standard mathematical terms (i.e., "statement" and "proposition") were rarely used; only one subject did so consistently. In part, this use of "problem" reflected the language of the classroom; the course assigned "homework problems," and the instructor we observed also used this term to refer to the proof tasks he presented. But it also represents a strong connection with students' prior work in mathematics. We found a similar pattern for "answer." There was no consistent pattern in how students described the form of successful solutions to proof tasks. References to "proofs" were relatively common, but so were references to "answers." S1 spontaneously used that term to describe her solution to "problems," but then sounded embarrassed when the interviewer asked her how "answers" fit with proof problems. S10 indicated that an important difference in the course was that problems now often led to "a whole set of answers, not just one." We interpret students' use of these terms as indicating that their meanings for task and solution were in the process of change yet still retained some connection to the computationally-focused past.

Reasoning in Proof

A central objective of our work has been to characterize how students saw and carried out the reasoning required to produce acceptable proofs. By reasoning, we include: (1) Their thinking and notating prior to writing any well-formed statements, (2) their proof composing on paper, and (3) their final expression of their proofs in LaTeX. Though students saw and described their work in the course as different from their prior experience, many struggled to describe their reasoning on proof tasks and how it was different from prior work of applying procedures and computing answers.

Students' responses to questions about the nature of their proof reasoning fell into four main patterns. One set of student responses focused on what their reasoning was <u>not</u> like—that is, that the prior pattern of first recognizing the task, applying the appropriate procedure or approach, and then completing it step by step did not work. Most (S3, S6, S8, S11) recognized that they could not use their instructors' proofs as model solutions for other problems and that no general procedures for "solving" a wide variety of proof tasks existed. But one student (S7) did report that he tried to adapt his instructor's proofs to new problems, and another (S13) inspected model solutions to make sure that her proofs included relevant features and details she saw in them. A second set of students emphasized the first phase of "reasoning" above—that they could not begin their proof without first understanding the concepts involved and how they were related. One student (S12) emphasized the importance of conceptual understanding in terms of knowing how to use concepts in proof writing beyond just memorizing definitions of those concepts. A third set offered some characterization of reasoning common to proof tasks, but their descriptions were difficult to follow. For example, S11 described the work of proving as looking forward across a large number of steps (not just one) to assess if the approach looks as if it would work. S10 characterized that proof writing involved "working backwards," from the consequent back to the premises. A smaller fourth set described their reasoning in more standard terms as the search for a proof technique (e.g., proof by contradiction, induction, or contrapositive) that would "work" on the task at hand.

For the final step in their proof reasoning, most students spoke about the mandated use of LaTeX on assigned homework, either positively or negatively. Those who were more positive indicated that LaTeX forced them to be more careful and explicit with the mathematical language in their proofs. They seemed to accept the positive effect of that coercion. Those who were more negative cited the additional time required to submit via LaTeX or questioned the importance of or the need to improve the "neatness" of their work.

Specific Challenges, Sense of Success and Learning Activities

When we asked about students' sense of success in the course, most responded in terms of their final grade. Less frequently, they assessed their satisfaction with understanding the content or their sense of having mastered that content. For example, S7 stated that he felt he mastered "90% of the course content." As Table 1 indicates, half of the students received final grades of 4.0, and only one student received a 2.5. S1 reported that her 3.0 grade missed being a 3.5 by a single point. So overall, despite the new demands of the course, most of the students were successful by the traditional measure of final grades.

Most students described the work required to be successful in the course as more demanding than prior courses. Some were primed for challenge by characterizations offered by friends who had already taken it. Once in the course, they found that the homework carried substantial value in the grading scheme and that completing it took more than it did in prior courses—often much more time. Most students offered estimates of 7 to 10 hours for each weekly homework set, compared to 1 to 2 hours in prior courses. For students who composed their proofs completely on paper, translation into LaTeX added significantly to the time required.

All but two students explained that they employed different practices in completing their homework than they had in prior courses. In response to the demands of frequent and challenging homework, most indicated that they responded by starting earlier in the week and/or distributing the work across days. Four communicated with peers via e-mail, worked with a classmate, or asked questions in class for the first time in mathematics. Twelve described the importance of using the university's math learning centers—many for the first time—in interesting and diverse ways. Some cited the availability of tutors to answer their questions; others simply sought a physical space to go and work alone; others emphasized collaborative work with other students in the space designated for this course. For some (e.g., S10), productive discussion and community building happened with students from other sections of the course. That said, the use of social and peer resources was not universal; some students (e.g., S5) simply worked on their own, even as they worked harder and longer.

Conclusions, Limitations, & Next Steps

Our research is on-going, and subsequent steps will help refine our characterization of students' experience of the course. But this first round of research has been revealing, in expected and unexpected ways. First, we have found that successful graduates of the course are developmentally "in motion" between the prior focus on computation and the new focus on proof. Second, how they describe their learning in the course is also significant. Some offered relatively argument-focused descriptions that cited language, notation, and proof techniques, but others described their focus on learning why the mathematics they had learned before worked the way it did. These students described how proving made them focus on concepts and relationships and thereby explained mathematics in much more precise ways than before.

This study complements prior work that has focused on students' reasoning on specific problem-based tasks—typically proof construction and evaluation. In targeting students' experience, we have focused on broader and more general themes, especially what is different in elementary proof work and how students reorganize their learning to meet the challenge. These issues are important foci for all efforts to assist students in understanding the new challenges and supporting their own in addressing them.

Our next steps are two-fold. First, we plan to recruit and interview students who are currently taking the course in the early weeks of the semester to refine our view of the difference and challenge in the course and reach some students who are or will soon struggle. Second, we intend to interview course graduates who are enrolled in their first semester of proof-intensive coursework to learn whether and how their experience in the transition to proof course is supporting their work and success in those courses (or not).

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Principles for Designing Tasks that Promote Covariational Reasoning

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Researchers have demonstrated the importance of covariational reasoning for students' development of various mathematical ideas. Several researchers have also argued that creating and sustaining multiplicative objects is a necessary mental action to reason covariationally. In this report, we describe task-design principles we have found to be productive for investigating and supporting students' construction of multiplicative objects and their covariational reasoning. Drawing on our research investigating students' covariational reasoning, we include data that highlights how these principles have been productive in our research and teaching.

Key words: Task Design, Covariational Reasoning, Multiplicative Objects

Researchers have identified that quantitative and covariational reasoning are important to the development of mathematical ideas such as rate of change and function (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995; P. W. Thompson, 1994), with students' abilities to develop images of quantities' magnitudes being critical to their reasoning covariationally (Carlson et al., 2002; Saldanha & Thompson, 1998). Specifically, these images afford students opportunities to construct multiplicative objects; this notion of covariation involves an individual sustaining an image of two quantities' magnitudes such that the two quantities are coupled so that she "tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (Saldanha & Thompson, 1998, p. 299). Researchers examining students' covariational reasoning have designed tasks to provide students opportunities to reason covariationally—*City Travels* (Saldanha & Thompson, 1998), *Power Tower* (Moore, Silverman, Paoletti, & LaForest, 2014), *Over and Back* (A. G. Thompson & Thompson, 1996; P. W. Thompson, 1994), and adaptations of the *Bottle Problem* (Carlson et al., 2002; Johnson, 2015; Zeytun, Cetinkaya, & Erbas, 2010). From our ongoing work in developing models of students' thinking and our interpretations of these researchers' works, we describe task-design principles that we have found productive in investigating and supporting students' construction of multiplicative objects and their covariational reasoning in small group and whole-class research and teaching settings. Our task-design principles are (a) avoid using quantities that are time or monotonic in the situation, (b) use shapes strategically to match or not match the graphical shape, (c) use different representational systems or orientations, and (d) use varying segment magnitudes to represent a quantity's magnitude in a situation.

Background

In this paper, we draw on P. W. Thompson's (2011) and Thompson, Hatfield, Yoon, Joshua, and Byerley's (2016) extension of the definition of covariational reasoning offered by Saldanha and Thompson (1998). We note that covariation is not an inherent feature of a situation, graph, table, etc.; a student must conceive of a situation, graph, table, etc. as constituted by covarying quantities (P. W. Thompson, 2011). Thompson (2011) characterized covariational reasoning in terms of an individual conceiving two quantities' magnitudes (or values), x and y, each varying (by intervals of size ε) over conceptual time, t. He used ordered pair notation, $(x(t_\varepsilon), y(t_\varepsilon))$, to denote the cognitive uniting of these two magnitudes, the result of which is a multiplicative object. Thompson's use of multiplicative object stems from Piaget's notion of "and" as a multiplicative operator (Inhelder & Piaget, 1958). The creation of such a multiplicative object is essential for conceiving displayed graphs as constituted by covarying quantities, and Thompson et al. (2016) explained that an individual's conception of a multiplicative object is not restricted to or contained in a displayed graph. Instead, a graph is one way to represent a constructed multiplicative object. Thus, as we present our task-design principles, we focus on students' construction of multiplicative objects primarily, but not exclusively, in the context of graphing.

With regard to previous recommendations on task-design principles, our tasks follow Gravemeijer and Doorman's (1999) recommendation that tasks be experientially real—tasks involve a situation that students can imagine and understand in order to support conceptions of varying quantities. Additionally, our tasks include simplified versions of common situations, often in the form of dynamic videos or applets (Johnson, 2013, 2015; Saldanha & Thompson, 1998; P. W. Thompson, 1994). Both aforementioned principles are mentioned in Carlson, Larsen, and Lesh's (2003) list of principles they used to structure covariational reasoning within a model-eliciting task. Lastly, although Johnson (2015) recommended sequencing tasks that support students in progressing from numerical to non-numerical reasoning, we avoid providing numbers in our tasks because quantitative reasoning is fundamentally non-numeric in nature (Smith III & Thompson, 2008). Moreover, we do not explicitly include non-numeracy in our task design principles because quantitative reasoning is principally non-numeric.

Task Background

Because of space limitations, we illustrate the task-design principles using two tasks: *Going Around Gainesville* (*GAG*) and *Which One? (WO?)*. *GAG* is a modification of Saldanha and Thompson's (1998) *City Travels* task. A student watches an animation of a car traveling around Gainesville on the way from Atlanta to Tampa (Figure 1). After we ask the student to describe the situation, we ask her to create a graph relating the car's total distance traveled and distance from Gainesville during the trip (Part I). After the student addresses Part I, in which she chooses her axes orientation, we request a graph relating the car's distance from Gainesville and distance from Atlanta on a given set of axes (Part II) (see Figure 1 for a normative solution to Part II).

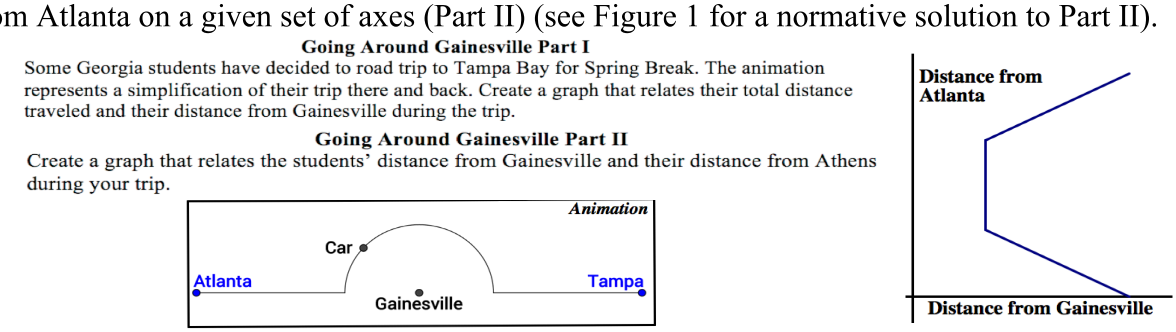
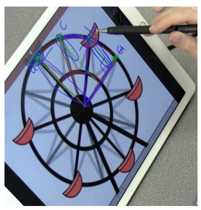
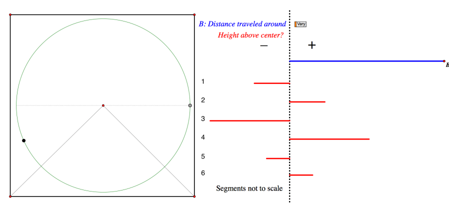


Figure 1: The *Going Around Gainesville* task and animation.

The *WO?* task (Figure 2) is an adaption of the *Ferris Wheel* task (Moore & Carlson, 2012). In this task, we present a student with an animation of a Ferris wheel rotating counterclockwise at a constant speed with a rider starting at the three o'clock position (see screenshot of animation in Figure 2a). We ask the student to describe the relationship between the height above the horizontal diameter of the wheel and the arc length the rider has traversed. We then present the student with a simplified version of the Ferris wheel situation with the position of a single rider indicated by a dynamic point on a circle. Beside the situation, there are seven directed horizontal line segments. We inform the student that the topmost line segment (shown in blue) represents the arc length the rider has traveled counterclockwise from the initial three o'clock position. Students can change the length of this topmost segment by dragging point B or by clicking the 'Vary' button. As the position of the point on the circle (i.e., the rider) moves, the length of the topmost segment varies appropriately. We then ask the student to determine which of the six red segments (labeled in Figure 2b for the reader), if any, accurately represent the rider's height above the horizontal diameter of the Ferris wheel as the rider's arc length traveled varies. The design of these six segments is in Table 1; segment 1 is a normative solution to the task.



(a)



(b)

Figure 2: (a) Ferris wheel animation (b) *WO?* animation with segment numbers.

Table 1
Behavior of red magnitudes with respect to B, the arc length traveled; all values relative to radius.

Seg.	$0 \le B \le \pi/2$	$\pi/2 \le B \le \pi$	$\pi \le B \le (3\pi)/2$	$(3\pi)/2 \le B \le 2\pi$
1	Increasing from 0 to 1 at a decreasing rate	Decreasing from 1 to 0 at a decreasing rate	Decreasing from 0 to –1 at an increasing rate	Increasing from –1 to 0 at an increasing rate
2	Increasing from 0 to 1 at a decreasing rate	Decreasing from 1 to 0 at a decreasing rate	Increasing from 0 to 1 at a decreasing rate	Decreasing from 1 to 0 at a decreasing rate
3	Decreasing from 1 to 0 at a decreasing rate	Decreasing from 0 to –1 at an increasing rate	Increasing from –1 to 0 at an increasing rate	Increasing from 0 to 1 at a decreasing rate
4	Decreasing from 1 to 0 at a decreasing rate	Increasing from 0 to 1 at an increasing rate	Decreasing from 1 to 0 at a decreasing rate	Increasing from 0 to 1 at a decreasing rate
5	Increasing from 0 to 1 at a constant rate	Decreasing from 1 to 0 at a constant rate	Decreasing from 0 to –1 at a constant rate	Increasing from –1 to 0 at a constant rate
6	Increasing from 0 to 1 at a constant rate	Decreasing from 1 to 0 at a constant rate	Increasing from 0 to 1 at a constant rate	Decreasing from 1 to 0 at a constant rate

Task-Design Principles and Illustrations

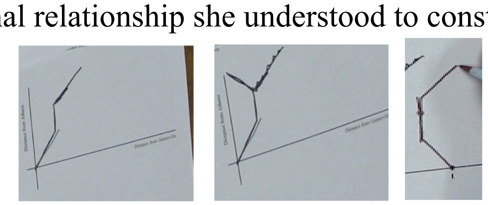
In this section we provide a description of each of the design principles and examples of student activity highlighting the productivity of these principles. The students were pre-service secondary mathematics teachers at a large university in the southeastern U.S. and were enrolled in or had completed a content course in a secondary mathematics education program. All students had completed a calculus sequence and at least two additional mathematics courses (e.g., linear algebra, differential equations, etc.) with at least C letter grades. We collected data using semi-structured task-based clinical interviews (Clement, 2000; Goldin, 2000) and teaching experiments (Steffe & Thompson, 2000), a methodology in which task design is critical and rooted in developing models of student thinking and seeking to engender shifts in student thinking based on those models; one's hypothetical models of student thinking and shifts in student thinking drives her or his design of tasks. The purpose of this paper is to report on our task-design decisions, and thus we point the reader to our other works for more information about our methods in the context of particular studies (e.g., Moore, Paoletti, & Musgrave, 2013; Moore, Paoletti, Stevens, & Hobson, 2016).

Avoid using quantities that are time or monotonic in the situation

One of the key features of considering a graph as representing a multiplicative object is to understand a coordinate point *simultaneously* represents *two* magnitudes (oriented orthogonally in the normative Cartesian system); this construction is non-trivial (Thompson et al., 2016). Complicating the matter, researchers (Carlson, Larsen, & Lesh, 2003; Johnson, 2015; Leinhardt, Zaslavsky, & Stein, 1990) have argued that students often reason uni-variationally when creating graphs, particularly if time is one of the quantities under consideration (i.e., students consider changes in one quantity without explicitly coordinating a second quantity). In our work, we have identified that students have a propensity to reason about a quantity varying with respect to experiential time if the second quantity under consideration varies monotonically (Paoletti, 2015), especially if this quantity is represented on the Cartesian horizontal axis; due to the monotonic variation of the second quantity, the student need not maintain explicit attention to its variation. Hence, our first task-design principle is to prompt students to consider quantity pairs such that neither quantity is time or varies monotonically.

In *GAG* Part II, we ask the student to create a graph using distance from Gainesville on the horizontal axis and distance from Atlanta on the vertical axis; distance from Atlanta is equivalent to total distance traveled until the car reaches Tampa, at which point distance from Atlanta begins to decrease as the car travels back towards Atlanta. This part of the task asks students to represent two quantities that are not varying monotonically, which has enabled us to determine the extent to which students sustain an explicit coordination of the two quantities under consideration (i.e., the extent to which students maintain understanding their graph as a multiplicative object) or if they (tacitly) represent another quantity (i.e., total distance traveled or time) during their construction of the graph in Part II.

To illustrate, throughout her activity in *GAG* Part II, Alicia accurately described the relationship between distances from the two cities. However, as she graphed the relationship she accurately described, she drew a graph that monotonically increased with respect to both axes (Figure 3a); we infer Alicia *tacitly* imagined the quantity on the horizontal axis as increasing. With respect to the situation Alicia understood that neither quantity varies monotonically, but to represent graphically the relationship she perceived in the Cartesian coordinate system, Alicia needed to create and sustain a multiplicative object of *both* distances. Alicia eventually did conceive her graph as representing a multiplicative object, evidenced by when she pointed to the bottom-left segment in Figure 3a and stated, "our distance from Gainesville should be getting smaller instead of bigger", and she was eventually able to adjust her graph (Figure 3b-3c) so that it represented the covariational relationship she understood to constitute the situation.



(a) (b) (c)

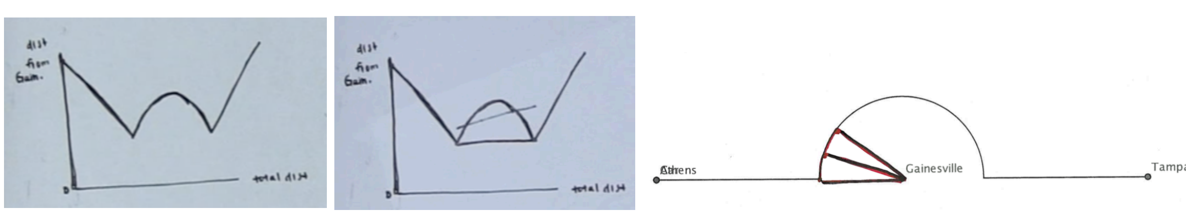
Figure 3: (a)-(c) Three stages of Alicia's solution to GAG Part II.

Use shapes strategically to match or not match the graphical shape

Several researchers (Clement, 1989; Leinhardt et al., 1990; Monk, 1992) have reported on students incorporating conceived iconic elements (i.e., visual features) of a task situation into graphical representations. We highlight that, in that moment, a student is not constructing and sustaining a multiplicative object of the form we discuss. Rather, the student is forming figurative associations between a perceived situation and graph (P. W. Thompson, 2016). Hence, as a second task-design principle, we design tasks such that perceived shapes in the situation do match perceived shapes of normative graphs, do not match perceived shapes of normative graphs, and a combination of the two.

For instance, in *GAG*, the simplified road is composed of linear segments and a semicircle. While the car travels on the straight paths, the car's distances from the two cities changes at a constant rate resulting in straight segments in the graph; while the car travels on the semicircular arc, however, the car's distance from Atlanta is changing while the distance from Gainesville remains constant, resulting in another linear segment in the graph. Including these different components in the situation allows us to gain insights into whether a student is reasoning with quantitative relationships or engaging in iconic translations. Further, we incorporate the aforementioned combination to examine whether students generalize such associations (e.g., "shapes always match graph" or "shapes never match graph").

To highlight the utility of this task-design principle, consider Alicia's activity in *GAG* Part I. After accurately describing the relationship between the car's distances from the two cities when drawing the segment from the vertical axis, Alicia engaged in an iconic translation as she drew a semicircular arc in her graph (Figure 4a). It was *after* creating the graph that Alicia thought about the result of that activity as a multiplicative object; she indicated that, to her, there was an increase and decrease in the distance from Gainesville while traveling along the semicircular path. Alicia then put concerted effort into justifying this part of her graph by comparing this claim with her image of the situation. Only when carefully attending to the car's distance from Gainesville in the situation, measuring it with a ruler (Figure 4c), did she conceive that this quantity did not change on the semicircle and replaced the arc in her graph with a straight segment (Figure 4b). She stated, "They're all the same! Why did I think it was changing? That's the radius…. So, we should be the same distance from Gainesville". Thus, Alicia compared the relationship she conceived her graph as representing to the relationship she constructed in her image of the situation, which resulted in her conceiving the same relationship as constituting both the graph and situation.



(a) (b) (c)

Figure 4: Two stages of Alicia's solution to *GAG* Part I (a-b) and her measurement work (c).

Use different representational systems or orientations

Previous researchers (e.g., Moore et al., 2016; Moore et al., 2014; P. W. Thompson, 2016) have illustrated that students' ways of thinking for graphs are often dominated by figurative activity (e.g., graphs "pass the vertical line test" or graphs are drawn left-to-right). This outcome can be explained by students having infrequent opportunities to construct images of covarying quantities and their having repeated, and possibly exclusive, experiences with graphs that have those figurative attributes (Carlson et al., 2002; Moore et al., 2016; Smith III & Thompson, 2008). Thus, for our third task-design principle, we suggest having students use various orientations of coordinate systems (e.g., changing axes orientations) or entirely different representational systems (e.g., alternative coordinate systems and representational systems like the ones described in the next principle). This design principle has supported us in gaining insights into the extent that students' actions are dominated by figurative activity or coordinating quantities. Moreover, we have been able to gain insights into the extent that students construct and sustain multiplicative objects across a variety of representational systems.

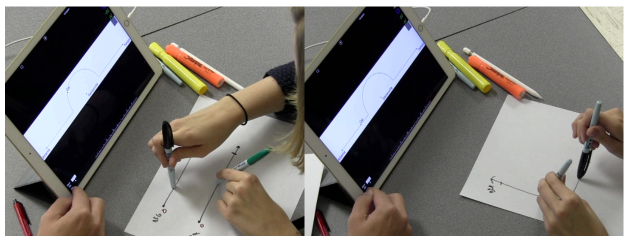
To illustrate, students may, from our perspective, be reasoning covariationally about two quantities in *GAG* Part I, but when given *GAG* Part II, they hesitate to draw a vertical line (see Figure 1) because they claim graphs cannot "look" like that. Other examples from *GAG* included students having difficulty constructing a relationship that is represented by a graph drawn "right-to-left". We have shown such issues to be problematic for students as they create graphs, particularly if the students do not persistently focus on covarying quantities (Moore et al., 2016). Further, we have found that using a multitude of coordinate systems supports students in reasoning covariationally in order to conceive graphs in different coordinate systems as representing an invariant relationship between quantities (Moore, Paoletti, & Musgrave, 2013). Hence, like the other principles, using different coordinate systems or orientations supports us in differentiating students who are conceiving of graphs as representing a multiplicative object versus those whose thinking is dominated by figurative thought; when changing coordinate systems or orientations, figurative aspects of a function's graph typically change.

Use varying segment magnitudes to represent a quantity's magnitude in a situation

The three preceding task-design principles have been helpful in allowing us to have a sense of the students who are reasoning covariationally within and across situations and graphical representations. As previously mentioned, constructing a multiplicative object does not require a coordinate system (P. W. Thompson et al., 2016). By working with varying magnitudes instead of prompting a student to create a graph (e.g., *WO?)*, we gain insights into students' reasoning while minimizing the influence of the ways of thinking they have developed for graphs (e.g., iconic translations, issues of function/dependency, ways of thinking based in figurative thought). Specifically, we are able to gain insights into the extent that a student constructs and sustains a multiplicative object with respect to a situation and the displayed magnitudes. Moreover, we are able to gain related insights into students' ways of reasoning about graphs by scaffolding tasks to support students in orienting the segments orthogonally, constructing a coordinate point, and imagining a trace representing all instantiated pairs of the covarying magnitudes (as seen in the "finger tool" explained in Lima, McClain, Castillo-Garsow, and P. W. Thompson (2009); P. W. Thompson (2002)).

Consider Lydia's reasoning in the *WO?* task. When referencing the first quarter rotation of the Ferris wheel (Figure 2a), she explained, "[A]s the arc length is increasing… [the] vertical distance from the center is increasing… but the value that we're increasing by is decreasing." After providing this accurate description of the situation, Lydia moved to select the appropriate segment in the *WO?* task (Figure 2b). She eliminated all segments except 1 and 5 (Figure 2b). Focused on the top of the Ferris wheel, Lydia said, "I think it is [segment 5], because it is decreasing at the same rate that I am increasing [referring to her moving the blue segment]." Despite having potential constraints from a graphing task removed, Lydia reasoned about the segments in a way that was not consistent (from the researchers' perspective) with her reasoning about the quantities in the situation. Lydia's contradictory statements in this task illustrate students' difficulties with constructing and sustaining a covariational relationship across representational systems that do not necessitate also constructing a coordinate point.

As Lydia continued in her teaching experiment, having repeated opportunities to reason about magnitudes in non-graphical situations was beneficial in her coming to understand graphs as representations of multiplicative objects. For instance, on *GAG* Part II, Lydia could not initially construct a graph as requested. However, she then imagined horizontal segments representing the varying magnitudes of each of the distances. As the animation played, Lydia used her pen tips, both oriented horizontally, to indicate how each quantity's magnitude varied (Figure 5a). Then, she engaged in the same activity but with her pen tips oriented orthogonally (Figure 5b); this activity supported her imagining and drawing a trace of the graph that she understood as uniting the two magnitudes into a point and trace (i.e., a multiplicative object) à la the "finger tool").



(a) (b)

Figure 5: Lydia reasoning with (a) horizontal magnitudes and (b) orthogonal magnitudes on *GAG* Part II.

Discussion

We presented several task-design principles intended to afford students opportunities to reason covariationally as they construct, maintain, and represent multiplicative objects in various representational systems. We have found that tasks designed with these principles provides students repeated opportunities to construct and compare relationships between quantities across a multitude of representational activities. Moreover, these design principles afford teachers opportunities to gain insights into their students' propensities to reason covariationally as well as to perturb students who engage in reasoning that is not attentive to covariational relationships. Hence, we believe that tasks designed using these principles will provide students with intellectual need (Harel, 2007); students will find covariational reasoning to be productive when engaging in these tasks. We conclude by emphasizing that we have found the last design principle most productive in our work with undergraduate students, particularly because of students' propensity to reason about graphing in ways that do not entail quantitative or covariational reasoning.

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Common Algebraic Errors in Calculus Courses

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College mathematics instructors often view the final problem solving steps in their respective disciplines as "just Algebra", but in reality, a weak foundation in Algebra may be the cause of failure for many college students. The purpose of this paper is to identify common algebraic errors students make in college level mathematics courses that plague their ability to succeed in higher level courses. The identification of these common errors will aid in the creation of a model for intervention.

Keywords: algebra, common errors, calculus

Introduction

As early as 1910 common errors related to arithmetic and rational number computation and the difficulties students were facing in learning mathematics were noted by De Morgan. Since then other researchers have catalogued common errors in computation and algebra (Ashlock, 2010; Benander & Clement, 1985; Booth, Barbieri, Eyer & Paré-Blagoev, 2014). Benander and Clement (1985) catalogued errors students made in basic arithmetic and algebra courses. Their work involved classroom observations and resulted in 11 categories of common errors including, basic problem solving skills, averages, whole numbers, fractions, decimals, percents, integers, exponents, simple equations, ratios and proportions, geometry, and graphing. Ashlock (2010) focused on the mathematics work of school-aged children and on helping instructors thoughtfully analyze their students' work in order to discover patterns in their errors for the purpose of improving instruction. Ashlock suggests that as students learn about mathematical operations and methods of computation, they often develop and adopt misconceptions and procedural errors. Teachers who understand that this occurs and are able to identify these problems in their students' work can develop strategies to help students. In a more recent study, Booth, et al. (2014) focused on the errors in algebra with school-aged students and identified errors that were "persistent and pernicious" given their predictive ability for student difficulty on standardized test items. Their study involved an in-depth analysis of students' errors during problem solving at different points during the year and resulted in the classification of these errors which include: *variable* errors, *negative sign* errors, *equality/inequality* errors, *operation* errors, *mathematical properties* errors, and *fraction* errors.

Drouhard and Teppo's (2004) work presented the idea of *denotation* and suggests that it is a developed sense about what one is writing and a lack of sense regarding denotation creates significant problems for students. They note "that students with poor capabilities to recognize this aspect of the meaning of an expression often make endless calculations because they do not know in what direction to go and when to stop" (p. 235). Harel, Fuller, and Rabin (2008) further comment on the idea of meaning and denotation indicating that students often cancel within problems without attending to the quantitative meaning of their action. For example, Harel et al. (2008) states "it is not uncommon for students to manipulate symbols without a meaningful basis that is grounded in the context in which the symbols arise; for instance, a student might write: $(\log a + \log b)/\log c = (a + b)/c$" (p. 116). In this case, students may be overgeneralizing their use of the distributive property and cancel "log" without considering the quantitative meaning of their action. Harel (2007) suggests that the lack of emphasis on mathematical meaning that students, and perhaps their teachers, apply to mathematical symbols creates what is referred to as a *non-referential symbolic* way of thinking and that this way of thinking can be tied to a myriad of algebra errors. Sfard and Linchevski (1994) believe that students must be motivated "to actively struggle for meaning at every stage of learning" (p. 225). They are concerned that "if not challenged, the pupil may soon reach the point of no return, beyond which what is acceptable only as a temporary way of looking at things will freeze into a permanent perspective" (p. 225). Mason (2002) in his framework: *Manipulating-Getting-a-sense-of-Articulating*, emphasizes that students must be given opportunities to make sense of situations. He believes that "students want, indeed need, confidence-inspiring familiar objects to manipulate and on which to try out new ideas so that they can literally 'make sense' of them" (p. 187). In Harel and Sowder's (2005) opinion "instruction (or curriculum) that ignores sense-making, for example, can scarcely be expected to produce sense-making students" (p. 46).

Although, research on students' difficulties with algebra in school has been well documented (e.g. Kieran, 1992; Hoch & Dreyfus, 2004; Stacey, Chick, & Kendal, 2004), studies on occurrence of these errors in college level mathematics courses is scarce. In response to this need a pilot study conducted by Stewart and Reeder (2017) revealed how the unresolved high school algebra misconceptions and shortcomings may create major complications in college mathematics courses. Figure 1 demonstrates the progress and complexities of mathematical ideas as students approach the calculus courses (Stewart, 2017, p. vii).

$$\frac{1}{2} + \frac{1}{16}$$
$$\frac{1}{x^2} + \frac{1}{x^4} = 50$$
$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{x^2}{16}\right)$$
$$\lim_{x \to 0} \frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{x}$$

Figure 1. The unresolved issues with fractions compound as students confront limits in calculus.

While it appears that many students follow the theories that are introduced in calculus courses, in many cases not having a rich algebra background prevents students from completing basic tasks. Many students become particularly frustrated as they realize that the fast pace of college mathematics lectures and new material are not going to wait for them to catch up. On the other

hand, the instructors become disappointed with students' performances as they note instances of algebra errors that should have been resolved years ago in high school.
The aim of this study was to identify and categorize students' common algebra errors in entry university mathematics courses. This paper does not deal with the question of why students are making such errors and continue to make them, but rather seeks to identify the type of errors that are most commonly made.

Method

Our research team included two mathematics educators working with three graduate students and a cognitive psychologist who specializes in children's algebra thinking process as our consultant. Informed by the previous work and findings from their pilot study, this research team collected data from entry level college mathematics courses from a university in the Southwest United States. Data were gathered from approximately 600 students' final exams and tests from the following mathematics courses: College Algebra; Pre-Calculus for Business, and Calculus and Analytical Geometry I during a single semester. For the purposes of this study, only the results from the Pre-Calculus for Business course will be discussed. Given the data for this research project were exams for actual college mathematics courses, the data had ecological validity. While the use of actual test and exam questions provided validity to the study in that participants were doing the mathematics they are and will be asked to do in their college level mathematics courses rather than working through problems developed for the purpose of the study that might invoke or invite certain errors, this created challenges for coding and predicting the kinds of errors students might make. The ecological nature of the data called for a process of open coding and examination of codes and coding again.

As in the case of Booth et al. (2014) we anticipated that the errors would be concept driven. For example, certain errors were present at the beginning of the semester and did not appear again until near the end of the semester. However, we were interested to see if the errors persisted and showed up again in the final exam. Hence, we collected data from students' tests 1, 2, 3 and the final examination, de-identified students' names and gave each student a number as well as their instructor's name in order to monitor their progress. For example, Jane Smith from Mr. Thomas's class was coded as: "SP16 Course Number Thom 1".

The data were scanned, and organized in a shared Dropbox for easy access by the members of the research team. The team met to review the findings of the pilot study and to discuss possible codes for common errors in student work. In order to ensure that the research team could work effectively to identify student errors, one set of exam questions was assigned to the group to analyze independently. The research team met the following week to discuss the coding categories and themes that emerged in an effort to determine an initial list of codes. This resulted in a list of the first nine codes Table 1. Further, the team agreed that beyond the initial nine codes, each person would add codes as needed. The data were then assigned to five members of the research team such that each set of test questions were analyzed independently by two researchers. Each exam question was analyzed independently and coded using the nine pre-established codes combined with a process of open-coding. Following this analysis and coding, the team met to discuss the codes and establish what word would represent the kinds of errors that were emergent in the data (Rossman & Rallis, 2012) and what was collectively understood when that code was used. This discussion resulted in the addition of several new codes and the further definition of some of the codes already in use. Following this discussion, the team once again analyzed the data to re-code where necessary and open code as needed. This process resulted in the coding list presented in Table 1. When the coding process was complete, 4,328 test and final exam questions had been analyzed from students (N=163) in two instructors' sections of Pre-Calculus for Business. Table 1 represents the percentage of each of those identified errors by code or error type. The results revealed that *Simplifying*, *Sign Errors*, *Log Properties* followed by *Distributing*, *Isolating Variable* and *Exponents* were the six highest percentages of the errors.

Table 1. Percentage of errors by coding type.

Code and Description	**%**	**Code and Description**	**%**
1. Isolating Variable (Balance point) - Students are unable to correctly work with variables on both sides of an equation.	9.09	11. Substitution - Students substitute variables or values incorrectly,	1.67
2. Simplifying - Students are unable to simplify or do not simplify when needed.	12.86	12. Absolute Value - Student do not interpret absolute value correctly.	0.12
3. Exponents - Mistakes are made with exponents.	9.03	13. Function Notation - Students do not interpret function notation correctly.	4.08
4. Sign Errors - An error made with signs.	10.76	14. Mystery Zero - Students replace a variable with zero.	1.61
5. Fractions - Mistakes with computations with fractions or with working with variables within fraction notation.	1.55	15. Quadratic Equation - Unable to solve a problem using the quadratic equation.	1.55
6. Distributing - Misuse of or ignoring the distributive property.	9.96	16. Computational Error - Simple addition or other computation mistake.	1.36
7. Cancelling - Cancelling when it is not appropriate.	3.83	17. ln/e Conversion - Students convert between logarithmic and exponential forms of an equation incorrectly.	2.97
8. Radicals - Misuse of the radical sign or inability to convert the radical sign to an exponent representation.	8.97	18. Log properties - Students incorrectly combine or expand logarithmic properties.	10.95

9. Factoring - Incorrect factoring by not removing the greatest common factor or only factoring from some but not all terms.	3.53	19. Miscellaneous	6.00
10. Inequalities - Mistake with inequality or with changing signs when dividing or multiplying across an inequality.	0.12	20. DNS-Student did not sit the exam (maybe T1, maybe T2, T3 and Final; maybe T3 and Final; maybe just Final).	

The followings (see Table 2) are a sample of questions from tests 1, 2, 3 and the final exam. The questions were unified across all the sections and designed by the course coordinator. The tests and final exams contained multi-choice questions as well as long-answer questions. For the purpose of finding the algebraic errors, we only considered the long-answer questions.

Table 2. Sample Questions from Tests/Finals.

Tests & Final	**Sample Questions**
T1	1) Solve the formula: $Y = \frac{T - m\,x}{k} - 12$ for the variable x. 2) Factor completely: $2a^2x^3 - 3a^2x^2 - 18x + 27$ 4) Find the complete solution set for: $\frac{1}{x^4} + \frac{1}{x^2} = 90$
T2	1) If the average rate of change of $f(x) = x^2 + 3x + 7$ from $x = 2$ to $x = k$ is equal to 13, then find the value of k 2) If $f(x) = 8 - 9x - 10x^2$, then find and simplify $\frac{f(x+h) - f(x)}{h}$
T3	1) If you place $100,000 into an account drawing 7.5% interest compounded continuously, then how many years would it take to have $350,000 in that account? [round to the nearest year] $[A = P(1 + \frac{r}{n})^{nt}]$ $[A = Pe^{rt}]$
Final	**8) Find the complete solution set for this equation:** $\log_3(x - 5) + \log_3(x + 3) = 2$

Results

The result of this research show that college students carry with them misunderstandings and challenges related to algebra from their high school years into their entry level college mathematics courses. The types of errors they made varied and yet making errors was persistent and plagued the students' abilities to learn new mathematics concepts throughout the semester from Test 1 to the Final examination. This study revealed that while algebra related errors are evident in student work throughout the semester, the type of errors made are often dependent on the type of mathematics problems they are asked to solve. For example, errors with fractions may appear to be resolved as the semester moves along but it may well be that students are simply not asked to work problems that involve fractions near the end of the semester. Figures 2 and 3 below are examples of some of the error types resultant from this study. Figure 2 highlights the kind of error that was coded as Isolating Variable given that the student struggled to isolate the variable on one side of the equation. Nearly 10% of the errors found in this study were of this type. Figure 3 below provides an example of a students' error with function notation.

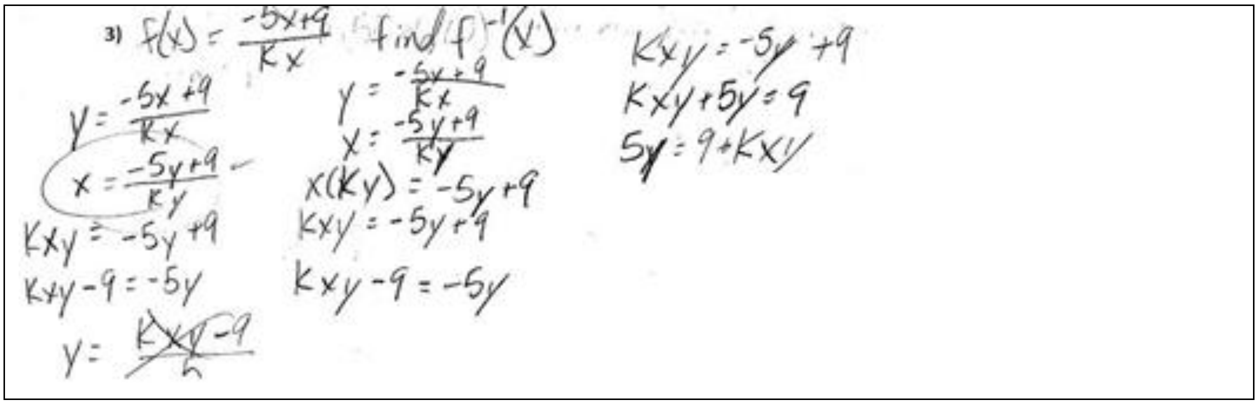

Figure 2. Isolating Variables (Balance Point).

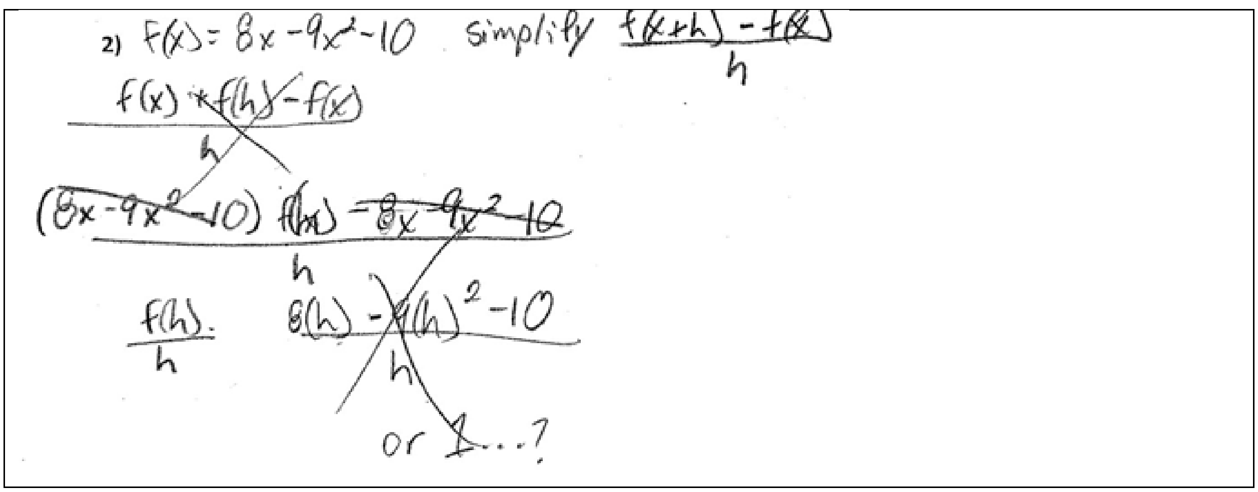

Figure 3. Function Notation.

Discussion and Concluding Remarks

This study investigated test and exam questions performed by 163 Pre-Calculus for Business students, and examined more than 4000 problems, in order to categorize the most common types of algebraic errors that college mathematics students make. Although, it is perceived that the type of exam questions posed, invited certain types of errors, the frequency of errors is persistent throughout the semester regardless of problem type. The students consistently make mistakes and errors with algebra from Test 1 to the final examination.

While analyzing the data, we witnessed over and over again how algebra errors rapidly terminated the flow of the problem solving sequence for many students and resulted in incorrect solutions or no solutions. Many mathematics instructors believe that it is not our responsibility to teach or re-teach school algebra in college level courses. Realistically, going over algebra misconceptions is not a possibility and we have no time to repair them. Many of the errors found in this study reflect Drouhard et al.'s (2004) finding that students make endless calculations when they do not know what direction to go and Harel's (2007) point about non-referential symbolic way of thinking. Booth et al. (2014) suggests that "the misconceptions underlying specific persistent errors are not corrected through typical instruction and may require additional intervention in order for students to learn correct strategies." (p. 21).

Future research

Although, it is believed among many instructors that these types of algebraic errors should have been resolved years ago in high school and maybe nothing can be done at this stage, we believe that pinpointing the type of errors will help in creating interventions that remedy the algebraic errors. We continue to refine our common error types as more data become available from College Algebra and Calculus and Analytical Geometry I courses. Our next steps are to interview the instructors who taught these courses and seek more information from them in order to create the most effective interventions to help future calculus students.

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Completeness and Sequence Convergence: Interdependent Development in the Context of Proving the Intermediate Value Theorem

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As a part of a larger RME-based instructional design project for advanced calculus, this paper reports on two students' reinventions of formal conceptions of sequence convergence and the completeness property of the real numbers in the context of developing a proof of the Intermediate Value Theorem (IVT). Over the course of ten, hour-long sessions I worked with two students in a clinical setting, as these students collaborated on a sequence of tasks designed to support them in producing a proof of the IVT. Along the way, these students conjectured and developed a proof of the Monotone Convergence Theorem. Through this development I found that student conceptions of completeness were based on the geometric representation of the real numbers as a number line, and that the development of formal conceptions of sequence convergence and completeness were inextricably intertwined.

Key Words: RME, instructional design, real analysis, limit, completeness

Introduction

The transition from lower-division mathematics courses, where the emphasis is often on calculational approaches, to upper-division courses, primarily concerned with proof and more abstract mathematics, is a challenging one for many undergraduate students. There has been growing interest in developing research-based, student-centered curricula for undergraduate mathematics to support students in making this transition in the areas of abstract algebra (TAAFU: Larsen, 2013; Larsen & Lockwood, 2013), differential equations (IO-DE: Rasmussen & Kwon, 2007), geometry (Zandieh & Rasmussen, 2010), and linear algebra (IOLA: Wawro, et al., 2012). The data presented in this paper comes from early efforts at similarly-motivated instructional design efforts for advanced calculus. One of the central ideas underpinning all areas of advanced calculus is that of limits and convergence. One of the features of the real numbers that makes limits and convergence so important (indeed, possible) is that of completeness. While a large body of research exists about how students think about limits and how that thinking develops in formality, there is a dearth of research dealing directly with students' conceptions of the completeness of the real numbers.

This paper reports on student strategies that anticipated the concept of completeness, as those strategies emerged in the context of a teaching experiment. This experiment was part of an instructional design effort to develop the proof of the Intermediate Value Theorem (IVT) as a starting point for inquiry-oriented advanced calculus. The students in the teaching experiment began by approximating the (irrational) root of a polynomial using the principles behind the IVT. They developed a sequence of approximations by looking for the sign change of the function on smaller and smaller intervals. As the IVT (and many other facets of convergence) implicitly depend on the completeness of the real numbers, I expected that investigations of these kind would give insight into students' informal conceptions of completeness, as well as insight into how students might be supported in reinventing formal characterizations of completeness.

In this paper I will detail how early student justifications anticipated the Monotone

Convergence Theorem (MCT)[1], and how the proof of that theorem became a powerful context for the interdependent development of more formal conceptions of sequence convergence and completeness.

Literature Review

A great deal of research has investigated student understanding of the concept of limit. The focus of these investigations has shifted over the last few decades. Initially, a large number of studies sought to describe the difficulties students encountered when trying to work with limits (Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Moru, 2009; Sierpińska, 1987; Szydlik, 2000; Tall, 1980; Tall & Schwarzenberger, 1978; Williams, 1991). Gradually, more and more studies have investigated how student conceptions might develop in formality (Cottrill, et al., 1996; Oehrtman, 2009; Oehrtman, Swinyard, & Martin, 2014; Swinyard & Larsen, 2012). One important feature of formal work with limits, first given prominence by Swinyard & Larsen (2012), is the shift from *finding limits* to *verifying limit candidates.* While limit problems in calculus are often centered around the use of algebra to *find* limits, formal activity with limits is usually centered around using formal definitions to prove that a limit exists or to prove general properties about limits. What has received almost no study is what the process of developing such formal definitions of limits looks like in the context of proving, or what role completeness plays in this process of formalization. While the research reported here informs both of these topics, this proposal will focus on the latter.

While Cauchy is widely recognized as one of the fathers of real analysis, his proofs conspicuously lack any mention of the completeness of the real numbers (Grabiner, 1981; Lützen, 2003). In fact, one of the first explicit treatments of the completeness[2] of the real numbers was Dedekind's "Continuity and Irrational Number" essay, originally published in 1872 (1901), over fifty years after Cauchy's *Cours d'Analyse*. However, it is a critical component of the proof of the Intermediate Value Theorem, and so identifying student strategies that anticipate formal conceptions of completeness in this context will be critical in developing instruction for advanced calculus using this context as a starting point.

As such, the specific questions that guided this component of this design research project were the following:

1. *In the context of proving the IVT, what student strategies anticipate the concept of completeness?*
2. *In what ways do the developments of completeness and sequence convergence support one another?*

Theoretical Framework

The instructional design heuristics of RME have guided the development and implementation of this design project. They have also been indispensable as tools for analyzing student thinking and activity. In particular the heuristic of *emergent models* provides language and tools for describing students' activity at the informal level and also for describing the development of their activity toward greater formality and rigor. In RME, these models emerge from student activity, in the sense that the models provide a way for a teacher/researcher to describe student activity (Larsen & Lockwood, 2013).

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1 MCT: *If $\{a_n\}$ is a bounded, monotonic sequence, then it converges.*

2 The German word that Dedekind used was the word for "continuity", but it is clear that he was describing the modern concept of the completeness of the real numbers.

In emergent models, the model-of/model-for transition captures large, significant developments in student activity and thinking (Gravemeijer, 1999). For describing more local development of these models, Rasmussen and Marrongelle described the construct of a *transformational record* (2006). Such a record is an inscription or notation recorded by the students, or used by the teacher to capture student thinking, that later is used by the students for further mathematical development. This construct can be particularly useful for teachers in supporting the development of emergent models. In the Results section I will illustrate how I used a transformational record to support students in developing a more formal understanding of sequence convergence.

Transformational records can also be described using the RME construct of record-of/tool-for (Johnson, 2014; Larsen, 2004; Larsen, 2013). A record-of student activity generally refers to an inscription or notation that represents one form of the larger emergent model. This record-of becomes a tool-for when students use it for further mathematical development. This *transformation* of the record at a local level represents a development of the larger emergent model. A model-of students' activity can be evidenced by many different forms. One way to describe the development of this model toward a model-for is through "students' increasing ability to reason with various forms of the model" (Johnson, 2014). While not as significant as a model-of/model-for transition, which represents the students becoming aware of and using the model as a whole, these record-of/tool-for transitions nonetheless represent important developments in student activity.

In the study reported here, we will see how the concept of completeness emerged as a global model-of student reasoning about the convergence of an approximation algorithm. Later least-upper bounds, as one form of that global model, emerged as a record-of student thinking. This form of the model then developed into a tool-for more formal activity as student used least-upper bounds in two important developments: 1) formally defining a specific mode of sequence convergence, and 2) completing a proof of the Monotone Convergence Theorem. After describing the structure and implementation of the teaching experiment, as well as the manner in which I analyzed the data, I will briefly describe how completeness emerged as a model-of student activity, and how that model developed through the course of the teaching experiment.

Methods

As a part of the early stages of an instructional design project, I ran two separate teaching experiments over the course of a year, a little more than six months apart. Each teaching experiment consisted of 10, hour-long sessions with myself as teacher/researcher and a pair of students working at a chalkboard at the front of the room. These students were volunteers selected from courses that were direct prerequisites to advanced calculus/elementary real analysis courses, and who had expressed an intention to take advanced calculus in the near future. All four participants had completed the calculus sequence, differential equations, and at least one proofs-based course, prior to participating in the teaching experiment.

The data for this paper comes from the second teaching experiment, with students who will hereafter be referred to as Dylan and Jay. With the first teaching experiment, my attention was primarily focused on issues of convergence. It was not until retrospective analysis that I discerned the importance of the role that completeness could play in this context. For this reason I modified the task sequence for the second teaching experiment, which resulted in an abundance of data relating to student understanding of completeness. For these reasons this paper focuses on the experiences of Dylan and Jay.

During the implementation of the teaching experiment there were anywhere from three days to an entire week between sessions. During that time I watched the videos of the previous session, creating written session summaries, and tried to identify student statements and strategies that begged for further investigation. For example, Dylan and Jay justified the convergence of a particular sequence by appealing to the fact that the sequence was increasing and was bounded. But it was not clear from their statements whether they thought such a sequence *had* to converge to the given bound. To start the next session I gave them exactly this conjecture and observed their discussion.

After the conclusion of each of the teaching experiments, I performed a retrospective analysis of the data as a whole. I watched all of the videos again, transcribing segments I had flagged during the ongoing analysis, looking for student strategies that anticipated the completeness of the real numbers. For each of these I sought to explain what elicited these strategies. Finally, I followed these strategies through the data and using the design heuristics of RME I sought to explain how these strategies were leveraged to support the development of more formal ideas, or how they might be leveraged in future implementations of the LIT.

Results

In the context of developing their own proof of the IVT, I found that characterizations of the completeness of the real numbers emerged from Dylan and Jay's activity. This suggested to me that completeness, as a collection of varied but equivalent characterizations, could be seen as a model-of students' activity. Using primarily the emergent models design heuristic, I will describe how completeness emerged as a model-of their thinking, and how aspects of this model became a tool-for more formal reasoning about convergence.

Justifying Convergence

When tasked with approximating the root of a polynomial, Dylan and Jay used the sign-change of the function over successively smaller intervals to construct a sequence of approximations to that root. The transcript excerpt below came from a conversation in which Dylan, Jay, and I were discussing how they knew that their algorithm would find a root of the function in question. They had recently established that, if the root were irrational, their Decimal-Expansion algorithm would *never* give them the exact root.

I: So how do you know that there is such a number?

D: As long as we can recursively show that every time we step our function forward it gets a little bit closer to zero. This is how you do the limit in general: every time you step it forward, every time you know you move forward a little bit, you get closer to the number you think the limit is.

There are a few problems with Dylan's characterization of a "limit in general". For one, he is a describing convergence in a monotonic fashion, and so is not truly giving a general description. Second he is characterizing the convergence of their sequence of approximations using the monotonicity of the *outputs* of the function. This held true of the specific polynomial with which their investigations had started, but it was not necessary in general for their algorithm to work.

While there were many details to be worked out, Dylan's statement represented very promising reasoning. Here we can see that the concept of completeness, taking the form of the Monotone Convergence Theorem, served as a model-of Dylan and Jay's explicit justifications for the convergence of their sequence of approximations. More specifically, his statement suggested that he believed an increasing sequence, that was bounded above, should converge. Whether or not he really believed that this characterized limits in general was immaterial at that moment. This emergent model suggested that codifying and analyzing Dylan's justification could be very fruitful.

On the board Dylan had written:

$$f(x_i) < f(x_{i+1}) < 0 \qquad (1)$$

In an attempt to draw their attention away from the outputs for a moment, I asked Dylan and Jay whether a similar statement could be made about the inputs. I did this because I wanted to have them analyze Dylan's statement, but I did *not* want considerations about the behavior of the function to muddy the water. Without any discussion, Dylan wrote:

$$x_i < x_{i+1} < \overline{x} \qquad (2)$$

(where $\overline{x}$ was the conjectured root). Then he and Jay explained why the second compound inequality might be preferable.

Jay: We're controlling this [gestures at (2)] more than we're controlling this [gestures at (1)]. We can't control the outputs, but we can control the inputs.

Dylan: Right. I guess we just observe this [gestures at (1)] for this particular function.

With Dylan and Jay in agreement with the statement about the monotonicity and boundedness of the *inputs*, we were ready to consider their justification as a conjecture. Completeness, manifested as the Monotone Convergence Theorem, was an even clearer model-of their thinking about the convergence of their sequence of approximations. In order to support the development of their thinking, I set Dylan and Jay tasks that would have them engage in vertical mathematizing, by having them reflect on and analyze their own reasoning about convergence.

Least-upper Bounds

Another important development arose when Dylan and Jay considered conditions under which a bounded, monotonic sequence would converge to an upper bound. Dylan put forth the following explanation:

> "Because if you can pick a value, some a, between x_{i+1} and b...and...x_{i+1} passes every value of a...like every possible value of a...and passes b...wait, if this is true, so it doesn't pass b. So worst case scenario it converges to b."

It appeared that what Dylan described was essentially a characterization of b as the least-upper bound of the sequence. He seemed to suggest that if we could choose a to be any arbitrary value less than b, and then we knew that a value of the sequence $\{x_i\}$ passed that value of a, then the sequence would have to converge to b. So there was no value of a less than b that was also an upper bound for the sequence.

This condition, that the sequence passes every value of a less than b, but never passes b, proved to be pivotal in Dylan and Jay's developments of both completeness and convergence. A short while after this, I incorporated this condition as an added hypothesis to their MCT and had them consider it; in this way it became a record-of their thinking, and also represented one form of the larger completeness model. Subsequent to that discussion Dylan explicitly leveraged the condition to define "decreases to zero".

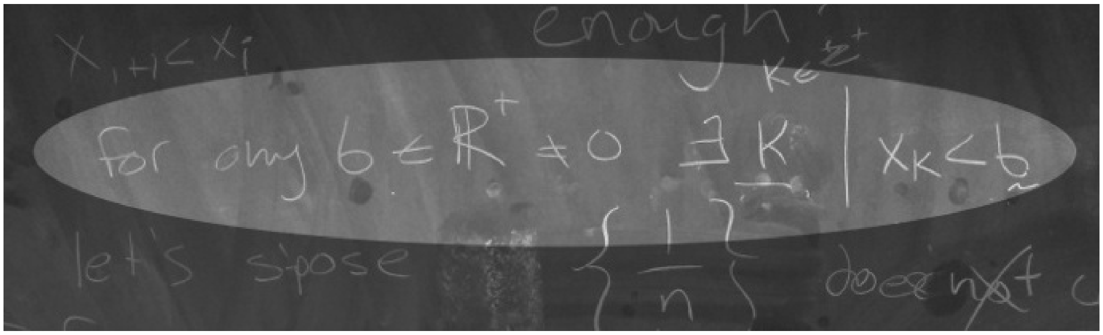

Figure 1: Students' definition of a sequence "decreasing to zero"

A Transformational Record

A little later Dylan and Jay set about to define formally what it meant for a sequence to "decrease to zero" (Figure 1).

In the following exchange Dylan explained the genesis of their definition.

Jay: How'd you get that?

Dylan: Basically going from our last idea that if a number converges to...to b, I guess. So, ...this is kind of formally writing out that...for every b that's less than a, or- which is zero in this case, I guess- is between where we're starting and the boundary.

Dylan made a connection back to their work with a previous conjecture: that if a monotonically increasing sequence was bounded by b, but passed every a less than b, then it must converge to b. Though the roles of the variables have been reversed, Dylan has described adapting their idea of "passes every a less than b" to this case of a sequence monotonically decreasing to zero.

Dylan went on to explain his definition in more detail.

> "So the boundary we know we want is zero. So we're going to talk about all the numbers that aren't zero, above zero...So we know this [sequence] is always getting smaller. Down to some...you know, whatever. It goes off to somewhere. But do you know it goes to zero? And you do as long as you can pick any of these [positive real] numbers and just keep going through until you find some k [sic] that's smaller than it."

Here we see the results of the students successfully leveraging a transformational record. Earlier in the experiment, when reasoning about the conditions under which a monotonic sequence might converge to its bound, Dylan's made the statement "passes every a less than b". A little later I recorded this reasoning, presenting it back to the students as an additional hypothesis to their MCT; in this way this characterization of least-upper bounds served as a record-of their thinking. And above we saw how this record became a tool-for solving the problem of defining the convergence of a sequence decreasing to zero. In this way an informal strategy of the students developed into a tool-for reasoning more formally about limits. More specifically, my presentation of their strategy as a conjecture acted as a *transformational record*, which they used to solve the problem of defining a sequence decreasing to zero.

Discussion

In the context of developing a proof of the Intermediate Value Theorem, the informal strategies of two students anticipated formal characterizations of completeness. The Monotone Convergence Theorem (MCT) emerged as the students considered the convergence of their (monotonically increasing, bounded) sequence of approximations to a root. Initially the convergence of such a sequence was intuitively clear to them. Digging deeper into *why* that might be the case led to the investigation and development of a proof of this idea. Least-upper bounds also emerged from their activity, for the first time as they tried to identify what conditions would guarantee that a monotonically increasing sequence would converge to an upper bound. Dylan and Jay subsequently utilized this idea to define what it meant for a sequence to converge to zero. Finally, though not detailed in this proposal, in constructing a proof of the MCT Dylan and Jay debated the existence of least-upper bounds, ultimately accepting their existence as a consequence of the real number line. The existence and properties of least-upper bounds were the key ideas in the ultimate completion of their proof of the MCT.

One way to frame this development is using the RME construct of emergent models. In justifying the convergence of their sequence of approximations, the students' thinking could be modeled by the larger concept of the completeness of the real numbers. Dylan and Jay were eventually able to use aspects of this model as a tool-for reasoning more formally about convergence. The development of this model was inextricably tied up with the development of their understanding of convergence. It was in the process of defining sequence convergence that least-upper bounds emerged as a record-of their thinking. Through the process of reflecting on and formally defining "maximum" (which helped them solidify their definition of sequence convergence), completeness, specifically as signified with least-upper bounds, began to transition to a tool-for the students to reason more formally about convergence *and* to complete a proof of the MCT. In this way the development of their understanding of completeness supported and was supported by the development of their understanding of sequence convergence.

Future research will investigate further the nature of this interdependent concept development in this context. For example, in my first teaching experiment, the pair of students considered a sequence of nested, shrinking intervals, rather than a sequence of approximations to a root. This thinking could also be modeled by the larger concept of the completeness of the real numbers, as it is essentially the Nested Interval Property. From an instructional design perspective, it will be important to investigate the constraints and affordances of developing a proof of alternative characterizations of completeness. This research suggests a promising instructional design approach, and knowing more about how these two concepts develop together would be invaluable for these efforts.

This proposal contributes to our understanding of how students think about the completeness of the real numbers. Specifically in the context of the Intermediate Value Theorem, there is strong evidence that completeness can be a powerful model, first as a record-of student thinking about convergence, and then for use by the students as a tool-for developing more formal conceptions of sequence convergence and completeness itself. This further suggests that intuitive notions of completeness could support students in developing their understanding of sequence convergence and completeness in other contexts, as well; for example, in an IBL (Inquiry-Based Learning) or even a traditional lecture-based advanced calculus course. There is also evidence that these informal student characterizations of completeness are rooted in representations of the real numbers as a number line; the historical development of completeness lends credence to this idea (Dedekind, 1901). While there is still much to uncover about how students think about completeness and how that thinking might progress, it is evident that there are important connections between completeness and convergence in students' minds.

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Angle Measure, Quantitative Reasoning, and Instructional Coherence: The Case of David

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This paper reports findings from a study that explored the effect of a secondary mathematics teacher's level of attention to quantitative reasoning on the quality and coherence of his instruction of angle measure. I analyzed 37 videos of an experienced teacher's instruction to characterize the extent to which he attended to supporting students in reasoning quantitatively, and to examine the consequences of this attention (or lack thereof) on the quality and coherence of the meanings the teacher's instruction supported. My analysis revealed that the incoherencies in the teacher's instruction were occasioned by his inattention to quantitative reasoning. This study therefore demonstrates that when teachers do not possess a disposition to attend to quantities and their relationships, the circumstances are ripe for instruction that emphasizes inconsistent, incoherent, and sometimes incompatible, mathematical meanings.

Key words: Mathematical Knowledge for Teaching, Quantitative Reasoning, Angle Measure, Radical Constructivism, Pre-Service Teacher Preparation

Introduction

Thompson (2013) argued that the quality of mathematics instruction in the United States suffers from "a systemic, cultural inattention to mathematical meaning and coherence" (p. 57). While a number of empirical studies demonstrate that Thompson's accusation applies to a spectrum of mathematics courses and topics (e.g., Ma, 1999; Stigler & Hiebert, 1999), several researchers have noted that pre- and in-service teachers' personal understandings of trigonometry, as well as their instruction, tend to be particularly lacking in coherence and conceptual meaning (Akkoc, 2008; Moore et al., in press; Tallman, 2015; Thompson, 2008; Thompson, Carlson, & Silverman, 2007). Others have observed that trigonometry is a notoriously difficult subject for students (Moore, 2012, 2014; Weber, 2005). Identifying the factors that contribute to widespread incoherent instruction of trigonometry is therefore a priority for improving students' learning of the subject.

A growing body of research (e.g., Castillo-Garsow, 2010; Confrey & Smith, 1995; Ellis, 2007; Moore, 2012, 2014; Moore & Carlson, 2012; Oehrtman, Carlson, & Thompson, 2008; Thompson 1994) has identified *quantitative reasoning* (Smith & Thompson, 2007; Thompson, 1990, 2011) as a powerful way of thinking that supports students in constructing a meaningful understanding of a wide variety of mathematics concepts. Several researchers have noted that quantitative reasoning is especially foundational for supporting students' conceptual learning of angle measure and trigonometric functions (Hertel & Cullen, 2011; Moore, 2012, 2014; Tallman, 2015; Thompson, 2008). However, the instructional consequences of teachers' attention to quantitative reasoning (or lack thereof) are less frequently documented. By *instructional consequences*, I do not mean the understandings students construct while engaged in instructional experiences that emphasize (or fail to emphasize) quantitative reasoning. I am instead referring to the characteristics of the instruction itself, namely the nature of the understandings it promotes as well as its coherence.

Discerning how teachers' attention to quantitative reasoning affects the quality and coherence of their trigonometry instruction has the potential to inform instructional and curricular innovations that seek to improve the quality of this instruction, both in the United States and internationally. For this reason, I designed the present study to achieve such discernment. Specifically, I explored the effect of a secondary mathematics teacher's level of attention to quantitative reasoning on the coherence of the meanings of angle measure his instruction supported.

Quantitative Reasoning

I leveraged Smith and Thompson's (2007) and Thompson's (1990, 2011) explicit formalizations of *quantitative reasoning* in the design of this study and in my analysis of its data. Quantitative reasoning is a characterization of the mental actions involved in conceptualizing situations in terms of *quantities* and *quantitative relationships*. A *quantity* is an attribute, or quality, of an object that admits a measurement process (Thompson, 1990). One has conceptualized a quantity when she has identified a particular quality of an object and has in mind a process by which she might assign a numerical value to this quality in an appropriate unit (Thompson, 1994). It is important to emphasize that quantities do not reside in objects or situations, but are instead constructed in the mind of an individual perceiving and interpreting an object or situation. Quantities are therefore conceptual entities (Thompson, 2011; Thompson et al., 2014).

Conceptualizing a quantity does not require one to assign a numerical value to a particular attribute of an object. Instead, it is sufficient to have a measurement process in mind and to have conceived, either implicitly or explicitly, an appropriate unit of measure. *Quantification* refers to the mental actions involved in conceptualizing an appropriate unit of measure as well as a measurement process, and results in an understanding of "what it means to measure a quantity, what one measures to do so, and what a measure means after getting one" (Thompson, 2011, p. 38). I emphasize that one need not measure an attribute of an object to have quantified it, but must have in mind a process by which she might do so (Thompson, 1994).

The quantities one might construct upon analyzing a situation are not limited to those whose numerical values are attainable from direct measurements. Defining a process by which one might measure a quantity often involves an operation on two or more previously defined quantities. In such situations, we say that the new quantity results from a *quantitative operation*—its conception involved an operation on other quantities. Quantitative operations result in a conception of a single quantity while also defining the relationship between the quantity produced and the quantities operated upon to produce it (Thompson, 1990, p. 12). It is important to draw attention to the distinction between a quantitative operation and a numerical or arithmetic operation. Arithmetic operations are used to calculate a quantity's value whereas quantitative operations define the relationship between a new quantity and the quantities operated upon to conceive it (Thompson, 1990).

Methods

The sole participant for this study was an experienced secondary mathematics teacher, David, who taught Honors Algebra II at a large suburban high school in the Southwestern United States. David used the *Pathways Algebra II* (Carlson, O'Bryan, & Joyner, 2013) curriculum materials in this course. The *Pathways Algebra II* materials are organized into modules, each of which contains a number of investigations that students are expected to work on in small groups during class sessions.

I collected data throughout David's instruction of Module 8 of the *Pathways Algebra II* curriculum. This module focuses on a variety of ideas related to trigonometric functions including angle measure, the output quantities and graphical representations of various periodic functions, periodic function transformations, and inverse trigonometric functions. In this paper, I present only the results of my analysis of David's instruction of angle measure. I do not discuss in detail my analysis of David's instruction of other topics addressed in Module 8 of the *Pathways Algebra II* curriculum since the conclusions I drew therefrom are consistent with those I present below.

David taught two sections of Honors Algebra II every weekday during the spring semester of 2014. I video recorded both classroom sessions over a seven-and-a-half-week period of this semester, which resulted in 37 videos of David's instruction. The only class sessions that I did not videotape were those during which students were testing or those in which David was teaching content unrelated to trigonometric functions. In addition to the video recordings of David's instruction, I generated field notes during the class sessions that focused on characterizing the extent to which David supported students in reasoning quantitatively, and on documenting the mathematical meanings David's instruction promoted.

The procedures I used to analyze the video data are consistent with Strauss and Corbin's (1990) and Corbin and Strauss's (2008) grounded theory approach. I began my analysis of these videos by making an initial pass of open coding during which I identified instances that David conveyed some way of understanding. I coded these occasions for the specific category of understanding David communicated. I then made a pass of axial coding in which I verified and refined my initial codes. After having coded the 37 videos of David's classroom teaching, I produced a 57-page document entitled, "Post-Analysis Memos" wherein I summarized each coded instance of the videos and included selective transcriptions of what appeared to be particularly revealing moments of David's instruction. These memos also focused on characterizing for each coded instance the extent to which David supported his students in reasoning quantitatively. In particular, I documented the degree to which David's instruction supported students in: (1) identifying quantities, (2) attending to units of measure, (3) constructing quantitative relationships, and (4) interpreting mathematical symbols and expressions as representing the values of quantities. I carefully read through these memos and organized the coded segments of video into themes. I then examined the data within each theme and characterized the extent to which the quality and coherence of the meanings David's instruction promoted was facilitated/impeded by his level of attention to supporting students in reasoning quantitatively.

Results

Meaningfully assigning numerical values to the "openness" of an angle requires that one has identified a quantity to measure and has specified a unit with which to measure it. David's instruction was often inconsistent with regard to the quantity one measures when assigning numerical values to the "openness" of an angle. On some occasions David supported students in conceptualizing angle measure as the length of an arc the angle subtends, while on other occasions he explained that measuring an angle involves determining the fraction of the circle's circumference subtended by the angle. These meanings are not the same. Understanding the fraction of a circle's circumference that an angle subtends as a measure of subtended arc length involves conceptualizing the circle's circumference as a unit of measure for the length of the subtended arc. Specifically, one must recognize that the resulting fraction represents a multiplicative comparison of the quantity being measured (subtended arc length) and the unit of measure (circumference). For example, to say an angle subtends 59/360ths of the circumference of a circle centered at its vertex is to say that the length of the subtended arc has a measure of 59/360 in units of one circumference. The two meanings of angle measure David conveyed were distinct since he did not support students in conceptualizing the circumference of the circle centered at the angle's vertex as a unit of measure for the length of the subtended arc. Due to space limitations, the following paragraphs illustrate only two occasions in which David's instruction supported inconsistent meanings of angle measure (from my perspective). I emphasize that the events discussed here are representative of several instances from David's teaching in which he promoted discrepant meanings.

Lessons 1 and 2

David began the first lesson of Module 8 by asking a student to draw two angles on the whiteboard. The student drew one angle above the other. David then explained that the measure of the angle on top is larger than the measure of the angle on bottom because, if one were to construct two circles of equal radii respectively centered at the vertex of each angle, the angle on top would subtend an arc that is longer than the arc subtended by the angle on bottom. Immediately following this explanation, David asked the question in Line 1 of Excerpt 1.

Excerpt 1

1. *David*: When we measure an angle what are we really measuring? I mean it's not like we're measuring a length, right? How would we describe the thing that I'm measuring when I just look at these two angles? …
2. *Student*: The openness of the angle.
3. *David*: Yeah. Which is weird. How do you measure openness? … I'm not measuring length. … We have to think about what we are actually measuring.

While David previously compared the openness of two angles by attending to the respective arc lengths these angles subtend, he claimed in Lines 1 and 3 of Excerpt 1 that quantifying the openness of an angle does *not* involve measuring a length. Following the dialogue in Excerpt 1, David explained that two angles have the same measure if "the length of the [subtended] arc is the same, as long as I made the circle have the same radius and it was centered at the vertex." David therefore supported contradictory meanings of angle measure during the first lesson; he pronounced that measuring an angle is not a process of measuring a length and then proceeded to compare the openness of two angles, as well as define what it means for two angles to have the same measure, by attending to the arc lengths the two angles respectively subtend. In other words, when speaking of angle measure David did not consistently reference the same quantity being measured.[1]

A few minutes after David's remark in Line 3 of Excerpt 1, he projected the image displayed in Figure 1 on the whiteboard and asked his students the question in Line 1 of Excerpt 2.

[1] Understanding angle measure quantitatively involves conceptualizing a specific attribute to measure as well as identifying an appropriate unit with which to measure it. When measuring an angle, the attribute one measures is the length of the arc the angle subtends. This subtended arc length must be measured in units that are proportionally related to the circumference of the circle that contains the subtended arc so as to make the size of this circle inconsequential to the measure of the angle. It is important to note that this condition on the unit of measure does *not* change the quantity being measured: *subtended arc length*.

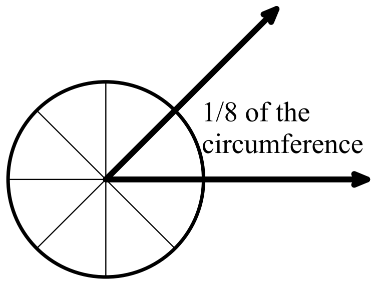

Figure 1. Angle measure as a fraction of the circle's circumference

Excerpt 2

1. *David*: So the angle subtends 1/8th of the circumference of the circle. (*Pause*) Now do units matter here? … Why do units not matter here?
2. *Student*: 'Cause you're using a proportion.
3. *David*: Why does that matter? …
4. *Student*: Because even though you're making the radius larger you're also making the whole circle larger.
5. *David*: So what happens when you do your proportion? Think in science class. (*Long pause*) 'Cause we're comparing it to our circumference, right? We're comparing arc length to circumference? What would happen to the units then? (*Long pause*) So lets just say for the sake of argument 1/8th could be a circumference of, uh, a circumference of 16, that would mean that the arc length would be two, if it's an eighth. So two inches divided by 16 inches is?
6. *Student*: One-eighth.
7. *David*: One-eighth. What are the units now? (*Long pause*) What happens when you put—and again think in terms of science class—what happens when you put two inches divided by 16 inches (*writes* "2in/16in"), your science teacher would say that's 1/8th. What are the units?
8. *Student*: It doesn't matter.
9. *David*: It does matter. What are the units?
10. *Student*: Inches.
11. *David*: Inches divided by inches give you inches?
12. *Student*: No.
13. *David*: What does it give you?
14. *Student*: One-eighth.
15. *David*: What are the units?
16. *Student*: It doesn't have units.
17. *David*: It doesn't have units? Why not?
18. *Student*: Because the inches cancel.
19. *David*: 'Cause inches cancel inches! … 'Cause I'm not just measuring arc length. What am I measuring? I'm measuring arc length and comparing it to what?
20. *Student*: Circumference.
21. *David*: Circumference! How am I comparing them?
22. *Student*: By length.
23. *David*: By length? What operation is going on here? Am I subtracting the circumference? (*Pause*) It's division! We're creating a ratio! Then do the units matter?
24. *Student*: No. …

25. *David*: What happens when we do the ratio? The units stop mattering, right? Because the units end up canceling. We're interested in the ratio. We're not interested in the units from the ratio because the units are going to reduce.

After acknowledging in Line 1 of Excerpt 2 that the angle in Figure 1 "subtends 1/8th of the circumference of the circle," David exclaimed that the measure of the angle is a value without units. In particular, David explained that if one measured the subtended arc length and circumference in inches, the ratio of these quantities is unit-less because the inches "cancel" as a result of the division. Moreover, by claiming, "I'm not just measuring arc length. What am I measuring? I'm measuring arc length and comparing it to what? … Circumference!" David did not support his students in seeing the ratio of subtended arc length to circumference as the length of the subtended arc measured *in units of the circumference*. Generally speaking, David's statements and questions in Excerpt 2 did not provide students with an opportunity to interpret the ratio of subtended arc length to circumference as a quantitative operation but rather as an arithmetic operation; that is, he did not communicate the division of these quantities as a measure of subtended arc length in units of circumference but simply as the ratio of two lengths. Such an emphasis is necessary if one is to support students in conceptualizing angle measure quantitatively (i.e., as a measure of some attribute in some unit). Therefore, while David's instruction during the first lesson overtly emphasized angle measure as a fraction of the circle's circumference subtended by the angle—and implicitly conveyed angle measure as the length of the subtended arc—he did not encourage students to see the former meaning as an application of the latter by failing to support them in conceptualizing the ratio of subtended arc length to circumference as a quantity that represents the length of the subtended arc measured in units of the circumference. In fact, David suggested that these meanings were incompatible by continually asserting that the process of measuring an angle is not one of measuring length.

David's instruction of angle measure did not consistently support students in conceptualizing the quantity one measures when assigning numerical values to the openness of an angle. On some occasions David conveyed that the process of measuring an angle is one of determining the length of the arc an angle subtends, while on other occasions he explained that measuring an angle involves determining the fraction of a circle's circumference subtended by the angle. While in certain circumstances one of these ways of understanding might be more natural than the other, David did not support his students in making the connection between these ways of understanding. In other words, David did not provide opportunities for students to see these meanings as two instantiations of the same quantification process (measuring the length of the subtended arc in units proportional to the circumference of the circle containing the subtended arc) because he did not support students in conceptualizing the circumference of the circle centered at the angle's vertex as a unit of measure for the length of the subtended arc.

Summary of Lessons 1-9

While David's instruction during the first lesson emphasized the understanding of angle measure as the fraction of the circle's circumference subtended by the angle, he also discussed angle measure as the length of the arc an angle subtends, but then suggested that these two meanings are incompatible by explaining that the process of measuring an angle is not one of measuring a length. The majority of David's instruction during Lessons 2 and 3 emphasized angle measure as the fraction of the circle's circumference subtended by the angle. Specifically, David's teaching promoted the mental imagery of imagining the circumference of the circle centered at the vertex of the angle being split into a number of equal pieces, and then attending to the ratio of the number of these pieces subtended by the angle to the number of these pieces contained in the circumference of the circle. However, on several occasions during Lessons 4-9, David discussed angle measure as the length of the subtended arc measured in units of the radius of the circle containing the subtended arc. Moreover, David's instruction throughout Lessons 4-9 was often contradictory in that he encouraged meanings that on other occasions he did not accept and devalued meanings that he elsewhere endorsed.

Discussion

The results of this study demonstrate that a secondary mathematics teacher's (David) inattention to quantitative reasoning contributed to his conveying incoherent meanings of angle measure and trigonometric functions. In particular, by not maintaining a consistent emphasis on supporting students in: (1) identifying quantities, (2) attending to units of measure, (3) constructing quantitative relationships, (4) interpreting mathematical symbols and expressions as representing the values of quantities, and (5) performing quantitative—rather than arithmetic—operations, the ways of understanding David's instruction promoted were often inconsistent and even incompatible. Moreover, the meanings David supported varied by context because they were not consistently governed by, or in the service of promoting, a particular way reasoning. These findings suggest implications for secondary mathematics teacher preparation.

While the well-documented affordances of quantitative reasoning on students' conceptual mathematics learning are enough to justify its emphasis in pre- and in-service teacher education, the results of the present study provide an additional incentive for mathematics educators to engage teachers in experiences that advance their ability to reason quantitatively, as well as support them in leveraging quantitative reasoning in their teaching of specific concepts. Specifically, the results of this study suggest that teacher educators should design instructional experiences that allow pre- and in-service teachers to develop the disposition to support students' identification of quantities and quantitative relationships, and their interpretation of mathematical symbols and expressions as representations of a measure of an attribute of some object in some unit, particularly in the difficult context of trigonometry. Equipped with such a disposition, teachers' instructional actions may consistently be in the service of leveraging a powerful way of reasoning to support students' learning of various mathematical ideas while simultaneously promoting the way of reasoning itself. The results of the present study demonstrate that when teachers do not possess a disposition to attend to quantitative reasoning, the circumstances are ripe for instruction that emphasizes inconsistent, incoherent, and sometimes incompatible, mathematical meanings.

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Virtual Manipulatives, Vertical Number Lines, and Taylor Series Convergence: The Case of Cody

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Evidence from recent Taylor series studies suggests that well-designed virtual manipulatives can support calculus students in developing an understanding of Taylor series convergence consistent with the formal pointwise convergence definition. In particular, virtual manipulatives depicting convergence along vertical number lines (VNLs) provide graphical representations of quantities necessary for pointwise convergence. We detail one student's reasoning about Taylor series convergence before and after a VNL was revealed in a Taylor series graph. Prior to the VNL the student had produced very accurate Taylor polynomial graphs based on visually perceptual clues but had omitted notions of pointwise convergence. After a VNL was revealed, the student's reasoning now included quantities along the vertical as he responded to approximation tasks. We believe that such reasoning can later support the student in developing an understanding of pointwise convergence.

Keywords: Taylor series, virtual manipulatives, quantitative reasoning, calculus, approximation

A recent survey collected from 225 students suggests that well-designed virtual manipulatives (VMs) can help students in developing an understanding of Taylor series convergence consistent with notions from pointwise convergence (Martin, Thomas, & Oehrtman, 2016). In particular, survey results found that students from classes using VMs were significantly more likely than students from classes not using VMs to draw Taylor polynomial graphs with the correct shape, observing general trends indicative of Taylor polynomials becoming better approximations for increasing degrees and Taylor polynomials drawn tangent to the function being approximated. Furthermore, this study observed gains in the number of students mentioning convergent behavior for particular values of the independent variable, x. It is the attention to convergence for particular values of x that is especially important when considering the formal pointwise convergence definition. Graphically, Taylor series convergence for particular values of x can be conceived along vertical number lines (VNLs) to indicate attributes, such as "estimates" and "error," of pointwise convergence. In particular, these attributes are quantities (Thompson, 1994, 2011) of a VNL that the student can anticipate as having values (magnitudes) that can be compared using approximation and error analysis. We ask the question, just because students from VM classes were drawing more accurate graphs depicting Taylor series convergence, does this mean that they were reasoning about Taylor series convergence in a way that was consistent with pointwise convergence, such as reasoning about needed quantities?

In this paper, we expand upon the analysis of the survey and detail one student's reasoning as he interacted with multiple images of Taylor series convergence, including images containing a VNL and a VM. While not necessarily representative of all calculus students, we found that that this student demonstrated similar patterns of understanding displayed by many students from our small sample of interviewed students. His story provides an example of a possible progression of reasoning as he interacted with a VNL and subsequently identified quantities related to pointwise convergence in the context of Taylor series convergence. We use this student to detail possible ways that VNLs might support students in conceiving of and relating relevant quantities related to pointwise convergence and identify how a VM might further support such reasoning.

Taylor Series Literature

A Taylor series for an infinitely differentiable function, f, centered at $x = a$ is given by $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$. If f is real analytic, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ for all x within the interval of convergence. Taylor series convergence is an example of pointwise convergence where a Taylor series converges to f over an interval of convergence I iff $\forall x \in I$, $\forall \varepsilon > 0$, there is some natural number $N > 0$ such that for all natural numbers $n > N, |f(x) - T_n(x)| < \varepsilon$. While we do not see a need for a calculus student to be able to quote the formal definition of pointwise convergence, our ideal first-year calculus student is a student who can meaningfully discusses Taylor series convergence in a way that exemplifies both symbolically and graphically the quantities and relationships seen within the formal pointwise convergence definition.

Studies investigating students' reasoning about Taylor series repeatedly document student struggles to comprehend this topic (e.g. Champney & Kuo, 2012; Kidron & Zehava, 2002; Kung & Speer, 2010; Martin, 2013; Martin & Oehrtman, 2010). Lack of attending to particular values of the independent variable distinguishes student understanding from expert understanding (Martin, 2013). In addition, students tend to focus on isolated facts and formulaic procedure when reasoning about Taylor series (Martin, 2013). When viewing graphical representations, students often attend to general behavior indicated by the most salient of visually perceptual clues lacking reference to quantities necessary for pointwise convergence (Habre, 2009; Kidron, 2004; Kidron & Zehavi, 2002; Martin, 2013). For example, students might describe Taylor polynomials as "getting closer" to the approximated function without alluding to any meaning of "close" as a measurable distance. In some cases, these perceived attributes are inconsistent with pointwise convergence. For example, while viewing Taylor series graphs, students have been observed noting that a Taylor polynomial and the approximated function were identical over an interval since the two graphs were visually "touching," Oehrtman (2009) referred to this perception as a collapse of error. Instead, a model of limit where Taylor polynomials evaluated at particular values of the independent variable, $T_n(x)$, are viewed as approximations to $f(x)$ with an associated error, $|f(x) - T_n(x)|$, and bound on the error can focus student reasoning on those quantities related to pointwise convergence (Martin & Oehrtman, 2010; Martin, Oehrtman, Roh, Swinyard, & Hart-Weber, 2011; Oehrtman, 2009).

Virtual Manipulatives (VMs) and Taylor Series

Creating Taylor series graphs can be especially time consuming during class. We believe that interactive computer representations, which we call VMs, show potential for not only supporting first-year calculus students in making sense of Taylor series but VMs might also alleviate some of the time constraints felt by instructors attempting to provide their students a much needed graphical understanding of Taylor series.

VMs have already been shown to provide students with a tool to explore and develop mathematical understanding (Moyer-Packingham & Westenskow, 2013). In particular, it has already been demonstrated that VMs can reinforce quantitative relationships that students can recall months later (Cory & Garofalo, 2011). Yet, as already noted concerning graphical images of Taylor series, VMs not using VNLs seem to have little impact on and may reinforce students' persistence in using visually perceptual clues ungrounded in quantitative reasoning (Habre, 2009; Kidron, 2004; Kidron & Zehavi, 2002). We hypothesize that well-designed VMs with VNLs coordinated with approximation tasks can move students away from merely attending to perceptual clues and include reasoning about quantities along the vertical.

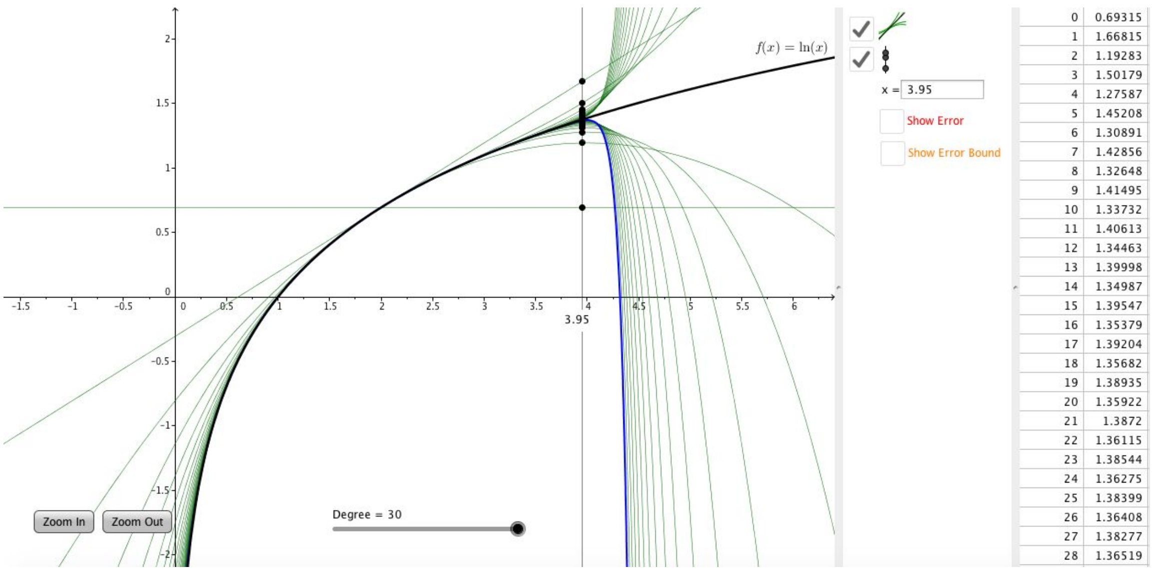

Figure 1. Screenshot of Taylor series VM

Since very little has been done to investigate the effects of VMs on student learning beyond 6th grade (Moyer-Packingham and Westenskow, 2013), research into video lectures provides additional detail that can inform on how to design VMs well. For example, irrelevant details should be removed (Harp & Mayer; 1998; Mayer, Griffith, Jurkowitz, & Rothman, 2008), conceptual difficulties can be explicitly addressed (Boesdorfer, Lorsbach, & Morey, 2011; Muller, 2008; Muller, Bewes, Sharma, & Reimann, 2007), and a series of still frames are preferable over animations in certain situations (Mayer, Hegarty, Mayer, & Campbell, 2005). The VM's provides the student control to show and hide relevant and irrelevant details, to progress through "still frames," and the zoom ability of the VMs was included to explicitly combat a collapse notion of error.

Methods

Cody was one of 225 students who participated in a Taylor series survey (Martin, Thomas, & Oehrtman, 2016), and one of five students who participated in two task-based individual interviews lasting around 60 minutes each. Cody was from a class using VMs to investigate Taylor series convergence during group (referred to as labs) and individual homework approximation exercices (Oehrtman, 2008). Our analysis revealed that in many cases, Cody's reasoning typified the group of interviewed students. For the purpose of this report, we focus exclusively on Cody's evolution of ideas as he progressed through the interview tasks (Figure 2) during the first interview.

1) Using the graph of sin(x) below, on the same axes sketch three different Taylor polynomials for sine. [included a graph of y=sin(x) over the interval [-4π,4π]]
2) Using the graph of sin(x) below, on the same axes sketch the Taylor series for sine. [included a graph of y=sin(x) over the interval [-4π,4π]]
3) The following graph represents a function $f(x)$. [see bottom graph in Figure 4] Graph the first, second, and third degree Taylor polynomials of $f(x)$ centered at x=2. If n is the degree of the polynomial you are graphing, label it $T_n(x)$. (E.g. $T_1(x)$ is the degree one polynomial).
4) What might $T_{1000}(x)$ look like? Add this to the graph above and label correctly. [used same graph as Task 3]
5) What if the series was centered at x=0? On the graph below, graph three Taylor polynomials for the Taylor series centered at x=0 and label correctly. [included a reproduction of the graph seen in Task 3]
6) Explain in detail what it means for a Taylor series to converge.
7) List all the ways in which Taylor series convergence is related to sequence convergence and series convergence if at all. Make sure your explanations reference a. Formulas when appropriate and b. Includes a graphical explanation that highlights sequences and/or series on your graph. (That is, add to the graph to appropriately highlight sequences and/or series convergence as it relates to Taylor series convergence.)
8) [Given graphs including a sequence of Taylor polynomials] The Taylor series for $g(x) = ln\, x$ centered at x=2 is given by $ln\ 2 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot 2^k}(x-2)^k = ln\ 2 + \frac{1}{1\cdot 2}(x-2) - \frac{1}{2\cdot 4}(x-2)^2 + \frac{1}{3\cdot 8}(x-2)^3 \mp \cdots$ and the degree n Taylor polynomial is given by $T_n(x) = ln\ 2 + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k \cdot 2^k}(x-2)^k = ln\ 2 + \frac{1}{1\cdot 2}(x-2) - \frac{1}{2\cdot 4}(x-2)^2 \pm \cdots + \frac{(-1)^{n+1}}{n \cdot 2^n}(x-2)^n$ a. The radius of convergence of this Taylor series is (0,4]. Speculate on why it diverges beyond this radius. b. Use the graph to explain $T_n(3.98)$. c. Use the graph to approximate ln (3.98).

Figure 2. Sample interview tasks.

Question 1.	What are you approximating?
Question 2.	What are your approximations? Which are overestimates? Which are underestimates?
Question 3.	What is the error for any one approximation?
Question 4.	What is error bound?
Question 5.	Can you find an approximation accurate to within 0.04 of what you are approximating?

Figure 3. Common approximation question repeated throughout images.

Prior to Task 8, Cody only produced graphs, but during Task 8, Cody was provided graphs as he advanced through static, animated, and VM depictions of Taylor series convergence while responding to approximation questions adopted from Oehrtman (2008) (Figure 3). In addition, Cody first viewed three static images, starting with a standard Taylor series graph and progressed through two more images with VNLs (one without error and one image with error depicted). Likewise, the VM did not initially show a VNL but Cody had the option to make a VNL visible. While viewing each depiction, Cody was repeatedly asked to "show" the interviewer on the image his responses to the approximation questions. The data was coded for indicators of quantitative reasoning and changes in how Cody was reasoning were tracked relative to the type of depiction (e.g., VNL or no VNL) he was viewing. In addition, commonalities and differences in reasoning employed by Cody between the VNL depicted on the sequence convergence VM and the any VNL depicted in Taylor series convergence was noted.

Results

Thinking Prior to VNL and VM

When first encountering the Taylor series' tasks, Cody's actions were dominated by often unfruitful attempts to remember facts and relationships from class. His remembrances were frequently peppered with comments expressing uncertainty, such "I guess" or "I'm not really sure." When graphing Taylor polynomials, Cody consistently produced linear, quadratic and polynomial shapes (Figure 4). He noted general trends, such as even and odd degree Taylor polynomials alternate between being "above" and "below" the the approximated function or that as "as you go further on in the exponents [while making a rightward chopping motion gesture with right hand] you start to, you add a curve each time [...]"

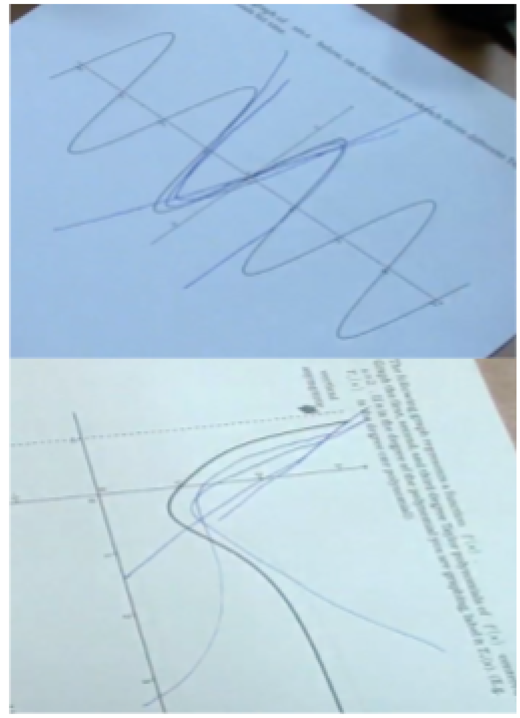

Figure 4. Images of Task 1 (top) and Task 3 (bottom)

For Cody, convergence was achieved by Taylor polynomials "approaching" the approximated function via two methods: a vertical transformation and an increasing interval over which polynomials become more accurate. Thus, small degree Taylor polynomials need not be tangent to the approximated function (note the Task 3 Image in Figure 4) but as degree increases, a vertical transformation causes the the larger degree polynomials to better approximate the given function. For example, for T_{1000} Cody drew a polynomial that traced the approximated function over a large interval. It should be noted that throughout the early interview tasks, Cody did not attend to convergent behavior for particular values of x. When explicitly asked to explain what it meant for a Taylor series to converge, he expressed confusion about the object to which the Taylor series converges, "Taylor series converging is, I guess, when it's approaching one certain...not a point...but it's approaching a number." Although he seemed to indicate convergence to a point, at no moment were his references explicitly coordinated with particular values for x, and his later choice to use "line" as a reference to the function instead of "point" or "number" suggests that his uses of "point" and "number" were purely metonymic. "As you go further and further, you go up to infinite number, or infinite powers you would start approaching that number that you're ideally trying to estimate, or the line that you're trying to estimate." Without any vertical number line visible, he continued to describe Taylor series convergence using general graphical trends.

Introducing the VNL in a Static Image

Once a VNL was visible, Cody's reasoning became structured by the vertical as evidenced by his gestures becoming almost entirely restricted to either points or quantities contained within the VNL. When presented with the first static image displaying a VNL, Cody immediately identified the VNL as providing vertical distances from the graph of ln(x). He then began to indicate points as answers to previous tasks. For example, “I guess this is, this point right here [pointing to where ln(x) and the vertical number line intersected] is ln(3.98). I guess we are approximating just that point.” When asked how to determine numerical values for the approximations, he connected the formula to the graphical representation by first gesturing vertically between the appropriate points on the VNL and saying “as you have more of these terms involved [points to the formula] it becomes closer and closer, you’d eventually will get pretty damn close, but I can’t tell exactly what the point is here [gesturing horizontally from the point (2, ln 2) to the location of ln 2 on the y-axis].” He continued with his horizontal gestures as he indicated approximations as points on the vertical number line paired with values on the y-axis.

When prompted to discuss error, he immediately stated that error was the “distance between the approximation and the actual” and then included a graphical quantity by pointing at the point (3.98, ln(3.98)) on the graph and described error as the distance between that point and “any of these points,” gesturing towards all the other points on the VNL. When asked whether it would be possible to approximate the value of ln(3.98) to within 0.04, Cody responded that one could “keep adding” terms, gesturing to the right to indicate a formula growing as if he were appending terms, causing the Taylor polynomials to “get closer and closer” while gesturing vertically over the vertical number line. While tracing a curve, Cody said “I guess, for it to converge it would have to approach the equation you are trying to estimate or for the point to approach, [now focusing on the vertical number line] the point you’re, so yeah, to approach the point you’re trying to estimate. So as these get closer and closer, [waving over the approximation points on the VNL] and that would be the convergence for the series as it opposed to that [referring to his previous answers on Tasks 5 and 6].”

Introducing the VM

Once Cody was interacting with the VM, he began exploring by clicking through the check boxes and then changing the degree of the Taylor Polynomial. He changed the x-value for the VNL to 3.98 and changed the value for the error bound (but not initially to 0.04). He later zoomed in on the graph at $x = 3.98$, and then changed the value for the error bound again, this time to 0.2. He expressed concern about the error bound, saying that he didn’t “know how you determine...the length of the orange line.” After more investigation, he became more comfortable with the orange line, noting “that it helps gage how far apart some of these are,” essentially describing an error bound despite his discomfort with the notion of an error bound as anything other than a difference between an overestimate and an underestimate. When asked whether he could find an approximation to within 0.04, without hesitation he changed the value in the error bound to 0.04, and pointed to the points within the boundaries set by the value of the error bound.

Earlier in the interview while looking at a static image containing a VNL, Cody’s language seemed indicative of a collapse of error, but when explicitly asked by the interviewer if the Taylor polynomial ever became the approximated function, Cody clearly stated that approximations “never fully reach” the exact value and that nonzero error remained. Later, Cody used Geogebra ability to zoom with images of large degree Taylor polynomials to visually justify that error remained.

Revision of Definition of Convergence

On his own initiative, Cody revisited his explanation to what it meant for a Taylor series to converge, Task 6. Cody now attempted to reconcile function convergence with his emerging notion of VNL convergence.

> "I guess, for it [tracing over $y = \ln(x)$] to converge, it would have to approach the equation you are trying to estimate, or for the point to approach [focusing on the VNL] [...], to approach the point you're trying to estimate. So as these get closer and closer [waving over the approximation points on the VNL], and that would be the convergence for the series, as it opposed to that [Taylor series]. [Cody added a VNL to a Taylor series graph he produced when first responding to Task 6.]"

Instead of repeating prior explanations, Cody indicated function convergence as approaching "the point you're trying to estimate" and made a change from a horizontal gesture of tracing over $y = \ln(x)$ to a vertical gesture for convergence at a point by waving over approximation points on the VNL. In addition, his use of "the point you're trying to estimate" might be indicative of a recognition that different values of the independent variable, x, would influence convergence. Furthermore, Cody related convergence to a point with merely a "series" and not a Taylor series, indicating that he was conceiving of graphical, and not merely symbolic, connections between series and Taylor series.

Discussion

After seeing the VNL, Cody framed his thinking in terms of the visible "dots" on the VNL. Even when his gestures included horizontal movements to the y-axis, his gestures began with the approximating "dots." While Cody was indicating the corresponding y-values for these "dots," there was no evidence that he was thinking of these locations as quantities in the sense of being a vertical distance from the x-axis. Vertical distances appeared in the context of error and error bound as Cody described "distances" from the graph of $y = \ln(x)$ to the "dots" on the VNL. And although Cody had not indicated these "dots" as vertical distances, the "dots" served a references for establishing quantities.

In addition to using the zoom feature to combat collapse, the impact of the VM on Cody's conception of convergence became clear in the second interview when Cody was asked to find an approximation to within 0.04 of the actual value while viewing a static image. Cody responded that he could, if he had a "slider." While his description of Taylor series convergence had clearly improved, in that moment, the VM was a part of his mental image of convergence.

Cody's experiences with the VNL and VM provide complementary evidence to that presented previously by Martin, Thomas, & Oehrtman (2016). While this work indicated that students in classrooms with VMs were more likely to answer questions about Taylor series convergence correctly, Cody's experiences give us some notions of how these VMs may be transforming the students' conceptions of convergence. While Cody's reasoning has room to grow, he provides insight into the kinds of positive learning experiences that can be fostered using these VMs.

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Developing Students' Reasoning about the Derivative of Complex-Valued Functions with the Aid of *Geometer's Sketchpad (GSP)*

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In this paper, I share results of a case study describing the development of two undergraduate students' geometric reasoning about the derivative of a complex-valued function with the aid of Geometer's Sketchpad (GSP). My participants initially had difficulty reasoning about the derivative as a rotation and dilation. Without the aid of GSP, they could describe the rotation and dilation aspect of the derivative for linear complex-valued functions, but were unable to generalize this to non-linear complex-valued functions. Participants' use of GSP, speech, and gesture assisted with discovering function behavior, generalizing how the derivative describes the rotation and dilation of an image with respect to its pre-image for non-linear complex-valued functions, and recognizing that the derivative is a local property.

Keywords: Amplitwist, Complex-valued function, Derivative, Dynamic Geometric Environments (DGEs), Gesture

Introduction

In the calculus reform era, a main goal was to develop students' conceptual understanding of calculus by integrating algebraic and geometric reasoning (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Lauten, Graham, & Ferrini-Mundy, 1994; Meel, 1998). While similar research has been conducted in other mathematical content domains, there is less research in complex analysis, which allows potential for geometric reasoning. The purpose of this research was thus to address two research questions: What is the nature of students' reasoning about the derivative of complex-valued functions, and what role does *GSP* play in developing students' geometric reasoning about the derivative of complex-valued functions (i.e., the *amplitwist*)?
For my work, I adopted the National Council of Teachers of Mathematics' (NCTM, 2009) *reasoning* definition, which is "the process of drawing conclusions on the basis of evidence or stated assumptions" (p. 4), and is categorized as algebraic or geometric. *Algebraic reasoning* involves symbolic manipulation, and *geometric reasoning* involves spatial elements. I also refer to inscriptions, which Roth and McGinn (1998) define as "signs that are materially embodied … and because of their material embodiment, inscriptions (in contrast to mental representations) are publicly and directly available, so that they are primarily social objects" (p. 37).

Literature Review

In this section, I summarize literature on students' reasoning in the realm of complex analysis, and the benefits of DGEs on students' reasoning. As these domains draw on gesture as evidence for reasoning, I conclude this section by synthesizing related gesture research.

The Teaching and Learning of Complex Numbers

There has been a recent increase in educational studies focusing on complex numbers (Danenhower, 2006; Harel, 2013; Karakok, Soto-Johnson, & Anderson-Dyben, 2014;

Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012; Panaoura, Elia, Gagatsis, & Giatilis; 2006; Soto-Johnson, 2014; Soto-Johnson & Troup, 2014). The results of these studies show that participants struggle to reason geometrically about complex numbers. In brief, Panaoura et al. found that high school students had difficulty transitioning between algebraic and geometric reasoning, while Danenhower showed undergraduate mathematics majors could not convert $\frac{a+ib}{c+id}$ to exponential or Cartesian form. Applying Sfard's (1991) duality principle. Karakok et al. saw that teachers displayed a process/object duality of the Cartesian form, but only an operational level of the exponential form. Harel (2013) noted that teachers were unable to attach a geometric meaning to the addition and multiplication complex numbers.
However, reasoning via embodied cognition may help students reason geometrically about complex analysis. Nemirovsky et al. (2012) observed that through the usage of a "floor tile" representation of the complex plane and stick-on dots and string to represent complex numbers, pre-service teachers discovered multiplying by i corresponds to a rigid 90° rotation. Soto-Johnson and Troup (2014) also found that undergraduates integrated their algebraic and geometric reasoning while drawing a representative diagram.

Dynamic Geometric Environments

In contrast with complex analysis research, research on educational technology is abundant. Some suggest that technology can cause harm by promoting overgeneralizations (Clements & Battista, 1992; Olive, 2000) or an over-reliance on technology (Salomon, 1990). Alternatively, technology can help students and teachers refine mathematical ideas (Arcavi & Hadas, 2000; Barrera-Mora & Reyes-Rodríguez, 2013; Heid & Blume, 2008; Hollebrands, 2007; Jones, 2000; Olive, 2000; Tabaghi & Sinclair, 2013; Vitale, Swart, & Black, 2014) without relying on mathematical authorities such as teachers or textbooks. DGEs such as *Geogebra* have been shown to help develop concepts related to real-valued differentiation (Hohenwarter, Hohenwarter, Kreis, and Lavicza, 2008; Ndlovu, Wessels, and De Villiers, 2010). Salomon (1990) suggested that DGEs may help students by providing multiple representations of mathematical objects. Hollebrands (2007) and Olive (2000) contend that *GSP* makes abstract ideas appear more concrete, which could help students ground reasoning in the physical environment. For example, Tabaghi and Sinclair (2013) found that while interacting with an eigenvector sketch in *Sketchpad*, students reasoned via gesture.

Gesture

Many (Alibali & Nathan, 2012; Goldin-Meadow, 2003; Keene, Rasmussen, & Stephan, 2012; Roth, 2001) believe "gestures can be used as a window into what students in a classroom are thinking" (Keene et. al., p.367). Others suggest diagrams and gestures inform each other (Châtelet, 2000; Chen and Herbst, 2013). For example, a vector could express the result of the multiplication of two complex numbers and a flick of the wrist could convey the associated rotation and dilation (Soto-Johnson & Troup, 2014). Some also state gesture and speech form a single, integrated system to support both visual and verbal content (Alibali & Nathan, 2012; Goldin-Meadow, 2003; Keene et al., 2012; Roth, 2001). For example, Goldin-Meadows found gesture aids memory and the ability to describe it, while restricting gesture hampers this ability. Some literature suggests gesture can transform over time from primarily representative to primarily pointing as students become more familiar with mathematical procedures (Alibali & DiRusso, 1999; Soto-Johnson & Troup, 2014; Garcia & Engelke, 2012; Marrongelle, 2007; Soto-Johnson and Troup, 2014). Garcia and Engelke (2012) also noticed that their undergraduate participants gestured more frequently when they were stuck on a task. Vitale et al.'s (2014) participants initially gestured to remind themselves of a geometric concept, but transitioned into using gesture as a validation tool. The students additionally interacted with virtual representations of spatial gestures, which resembled their "real" gestures.

Given that gesture may result from explorations with technology (Tabaghi & Sinclair, 2013) and that gesture can arise organically, it was critical to adopt a framework that would be sensitive to this phenomenon. Thus, I employed embodied cognition as my theoretical perspective, which allowed me to interpret my participants' reasoning via gesture and actions taken within a DGE. In the next section I summarize my interpretation of embodied cognition.

Theoretical Perspective

Embodied cognition is mainly concerned with the relationship between reasoning and actions within the physical environment (Anderson, 2003), though the nature of this relationship differs between researchers. While some suggest embodied cognition is a tool for mental cognition, (Alibali & Nathan, 2012; Lakoff & Nuñez, 2000; Wilson, 2002), others dispense with mental models entirely by equating reasoning with body-based activity, such as is found in Nemirovsky's (2012) fluid "realm of possibilities." Under the second view, bodily experience is itself an inextricable part of the learning process. So, one view is concerned with mental models influenced by embodied action, while the other focuses on personal experience. These views do not seem entirely incompatible; an impersonal description of pain as "the firing of C-fibers" (Nemirovsky et al., 2012, p. 2) neither invalidates nor takes precedence over someone's personal experience with pain. Similarly, postulations about the cognitive way the mind responds to bodily actions may not be contradictory to inferences made about a learner's experience. I align more closely with this second viewpoint. Similar to Soto-Johnson and Troup (2014c), I interpret embodied cognition to include bodily actions taken within the physical environment. I believe perceptuo-motor activity (including DGE activities) can influence reasoning and that reasoning can influence bodily actions. Rather than postulate any cognitive mechanisms, I simply suggest a relationship between how my participants utilized DGEs, gestures, and inscriptions. Particularly, participants reasoned with *GSP* via construction and manipulation of geometric inscriptions and via the production of virtual gesture through mouse movement. Thus, I take this reasoning and participants' usage of *GSP*, gestures, and inscriptions to be inseparable.

Methods

This report is a summary of my Ph.D. thesis. Thus, in this section, I describe some of the aspects of research performed, including participant selection, task design, and data analysis.

Setting and Participants

To help with participant selection and triangulation, I attended complex analysis class sessions related to the derivative. I video-recorded these classes and took notes paying particular attention to gestures employed. Four students agreed to participate in my study, who I refer to as Christine, Zane, Edward, and Melody. I conducted two sets of interviews, one with Christine and Zane, and another with Melody and Edward. Each group worked on tasks over four non-consecutive days. Each interview lasted two hours.

Concept Analysis of *Amplitwist*

Within the context of real-valued functions, the derivative function has a well-recognized geometric interpretation as the slope of a tangent line. However, as the graph of a complex-valued function is four-dimensional, the generalization of this concept is not straightforward. To help overcome this problem, one can represent the graph of a complex-valued function as two sets of axes: one for domain and one for range. Thus, graphing a complex-valued function can be represented as a transformation $f: C \rightarrow C$. In describing the derivative of a complex-valued function geometrically, Needham (1997) considers how a complex-valued function maps an extremely small circle centered about the point z in the domain. "The length of $f'(z)$ must be the magnification factor, and the argument of $f'(z)$ must be the angle of rotation," (p. 196) a concept that Needham refers to as an *amplitwist*. Needham further elaborates that the derivative of a complex-valued function provides a linearization that locally approximates the function, as "'expand and rotate' is precisely what multiplication by a complex number means" (p. 196).

The Tasks

The overall goal of the tasks I developed for my interview sequences was to encourage reasoning about the derivative of a complex-valued function as described by Needham's (1997) concept of an amplitwist. I utilized tasks developed from a previously conducted pilot study.

In Tasks 1 and 2, students followed instructions on a lab worksheet to construct the function $f(z) = z^2$ and $f(z) = e^z$ with the aid of *GSP* and predict how the function maps points, lines, circles, and the complex plane (see Appendix B). The goals of this task were for participants to establish proficiency with *GSP* and determine the mapping of circles under a complex-valued function. For Task 3 I prepared the linear complex-valued function $f(z) = (3 + 2i)z$ with a complex-valued derivative in order to reduce the likelihood that students reasoned that the real part of the derivative is a dilation factor and the imaginary part of the derivative is a rotation factor. I asked participants to describe their geometric reasoning about the derivative of a complex-valued function both with and without *GSP*, and to use $f(z) = (3 + 2i)z$ to demonstrate. I re-introduced *GSP* later to allow them to test their conjectures and continue to explore their reasoning. The purpose of this task was to help participants relate the magnitude and argument of the derivative to the way the linear function dilates and rotates a circle. In Task 4, I asked participants to generalize their reasoning from Task 3 to the functions from Tasks 1 and 2 as well as the functions $f(z) = z^2$, $f(z) = e^z$, and $f(z) = \frac{1}{z}$. The goal of this exercise was to support students' efforts to generalize the geometric reasoning they developed in Task 3 to the general case. Finally, in Task 5, I asked students to determine the value of a derivative at a particular point for the rational transformation $f(z) = \frac{(2z+1)}{(z+i)(1-z)}$. I gave them only geometric information about a rational transformation, and asked them to use this information to reconstruct the algebraic formula. The goal of this task was to encourage students to develop reasoning about the derivative as a local property and to develop reasoning about points of non-differentiability as they relate to the amplitwist concept.

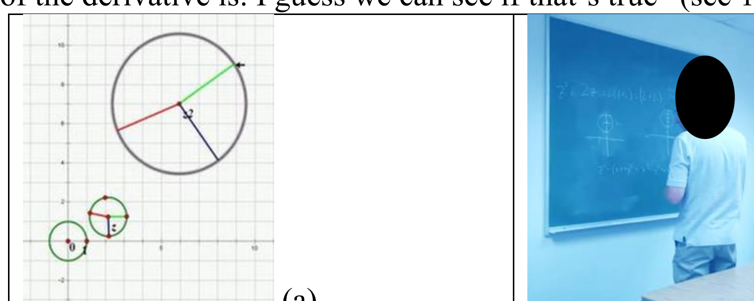
Data Collection and Analysis

To obtain data, I video-recorded class sessions the instructor deemed relevant. I recorded all interviews with both a camera to capture gesture and screen-capture software to record actions taken with *GSP*. My analysis began by watching the videos in conjunction with the screen-captured *GSP* recordings to determine where the participants appeared to be making progress

toward a conception of the derivative of a complex-valued function as a local linearization. I transcribed all recorded gesture, speech, and usage of inscriptions, after which I coded lines as algebraic or geometric based on this data. Then, I wrote summaries of each day for each set of participants wherein I detailed the progression of selected events.
Finally, I performed both a cross-case and within-case (Merriam, 2009; Patton, 2001) analysis on the written summaries, referencing both the Excel spreadsheet and the actual raw data as needed.

Results and Discussion

My findings suggest that *GSP* helped my participants generalize how the derivative describes the rotation and dilation of an image with respect to its pre-image for non-linear complex-valued functions. Furthermore, asking participants to construct an algebraic formula from geometric data served as a reminder that the derivative is a local property.
For the first two tasks, I did not ask the participants anything about the derivative. Rather, I asked them to construct the function $f(z) = z^2$ for Task 1, and $f(z) = e^z$. In the process, I answered their questions about how to use *GSP*, and asked them questions about how the function mapped various circles. During this questioning, both groups noted that circles mapped to other roughly circular shapes, and recalled that t points "rotate" and "dilate" or "stretch" when multiplied by a complex-valued function. They did not appear to say anything about rotation and dilation in conjunction with the derivative of a complex-valued function, however. Rather, Christine states simply, "I don't know like what slope means in complex world,"
When participants reasoned about the geometry of a linear complex-valued function $f(z) = (3 + 2i)z$ in Task 3, they verbalized that the function maps a circle to another circle which is rotated by the argument of $3 + 2i$) and dilated by $|3 + 2i|$ with respect to the original circle. While Melody and Edward made this observation in Task 3, Christine and Zane did not verbalize this same point until Task 4, although they did mention that the "stretch" and "rotation" factors do not change (see **Error! Reference source not found.**(a)). Melody and Edward observed, "the derivative is rotating this consistently wherever it is and then it's expanding it out whatever the length of the derivative is. I guess we can see if that's true" (see 1(b)).



(a) (b)

Figure 1: $\boldsymbol{f(z) = (3 + 2i)z}$ transforms green input circle and spokes to blue output circle and spokes of corresponding color (a) and Edward transforms a circle under $\boldsymbol{z \to 2z}$ (b).

Later, during Task 4, Melody adds gesture to their description:

Melody: Like the center (points at $(2,0)$) would be at 2? So the center of the circle doesn't necessarily depend on the derivative. Like where the output one is located doesn't depend on the derivative (curls fingers slightly into claw (see Figure 2(a)), beats toward screen while moving arm counterclockwise in an upper circular arc) The output one is just the size (hand makes claw shape, extends fingers outward and back in) of it and (twists hand first clockwise like a doorknob (see Figure 2(b)) then back counterclockwise) the rotation not the, like the location would depend on the z squared.

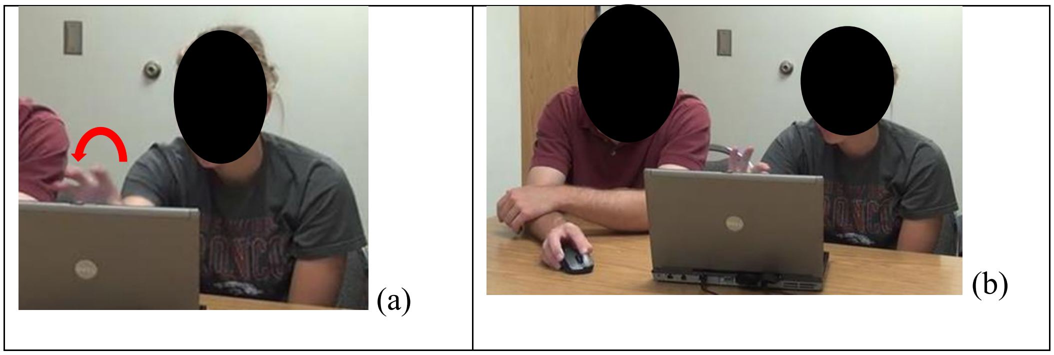


Figure 2: Melody produces a “claw-like” gesture for dilation (a) and produces a “doorknob” gesture for rotation (b).

One of the most interesting findings resulted from Task 5, wherein participants utilized geometric data to construct an algebraic formula for a rational function. In this task, Edward and Melody explicitly connected strange output behavior to non-differentiable points, stating, “that can’t be differentiable there…It’s weird” (see Figure 3). At this point, Edward was able to verbally reason geometrically about what made a point differentiable, by stating, “okay, so to know if this is differentiable, we want to kind of know when, where, goes to, small circle goes to small circle.” This utterance is an atypically precise geometric description of the fact that the derivative is a *local* property. It was during this task that participants most precisely characterized the derivative as a local property. Note that only Edward and Melody accomplished this task, as Zane and Christine progressed more slowly through the tasks.

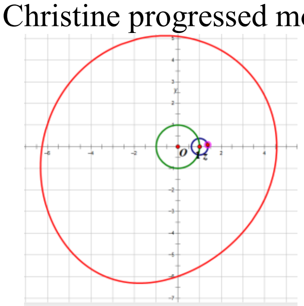


Figure 3: Edward and Melody zero in on a non-differentiable point.

Overall, I answer the two research questions listed above in the following ways. To answer the first, I argue that my participants reasoned about the derivative of a complex-valued function via embodied cognition in three distinct ways. In particular, they grounded their algebraic and geometric inscriptions via gesture and speech as in Goldin-Meadow (2003) (see leftmost cycle in Figure 4), integrated their algebraic and geometric reasoning methods via these inscriptions as in Soto-Johnson and Troup (2014) (see center cycle in Figure 4), and grounded these reasoning methods in both the real and virtual environments as in Hollebrands (2007), Olive (2000), and Tabaghi and Sinclair (2013) (see rightmost cycle in 4. To answer the second question, I detail three developments in reasoning which arose during their work with *GSP*, and seemed critical to my participants’ reasoning about the derivative.

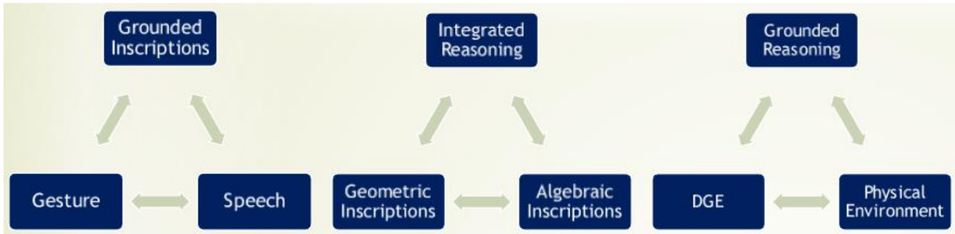


Figure 4: Integration through Embodied Cognition.

These three developments are as follows. First, my participants recognized a need for a geometric characterization of linear complex-valued functions, which was supported by grounding algebraic and geometric inscriptions via gesture and speech. Many of these inscriptions were displayed by *GSP*. This development allowed participants to begin extending their reasoning about real-valued functions to the complex-valued case. Second, my participants reasoned geometrically that linear complex-valued functions rotate and dilate every circle by the same amounts $Arg\left(f'(z)\right)$ and $|f'(z)|$, respectively. Finally, the participants observed small circles map to small circle-like objects under any complex-valued function.

Concluding Remarks

Thus, it appears that in my case study, gesture, DGEs and speech really did work together to aid students in developing reasoning about the derivative of complex-valued functions. I conclude by describing some possible teaching implications and directions for future research.

Teaching Implications

The findings described above suggest a few implications for teaching the derivative of a complex-valued function. First, it demonstrates potential learning trajectory for students seeking to develop their geometric reasoning about the derivative of a complex-valued function. In particular, my students first developed reasoning about the geometry of lines in $\mathbb{C}$, then reasoned geometrically about a constant derivative in terms of rotation and dilation, and finally reasoned about the need to reason specifically about small circles. Second, it suggests that my students worked beneficially with DGEs when placed in pairs and allowed some opportunity for free exploration of related algebraic and geometric reasoning, as in Olive (2000). Finally, my research suggests that it may be helpful to direct students' focus to key points over the course of their mathematical investigations, even when aided by a dynamic geometric environment (DGE).

Future Research

One possible direction for future research is to increase the breadth of these results by implementing dynamic geometric environments (DGEs) on a large scale in real classrooms and collecting quantitative data on student performance on tasks related to reasoning about the derivative of a complex-valued function as an amplitwist. Such research would theoretically allow these or related results to achieve some level of generality. Another possible direction is to increase depth of the results in this case study by iterating on the last task specifically. My participants reported beneficial effect from constructing algebraic information from exclusively geometric information from *GSP*, so further research on the effects of similar tasks for other students could prove highly interesting.

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Calculus Students' Reasoning about Slope as a Ratio-of-Totals and its Impact on Their Reasoning about Derivative

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Although studies have shown that students have difficulty with slope and derivative concepts, little is known about connections between these difficulties. In this study, written surveys and clinical interviews were used to examine students' understanding of both slope and derivative in real-life contexts. The dominant incorrect reasoning was thinking of slope as the ratio-of-totals $\left(\frac{y}{x}\right)$ *instead of the ratio-of-differences* $\left(\frac{\Delta y}{\Delta x}\right)$. *This incorrect thinking about slope influenced students' understanding of derivative. Thinking of slope as a ratio-of-totals implies that all linear relationships are directly proportional. Students interpreted slope as something that can be used to calculate values of dependent variables (by multiplying them by the value of the independent variable). As a result, they often interpreted the derivative as something that could be used to find the value of dependent variables (by multiplying the derivative by the value of the independent variable), implicitly using the incorrect relationship* $f(x) = f'(x) * x$.

Key words: Calculus, Application of Derivative, Rate of Change, Slope, Student Understanding

Introduction and Research Questions

A robust understanding of derivatives and instantaneous rates of change in calculus requires understanding slope and average rate of change (Hackworth, 1994). Calculus students may not have the robust understanding of slope and rate of change that instructors assume, which has consequences for their learning. It is thus important for instructors to be alert to students' understanding of slope coming into calculus so derivative instruction can be designed that expands on that knowledge. Furthermore, students must understand not only instantaneous rates of change but also continuously changing rates. This covariational reasoning is essential for interpreting dynamic function situations (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002).

My work builds on research about student understanding of slope and rate of change (Barr, 1980, 1981; Orton 1984; Stump, 2001); rates of change involving and not involving time (Stump, 2001); derivatives (Bezuidenhout, 1998; Ferrini-Mundy & Graham, 1994; Zandieh, 2000); rates of change of linear and non-linear functions (Orton, 1983), and how students' knowledge of rate of change was affected by derivative instruction (Hackworth, 1994). Findings from these studies indicate that students have difficulty understanding slope as a constant rate of change and derivative as an instantaneous rate of change.

This study investigates students' interpretation and use of slope and derivative in real life contexts. Such applications problems require students to translate from the context to the abstract level of calculus and then back to the context, a process that requires conceptual knowledge (White & Mitchelmore, 1996). Educators have emphasized the utility of these sorts of problems, noting that "not only do real-world situations provide meaningful opportunities for students to develop their understanding of mathematics, they also provide opportunities for students to communicate their understanding of mathematics" (Stump, 2001, p. 88).

There has not been much research on students' interpretation of the derivative as a rate of change, students' interpretation of slope as a constant rate of change, or students' understanding of the differences in making predictions involving constant and instantaneous rates of change.

Knowing more about students' understanding of slope and derivative as rates of change can help improve our instruction by helping educators build better derivative instruction based on students' knowledge of slope. Improving instruction is one way to retain more STEM majors, an overall goal of the mathematics education community (Holdren & Lander, 2012)

Building on prior research on students' difficulties with slope and derivative, this study focuses on the connections between students' verbal interpretation of slope and their verbal interpretation of the derivative. It also probes students' abilities to critique the reasoning of others, a Standard of Mathematical Practice in the Common Core (National Governors Association Center for Best Practices, 2010). These were assessed with questions focusing on the appropriate use of rate of change to make a prediction. Both are new research areas at the college level, and led to the following research questions:

- Is there a relationship between calculus students' understanding of slope and their understanding of derivative? Specifically, do students' abilities to correctly interpret the slope as a constant rate of change make them more likely to be able to interpret the derivative as an instantaneous rate of change?
- Given predictions based on slope and derivative, can students appropriately critique the reasoning used to make the predictions?

Methodology

I collected written solutions from 69 students enrolled in differential (i.e., first semester) calculus classes at a public university in the northeast. Students were approximately 80% through the course. I then performed follow-up clinical interviews (Hunting, 1997) with thirteen students during the first half of second-semester calculus; eight had completed the written survey the previous semester. My research approach, an analysis of student understanding gained from direct students responses, is consistent with a cognitive theoretical perspective and is well established in the mathematics education community (Byrnes, 2000; Siegler, 2003).

The survey and interview questions were based on linear and non-linear one-variable relationships, concepts likely familiar to first-year calculus students. Sample questions are shown in Figure 1. The questions about linear relationships were posed to gain insight into students' knowledge of predictions based on linear change. These questions were adapted from a general education textbook written to emphasize conceptual understanding (Franzosa & Tyne, 2010). To answer these questions, students must understand linear change as a constant rate of change. The questions about nonlinear relationships were more complex, and were adaptations from *Calculus, 6th edition* (Hughes-Hallet et al., 2013). To answer these questions, students must understand the derivative as an instantaneous rate of change that can be used to predict marginal change, and that the derivative *cannot* be used to make predictions at other input values.

Answers were coded as correct or incorrect. To investigate possible relationships between performance on linear and non-linear questions, 2x2 contingency tables were used to examine combinations of right and wrong answers. A sample table is shown in Table 1, which presents the survey data comparing the critiquing linear questions and the critiquing non-linear questions.

The focus in the contingency tables is on the shaded diagonal cells, namely those students who got exactly one of the questions correct. I performed McNemar's test on each table, testing whether there was an asymmetry in the success levels of students (Agresti, 2007). The outcome of this test reveals relationships between correctness on the two questions. In the case of the example in Table 1, the significant McNemar test results signify that students performed significantly better on the linear slope critiquing questions as compared to the non-linear derivative critiquing questions. After performing McNemar's test, I then used a modified Grounded Theory approach (Strauss & Corbin, 1990) to categorize common incorrect responses from the unit, interpretation, and critiquing questions (Figure 1). The categories that emerged from this analysis formed the basis of my interview questions.

For certain drugs, the amount of dose given to a patient, D (in milligrams), depends on the weight of the patient, w (in pounds).

A. Assume that D(w) is a linear function with a slope equal to 2 (m = 2).
 0. On the graph below, give a rough sketch of what the function D(w) looks like. Label the axes, but no need to scale them.
 1. What are the units on the slope, m = 2?
 2. Explain what this slope (m = 2) means in the context of the problem.
 3. Using the slope (m = 2), Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds. How much confidence do you have in her reasoning? (circle one and provide explanation)
  Very Confident   Somewhat Confident   Not Confident
 4. Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is twenty pounds heavier and she reasons that she must increase the dose by 40 mg (2 mg for each pound of weight). How much confidence do you have in her reasoning? (circle one and provide explanation)
  Very Confident   Somewhat Confident   Not Confident

B. Now, assume D(w) is a non-linear function.
 0. On the graph below, give a rough sketch of what the function D(w) might look like.
 1. What are the units on $\frac{dD}{dw}$? (also known as $D'(w)$)
 2. Explain the meaning of the statement $D'(140) = 2$ in the context of the problem.
 3. Using the fact that $D'(140) = 2$, Nurse Jodi predicts that a patient's dose will increase by 2 mg when the patient's weight changes from 140 pounds to 141 pounds. How much confidence do you have in her reasoning? (circle one and provide explanation)
  Very Confident   Somewhat Confident   Not Confident
 4. Nurse Jodi accurately doses a 140-pound patient using the model. Her next patient is 160-pounds and she reasons that since $D'(140) = 2$, she must increase the dose by 40 mg (2 mg for each pound of weight). How much confidence do you have in her reasoning? (circle one and provide explanation).
  Very Confident   Somewhat Confident   Not Confident

Figure 1. The interview instrument.

	Non-Linear Context			
		Correct	Incorrect	Total
Linear Context	Correct	30%	39%	69%
	Incorrect	3%	28%	31%
	Total	33%	67%	N=69

Table 1. 2x2 contingency table for the critiquing questions, McNemar's test significant ($p<0.001$)

Based on findings from the analysis of survey data and on pilot interviews, I created questions to explore students' understanding of the appropriateness of using the slope and derivative to make predictions. In differential calculus, students are taught what the derivative represents, but rarely are asked when it is appropriate to use the rates of change to make predictions. During the interviews, which were recorded with LiveScribe™ technology, students were asked to "think out loud" while they worked through problems. Interviews allowed me to probe student thinking more deeply, especially focusing on the themes that emerged during the survey data analysis. Analysis of interview data was done using both the categories developed from the survey data analysis and techniques from Grounded Theory when additional category identification was needed. A major goal of the analysis was to further illuminate students' reasoning about slope, derivative and how they are used and interpreted.

Results

With regard to research question #1, in both the surveys and interviews, students who interpreted the slope correctly were no more likely to interpret the derivative correctly than those who did not interpret slope correctly (questions A2 and B2; Figure 1). Success rates were low for interpreting both the slope and derivative (Table 2).

	Slope Interpretation	Derivative Interpretation
Surveys (N=69)	17%	13%
Interviews (N=13)	31%	38%

Table 2. Comparison of success rates for the survey and interview interpretation questions

For the slope interpretation, the dominant incorrect method was a ratio-of-totals approach $\left(\frac{y}{x}\right)$, implying a directly proportional relationship of the form $y = mx$ with a y-intercept of zero (Table 3). For example, Missy and Jackie were asked the meaning of a slope of 2 mg per pound and both gave a ratio-of-totals response. Missy said, "You multiply [the weight] by 2 to get the dosage." Jackie said "it means that for every pound, 2 milligrams of the dose." The correct response is "for each additional pound of weight, the patient would need 2 additional mg of drug." Characteristic of this type of response, neither Missy nor Jackie included "additional" in their response.

	Ratio-of-Totals Approach
Surveys (N=69)	39%
Interviews (N=13)	54%

Table 3. Rates of incorrect ratio-of-totals reasoning for the slope interpretation questions

It is important to note that the difference between the correct answer and a "ratio-of-totals" answer is very subtle but it appeared that students were not just inadvertently leaving out the word "additional." Most students who interpreted the slope incorrectly as a ratio-of-totals interpreted the derivative similarly as $f(x) = f'(x) * x$. That is, they treated the derivative as something that could be used to find the value of the dependent variable. For example, when asked what $D'(140) = 2$ meant, Dawn answered using ratio-of-totals, stating, "it would tell you how much drug to get. So, if it is 2 mg/pound, with a 140-pound patient, your dosage would be 2 times 140, or 280 mg.” In the interviews, six of the seven students who interpreted the slope as the ratio-of-totals went on to interpret the derivative similarly (as a rate of change that can be used to calculate the total). Students’ ways of thinking about this appear to be quite stable and four of the five students who did interpret slope correctly went on to interpret the derivative correctly. Rates of the ratio-of-total approach for derivative interpretation questions are shown in Table 4.

	$f(x) = f'(x) * x$ Approach
Surveys (N=69)	16%
Interviews (N=13)	54%

Table 4. Rates of incorrect $\boldsymbol{f(x) = f'(x) * x}$ reasoning for the derivative interpretation questions

With regard to research question #2, most students had difficulty critiquing others’ reasoning about slope and derivative predictions. Considering that students were not particularly successful interpreting slope and derivative, it is not surprising that students struggled with critiquing the reasoning of others, many showing little understanding of the covarying nature of the derivative. In particular, they had difficulties with the idea that the derivative is an instantaneous rate of change whose value changes depending on its input. Students applied the dominant incorrect way to interpret the slope and derivative (ratio-of-totals) to the critiquing questions, where many concluded that they could calculate the total by using the derivative ($\text{e.g.}, f(x) = f'(x) * x$). Seven of the thirteen interviewed students said that they did not agree with Nurse Jodi because they needed another derivative (the derivative at 141 or 160 pounds) to calculate the total dosage (multiplying the derivative times the weight to yield total dosage). Six of these seven students had interpreted slope as a ratio-of-totals, thus extending their incorrect ratio-of-totals interpretation to derivatives and concluding that $f(x) = f'(x) * x$. Table 5 shows a comparison of success in critiquing for the surveys and interviews.

	Correct	Need another derivative
Surveys (N=69)	16%	14%
Interviews (N=13)	31%	62%

Table 5. Response rates for critiquing questions in the non-linear context for surveys and interviews

Discussion

The findings related to research question #1 support prior research that showed that rates of change are hard for students (Carlson, 1998; Hackworth, 1994). The findings also extend findings about high school students’ difficulties with slope (Stump, 2001) to college students. Research question #2’s findings support prior research that highlighted students’ difficulties using calculus to analyze dynamic situations (Carlson, 1998) and students’ struggles with use of the derivative to approximate the function near a point (Asiala et al., 1997). These finding extend prior research by identifying students’ common incorrect use of the derivative as a ratio-of-totals to be used to calculate the function value.

In summary, findings suggest one common incorrect way of thinking, anchored in students’ misunderstandings of slope as a ratio-of-totals. This influenced students’ thinking about calculus in that they incorrectly generalized it to the derivative.

Implications for Teaching

Expanding on previous findings that have shown that students lack a solid understanding of rates of change in general (Hackworth, 1994; Orton, 1984), present findings suggest that students do not have a robust understanding of what slope and derivative mean as a rate of change in the context of modeling situations, nor do they understand appropriate uses of slope and derivative to make predictions. Findings also suggest that students' incorrect interpretation of slope (as a ratio-of-totals which translates to a directly proportional relationship of the form $f(x) = m * x$) seems to influence their incorrect interpretation and use of a derivative (as $f(x) = f'(x) * x$). These findings have instructional implications for calculus instructors and pre-service teachers.

Calculus instructors could benefit from knowing what beginning calculus students' understanding is of slope and their interpretation of slope in modeling contexts. By asking questions similar to those in this study, instructors can assess students' initial understandings. Findings suggest that when students answer slope interpretation questions using a ratio-of-totals approach, they are not merely leaving out the word "additional." These answers are symptomatic of their under-developed understanding of slope. To address these shortcomings, instructors can help students revisit the middle school concepts of slope, linear relationships, and directly proportional relationships (a subset of all linear relationships). Based on the results of this study, which showed a large percentage of the students having a ratio-of-totals approach, it may be valuable to take time at the start of our calculus course to help ensure that all students understand the differences between directly proportional relationships and other linear relationships, and how the interpretation of slope is different in each. This could be addressed when ideas of average rates of change are introduced. When interpretations of the derivative are addressed later on in the course, questions about linear equations and slope could be revisited, and distinctions made between the two to help combat students' incorrect generalizations.

Second, it could be productive to focus on not just what derivatives can be used for (linear approximation, marginal cost, etc.), but also stress their limitations in making predictions. Students could be given opportunities to compare the estimated values (from the derivative) with the actual value (from the original function), both by calculating the values and showing the error graphically. Discussions around how much error might be appropriate in different contexts could be beneficial.

Findings from this study suggest that many calculus students do not know how to appropriately use a derivative to make an estimate. Opportunities to discuss the appropriateness of predictions are not extensive; instead students are asked to make estimates without being asked to consider how far away the input value is, or how rapidly the function is changing around the point of interest. It could be valuable to provide students with more opportunities to reflect on the appropriateness of their predictions.

Concepts related to slope are addressed at the middle school level, and findings from the present study have several implications for the pre-service education of teachers. First, given the similarity in populations, pre-service teachers may have the same understandings about slope and rates of change as were displayed by participants in the present study. In addition, pre-service teachers could benefit from becoming aware of how ideas about directly proportional relationships can influence student understanding about slope and be given opportunities to consider ways of addressing this with their future students. For example, future teachers could provide more extensive opportunities for students to understand that directly proportional relationships are a subset of all linear relationships, and that not all linear relationships are of the form $f(x) = mx$. For example, students could calculate both $\frac{y}{x}$ and $\frac{\Delta y}{\Delta x}$ for linear relationships of the form $f(x) = mx$ and $f(x) = mx + b$ and make conclusions about the slope based on the results. Extending to slope interpretation, students could be given the opportunity to interpret slope as a rate of change via questions similar to those from this study, but specifically focusing on interpreting slope of directly proportional relationships versus those that are not. By highlighting differences between directly proportional relationships and other linear relationships, students can have opportunities to understand how interpretation of slope differs.

We also know that current curricula "provide little opportunity for developing the ability to: interpret and represent covariant aspects of functions, understand and interpret the language of functions, interpret information from functional events, etc." (Carlson, 1998, p. 142). As a mathematics education community, we need to continue to improve our instruction and focus our attention on the topics (such as covariational reasoning and interpretation) that research points to as critical to student success.

Implications for Future Research

More research is needed on students' interpretations of both slope and derivative as rates of change. Findings from the present study suggest that students' most common incorrect interpretation of slope as a ratio-of-total (as $\frac{y}{x}$ instead of $\frac{\Delta y}{\Delta x}$) is influencing their ability to understand the derivative. How early do these ways of thinking form? Student thinking about slope as a rate of change can be researched at the middle school level to see whether the ratio-of-totals interpretation is prevalent in those early years. We know that "full concept development appears to evolve over a period of years" (Carlson, 1998. p.143). The concept of slope is first introduced in middle school, utilized throughout high school, and then expanded on in calculus with the introduction of the derivative. Research could examine how students' understanding of slope evolves over these key years to further illuminate the role of this way of thinking.

At the college level, one could also focus on the effectiveness of instruction that is specifically designed to address these issues via design or teaching experiments. Differential calculus students' knowledge of slope as a rate of change could be assessed at the start of the course, and then instruction on slope and interpretation of slope (for linear functions of the form $f(x) = mx + b$ and $f(x) = mx$) could occur prior to starting derivatives. After instruction on derivatives, a post-test could measure student gains on interpreting both slope and derivative, to see whether the targeted instruction on slope aids in students' understanding of derivative.

Knowing more about students' understanding of slope and derivative as rates of change can help improve instruction. The concept of slope is fundamental to the mathematics curriculum from the middle grades and on; we must do better to ensure that our students have solid conceptions of this rate of change. If we can help pre-service teachers support develop these understandings when the concept is first introduced in middle school, and address difficulties they bring forward to high school and beyond, we can ensure that students in our calculus course have the solid foundation necessary to understanding the complexities of the derivative.

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Students' Understanding of Vectors and Cross Products: Results from a Series of Visualization Tasks

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Previous studies have explored student understanding of vectors in physics, engineering, or linear algebra settings, but there has been scant research on student understanding of vectors in a multivariable calculus context. In this study, we begin to explore how students think about vectors and cross products by analyzing student responses to open-ended questions from an online, conceptually-oriented multivariable calculus cross product activity. We identify several themes consistent with previous research on physics students including confusion between the cross product and its magnitude as well as difficulty identifying or communicating the direction of the cross product vector. This preliminary research begins to develop categories that could outline a conceptual model of student understanding of vectors and cross product. The analysis also informs several recommendations for improving the cross product activity.

Key words: [vectors, cross products, student understanding, visualization]

In mathematics, engineering, and physics, vectors play a foundational role. Vectors are used extensively throughout physics and engineering within mechanics topics such as force, velocity, and acceleration, and in mathematics coursework vectors appear in multivariable calculus, geometry, linear algebra, and differential equations. While basic vector concepts, representations, and operations with vectors are presented in both high school and preliminary college mathematics, students often will not be introduced to vector dot and cross product operations until they undertake college-level calculus coursework. Despite the regular occurrence throughout the curriculum, students continue to have significant conceptual difficulties with vector concepts and manipulations.

One approach conjectured to improve student understanding of three-dimensional topics is to improve students' visualization skills using computer exercises (Sorby & Baartmans, 2000). CalcPlot3D is an online, freely available 3D-graphing applet that allows students to visualize and manipulate concepts including vectors, vector fields, parametric curves, surfaces, and gradients. In addition to the graphing calculator feature, CalcPlot3D offers discovery-learning activities for students to explore multivariable calculus concepts (Seeburger, 2016). This study analyzes student responses to Seeburger's cross product activity.

The purpose of this study is to investigate students' transitional conceptual understanding of vectors, and more specifically the cross product of vectors, as they work through an exploration activity with embedded questions while using the CalcPlot3D visualization tool. This research is partially supported by the National Science Foundation under Grant Numbers 1524968, 1523786, 155216.

Background Literature

There is extensive literature that explores students' misconceptions when confronted with problems involving force and motion (i.e., mechanics), both of which are represented by vectors (Aguirre & Rankin, 1989; Barniol, Zavala, & Hinojosa, 2013; Flores, Kanim, & Kautz, 2003; Hestenes & Wells 1992; Hestenes, Wells, & Swackhamer, 1992; Miller-Young, 2013). Both force and motion utilize vector concepts; however, the students' misconceptions regarding vector concepts, properties, and fluency in vector operations are not explored directly. Rather, these concepts are embedded within the application. For instance, Hestenes, Wells, and Swackhamer (1992) utilize a Force Concept Inventory to assess student understanding of Newtonian physics, but the inventory does not directly assess students' understanding of vectors.

Others (Barniol & Zavala, 2014; Knight, 1995; Nguyen & Metzler, 2003; Van Deventer & Wittmann, 2007; Wang & Sayre, 2010; Zavala & Barniol, 2010) provide more explicit consideration of students' understanding of vector concepts, representations, and operations outside of a kinematic or mechanics context. Knight (1995) found that approximately 40% of students in an introductory calculus-based physics course had no idea what a vector was. About 50% of the students could add vectors correctly; however, none of the students were able to evaluate a vector cross product. Barniol and Zavala (2014), using their Test of Understanding of Vectors (TUV), examined the knowledge of university students who had completed an introductory calculus-based physics course. The TUV contains non-contextual multiple-choice problems covering vector properties and basic vector operations. Three of these problems address the cross product. Two are computations and the third asks the students to select an appropriate geometric interpretation of the cross product from a list of options. The percentage of students who could correctly answer this problem was 57% (Barniol & Zavala, 2014).

Research on student understanding of vectors in college-level mathematics courses tends to focus on the transitional proof courses such as linear algebra and geometry. For instance, Stewart and Thomas (2009) developed a framework for vectors in linear algebra that combines an action-process-object schema (Dubinsky, 1991) for vectors with Tall's (2004) categorization of three mathematical ways of thinking: embodied world, symbolic world, and formal world. Kwon (2013) presents a new framework for conceptualizing vectors in college geometry that identifies three representations of a vector: vector as a translation; vector as a point and point as a vector; and geometrical vector sum.

Theoretical Framework

While many studies highlight student misconceptions, this study focuses on students' transitional conceptions, since understanding is not necessarily static. *Transitional conceptions* relate to students' current notions of a concept that are cued by the task at hand and that may include what some would call misconceptions. Transitional conceptions are not fully integrated in a coherent manner, and hence tend to be in flux. Yet, they do result from a sense-making activity even though they may only address some (but not all) aspects of the concept and may be productive in some (but not all) contexts. Studying transitional conceptions can potentially lead to a better or more accurate view, perhaps even to a conceptual model, of student understanding of the concept (Moschkovich, 1999). To understand students' meaning-making processes, it is imperative that instructors consider the transitional conceptions that occur when students are engaged in learning new concepts (Wolbert, Moore-Russo, & Son, 2016). Many have begun to consider college students' transitional concepts (Chiu, Kessel, Moschkovich, & Muñoz-Nuñez, 2001; Cho & Moore-Russo, 2014; Nagle, Casey, & Moore-Russo, 2015; Wolbert, Moore-Russo, & Son, 2016) in various areas of post-secondary mathematics. However, there is little research on transitional conceptions specific to vectors in multivariable calculus classes. This study aims to add to the existing body of knowledge, focusing, in particular, on students' transitional conceptual understanding of the vector cross product.

Conceptual understanding encompasses both "what is known (knowledge of concepts)...[and] the way that concepts can be known (e.g. deeply and with rich connections)" (Star, 2005, p. 408). It can be considered as "a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (Hiebert & LeFevre, 1986, p. 4). When concepts are first learned, students' understanding can be fragmented and lack either organization or connections to related concepts (Schneider & Stern, 2009).

Both intra- and inter-connections are possible for a concept. Tall and Vinner (1981) described an individual's concept image as the entire cognitive structure related to a particular concept. Past research has considered the many "conceptualizations," "notions," or "connected components" associated with a particular concept in mathematics (e.g., work on slope by Moore-Russo, Conner, & Rugg, 2011; Nagle & Moore-Russo, 2013); these are the intra-connections for a single concept. There are also inter-connections between different topics (e.g., Zandieh and Knapp's (2006) work that inter-relates rate, limit, and function to look at how students come to understand derivatives); however, this paper focuses only on intra-connections.

Understanding is dynamic, and students' conceptions regarding a topic are transitional. As students advance in mathematics, there are instructional expectations that they will develop internal networks that are rich in relationships where they are able to move flexibly among representations and notions of the concept as they advance in the learning of a topic. Assessment to confirm that students are making such intra-connections to form robust, flexible concept images is important since conceptual understanding, in combination with procedural fluency, is necessary for success in mathematics (Hiebert & Carpenter, 1992).

Methods

This study is situated within a larger research program to determine if student understanding of cross products can be enhanced through visual explorations. Before being able to assess the impact of visual explorations on student understanding, it is necessary to build a model of student understanding of the cross product. As a first step to this goal, we sought to begin to characterize students' transitional conceptual understanding of cross products by examining their responses to four open-ended questions on a conceptually-oriented cross product assignment.

Subjects and Setting

The data analyzed were from electronic responses of 434 college-level multivariable calculus students to four open-ended questions from an online assignment. The data was collected over four years from students from community colleges, four-year private colleges, and four-year public colleges. Each student completed a pre-test, exploration assignment, and post-test.

The exploration assignment consisted of 10 open-ended and 2 multiple choice questions about vectors and cross products. The students were directed to a visual applet that contained two vectors (one red and one blue) along with their cross product. The red and blue vectors were graphed with initial points situated at the origin in the *xy*-plane. Students could manipulate the length and direction of the two vectors on the *xy*-plane, and based on the students' input, the applet automatically redrew the cross product, computed the magnitude of the cross product, and indicated the angle between the two given vectors.

Previous research on the pre- and post-test multiple choice questions indicate some knowledge gain on the relationships between the angle between two vectors or between the length of two vectors and their cross product through the use of the exploration (Seeburger, 2009). Here we examine student responses to the first four open-ended questions embedded in the pre-existing exploration to gain a better view of how students understand cross product. The remaining questions will be examined in a follow-up study. The questions examined here are:

Q1. What is the geometric relationship between the cross product vector and the two vectors that form it? (Hint: This is NOT a formula.)

Q2. How is the cross product vector geometrically related to the two vectors that form it? (Hint: This is NOT a formula.)

Q3. For vectors of fixed length, but varying the direction of one of the vectors, when is the magnitude of the cross product at a maximum?

Q4. For vectors of fixed length, but varying the direction of one of the vectors, when is the magnitude of the cross product at a minimum?

Analytical Method

The items were examined for emerging themes through a general inductive analysis. According to this method, the researcher does not begin with a preconceived structure but allows categories to emerge from the data. The researchers utilize categories to make sense of what is observed (Thomas, 2006). To identify emerging categories, one member (the first author) of the research team began the task of reading all responses to the four items to note what was observed. These emerging categories were shared with another member of the research team (the fourth author) for general consensus. A third member of the research team (the second author) then read all the responses and tried to see if the initial categories could be collapsed into unifying topics. She shared her findings with the third author until consensus was reached; both agreed on all the identified categories, but refinements were made to the topic descriptions. The list of categories was then shared and discussed with the first author, who also agreed with the identified categories and then helped further refine the topic descriptions. All, but the miscellaneous incorrect category and the blank category, were coded on three levels: as not being present (0), being present but with a developing or transitional understanding (-), or being present with a correct or accurate (+), but perhaps not complete, understanding. Finally the first and second authors coded all the items using the category descriptions in Table 1.

After coding was completed, interrater reliability statistics for each code were computed with ReCal (Freelon, 2013). With the exception of the miscellaneous category, all codes resulted in Krippendorff's alpha above 0.80. The first and second authors then came to a consensus on the few instances in which they were in disagreement.

Table 1
Categories and Codes with their Descriptions

Category and Codes		Description
Angle between two vectors	V+	angle must be between 0 and 180 degrees
	V-	angle between two vectors can be negative or greater than 180 degrees
Orthogonality	O+	cross product vector is orthogonal/perpendicular/90 degrees from the two vectors that form it
	O-	developing, but not correct/precise, statement about orthogonality
Right-hand Rule	R+	Mention of the right hand rule; Statement that correctly addresses the need to attend to orientation of one vector relative to another
	R-	developing, but not correct/precise, statement about the right hand rule
Parallelogram	P+	magnitude of cross product is area of parallelogram formed by two vectors
	P-	developing, but not precise, statement about parallelograms or area (e.g. “cross product forms a parallelogram.”)
Formula for magnitude of the cross product	F+	magnitude of cross product = $\|\|axb\|\|=\|a\|\|b\| \sin\theta$; correct statement involving sine and vectors
	F-	incomplete or incorrect statement involving multiplication, sine, cosine, or other formulas.
Angle impact on magnitude of the cross product	A+	correct statement describing how the angle between vectors influences the magnitude of the cross product
	A-	incorrect or vague statement about how changing the angle between two vectors will affect the length of the cross product (e.g. “cross product depends on the angle between the two vectors”)
Length impact on magnitude of the cross product	L+	correct statement describing how the length of the two vectors influences the magnitude of the cross product
	L-	incorrect or vague statement about how changing the length of one of the vectors will affect the length of the cross product
Other incorrect	I	incorrect statement involving x and y coordinates; vector addition; statement involving quadrants or planes; other nonsensical statements
Blank	B	no response; “I don’t know”

Table 2
Number of Responses in Each Category Code for Questions 1-4 (Q1-Q4) of 434 Participants

	Codes															
	V+	V-	O+	O-	R+	R-	P+	P-	F+	F-	A+	A-	L+	L-	I	B
Q1			217	8	4	2	23	52	4	19	6	8		5	29	81
Q2		7	256	10	10	4	30	40	4	15	25	17	1	6	14	56
Q3	1	29				2				3	371	17		7	1	44
Q4		29				5				4	348	37	6	1	7	44
Tot.	1	65	473	18	14	13	53	92	8	41	750	79	7	19	51	225

Results

In Table 2 we summarize the results of the coding by question for the 434 students. While some of the codes were rarely assigned (e.g. L+, L-), the complete data set contains responses to other questions in which these themes are more prevalent. The authors plan on applying the methods tested in this initial analysis to the remaining questions in the exploration in the next phase of research.

Overall students evidenced some understanding of the cross product. The percentage of students providing a correct (+) response which may, or may not, be complete, for either Q1 or Q2 was 75% and for Q3 or Q4 was 88%. Note that some of these students may have also indicated additional incorrect or transitional understanding (-) of another aspect of the cross product in their responses as well.

Although the first two questions are nearly identical, some students answered them differently. Another theme we noted in Q1 and Q2 answers was that only 4% of all students referred to the right-hand rule or the orientation of the cross product to the two vectors that formed it (in either a correct (+) or transitional (-) way). The students who did describe the orientation of the cross product to the two vectors that form it appear to be recalling previous knowledge since the right-hand rule is not explicitly referred to in the exploration until later on. Nearly all of these students only quote it by name: "These vectors also form what is called the Right Hand Rule." Only a few students made an attempt at describing the right-hand rule in their own words. For example:

> *The direction of the cross product is found by using the right hand rule which involves placing your thumb on the first vector of the cross product and your forefinger on the second vector and seeing where your middle finger is pointing. The direction it is pointing is the same as the direction of the cross product.*

Furthermore, when considering the relationship between the two given vectors and their cross product, students more often only describe how the angle between the two given vectors impacts the length of the cross product, but do not consider how the length of the two given vectors would affect the magnitude of the cross product.

Discussion

Although concentrated to the analysis of four open-ended questions concerning the cross product, this study reveals difficulty in student understanding of cross products consistent with the literature. In particular, students have difficulty with the right-hand rule, tend to rely on formulas, and confuse the cross product with its magnitude. Further analysis of the remaining cross product exploration questions along with this preliminary study will inform recommendations for future versions of the CalcPlot3D concept exploration and provide insight into the intra-connections of vector cross products that students are and are not understanding.

While 65% of students correctly stated that the cross product was orthogonal to the two vectors that form it in responses to the first two questions, only 4% of students made any reference to the right-hand rule, orientation, or noncommutativity of the cross product vector. Similar results are found in a physics context by Zavala and Barniol (2010) and by Scaife and

Heckler (2010). Note, however, that in their research the students were required to compute the cross product while in our study the cross product was provided to the students graphically. Barniol and Zavala found that 44% of the students were able to interpret the cross product (AxB) as a vector perpendicular to both A and B, although only 22% of the students identified the correct direction. In the context of magnetic force, Scaife and Heckler (2010) saw that in a series of four similar questions, 40% of students made a sign error for the cross product at least once. They conjecture this is due to confusion about the application of the right-hand rule and failure to recognize that the cross product is a noncommutative operation.

Another theme found in our data was that 5% of the responses to Q1 and Q2 in our study included some (either correct or incorrect) reference to a formula, even though both questions explicitly state "Hint: This is NOT a formula." This supports research of Zavala and Barniol (2010) who found 9% of third semester physics students studied referred to a formula for the cross product in their open-ended responses to interpret the cross product. This additionally advances the notion, witnessed by Miller-Young (2103) in the context of an engineering class, that students often rely on memorized equations and procedures, even when instructed not to do so.

Zavala and Barniol's (2010) analysis showed 32% of students did not distinguish the cross product magnitude from the cross product itself. In our data set, this same trend is present in many of the (P-) responses similar to "The area of the two vectors equals the cross product." On the other hand 50% of students in our study focused on the direction of the cross product and did not describe its magnitude. This might suggest that students did not have a robust concept image, or they might not have seen the need to report on all connected components of the concept.

Future Recommendations

One of the limitations of the CalcPlot3D exploration is that the applet does not allow students to move the vectors off of the origin or off of the *xy*-plane. This may have led to incorrectly over-generalized responses; for example, "[The cross product] point[s] in the z axis direction." Marton and Booth's Variation Theory suggests that activities be structured to ensure students experience a diversity of examples (Lo, 2012). In this case students should experience vectors in a variety of orientations, not just those on the *xy*-plane.

The exploration provides students the opportunity to communicate and describe the geometric features of vectors and their cross product. Some students had difficulties accurately and precisely describing these features which may contribute to only a limited understanding of the relationships. For instance, in Q4, 86% of the students who described the angle between vectors as negative or larger than 180 degrees also demonstrated only a transitional understanding of the relationship between the angle of the vectors and the length of the cross product. Adding more examples of verbal descriptions of the geometric features of vectors may provide students with scaffolding to better communicate mathematics in open-ended responses.

The next step for the research team is to complete the analysis of all of the responses to the remaining exploration questions to gain more insight into what intra-connections of cross product are being made by the students and how the CalcPlot3D explorations can be improved to better address student difficulties.

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Variations in Precalculus Through Calculus 2 Courses

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In this paper we analyze variations in the structure of courses designed for the Precalculus through Calculus 2 (P2C2) sequence. We examine the nature of such variations, frequency nationally, and how DFW rates and instructional approach compare to the standard courses. While most identified variations in course structures have on average lower DFW rates when compared to the national average, a comparison within institutions indicates that these alternative course structures have higher DFW rates when compared to the standard P2C2 sequence offered at the respective institution. In addition, we observed that course variations which allow for increased instructional time have greater amounts of active learning techniques as part of the instructional format. Results from these findings along with their implications for the next phase of the Progress through Calculus project are discussed.

Keywords: Calculus, Precalculus, Curriculum, Course Variations

There is abundant literature highlighting the importance of student success in introductory undergraduate mathematics courses, often pointing to how such courses are a hurdle for students intending to continue into STEM majors and future related careers. Even for those students who do not choose to major in a STEM field, success in entry-level undergraduate mathematics courses such as calculus can impact a student's overall persistence in postsecondary education. Research examining students' success in introductory mathematics courses consistently indicates that students are not learning the intended curriculum (Tallman, Carlson, Bressoud, & Pearson, 2016; Thompson, 1994), resulting in lower preparation for subsequent courses (Carlson, 1998; Selden & Selden, 1994) and a marked decrease in their desire to pursue a STEM degree (Bressoud, Mesa, & Rasmussen, 2015; Ellis, Fosdick, & Rasmussen, 2016; Seymour & Hewitt, 1997). These students often cite the poor instructional experiences in introductory level courses as the primary reason for their departure from the STEM fields.

Institutions of higher education and departments of mathematics are becoming increasingly attuned to the particular challenges faced by these students and are seeking new ways of supporting student success in introductory mathematics courses. Rasmussen et. al (2016) found that most departments are aware and value characteristics of more successful calculus programs, yet they are not always successful at implementing these features at their institutions. As such, departments have responded in a number of ways to support student success in the P2C2 sequence. Departments are utilizing local data to inform placement procedures, using active learning pedagogy techniques, creating course coordination systems, and developing or improving graduate teaching assistant training. Another departmental response, which is the focus of this report, is to offer alternative P2C2 course structures. An alternative course structure reimagines the "standard" P2C2 sequence, which we take to be three, one semester courses consisting of Precalculus, Calculus I, and Calculus 2. An example of an alternative course structure is to offer, in addition to the standard sequence, a "stretched out" Calculus 1 course that takes two semesters and infuses additional algebra and trigonometry in the context of the calculus. In this report we address the following research questions:

1. What variations to the standard P2C2 course sequence are currently in place and how common are they nationally?
2. What are the relationships between variations to the standard P2C2 course sequence, instructional approach, and student success?

Data for this analysis comes from a national census survey focused on the P2C2 course sequence. This census survey is Phase 1 of a five-year project which began in 2015, and was distributed to all mathematics departments that offer a graduate degree in mathematics. These institutions were selected because they produce the bulk of STEM graduates while often struggling to find a balance between the demands of research and teaching. Phase 2 will consist of longitudinal case studies of selected institutions.

Theoretical Background

We address our research questions and frame alternative P2C2 course offerings in terms of the intended, enacted, and assessed curriculum (Kurz, 2011; Porter, 2006; Webb, 1997). The intended curriculum refers to the knowledge and skill targets for students (Porter, 2006), in this case the course objectives for precalculus and mainstream calculus. In addition, this includes variations of the intended curriculum deigned as specialization for service disciplines (engineering, biosciences, etc.). The enacted curriculum refers to the knowledge and skills delivered during instruction which varies based on instructional time (allocated time for instruction), content coverage (amount and variety of academic standards), and instructional approach (Kurz, 2011). For example, a stretched out Calculus 1 course might be configured to allow for more active learning, but this is an empirical question and one that we address in this report. We also investigate how institutional response to the alignment between the intended and enacted curriculum affects the assessed curriculum. We use DFW rates (grades of F, drops and withdrawals) as a proxy for the assessed curriculum.

Figure 1 captures how we make use of the intended, enacted, and assessed curriculum in this study. In particular, the survey data allows us to detail intended curriculum aspects such as course objectives and intended audience, and the enacted curriculum aspects such as instructional time, content coverage, and instructional approach.

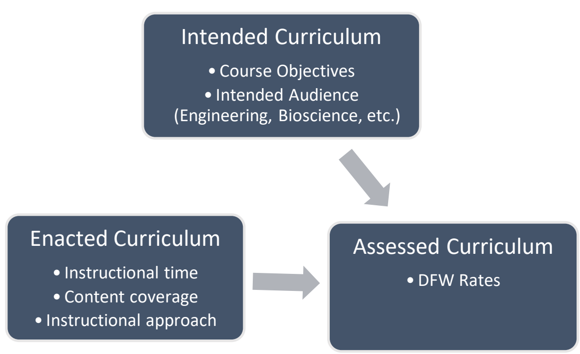

Figure 1. Observed components for the intended and enacted curriculum, and their relation on the assessed curriculum.

Methods

In the United States there are a total of 330 departments that offer either a Masters or PhD in mathematics. All 330 institutions (178 Doctoral and 152 Master's) were surveyed yielding a 68% response rate. We designed the census survey to gather information on the implementation of the features of successful programs identified by the CSPCC project and to gain an understanding of the variety of P2C2 programs being implemented across the country, the prevalence of such programs, and what institutions are doing to improve their programs. The survey consisted of three main parts. Part I asked for a list of all courses in the mainstream P2C2 sequence. Mainstream refers to any course in this sequence that would be part of student preparation for higher-level mathematics courses such as a first course in differential equations or linear algebra. Part II asked about departmental practices in support of the P2C2 sequence. Part III asked for detailed information about each course in the mainstream P2C2 sequence, including enrollment data and details about course delivery.

Survey responses were then cross-referenced and updated to the extent possible by a comprehensive search of publicly-available course catalogs and department websites. This led us to a collection of 1108 courses from 223 institutions, with details supplied for 895 of these courses by 205 institutions. We used a grounded theory approach (Corbin & Strauss, 2008) to code and categorize variations to the standard P2C2 course sequence. Based on this analysis, 11 variants of the standard P2C2 courses were identified and are outlined below. We choose to highlight these variations since they demonstrate the response from institutions to support student of varying preparedness levels and interest.

Results

We present results according to the two research questions. We first describe existing course variations in terms of three main themes: intended audience, instructional time, and content coverage. We then provide a summary table (Table 1) specifying the frequency in which these variations are present. We then address the second research question by analyzing how course variations fare in terms of DFW rates and instructional approach.

Intended Audience

Several institutions have specialized their mainstream calculus sequence to tailor the intended curriculum to service different disciplines. The most common variation is *calculus for the biosciences*, which is a mainstream calculus course designed explicitly for students in biological or life science majors. Often time the course includes applied topics in biological modeling and investigation of real-life phenomenon. *Calculus for engineering* is another course variation on the mainstream calculus course that is intended for students in engineering majors. Often this course includes emphasis on physical applications to engineering and computational techniques. Seymour and Hewitt (1997) found that a disconnect between calculus content and intended major to be a major contributing factor in students' decision to leave a STEM field. Such variations have the potential to address such concerns.

In addition, we identified institutions that offered a mainstream calculus for another subject, which was specifically designed for students in a non-STEM major (e.g. Calculus for Economics). Mathematics departments often have to provide non-mainstream introductory mathematics courses for students in other disciplines, yet these students face a subsequent challenge if they intended to switch to a STEM intending degree. Some institutions have responded to this potential audience by providing a transition to mainstream course variation. This course serves as a bridge between a non-mainstream calculus course and a mainstream calculus course or upper-division mathematics course that has a lower credit load, and does not require the student to retake the entire mainstream P2C2 sequence.

Instructional Time

The most common course variation observed alters the standard curriculum by allowing the course content to be covered over a longer duration of time, which addresses another major factor that students cite for leaving a STEM major – overpacked courses taught at too fast a pace (Seymour & Hewitt, 1997). For the preparation for calculus courses we observed institutions that offered the choice between a single course covering the requisite skills necessary for enrolling into calculus or a two (or more) course sequence, typically consisting of a course in trigonometry and a course in college algebra. We denoted these options as *modular precalculus* and *modular stretched-out precalculus.* We use the term modular since students are able to modularize or choose between the necessary sequence. In addition, we observed institutions that offered only a single course option or only a two course sequence as preparation for calculus. We denoted these options as *standard precalculus* and *standard* stretched-out *precalculus* .

For single variable calculus we observed two unique course variations that allowed for the formal curriculum to be covered over an extended length of time. The *stretched-out calculus* course takes the traditional one-term course and stretches the content over a two-course sequence, often including additional requisite material when appropriate. This option is usually intended for students who would be at-risk in the traditional calculus sequence and would benefit from a slower paced delivery of the material. The *stretched-out Calculus 1&2* is a variation on the stretched-out calculus where three courses, when taken together, are the equivalent to the standard two-course single variable calculus sequence.

Content Coverage

While some variations seek to allocate more time for course delivery by spreading the curriculum over extra courses or terms, another set of calculus course variations are designed to provide background content and requisite skills as supplements to course offerings. The most common variation in this category is *calculus infused with precalculus,* which is a single-term calculus course (typically with more credit hours) designed to include requisite pre-calculus topics covered throughout the course duration. A similar course variation, referred to as *co-calculus*, is a one-term course taken concurrently with a single variable calculus course that covers selected precalculus topics, coordinated with the content of the calculus course. This course variation is intended for at-risk students who can be identified early in the term through low-course performance and subsequently enrolled in the co-calculus course to provide supports for study skills and the coverage of precalculus content.

While the previous two course variations are typically intended to support students in the first course in single variable calculus (Calculus I), we also observed one course variations intended to specially support students in the second term of single variable calculus (Calculus 2). *Accelerated calculus* is a calculus course explicitly designed for students who have taken calculus in high school (usually with AP credit). These courses cover mainly material that would be considered Calculus 2, but also include Calculus 1 material that may not have been covered in sufficient depth in an AP course. All variations in the "content coverage" category involve strategies to supplement the mathematics content of regular course *while keeping students on-track* in terms of time to graduate.

Other

In addition to previously mentioned variations we also observed several unique course offerings that failed to warrant their own code, but were identified as *other* unique course structures. This category included: courses (or course sections) explicitly designed for students who have not seen calculus before; a course designed to divert less-prepared students mid-term; precalculus courses which include a preview of calculus topics; courses designed for transfer students; applied courses in technology; courses offered only in summer as preparation; etc.

Course Variation Summary

Overall, 125 (56.3%) of the institutions have at least one course variation and excluding the most common variation, for modular precalculus and modular stretched-out precalculus, 75 (33.8%) of the institutions have at least one calculus course variation. The frequency for which we observed each of the course variations at surveyed institutions is presented in Table 1. The remaining results present a descriptive account of the survey responses describing the enacted curriculum (instructional approach) followed by measures of the assessed curriculum (DFW rates) stratified by the observed course variations. The descriptive statistics are only provided for variations which had data from three or more observations.

Table 1. Overview of course variations offered nationally

Category	**Variation**	**Institution (n=223)**	**Percent (%)**
Intended Audience	Calculus for biosciences	15	6.8
	Calculus for engineering	14	6.3
	Calculus for other subject	3	1.4
	Transition to mainstream	3	1.4
Instructional Time	Modular Stretched-Out Precalculus/ Modular Precalculus	62	27.9
	Stretched-out calculus	20	9.0
	Stretched-out calculus 1&2	7	3.2
Content Coverage	Calculus infused with Precalculus	11	5.0
	Co-calculus	3	1.4
	Accelerated calculus	14	6.4
Other	Other	10	4.5

Course Variation DFW Rates and Instructional Approach

For each of the P2C2 courses provided, respondents were asked to indicate the primary instructional format during the regular class meetings. The response options along with the percentage for which they occurred for each of the course variations is presented in Figure 2. We observed the highest amounts of active learning techniques implemented in course variations that increase the amount of instructional time, with the greatest observed amounts in the stretched-out Calculus 1&2 variation, followed by the stretched-out calculus variation. In addition, we observed greater amounts of computer based instruction in calculus for engineering and the stretched-out Calculus 1&2 course variation.

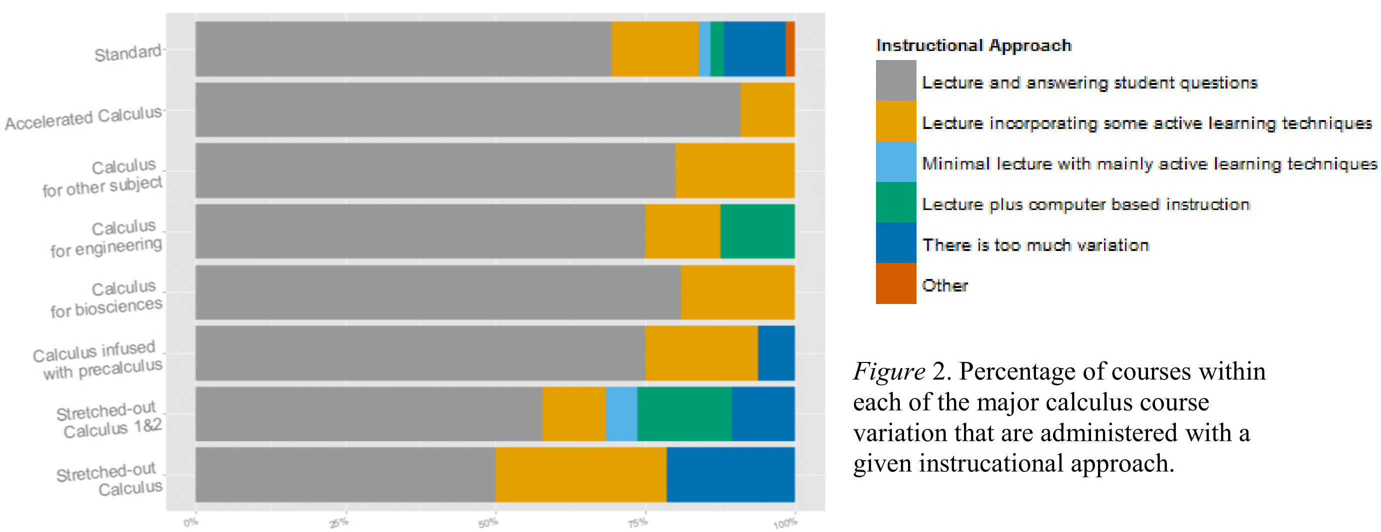

Figure 2. Percentage of courses within each of the major calculus course variation that are administered with a given instrucational approach.

Respondents also indicated the typical drop, fail, or withdraw rate for each of the courses they listed as part of the mainstream P2C2 sequence. We observed that the DFW rates between the variations of the preparation for calculus courses (Figure 3) are markedly similar, with no significant differences.

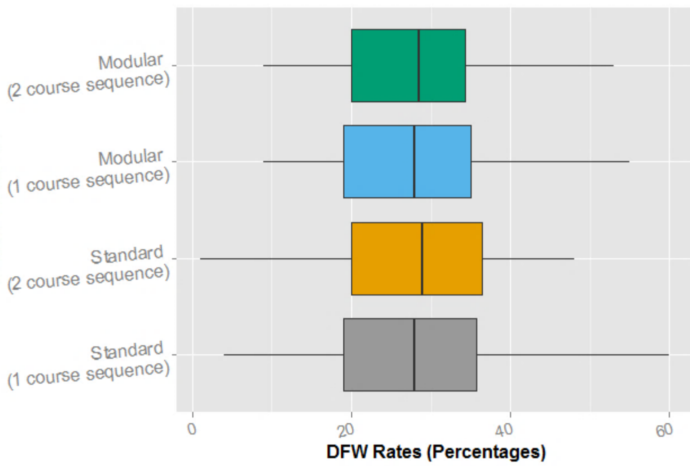

Figure 3. Box-and-whisker plot depicting DFW rates for the four major course variations of preparation for calculus along with the reported averages: modular two course (n 60, 27.6%), modular one course (n = 47, 27.2%), standard two course (n = 34, 27.1%), and the standard one course (n = 90, 27.5%).

The DFW rates for course variations in mainstream calculus (Figure 4) however do show major differences when compared to the national average. We computed the average DFW rate for any standard calculus course (24.1%), but isolated those courses intended for honors students, as these courses are typically intended for high-achieving students. We observed that each of the course variations had on average lower DFW rates compared to the standard course structure, except stretched-out calculus which had a similar yet higher DFW rate (25.3%). Indicative of our qualitative descriptions the lowest DFW rates occurred in course variations intended for high achieving students with strong prior backgrounds (Stretched-out Calculus 1&2, and Accelerated Calculus).

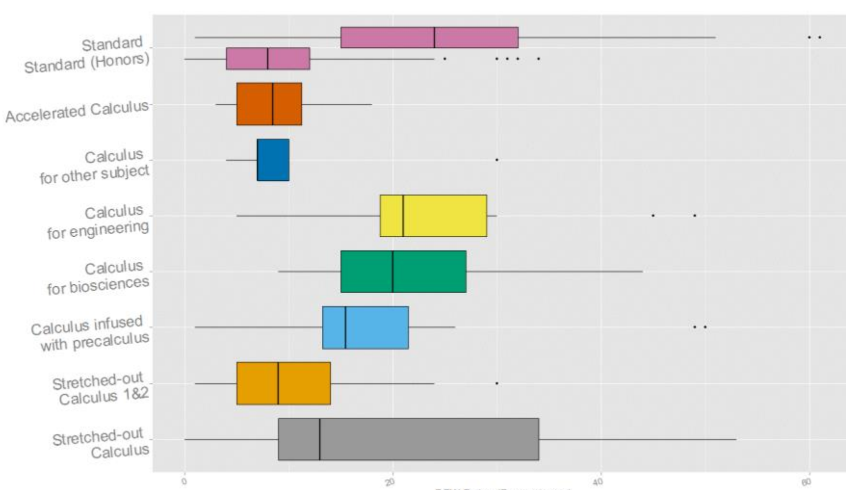

Figure 4. Box-and-whisker plot depicting DFW for the major course variations of calculus along with the reported averages:
Standard (n = 383, 24.1%), Standard (honors) (n = 68, 9.6%), Accelerated Calculus (n = 8, 9.12%), Calculus for engineering (n = 16, 23.6%), Calculus for biosciences (n = 19, 22.0%), Calculus for other subjects(n = 3, 14.6%,), Calculus infused with Precalculus (n = 14 18.9%), Stretched-out Calculus 1&2 (n = 17, 11.0%), Stretched-out Calculus (n = 38, 25.3%).

In addition to comparing DFW rates against the national average, we also analyzed the DFW rates for a given course variation against the DFW rates for the standard course sequence offered at that given institution (Table 2). While accelerated calculus has the same trend in comparison to the national average, the other course variations present an entirely different picture with regards to the DFW rates. We observed a larger amount of comparisons in which the alternative course variation had higher DFW rates compared to the equivalent standard course sequence at that particular institution.

Table 2. Institutional comparison of DFW rates for alternative and standard course sequences in single variable calculus.

Course Variation	**Avg. DFW rate for variation**	**Avg. DFW rate for standard**	**Institutional comparisons where DFW rates were higher for:**		
			Variation	*Standard*	*Identical*
Accelerated Calculus	8.2	20	0	3	1
Calculus for other subjects	10.2	8	2	2	2
Calculus for Engineering	25.3	26.1	11	3	2
Calculus for Biosciences	22.1	23.6	11	6	2
Calculus infused with Precalculus	19.0	13.1	7	3	1
Stretched-out Calculus 1&2	12.8	11.6	5	0	0
Stretched-out Calculus	22.4	19.6	7	3	2

Conclusion

Our project presents the first description of the variety (and frequency of) nonstandard curricular structures for P2C2 courses across the nation. In particular, we are able to paint a picture of how institutions across the country are attempting to address concerns about student success in introductory mathematics courses by addressing the intended and enacted P2C2 curriculum. We note that relatively few departments across the country are experimenting with these innovations, but our results indicate that these innovative structures have some potential at impacting the assessed curriculum. Given that this information comes from Phase 1 of a multi-year project, we find ourselves with the unique opportunity to further investigate the implementation of alternative course structures during Phase 2 of our case study analysis. The in-depth institutional analysis will provide a richer understanding of the surrounding programs and context in which such structures support student success, and to explore more nuanced measures of success and student understanding.

While Phase 2 of our research will reveal much more about the nature of alternative course structure, we can begin to comment on some of the potential advantages of these non-standard offerings. First, we note that many variations involve increasing the time dedicated to courses – be it stretched over extra terms or supplemented with co-requisite units. This increase in contact hours may free up class time in order to leverage active learning while alleviating concerns about content coverage. Secondly, many alternative course structures are aimed at supporting students who are underprepared for calculus. While the DFW rates in these courses tend to be higher than their non-standard counterparts, we expect this is related to the proportion of underprepared students in each – though further exploration is needed to conclude this with certainty. As failure in calculus is akin to an exit from STEM, supporting these students and bringing them up to speed with their peers may be an important piece of stopping the leaking pipeline.

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[1] The Progress through Calculus PI team consists of Linda Braddy, David Bressoud, Jessica Ellis, Sean Larsen, Estrella Johnson, and Chris Rasmussen. Graduate students include Naneh Apkarian, Jessica Gehrtz, Dana Kirin, Matthew Voigt, Kristin Vroom.

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Using Expectancy Value Theory To Account For Students' Mathematical Justifications

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There is a robust body of research demonstrating that when students are asked to justify a mathematical assertion, they will frequently generate empirical arguments to do so. They also sometimes claim a deductive argument does not supply them with certainty that the assertion is correct. Mathematics educators frequently attribute this to students having deficient standards of conviction. In this paper, we illustrate another theoretical account. Students might believe that they lack the cognitive capacity to produce a superior argument to an empirical argument or to verify that a deductive argument is correct.

Key words: Expectancy value; Justification; Proof

In this paper, we consider two robust findings from the research literature on proof. First, when students are asked to justify a mathematical assertion, they frequently do so by generating an empirical argument (e.g., Coe & Ruthven, 1994; Healy & Hoyles, 2000; Recio & Godino, 2001). Second, when some students are presented with a valid deductive argument (i.e., a proof) in support of a mathematical assertion, they will claim not to be certain that the claim is correct (e.g., Chazan, 1993; Fischbein, 1982; Morris, 2002). The research questions addressed in this proposal question why is this the case.

Theoretical Perspective

The Proving as Convincing Research Paradigm

According to Stylianides, Stylianides, and Weber (in press), much of the vast mathematics education on justification and proof uses the metaphor that proving is tantamount to convincing, in which a critical goal of the Proving as Convincing paradigm is to identify the types of arguments that students find convincing. A common methodology to do so is to provide students with justification tasks, classify the students' justifications by the types of arguments that they use to support the assertion (e.g., empirical generalizations, logical deduction, appeals to authority), and then infer that students presented these arguments because those are the types of arguments that are most convincing to students. Consequently, when students justify claims by empirical arguments, these researchers infer that students gain certainty on behalf of empirical arguments, or at least that empirical arguments are students' preferred mode of justification. Researchers find this inference undesirable on the grounds that mathematicians obtain conviction from deductive arguments or proofs, not from empirical arguments (c.f., Harel & Sowder, 2007). Another common methodology in the Proving as Convincing paradigm is to present students with a proof of an assertion and then see if they see if they behave as if they still harbor some doubt about that assertion. When students do, researchers also interpret this as undesirable because mathematicians, unlike these students, gain certainty from proof (c.f., Harel & Sowder, 2007).

Recently, the methodologies described above, as well as the epistemological assumptions that underpin them, have been challenged by some researchers. We briefly review some of the challenges here. First, the notion that mathematicians always gain certainty in mathematical assertions on behalf of proofs and that mathematicians never gain certainty on behalf of empirical arguments has been shown to be an oversimplification that does not accord with modern mathematical practice (e.g., Weber, Inglis, & Mejia-Ramos, 2014). Second, studies in which students are interviewed by their purportedly non-normative judgments, such as claiming to be convinced by an empirical argument or retain doubts in a statement that has been proven, have found students to appear more sensible and mathematically normative than the common interpretations for the literature suggest (e.g., Bieda & Lepak, 2014; Stylianides & Al-Mourani, 2010; Weber, 2010). Third, the results from some empirical studies have suggested that assertions that are difficult to prove questions were more likely to elicit empirical arguments because proofs might be too difficult for students to generate (e.g., Knuth, Choppin, & Bieda, 2009; Stylianides & Stylianides, 2009).

Our Alternative Expectancy Value Account

In the educational psychology literature, expectancy value theories are broad theories of motivation that study the relationship of beliefs, values, and goals with action (Eccles & Wigfield, 2002). The central goal of this paper is to show how three central constructs in expectancy theories—values, costs, and likelihood of success—can be used to qualitatively account for students' willingness or refusal to seek proofs of conjectures that they form.

Before sketching out our theory, we highlight an important distinction made by Bandura (1997) on two different expectancy beliefs, a difference that until now generally has not been taken accounted for in the research literature on proof. Bandura distinguished between *outcome expectations*—beliefs that certain behaviors will lead to certain outcomes—and *efficacy expectations*—beliefs that whether one has the capacity to perform the behaviors to achieve these outcomes. When students produce an empirical argument, we argue that we need to distinguish between two accounts: (i) the student believes the empirical argument provides certainty and (ii) the student believes that he or she does not have the capacity to produce a better argument. (i) is an outcome expectation and (ii) is an efficacy expectation. As we noted earlier, (i) is a common inference in the literature and suggests that students are not aware of the limitations of empirical arguments. However, we will argue that (ii) is sometimes the more accurate account. Similarly, if students are not certain of a claim after reading a proof of the claim, two possible accounts are (i) the student does not think that proofs can provide certainty or (ii) the student questions whether he or she has the ability to be certain that a proof is correct. Again, researchers often infer (i), an outcome expectation, but we believe that (ii), an efficacy expectation, often provides a more accurate account.

Value. When students are asked to justify a mathematical assertion, we assume that students want to achieve a high level of conviction for why an assertion is true. Our premise is that the *value* that students put on obtaining high levels of conviction will influence how hard students will work to obtain a better justification and whether they will settle for a non-optimal explanation as good enough. These values can be externally or internally imposed. Externally, submitting a false conjecture for an assignment (or a true conjecture that is not adequately supported) can result in the student receiving a negative outcome, such as a low grade for a course, or otherwise result in some other consequence that would hurt their academic or career goals. Eccles and Wigfield (2002) referred to this as Utility Value. Internally, students may wish to resolve a genuine curiosity as to whether a conjecture is true or false; Harel (1998) referred to as Intellectual Need. Also, a student may wish to avoid presenting a false conjecture to his or her peers or teacher as this would harm his or her mathematical self-image; Eccles and Wigfield (2002) referred to as Attainment Value. The higher the perceived value of determining the truth of a conjecture, the more likely a student is to work to produce a justification that bestows a high level of conviction.

Cost. Once a student has an imperfect justification for why a mathematical statement is true, the time spent searching for a superior justification, including a proof, is time that could be spent on other enjoyable activities, such as talking to friends. Here it is important to observe that the cost of searching for a proof may depend heavily on the individual and the situation. For many students, attempting to write a proof is a painful activity that brings up feelings of intellectually inferiority (e.g., Weber, 2008). Other students and mathematicians may *enjoy* seeking a proof, so engaging in the activity of proving is an end in and of itself—having what Eccles and Wigfield (2002) referred to as a high Intrinsic Value. In the latter case, individuals will *avoid* having someone else show them a proof because it would deny them the opportunity to search for that proof themselves. The higher the cost, the less likely it is that a student will work to produce a proof.

Likelihood of success. In deciding whether to pursue an objective, students will make a subjective estimation of how likely they are to achieve that objective. If students have an imperfect justification for why a mathematical statement is true, they are unlikely to seek a proof unless they believe that there is a reasonable chance that their search will be successful. If students think it is highly unlikely that they can prove a statement, they may simply not seek one and settle for an empirical justification, even if they are aware that a proof, in principle, bestows more conviction than the empirical argument.

Earlier, we argued that when a student submitted empirical arguments, researchers have often drawn the conclusion of that student having a limited epistemology. The researcher made the inference that the student believed that empirical arguments can bestow certainty, or at least more conviction than deductive arguments. Here we offer three other possibilities: (i) students might not be interested in being certain of the conjecture in question, (ii) they might find searching for a proof to be an unpleasant endeavor that is not worth the effort, or (iii) they might settle for the empirical argument because they believe that they lack the capacity to find a proof.

Methods

Rationale

In this study, we asked prospective and inservice secondary mathematics teachers to work on challenging problems that invited students to use empirical reasoning to make conjectures. We asked participants to share with us both their answer to the problem and a justification for why they believed their answer was correct. Following Stylianides and Stylianides (2009), we also asked participants how confident they were in their answers on a scale of 0 through 100. In cases where participants gave a response of less than 100, we asked them why they retained doubt about their answer, what further evidence could give them more confidence that their solution was correct, and why they were not seeking that evidence.

The goal of this study was twofold. The first is to argue that the Proving as Convincing research paradigm often does not offer an accurate account of why students will justify their conjectures with empirical arguments. Consistent with the research literature, in our study, we found that participants frequently justified their answers with empirical arguments. If participants were doing so because they held undesirable standards of conviction, we would expect the participants to have certainty, or at least a high degree of confidence, in their answers on behalf of these empirical arguments. At a minimum, the participants should aver that the empirical arguments provided them with at least as much confidence as a proof could provide. However, this is not what we observed. Second, we want to illustrate how our expectancy value model can explain why participants offered the empirical arguments that they did. Taking into account the participants' perceived value of obtaining complete conviction, cost anticipated in searching for a proof, and likelihood of success in finding a proof allows us to provide coherent accounts of why students produced and settled for empirical arguments. Indeed, we will further argue that in many cases when participants offer empirical arguments, participants are neither behaving irrationally nor inconsistent with mathematical practice.

Research Context

This study occurred at a large state university in the northeast United States. The data was collected in the first six weeks of a content-based course for prospective and practicing teachers of secondary mathematics focusing on problem solving. The aim of the course was for these teachers to develop the mathematical knowledge and dispositions to solve mathematical problems effectively and to enable these teachers to help their future students to do so. The course met weekly and was co-taught by the first two authors of this paper.

For the first six weeks of the course, class meetings were comprised of the students solving challenging problems in groups for about one hour, presenting their solutions to the larger class, and then reflecting on how this experience related to the nature of mathematical problem solving and the teaching of mathematics. The data from the paper was collected in four of these six meetings.

Our analysis focuses on the 11 prospective or practicing mathematics teachers who completed the course, all of whom claimed to have completed some courses in advanced mathematics. The students worked in four groups, which we call Group A, B, C, and D, which were mostly stable throughout the semester.

Materials. The data was collected when the students worked on four problems, two of which have multiple parts. These problems were chosen such that the answer to the questions could be conjectured via empirical reasoning but each answer could also be justified by deductive argumentation or proof. Problems 1, 2, 3, and 4 were given in the first, second, fourth, and sixth week of the course, respectively. (The problems in week 3 and week 5 were a geometry and a modeling problem respectively. These were not included in our analysis because they did not permit empirical generalizations).

Procedure. For the first six weeks of the semester, class began by having students work collaboratively on the mathematical problems. They were given one hour to work on the problems. If a group had agreed upon an answer to the problem, they could raise their hand to discuss the answer with one of the two course instructors. When this occurred, the instructors would interview the students about their solutions. Otherwise the instructors would circulate the room observing the students' behavior and answering questions that the students may have had about the meaning or interpretation of the problem. The instructors would not, however, provide hints or assistance to the student or confirm that their answers were correct or that the students were on the right track. After 60 minutes had elapsed, if a group did not discuss their solution with the instructors (which was usually the case), the instructors would then go to each group and ask them to discuss the answer that they obtained.

The interviews between each group of students and a course instructor were semi-structured and audiotaped. The instructor photographed the students' written solution. The instructor first asked the student what their solution was and then asked the students to explain and justify their solution. At this stage, the instructor sometimes asked for clarification if he or she was not sure that she understood the students' arguments. The interviewer then asked the students to state how certain their solution was correct on a scale of 0 through 100. If any students gave a response of less than 100, they were asked why they were not certain in their solution, what further evidence could give them more confidence in their answer, and why they were not seeking this evidence. In these discussions, the interviewer carefully distinguished between students' *answers* and their *justifications*. The interviewer would explicitly say that he or she wanted the students' confidence about the correctness of the answer, not the correctness or permissibility of the justification. Interviews typically lasted between five and ten minutes.

Problem 3 had two sub-questions and Problem 4 had three sub-questions. For these class meetings, each sub-question was treated as a separate problem. (i.e., the instructor would go through the protocol for each explanation that each group provided for one of the problems). Hence, each group was given the opportunity to supply seven answers—one for Problem 1, one for Problem 2, two for Problem 3, and three for Problem 4. For some problems, the groups disagreed on the answer and offered multiple justifications. In these cases, each answer was treated separately. In total, 31 answers were offered.

Analysis. All of the audio-recorded data was transcribed. The analysis proceeded through three stages. In the first stage of the analysis, we categorized each justification as empirical or deductive. For the sake of brevity, we do not discuss how we assigned these codes here. In the second stage, we determined the confidence that individual students had in their group's answer. In the third stage, we engaged in thematic analysis (Braun & Clarke, 2006) to categorize the reasons that (1) students expressed certainty in an answer, (2) why students expressed some doubt in their answer (i.e., gave a conviction score of less than 100), and (3) why they did not seek more evidence when they gave a conviction score of less than 100.

Results

Table 1 presents a summary of the arguments that students' produced as well as how confident that the students were that the argument was correct. Table 1 reveals that the majority (18 of 31) of students' justifications for their answers were empirical in nature, replicating the result from the research literature that students frequently justify by empirical arguments. However, when students justified their answer with an empirical argument, they usually retained some doubt about the correctness of their answer, giving a confidence rating of less than 100 in 32 of their 39 (82%) of their ratings. Consequently, these data indicate that students' propensity to justify their mathematical claims with empirical evidence does not imply they believe that empirical evidence can provide certainty that a mathematical claim was true.

	Deductive Justification	Empirical Justification
Number of justifications	13	18
Students with a confidence level of 100 for the justification	27	7
Students with a confidence level of less than 100	6	32
Average confidence level	99.4	68.9

Table 1. Confidence level of students in deductive and empirical justifications.

To understand why students gave the responses that they did, consider Group B's solution and justification for Problem 1, where Billy offered the incorrect answer of ¾.

Problem 1: Suppose there are a row of squares with each square numbered 1, 2, 3, and so on. You start on square 1. You flip a coin. If you get heads, you move up one space. If you get tails, you move up 2 spaces. You repeat this process indefinitely. What is the probability that, at some point, you are on square 25?

[1] Int1: Okay. On a scale of 0 through 100, how confident are you that your solution is correct?

[2] Int2: That the answer is ¾.
[3] Billy: ¾. [Laughter]
[4] Int2: 75. Is that a fair number?
[5] Bob: Wasn't it like the same pattern when he worked it out with 3? So first he worked it out with like 3 and then he tried from like 23 to 25 and then he got like the pattern was still continuing. So that kind of convinced me too.
[6] Brenda: I'd say 98.
[7] Bob: I'm confident. So I'd say 98 too.
[8] Int2: So none of you said 100. So why didn't you say 100? What doubts do you still have?
[9] Billy: For me, I'm a little doubtful because I know like the long way of finding the answer of finding the exact, like the number of heads, the number of tails, the number of choices, blah, blah, blah, but that's way too long. So I'm trying to shorten my way and find the answer. Which means I'm like eyeballing it to say. Because of that, I'm not 100% sure but I'm pretty confident that this is the right way.
[10] Int2: So are you saying you are willing to take a slight risk that this might not be right to avoid the longer process… I'm sorry do you have anything that you would like to add to that Brenda?
[11] Brenda: No, I mean, I'm pretty confident. I would say, because I don't know, if this was a question, I would probably stop at that and be like, I'm going with this question. If it was like an exam and it was pass or fail, this would be my final answer.
[12] Int2: Would there be any evidence or further work that would make you more confident?
[13] Brenda: I guess like what Billy was saying, maybe doing the whole long way, going out but realistically, I don't know. I'm kind of lazy in that aspect.
[14] Int2: You started to answer this, but why wouldn't you seek that evidence…
[15] Billy: It's too long.
[16] Brenda: Yeah.

In this excerpt, we again see the students claiming that they did not find the empirical argument to be fully convincing [lines 6-8]. Billy and Brenda said they would be more convinced by a deductive argument [9, 13] yet both said that they would not seek the deductive argument because it would not be worth the effort [15-16]. What is striking is that Billy claimed that he knew how to produce a proof [9] and he felt that the likelihood of obtaining a proof was high [9]. Even though Billy's confidence in the answer that he offered was relatively low [3], he did not produce this proof. For Brenda and Billy, the value of raising his confidence level was not worth the cost of seeking a lengthy proof.

Group B was told that their answer was incorrect and they were asked to continue working on the problem. Later, Group B obtained the correct answer, justifying it with a deductive argument (the one surveyed by Billy in [9]).

[17] Int: Let me ask you the same question. How confident are you on a scale of 0 through 100 that that is correct?
[18] Billy: [pause] I'll go with 100 for this one.
[19] Brenda: 100.
[20] Bob: I'll go with 99.
[21] Int: 100. Why are you certain?
[22] Billy: For no reason. [laughs] Because I worked, I calculated, I've come up with a formula then I applied it to getting to 3 from 1 and getting to 5 from 1 and it worked out.
[23] Int: So you had the logic for your formula…
[24] Billy: Yeah.
[25] Int: And you checked it for small cases.
[26] Billy: Yes.
[27] Int: Bob, is there a reason that you still have a scintilla of doubt?
[28] Bob: I don't know. [pause] It's just I'm kind of shaky about the whole choose thing. But the way he kind of presented it, I kind of like, did it in an old-fashioned kind of list and I got the same maximum of

like 24 and minimum of like 12 flips. So like I'm agreeing with what he's saying. What's kind of stopping me is the whole choose thing and then we're assuming that order doesn't matter, right?

We observe two things in this transcript. First, Billy claims certainty in his answer [18], but this is not just because he verified the logic in his argument. It is also because he checked that his argument worked in the case of a simple example [22]. This was a common occurrence in this study. Second, Bob obtains a high degree of confidence but not certainty in Group B's answer [20] even though he can follow Billy's argument [28]. Our interpretation of the explanation in [28] is that Bob felt that he lacked the background in combinatorics ("I'm kind of shaky about the whole choose thing") to ensure that there was not a mistake in Billy's reasoning.

In the full paper, we will illustrate that the themes discussed here—that students settled for empirical arguments due to a lack of motivation, that they obtained certainty in deductive arguments by coordinating them with empirical evidence, and that students who expressed doubt in a proven statement did so because they were not confident in their ability to verify a proof, were common occurrences in this study.

Discussion

In our study, we found that, consistent with the research literature, students often justified their mathematical assertions with empirical arguments and sometimes expressed doubt in a statement after reading (and accepting) a proof of that statement. The Proving as Convincing perspective in the mathematics education literature (as described by Stylianides, Styliandies, & Weber, in press) can account for some of these occurrences. There were instances in which a student genuinely appeared to gain certainty from an empirical argument. However, for most instances, the participants submitted empirical arguments because they lacked the motivation to seek a proof or doubted that they could produce a proof if they sought one. Consequently, we propose that our broader Expectancy Value framework can offer a more accurate account for why students offer the justifications that they do. One implication for mathematics education researchers is that they should ask students to express the level of confidence that they have in their solution after they supply a justification (e.g., Stylianides & Stylianides, 2009; Weber & Mejia-Ramos, 2015).

As a final caveat, we note that our study occurred with mathematics teachers who had experience in proof-based mathematics courses. This population appears to differ from other populations in how they construct and evaluate justifications (e.g., Iannone & Inglis, 2010; Weber, 2010). Hence, we do not claim at this stage that our empirical findings will generalize to these populations, only that the theoretical account that we provide in this paper should be taken into account when conducting research with this population.

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Expert vs. Novice Reading of a Calculus Textbook: A Case Study Comparison

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We present case studies of a student and a non-mathematics professor reading an excerpt from a calculus textbook. We use the ideas of sense-making frames and gaps and the implied reader to compare their reading experiences. In particular, we attempt to distinguish the role of calculus background knowledge from reading expertise in making sense of the text.

Key words: calculus, textbooks, sense-making, implied reader, expert vs. novice

Textbooks have long been a staple of mathematics classes, and recent trends toward blended instruction mean that students are increasingly expected to learn independently from a variety of text materials (e.g., Maxson & Szaniszlo, 2015). However, research has suggested that mathematics students struggle to read their textbooks effectively (e.g., Shepherd, Selden & Selden, 2012). Consequently, it is important to understand how students make sense of and learn from reading mathematics text materials.

There has been relatively little research investigating how students read and comprehend mathematical texts. Osterholm (2006) found that students were less successful at interpreting passages that were written in "mathematical English." Shepherd, Selden, and Selden (2012) found that undergraduate calculus and precalculus students struggled to read their textbooks effectively. Shepherd and van de Sande (2014) compared the reading practices of first-year mathematics students, mathematics graduate students, and mathematicians, and differentiated their reading strategies by background knowledge, self-monitoring, and resource use.

This paper presents a case study of two people reading an excerpt from a calculus textbook. We build on prior research by investigating the sense-making practices of the readers and the interaction of these practices with the readers' background knowledge. In order to highlight the different roles played by sense-making practices and background knowledge, we compare reading episodes for a calculus student and a non-mathematician STEM professor.

Theoretical Framework

To capture the interactive nature of the reading process as well as the constraints inherent to the text, we use two interpretive tools: sense-making frames and gaps (e.g., Dervin, 1983; Klein, Phillips, Rall, & Peluso, 2007) and the implied reader (e.g., Weinberg & Wiesner, 2011).

Sense-Making Frames and Gaps

During the reading process, a person experiences and seeks to organize a collection of phenomena. Based on prior knowledge and experience, the person selects a *sense-making frame*: "a mental structure that filters and structures an individual's perception of the world by causing aspects of a particular situation to be perceived and interpreted in a particular way" (Weinberg, Wiesner, & Fukawa-Connelly, 2014, p. 169). The reader then encounters *gaps*, which are questions that the reader (explicitly or implicitly) asks in order "to construct meaning for the mathematical situation" (Weinberg, Wiesner, & Fukawa-Connelly, 2014, p. 170). *Bridges* are the answers that the reader constructs in response to these questions.

Weinberg, Wiesner, and Fukawa-Connelly (2014) identified four types of sense-making frames: *content frames* focus on the meaning of the mathematical content of the text; *communication frames* focus on the text's organizational structure; *situating-mathematics frames* focus on the mathematical significance of the text; and *situating-pedagogy frames* focus on pedagogical choices that reveal the meaning or significance of the mathematical ideas.

The Implied Reader

To realize the learning opportunities afforded by a text, there is a limited range of productive interactions with—and interpretations of—the text. Weinberg and Wiesner (2011) defined the *implied reader* as "the embodiment of the behaviors, codes, and competencies that are required for an empirical reader to respond to the text in a way that is both meaningful and accurate" (p. 52). For example, competencies for reading a calculus text that describes an application of integrals might include understanding integrals as accumulation. Codes might include recognizing that an image next to text implies that the text describes the image. Behaviors of the implied reader could include filling in logical details of an argument or derivation.

Drawing on this framework, our research questions are:

1. What gaps does each reader experience and how do they bridge those gaps?
2. How can the reader's gaps and bridges be characterized in terms of their sense-making?
3. How can the reader's gaps and bridges be explained by the implied reader of the text?

Methods and Methodology

We invited two groups to participate in the study. For the first group, we invited all 20 students in a second-semester calculus class (at a midsized undergraduate institution in the northeast) to participate in the study; five students volunteered and all participated in the interviews. For the second group, we personally invited (from the same institution) three faculty members in the physics department and one each in the chemistry, biology, computer science, and economics departments; all agreed to participate.

For this paper, we selected one student, "Peter," and one professor, "Professor M," to use as case studies. At the time of the interview, Peter was a sophomore with a major in architectural studies and a minor in mathematics. He had earned a B in Calculus I and ultimately earned an A- in Calculus II. Prior to the interview, Peter had regularly read sections from his textbook (Hughes-Hallett et al., 2012) and completed related activities, such as writing summaries and solving problems. Professor M was a tenured professor with a PhD in chemistry, and had taken two semesters of calculus in the early 1990's. He indicated that he did not use calculus ideas in his research or teaching but regularly performed algebraic manipulations as part of teaching.

In order to have the students match the implied reader's competencies as much as possible, we selected textbook excerpts for interviews where the students had been taught the prerequisite knowledge. Both interviewees read two excerpts from *Section 8.2: Applications to Geometry in Calculus* (Hughes-Hallett et al., 2012). The first excerpt, shown in Figure 1, gave a description of the general computational strategy used in the section. The second excerpt, shown in Figure 2, included a derivation of the arc length formula.

At the beginning of the interview, we asked participants to describe their knowledge of integrals and Riemann Sums and gave them a problem about interpreting an integral for a graph of electrical current vs. time to assess their understanding of integration concepts. Next, they were asked to read an excerpt from the textbook using a message q/ing protocol (e.g., Dervin, 1983): we instructed the participants to hold a pen next to the line they were reading, and identify places where they thought the text was unclear or confusing. After completing the reading, participants were asked to describe the main ideas of the section. The interviewer then used an *abbreviated timeline* protocol (e.g., Dervin, 1983), revisiting each paragraph in the text and asking the participants whether they felt that any aspect of the paragraph was unclear.

In Section 8.1, we calculated volumes using slicing and definite integrals. In this section, we use the same method to calculate the volumes of more complicated regions as well as the length of a curve. The method is summarized in the following steps:

To Compute a Volume or Length Using an Integral

- Divide the solid (or curve) into small pieces whose volume (or length) we can easily approximate;
- Add the contributions of all the pieces, obtaining a Riemann sum that approximates the total volume (or length);
- Take the limit as the number of terms in the sum tends to infinity, giving a definite integral for the total volume (or total length).

Figure 1. Section 8.2 Overview

A definite integral can be used to compute the *arc length*, or length, of a curve. To compute the length of the curve $y = f(x)$ from $x = a$ to $x = b$, where $a < b$, we divide the curve into small pieces, each one approximately straight.

r Eight USING THE DEFINITE INTEGRAL

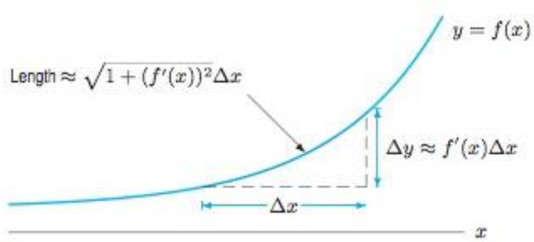

Figure 8.27: Length of a small piece of curve approximated using Pythagoras' theorem

Figure 8.27 shows that a small change Δx corresponds to a small change $\Delta y \approx f'(x)\,\Delta x$. The length of the piece of the curve is approximated by

$$\text{Length} \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \approx \sqrt{(\Delta x)^2 + (f'(x)\,\Delta x)^2} = \sqrt{1+(f'(x))^2}\,\Delta x.$$

Thus, the arc length of the entire curve is approximated by a Riemann sum:

$$\text{Arc length} \approx \sum \sqrt{1+(f'(x))^2}\,\Delta x.$$

Since x varies between a and b, as we let Δx tend to zero, the sum becomes the definite integral:

For $a < b$, the arc length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

Figure 2. Arc length introduction, diagram, and formula derivation

All interviews were video-recorded to capture the discussion and participants' gestures. The audio portions of interviews were transcribed, and notations were added describing participants' gestures. We then collaboratively identified the following aspects of each interview:

- a description of each gap, summarizing the question that the reader experienced;
- our hypothesis about the source of the gap (e.g., mismatches with the implied reader);
- a description of each bridge and a hypothesis about how the reader formed the bridge;
- the sense-making frames the interviewee used while experiencing each gap and bridge.

Results

Case Study 1: Professor M

We identified 12 distinct gaps that Professor M experienced while reading. Here we describe four gaps, representing the variety in Professor M's gaps, bridges, and sense-making frames. The italicized questions below each represent a gap as summarized by the research team.

Background Knowledge. When asked what he knew about integrals and Riemann sums, Professor M recalled some ideas related to integrals: "taking areas under curves" and "doing areas of three-dimensional shapes." However, when working on the pre-reading problem, he appeared to confuse integrals and derivatives. He indicated no knowledge of Riemann sums beyond recognizing the phrase.

What is a Riemann sum? Professor M gave a summary of the first excerpt (Figure 1):

> We started learning how to calculate volumes of fairly simple shapes, now we're going to use what we learned to do more complicated shapes by cutting these complicated shapes into pieces that are easy to approximate. And then add them together by taking what's called this Riemann Sum, I think.

Professor M's phrasing—"what's called this Riemann sum, I think"—suggested that he experienced a gap about the meaning of a Riemann sum or how it was being used in the text. He appeared to think of this gap as related to terminology knowledge (suggesting a content frame); this was consistent with his self-reported knowledge of Riemann sums and reflected a missing calculus competency that was part of the implied reader.

Professor M appeared to initially guess about the role of the Riemann sum. However, by the end of the interview, he had bridged this gap: he thought of Riemann sums as a sum of small lengths that approximated the length of the curve:

> So I think the Riemann sum is basically taking all these, they're breaking this curve up into very tiny, little lines. Right? And then they're measuring the length of each of those little lines and adding them together.

Professor M also described the relationship between a Riemann sum and an integral:

> The arc length is taking a definite number of smaller lines and adding them together. So you get an approximation of what the length of this whole curve is. This [the integral] is sort of doing the same thing, but with an infinite number of points.

Finally, Professor M responded to a question about why Riemann sums appeared in the text:

> Because that's conceptually easier, I think, to understand…. Like take smaller steps, make smaller ones, and then—conceptually, I think that's a lot easier to, um... process, um… I think—connect that to something real.

Professor M's responses suggest that he built a bridge by coordinating multiple pieces of the text, including the section overview and the arc length introduction. Although his gap appeared to have emerged from a content-oriented frame, his final response suggested that he also employed a situating-pedagogy frame. By coordinating these frames, Professor M was able to make sense of the Riemann sum's role in the derivation process.

Does the diagram go with the first paragraph? What does the first paragraph mean? Reading the second excerpt (Figure 2), Professor M initially experienced two related gaps:

> "To compute the length of the curve y".... Is—Is this [points to diagram] the curve that they're talking about? I guess.... The graphic, this here [points to diagram], is supposed to accompany this [points to first paragraph]? Oh wait [points to 'Figure 8.27' in paragraph after diagram], no. Okay, not really.... Alright, let me then go back. It looks like they're referring to the figure here [points to "Figure 8.27" in paragraph after diagram].

Professor M first expressed a gap about whether or not the first paragraph in Figure 2 referenced the diagram below it. He appeared to initially expect that the "curve" in the first paragraph referred to the diagram. He bridged his gap by examining the text further, locating the text "Figure 8.27" in the paragraph below the diagram, and—based on this—deciding that the diagram was *not* in reference to in the first paragraph. His focus on the organization of the text indicates that he was employing a communication frame.

Resolving this gap prompted Professor M to re-examine the first paragraph in Figure 2. His verbal hesitations suggest that he experienced a gap about the meaning of the mathematical objects described in the first paragraph; he then bridged this gap as he continued to read:

> Okay, 'Compute the length of the curve *y=f(x)*... from *x=a* to *b*.' Okay, *a* to *b*. This [points to *x*-axis on diagram] is the *x*-axis, where *a* is LESS than *b*. Okay. We divide the curve into small pieces. Okay. Alright, alright. Each one approximately straight. So we're taking a bunch of little straight lines [drags pen across diagram as he makes "ch ch ch ch" sounds] and putting them together.... Alright.

In this excerpt, Professor M switched to a content frame, focusing on the mathematical content of the text. Although he identified the diagram as accompanying the paragraph following it, Professor M used the diagram to create his own instantiation of the content of the first paragraph by adding an "*a*" and "*b*" to the "*x*-axis" and drawing lines on top of the diagram.

What is happening algebraically in the "Length" approximation? As Professor M read the "Length" approximation (Figure 2), he appeared to experience and quickly bridge a series of gaps about the algebraic manipulations involved, consistent with a content frame:

> Okay... [moves pen across the equations] Okay.... So this is simple, sort of algebra here. Take the Pythagorean Theorem, plugging in this [points to $f'(x)\Delta x$] for the change in y, combining like terms and so forth, getting that [points to $\sqrt{1+(f'(x))^2}\Delta x$].

His speech suggests that he was implicitly asking and answering questions about how the algebraic expressions were connected. Since the text did not explicitly address how each expression on the line was connected, the process of questioning this line of text and filling in reasoning was a behavior of the implied reader. As illustrated by Professor M's work, effectively doing so required coordinating information from the diagram (about the Pythagorean theorem) and from the previous line of text, as well as drawing on knowledge of algebraic manipulations.

Case Study 2: Peter

We identified nine distinct gaps that Peter experienced and evidence of content, communication, and situating-pedagogy frames during Peter's reading. Below, we describe several of the gaps that Peter experienced, which reflect common themes in his reading experience as well as overlap with the gaps Professor M experienced.

Background Knowledge. Peter described an integral as the area under a curve and indicated that a Riemann sum "splits up the area." While working on the pre-reading question, Peter determined that an integral and a Riemann sum would both measure the total number of electrons passing by a point in a circuit over a certain period of time. He then expressed uncertainty about his conclusion and about whether an integral and a Riemann sum would measure the same quantity. He also indicated confusion about the purpose of a Riemann sum:

> They always just, like, take the Riemann sum and they just have a formula—the formula doesn't change—and then they just put it in an integral. So I was kinda, like, thinking on how.... Why couldn't we just kinda skip the Riemann sum step, 'cause they never really

go over that too much in a book and… Whenever like teachers try to explain it, it's kinda just, like, so you know the change in the A or whatever the change in value could be.

Peter's experience appeared to be that one puts a formula into a Riemann sum and then changes the summation symbol into an integral, but he did not know the purpose of doing this.

Where are a and b in the diagram? Like Professor M, Peter experienced a gap while trying to connect the first paragraph and the diagram in Figure 2:

> What threw me off was that in the picture—the graph—there wasn't really an *a* or a *b*. So I was kinda looking for those—I was kinda assuming that there would be an *a* here and a *b* here [traces his pen from the curve down to the *x*-axis in two different spots] but, um.... So I'm thinking those come in later, but I don't know what they had to do with it for now.

Peter used a communication frame to focus on the physical layout of the text and employed a formatting code that graphics should be associated with adjacent text. This code, misapplied to the text, created a gap for Peter about the lack of association that he did not resolve.

What is happening in the "Length" approximation? Peter articulated several distinct gaps while reading the string of equalities and approximations for "Length" in Figure 2. He first indicated confusion about the "1" under the square root sign: "But then they kinda just put one for the change of x squared." Then, he expressed confusion at the appearance of the derivative and the move of Δx: "And then they just put derivative of x squared and took out the change of x"; and finally the relationship between Δy and $f'(x)$: "Like yeah, they kinda just threw that [$\Delta y \approx f'(x)\ \Delta x$] in there without really explaining it."

Peter's attempts to identify the justification for each new term in the derivation suggests that he was using a content frame. He enacted the behavior of the implied reader to question how the algebraic expressions were connected, but he didn't implement the needed competencies to identify the mathematical reasoning for these connections and continued reading without evidence of having bridged any of these gaps.

Why is it important to use a Riemann sum? After reading the integral formula for arc length (in Figure 2), Peter stated:

> It was saying before with the Riemann sum. Like they put the formula that they got here [points to "Length" formula in Figure 2] into the Riemann sum and that's fine. Um.... And then they kinda just say they're putting it into a definite integral.

After completing the reading, the interviewer asked Peter why he thought a Riemann sum was used before an integral in the reading. Peter responded:

> For finding the length of the curve, I don't really understand entirely yet. I was under the impression that Riemann sums were there kinda just to show the understanding that um.... As the change of x gets smaller, then you're, um, you're gonna have a more accurate answer. But change of x in this case [pointing to the diagram] is the... I guess, length of this triangle they're trying to make. And as that gets smaller, it's not like you're getting a more accurate answer, but you're just gonna get a smaller answer for the curve.

Peter hadn't coordinated the sum with the individual terms in the sum, failing to possess the implied reader's fluency with Riemann sums. He largely employed a content frame in these excerpts, focusing on how the Riemann sum fit into the derivation of the integral formula for arc length. However, his attempt to link Riemann sums to larger pedagogical goals indicated he was also using a situating-pedagogy frame. Although this frame could potentially have provided a foundation to make important conceptual connections, Peter's limited mathematical competencies kept him from effectively creating a bridge by coordinating the two frames.

Discussion

The readers described here experienced related gaps about the same parts of the text. However, differences in the ways that they experienced and responded to these gaps gives insight into the role of sense-making frames and background knowledge in the reading process.

Both readers experienced a gap about the diagram in Figure 2 in relation to the text. From the researchers' perspective, the diagram had three potential roles: a "zoomed-in" segment of a generic curve (explicitly referenced in the text), an exemplar of the curve *f(x)* described in the first paragraph, and a display of the result of the "Length" calculation. Each reader's gap was a product of their attempt to employ a formatting code—proximity between text and figures indicates they are related—and the undifferentiated roles of the diagram in the excerpt. However, Professor M experienced this gap as a questioning of the initial formatting code and used another formatting code—texts reference figures where they are being used—to resolve the gap productively. In contrast, Peter expected the issue to be addressed by the text and moved on.

Both readers experienced multiple gaps while reading the "Length" derivation in Figure 2. These gaps were compatible with the implied reader rather than the result of a mismatch between implied reader and actual reader: Here, the text provided little justification for the algebraic derivation, requiring the reader to interrogate the text and to have the appropriate background competencies to fill in reasoning. Professor M was more successful in addressing these gaps, exhibiting a closer match to the behaviors and competencies of the implied reader.

Both readers also experienced a gap about the role of the Riemann sum in the derivation of the formula for arc length. Here, missing or incomplete competencies about Riemann sums, on the part of each reader, played an important role. However, the extent of the mismatch between the implied and empirical readers did not fully explain the degree to which the readers bridged this gap. Peter arguably knew more about Riemann sums (and Calculus in general) than Professor M. Peter attempted to draw on his previous experience with the textbook and instructors' input, but was not able to bridge the gap. Moreover, Peter did not use the reading itself as a source of insight for his confusion. In contrast, Professor M was able to bridge this gap by drawing on different parts of the reading—as well as coordinating pedagogical and content sense-making frame—to deduce what Riemann Sums were mathematically and why they might be pedagogically useful.

The two readers also differed in how effectively they employed non-content sense-making frames. In particular, Professor M used a communication-oriented frame as a launching point for understanding the introduction to arc length, and he ultimately drew on this description as part of his understanding of Riemann sums. Professor M also exhibited a situating-pedagogy frame while describing Riemann sums. Although Peter used communication and situating-pedagogy frames, his use of these frames was not as sustained or reflective and limited their effectiveness.

A mismatch between the implied and actual readers explained some but not all of the gaps that the readers experienced. This suggests that a focus on teaching "content" is insufficient to help students read textbooks more effectively. Moreover, these case studies highlight that some gaps are inherent to the reading process (such as the algebra gaps associated with the "Length" calculation). Normalizing this experience for students and helping them to distinguish between gaps that are inherent to the text and gaps that are related to background competencies may be important for increasing the effectiveness of textbook reading. Lastly, these case studies highlight that an implied/actual reader mismatch does not necessarily prevent the reader from creating valid meaning from the reading; effectively implementing non-content sense-making frames may be one way that readers can effectively overcome this mismatch.

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Mentor Professional Development for Mathematics Graduate Student Instructors

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Abstract

To develop graduate student instructors' (GSIs) skills and abilities as collegiate mathematics instructors, researchers at two universities implemented a peer-mentorship model where experienced GSIs completed a 15-week professional development (PD) to learn how to mentor novice GSIs in teaching undergraduate mathematics. Using pre-survey, post-survey, and semi-structured reflective interviews, we studied changes in 11 mentor GSIs' perspectives on teaching and learning practices and what aspects of the mentor PD were deemed valuable by the mentors. Results suggest that this mentor PD, as a peer-mentorship model, helped GSIs deconstruct the dichotic mathematical paradigm of statements being true or false when discussing teaching. Moreover, mentor GSIs valued how the mentor PD helped guide them to facilitate novice GSI post-observation discussions.

Key words: Graduate Student Instructors, Professional Development, Peer-Mentoring

A key ingredient in a successful collegiate mathematics department is the "effective training of graduate teaching assistants" (Bressound, Mesa, & Rasmussen, 2015, p. 117). This training is crucial because graduate student instructors (GSIs)[1] serve as instructors of record for hundreds of thousands of undergraduate mathematics students each semester (Belnap & Allred, 2009; Lutzer, Rodi, Kirkman, & Maxwell, 2007) and significantly impact the quality of collegiate mathematics instruction across the US. Despite their prevalent role as instructors of undergraduate mathematics, GSIs typically lack guidance and support to teach undergraduate students effectively (Latulippe, 2009; Rogers & Steele, 2016; Speer, Gutmann, & Murphy, 2005; Speer & Murphy, 2009).

When an instructor lacks teaching support, they draw on their teaching beliefs, attitudes, and dispositions to define their pedagogy (Welder, Hodges, & Jong, 2011). Research has shown that an instructors' initial experiences with teaching develops their beliefs and practices of teaching that may last their entire career (Lacey, 1997; Lortie, 1975; Zeichner & Tabachnick, 1985). Thus early teaching experiences of GSIs may shape how GSIs teach in the short term and in the long term as potential future faculty members. Without guidance, for instance, GSIs may struggle to distinguish between how their graduate-level mathematics courses are taught and how they should teach their undergraduate mathematics courses (Speer, King, & Howell, 2014). In order to address this critical need for early support in GSIs' development as effective teachers, this study generated and implemented a mentor professional development (PD) at two American universities to develop experienced GSIs into mentors for novice GSIs (protégés). This paper focuses on the effect of the mentor PD[2] on mentor GSIs' views of teaching and learning.

[1] GSI was used instead of TA (Teaching Assistant) because GSI targets the specific set of graduate students who are full instructors of record.

[2] The implementation of this mentor PD was supported by a National Science Foundation grant at two universities (NSF GRANT 1544342 &1544346).

Related Literature

GSI Guidance and Support

In K-12 teacher education, the critical role of student teaching with a mentor teacher has been recognized as a vital precursor to fully instructing a course (Council for the Accreditation of Educator Preparation, CAEP Standard 2). At the collegiate level, no such standard precursor exists across doctoral granting institutions (Speer et al., 2014). This is due, in part, to the wide variety of roles graduate students may be assigned (e.g. tutors, graders, recitation instructors, or instructors of record) and the limited resources within mathematics departments.

Consequentially, researchers have determined that GSIs are unable to articulate their pedagogical decisions clearly. Rogers & Steele (2016) studied teaching mathematics content courses for preservice elementary teachers and found that GSIs:

> did not demonstrate strong abilities to articulate the reasons for their instructional decisions or identify alternative pedagogical moves that may have led to different outcomes; instead, their interview comments often focused on sufficiently covering the mathematical content and the importance of preparing PSTs with strong content knowledge to be able to teach their future elementary students. (page number)

This lack of explicitly articulating pedagogical decisions could be due, in part, to the fact that mathematics graduate students often start their doctoral program with limited pedagogical course experience (Speer et al., 2014).

Although there can be many individuals who offer general advice about teaching to graduate students, including mathematics faculty members and course coordinators, rarely is this advice individualized enough for the GSI to justify and reflect on their pedagogical decisions (Speer et al., 2014). Shulman (1986) reminds us that for GSIs to justify pedagogical decisions within mathematics requires an understanding of pedagogical content knowledge, "which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (p. 9). The links between content and pedagogy do not develop automatically in teaching (Ball, Thames, & Phelps, 2008) and teachers need to be aware of the goals orienting their decisions from the teacher's and student's perspective (Schoenfeld, 2010). This is where a mentor teacher can be helpful by offering specific advice and justifying that advice with the necessary pedagogical decisions. Many universities have used faculty as mentors for GSIs, however Johnson and Nelson (1999) found that such relationships are ethically complicated and multifaceted because of other hegemonic roles faculty must play, such as doctoral advisors and qualifying exam evaluators. We posit that to be genuinely aware of the individualized pedagogical decisions requires a mentor closely in tune with a protégé's current experiences. To that end, we focused on a mentor PD for experienced GSIs to guide and support protégés.

Mentoring GSIs

Research has indicated that mentoring has social and cultural benefits if mentor GSIs were inclined to help protégés learn how to teach. Johnson and Nelson (1999) indicate that mentoring is central to "quality graduate education" (p. 205), a key component of a successful mathematics department (Bressound et al., 2015). Crisp and Cruz's (2009) meta-analysis of mentoring literature from 1990 through 2007 found that certain subgroups (minorities and females) benefited greatly from peer mentoring, as mentors offer support to socialize professionally, work, navigate, reflect on academic discourse, and help alleviate stress within doctoral programs. Zaniewski and Reinholz (2016) looked at mentoring from a cultural perspective of identity in a peer mentoring program. Admittedly, Zaniewski and Reinholz's study focused on a different population, experienced undergraduates mentoring freshman undergraduates in the physical sciences, but their results demonstrated positive psychosocial and academic interactions resulting in friendships that generated a *community of practice* (Kensington-Miller, Sneddon, & Stewart, 2014) amongst certain majors. Such results are desired within doctoral programs. Thus the literature supports the design and implementation of peer-mentoring for GSIs, yet raises the question: How do we mentor the mentors?

Mentoring curricula

Although teaching experience is necessary, it is not sufficient for mentoring because mentors need to understand *their* role and purpose in facilitating meaningful pedagogical decision-making conversations with protégés (Rogers & Steele, 2016). Consequentially, the design of a mentor PD curriculum should focus training mentors to guide and support protégés' understanding of their pedagogical decisions. Despite the small body of literature structuring mentor PD curricula (Crisp & Cruz, 2009), we note Boyle and Boice's (1998) seminal work on mentor PD that studied mentoring both novice faculty and novice GSIs with tenured faculty where they considered mentoring as the "cousin of faculty development" (p. 158). These researchers compared spontaneous mentoring (talk to the mentor if there are problems) and systematic mentoring (meeting regularly every week) and found that the systematic mentoring was more effective in supporting GSIs and faculty because the mentor could not prepare appropriately when it was spontaneous. Boyle and Boice also observed the topics that dominated mentor meetings in decreasing order of frequency were (1) discussions of undergraduates, (2) teaching styles, (3) teaching-related goals, (4) grading issues, and (5) course preparation.

Boyle and Boice's (1998) results informed the framework of our mentor PD because all five frequented topics could be discussed within the mentor PD through two main responsibilities: observing protégés teach and running small group protégé meetings systematically (not spontaneously). Thus our mentor PD curriculum revolved around observing protégés (including post-observation discussions) and facilitating small group discussions. A natural next question becomes, what impact did the mentor PD have on experienced GSIs? This study examines the results of a 15-week mentor PD around two research questions:

(RQ1) How did the mentor PD change mentors' perceptions of student behavior, student learning, and effective teaching?

(RQ2) What specific aspects of the mentor PD curriculum did the mentors view as most valuable in their preparation as mentors?

Method

Participants

Experienced graduate students at two universities applied and were selected to be mentors by the researchers based on their teaching experiences (aptitude for implementing student-centered techniques), their pedagogical accolades (teaching awards and student evaluations), and most importantly their desire to help novice GSIs to improve teaching at their university (essay responses were required). The number of participants was determined by the average size of each university's mathematics GSI program. Eleven mathematics and statistics doctoral candidates were selected to participate in the mentor PD seminars (four from one university and seven from another).

Mentor PD Curriculum

The goal of the 15-week mentor PD was to equip the 11 experienced GSIs to be effective peer-mentors. The participants and a mathematics education researcher met for 50 minutes once a week to discuss the responsibilities of the mentor as well as to generate frameworks and perspectives necessary for mentoring. Building on the GSI mentoring literature, the Mentor PD curriculum was structured around two main mentor responsibilities: (1) observing protégé GSIs including feedback, and (2) facilitating bi-weekly small group meetings (one mentor with four protégés) which provided a space to discuss teaching.

The first two weeks of the mentor PD incorporated a review of aligning lessons plans, goals, and assessments that mentor GSIs had previously learned in their mathematics teaching pedagogy courses. The next month (weeks 3-6) focused on the GSI Observation Protocol (GSIOP), which was generated by building off of the MCOP2 (Zelkowski, Gleason, & Livers, 2016) and forms used for GSI teaching evaluations. This work prepared the mentors to observe each of their four designated protégés classes three times during the protégé's first course.

The middle of the Mentor PD (weeks 7 & 8) introduced the notion of different colored flags to help mentors prioritize observational feedback for the protégés to digest easily. Although the GSIOP offers a thorough observation, feedback was focused to not overwhelm protégés. No more than two specific issues of each color were flagged (green flag: good, yellow flag: area for growth, red flag: immediate area of concern). Flagging was revisited in the mentor PD (weeks 13 & 14) with hypothetical scenarios in which the mentors role-played post-observation discussions to understand how to approach different teaching styles and perspectives without sounding overly critical or evaluative.

Another major topic addressed was how to effectively design, organize, implement, and facilitate biweekly small group meetings (weeks 9-12). By analyzing written scenarios, mentors learned how to structure and guide their small group meetings by determining who is driving the discussion and how it is being driven. Finally, week 15 brought many ideas together around the theme of *critical reflection* (Brookfield, 1995) by focusing on curricular hegemony and pedagogical efficacy over efficiency. Although researchers collected rich observational data from the mentor PD, space limitations mandated this paper remain focused on the results that would answer our research questions.

Mentor PD Data Collection

At the beginning and end of the mentor PD, the 11 mentor GSIs answered a survey (adapted from Jong, Hodges, Royal, & Welder, 2015) to examine their attitudes, experiences, and conceptions about teaching collegiate mathematics. Since the original survey drew upon the Mathematics Experiences and Conceptions Survey focusing on preservice teachers, we modified Jong and colleagues' instrument to focus on the tertiary instructors. The pre- and post-surveys shared a group of questions that asked the mentors to rate how strongly they agreed with statements in three categories: (a) beliefs about students (15 statements), (b) teacher characteristics (11 statements), and (c) lesson design (3 statements). That is, on a scale of one to five, with one being strongly disagree and five being strongly agree, participants rated their agreement with statements such as: (a) students should use multiple ways to represent concepts and solve problems (beliefs about students), (b) as a teacher I provide wait time and think time regularly (teacher characteristics), and (c) the structure of my lesson must be well organized to effectively achieve its goals (lesson design). We analyzed the mentors' pre- and post-survey responses to help address RQ1. Additionally, we included a question unique to the post survey: "List up to three things you learned from the mentor PD that you believe will help you as a mentor. Please list them in order of significance and briefly explain why." All 11 mentors listed what they learned most and second most while only nine listed a third lesson learned. Our analysis of these responses informs RQ2.

After the conclusion of the mentor PD, an external evaluator conducted 1-hour, semi-structured reflective interviews with each mentor. The mentors were given a copy of their pre- and post-survey responses and asked to elaborate on what they saw in their own responses to triangulate the data. Mentors' responses about how and why their attitudes changed throughout the mentor PD also informs RQ1.

Mentor PD Data Analysis

We first examined the quantitative data by analyzing the pre- and post-survey questions. This quantitative analysis informed the design of the semi-structured reflective mentor interviews, which we qualitatively coded relative to the mentor's responses to the attitudes, experiences, and conceptions on teaching questions. For all 29 pre and post questions, t-tests were used to determine variance before and after the mentor PD. Aggregate quantitative analyses of the pre- and post-survey data were shared during the semi-structured interviews to help answer RQ1. The additional post-survey question (top three things learned) was qualitatively analyzed relative to the topics and course design to determine what aspects of the mentor PD were deemed valuable by the mentors (RQ2).

Results

Change in Attitude (RQ1)

Paired sample t-tests were implemented on all 29 pre and post survey questions to look for variance (N=11 with *alpha*=5%). Although, no variance was significant (which may be due to the limited number of participants, N=11), a few descriptive statistics on change in mean offer insight into how the mentor PD affected specific mentors' attitudes. Due to limited space, we discuss the survey question with the greatest change in attitude. Mentors attitudes that "students' success in mathematics depends primarily on how hard they work" had the largest average negative value change after the mentor PD (ΔM=-0.64, ΔSD=1.43). This indicates that, on average, mentors agreed less with this statement after the mentor PD (not statistically significant however with *alpha*=5%).

In the reflective interviews, 9 of the 11 mentors did explicitly discuss their negative change in attitude on the survey relating student success and hard work. One mentor said,

> When I first took this [pre-survey], I strongly agreed because it sounded right, the way it should be, but what caused me to change was the word 'primarily' because success in mathematics is *in part* how hard they work…but a good teacher *certainly* makes a difference, the resources available *certainly* makes a difference. If not, we are assuming students…are lazy and just don't work hard, which is not true in my opinion.

A second mentor corroborated this perspective in their interview by directly connecting his negative change in attitude to the mentor PD.

> I probably changed my answer after the seminar because I had seen poorer examples of instruction in the mentor PD, which leads me to believe that, despite what I would like, the quality of instruction plays a role in how well they learn the subject material.

Another mentor focused on deconstructing the word success.

> The hard part for me to really piece together about this question is 'student success' in mathematics. If I have a student who comes in with really strong [mathematical] background

comes in and aces all the homework, aces all the exams, versus the student who improves greatly but does not get as good of a grade, what is that in terms of success?

Fundamentally, we see the mentors critically reflecting (a topic from week 15 during the Mentor PD) during these interviews because the mentors are deconstructing the meaning of certain words such as "primarily" and "success" that were taken for granted prior to the mentor PD.

Most Valuable Topics from Mentor PD (RQ2)

In the post-survey, mentors listed the three most important things they learned from the mentor PD in free-response form. We qualitatively coded their responses relative to the curricular topics. For example, a mentor stated that they most valued learning "methods for offering constructive criticism", which was coded under facilitating post-observation discussion. Curricular topics are listed in order of ranking by mentors in Table 1.

Table 1

Top Three Things Mentors Learned from the Mentor Professional Development (PD)

Mentor PD Curricular Topics (Weeks of the Mentor PD Seminar Curriculum)	Number of Mentors who Rated This Topic		
	Highest	Second Highest	Third Highest
Facilitating post-observation discussion (Weeks 7, 8, 13, & 14)	6	2	2
Designing, organizing, & implementing small group meetings (Weeks 9 & 10)	2	2	3
Designing, organizing, & implementing GSIOP (Weeks 3-6)	2	1	3
Facilitating small group meeting discussions (Weeks 11 & 12)	0	4	0
Critical reflection during small group meetings (Week 15)	1	1	0
Lesson goals, assessments, & mathematical task alignment (Weeks 1 & 2)	0	0	0
OTHER-It is okay for people to have different teaching beliefs	0	1	0
OTHER-Teaching is difficult and messy	0	0	1

Two mentors stated comments that spanned multiple curricular topics and were thus listed as "other" in Table 1. As no mentor was coded as indicating the same curricular topic more than once, Table 1 shows that 10 of 11 mentors valued learning how to facilitate post-observation discussion, but critical reflection during small group meetings was mentioned by only two mentors.

Discussion

Summary

In sum, our study provides valuable information about how a peer-mentorship model influenced mentor GSIs' perspectives on teaching and learning (RQ1) and identifies aspects of the model that mentor GSIs found valuable in learning (RQ2). Although the t-tests indicated no significant variance in mentors' perspectives on teaching and learning, the reflective interviews indicated qualitatively that the mentor PD resulted in mentors thinking about certain terms, such as "success" as relative to courses and students they were currently teaching. Thus mentors were able to deconstruct the dichotic paradigm (true/false) prevalent in mathematical statements but not mathematics education. These results align with the results for RQ2 because the mentors suggested that they greatly valued focusing on post-observation discourse. When mentors provide constructive feedback to protégés after observing their classroom teaching, it is crucial that they address teaching concerns with subjective understanding of words such as “success” so as not to indicate to protégé GSIs that there is an absolute correct way of teaching or defining student “success”.

Implications for Research

This study informs the field’s knowledge of GSI guidance and support by illustrating how this mentor PD in the context of a peer-mentorship model can influence experienced GSIs’ understanding of mentorship, teaching, and learning. Through the peer-mentorship model, our research illuminated that the mentor PD did influence GSIs ability to justify their pedagogical understanding of certain terms. Although this does not directly indicate that mentor GSIs were able to justify pedagogical decisions they made in their own classes, their choice to value discursive facilitation and their ability to deconstruct terms such as “primarily” and “success” indicate their ability to consider multiple factors needed to reason and justify pedagogical decisions, a crucial concern of the current literature (Rogers & Steele, 2016).

Additionally, the results of RQ2 corroborate Boyle and Boice’s (1998) recommendation that mentor meetings focus on productive discourse. That is, Boyle and Boice identified that discussion of undergraduates and teaching styles were the most frequented discussions in mentor meeting and Table 1 indicates that facilitating post-observation discourse and facilitating group meeting discussions were the most popular of the highest and second-highest learned topics, respectively. This emphasis on discourse was the foundation of Smith and Stein’s (2011) canonical work on discourse best-practices, which was used in the topics from the review weeks (1 & 2) and creation of the peer-mentorship model.

Implications for GSI Programs

The emphasis on discourse illustrated by the results of this study is an ideal way to begin in developing peer-mentoring programs for GSIs because from this discourse stems the need and value of a community of practice amongst GSIs within mathematics departments. In their research on how mathematics teaching practices shifts undergraduate instructors’ academic identities, Kensington-Miller et al. (2014) emphasize the need for a community of practices for an improvement in teaching practices to take place. These researchers define a community of practice as “a place of collaborative inquiry where various approaches to teaching can be tested through a reflective sharing process . . . . [A community of practice] can contribute to deeper levels of awareness and achieve new learning that can, in turn, lead to significant change” (Kensington-Miller et al., 2014, p. 829) . To work through reflective sharing for achieving awareness, a community requires a safe environment with a knowledgeable facilitator for productive discourse, which was an aspect of our mentor PD that the mentor GSIs greatly valued. Thus our data corroborates and aligns with prior research (Boyle & Boice, 1998; Kensington-Miller et al., 2014; Smith & Stein, 2011), by underscoring the need for mentors to be skilled at facilitating productive discourse in a collaborative environment and systematically organizing mentor meetings in hopes of generating a sustainable community of practice.

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Order of Operations: A Case of Mathematical Knowledge-in-Use

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I describe reactions of secondary school mathematics teachers to the following assertion: "According to the established order of operations, division should be performed before multiplication". I use the notions of local and nonlocal mathematical landscape (Wasserman, 2016) to analyze teachers' responses to the convention of order of operations in general and the presented assertion in particular.

Key words: order of operations, local and nonlocal mathematical landscape, knowledge at the mathematical horizon, associativity

In Canadian schools the acronym BEDMAS is used as a mnemonic, which is supposed to assist students in remembering the order of operations: Brackets, Exponents, Division, Multiplication, Addition, and Subtraction. In American and British schools the prevailing mnemonic is PEMDAS, where "P" denotes parentheses, and it further assists memory with the phrase "Please Excuse My Dear Aunt Sally". Note that while "parentheses" and "brackets" are synonyms, the order of division and multiplication (D and M) is reversed in PEMDAS vs. BEDMAS.

While researchers and educators argue against the use of mnemonics, as it does not support conceptual understanding and may lead to mistakes (e.g., Ameis, 2011, Hewitt, 2012), it is still a shared practice among many teachers. Ongoing discussions related to the order of operations attend to how this order is interpreted by various computational devises, noting considerable inconsistency. My study is concerned with interpretation and explanation of the *order* in the conventional order of operations by secondary school teachers.

Mathematical Landscape and Knowledge-in-Use: Theoretical Underpinning

Wasserman (2015) described how knowledge of "advanced mathematics may positively impact instruction" (p. 29). ("Advanced mathematics" here refers to topics beyond school curriculum, noting significant similarities in curricula across the world). He focused on topics and ideas of abstract algebra and demonstrated not only how these are connected to school mathematics, but also how they can shape or alter the teaching of school mathematics. Arithmetic operations and their properties were among the identified topics, for which abstract algebra can impact teaching. Wasserman included particular examples of planned teaching activities where abstract algebra experience transformed teachers' perceptions, and ultimately their teaching.

In his more recent work, Wasserman (2016) introduced the topological metaphor of *mathematical landscape*. He considered the *local mathematical landscape* to be the mathematics being taught and the *nonlocal mathematical landscape*, as consisting of "ideas that are farther away" (p. 380). He suggested that this division "tackles the notion of mathematical knowledge beyond what one teaches" (p. 380). These metaphors are linked to the notion of "knowledge at the mathematical horizon", focusing on teachers' (rather than students' or curricular) horizons, which Zazkis and Mamolo (2011) re-conceptualized as advanced mathematical knowledge used in teaching.

Wasserman distinguished three perceptions on the significance of exposure to nonlocal (advanced) mathematics: advanced mathematics for its own sake, advanced mathematics as connected to school mathematics, and advanced mathematics as connected to the *teaching* of school mathematics. He asserted that "teachers' development of and understandings about nonlocal mathematics must not only relate to the content of school mathematics, but to the teaching of school mathematics content" (p. 386). This is because exposure to advanced mathematics helps teachers in developing Key Developmental Understandings (KDUs) (Simon, 2006), which change perceptions about content and influence mathematical connections, so in turn, have an impact on teaching.

In what follows I present a story where nonlocal mathematical knowledge can influence teaching in a 'situation of contingency'. I borrow the notion of contingency from the "Knowledge Quartet" framework (Rowland, Huckstep & Thwaites, 2005), in referring to an event that is unforeseen or deviates from the agenda when planning for instruction.

A Story in Two Accounts

I follow the narrative inquiry methodology, where "narrative inquiry is aimed at understanding and making meaning of experience" (Clandinin & Connely, 2000, p. 80). In presenting the story I rely on Mason (2002) in distinguishing between *account-of* and *accounting-for*. The term 'account-of' provides a brief description of the key elements of the story, suspending as much as possible emotion, evaluation, judgment or explanations. This serves as data for 'accounting-for', which provides explanation, interpretation, value judgement or theory-based analysis.

The story is situated in a course "Foundations of Mathematics" for secondary mathematics teachers (n=16), which is a part of the Master's program in mathematics education. Building and strengthening connections between advanced mathematics and school mathematics was an explicit goal of the course.

Account of – Part 1. Background: Conventions Task

One of the assignments for secondary mathematics teachers (here also referred to as 'students') was to consider mathematical conventions. This assignment followed discussion on the choice of a particular mathematical convention, the use of superscript (-1) in different contexts. In prior research, prospective secondary teachers' explanations of the "curious appearance" of superscript (-1) in the two contexts – inverse of a function and reciprocal of a fraction – were studied by Zazkis and Kontorovich (2016). It was found that the majority of participants do not attend to the notion of 'inverse' with respect to different operations, that is, do not view "reciprocal" as multiplicative inverse. Rather, the differences between the contexts were emphasized and analogies were made to other words and symbols, whose meaning is context dependent.

In the conversation with students about the superscript (-1) similar ideas were initially voiced, but later an agreement converged towards a group-theoretic perception of inverse, as exemplified in two different contexts. This provoked interest in the choice of other mathematical conventions, conventions that are often introduced and perceived as arbitrary, rather than necessary (Hewitt, 1999), without any particular explanation. The "Mathematical conventions task" was designed to address this interest.

The idea behind this task was to extend a conversation on the choice of conventions, and acknowledge either the arbitrary nature or the reasoning underlying some of these choices. The students were asked to write a script for a dialogue between a teacher and students, or between students, where interlocutors explore a particular mathematical convention and a reason behind it. The particular conventions were left for the students' choice. The detail of the task is found in Figure 1.

> Choose a mathematical convention and consider possible explanations for the particular choice.
>
> IN YOUR SUBMISSION:
>
> 1. Reflect on the process of choosing the particular mathematical convention for this task. Share alternative conventions that you considered for this task and explain why they were not chosen. (1-2 paragraphs)
>
> 2. Write a script for a dialogue in which interlocutors consider possible explanations for the convention you explored. The dialogue should reflect possible doubts, uncertainties and arguments regarding the suggested explanations. The dialogue should end either with an explanation that interlocutors accept or a summary of the disagreement between the characters. (3-5 pages). The dialogue can begin in the following way:
>
> Sam: Hey Dina, have you ever noticed that (the chosen convention)?
> Dina: Well, everybody knows that.
> Sam: Yes, but did you ever think about why it is so?
> Dina: Why should I think about it? It's a convention.
> Sam: But, still… Can you propose an explanation?
> Dina: Maybe, this is because…
>
> Feel free to modify the proposed beginning of the dialogue.
>
> 3. What have you learned, if anything, from completing this task? (1-2 paragraphs)

Figure 1: Mathematical conventions task

Account of – Part 2. Order of Operations Convention

One of the repeated examples for a convention (chosen by 3 out of 16 students) was order of operations when performing arithmetic calculations. Below is an excerpt from the script written by Andy, who describes a conversation occurring in Grade 8 class.

Sam:	Hey Mr. X, a couple of us can't decide on answer to the following question: $25 + 5 \times 7 - 2 \times 10 \div 5$
Mr. X:	What do you mean?
Mary:	I bet them a dollar that they couldn't get the correct answer to a question I made: $25 + 5 \times 7 - 2 \times 10 \div 5$
Sam:	Well I got 40. Jane says it's 56. Tom believes it's 436, and no one can agree on a solution.
Tom:	Mine is correct! I know it.
Mr. X:	Tom why do you say that?
Tom:	I had a process of how I did mine.
Mr. X:	How so?
Tom:	I just did one operation after another: 26 plus 5 times 7 and so on. See: $25 + 5 \times 7 - 2 \times 10 \div 5$

	$30 \times 7 - 2 \times 10 \div 5$ $210 - 2 \times 10 \div 5$ $218 \times 10 \div 5$ $2180 \div 5$ 436
Sam:	I did something similar, but I started on the right side of the problem: $25 + 5 \times 7 - 2 \times 10 \div 5$ $25 + 5 \times 7 - 2 \times 2$ $25 + 5 \times 7 - 4$ $25 + 5 \times 3$ $25 + 15$ 40
Mary:	You guys did the operations in the wrong order.
Jane:	I agree with you Mary.
Mr. X:	What order would you suggest?
Jane:	Well I did the division first followed by multiplication, addition and subtraction. $25 + 5 \times 7 - 2 \times 10 \div 5$ $25 + 5 \times 7 - 2 \times 2$ $25 + 35 - 4$ $60 - 4$ 56
Tom:	I don't understand why you started with division. Why would you start there?
Jane:	Everybody knows that's the proper order to do operations.
Sam:	Mr. X is that correct?
Mr. X:	Jane is correct. That is the correct order to do those operations.
Sam:	But why?
Mr. X:	A long time ago a group of people had a very similar situation that we have now. They were confused and couldn't figure out who had the correct solution to a problem that involved the very operations you are having problems with. It happened around the early 15th century in a small European kingdom, it was called the Kingdom of Math. The King of Math, as it were, was a very intelligent leader and believed that his people should always come together to solve their problems.
Jane:	Really, Mr. X, a kingdom of math?
Mr. X:	Oh yeah, they were a very progressive country. Several of the King's subjects had come to him to settle a problem that they were having. They couldn't decide on an order of the operations that needed to be used. Sound familiar?
Mary:	Very funny, Mr. X.
Tom:	So what did the King do?
Mr. X:	The king commanded his most trusted advisors, members of the Order of Knowledge, to look into the problem. It took several months before the Order had a response for the King. They proposed that the only way to solve this problem was for the King to proclaim an order to the operations so that everyone would know the correct way to solve the mathematical problem.
Sam:	That makes sense. Then everyone would follow the same order and no one would be confused about what steps to do first.

Andy offered the following comment at the end of the assignment:

"I felt that there was only one reason that I could students: "We need to have an order that everyone follows so we can be consistent". "This is the way we all do it". "We" being us in the math community. Whether you're in France, New Zealand, or Canada it's the same. This is because we've all agreed to use the same order so as to have the same understanding of the operations. I tried to find an actual history of the order of operations, but couldn't find anything concrete. So I decided to make up a story that would hopefully give them some connection to the problem and some entertainment along the way."

Accounting for – Mathematics of "Division First"

It is clear from Andy's commentary that accompanied the script that he perceives the convention of order of operations as an arbitrary decision. The reasoning behind this choice, other than the need for consistency, was unclear to Andy and was not found when sought. Other secondary school teachers agreed with this view.

The teacher-character's agreement with the student statement, "division first followed by multiplication, addition and subtraction" could have been overlooked, as the result was correct. It is further unclear from the script whether the listed order refers to the general convention, or to the particular case. Nevertheless, both the claim of "division first", and the order in which the operations were performed in Jane's example, attracted my attention. For simplicity, let us consider only the last short computation that involves multiplication and division only,

$2 \times 10 \div 5$

Performing "division first" means interpreting this calculation "as if" there are parenthesis around the operation of division

$2 \times (10 \div 5)$

But actually, "division first" and "in order of appearance" yields the same result:

$2 \times 10 \div 5 = 20 \div 5 = 4$ and $2 \times (10 \div 5) = 2 \times 2 = 4$

Is this a coincidence? In other words, is it a "general case" that

$\mathrm{a} \times \mathrm{b} \div \mathrm{c} = \mathrm{a} \times (\mathrm{b} \div \mathrm{c})$?

The situation is easily resolved attending to (a) division is an inverse operation of multiplication and (b) multiplication is associative. Therefore, division can be performed "out of order", as

$$\mathrm{a} \times \mathrm{b} \div \mathrm{c} = \mathrm{a} \times \mathrm{b} \times \frac{1}{c} = \mathrm{a} \times \left(\mathrm{b} \times \frac{1}{c}\right) = \mathrm{a} \times (\mathrm{b} \div \mathrm{c})$$

Account of – Part 3. Addressing "Division First" Assertion

As a consequence of "division first" suggestion in Andy's script, the following assertion was presented to a class discussion:

> Assertion: "*According to the established order of operations, division should be performed before multiplication*."

It was presented a student's claim, for which a teacher's response was sought.

Four students (out a class of 15) agreed with the claim, while others insisted on the "left to right" order, when only division and multiplication appear in a computation. BEDMAS was the presented argument that supported the assertion*. However, majority of students claimed that "division first" was wrong and attempted to find a counterexample, where giving priority of

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* Of interest is that in a class of prospective elementary school teachers, about 70% agreed with the assertion based on BEDMAS

division over multiplication vs. performing these operations in order they appear will lead to different results. (An analogous idea of "multiplication first" or "order does not matter" was suggested, but immediately rejected by a counterexample.)

When "simple" computations did not lead to a counterexample, students turned to more complicated examples. These examples included a longer chain of computations, fractions, and negative numbers. An additional conjecture was voiced that "division first" works only in case there is divisibility between the chosen numbers for division. This resulted in more complicated examples, but the conjecture was refuted after several tests.

In a class session, a search for a counterexample lasted for about 25 minutes. There were occasional exclamations of "Eureka!", which eventually resulted in double checking that uncovered computational errors. A failure to come up with a counterexample, resulted in a conjecture that prioritizing division over multiplication will always work.

Considerable scaffolding was needed to prove this conjecture. When someone suggested that "it works" because "division is just an inverse of multiplication", I countered the claim with "multiplication is just an inverse of division" and "it doesn't work". The suggestion of associativity was voiced only after students were asked explicitly to consider in what ways division and multiplication are different. As a result of focusing on this difference, the assertion was rephrased: *Division can be (rather than should be) performed before multiplication*.

Accounting for – Analysis of Responses

The assertion presented for discussion for secondary school teaches caused a cognitive conflict to both supporter and objectors. Those who supported the conjecture based on BEDMAS were surprised to find out that performing addition before subtraction (A before S) does not lead to an expected result. Those who believed that conjecture was false, and claimed that division and multiplication have the same priority and should be performed in left-to-right order, were surprised to find out that giving priority to division indeed "works".

Each group exhibited a robust "strength of belief" (Ginsburg, 1997), based on knowledge that was entrenched and never questioned, as evident in a lengthy search for a counterexample. Extending the example space in search for counterexamples indicates, in accord with Zazkis and Chernoff (2008) that different examples have different convincing power.

The justification and reformulation of the assertion, based on associativity of multiplication, was readily accepted, and even came with an "AHA!" experience for some teachers. As such, it is curious why such an argument was hard for teachers to find on their own. The theoretical constructs presented above provide a possible explanation.

I suggest that the notion of associativity, even if not so "advanced", does not belong to secondary school teachers' local mathematical landscape. That is to say, they do not teach associativity, and even when this property is acknowledged together with other properties of arithmetic operations, it is mentioned together with commutativity. To elaborate, operations discussed in school mathematics are either commutative and associative, or neither commutative nor associative, which results in frequent confusions between the two (Hadar & Hadass, 1981, Zaslavsky & Peled, 1996). Associativity appears as a property "on its own" when considering groups and their structure. As such, while the notion itself does not require advanced background, knowledge of advanced mathematics reshapes how associativity is perceived. For teachers, associativity appeared to be found in the nonlocal environment, and the connection between local and nonlocal mathematics was not immediately articulated. Furthermore, I suggest that the discussion of the assertion helped in connecting nonlocal mathematics (associativity) to local mathematics (order of operations) in a potential situation of contingency in their teaching.

Discussion

With respect to the conventional order of operations in arithmetic, should division have priority over multiplication? If yes, why so? If not, does giving priority to division lead to an incorrect result? These questions, and unexpected answers, were explored with a group of secondary mathematics teachers. It was concluded that while teachers indeed possessed nonlocal knowledge needed to address these questions, it was not exploited in connection to a contingency situation that may appear in their teaching.

In relation to teachers' mathematical knowledge Wasserman (2016) uses the terms 'nonlocal' and 'advanced' as almost synonymous, referring to knowledge beyond what is taught in school. The example of order of operations demonstrates that 'nonlocal' is not necessarily 'advanced', but situated beyond teachers' "active repertoire" of knowledge used in teaching.

I concur with Wasserman (2016) that "knowledge of nonlocal mathematics becomes potentially productive for teaching at the moment that such knowledge alters teachers' perceptions of or understandings about the local content they teach" (p. 382). While Wasserman (2015) and Wasserman and Stockton (2013) exemplified how planning of instructional sequence and instructional examples can be influenced by teachers' exposure to advanced mathematics, I demonstrated how such exposure can be useful in the situation of contingency.

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Can/Should Students Learn Mathematics Theory-Building?

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Mathematicians commonly distinguish two modes of work in the discipline: Problem solving, and theory building (Gowers, 2000). Mathematics education offers many opportunities to learn problem solving. This paper explores the possibility, and value, of designing instructional activities that provide opportunities to learn mathematics theory-building practices. It begins by providing a definition of these theory-building practices on the basis of which to formulate principles for instructional designs. The paper argues that theory-building practices serve not only the synthesizing role that they play in disciplinary mathematics, but they also have the potential to enrich learners' reasoning powers and to enhance their problem solving skills. These instructional designs offer a new approach to supporting student work on generalization and abstraction. They have been piloted with preservice and practicing secondary teachers.

Key words: theory building, problem solving, abstraction, instructional design.

The Common Core has promoted the teaching of mathematical practices. I take *the practices of an occupation to be the things one does when so occupied.* So what do (research) mathematicians do? Well they commonly distinguish two modes of doing their work: problem solving and theory building (Gowers. 2000). Does (or should) this duality find any expression in mathematics education? Certainly problem solving has a robust presence (Polya, 1957; Schoenfeld, 1994); what about theory building? I argue here that it does make sense, and have value, to give students opportunities to learn what I call (and shall define) to be *mathematics theory-building practices*, and I will present some instructional designs to support this.

Mathematics Theory-Building Practices

Mathematical knowledge is cumulative. Rarely are ideas discarded. What saves it then, after several millennia of relentless growth, explosively in the last two centuries, from sinking under its own weight into some dense mass of ideas? This is achieved through ongoing processes of abstraction, whereby diverse phenomena are made to all appear as particular manifestations of a single unifying theoretical construct. The latter is essentially a piece of conceptual invention with no *a priori* existence. It is born as a distillation, and naming, of what is common to the variety of its existing incarnations. The large-scale cumulative products of these processes are mathematics theories.

Definition

Based on this, I define (***mathematics*[1]*) theory-building practices*** *to be creative acts of recognizing, articulating, and naming something mathematically substantial that is demonstrably common to a variety of apparently different mathematical situations,*

[1] In science, in contrast, Bereiter (2012) writes, "Theory building starts when an explanatory idea is modified or further developed to produce a better explanation."

something that, at least for those engaged in the work, might have had no prior conceptual existence. I refer to an instance of engagement in such a theory-building practice as a ***theory-building act***.

It is important to distinguish mathematical theories from the theory-building processes that produce them, just as proofs are different from the practices of proving. The most familiar example of such abstraction is that of number itself. For example, what is the number seven? We cannot point to it, or say directly what or where "seven" is in the natural world. Rather we can point to many collections of seven things; and we can say concretely what it means for two such collections to have the same cardinal. "Seven" then is a quality that all of these diverse collections have in common, but we can specify concretely only what it means for this quality to be common.

The main products of theory-building practices are of course theories. Some large-scale examples are the theories of groups, topological vector spaces, manifolds, and categories. A college level example might be linear algebra, in which the solving of systems of linear equations is eventually generalized and abstracted in terms of vector spaces, over a field of scalars, and linear transformations. Theories are general, and so achieve a kind of cognitive rescaling or compression of the knowledge edifice. Complex mathematical structures are thus identified and named so that they become, for the expertly initiated, as mentally manipulable as counting numbers are for a child. At the same time theories are sufficiently specific to be susceptible to productively detailed and incisive theoretical exploration. Note that the examples above are the names of theories, but nothing is said here about the rich history of how each of them evolved and came to be formalized, through a long trajectory of theory-building actions.

I will use the above definition of theory-building practices to formulate general principles for the design of some instructional activities that provide learners with opportunities to engage in theory-building practices. I present examples of such instructional designs based on these design principles. These designs involve seeking *relations among a given set of mathematics problems*. Learners are asked to not only solve the problems, but, importantly, to find and articulate *structural* connections among the problems. The designs have been piloted with groups of students and teachers.

The premise of this work is that theory-building, not unlike problem-solving, is a high leverage mathematical practice that, suitably adapted for instruction, is of value for all students, not only those who pursue mathematics as a career.

In a pilot of these problem designs, most students, in their reflections: 1. Professed to enjoy the activity; 2. Found that it would be useful in their teaching; and 3. Noted that, in none of their previous math courses, had they ever been asked questions like this.

A Sample of Related Literature

Mathematical theory-building practices involve a high level of abstraction. Abstraction has been much studied since the work of Piaget (1994) and Vygotzky (1997), and more recently in the study of student assimilation of already formalized abstract concepts, for example (Hershkowitz et al., 2001; Scheiner, 2016).

Some curricular interventions, e.g. the "New Math," introduced students axiomatically to some existing theories. This of course is distinct from theory building.

Early work on structural relations between different problems comes from cognitive psychology studies of transfer, to see whether knowledge of how to solve problem A transfers to the ability to solve an "isomorphic" problem B (Simon et al., 1976; Siegler, 1977). Here A and B are said to be *isomorphic* if there is a correspondence between the objects, relations, and operations of A to those of B so that any solution strategy of A transforms into one of B. Informally, A and B have the same structure, and differ only in superficial features of context. It was found that transfer generally does not occur.

Silver (1979), inspired by Polya's heuristic, "think of a related problem," and building on earlier work of Kruteskii (1976), conducted a study of students' perceptions of relatedness of families of word problems, designed to vary on dimensions of relatedness, notably mathematical structure and context. Among his findings, students with high proficiency levels, by a variety of measures, tended to sort problems by structure, while those with lower levels focused more on context. This reinforced similar findings of Kruteskii.

In Mason (1989) and Mason, et al. (2009) one finds strong arguments for the value of developing a sensitivity to mathematical structure, as well as problem-solving contexts in which there are rich opportunities for structural thinking and exploration.

Maher et al. (2010) report on one of the few longitudinal studies of children's development of high level mathematical practices, including development of schemas for connecting structurally related problems.

In 1968 Zal Usiskin published, in *The Mathematics Teacher*, a brief paper, "Six nontrivial equivalent problems." Although addressed to teachers, the paper did not discuss instructional uses. What unifies these problems is the fact that they can all be modeled by simple algebraic variants on a single Diophantine equation: Find all whole number solutions (n,m) of, $1/n + 1/m = 1/2$. In 2015 I expanded Usiskin's list to a set of thirteen problems, spanning several mathematics domains, all sharing the same mathematical model. This is accompanied by a lesson plan for its use in teacher professional development.

Instructional Design Principles

I turn now to a set of principles for the instructional design of opportunities to engage in theory-building practices. The designs comprise three components: a *problem set*; the *task* presented to the learners; and a sketch of the *instructional enactment*, or "lesson plan." The design principles guide the construction of these components.

Problem Set Design Principles

The definition of theory-building practices calls for, "recognizing, articulating, and naming something mathematically substantial that is demonstrably common to a variety of apparently different mathematical situations." Here I take a "mathematical situation" to refer to mathematics problem. I take "something mathematically substantial that is demonstrably common" to mean a mathematical structure that is centrally involved in each problem, in the sense that the "problem space," or "solution space" of the problem can be described using that structure.

With these specifications I can now state the *problem set design principles*:

1. An assigned problem set should consist of about 3 - 8 problems.

2. Individual problems should be substantial, yet accessible without formidable challenge for the given community of learners.
3. The set of problems should be significantly diverse, for example with variable contexts, or even belonging to different mathematical domains.
4. It must be possible to demonstrate that substantial subsets of the problems involve a common mathematical structure.

Rationale: #1is a practical consideration; #2 is to keep the primary focus on structural relations between different problems, not individual problem solutions; #3 captures the "apparently different" aspect of the definition; #4 is central to the concept of theory-building practice.

The Learners' Task Design Principles

Two formats are proposed.

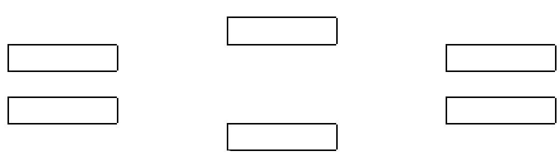
- The **discernment format**: Learners are asked to simply discern and explain relations among a set of problems that do not all involve a common structure.
- The **common structure format:** Learners are asked to identify and articulate a mathematical structure and demonstrate its presence in each of the problems.

The discernment format presents the task as in Table 1.

Table 1
Discernment Assignment Format

Below are problems, labeled A,B,C,D,E, . . . The object is to place the letter of each problem in one of the boxes below.

- *Put letters in the same box if they are mathematically the "same" problem, apart from superficial differences of context.*
- *If problems in different boxes are closely related mathematically, connect their boxes by a line, or by a double line if the connection is very strong. (Note, you need not use all of the boxes, and you may reasonably answer this question even if you have not completely solved the individual problems.)*
- *Work on this individually for a few minutes. Then compare answers in your group.*
- *Try to come to some consensus on how to* ***explain*** *(to the whole group) your choices, in particular the nature of the connections.*



The products in this format will be "connection networks," and these can be quite variable. The processes of explaining them, and then efforts to reconcile differences within and across different groups can help to develop a probing discourse about mathematical relations among problems.

The common structure format assigns the following (interactive) phases:

Phase 1. Solve the individual problems.
Phase 2. Identify and articulate a common mathematical structure, and demonstrate how it is involved in each of the problems.
Phase 3. Notice ways in which the problems are significantly different, and analyze some of the consequences of these differences.

Phase 2, expressly capturing the theory-building component of the task, appears to be the most novel and challenging aspect for the learners. It is not likely that learners will gain high levels of proficiency with theory-building practices from a limited exposure to this kind of work. But it is still possible, even with limited exposure, to awaken a sensibility to the kinds of questions posed, a foundation that can be built upon.

Phase 3 is not directly entailed in the definition of theory-building practices, but pilots of these designs show that a given mathematical structure can be situated in problems in ways that significantly vary the mathematical sense and cognitive demand of the problem. Awareness of this kind of variation is an important aspect of discrimination in theory-building practices.

Instructional Enactment Design Principles

The demands of these designs can be substantial, complex, and novel: solving a substantial collection of non-trivial and diverse problems; finding and articulating a substantial mathematical structure common to the different problems; and noticing significant differences in the structurally related problems.

I have found that the instruction works best when the students work in collaborative groups (of 3-6 students), and that the work extends over a reasonable amount of time, at least two sessions. The individual groups can pool diverse ideas, distribute the work, and reconcile different outcomes. I then have each group prepare a formal presentation of its work to the whole group, allowing then for whole group discussion.

Example Designs

Figure 2.

The 3-permutation set (Discernment)

A. What are all three-digit numbers that you can make using each of the digits 1, 2, 3, and using each digit only once?
B. In a group of five students, how many ways are there to pick a team of three students?
C. You are watching Angel, Barbara, and Clara on a merry-go-round. At each moment you see them in some order – left, middle, right. As the merry-go-round turns, what are all the different orders in which you see them?
D. If Angel, Barbara, and Clara have a race, and there are no ties, what are all possible outcomes: first, second, third?
E. From a bag full of many pennies, nickels, and dimes, I randomly choose three coins. What are all possible coin combinations that I might have?
F. In a 3 x 3 grid square, color some of the nine (unit) squares blue, in such a way that there is exactly one blue square in each row and in each column. What are all ways of doing this?
G. What are all the symmetries of an equilateral triangle?

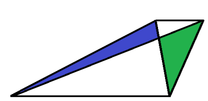
Here B and E are outliers, not deeply related to the other problems, or even to each other, though they both have ten solutions. A structure involved in each of A, C, D, F, G, is the set of permutations of three objects and the solution to each of these problems is in fact the full set of six permutations. In A, D, and F this outcome is demonstrably inherent in the problem, though this fact is most subtle for F. However it can be argued that in C and G there is no *a priori* guaranty that all permutations will be achieved. In fact, when students were asked to reformulate the problems with 4 in place of 3, they found that the new C and G did not achieve all 4! permutations.

Figure 3.
The 8-choose-3 set (Discernment)

A. A taxi wants to drive from one corner to another that is 5 blocks north, and 3 blocks east. How many possible efficient routes are there to do this?
B. On the number line, starting at 0, you are to take 8 steps, each of which is either distance 1 to the right, or distance 1 to the left, and in such a way that you end up at -2. How many different such walks are there?
C. The home team won a soccer game 5 to 3. How many possible sequences of scoring were there as the game progressed?
D. You have coins worth 3¢ and 5¢. With 8 such coins, how many different values can you obtain?
E. From a group of 8 students, you need to select a (5-person) basketball team. How many different ways are there to do this?
F. You are to cut a 9-inch ribbon into six pieces, each of length a whole number of inches. How many ways are there to do this?
G. In the expansion of $(1 + x)^8$, what is the coefficient of x^3?

Here D is an outlier. The solution space of all of the other problems is represented by the set of all "binary sequences of length 8, with 3 terms of one type."

Table 4
The Measure Exchange set (Common Structure)

1. (Tea & Wine) I have a barrel of wine, and you have a cup of green tea. I put a teaspoon of my wine into your cup of tea. Then you take a teaspoon of the mixture in your teacup, and put it back into my wine barrel. Question: Is there now more wine in the teacup than there is tea in the wine barrel, or is it the other way around?
2. (Heads Up) I place on the table a collection of pennies. I invite you to randomly select a set of these coins, as many as there were heads showing in the whole group. Next I ask you to turn over each coin in the set that you have chosen. Then I tell you: The number of heads now showing in your group is the same as the number of heads in the complementary group. Question: How do I know this?
3. (Faces Up) I blindfold you and then place in front of you a standard deck of 52 playing cards in a single stack. I have placed exactly 13 of the cards face up, wherever I like in the deck. Your challenge, while still blindfolded, is to arrange the cards into two stacks so that each stack has the same number of face-up cards.
4. (Triangle Medians) In a triangle, the medians from two vertices form two triangles that meet only at the intersection of the medians. How are the areas of these two triangles related? More precisely, let ABC be a triangle. Let A' be the mid-point of AC, B' the mid-point of BC, and D the intersection of AB' and BA'. How are the areas of AA'D and BB'D related?
5. (Trapezoid Diagonals) The diagonals a trapezoid divide the trapezoid into four triangles. What is the relation of the areas of the two triangles containing the legs (non parallel sides) of the trapezoid? 

A structure involved in each of these problems is a simple principle of measurement: *If two quantities have equal measure, and you remove from each what they have in common, then what remains of each of them will still have equal measure.*

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How Limit can be Embodied and Arithmetized: A Critique of Lakoff and Núñez

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In Where Mathematics Comes From, Lakoff and Núñez (2001) describe how the notions of infinity, continuity, and limit can be constructed through metaphorical extensions of embodied experiences. This paper will critique their historical and psychological analysis, revealing an unresolved tension between a simplified, geometric "approaching" conception and the arithmetization of calculus by Weierstrass. A proposal of how to rectify this conflict through acknowledging how novices can metaphorically tie these concepts together is discussed.

Key words: Embodied Cognition, Limit, Infinity

The mathematical definition of limit has gone through various refinements throughout history (Kleiner, 2001), but its current form, for two-dimensional limit, requires the coordination of two small intervals. One interval's radius, signified by the Greek letter delta (δ), surrounds the input or x-value, while the second interval's radius, represented by the Greek letter epsilon (ε), surrounds the output or the result of the function applied to the x-value. The coordination of the two intervals is accomplished through the logical quantification portion of the definition: for any epsilon, there exists a delta, such that if the input value is within the delta interval, then the output value must be within the epsilon interval. These intervals are symbolized through the distance interpretation of the absolute value, while the logical quantification uses common logical symbols for "for any," "there exists," and the "if …, then …" structure.

Limit can be mathematically conceptualized in two different ways, regardless of how a student could come to understand the concept: either in a dynamic way or in a static way (Cornu, 1992). The dynamic conception is a motion-based idea of limit, frequently expressed using the word "approaching" (or similar words describing movement), or graphically demonstrated by showing a point moving along a graph towards another point. This was also the first way in which limit was historically conceived by Newton and Leibniz (Kleiner, 2001). Students thinking in terms of a dynamic conception would describe, through words and gestures, a limit as a point traveling along a curve.

The static conception of limit is an arithmetic-based idea of limit which coordinates static intervals. This conception matches the formal definition devised by Cauchy and Weierstrass (Kleiner, 2001), and can also be thought of as a way to precisely describe closeness. Students' thinking in terms of a static conception would describe, again through words and gestures, limit as coordinated intervals. In summary, the dynamic conception emphasizes motion in describing limit, while the static conception is devoid of motion and instead emphasizes closeness.

The trajectory of mathematics education research of limit began with an initial focus on establishing misconceptions in undergraduates' understanding of limit (Bezuidenhout, 2001; Davis & Vinner, 1986; Tall & Vinner, 1981), which then evolved into proposals on how students might build appropriate conceptions of limit (Cottrill et al., 1996; Lakoff & Núñez, 2001; Williams, 1991, 2001). These proposals brought about several more studies yielding successful interventions into students' limit conceptions (Boester, 2010; Oehrtman, 2009; Roh, 2008, 2010; Swinyard & Larsen, 2012). However, while these most recent studies expanded the underlying framework for the learning of the limit concept, issues with how general theories of advanced mathematical thinking were initially applied to explain limit thinking in novices (students) remain. In particular, an unresolved tension exists between how Lakoff and Núñez explain the transition between a geometric "approaching" conception of limit and how Weierstrass arithmetized Calculus to produce our modern, static conception of limit.

The Problem of Conceptualizing "Approaching" Through the BMI

Lakoff and Núñez introduce limit in *Where Mathematics Comes From* (2001) by first discussing one-dimensional limit: an infinite sequence along a number line that approaches a particular number (the limit). To demonstrate the process of "approaching" a limit, and how this utilizes the BMI, or Basic Metaphor of Infinity (p. 155), they give an example sequence $\{x_n\} = n/(n+1)$. As n increases, the value x_n gets closer and closer to 1 (p. 187).

The authors choose to express this notion of "approaching" by utilizing the notation of the formal definition of limit in terms of sequences: the sequence $\{x_n\}$ has L as a limit if, for each positive number ε, there is a positive integer n_0 with the property that $|x_n - L| < \varepsilon$ for all $n \geq n_0$ (pp. 189-90). Unfortunately, this definition cannot use the BMI directly, because there is nothing being iterated. In order to bridge the gap, they decide to express "approaching" in terms of nested sets, whose iterative quality can be directly used by the BMI: $0 < r < |x_n - L|$, where R_n is the set of all values r bounded between zero and $|x_n - L|$. As $|x_n - L|$ gets smaller, the range of values in R_n gets smaller. Since the largest value of R_n is decreasing, these sets can be nested: $R_{n+1} \subset R_n$. This chain of nested sets is then used in the BMI to obtain the limit (where the "last" nested set, R_∞, would be the empty set). Lakoff and Núñez take the expression $0 < r < |x_n - L|$ to be synonymous with $|x_n - L| < \varepsilon$ so that we can then extend the BMI to cover the standard formal definition.

Lakoff and Núñez are required to use the BMI to explain the concept of limit, because limit utilizes the concept of infinity. Because infinity does not exist in reality, this presents a critical challenge to a theory of mathematics that is entirely based on real-world, embodied experiences, hence the necessity of the BMI. Since the standard formal definition of sequences doesn't use an indefinite iterate process contained within the BMI, they need to formulate a new definition which does, hence nested sets. Since nested sets can be plugged into the BMI as an indefinite iterative process, they can thus be used to explain our conception of limit.

While this explanation follows the standard embodied cognition model of linking metaphors (including the BMI) built up from grounded metaphors based on embodied experience, there are several problematic issues with this explanation of how we comprehend limit. First, $0 < r < |x_n - L|$ is technically not synonymous with $|x_n - L| < \varepsilon$. $0 < r < |x_n - L|$ can be used when x_n is already known to be approaching L, while $|x_n - L| < \varepsilon$ is essentially checking to make sure. For example, if the actual limit of the sequence is L_t, where L_t falls within the neighborhood defined by $|x_n - L|$ for all n, then the values of $0 < r < |x_n - L|$ can still create a chain of nested sets that follow $R_{n+1} \subset R_n$ due to the density of the real numbers, however R_∞ would no longer be the empty set (instead, it would measure the distance between L and L_t). This distinction is not possible with $|x_n - L| < \varepsilon$, as for some ε, $|x_n - L| \geq \varepsilon$. $0 < r < |x_n - L|$ assumes that L is the actual limit, while $|x_n - L| < \varepsilon$ proves it is so.

More importantly, however, in their attempt to mathematize the concept of "approaching", they are completely ignoring the natural, physical-ness of approaching. What Lakoff and Núñez are trying to do is characterize the distance between you and the limit: the distance becomes smaller as you approach the limit. However, they use a complicated mathematical concept (nested sets) encoded in a complicated mathematical notation, when a description of "approaching" as simply going towards something would suffice. This notion of "approaching" more closely resembles one of the four grounding metaphors (p. 50), arithmetic as motion along a path. However, "approaching" might even be considered a grounding metaphor for limits, because it ties the concept of limit back to the image schema of motion along a path.

In fact, thinking of "approaching" in this simple, physical way may even help students understand one very important piece of the formal limit concept. The same linguistic distinction between "jumping" and "swimming" can be made here: "approaching" is an imperfective aspect (p. 156), because it does not inherently mean that you arrive at what you are approaching. This is actually helpful because it matches the definition of limit (in that a function may not exist at the limit). You may never reach the limit, because it may not exist, even though you are approaching it. Reaching a limit is a consequence of the BMI – in order to have a limit, you must "reach" it at the infinite step (at least potentially, if not actually). While it may be important sometimes to know what happens at the limit (for example, when considering continuity), this is not required when finding the limit itself.

Oversimplifying "Approaching" by Ignoring Complicated Approaching

While Lakoff and Núñez overcomplicate matters by using nested sets, they completely oversimplify the ways in which limits can be approached that are covered through the absolute value-based definition. Recall the example sequence $\{x_n\} = n/(n+1)$. Notice that this example is strictly monotonic, in that it creeps up on the limit in one direction, always getting closer and closer to 1, never farther away. This matches the smooth, motion based "approaching" conception of limit. Taking smaller and smaller steps towards a wall would be a physical example of this.

However, sequences (and functions) can approach limits in far more complicated ways than simply monotonically. The authors make an attempt to fix this by introducing sequences which converge indirectly but still have a limit. For example, they give a "teaser sequence" 3/6, 4/6, 5/6, 9/12, 10/12, 11/12, 15/18, 16/18, 17/18, 21/24 … which bobs up and down while generally trending towards 1 (pp. 192-3). This improves their coverage of possible types of limit convergence, but does stretch the metaphorical interpretation of limits as "approaching". This would correspond to a physical example where, for each step taken towards a wall, a step a fraction of that size is taken going away from the wall (remembering that the steps towards the wall are getting smaller and smaller).

The physical interpretations of both monotonic and indirect convergence also illuminate another way that students may think of limit, as a barrier that cannot be crossed (Davis and Vinner, 1986). Mathematically, this is called a bound. Both examples presented are bounded at the limit because they only approach from one direction. You do not have to consider incredibly complicated sequences to find ones that approach a limit from both sides (even though Lakoff and Núñez (1997, 2001) restrict this property to a selection of "monster" functions that inspired the transition to the arithmetization of Calculus). There are many examples of sequences that do this, such as $\{x_n\} = (-1)^n / n$, which approaches zero from both sides.

"Approaching" from either side of a limit is covered in the absolute value portion of the formal definition: by making a range around L, it doesn't matter if you are above L or below it, as long as you are no more than ε away. Lakoff and Núñez also cover this possibility in their definition when they utilize absolute value notation for nested sets, although they do not comment about the implications of such notation. Remember that the nested sets R_n are defined as the sets of the values r can take when $0 < r < |x_n - L|$. Why should the absolute value be used here, since in the examples presented, all instances of x_n are less than L (thus simply writing $0 < r < L - x_n$ would suffice)?

While the examples only require $0 < r < L - x_n$, they use $0 < r < |x_n - L|$ because using the absolute value is necessary in the formal definition. If their definition did not cover limits which approach from both sides, they would be defining "approaching" in a lopsided way which was fundamentally different from the formal definition. In order to motivate using $0 < r < |x_n - L|$, they should have used an additional, more complicated example, but this would have stretched the physical interpretation of "approaching" past its breaking point. While it is still possible to stretch this motion-based concept to include approaching from both sides, there are some important implications of this.

First, you will be approaching the limit from both sides, which is difficult to conceptualize using the types of physically-based examples used to ground the "approaching" concept. Using the same strand of physical examples from above, it would be like trying to approach a wall from both sides simultaneously. Second, you would need to abandon the idea that the limit is a barrier or bound, because you would be (repeatedly) passing it. You could not conceptualize the wall in our example as a barrier. Third, the sequence or function may equal the limit while it is "approaching" the limit. In taking the limit of the function $y = x\sin(1/x)$ as x approaches zero, there are many times where the function has x-values near zero which yield y-values that are zero. In fact, we don't even need to turn to something this complicated to see this. A constant sequence or function will equal its limit everywhere. This really challenges the concept of "approaching" the limit, since, for a constant function, you don't actually move.

These three implications pull the "approaching" metaphor away from the natural, motion-based conception of limit stated at the very beginning of this section. They also create problems for the nested-set conception of "approaching". A set of points on one side of a limit could not be nested inside of a set made from points on the other side of the limit (without resorting to some argument either about the size of the set rather than its actual contents or using sets which straddle the limit and are not bounded at either end).

Conflating "Approaching" With the Formal Definition

If Lakoff and Núñez want to stay with the embodied, motion-based "approaching" conception of limit, they need to stick with simple examples which are not stretched too far from normal experiences (thus avoiding the implications above). However, in mathematizing these simple examples, they should not use the absolute value, because it is not necessary. This means that whatever mathematical definition they try to establish, based on the examples, it cannot use the absolute value, which means that it cannot duplicate the formal definition (figure 1, argument one).

If they want to motivate the formal definition, then they must use the absolute value as it appears in the formal definition when using the BMI. In order to motivate the usage of the absolute value, they need more complex examples which approach the limit from both sides or converge indirectly. In doing this, they then must use much more complicated explanations of "approaching" than those which naturally arise because of the implications discussed above (figure 1, argument two).

The authors try to do both. Their top-down argument originates with the desire to motivate the formal definition using the BMI. They cannot do this without using the absolute value. However, they examine the "approaching" conception of limit and use examples which naturally tie into this conception. Unfortunately, there is no reconciliation between the simple examples and their usage of the absolute value (figure 1, Lakoff and Núñez).

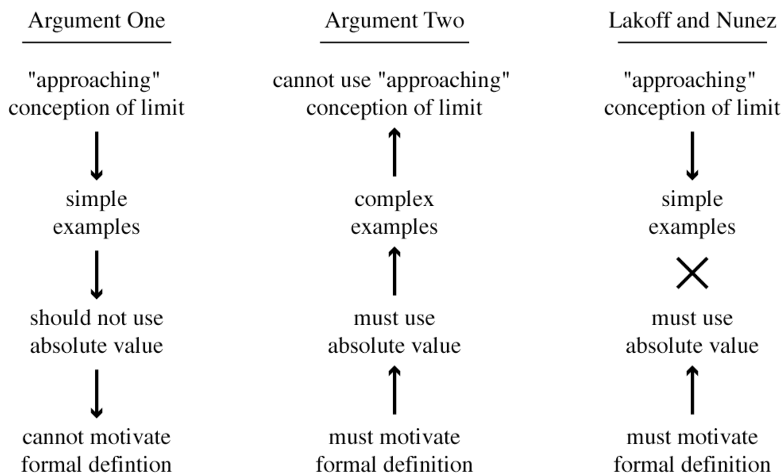

Figure 1. Linking the informal, motion-based conception of limit with the formal definition.

The authors make what, at first glance, appears to be a minor oversight. Their examples of "approaching" a limit do not require using the absolute value to characterize their behavior. When they formalize the "approaching" conception of limit through their usage of conceptual metaphor, they use the absolute value. Unfortunately, by missing the true importance of the absolute value and its cognitive implications in connection with the formal definition and the "approaching" conception of limit, their argument that this is the way in which we understand limits falls apart.

The underlying reason why this merging of the "approaching" conception of limit and the formal definition doesn't work is because the informal definition is based on motion, while the formal definition is devoid of any sense of motion because of its basis on the concept of range or proximity. The BMI uses definite iterative processes to explain indefinite continuous processes, but the formal definition is not a process as such: it is a static entity, not a dynamic one. Lakoff and Núñez had to build a stop-gap formal definition which used an indefinite iterative process (nested sets) in order to apply the BMI to the formal definition. While the BMI can explain the "approaching" conception of limit because of its basis in a dynamic process, this usage simply cannot be connected to the formal definition. In attempting to leap the gap between an "approaching" conception of limit and the formal definition through their unmotivated usage of the absolute value, it is clear that they are missing this vital distinction. Their mixing of the formal definition with the approaching conception is thus fundamentally flawed, and taints both interpretations of the limit concept.

Lakoff and Núñez (1997) attempt to get around this by stating that the formal definition actually has noting to do with a static or a dynamic conception of limit, but rather that the quantification (the logical structure) in the formal definition is the important part. They claim:

> Many students of mathematics are falsely led to believe that it is the epsilon-delta portion of these definitions that constitutes the rigor of this arithmetization of calculus. The epsilon-delta portion actually plays a far more limited role. What the epsilon-delta portion accomplishes is a precise characterization of the notion of "correspondingly" that occurs in the dynamic definition of limit where the values of $f(x)$ get "correspondingly" closer to L as x gets closer to a. That is the only vagueness that is made precise by the epsilon-delta definition. (Lakoff & Núñez, 1997, p. 71).

While it is true that mathematicians must formalize the concept of "correspondingly", that, in and of itself, does not formalize "closeness". Another way to express the "approaching" metaphor is to ask, as $f(x)$ gets closer to L as x gets closer to a, just how close do we need to be? The correct answer is: as close as we want. It is this statement, "as close as we want", that also needs to be formalized, and this is not covered by simply formalizing "correspondingly".

To demonstrate how formalizing "correspondingly" fails to include a formalization of "closeness", we can examine Lakoff and Núñez's "dynamic epsilon-delta limit" definition: "$\lim_{x\to a} f(x) = L$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that as x moves toward a and gets and stays within the distance δ of a, $f(x)$ moves toward L and gets and stays within the distance ε of L" (Lakoff & Núñez, 1997, p. 71). Here they retain the quantification of the formal definition, but express the rest in terms of "approaching". This mixture of the dynamic ("moves towards") and the static ("gets and stays within a distance") conceptions of limit results in a definitional overkill. Once one has the static pieces of the definition, the dynamic pieces are technically unnecessary. If you simply have the dynamic pieces, then you do not necessarily have all of the pieces of the formal definition (because of the nature of "approaching"). In their attempt to show how a formalization of "correspondingly" is the key component of the formal definition, they are unable to avoid including pieces of the static definition. Thus, while quantification is important, it is not the only difference between the formal definition and the "approaching" metaphor.

A Proposed Resolution: Connecting "Approaching" With the Formal Definition

The "approaching" conception of limit and the formal definition cannot be reconciled the way Lakoff and Núñez propose. Instead of the argument presented, the authors could have simply kept the motion-based conception and the formal definition initially separate, instead of conflating the two. First, they could have presented the motion-based, "approaching" conception of limit, the examples which match this, and a conceptual metaphor which does not use the absolute value. Then later, they could have shown how the motion-based, "approaching" conception of limit breaks down when faced with more complicated types of limit convergence, how this ultimately lead Cauchy (Grabiner, 1992a, 1992b) to move to a new conceptual metaphor of proximity or range, which was later refined by Weierstrass (Lakoff and Núñez, 2001, p. 308) and how this leads to the absolute value being used in the formal definition.

By mixing a motion-based definition with an static-based definition, Lakoff and Núñez cloud the real issue: how do people move from the intuitive, grounded, dynamic conception of limit to the formal, static definition? This is the central pedagogical question that we need to be asking, but through their casual use of absolute value, they gloss over it. The authors attempt to begin with the "approaching" conception of limit, which students use, and try to directly connect it to an understanding of the formal definition. However, while the formal definition is an important concept for students to understand, it is based on a completely different metaphorical foundation than the motion-based conception of limit. This implies that, when it is time for the transition from the "approaching" conception to the formal definition to take place, a new metaphor needs to be introduced.

The formal definition was created to solve and explain cases of limit convergence that the "approaching" conception cannot explain, thus no amount of twisting the informal conception of limit will suffice. This has happened with other prominent scientific concepts as well. For example, Einstein's general theory of relativity was created to explain not only the normal cases explained by Newtonian mechanics, but also cases where those ideas break down (particularly situations concerning the very large and the very small). Thus, a new metaphor was created which encompasses the old one, but does not build off of it. Newtonian mechanics cannot be sufficiently extended in order to link to the general theory of relativity; however, the general theory of relativity can be used in situations where Newtonian mechanics works. The same principle should be attempted here: because the "approaching" conception simply cannot be extended to cover all cases of limit convergence, a new metaphor which explains the formal definition needs to be introduced to students, at the same time making clear that, while the formal definition works in all cases, the "approaching" metaphor still works in simple cases. At the same time, the informal dynamic conception should not be abandoned, as there is value in supporting such dynamism, even if the expert view needs to contain the formal definition (Marghetis & Núñez, 2013).

Based on classroom evidence, Boester (2010) suggests that students at first connect these distinct metaphors by assuming the "approaching" conception is the correct one, and that the static conception of the formal definition is a special case, as shown by putting the synchronized ranges created in the definition into motion. This would imply that students are using the pieces of the "approaching" conception as the source domain and the static conception as the target domain, thus mapping the dynamic aspect of motion along a line to the static (now dynamic) aspect of the range. However, some students evolved past this connection, recognizing that the formal definition conception is the correct one, and that the "approaching" conception is the special case (analogous to an expert view). This would imply that the students are using the static conception as the source domain and the dynamic conception as the target domain, thus mapping the static ranges to the dynamic (now static) aspect of approaching.

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Those Who Teach the Teachers: Knowledge Growth in Teaching for Mathematics Teacher Educators

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This theory-based report gives evidence and builds a conceptual framework for a construct called "mathematical knowledge for teaching future teachers" (MKT-FT). Mathematics teacher educators construct MKT-FT as they teach courses for pre-service teachers. Connections to mathematical knowledge for teaching (MKT) are discussed, with an emphasis on the complex relationships between aspects of pedagogical content knowledge in MKT-FT and MKT.

Key words: Mathematical knowledge for teaching, Discourse, Teacher educators

In the 30 years since Shulman's (1986) seminal speech on the importance of pedagogical content knowledge, a variety of theories about such knowledge have emerged (Depaepe, Verschaffel, & Kelchtermans, 2013). Among the most well known in the U.S. is at the heart of a primary-grades-focused model of *mathematical knowledge for teaching* (MKT) introduced by Hill, Ball, and Schilling (2008). The subject matter knowledge (SMK) and pedagogical content knowledge (PCK) components of Ball and colleagues' model of MKT are illustrated in Figure 1.

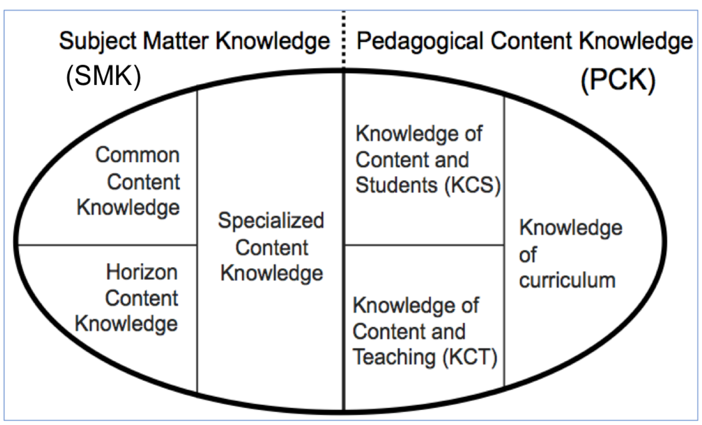

Figure 1. Model of mathematical model for teaching (Hill et al., 2008)

In the context of more advanced mathematics, others have explored how the idea of MKT may be productively refined for use in research and development in secondary and post-secondary settings (Hauk, Toney, Jackson, Nair, & Tsay, 2014; Speer, King, & Howell, 2015). Speer and colleagues considered college instructional questions such as: What are the types of specialized, horizon, and common mathematical knowledge for teaching calculus? While Hauk and colleagues have tackled: How does one productively unpack the aspects of PCK – knowledge of content and curriculum, content and teaching, content and students – when the teaching is in a college, the students are adults, the collections of mathematics experiences brought to the classroom are larger, and the sociocultural relationships between student and teacher are quite different from those assumed in the K-8 foundations of the initial framing of

MKT? Hauk and colleagues (2014) developed an expanded model for college teacher PCK. They used the PCK components in Figure 1 as the vertices of the base of the tetrahedron and added a fourth vertex which they call *knowledge of discourse* (knowledge about the nature of discourse, including inquiry, socio-mathematical norms, and forms of communication in mathematics within and outside of post-secondary educational settings) – see Figure 2.

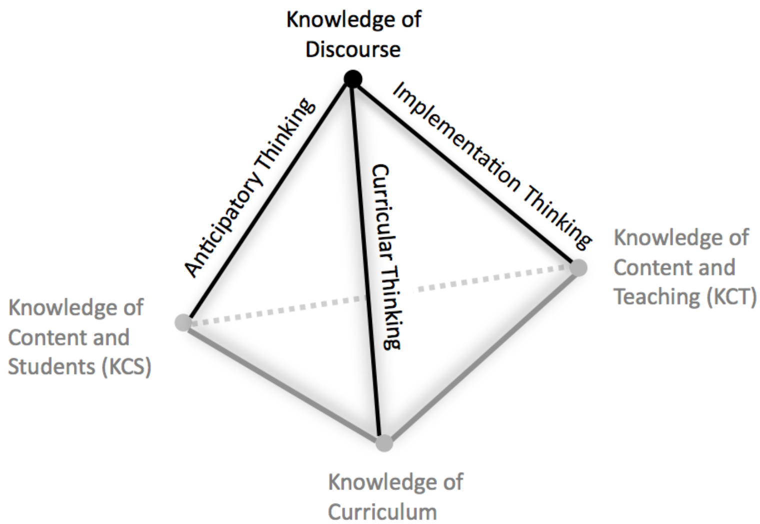

Figure 2. Tetrahedron model of PCK (Hauk et al., 2014).

Here we consider a related question: *What is the nature of "mathematical knowledge for teaching" for college instructors who teach mathematics for pre-service elementary teachers?* Such college teachers are teaching adults in post-secondary settings where the mathematical content is in the context of elementary mathematics (rather than advanced) *and yet* the content is itself linked to K-8 MKT of common, specialized, and horizon subject matter knowledge.

Indeed, Gallagher, Floden and Gwekwerere (2012) note that we know little about what skills are required to be an effective mathematics teacher educator nor do we know much about how those skills develop. Here, by *mathematics teacher educator* we mean anyone who provides guidance, mentoring, or professional learning opportunities to prospective or in-service teachers. The current paper focuses on the subpopulation of mathematics teacher educators who teach mathematics-content-rich courses where the students are pre-service K-8 teachers.

A natural question arises: Why should the Research in Undergraduate Mathematics Education (RUME) community have an interest in examining the knowledge required of mathematics teacher educators to perform their jobs effectively? First and foremost, much of the mathematical preparation for teaching among future K-8 teachers happens in colleges, most at the undergraduate level (Masingila, Olanoff, and Kwaka, 2012). In fact, Masingila and colleagues found that 88% of the teaching of "mathematics for elementary teachers" college courses happens in mathematics departments, with between 27% and 43% of faculty in these departments holding a Ph.D. in either mathematics or mathematics education. Across institution types (2-year, 4-year, and advanced-degree granting) there were more faculty with mathematics Ph.D.s teaching these courses (as opposed to those with Ph.D.'s in mathematics education).

We know that early learning experiences are formative and that children who learn to see themselves as mathematical agents do better in secondary school and beyond (Aud, et al., 2013, Shim, Ryan, & Anderson, 2008; Woodward et al., 2008). We know teaching that supports children in building skills with mathematical process, practices, and content is socio-culturally rich and responsive to societal as well as community needs (Aud & KewalRamani, 2010; Gay, 2010; Khisty & Chval, 2002; Téllez, Moschkovich, & Civil, 2011). We know that future teachers have greater resources to draw on and are more likely to offer children what they themselves have experienced as learners (including the undergraduate learning experiences that are most proximal to their launch as teachers; e.g. Ball and Bass, 2000; Conference Board of the Mathematical Sciences, 2001; Hodgson 2001). There is a need for mathematics faculty who are prepared to teach mathematics content courses for pre-service elementary teachers (PSETs) in ways that resonate with the kinds of classrooms those future teachers are expected to sustain.

In the U.S., the current population of instructors for such courses includes adjuncts, graduate students, and full time tenure- and non-tenure-track track mathematics faculty (Masingila, et al., 2012). Large segments of this instructor population have difficulty teaching courses for PSETs (Flahive & Kasman, 2013; Greenberg & Walsh, 2008). Though instructors in mathematics departments usually have a deep mathematical background, they often face challenges teaching content that is relevant and has utility for PSETs, unaware of the "cognitive and epistemological subtleties of elementary mathematics instruction" (Bass, 2005, p.419).

Given this state of affairs, Masingila, Olanoff, and Kwaka (2012) advocate for the design and implementation of professional development for mathematics teacher educators. Indeed, Masingila and colleagues note that many faculty who participated in their study asked the researchers where they could find professional learning resources! The RUME community includes experts on such matters: any design and implementation of effective professional development for mathematics teacher educators must involve attention to identifying the types of knowledge that faculty use and need in their regular practice of teaching future teachers.

For these reasons, we propose that college instructors possess a specialized constellation of knowledge to be studied: *mathematical knowledge for teaching future teachers* (MKT-FT). We posit that like MKT, MKT-FT is largely individually constructed for mathematics teacher educators, though often socially mediated. Seaman and Szydlik (2007) discussed the necessity but insufficiency of the early model of MKT for college mathematics instruction, particularly in the context of teaching future teachers. Several authors have noted the existence of what we see as components of MKT-FT. Zopf (2010) and Olanoff (2011) argued that effective teaching of future teachers requires mathematical knowledge of the work of teaching K-8 mathematics and awareness of the complexities of K-8 MKT itself.

According to Rider and Lynch-Davis (2006) and Smith (2003), the mathematical knowledge needed for teaching future teachers attends to the fact that one is teaching adult learners who have some familiarity with the mathematics (as opposed to teaching children who may be learning content for the first time). And, we note, there is a perceived autonomy of the learner in the post-secondary setting that is largely absent in K-8 and high school contexts. Smith (2005) has claimed that faculty who work effectively with future teachers have some (perhaps implicit) knowledge of educational theory and K-12 practice, as well as knowledge resources for connecting ideas and concepts in ways that prepare pre-service teachers to review, select, and engage with the wide array of curricular decisions that must be made by a teacher (e.g., decisions regarding which resources, worksheets, texts, and activities to use or avoid). Olanoff (2011) points out that Deborah Ball herself considers MKT to be the analogue of "common content knowledge" for faculty when considering what it might mean for an instructor to have MKT for teaching teachers.

Research and development on the preparation of teacher educators has long assumed a nesting of types of knowledge. One representation of that can be seen in Carroll and Mumme's work (2007). Figure 3a represents the nesting of mathematical content as subject matter knowledge (orange disk), linked to (future) teacher and elementary student within the larger context of the classroom (yellow disk). Similarly, in Figure 3a, mathematical knowledge for teaching (the stuff in the yellow disk) is linked to (future) teacher and mathematics teacher educator ("leader") within the larger context of teacher professional learning (green disk).

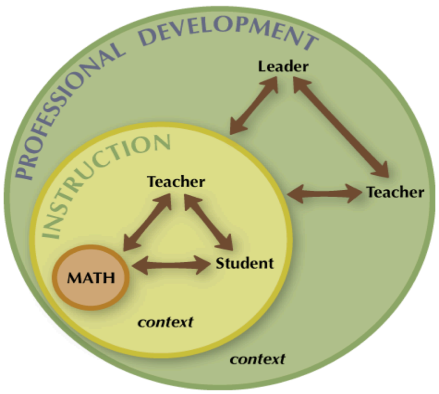

Figure 3a. Nesting of teaching and learning connections (Carroll & Mumme, 2007).

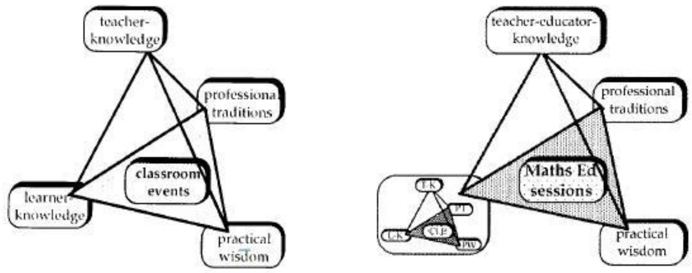

Figure 3b. Tetrahedron models for teacher-educator knowledge (Perks & Prestage, 2008).

Now we have a highly multi-dimensional situation. For each disk in Figure 3a there is an associated set of specifications for what counts as the context and for what constitutes the "content" about which one has "pedagogical content knowledge." Perks and Prestage (2008) made the case that knowledge for teaching teachers operates on several levels with a partially-nested self-similar design for their model of *teacher-educator knowledge* (Figure 3b).

The model for MKT-FT proposed in Figure 5 blends features of the three models discussed above. At each vertex are both K-8 mathematical content in the college class *and* "content" that is Knowledge of Content and Students in K-8 (illustrated for just the KCT vertex as magnified and highlighted, lower right, in Figure 5). We claim a similar cascade of knowledge structures, related to Content & Students, Curriculum, and Discourses in K-8 are related to the PCK aspect of college mathematics instructor MKT-FT (illustrated by similar "mini" tetrahedral at each of the other vertices in Figure 5).

While the nesting of knowledge structures within others is represented as geometrically self-similar, a fractal structure, the knowledge and thinking represented at each vertex is not identical. Each vertex of the "large" tetrahedron for PCK in the mathematical knowledge for teaching future teachers has a four-to-one mapping. For instance, the MKT-FT vertex for *knowledge of content and students* (KCS) is defined a la Ball as "content knowledge intertwined with knowledge of how students [who are future teachers] think about, know, or learn this particular content" (Hill et al., p. 375). In Figure 5, KCS includes knowledge of how college students engage with learning the MKT they will need in the future as teachers as well as (from the "smaller" tetrahedron) MKT related to K-8 teaching and learning. Knowledge of content and teaching in the MKT model is about teaching moves for working with K-8 mathematics students. In the MKT-FT model, knowledge of content and teaching includes a knowledge of teaching moves for mathematics as a part of teaching the college course for future teachers as well as of teaching moves for including attention to K-8 MKT in teaching that college course.

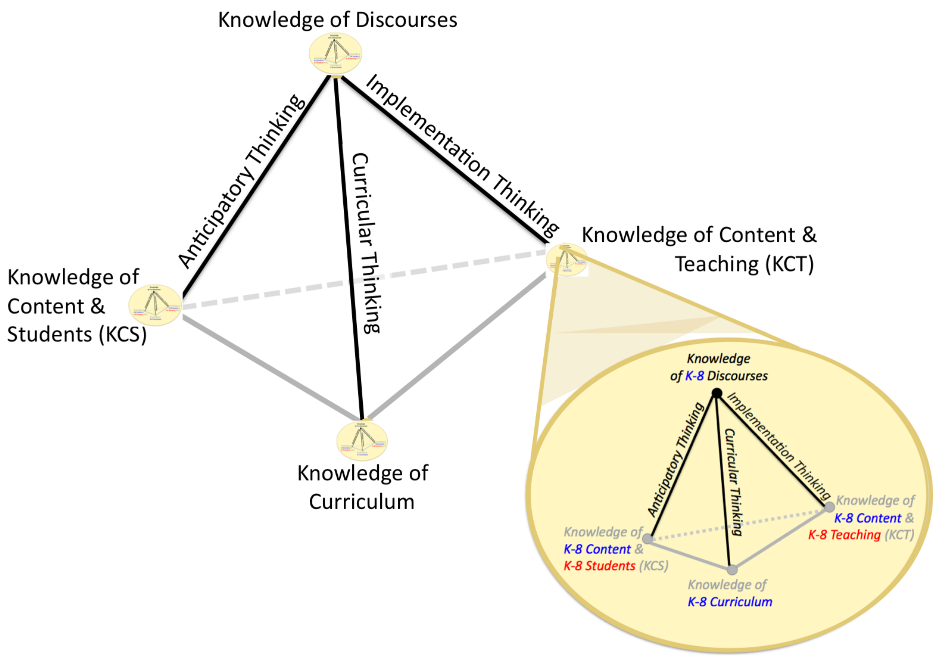

Figure 5. Model of MKT-FT pedagogical content knowledge – each vertex itself contains knowledge structures related to K-8 MKT.

An Example: MKT-FT and Task Design

Teacher education has long noted that task design may be a significant component in the development of knowledge for teaching pre- and in-service teachers. For example, Stylianides and Stylianides (2006) asserted that it is crucial for activities to be *teaching-related mathematics tasks,* so future teachers learn important school mathematics while at the same time making connections between how learning the content relates to its teaching. For Seaman and Szydlik (2007), well-designed tasks deepen the "mathematical sophistication" of future teachers, which they define as occurring as a result of enculturation into the mathematics community signified by teachers exhibiting the ways of knowing and values of mathematicians as their own. A broader example comes from the *Journal of Mathematics Teacher Education* issue devoted to the important topic of task design (Zaslavsky, Watson, & Mason, 2007). Papers in this special issue discussed different aspects of good task design in courses for teachers. For instance, according to Chapman (2007), effective tasks facilitate new understandings of familiar concepts and prompt reflection and discourse, while Bloom (2007) argued that quality tasks enhance mathematical habits of mind among college learners who are future teachers.

Yackel, Underwood, and Elias (2007) demonstrated the profound effect that attention to task design and reflection on task implementation can have on MKT and MKT-FT development of those who teach future teachers. One of their instructors commented,

> I found it interesting that adult students also go through some of the same progressions that children do. In particular, I often noticed that many students initially needed to use [iconic representations] to perform calculations, such as explicitly drawing boxes, rolls, and pieces…Having never taught young children, I had never seen this first hand. *Base 8 gave me the opportunity to experience this part of children's learning*

[emphasis added]. I think this is valuable to college instructors because most, like myself, will never have an opportunity to work with elementary school children closely. (p. 364)

Hence, we see that task design becomes a part of Knowledge of Content and Teaching for teacher educators: it is part of a teaching move designed to facilitate PSET learning of MKT. This example also reinforces the nonlinearity of our model for MKT-FT, as the instructor above mentions building the Knowledge of Content and Students (both for primary and adult learners) by engaging with their Knowledge of Content and Teaching.

As another example of the pluralistic nature of MKT-FT, consider subject matter knowledge itself: as Ball noted, MKT becomes common content knowledge for mathematics teacher educators. However, MKT-FT specialized content knowledge includes an awareness of and responsiveness to the educational literature as a means of helping PSETs understand why certain mathematical practices or pedagogical practices are favored (e.g., in the Common Core Standards). Horizon knowledge for teacher educators includes recognition of district, state, and national mathematics standards.

And what about the knowledge of discourse discussed earlier? Well, for mathematics teacher educators, MKT-FT knowledge of discourse subsumes the same knowledge of discourse that teachers have, and draws on knowledge of communicating about MKT and the teaching of mathematics in different sociocultural situations.

As Hauk, et al. (2014) point out, the literature on PCK includes both stable and dynamic features. To account for this, they use the edges in their tetrahedron to represent the *ways of thinking* about teaching mathematics that teachers use in practice. These ways of thinking are enacted in the classroom as teachers adapt to varying sociomathematical and cultural contexts that arise over time. In like manner, effective mathematics teacher educators also possess ways of thinking about teaching mathematics and about teaching MKT that change as the social, mathematical, and cultural climates change in their courses for PSETs. Hence, the edges in our fractal tetrahedron also represent these dynamic ways of thinking for faculty who teach future teachers.

At the Conference

In pursuit of applications of this model in current data analysis and in future research designs, at the conference we will present several examples of both knowledge and thinking as we envision them in the model. We have these questions for RUME participants in the session:

1. What would make a compelling argument for you about the connections among the ideas?
2. What kinds of data provide evidence for each?
3. How might we design a study to focus on a subset of the “small” or “large” tetrahedra?
4. Based on your experience, what connections among the ideas in the model are central?
5. How would knowing the answer to the questions we ask help faculty preparation and development? Or, inform practice of teaching adults who are in- and pre-service teachers?
6. What other questions are coming to mind, now that we have had these questions?

Acknowledgement

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Research in Courses before Calculus Through the Lens of Social Justice

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The terms equity, diversity, inclusion, and social justice have entered the research lexicon. Yet, researchers face significant challenges in gaining a nuanced understanding of the various ideas associated with these words. This theory-focused report presents some recent policy efforts to generate a shared meaning for "social justice" in mathematics education and offers a framework for making sense of (and making sense with) intercultural interactions as an essential component of rigorous research. To anchor discussion, we focus on research on teaching and learning in the courses before calculus (e.g., algebra, pre-calculus, liberal arts math, math for pre-service elementary school teachers, algebra-based statistics).

Keywords: Social justice, Cultural competence

Race.
Yes, that kind, the White and Black and Red and Yellow and Brown kind.
Go ahead.
Experience the shock.

Take a moment for a long, uncomfortable silence.
Yes, we just touched the third rail in American mainstream culture.

Breathe.
Here it comes again.
Race.
You know, of course, that if we have said "race" that "racism" is not far off.
Hovering, stage-right, there it is: Racism.
Its companions join from stage-left:
    Sexism,
    Ableism,
    Genderism,
    Ethnocentrism,
    Classism,
    Fill-in-the-blank-ism
They come together when we consider:
    Social justice.

In the past year two organizations, TODOS: Mathematics for All and the National Council of Supervisors of Mathematics (NCSM; 2016), issued a position paper entitled *Mathematics Education Through the Lens of Social Justice: Acknowledgement, Actions, and Accountability*. In it, social justice in mathematics education includes "fair and equitable teaching practices, high expectations for all students, access to rich, rigorous, and relevant mathematics, and strong family/community relationships to promote positive mathematics learning and achievement." Underlying all of these is a call for attention to the ways power, privilege, and oppressions contribute to and maintain an inequitable learning system.

There is a distinction to be made between perpetuating an "ism" and dealing with the fact that it exists. For example, *racism* refers to the ways in which avoidable and unfair inequalities are perpetuated based on ethnic, cultural, religious, and other characteristics associated with the concept of "race" at interpersonal, institutional, and societal levels (Berman & Paradies, 2010). By comparison, *racialization* refers to the processes by which characteristics identified as "racial" become meaningful in different social situations (Delgado & Stefancic 2001; Walton, Priest, & Paradies, 2013). The two are often conflated, resulting in the contention that any mention of race is racist. This conflation of terms can derive from a variety of views, from a belief in "color-blindness" (Apfelbaum, Norton, & Sommers 2012), a drive to be "color-mute" (in which race-talk is actively silenced or removed in social interactions or written documents; Pollock, 2004) or from a stance that society is past race-based discrimination and can be considered "raceless" (Ono, 2010). Color-muteness and racelessness draw on color-blindness but for different purposes. Paradoxically, to be color-mute is to recognize racial difference in order to actively remove mention of it, while racelessness is a "post-racial" approach, in which it is assumed that race no longer matters (Kempf, 2012).

Racialization, or genderization or other fill-in-the-blank-izations that use language to acknowledge inequities, can be valuable support for making bias explicit (rather than implicit). Research suggests that goals for equity, inclusion, and social justice are undermined when biases remain unexamined, implicit, or "unconscious" (Warikoo, Sinclair, Fei, Jacoby-Senghor, 2016).

Discerning difference, recognizing pattern, and anchoring new knowledge in those already noted differences and patterns are at the core of all human cognition. In other words, making "bias" explicit and challenging it are the essentials that allow humans to think, know, and learn.

Each of the authors has taken a different route in coming to mathematics education as a professional focus. Our experiences are rich in dealing personally with institutional "-isms," both academic and societal. In this theory-focused report we explore tools to increase researcher capacity for nuanced noticing of "isms" while tackling the truth of their existence. To attend to social justice in and through research in collegiate mathematics education means addressing our own needs (as humans who are researchers) for language, concepts, and awareness-building to support intentional decision-making. At the same time, we acknowledge the related, inevitable, struggle, of engaging in challenging conversations. Toward that end, we offer key ideas from education theory and intercultural sensitivity development along with a few examples.

Conversational Tools

As people trained in mathematics, those who do research in undergraduate mathematics education (RUME) know that mathematical sense-making is more than "identify the problem and solve it." As citizens of a first-world country in the 21st century, we are aware of societal injustice. Thus, as a community of researchers, we have an opportunity to dig deeper into a line of thought that has been emerging for a while (e.g., Aguirre & Civil, 2016; Adiredja, Alexander, & Andrews-Larson, 2016; D'Ambrosio et al., 2013; Davis, Hauk, & Latiolais, 2010; Gutiérrez, 2013; Nasir, 2016).

Over the past 10 years, Singleton and Linton's (2006) courageous conversations framework has become a cornerstone in the professional development of teachers around race. The framework is built on four agreements. Each can contradict some tightly held cultural norms relating to race talk. To participate in a "courageous conversation" people agree to: stay engaged, expect to experience discomfort, speak their truth, and expect and accept a lack of closure.

Being courageous about the risk of engaging in "--izations" involves navigating our own meanings for comfort, safety, and bravery. One tool we have found helpful is the Venn diagram shown in Figure 1. The diagram can support being self-aware and communicating with others about how each of us experiences risk in a conversation. We have used the diagram in a variety of settings: with pre-service K-8 teachers or developmental algebra students in creating classrooms where the challenging conversations are about mathematics (and its teaching) as well as with other researchers when the challenging conversations are about the role of race in research design. For example, a person may rarely feel safe having a conversation with people or about people from races other than their own, but can be brave and handle the expected discomfort in order to stay engaged in a valued exchange of ideas.

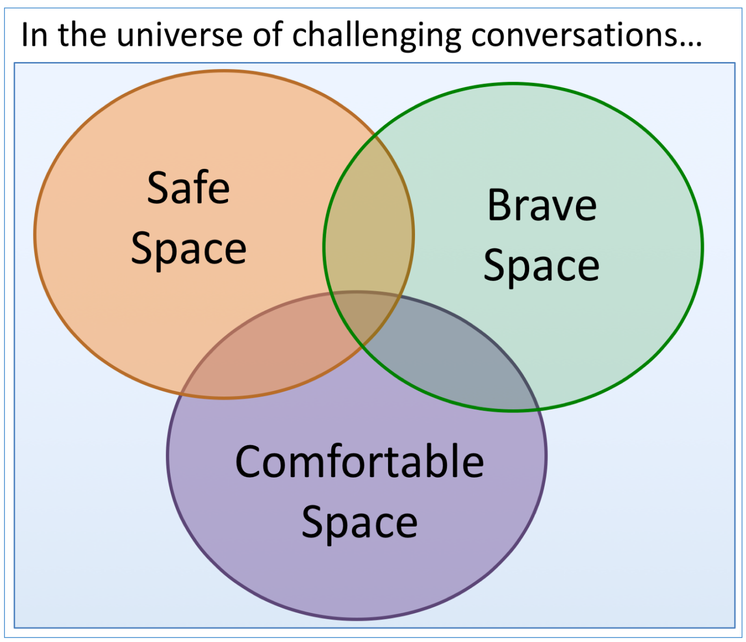

Figure 1. Juxtaposition of three types of experience related to taking risk (personal or professional) during a challenging conversation.

Along with the Venn diagram, we also have found the diagram in Figure 2 (next page) to be a useful tool. Qualitative researchers will recognize the interactions pictured among framework, experience, and reflection. What we have added are specifiers about being aware (mindful) during experience, being purposeful (focused) in reflection, and being rooted in human relationships (intercultural) when framing what we do. In fact, Figure 1 is a tool to support mindful experience. Generating this report resulted from focused reflection and an effort to share our, sometimes stumbling, journey into rehumanizing our work as researchers in collegiate mathematics education.

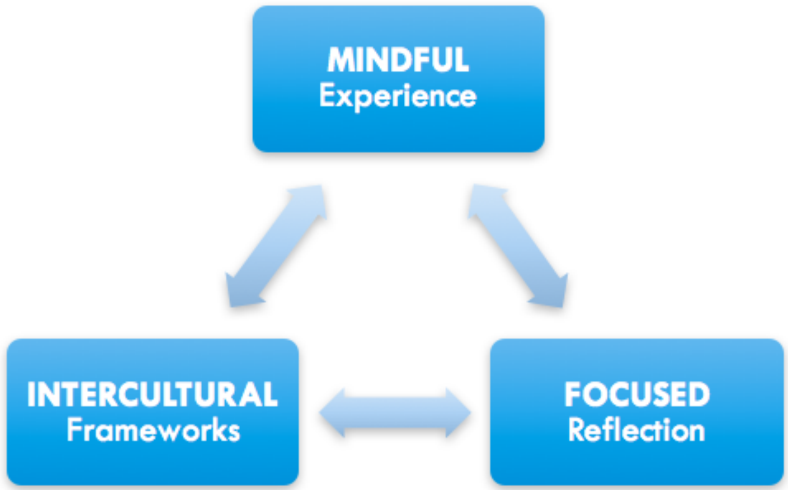

Figure 2. Descriptions and relationships among aspects of building self- and other-awareness.

In what follows, our focus is how the idea of *intercultural frameworks* can support social justice in RUME. We encourage the reader to stay engaged, while also being mindful and expecting to experience discomfort. We will "speak our truth" by illustrating ideas with examples. In offering these ideas, just as each reader may face different degrees of professional and personal vulnerability in reading it (i.e., different experiences of safety, comfort, and bravery), so has each author in creating it. Finally, this theory-based piece is an opening, so we expect and accept a lack of closure.

Moving From Personal Reflection to Professional Action

According to the TODOS-NCSM (2016) position paper, three conditions are necessary to establish socially just and equitable mathematical education for all learners:

(1) **acknowledging** that an unjust social system exists,
(2) **taking action** to eliminate inequities and establish effective policies, procedures, and practices that ensure just and equitable learning opportunities for all, and
(3) being eager for **accountability** so changes are made and sustained.

Taking the three tenets as foundational, we can ask: What is the role of social justice in RUME? What is the role of RUME in social justice? When we conduct research in the U.S. we make decisions about who participants are. What would be different if decisions in a project you know about had included overt and repeated attention to the three tenets?

Read through the three conditions again and think about it for a moment.

How would acknowledging that the social system is unjust shape selection of sites? Participants? Topics? Interpretations of actions and words? How might our next research question be framed so the answer would be evidence to support action to eliminate an inequity? How do we do that? How might the research design or analyses need to be different if the results of the work were held accountable by research peers *and* judged in a court of stakeholder opinion that valued equity as much as excellence in mathematics education?

Short, possibly terrifying answers to these questions exist: observe the world; be purposeful in inviting people into your research who do not look like you or act like you or think like you. But the short answers are deceptive. Such calls for courageous action in the face of our own ignorance can feel motivating *and* can be recipes for disaster. Entering the uncomfortable, not necessarily safe space of research that pays attention to social justice can be rewarding – but we must also stick it out even when it blows up in our faces.

Mathematical training has prepared us for this. The colossal failures before generating an elegant proof are well known to those who have done advanced mathematics. Some may ask: are the stakes as emotionally high in creating a proof? Mathematicians will say yes. For others, a comparable situation might be the personal and professional status that rides on completing and reporting well on a piece of educational research. So why do we do it? In short, to keep our jobs. But, we also may do it because something fulfilling comes from the process that is unavailable elsewhere. The reason we get on our particular professional roller coaster is because we want the thrill of the ride and we feel safe enough, or pressured enough, to be brave enough to take the risk. The idea of social justice in research adds complexity to the roller coaster. The main emotional aspects may be trepidation along with meta-affect of hope, a hope based on a (perhaps hazy) vision of a socially just world. The ideas and language of intercultural competence can help. They provide tools for meta-cognition about the affect and meta-affect.

Framework for Intercultural Awareness and Competence Development

Interactions with other people are shaped by our orientation to noticing and engaging with difference and the approach we take to the interaction itself. Do we anticipate great risk? Do we have the privilege of assuming safety? Social justice-aware research is shaped by what a researcher knows or anticipates about others' experiences. Below, we first offer language and structure for the research community for noticing and discussing difference. Then, we give a vignette and examine it in light of the suggestions.

The ways we are aware of and respond to others, including the particular challenges we each face dealing with the societal realities of racism, sexism, and other inequity-preserving structures, is a consequence of our *intercultural orientation*. This is neither a reference to our beliefs about culture or race nor about views on researching in mathematics education. Rather, intercultural orientation is the perspective about human *difference* each person brings to interacting with other people, in context. For researchers, it includes perceptions about the differences between their own views and values around various types of work in mathematics education, the views of their colleagues, and the views of various stakeholders in research. Intercultural competence is the capability to shift perspective and appropriately adapt behavior to socio-cultural differences and commonalities (Hammer, 2009).

To build skill at establishing and maintaining relationships in, and exercising judgment relative to, cross-cultural situation requires the development of intercultural sensitivity (Bennett 2004). The developmental continuum for intercultural sensitivity has five milestone orientations to noticing, making sense of, and developing response to difference: *denial*, *polarization*, *minimization*, *acceptance*, and *adaptation*.

With mindful experience a person can develop from ethnocentric ignoring or *denial* of differences, moving through an equally ethnocentric *polarization* orientation of an us-versus-them mindset. With growing awareness of commonality, a person enters the less ethnocentric orientation of *minimization* of difference, which may over-generalize commonalities. From there, development leads to an ethnorelative *acceptance* of the existence of intra- and intercultural differences. Further development aims at a highly ethnorelative *adaptation* orientation in which differences are anticipated and responses to them readily come to mind. This summary is represented in Figure 3 and elaborated below.

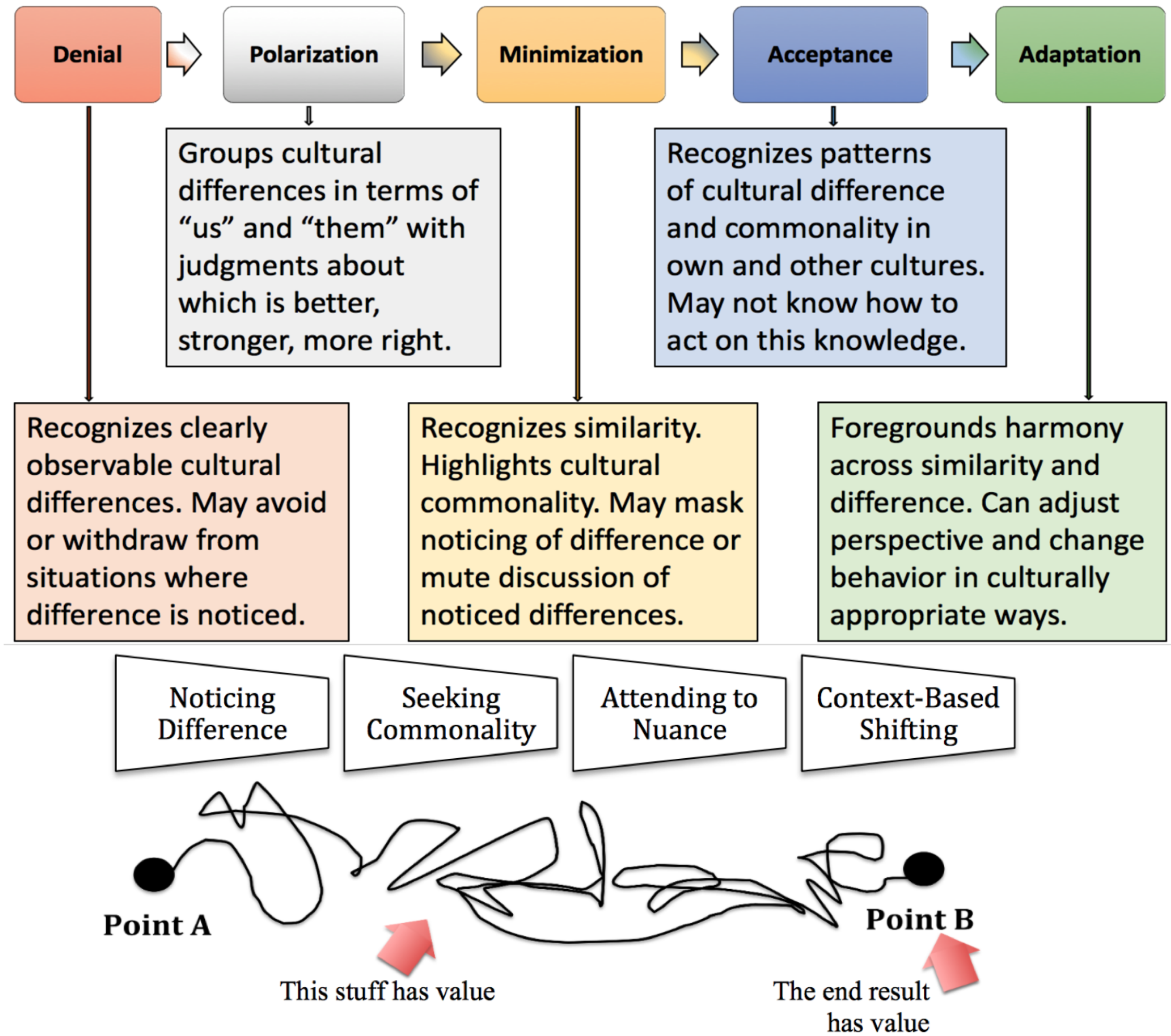

Figure 3. Diagram summarizing intercultural competence developmental continuum. One may start at point A and eventually develop to point B but the path is not linear.

Denial. A central part of awareness is to observe self and others. In the denial orientation, little observation of others happens. Such an orientation is not denial in the sense of “I'm going to say it is not there” but denial as in “I can't even see it.” The view is “we’re all researchers and we all do our work” without attention to what “our work” might mean to others.

Polarization. The polarization orientation might be characterized as: “There's a RIGHT way to do things and there's a WRONG way to do things. And we're going to make sure we use the right way.” For example, depending on the experience and values of the conversant, the “right” way to do research in collegiate mathematics education may or may not include education discourse or the language of assessment, statistical analyses, curriculum, program, or teacher development. Nonetheless, enacting a polarized orientation in mathematics education research would mean seeing that a practice is happening or noticing a norm being developed.

Minimization. From a minimization orientation, people may focus attention on how others' approaches are similar to their own valued ways of designing, conducting, and reporting research. For some, this might be characterized as, "Look how this stuff called equity in math ed is LIKE research on effective mathematics teaching. It has a lot in common with it, even if the way it is said is a little different, I say effective, they say equitable. Let's talk about how it is similar. Let's leverage the fact that we have seen this before." From this perspective, any research in mathematics education is similar to all other research in mathematics education – whether one is reflecting on teaching a class, examining inquiry-based project learning involving the mathematics of institutionalized racism, researching how teachers take up the idea of brave conversations about race, sex, and poverty in professional development workshops, or in researching how graduate students validate proofs.

Acceptance. Through increased attention to nuance in the differences that exist among commonalities, one begins the transition from an orientation focused on minimization of difference to one of acceptance of difference. Here, the word "acceptance" is used in its socio-cultural sense – the action or process of consenting to receive (rather than its psychological one – believe or come to recognize as valid or correct). From this orientation, it might be characteristic for a researcher to notice various ways of determining the "quality" of research and suggest colleagues use whichever makes most sense for them. Additionally, the researcher might encourage peers to accept and understand the differences that exist in ways of prioritizing measures of "quality" in research.

Adaptation. Beyond accepting that there are these differences, adaptation-oriented people seek for themselves, and find ways to give colleagues, opportunities in noticing, articulating, and responding to differences. This might be characterized by statements such as, "I am looking for ways to work with colleagues. I don't have to assert or defend many, or even one method. Quality in research is a relative thing. How my colleagues and I connect ideas and access, organize, or value ideas is not necessarily strictly limited to the ways valued by my perspective." In adaptation, a person can converse well with people of differing mindsets (and with other orientations), understanding and appropriately using discourse familiar to discussion partners.

Intercultural competence is developed non-linearly. And, it is not static. The higher the stakes of a situation the more challenging it can be to maintain what might be one's usual, everyday orientation, towards difference. When the potential for conflict or challenge emerges in a situation (like reading this paper), affect and meta-affective pressures mean we may fold back to earlier developmental levels (e.g., the wandering curve in Figure 3). Being mindful of ourselves – our reactions and orientations – can support developmental growth.

Vignette

A RUME researcher is creating a professional statement that situates her work in the context of social justice in teaching future teachers. Her aim is to give background that explains the trajectory of her professional development and research goals.

As a teacher I want to foster a learning environment in which everyone has equal opportunity to learn. And, I want to foster a research environment where everyone has equal opportunity to contribute to the research. It is about bringing a broader perspective to new researchers that encourages people to change their focus from within to around. I have looked, but haven't really found the research on culture and teaching very useful, so I don't read it much. Just too much blame and shame there.

I recently conducted research on the implementation of a new curriculum for developmental math students. The curriculum had relevant, real-world activities that helped make clear why algebra can be a useful tool. When I am researching general mathematics, I am focused on what is offered by a curriculum that changes mathematics "from a gatekeeper to a gateway." I want students to see the ways society places workers in classes based on how much math they know and can use. We see this in K-12 education, and it is something most students have experience with. I want my research to recognize and capture information on how and if students see that the mathematics they learned in grade school had an impact on their options at the college level and beyond. Getting tracked into a lower level of mathematical learning does not have to limit their expectations and opportunities in life, but it does mean they will need to work harder as an adult to get to where they want to be. When I research lower level college mathematics courses, the enrolled students are often (but not always) those who were tracked into lower level mathematics courses in high school. The tracking students experience in grade school can have long-standing negative effects on the opportunities they see in the future.

Discussion - Connecting the Framework to the Vignette

Given our experiences in conversations with research colleagues, for this report we selected material for the vignette to highlight the minimization orientation. In the vignette, the narrator discussed intentional efforts to notice and include the three tenets of the social justice view. At the same time, her own developmental level of intercultural competence for noticing and responding to difference meant she saw students' opportunities to learn as affected by previous tracking without noticing that the students themselves were participants in whether something was an "opportunity." Her next round of research could include probing students' and instructors' perceptions of scenarios, finding out what is perceived as an opportunity, by whom.

Students in "lower level math" courses are much more likely to have had regular encounters with disruptions to intimacy with mathematics. Every interaction with mathematics may be a risk, with the type and level of risk varying widely depending on the many years of previous experience of adult students. What the researcher or classroom instructor might perceive as an equal opportunity to increase intimate connections with mathematics may not be fair if the risks are vastly different for different students. The narrator had not investigated the literature because it was uncomfortable to read. She has an opportunity to take the kinds of risks that her new curriculum asks students to embrace.

Another opportunity for the researcher is expanding on her reflection that: "Getting tracked into a lower level of mathematical learning does not have to limit their expectations and opportunities in life, but it does mean they will need to work harder as an adult to get to where they want to be."

Conclusion

The feelings and thoughts experienced when reading the first page of this piece may have some indicators for you about where you are on the developmental continuum from denial to adaptation. A large body of research in intercultural orientation has suggested that the majority of the college-educated U.S. population interacts with the world from a perspective somewhere between polarization and minimization. Our own research, among researchers in collegiate mathematics education, mathematicians, and U.S. public school teachers in grades 6 and up has echoed this result several times (Hauk, Toney, Nair, Yestness, & Troudt, 2015; Hauk, Yestness, & Novak, 2011). We take this as a starting point and seek ways to move along the developmental continuum. That means asking questions and finding resources that can support growth of intercultural sensitivity while being aware of the energy and care needed, for ourselves and others, to make brave decisions and to explore beyond the bounds of what we consider comfortable or safe.

The more that researchers in cognition use critical theory features and vice versa, the closer the research community may come to an equality in talk. The problem in this over-reliance on commonality is that equality in discourse style is not equity in discourse. As Marilyn Cochran-Smith and colleagues have recently described it, “With the former, the valence of the terms is primarily about sameness (equality) or difference (inequality), while with the latter, the valence of the terms has primarily to do with fairness and justice (equity) or unfairness and injustice (inequity)” (Cochran-Smith et al., 2016, p. 69).

Remember, we said to expect an absence of closure. To support that, we offer some suggestions for questions to drive a conversation in your near future. Find someone to talk with who agrees to Singleton and Linton’s (2006) four standards and discuss some/all of these questions. The set of questions asks you to do some *acknowledging*, *take some action* (e.g., step into a brave space to read) and be *accountable* at least to yourself and a conversational partner.

(1) Read Bennett’s (2004) article. What new goals do you have for your own intercultural competence development? Review the references below, what might you read next to help make progress on those goals?

(2) Read the article by Aguirre and colleagues (2017). How are your answers to the questions in #1 altered? Why?

(3) Questions for a focused reflection on research identity: What is going on in my community? What do I mean by "my community"? Research thought partners? Fellow faculty? Geographically defined? Socio-culturally defined? Why that?

(4) Questions for a focused reflection on the intersection of research and social justice: If I were conducting research that attended to social justice in college mathematics, what data would be collected? What is problematic about using that data? For example, what does a social justice lens offer in thinking about what constitutes a valid assessment of mathematical thinking, knowing, and/or understanding? Of mathematical knowledge for teaching?

(5) Questions for generating an intercultural framework: Given the potential variability in intercultural development of students, instructors, and researchers, of textbook authors and administrators, and acknowledging institutional inertia, how might we create a developmental continuum for social justice in mathematics education?

(6) Questions to prepare for mindful experience: What is my position within the social justice framework? If you have not already, read Aguirre et al., (2017) and consider: What are next steps I am ready for now? Three months from now? One year from now?

(7) Questions to connect social justice to a framework for research: Read the TODOS/NCSM (2016) position statement and Aguirre et al. (2017) article. The TODOS/NCSM position statement details steps to take to implement the recommendations of acknowledgement, actions, and accountability. On the next page is an example of how the ideas might be translated to RUME. Discuss and create similar tables of parallels, one table for each of the other aspects in the position paper.

Example: Belief Systems and Structures

Actionable Items for Changing Educational Practices in Mathematics (p. 4)	***Potential Actionable Items for Changing Research Practices***
• Interrogate individual and societal beliefs underlying deficit views about math learning and children with specific attention to race/ ethnicity, class, gender, culture, and language.	• Interrogate individual and societal beliefs underlying deficit views about mathematics learning in college with specific attention to race/ethnicity, class, gender, culture, and language.
• Refrain from using deficit discourse in instructional decision making (e.g., placement decision, course offerings).	• Refrain from using deficit discourse in research decision making (e.g., sample, instrument design, interview strategies).
• Show evidence that course taking patterns are changing, remedial/ intervention courses reduced, and advanced mathematics offerings are more robust and plentiful.	• Conduct research to provide evidence about course taking patterns, including remedial/ developmental courses, and report on context for all, from basic to advanced mathematics.
• Increase recruitment and retention of mathematics teachers and leaders from historically marginalized groups.	• Increase recruitment and retention of researchers and research participants from historically marginalized groups.
• Require professional development opportunities that focus on social, cultural, linguistic, contextual, and cognitive facets of mathematics and mathematics learning.	• Require professional learning opportunities that focus on social, cultural, linguistic, contextual, and cognitive facets of mathematics, mathematics learning, and research thereon.
• Create a mathematics vision with accountability mechanisms for the classroom, school, and district that uplifts students to learn rigorous and relevant mathematics.	• Create a research vision with accountability mechanisms for individuals, projects, and professional groups that uplifts research on the learning of rigorous and relevant mathematics.

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What Should Undergraduate Mathematics Majors Understand About the Nature of Mathematical Knowledge?

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A primary function of mathematics education is that students understand the subject matter of mathematics. That is, students are supported in understanding mathematical concepts and attaining mathematical knowledge. But there is another function of mathematics education, often unaddressed in research, which deserves more attention. In addition to learning content, students must be supported in developing informed views about the human processes by which mathematical knowledge is produced and the unique characteristics of that knowledge. Through an exploration of humanistic philosophy of mathematics, the purpose of this paper is to identify characteristics of the nature of mathematical knowledge that may be important for undergraduate mathematics majors to know and understand. Four characteristics are discussed: mathematical knowledge is subject to revision; mathematical knowledge is socially validated; proofs are bearers of mathematical knowledge; and informal mathematical work is the foundation of formal knowledge.

Key words: Nature of Mathematics, Nature of Mathematical Knowledge, Philosophy of Mathematics

Students and teachers often have a limited view of the nature of mathematics and may believe mathematics is a static body of knowledge consisting of arbitrary rules and procedures (Beswick, 2012; Erlwanger, 1974; Thompson, 1992). These naïve views may negatively affect the teaching (Thompson, 1992; White-Fredette, 2010) and learning (Erlwanger, 1974) of mathematics. In contrast to these naïve views, many mathematics education scholars view and describe mathematical knowledge as a dynamic human product (Boaler, 2016), and emphasize the human aspects of mathematical work such as creativity (Burton, 1999) and fallibility (Ernest, 1991). These modern views are influenced by cultural approaches to mathematics (Bishop, 1988), humanistic philosophy of mathematics (Ernest, 1991), or perhaps scholars' own experiences doing mathematical work (e.g. Hersh, 1997). The gap between uninformed views of mathematics (largely held by students and teachers) and the informed cultural-historic perspectives held by scholars needs to be addressed within mathematics education.

Lessons from Science Education

In science education, scholars have argued that students and teachers not only need to understand the facts of science, but also need to possess a general understanding of science as a discipline, and the nature of scientific knowledge (McComas, Almazroa & Clough, 1998). Proponents for teaching the nature of science (NOS) are concerned with providing students and teachers a general understanding of the scientific enterprise. As Hurd (1960) noted, "A student should learn something about the character of scientific knowledge, how it has been developed, and how it is used" (p. 34). Driver, Leach, Miller, and Scott (1996) noted "such understanding is necessary in order to appreciate science as a major element of contemporary culture" (p. 19). Research in science education indicates that understanding NOS assists students in learning science content (McComas, Almazroa & Clough, 1998, p. 517). Furthermore, students enjoy learning about NOS and lament when social/historical aspects of science are left out of instruction. McComas and colleagues (1998) wrote, “Incorporating the nature of science while teaching science content humanizes the sciences and conveys a great adventure rather than memorizing trivial outcomes of the process” (p. 519). Perhaps similar positive outcomes would be seen if mathematics students had the opportunity to learn about the nature of mathematical knowledge, but little work has been done in this area (Jankvist, 2015).

Beginning a Systematic Inquiry into the Nature of Mathematics

In science education, scholars have done significant work conceptualizing the construct of nature of science (NOS), and have conducted research to understand how NOS can be taught to students and teachers (Irzik & Nola, 2014). Research on the teaching and learning of nature of science is guided by frameworks or lists that explicitly outline an informed conception of NOS and goals for learning (Lederman & Lederman, 2014). Perhaps a first step in conducting a systematic inquiry into the teaching and learning of the nature of mathematics is to consider the characteristics of the nature of mathematics that students or teachers should understand, and compile them into a framework/list with the purpose of conducting further research into student and teacher understandings of those characteristics.

In this paper we draw from the literature on philosophy of mathematics to formulate an initial list of possible goals for undergraduate mathematics majors’ understandings of the nature of mathematics. Such an understanding will certainly be valuable for students who pursue graduate studies in mathematics or become secondary teachers. I agree with Fried (2014) that those who are “mathematically educated must feel at home with mathematics, appreciate its power, and know it as a part of one’s culture” (p. 30). The conceptions of mathematics that future teachers develop in the university will likely stay with them as they begin to teach, and will likely have an influence on their students’ understanding. If the nature of mathematics cannot be taught at the university, the home of disciplinary mathematics, then where can it be taught?

What is the Nature of Mathematics?

Note that the question “What is the nature of mathematics?” is a philosophical one. Ernest (1991) noted, “The philosophy of mathematics is the branch of philosophy whose task is to reflect on, and account for the nature of mathematics” (p. 3). Philosophers of mathematics primarily seek to understand and describe the discipline of mathematics and the nature of its corresponding knowledge (Kitcher, 1983). Although there are many philosophies of mathematics, humanistic philosophy of mathematics has had a profound influence within mathematics education (Lerman, 2000; Toumasis, 1997). Humanistic approaches are unique in that they take as foundational the notion that mathematical knowledge is a human product. As Hersh (1997) wrote, “To the humanist, mathematics is *ours*—our tool, our plaything” (p. 60). Humanistic philosophy of mathematics (Ernest, 1991; Hersh, 1997; Lakatos, 1976) informs the characteristics of the nature of pure mathematics that are discussed in this paper. When describing these characteristics, I will also contrast humanism with Platonism and formalism. These philosophies have typically served to proliferate the idea that mathematical knowledge is absolute and value-free (Dossey, 1992).

Steen (1988) called mathematics the science of patterns. She described three branches of the mathematical sciences: core (pure), applied, and statistical. There are also many other types of mathematics such as artisanal or commercial-administrative forms (Harouni, 2015).

Mathematical knowledge is certainly not limited to western academic knowledge (Bishop, 1988). Nevertheless, pure mathematical knowledge is particularly relevant to much of the research in undergraduate mathematics education (especially regarding the teaching and learning of proof). In this paper I will narrow our focus and consider what aspects of the nature of pure mathematical knowledge undergraduate mathematics majors should know and understand.

The Nature of Mathematical Inquiry versus the Mathematical Nature of Knowledge

In the domain of pure mathematics, I conceive of two aspects of the nature of mathematics (NOM) that may be fruitful to distinguish: the nature of mathematical inquiry (NOMI) and the nature of mathematical knowledge (NOMK). NOMI refers to the practices that mathematicians engage in when creating knowledge (e.g. conjecturing, proving, communicating, etc…) and the human experience of such activity (e.g. emotion). NOMK refers to the nature of the knowledge that mathematician's produce (e.g., is mathematical knowledge absolute or subject to revision?). It should be noted that this distinction is not always clear cut. For instance, would an understanding of conjecturing be categorized as NOMI or NOMK? One can make the case for NOMI—conjecturing is an important mathematical practice that plays a role in the creation of mathematical knowledge. On the other hand, established theorems were once conjectures. This would place conjectures in the category of NOMK. Although the distinctions between knowledge and practice are not always clear cut, it is important to make the distinction when possible as there has been confusion in science education when scholars have conflated the nature of scientific knowledge and scientific inquiry when discussing nature of science (Lederman & Lederman, 2014). Due to length constraints, the focus of this paper shall be the nature of mathematical knowledge (NOMK).

The Nature of Pure Mathematical Knowledge

Mathematical Knowledge is Subject to Revision

Humanistic philosophers work from the simple assumption that mathematics is a human activity and product (Hersh, 1997). As a human product, mathematical knowledge is necessarily imperfect, fallible, and subject to revision (Ernest, 1991). Imre Lakatos is typically credited with the fallibilist view of mathematical knowledge (Kitcher, 1983). Through the story told in *Proofs and Refutations,* Lakatos (1976) demonstrates the fallible, revisionary nature of mathematical knowledge—as counterexamples are found to what are believed to be solid proofs and theorems, mathematics grows and changes. Fallibilism stands in contrast to Platonism and formalism.

Platonism. According to Dossey (1992), "Plato took the position that the objects of mathematics had an existence of their own, beyond the mind, in the external world" (p. 40). Brown (2008), a modern Platonist, wrote "Mathematical objects are perfectly real and exist independently of us" (p. 12), and we gain access to these objects through "the mind's eye" (p. 14). If mathematical objects are conceived to have a transcendental existence, then mathematical truth exists independently of humans and awaits human discovery.

The Euclidean (Deductivist) paradigm. The Platonic perspective formed the foundation of the Euclidean paradigm that dominated mathematics for 2,500 years (Ernest, 1991). This view is that humans can arrive at certain mathematical knowledge by following the deductive process. A few self-evident truths called postulates are assumed along with some definitions of mathematical terms. Then, beginning from these postulates and definitions, one can proceed by logical deduction to arrive at other certain truths. But mathematicians eventually found they could create new bodies of useful mathematical knowledge, non-Euclidean geometries, by withholding Euclid's assumption about parallel lines (Ernest, 1991). Thus, what were considered postulates could no longer be considered obvious truths, as a different "truth" could be arrived at when one began with a different set of assumptions. According to Hersh (1997), "Geometry served from the time of Plato as proof that certainty is possible in human knowledge…. Loss of certainty in geometry threatened loss of all certainty" (p. 136). Mathematicians responded to this loss of certainty by attempting to ground the foundation of mathematics in logic, arithmetic or set theory, but these attempts led to contradictions (Hersh, 1997). Formalism, the attempt "to characterize mathematical ideas in terms of formal axiomatic systems" (Dossey, 1992, p. 41) emerged in response to these conundrums.

Formalism. Under the formalist paradigm, mathematics begins with a collection of formal axioms and definitions. From these axioms and definitions theorems are logically deduced. If we adopt *Axioms X, Y, Z* we arrive at one branch of mathematics; or we can adopt *Axioms W, X, Y* and arrive at a different branch. Under formalism, neither branch of mathematics is *truer* than the other. In describing today's formalist, Hersh (1997) wrote,

> For him, all mathematics, from arithmetic on up, is a game of logical deduction. He defines mathematics as the science of rigorous proof… All mathematicians can say is whether the theorem follows logically from the axioms. Mathematical theorems have no content; they're not *about* anything. On the other hand, they're absolutely free of doubt or error, because a rigorous proof has no gaps or loopholes. (p. 163)

Hersh (1997) claimed that the working mathematician is caught between Platonism and formalism. "[W]hen he is doing mathematics, he is convinced that he is dealing with an objective reality… But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all" (p. 11). Furthermore, "To abandon both, we must abandon absolute certainty, and develop a philosophy faithful to mathematical experience" (p. 43).

Both Platonism and formalism (or a melding of the two) perpetuate the absolutist view that mathematical knowledge "consists of certain and unchallengeable truths" (Ernest, 1991, p. 7). Humanists adopt the fallibilist view: mathematics is an imperfect human activity, and so mathematical knowledge is subject to revision. According to humanistic philosophers, we need not hold onto the idea that mathematical knowledge is absolutely certain, or that mathematics is restricted to axioms, definitions and proof. Hersh (1997) noted, "Mathematical knowledge isn't infallible. Like science, mathematics can advance by making mistakes, correcting and recorrecting them" (p.22). Lakatos's (1976) *Proofs and Refutations* highlighted the way mathematical knowledge can be revised over time as mathematicians find new counterexamples to established proofs (even proofs that were widely regarded as correct may be eventually be refuted). As Hersh (1997) noted, "*For two millennia, mathematicians and philosophers accepted reasoning that they later rejected*. Can we be sure that we, unlike our predecessors, are not overlooking big gaps? We can't. Our mathematics can't be certain" (p. 45).

Burton (1995) noted that if we consider mathematics to consist of a body of absolute truths then "the purpose of education is to convey [the truths] into the heads of learners" (p. 276). When mathematics is presented as "information which should not be questioned" (p. 276), some learners may perceive that mathematics is a subject that only a few people have the ability to understand. Thus an absolutist conception of the nature of mathematics, in addition to being philosophically indefensible (Ernest, 1991; Hersh, 1997), also disempowers learners who do not immediately perceive the "truth" of what the teacher is trying to convey in the classroom. Burton

(1995) advocated for a humanist/feminist view of mathematical knowledge in school, claiming that "Re-telling mathematics, both in terms of context and person-ness, would consequently demystify and therefore seem to offer opportunities for greater inclusivity" (p. 280).

Mathematical Knowledge is Socially Validated (NOMK)

Philosophers presenting a humanistic view of mathematics (e.g. Ernest, 1991; Hersh, 1997; Lakatos, 1976; Tymoczko, 1988) are frequently cited and have been influential in mathematics education (e.g. Ball, 1988; Boaler, 2016; Komatsu, 2016; Lampert, 1990; Larsen & Zandieh, 2008; Weber, Inglis, Mejia-Ramos, 2014). A key feature of humanistic philosophy is the notion that mathematical knowledge is socially validated. Ernest has described in detail the general process by which mathematical knowledge is socially validated in the discipline of mathematics. Ernest contended that mathematical knowledge is created through a subjective/objective cycle. Individuals subjectively create knowledge, and it is validated inter-subjectively by other mathematicians so that it becomes objective knowledge accepted by wider communities (perhaps through publication). This objective knowledge can then inspire more individual thought as it is subjectively reconstructed, and this may lead to further subjective creations, which may then in turn become objective taken-as-shared knowledge.

Hersh (1997), an academic mathematician, emphasized importance of proof in the social validation of knowledge. Hersh noted that for mathematicians, the primary purpose of proof is to convince other mathematicians that a claim is true (Hersh, 1993). But the truth of the statement is not within the proof itself, but in the refereeing. Hersh (1997) claimed, "What mathematicians at large sanction and accept *is* correct mathematics" (p. 50). Furthermore "There are different versions of proof or rigor, depending on time, place, and other things". (p. 22).

How might students come to understand the social function of proof and the socially validated nature of mathematical knowledge? Some authors have designed instructional activities so that students have authority to judge what is correct through classroom discussions, and to negotiate standards of rigor and proof (cf. Ko, Yee, Bleiler-Baxter, & Boyle, 2016). Perhaps students will understand that mathematics is socially validated if they have the opportunity to participate in an inquiry-oriented classroom in which the students (rather than the teacher) are responsible for justification and validation. Speaking on findings from an inquiry-oriented differential equations course, Rasmussen and Kwon (2007) noted, "social norms that empower students to be creators of mathematical ideas, along with the explanations and justifications that support these ideas, provide an opportunity for learners to develop desirable beliefs about the nature of mathematics" (p. 192). But are these students developing desirable beliefs about the nature of mathematics in general, or just what "constitutes mathematical activity in [their particular] classroom" (Yackel & Rasmussen, 2002, p. 324)? Research in science education has found that "doing science" in an inquiry setting is not sufficient for students to develop informed understandings of the nature of scientific knowledge (Bell, Blair, Crawford, & Lederman, 2003). The outcome may be different for mathematics instruction, but we need more research to investigate this question.

Proofs are Bearers of Mathematical Knowledge.

Central to humanistic philosophy of mathematics is the distinction between descriptive and prescriptive philosophy (Ernest, 1991). Philosophy of mathematics traditionally was driven by the notion of what mathematics *ought to be*. For a Platonist or formalist, mathematics is (or should be) the most certain and absolute of human knowledge; thus the purpose of philosophy of mathematics is to justify its absolute nature. For instance, even if most published mathematical proofs contain gaps in logic (Ernest, 1991; Hersh, 1997), proof *ought* (from a prescriptive perspective) to be a rigorous deductive argument from accepted premises to conclusion. Humanistic philosophers work from a descriptive perspective, basing their philosophy on what mathematicians actually do, rather than an ideal vision. Thus humanistic philosophy of mathematics is distinct from traditional philosophy because it incorporates sociological, anthropological, or historic studies of mathematics. Indeed the classroom discussions in Lakatos's (1976) *Proof and Refutations* paralleled an account of the historical events surrounding the development of proof of an Eulerian conjecture. Hersh (1997) wrote "A humanist sees mathematics as a social-cultural-historic activity. In that case it's clear that one can actually look, go to mathematical life and see how proof and intuition and certainty are seen or not seen there" (p. 48). The humanistic emphasis has relevance not only to philosophers, but also for mathematics education scholars who desire to understand the practices of mathematicians so that implications may be drawn to inform the teaching of mathematics.

In his interviews with professional mathematicians, Weber (2010) found that mathematicians read proofs in order to learn new methods and techniques that could be used in their own work, in essence, filling their mathematical toolbox with tools that can be valuable in their own work. For this reason, Hanna and Barbeau (2008) claimed that proofs are bearers of mathematical knowledge (Rav 1999). Hanna and Barbeau (2008) noted that like mathematicians, students can fill their mathematical toolbox as they work to comprehend unfamiliar proofs. Undergraduates can find value in reading classmate's proofs in order to get new ideas to aid in their own proof construction (Pair & Bleiler, 2015). Students should be aware that proofs are bearers of mathematical knowledge, and a key function of proof for mathematicians is the transmission of methods (Weber, 2010). Students may also benefit from learning about the other roles proof serves for the discipline of mathematics (de Villiers, 1990).

Informal Mathematical Work is the Foundation of Formal Knowledge.

An allegiance to formalism may result in the delegitimization and rejection of informal mathematics. For instance, Brown (1996) described a formalist mathematics instructor in a graduate course. This instructor explained that the only valuable objects in mathematics were the formalisms: axioms, definitions, and logically deduced theorems. During the writing of a proof at the board, he became temporarily stumped and resorted to draw a diagram. After using the diagram to obtain the insight needed to finish the proof, the teacher hurriedly erased his diagram and resumed the formalist presentation. Lakatos (1976) wrote, "This [deductivist] style starts with a painstakingly stated list of *axioms, lemmas* and/or *definitions.* The axioms and definitions frequently look artificial and mystifyingly complicated. One is never told how these complications arose" (p.142). Axioms are presented to students as having a divine status that is not to be questioned (Brown, 1996). Instead, students should have the opportunity to understand the concepts related to axioms and how mathematical axioms are formulated.

Hersh (1991) elaborated on the importance of informal mathematical work when he claimed that mathematics has a front and a back. The front is what is typically seen, in journals, and in textbooks (e.g. axioms, definitions, theorems, proofs). The front of mathematics is the polished, finished form of mathematics. But just as important and meaningful is the behind the scenes work, the creative emotional activity that serves as the basis for formal mathematical knowledge. Hersh noted that mathematics, being a human activity, is influenced by economic and social pressures. He explained that mathematics as an institution benefits from a presentation that hides the human struggle.

> The standard exposition purges mathematics of the personal, the controversial, the tentative, leaving little trace of humanity in the creator or the consumer. … If mathematics were presented in the style in which it's created, few would believe its universality, unity, certainty, or objectivity. These myths support the institution of mathematics. For mathematics is not only an art and a science, but also an institution, with budgets, administrations, rank, status, awards, and grants. (p. 38)

Students within a mathematics classroom, especially an undergraduate setting, should understand the nature of the informal work that ultimately leads to polished theorems. Lakatos (1976) claimed mathematics is a quasi-empirical discipline. De Villiers (2004) summarized the notion of quasi-empiricism: "[T]he objects in mathematics, though largely abstract and imaginary, can be subjected to empirical testing much as scientific theories are. Quasi-empiricism will, therefore, refer here to all non-deductive methods involving experimental, intuitive, inductive, or analogical reasoning". (p. 398). Lakatos (1976) showed how quasi-empirical methods (e.g. using a counterexample to refute a theorem statement), are implemented in practice and contribute to the development of mathematical knowledge. Mathematicians often use examples to look for patterns and make conjectures (de Villiers, 2004); it is this informal work that ultimately leads to the formal theorem. In interviews in which mathematicians were asked to determine if a proof was valid, Weber (2008) found that mathematicians used inductive examples to make sense of deductive inferences within a proof. Students should understand that mathematics is not only axioms, definitions and the following of deductive steps. Inductive methods also play a crucial role in creating (Lakatos, 1976) and validating (Weber, 2008) mathematical knowledge.

Summary and Implications

In this paper we have discussed four characteristics of the nature of mathematical knowledge: mathematical knowledge is subject to revision; mathematical knowledge is socially validated; proofs are bearers of mathematical knowledge; and informal mathematical work is the foundation of formal knowledge. Scholars in undergraduate mathematics education will find it valuable to consider whether these or other characteristics of the nature of mathematical knowledge may be worthy to serve as goals for student understanding. Research will be needed to identify and understand the instructional methods that can be used to support undergraduate mathematics majors in coming to know and understand these characteristics.

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Making RUME for Institutional Change

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Overwhelming evidence favors the use of active learning in undergraduate STEM classrooms. Thus, the issue faced by educators is no longer what to do in classrooms, but how to enact what is known to be effective. This poses a challenge, because faculty teaching is embedded in the context of departments, universities, and the broader disciplinary culture. Thus, improving education requires knowledge of how systems work and how to enact systemic change. While organizational change has studied these issues for decades in nonprofit and business settings, the application of this knowledge to higher education is relatively new. Accordingly, this theoretical paper provides an introduction to the organizational change literature in the context of higher education and provides an example of its application through Departmental Action Teams (DATs). By highlighting five principles from organizational change, this paper serves as a reference for change agents wishing to improve undergraduate mathematics education.

Keywords: Institutional change; Faculty development; Culture; Active learning

There is now considerable evidence for the use of active learning techniques in STEM classrooms. Broadly speaking, active learning aligns with sociocultural and constructivist views, which posit that learning involves constructing meaning through engagement in social practices (Lave & Wenger, 1998; Smith, diSessa, & Roschelle, 1993). As such, active learning courses involve students as active participants in classroom sessions, through activities such as: groupwork, peer instruction, class discussions, and personal response systems. Active learning is generally contrasted with "pure lecture," in which students passively listen to lecture and take notes. While active learning courses often involve some lecture (or mini-lessons), the distinction is that lecture is one of many modes used for instruction, rather than the only one.

A recent meta-analysis of 225 studies demonstrates the benefits of active learning; pure lecture increases failure rates in STEM courses by 55% when compared to active learning (Freeman et al., 2014). This evidence is so strong that the authors of the study described teaching solely through passive lectures as akin to educational malpractice. Moreover, the use of active learning can help reduce existing disparities between students from dominant groups and those historically marginalized in STEM classrooms (President's Council of Advisors on Science and Technology, 2012). Finally, these benefits appear to extend beyond just the courses incorporating active learning, to support students to do better in future courses as well (Kogan & Laursen, 2014). Thus, STEM education research has identified improved instructional techniques (i.e. active learning), scientifically proven to be more beneficial for students than traditional methods. As such, RUME (Research in Undergraduate Mathematics) is confronted with a new challenge: how to foster the use of active learning in undergraduate mathematics classrooms.

Research shows that simply providing faculty with evidence of the value of active learning is insufficient (Foertsch, Millar, Squire, & Gunter, 1997; Reese, 2014). This has been studied extensively in physics, where despite the wealth of instructional advances and widespread awareness of them, they are not widely used (Dancy & Henderson, 2010; Henderson & Dancy, 2007; Lutzer, Rodi, Kirkman, & Maxwell, 2005). Even when new pedagogies are adopted, sustainability is a challenge (Henderson, Dancy, & Niewiadomska-Bugaj, 2012). As such, recent calls for educational improvement have begun to shift their focus from developing new learning techniques to understanding how to scale and sustain the use of existing techniques (PCAST, 2012). This theoretical paper focuses on exactly this issue: how to enact and sustain educational change. Ultimately, I argue that change efforts should draw on the vast literature of organizational change. I highlight five principles from this literature to support change agents in their own local efforts, and provide an example of their application through Departmental Action Teams (DATs).

Theoretical Framing

Educational improvement requires attention to the university as a holistic system (Corbo, Reinholz, Dancy, Deetz, & Finkelstein, 2016). A recent meta-analysis of 191 STEM education improvement efforts showed that 85.3% of efforts focused on only a small part of the system, and that they were "clearly not effective" (Henderson, Beach, & Finkelstein, 2011). As such, researchers must draw from organizational change, to better understand how to change *systems*. Indeed, if the RUME community seriously considers the seven recommendations of the recent study of college calculus programs (Bressoud, Mesa, & Rasmussen, 2015), the need for systemic change is clear. Thus, change agents must expand their work with individual faculty members to consider how their efforts are embedded in departments, universities, and disciplinary culture. While there are interactions between each of these levels, the academic department itself is often considered the key unit of change, due to its relative coherence as a unit (AACU, 2014).

Given its roots in educational psychology, most educational research is grounded in experimental science. The logic of experimental science is that variations in treatments and contexts can be accounted for statistically, to generalize results across settings. In contrast, organizational change is better understood as an improvement science (Bryk, Gomez, & Grunow, 2011). Given the complexity of organizations, improvement scientists argue that context is too important to be "averaged out" statistically; instead, one must develop a "system of profound knowledge" for how to enact change within a given context (Lewis, 2015). While some educational research aligns with this perspective, such as in action research (Zeichner & Noffke, 2001) and design-based research (Cobb, Confrey, Disessa, Lehrer, & Schauble, 2003), these approaches have not yet been widely adopted.

What follows is a brief description of principles extracted from a synthesis of improvement efforts in higher education (AACU, 2014; Elrod & Kezar, 2015; Henderson et al., 2011), intended to provide undergraduate mathematics educators with powerful ideas that they can use immediately to support their own educational improvement efforts. For a general overview of this literature, consider the book *How Colleges Change* (Kezar, 2013). While it is beyond the scope of this paper, improvement science offers tools for assessing the impact of systemic change efforts, such as: program improvement maps, driver diagrams, and Plan-Do-Study-Act (PDSA) cycles (Bryk et al., 2011). In what follows, I describe five ideas from organizational change and their application to RUME.

Five Good Ideas

These ideas draw the reader's attention to concepts that are often overlooked in educational improvement. The five ideas are: (A) building a shared vision, (B) supporting agency and ownership of a change, (C) promoting the use of evidence, (D) creating opportunities for early wins, and (E) designing for sustainability. As the authors of a recent effort for systemic change on university campuses note, “almost all of these process – organizational learning, addressing politics, creating a shared vision and unearthing cultural assumptions – were extremely hard for STEM leaders…These processes are often messy and non linear” (Elrod & Kezar, 2015, p. 7). In other words, while these principles are supported by the organizational change literature, they can be difficult to enact, and are not yet widely used.

Building a Shared Vision

Suppose a group of faculty aims to improve student interactions in their department. A common approach is to generate a list of barriers, such as: large class sizes, an overstuffed curriculum, lack of department funds, and many students commuting to campus. Having identified these problems, the group identifies possible solutions (e.g., classroom response systems, curricular change) and debates their relative merits. Yet, this “problems focus” tends to result in a fixation on specific problems and preferred solutions to them. For instance, one group member may fixate on large lecture courses, and the use of classroom response systems as a “solution.” Most individuals have such preferred solutions, and this often leads to inflexibility.

Rather than operating in “problem-solving mode,” discussions are more effective when they focus on positive outcomes to be achieved (Cooperrider, Whitney, & Stavros, 2008). Suppose the same group of faculty works to generate a shared vision for student interactions in the department. They decide upon the goal: students will feel like a part of a community with their peers and work together productively to succeed as mathematicians. This opens the conversation to many other possibilities, including: improving department culture, running community events, and creating space for student collaborations outside of class. Such an approach builds flexibility, helps reduce conflict, and thus increases collaboration. An “outcomes focus” changes the nature of the conversation, allowing group members to see possibilities (e.g., creating a welcoming departmental culture) where before they saw only obstacles.

Agency

Change is not something that can be “done to others.” Yet, very often, educators have a new curriculum or teaching techniques that they would like others to adopt. In other words, the change agents would like to change others. However, as the research on dissemination approaches highlights (Henderson et al., 2011), this is generally not effective. Instead, a change agent should work with others, to help them achieve their goals. This process often begins with developing a shared vision for what the participants want to achieve, affording participants agency in the process. Agency relates to the ability of individuals to influence their circumstances (Bandura, 2006). When individuals have agency over a change effort, they are more invested in the work, as they develop ownership over their change effort. Because of this investment, the individuals are more likely to expend more effort, rather than giving up, when obstacles are inevitably encountered.

Evidence

Psychological research shows that individuals use shortcuts to make decisions (Kahneman, 2011). According to the availability heuristic, individuals usually rely on the most accessible or salient examples to make decisions, regardless of how representative of the larger population they are. Consider a faculty member trying to explain why a student is doing poorly in their class. The faculty member may notice a student has skipped a few class sessions, and infer that the student is lazy or unmotivated. In general, it is easier to center the locus of control within the student, than to consider systemic factors, such as departmental culture or the student's life circumstances (e.g., working a full-time job to pay for college).

The above explanations are ascriptions of motive, intended to describe the underlying causes of the student's behavior. These ascriptions can be placed on a spectrum from person- to system-focused (Blum & McHugh, 1971; Simpson & Vuchinich, 2000). Person-focused explanations (e.g., laziness) tend to be readily available, so faculty members are more likely to adopt them "by default." Yet, person-focused explanations are also outside of the faculty member's control, so adopting these explanations means that the faculty member has little agency to change the situation. In contrast, systemic factors (e.g., departmental culture) can be changed, so focusing on them increases faculty agency.

When conversations focus on "anecdata," they tend to revolve around person-focused explanations. Thus, to shift conversations towards systemic factors, change agents can use evidence that highlights the systemic nature of issues. In accordance with building a shared vision and promoting agency, change agents should help faculty gather data to answer their questions, rather than presenting data to argue for a preformed agenda.

Early Wins

Change is a time-consuming. For instance, work in teacher professional development shows that effective interventions are longterm and must be holistic (Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009; Wilson & Berne, 1999). Similarly, work at the department or campus level can be expected to take many years (Elrod & Kezar, 2015). Yet, when change work takes years to come to fruition, it is easy for participants to disengage or processes to become stalled. Thus, it is key to build in "early wins" to the process (Kotter, 1996).

The idea of early wins is simple; to begin a change process, a group identifies a vision for what it would like to achieve. Again consider the group that aims to build greater community for its students. While changing the community and the culture of the department is a many-year project, there are also many waypoints or markers of change that would provide evidence of improvement. For example, the group could: survey students about their experiences (collecting data), run community events, run faculty professional-development for inclusive teaching, or seek external funding. By conducting these activities, the group creates concrete "successes" towards the larger goal of building community. This is important internally, for the motivation of the group, and externally, with respect to department and campus politics.

While there may often be early wins on the way to a larger goal, from the perspective of a change agent, they can be strategically built into the change process. For instance, creating early wins is built into the cyclical nature of PDSA cycles in improvement science (Bryk et al., 2011). PDSA cycles are iterative improvement cycles that focus on implementing and analyzing ideas quickly, to enact efforts in a way that is sensitive to the local context. Thus, once a shared vision has been developed, PDSA cycles are one way to identify short-term goals that can be achieved in service of the larger goal, to ensure that the process does not stall.

Sustainability

Change agents often talk about "solving" educational problems, or having courses that are "transformed." This language implies that educational improvement is something that can be "done" and then it will be sustained. Yet, it is difficult to sustain even largely successful change efforts when external funding is removed (cf. Chasteen et al., 2015). Thus, efforts should focus on continuous learning and sustainability from the offset (Senge, 2006).

One key aspect of sustainability is focusing on culture. Culture is a "pattern of shared basic assumptions learned by a group…which has worked well enough to be considered valid and, therefore, to be taught to new members as the correct way to perceive, think, and feel" (Schein, 2010, p. 19). As such, a department's culture consists of these beliefs, values, customs, rituals, practices, artifacts, and institutional structures. These various components of culture interact with one another to provide a relatively coherent system. Thus, if one attempts to change only a single component of the cultural system, without addressing other components, it is unlikely that the change will be sustained over time.

As described above, dissemination efforts often do not result in sustained use of new teaching techniques (Henderson et al., 2012). Because these efforts focus only on practices, a single component of culture, there are numerous forces acting in opposition to the use of the new teaching techniques. For example, traditional beliefs about teaching and learning will influence *how* the practices are used, which may limit their efficacy. Or, reward structures may be such that the time required to learn to use the new techniques is not perceived as worthwhile.

Departmental Action Teams

Departmental Action Teams (DATs) were developed as one component of the STEM Institutional Transformation Action Research (SITAR) Project, which fostered and studied systemic change in STEM departments at one research-extensive university (Corbo et al., 2016). A DAT consists of 4-8 participants (primarily faculty) working collaboratively to improve education in a single department. DATs are externally facilitated; facilitators bring expertise in educational research, help coordinate logistics, and draw on principles from organizational change (i.e. the key ideas identified above). In what follows, the process of facilitating a DAT is described in more detail to provide change agents with concrete examples of how to implement organizational change principles in their own educational improvement efforts.

A DAT begins with members developing a shared vision around an issue of common interest in their department (key idea A). Participants have agency to choose an issue that is meaningful to them (key idea B); the role of the facilitators is to help the DAT work on the issue in the most productive way possible, not to tell participants what issue to work on. As the DAT works to achieve its vision, it gathers and analyzes relevant data (key idea C): so it can make informed decisions about potential actions, and so it can build political will from external stakeholders (e.g., a department chair). Along the way, the facilitators and DAT members think strategically about how to build early wins into the process (key idea D) so that progress does not stall. Finally, the types of issues a DAT addresses are crosscutting, and building sustainable structures is a goal from the offset (key idea E).

DATs in SITAR met regularly, typically for an hour every other week for multiple semesters. Between meetings, DAT members assign their own "homework," determining what needs to be done to continue moving the group forward. DAT members may also schedule additional meetings as necessary. Thus, while DATs are externally facilitated, they are departmentally-driven. To date, the DAT model has been used to facilitate 6 working groups through the SITAR project. In what follows, I provide examples of the five key ideas in action. Data are drawn from from four STEM departments: Alchemy, Potions, Prophecy, and Runes (actual department names redacted for confidentiality).

Shared Vision

Once the group has been established, a DAT begins by building a common vision for its work. To help participants build a common vision, the facilitators use a “sticky note” activity (adapted from http://serc.carleton.edu/departments/degree_programs/idealstudent.html). Each DAT member is given a pad of large sticky notes, and asked to write individually their responses to the following prompt:

> Imagine you are writing a letter of recommendation for a student graduating from your department. Ideally, what would you like to be able to say in response to the following questions: (1) what kind of person will they be? (2) what will they be able to do? (3) what will they know? (4) what skills will they have? (5) how will they behave? (6) what will they value?

After writing their responses, the DAT members stick their responses on the wall, the group organizes them looking for common themes, and then they have a whole group discussion about vision. These prompts are designed to help faculty focus on students (not just themselves), and to seek areas of overlap in what all participants value. These prompts appear to work effectively even with DAT participants that have very different views on education.

Agency

While the external facilitators help shape discussions, they do not tell DAT participants what to do. As described above, it is DAT members who determine the vision and direction of the group, not the facilitators. Moreover, the participants determine homework, whether or not to schedule additional meetings, and many other key features of how the work gets done. As a result, DAT members perceive the change effort as theirs. For example, as a participant in the Runes DAT described:

> I really think they did a fabulous job of letting all of us kind of speak our piece and keeping it harmonious and letting us kind of find our own way. I think- Like I said, I think, I'm hoping that everybody's as excited about this as I am, because I think we've struck on something that'll really work for our department.

While this is a quote from just one member of one DAT, it is generally consistent with the perceptions of other DAT members that they “owned” their efforts.

Evidence

Gathering evidence to support decisions was a common thread across DATs. In the Runes and Prophecy DATs, both focused on curricular integration, participants gathered and analyzed institutional data about the course taking patterns and success of their students. In the Potions and Alchemy DATs, both focused on diversity and inclusion in their departments, a wealth of data about the retention, success, and experiences of students from diverse groups were collected and analyzed. These data were used to determine plausible actions for the DATs, and on multiple occasions were used in presentations at faculty meetings or to departmental committees to gather support for the actions of the DATs.

Early Wins

The Potions DAT spent the majority of its first year analyzing data related to diversity in the major, which resulted in a detailed report to the department. Yet, beyond curating data, the DAT also engaged in a number of actions, such as: targeted recruiting of admitted students, building collaborations with other diversity organizations on campus, and leading the department's response to a campus diversity initiative. All of these actions supported the DAT to be seen as a positive force in the department, and ultimately supported it to be institutionalized in the form of a standing committee. Similarly, the Alchemy DAT has begun to focus on concrete actions in parallel to collecting and analyzing data.

Sustainability

Both the Runes and Potions DATs (the only multi-year DATs, to date) have successfully created new departmental structures to sustain their efforts. In Runes, new curriculum coordinator positions have been created (and funded by the department) to revisit curricular integration issues on an ongoing basis. In Potions, the DAT has been formalized as a standing committee. Moreover, the facilitation practices used by the external facilitators have been adopted, and the Potions DAT is continuing to use them in its ongoing work.

Summary

STEM education has identified research-based approaches for improving classroom learning. Yet, actually enacting these approaches remains a challenge. Fortunately, there is a wealth of organizational learning theory, traditionally applied to businesses and nonprofits, that can be adapted to support higher educational change. Accordingly, this paper has two primary aims: (1) broaden the focus of RUME to emphasize systemic change perspectives, and (2) provide practical tools that RUME practitioners can use to increase the impact of their work.

A systemic change perspective provides a new lens for RUME practitioners engaged in improvement efforts. For example, it highlights the systemic nature of educational improvement, operating at multiple levels: students, classrooms, departments, universities, disciplines, and society. As such, improving education requires thinking about these multiple levels. It also highlights the political nature of change, such as the need to develop processes that will effectively engage a variety of stakeholders. Finally, this perspective highlights the need for sustainability. When sustainability is built into a process from the offset, rather than considered as an afterthought, it is much more likely for continuous improvement to result.

This paper provides a number of practical tools that the RUME community can draw upon. For instance, simply organizing improvements around outcomes rather than problems, can result in much more productive conversations. Similarly, building in early wins can help make progress visible, rather than resulting in improvement efforts stalling. By affording participants with agency and the ability to make decisions around evidence, rather than anecdote, it is more likely that innovations in education will be used and sustained. In sum, organizational change provides useful theoretical background to promote systemic change. As researchers adopt this perspective, theory of organizational learning can be adapted and contextualized to RUME.

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Inquiry as an Access Point to Equity in the Classroom

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Although many policy documents include equity as part of mathematics education standards and principles, researchers continue to explore means by which equity might be supported. Teaching practices that include active learning have been proposed to address this issue (e.g., CBMS, 2016; NCTM, 2014). In this paper, we theoretically explore the ways in which active learning teaching practices that focus on teaching for inquiry (e.g., Inquiry-Based Learning (IBL) or Inquiry-Oriented Learning (IOL)) support equity in the classroom. Specifically, we claim that some characteristics of inquiry (Student Ownership, Knowledge Building, Peer-Involvement, Doing Mathematics, Student-Instructor Relationship, and Student Success) put forth by Cook, Murphy, and Fukawa-Connelly (2016) may align with the Four Dimensions of Equity (Access, Achievement, Identity, and Power) proposed by Gutiérrez (2009). Therefore, inquiry teaching may be a first step for a focus on equity without compromising the excellence (Gutiérrez, 2002) or material that is often prescribed in undergraduate mathematics courses.

Key words: Active Learning, Equity, Inquiry-based Learning, Inquiry-oriented Learning

Many policy documents and institutions both highlight the importance of equity and caution educators with possible consequences of not attending these issues in research and teaching. Most recently in their *Statement on Active Learning* (2016), the Conference Board of the Mathematical Sciences stated, "Pervasive problems caused by issues of equity and access, starting long before students begin post-secondary study, prevent or discourage many students from continuing in their study of mathematics and other STEM disciplines" (p. 3). More strikingly, Nasir, Shah, Gutiérrez, Seashore, Louie, and Baldinger (2011) reported that three decades after the release of *A Nation at Risk* (1983), which "cautioned that America's economic future depended on the mathematical and scientific literacy of all of its citizens" (p. 1), there are still "substantial disparities both in resources and in achievement" that are "organized along troublingly clear lines of race, ethnicity, and socioeconomic status" (p. 1).

Although many policy documents include equity as part of standards or principles of mathematics education, ways in which this goal can be achieved are not explicit. For example, at the K-12 level, the *Principles for School Mathematics* provided by the NCTM have included *Equity* since the early 1990's. The American Mathematical Association of Two-Year Colleges (2006) states, "All students should have equitable access to high-quality, challenging, effective mathematics instruction and support services" (p.10). Yet, as Gutiérrez (2007) indicates, "[m]ost members of the mathematics education research community would agree that equity is a valued goal, maybe even the reason behind their research. However, much less consensus arises when the question is raised: how do you think we should address equity?" (p. 2).

Teaching practices that include active learning have been proposed to address this issue (e.g., CBMS, 2016; NCTM, 2014). We explore the ways in which active learning teaching practices, with a focus on teaching for inquiry (e.g., Inquiry-Based Learning (IBL) or Inquiry-Oriented Learning (IOL)), can provide a pathway to equity in the classroom without compromising the excellence (Gutiérrez, 2002) or material that is often prescribed in undergraduate mathematics courses. We claim that characteristics of inquiry align with the Four Dimensions of Equity proposed by Gutiérrez (2009). That is, we claim these four dimensions explicate how inquiry pedagogy promotes equity in mathematics courses. This particular framing helps us to identify how the inquiry instruction can promote a more equitable experience for all students within the context of the curriculum that is usually required in undergraduate mathematics courses.

Equity

In general, equity teaching promotes a mindset where all students are capable of learning mathematics (Bullock, 2012; Gutiérrez, 2002; Jett, 2012). Equity research seeks to surface teaching practices that enable these mindsets (Gutiérrez, 2002) among instructors and students alike (Oppland-Cordell & Martin, 2015). It is important that instructors bracket prejudices about student participation and achievement levels based on race, gender, social class, proficiency in the dominant language, or ethnicity (Gutiérrez, 2002). Similarly, judgments based on a student's prior performance, particularly if s/he has performed poorly in the past should not be seen as personal weakness, rather as a consequence of the complex social, economic, and cultural factors (Frankenstein, 1983) that affect individual experiences while learning mathematics.

Gutiérrez (2009) argued that teaching for equity includes four dimensions: *Access*, *Achievement*, *Identity* and *Power*. *Access* and *Identity* are considered precursors to *Achievement* and *Power*, respectively. *Access* addresses the resources that students have available to them to participate in mathematics such as "quality of teachers, adequate technology and supplies, classroom environment that invites participation, infrastructure for learning outside the classroom" (p.5), and the opportunities to draw upon their "cultural and linguistic resources" (p. 5). *Achievement*, on the other hand, is an outcome affected by students' opportunities to learn and can be measured by "participation in class, course taking patterns, standardized test scores, majoring in math, having a math-based career" (p. 5). Adiredja, Alexander and Andrews-Larson (2015) summarized this description by offering that learning outcomes can range from the "knowledge on specific content to students' ability to productively use mathematics to participate in society" (p. 64).

On a different axis, *Identity* attends to the "balance between self and the global society and ways students are racialized, gendered and classed" (Gutiérrez, 2009, p. 5) and to pay "attention to whose perspectives and practices are 'socially valorized'" (p. 5). The goal is to "strike a balance between opportunities to reflect on oneself and others as part of the mathematics learning experience" (p.5). *Power* can mean to empower students towards high academic achievement, but Gutiérrez explained it as students using their math knowledge to reach "personal goals of excellence such as helping their community to solve a local problem" (p. 6). Adiredja et al. (2015) added that learning focused on this dimension attends to "disrupting the existing power distribution and dynamics in a society based on race, gender, and social class" (p. 64). To achieve this, students can be involved in decision-making on "what counts as productive mathematical knowledge" (Adiredja et al., 2015, p. 64), pacing of content (Laursen, Hassi, Kogan, and Hunter, 2011), and starting points for curriculum (Frankenstein, 1983). This type of learning requires a social transformation as measured by whose voice can be heard in the classroom and "opportunities to use math as an analytics tool to critique society" (Gutiérrez, 2009, p. 6).

Gutiérrez (2009) situated these four dimensions more broadly, namely, “in society” or in a “community” (p. 6). In discussion of power, Gutiérrez (2002) positioned the distribution of power in the contexts of the classroom, future schooling, everyday life, and the global society. In this paper, we use the classroom as a stepping-stone to discuss alignment of inquiry pedagogies to dimensions of equity. As such, we utilize these four dimensions of equity as a framework to discuss how active learning pedagogies, and inquiry learning in particular, have the potential to increase access, lead to higher achievement, provide opportunities for students to reflect on their identities, and attune students to power dynamics in their mathematical community: the classroom. We acknowledge that just using inquiry learning alone may not fully address equity, especially if there is not a change to the system outside the classroom or if students do not have opportunities to question power distribution and dynamics in the greater society. The purpose of our theoretical exploration is simply to investigate inquiry learning as an entry point towards a more equitable classroom, in order to move towards a more equitable society.

Active Learning

While this paper reports on teaching through inquiry, we see this pedagogy as a subset of a collection of pedagogies termed *active learning*. Although many different definitions exist, Prince (2004) noted “active learning is generally defined as any instructional method that engages students in the learning process” (p. 223). However, pedagogical techniques used in active learning vary between instructors and may include group work, think-pair-share, student presentations, project-based learning, and many other teaching techniques. Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt, and Wenderoth, (2014) reported that active learning techniques have a strong positive impact on student learning as a result of the meta-analysis of 225 studies in STEM education. In addition, Kogan and Laursen’s (2014) study indicates that “the benefits of active learning experiences may be lasting and significant for some student groups, with no harm done to others. Importantly, ‘covering’ less material in inquiry-based sections had no negative effect on students’ later performance in the major” (p. 197). There is even strong evidence that active learning promotes student engagement and achievement when coupled with lecture; lecture is defined as “continuous expositions of a speaker” (Bligh, 2000, p. 4) where student activity is “limited to taking notes and/or asking occasional and unprompted questions of the instructor” (Freeman, 2014, p. 5). Prince (2004) found that incorporating several short active learning segments into lecture on a topic in an engineering course improved students’ retention of the material and exam scores on that topic.

Overview of Inquiry Learning

Some major goals of inquiry learning are to “deeply engage [students] in rich mathematical tasks, [give students] ample opportunities to collaborate with peers (where collaboration is defined broadly)” (Academy of Inquiry-Based Learning, n.d.), “enable students to learn new mathematics through engagement in genuine argumentation, … empower learners to see themselves as capable of reinventing mathematics, and to see mathematics itself as a human activity” (Rasmussen and Kwon, 2007, p. 190). Despite the numerous studies on inquiry-based or inquiry-oriented learning, there is not a consistent definition of these pedagogies. However, one defining feature of inquiry learning seems to be the modified role of the instructor in the classroom, which Cook et al. (2016) label the *Student-Instructor Relationship*. Thus, in an inquiry course, part of the job of the instructor is to ask about student thinking (Kuster, Johnson, Keene, & Andrews-Larson, submitted). Laursen, Hassi, Kogan, and Weston (2014) found that students in IBL courses often reported being able to express their own ideas while the instructor listens. Another facet of inquiry is student participation in authentic mathematical experiences, which Cook et al. (2016) refer to as *Doing Mathematics*. Kuster et al. (submitted) argue that "questions that require students to engage in problem solving activity affords the instructor opportunities to inquire into student thinking and reasoning" (p. 8). Thus, doing mathematics contributes to the student-instructor relationship.

Cook et al. (2016) also describe *Student Ownership* as the action of encouraging learners to create, generate, and developing their own knowledge. This knowledge is built from their prior knowledge, which is labeled *Knowledge Building*. Kuster et al. (submitted) also see this as a fundamental part of IOL and they refer to it as "building on student contributions". As part of knowledge creation, students are given opportunities to provide explanations and justifications of their thinking while others listen to and attempt to understand the ideas being discussed or presented, termed *Peer Involvement* by Cook et al. (2016). In Laursen et al. (2014), students in IBL courses reported often participating in activities such as asking questions, evaluating other students' work, and working together in class. Kuster et al. (submitted) also identified students' "being engaged in one another's thinking" as a characteristic of IOL.

According to Cook et al.'s (2016) exploration of existing studies, an outcome of their aforementioned features of inquiry is that inquiry-based or inquiry-oriented learning is better aligned to how people learn. Similarly, in a study that gathered data from over 100 sections of IBL and non-IBL courses taught between 2008 and 2012, Laursen et al. (2014) reported higher "cognitive gains in understanding and thinking, affective gains in confidence, persistence, and positive attitudes about mathematics, collaborative gains in working with others, seeking help and appreciating different perspectives" (p. 409) in students from IBL courses compared to those in non-IBL sections of the same courses. Notably, Laursen et al. (2014) also found that in IBL courses, both men and women's attitudes about mathematics improved as well as their interest in pursuing mathematics, but the women had greater gains in these areas than men. Cook et al. (2016) categorized this as *Student Success*.

In surveying authors of the papers that they reviewed, Cook et al. (2016) identified three foci in participants' definitions of inquiry. The first focus is on the student-instructor relationship and how it is different from a lecture-based course, which is an extension to the aforementioned *Student-Instructor Relationship* theme. Participants stated that the students should not look to the instructor as the sole mathematical authority in the classroom. The second focus is similar to the theme *Peer-Involvement*; that is, the class should include opportunities for peers to interact. The last focus is different from the aforementioned themes as it highlights the importance of valuing and nurturing curiosity in the students. This last focus (valuing and nurturing curiosity) and the first five themes (*Student-Instructor Relationship, Doing Math ,Student Ownership, Knowledge Building*, and *Peer Involvement*) can be classified as Classroom Climate, whereas the last theme (*Student Success*) is related to an outcome of the classroom climate.

The themes that we consider in this paper are not an exclusive list of IBL/IOL teaching practices; they are still under development and undergoing revisions. Hence, our theoretical exploration is only a preliminary start of a framework that aims to explore the alignments between IBL/IOL features with the Four Dimensions of Equity by Gutiérrez (2009).

Alignment of IBL/IOL with the Four Dimensions of Equity

With this proposed framework, we put forth the claim that, as a pedagogical practice, inquiry learning can be used to promote equity by providing students access and chances to explore their identities, with the hopes of both a shift in both power and achievement in the course. Our exploration originated from several reports, particularly Laursen et al.'s (2014) assertion that

"IBL benefits all students even as it levels the playing field for women" (p. 415). Their study documented ways in which IBL can increase achievement and attitudes among students. To explicate how described features of IBL/IOL provide a more equitable experience for students studying mathematics, within the Four Dimensions of Equity we describe some selected related features of IBL/IOL.

Access

Gutiérrez's (2009) definition of equity included a "classroom environment that invites participation" (p. 5) as a tangible resource to access. Civil (2006) reinforced this definition and stated: "equity to me is related to access by **all** students to opportunities to engage in rich mathematics" (p. 56). IBL/IOL pedagogies revolve around a classroom environment that invites and encourages all students' participation in doing, discussing, and presenting mathematics (*Peer Involvement*). When all students are given opportunities to be active participants in the mathematical community of the classroom (*Doing Math*), students are given an additional access point to learn because they are given the chance to provide explanations and justifications of their thinking processes while others listen and attempt to understand the ideas being discussed or presented. We believe that these opportunities give all students the chance to be exposed to other ways of thinking which can result in richer learning experience for them.

Nasir et al., (2011) provided characteristics of classroom practices that support equity: "Powerful classroom practices include those that foster student-centered discourse, student exploration of mathematical ideas, and on-going feedback (Davis, et. al., 2007; Boaler, 2002b; Fullilove & Treisman, 1990" (Nasir et al., 2011, p. 17). Inherent in the on-going feedback is the *Student-Teacher Relationship*: the instructor's responsibility of inquiring into student thinking and "fostering and facilitating productive student discourse" (Nasir et al., 2011, p. 17).

Achievement

Gutiérrez (2009) refers to Achievement as a measure of "how well students can play the game called mathematics" (pg. 6). In other words, this dimension relates not only to student performance on exams and standardized tests, but also considers a student's mathematical "story." This can refer to measures such as whether students continue taking mathematics courses or whether they choose a mathematical career.

When all students are encouraged to create, generate, and develop their own knowledge (*Student Ownership*), confidence in doing mathematics and participation in class may be positively affected. Laursen et al. (2014) demonstrated that participation in IBL courses does increase student performance as well as other measures related to this definition of achievement. Learning gains were found in IBL sections over non-IBL sections of the same course; not only improvements in course performance, but gains in confidence, persistence, and enjoyment of mathematics (*Student Success)*. Some of these outcomes may lead to Gutiérrez's (2009) measures of Achievement, namely "course taking patterns, majoring in math, and having a math-based career" (p. 5). Kogan and Laursen (2014) also reported that all students in IBL courses were positively impacted to enroll in more mathematics courses.

Identity

We claim that the *Peer Involvement* theme of IBL/IOL aligns with Gutiérrez's (2009) definition of Identity. When students are actively engaged with each other and each other's thinking (*Peer-Involvement*), it can lead to a shift in mathematical identity. Hassi's (2015) qualitative study of students reflecting on their IBL learning experiences supports our claim. In that study, students talked about “the role of the social environment in an IBL class for gaining or verifying their self-esteem or self-confidence” (p. 60). In addition, Oppland-Cordell & Martin (2015) write that “the ways in which individuals continuously construct identities of participation and non-participation over time in [Communities of Practice] is related to how they position themselves, how others position them, and how such positionings are related to their histories and experiences in the broader contexts in which [Communities of Practice] are embedded” (p. 24). At the secondary level, Boaler and Greeno (2000) contrasted students who learned by working through rote problems in a textbook with students who learned through mathematical discussions (*Peer Involvement*). They found that in discussion-based classes, students were required to contribute more aspects of their selves (as compared to non-discussion-based), which can be done through reflecting on community participation and family relationships.

Power

Gutiérrez (2009) thinks of student voice as a fundamental part of the power dimension; inquiry is changing whose voice is primarily present in the classroom. Instructors are responsible for facilitating student discussion and presentation of the problems (Yoshinobu and Jones, 2012; Cirrillo, 2013). When given opportunities to provide explanations and justifications of their thinking while others listen to and attempt to understand the ideas being discussed or presented (*Peer Involvement*), power shifts to the students because they decide on “what counts as acceptable knowledge” (Adiredja et al., 2015, p. 66).

The instructor is the primary architect of the problems worked on (Laursen et al., 2011), and when the tasks assigned include problem-posing, students create and solve their own problems (*Doing Math*). In this scenario, students have power in deciding the curriculum.

The instructor’s main role is not as a problem-solver, but as an expert participant (Levenson, 2013) that guides students to generate, create, and develop their own knowledge (*Student Ownership*). In this way, the instructor signals that the students’ thoughts, beliefs and contributions are a valued part of the learning process and removes her/himself as the sole source of knowledge in the classroom. If we agree that *Doing Math, Peer Involvement*, and *Student Ownership* are components of inquiry teaching, then this represents a substantial shift of the power dynamic from instructor to students.

Future Steps

“Equitable classrooms are reflections of a pedagogical, political, and moral vision.”
(Lotan, 2006, p. 526)

We acknowledge that changing the curriculum to include ways for students to use mathematics to critically analyze the society in which they are gendered, raced, and classed extends past our theoretical framework to provide a richer equitable experience. However, for instructors who are not ready to (fully) change the curriculum of their class, we claim that by merely engaging in practices of IBL/IOL, we can start to move towards teaching for equity and thinking equitably. That is, engaging in practices of IBL/IOL is a good entry point for people who are ready to begin embracing equitable practices.

We also see room to frame IBL/IOL in terms of culturally responsive pedagogy. Hernandez (2013) reviewed literature by seminal culturally responsive pedagogy researchers (Banks, Gay, Ladson-Billings, Nieto, Villegas and Lucas) and found five main themes within the research. To demonstrate how the inquiry themes posited by Cook et al. (2016) or the dimensions of equity theorized by Gutiérrez (2009) may align with the five themes of culturally responsive pedagogy, we have added the inquiry themes in brackets to the definitions below:

- Content Integration: inclusion of content from many cultures, the fostering of positive teacher-student relationships, holding high expectations for all students, and the use of research-based instructional strategies [*Student Success*] that reflect the needs of a diversity of backgrounds and learning styles [*Knowledge Building*]
- Facilitation of Knowledge Construction: the teacher's ability to build on what the students know [*Knowledge Building*] as they assist them in learning to be critical, independent thinkers who are open to other ways of knowing
- Prejudice Reduction: teacher's ability to use a contextual factors approach [*Student-Teacher Relationship*] to build a positive, safe classroom environment [*Access*] in which all students are free to learn regardless of their race/ethnicity, social class, or language
- Social Justice: teacher's willingness ''to act as agents of change'' (Villegas), while encouraging their students to question and/or challenge the status quo in order to aid them in ''the development of sociopolitical or critical consciousness'' (Ladson-Billings) [*Power*]
- Academic: teacher's ability to ''create opportunities in the classroom'' (Villegas) [*Access*] that aid all students in developing as learners to achieve academic success [*Student Success, Achievement*] (p. 811-814) [emphasis added].

In preliminary efforts to give empirical evidence of our framework, we found some data that did not fit into the themes presented by Cook et al. (2016) describing IBL/IOL. For example, "limited involvement by instructor" or "instructor acts as facilitator/mediator" didn't seem to fit in the *Student-Instructor Relationship* category, which puts the instructor as an inquirer into student thinking or not as the sole authority. Additionally, the theme of *Doing Mathematics* seemed to intersect with the other themes so often that we may need to create a hierarchy of themes. Thus, we plan to continue refining the themes of inquiry, and the subsequent connections to equity, from student data we have collected and as other research emerges.

The theoretical framework we put forth in aligning inquiry pedagogies to equity teaching is merely a start to providing equitable experiences for all our students regardless of race, gender, ethnicity, social class, sexual orientation, or language. To deepen equity in the field of mathematics, educators can integrate content that uses mathematics to critically analyze the ways in which students are gendered, classed and raced. For example, to address the contentious "All Lives Matter" movement, students in a proofs or logic class can analyze the statement, "If you are black, then your life matters" to help people understand that saying "Black Lives Matter" does not mean that other lives do not matter. In the future, we aim to deepen our alignments of inquiry teaching and learning with equity through interviewing students who have experienced inquiry pedagogies. We also hope to explore pedagogical techniques to better integrate content that allows students to use mathematics to critically analyze social justice issues. Both future goals are considered with the intent to extend our theoretical framework beyond the classroom and towards the global society.

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Leveraging Real Analysis to Foster Pedagogical Practices

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Although it is frequently a required course, many secondary teachers view real analysis as unnecessary and unrelated to teaching secondary mathematics. In accord with a proposed model for improving the teaching of advanced mathematics courses for teachers, we implemented a course that framed real analysis content by 'building up from' and 'stepping down to' teaching practice. In this paper, we describe how this model was implemented in a single module and analyze secondary mathematics teachers' engagement in and reflections on the desired pedagogical aims, which provide evidence that they saw what they learned in the real analysis module as being useful for informing their pedagogical practice.

Key words: Real Analysis, Secondary Teaching, Teacher Education, Pedagogical Practice

In the United States, prospective secondary mathematics teachers usually are required to take a substantial number of courses in advanced mathematics, including real analysis. However, while mathematical organizations believe that a mastery of advanced mathematics is important for teaching secondary mathematics (e.g., CBMS, 2012), research has also shown that completing courses beyond a fifth course in university studies – which is where advanced mathematics courses fall – yields very minimal gains in a secondary mathematics teachers' efficacy (Monk, 1994). Indeed, many students do not perceive any relevance between advanced mathematics and the teaching of high school mathematics (e.g., Zazkis & Leikin, 2010). In this paper, we address the following broad question: Given that prospective teachers are required to complete courses in advanced mathematics, how can we design these courses so that they productively inform teachers' future pedagogy? We look specifically at one module in a real analysis course designed according to this aim, and consider how the prospective and practicing teachers (PPTs) engaged in and reflected on a particular pedagogical practice.

Literature and Theoretical Perspective

A Model for Teaching Secondary Teachers Advanced Mathematics

From our point of view, the belief that completing a course in real analysis will improve a PPT's ability to teach secondary mathematics has been based on a traditional view of transfer from the cognitive psychology literature (e.g., Perkins & Salomon, 2002). More specifically, there is an assumption that as a byproduct of learning advanced mathematical content, PPTs will better understand secondary mathematics content and will consequently respond differently to instructional situations in the future – a tenuously presumed "trickle down" effect (Figure 1a). Given the notorious difficulties in achieving this type of transfer, it is not surprising that PPT's experience in real analysis often does not improve their teaching. In Figure 1b, we propose an alternative instructional model (Wasserman et al., 2016). This model is based on two premises. The first is the knowledge that PPTs learn should be inherently practice-based and applicable to the actual activity of teaching (e.g., Ball, Thames, & Phelps, 2008). The second is that PPTs will be more likely to make connections between real analysis and pedagogical practice if what they learn is situated within the context of teaching (e.g., Ticknor, 2012). Our model is composed of two parts: building up from practice and stepping down to practice. To *build up from (teaching) practice,* the real analysis content is preceded by a practical school-teaching situation. The building-up portion provides a context that sets the stage for the study of real analysis content in ways that are both relevant to teachers' practices and particularly well-suited to being learned in real analysis. The second part, *stepping down to (teaching) practice*, then uses the mathematical ideas from real analysis as a means to reconsider the pedagogical situation that began the module, as well as other relevant pedagogical situations. Stepping down to practice explicitly clarifies the intended mathematical and pedagogical aims. In between building up from and stepping down to practice, the real analysis topics are covered in ways true to its advanced character with formal and rigorous treatment, but the tasks make explicit what connections and implications these have for both secondary mathematics and its teaching.

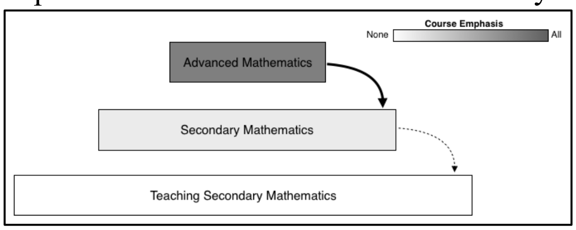

Figure 1a. Implicit model for advanced mathematics courses designed for teachers

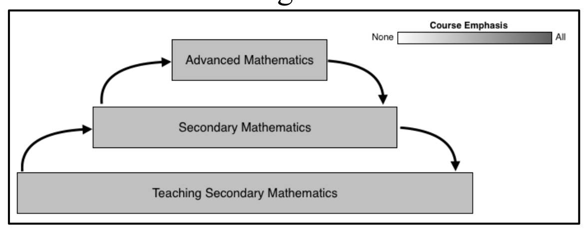

Figure 1b. Our model for advanced mathematics courses designed for teachers

An Example of our Model: Considering Derivative Proofs as "Attending to Scope"

In their professional work, teachers must explain content, practices, and strategies (e.g., TeachingWorks, 2016); we regard a particularly important facet of providing an explanation as being attentive to the scope of that explanation. For example, in trying to help elementary students understand subtraction, some teachers may state: "you cannot subtract a larger number from a smaller one." This explanation has a limited scope—it is only accurate when one is considering positive numbers – and can hinder students' future mathematical learning (e.g., Ball & Bass, 2000). Real analysis, with its rigorous proofs and its attention to the number sets to which a statement applies (e.g., the Intermediate Value Theorem requires the completeness axiom and consequently applies to **R** but not **Q**), is a domain in which attention to scope can be learned. In a real analysis course, students are expected to study and produce rigorous proofs of the common differentiation rules, such as the power rule and the product rule. (See, for instance, the textbooks of Abbott, 2015, and Fitzpatrick, 2006.) In the module that we describe, we use these real analysis proofs to highlight the importance of attending to the scope of a statement, and leverage them to help foster developing this desirable pedagogical practice.

Building up from practice. We began the module by presenting PPTs with the following situation: "Mr. Ryan teaches everything from Pre-Algebra to Calculus. The following scenes are snapshots from his classes at different times during the year." Using cartoons (which are not included for space purposes), two of Mr. Ryan's statements throughout the year were depicted: the *Exponents statement,* "Exponents are just repeated multiplication," and the *Power rule statement,* "If you see a function with an exponent, to take the derivative, you bring down the exponent to the front and subtract one from the exponent" [on the board was written $f(x)=2x^3$]. From this cartoon, PPTs were first asked to "Evaluate the pedagogical quality of each one of these explanations." We note that these two statements are *limited in scope*: exponents can only be viewed as repeated multiplication if the exponent is a positive integer, and the power rule only works for power functions. Next, students were shown a typical proof of the power rule for differentiation, using the binomial expansion of $(x+h)^n$. They were asked to identify for which sets the proof would be valid (**N, Z**, **Q,** or **R**), with the point being that this proof, which was likely familiar and is often presented to calculus students, is only valid for natural numbers.

Real analysis. The real analysis portion of the course consisted of the presentation of proofs of the power rule, the product rule, the quotient rule, the chain rule, and the inverse function rule. A progression of proofs of the power rule (for $f(x)=x^n$, $f'(x)=nx^{n-1}$) was presented that differed depending on the scope (i.e., for *natural*-numbered exponents, for (non-zero) *integer*-numbered exponents, etc.), and the proofs of the product rule, quotient rule, chain rule, and inverse function rule were provided so that they could be used for the power rule proofs.

Stepping down to practice. After the real analysis was presented, PPTs were invited to revisit the classroom scenario presented earlier. They were also asked on their homework to evaluate the pedagogical quality of two other explanations that were limited in scope, which we called the *Perimeter statement* ("The perimeter is just the sum of all the side lengths") and the *Add zero statement* ("Remember, to multiply a number by ten, just add a 0 to the end."). Finally, PPTs were asked write a journal entry in which they reflected on: "What, if anything, did you find helpful for your teaching in this week's class? If there were helpful aspects, specify in what ways they might influence your teaching – if nothing was helpful, explain why."

Methodology

Research Context, Participants, and Data Collection

We designed an experimental real analysis course in which each session was designed using the model presented in Figure 1b and described above. We implemented this course with 32 PPTs, 31 of whom agreed to participate in our research study. In this paper we focus on the Attention to Scope module, which took place across two 100-minute sessions. We collected and analyzed three sources of data: (i) we audio- or video-recorded all students collaboratively working on the module's activities; (ii) we collected their homework responses to the *Perimeter statement* and the *Add zero statement*; and (iii) we collected their reflective journal assignments for what they learned from these modules.

Analysis

We coded each source of data in the following way. For (i), to analyze PPT's in-class activity, we had recordings from five tables (T1, T2, T3, T4, T5), each containing about six students. When the groups were analyzing Mr. Ryan's exponent and power rule explanations, we used an open coding scheme in the style of Strauss and Corbin (1990) to capture the aspects of the classroom scenario to which the PPTs attended. When asked to cite the limitations of the proof of the power rule using the binomial expansion, we recorded each group's answers and the justification for their answers. For (ii), when coding the homework responses, we determine if the PPTs mentioned a limitation in scope for the statement and whether this limitation in scope was mathematically accurate. For (iii), we used an open coding scheme to document the category of responses that were present in PPT's reflective journal entries.

Results

We organize the presentation of results from our analysis in terms of their support for three particular claims: Claim 1) When evaluating the pedagogical quality of a teacher's statement, PPTs increased the attention they gave to the mathematical scope and limitations of a statement; Claim 2) PPTs valued the idea of attending to the scope and language of an explanation for their teaching; and Claim 3) Our model contributed to the goals of the module, particularly via the use pedagogical discussions to motivate the real analysis and vice versa.

To document Claim 1, we argue that PPTs showed limited attention the scope of Mr. Ryan's explanations at the start of the module, but that PPTs showed increased attention to scope on their homework assignment. When considering Mr. Ryan's explanations at the start of the module, we note that all tables were engaged with the task, and all evaluated the pedagogical quality of the exponents statement and the power rule statement negatively and gave reasons for their justification. At each table, a number of pedagogical concerns were raised. These included, for example, concerns about what students might understand 'repeated multiplication' to mean (e.g., "2x3x5x7" (T1)), and whether 'bringing down the exponent to the front' might be unclear (e.g., "It could be 23, not 6" (T3)). However, for some tables, this was the extent of their evaluations – focusing on the '*explanation* of the mathematics' and not the '*mathematical* aspects of the explanation.' Indeed, specifically in consideration of scope, only two tables (T3 and T4) attended to the limitations of scope for both statements, one table (T5) considered the limitation of scope with the exponents statement but not the power rule statement, and the other two tables (T1 and T2) did not attend to scope at all. That is, only 2 of 5 tables were consistent in their attending to the limited scope of both explanations. In contrast, however, on the homework assignment, both for the Perimeter and the Add Zero statements, all 31 PPTs correctly noted the limited scope of these statements. In addition, the quality of their attention to mathematical scope in the HW responses also increased. Even for the two tables that did so initially (T3 and T4), the exploration of the mathematical limitations of the statements was relatively narrow – they did not attempt to exhaust the possible scenarios. Each table only identified *one* instance where the power rule statement was limited (i.e., $\sin^2(x)$) – neither table discussed, for example, the derivative of e^x. In the HW exercises, however, the quality of the PPTs exploration of mathematical limitations was richer and more exhaustive. Both within individual responses as well as collectively across all PPTs, there was greater variety of limitations referenced (i.e., straight/curved, closed/open, exterior/interior lines, single/composite figures, 2D/3D) – indeed some of their discussions went beyond what was initially anticipated. We regard both the increased number of PPTs as well as the improved quality of responses as supporting Claim 1.

We document Claim 2 based on PPT's written reflections on what they learned from this module. In general, a common theme from PPT's written reflections was that they specifically valued the desirable pedagogical idea of attending to the scope and language of an explanation for their teaching – primarily addressing the mathematical precision of their language with students. Of the 27 PPTs who submitted reflections, 25 identified this idea as both: i) specifically stemming from the real analysis module; and ii) valuable for their teaching. We see both of these aspects in the following response that is representative of their reflections: "This lesson made me realize that as a teacher I must pay close attention to what I am saying. When I make statements that have errors, I need to know what loopholes or misconceptions held in my statement and be conscious of these as I create examples and answer questions" (S1). Here, it is worth reminding the reader that the PPTs were not obliged to say that they learned anything useful from the real analysis class (an option that some PPTs chose when reflecting on other modules). Within their statements, a few subthemes about implementation considerations arose: 1) scaffold definitions, beginning simple but getting increasingly rigorous (6 responses); 2) make sure explanations were not just procedural (5 responses); and 3) take into account the teaching context when considering the rigor and accessibility of explanations used (7 responses). In summary, we take this as evidence that PPTs saw pedagogical value in this module.

Lastly, we consider Claim 3 about the contribution of the model toward these aims. Notably, the data supporting this claim is anecdotal; however, we regard reflection on the model as important, and the responses from some students as suggestive about its contribution. First, we explore the possible value that the pedagogical situation may have added to the real analysis. Within PPTs' evaluation of Mr. Ryan's statements initially, pedagogical quality hinged somewhat on the mathematical scope and limitations. Thus, when transitioning to the real analysis proofs, PPTs appear to have given additional gravitas to considering mathematical limitations because of the related professional value, and their sense of the proofs may have been similarly tied to these limitations. As one instance, upon realization that the first power rule proof was limited to **N**, one student concluded: "This proof takes anything that I've ever believed in... Like here's a proof. Not anymore! Like this is the proof that I've always taught. And now, I'm like, everything about this is wrong" (T3). That is, the negative pedagogical evaluations appear to have prompted further mathematical motivation. Second, we explore the potential value that the real analysis may have added to accomplishing the desired pedagogical aims. Notably, the sequence of real analysis content explicitly modeled this attention to scope. And although one might do this without real analysis, at least some PPTs made this link and reflected on the value: "Seeing the connection between the analysis content and two very different concepts taught in high school was particularly useful.... the progression we took in the proofs from each set of numbers was a very elegant way of showing the different methods of proof, showing the flaws within each…" (S6). We see these – and other – comments as supporting the idea that the interaction between pedagogical discussion and real analysis was mutually beneficial to developing both.

Discussion and Conclusion

The analysis and reporting in this paper of one module from an experimental real analysis course – a single case study – sought to explore the broad issue of how advanced mathematics courses can be designed to inform PPTs pedagogical practice. In particular, the data from the study support the claim that, after engaging in the 'Attending to Scope' module from the real analysis course, the PPTs both increased their attention to (Claim 1) and valued (Claim 2) the desirable pedagogical practice of attending to the mathematical scope and limitations in teachers' explanations. By design, the real analysis content was both tightly connected to and framed by this pedagogical practice; however, as was evident from some of the tables of PPTs (i.e., T3, T4), one does not have to learn real analysis to be able to attend to the scope and limitations of secondary mathematics explanations. However, since a real analysis course already sort of inherently models this idea in both the precision of statements and progression of proofs, it seems sensible to exploit this connection for teachers. Indeed, the teachers in this study, overall, increased their attention to and valued this pedagogical practice. The anecdotal evidence for Claim 3 also seems to support at least one of the ways in which the model may have facilitated these goals. Thus, we see this as evidence that by framing real analysis content with pedagogical situations, in addition to learning real analysis, PPTs can also learn important teaching ideas. Further work studying how best to mathematically prepare secondary teachers is needed, including the degree to which this particular model is productive and/or needs refinement, and could help guide improved design and implementation of advanced mathematics courses for secondary teachers.

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Characterizing Mathematical Digital Literacy: A Preliminary Investigation

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This preliminary report offers initial results from a study designed to begin identifying characteristics of digital literacy in mathematics. Undergraduate students in a three-course honors calculus sequence were provided with tablet computers as part of a digital literacy initiative and digital tasks were integrated into the courses. Student work was analyzed and coded for type of ICT tool use and possible components of mathematical digital literacy. The specific types of tasks developed for and integrated into the class will be discussed below with specific illustrative examples highlighted. The aspects of mathematical digital literacy illuminated by student work will be outlined, with some initial conclusions and conjectures about the nature of digital literacy in mathematics.

Keywords: Digital Literacy, Technology, Calculus, Preliminary Report

Introduction and Background

The ever-increasing role of technology in everyday life and work prompts questions about the skills and understandings needed for effective use of that technology. The range of information and communication technology (ICT) tools grows ever greater, and the ability to obtain, manage, synthesize, analyze, and communicate information is constantly changing and adapting. As technological capabilities rapidly change, the accompanying skills and understandings necessarily shift in response. Competence and knowledge with technological tools is described by and named with a variety of terms, the most prevalent of which is *digital literacy*, a term first defined by Gilster (1997) as "the ability to understand and use information in multiple formats from a wide range of sources when it is presented via computers" (p. 1). It is now frequently used as an umbrella term with a variety of implications, though there is general agreement that digital literacy involves interaction and integration of a number of proficiencies, such as procedural competence with ICT tools, cognitive skills for using them effectively, and social and communication skills (Avriam & Eshet-Alkalai, 2006; Goodfellow, 2011). The use of the word "digital" is itself far from universal, with some sources variously referring to media literacy, digital and media literacy, ICT literacy, or related specialized terms. This paper will use the term "digital literacy" to encompass the wide variety of terms used in order to draw on the valuable contributions of multiple approaches.

Educational Testing Service (2003) characterized seven proficiencies that characterize general digital literacy: Define, Access, Manage, Integrate, Evaluate, Create, and Communicate. Specific applications of the term might include or alter those proficiencies within the context of a particular subject. In education, digital literacy has been increasingly emphasized in general (Gutierrez & Tyner, 2012) and in mathematics specifically (NCTM, 2000, National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), and has been shown to have a positive impact on student learning (Li & Ma, 2010). ICT tools are also increasingly integrated into the work of research mathematics (Monroe, 2014).

Despite the increased emphasis on and integration of ICT tools within mathematics, *mathematical digital literacy* is not well-defined. The competencies with ICT tools specific to mathematics would be of particular concern to educators, curriculum developers, and many other stakeholders within the field. This presentation describes an investigation into digital literacy among undergraduate students in an honors calculus sequence. By assessing how students engaged with digital tools that were often new and unfamiliar in order to solve mathematical problems and understand mathematical concepts, preliminary characteristics of mathematical digital literacy emerge.

Context and Methodology

Setting and Data Collection

Undergraduates in a three-course honors calculus sequence were provided with tablet computers as part of a digital literacy initiative at their university. These courses (Honors Calculus I - 22 students; Honors Calculus II - 22 students; Honors Calculus III - 18 students) covered the traditional material of the calculus sequences in a "late transcendentals" ordering. In the past, the mathematics program had not emphasized the use of digital tools, so integrating them into the work of the course provided an opportunity to observe emergent digital literacy in mathematics and investigate an initial characterization. The primary digital tools introduced to the students by the instructor were Wolfram Mathematica and the online Desmos graphing calculator.

An initial assignment allowed students to use any tools they might choose and consisted of problems for which digital ICT might be useful, but which focused on concepts already familiar to students. For example, finding the zeros of a sixth degree polynomial or determining the domain of a ratio of logarithmic functions. This served as an initial assessment of how students chose to use such tools. Throughout the semester, two types of digital tasks were used to assess student interaction with and use of ICT tools - digital assignments and digital exams. These were supplementary to the traditional written course content.

Digital assignments were primarily meant to provide students with base-line experience using digital tools to solve mathematical problems and were typically assigned as handouts or pdf files related to the content that had recently been discussed in lecture. A set of instructions led students through the use of Mathematica or Desmos (depending on the content) to visualize and solve a set of problems. Often the instructions would require students to choose parameters to create their own individualized problems. For problems that required more advanced coding, students would be provided with a template file to edit. During in-class digital assignments, the instructor would typically provide demonstrations and move around the classroom to help students with syntax and interpretation. Students submitted their work digitally as either a Mathematica Notebook file or a link to a Desmos graph. Though an indirect consequence of using these ICT tools may have been an increase in student understanding of content, the primary focus of digital assignments was on gaining literacy with digital tools and accessing new problems and information via their use.

Digital exams were completed in the class period following a written exam. The problems on the digital exam often required students to create digitally generated images/animations and to make computations that could not be completed by hand in a reasonable amount of time. The exams were "open resource" - students were allowed to use any digital resource at their disposal except for online help forums, including those not discussed by the instructor. This permitted analysis of the ways in which they chose to use digital tools. The TPACK framework (Mishra & Koehler, 2006) and Howland, Jonassen, and Marra's (2012) five dimensions of learning involving ICT tools served as general guides for designing the integration of digital tools into assignments and exams in each course.

Student work was collected for each digital assignment and digital exam. On the initial assignment and digital exams, students were asked to identify what digital tools they used. At the end of the course, a survey about their attitudes toward and use of digital tools gave additional information. The first digital assignment and the digital exams permitted students the most freedom in selecting and utilizing digital tools and were therefore the first to be analyzed for characteristics of digital literacy. Open coding (Strauss & Corbin, 1990) was used to develop coding schemes that described student use of and interaction with the digital tools.

Examples of Digital Tasks

An example of a digital assignment. Early in the Calculus I course, students were required to complete a digital "Desmos" assignment asking them to explore limits involving trigonometric functions. The first portion of the assignment asked students to consider the limit $\lim_{x\to 0}\frac{sin(ax)}{bx}$ using the function as graphed in an existing Desmos file. Students used the Desmos "sliders" to evaluate the limit for various values of *a* and *b*. Eventually, students choose their own unique values to verify the pattern. The assignment included a similar exploration for 3 other common trigonometric limits: $\lim_{x\to 0}\frac{ax}{sin(bx)}$, $\lim_{x\to 0} ax\, cot(x)$, and $\lim_{x\to 0}\frac{1-cos(ax)}{bx}$.

Apart from the digital assignment, students were provided with a rigorous proof of the first computation. As with many digital assignments, this one provided exploration, experience, and visualizations that would later support formal computations, theorems, and proofs. During the assignment, the instructor reminded students that the use of Desmos itself as a tool was only the secondary purpose of the assignment. The primary purpose of this assignment was to encourage students to begin using technology when presented with an apparently intractable problem.

An example of a digital exam. Figure 1 shows a Calculus II Digital Exam problem on Taylor series that is impractical to solve by hand. To illustrate the ways in which students used digital tools to solve problems and communicate their solutions, samples of student work on this problem are included in Figure 2 below and discussed later.

3. (5 pts) $g(x) = \int \sin(x^2)\,dx$ is not an elementary function but still has a convergent Taylor series. Assume $g(0) = 0$. Find the 15th Taylor polynomial centered at $x = 0$ of this funciton. Use this polynomial to approximate $g(3/2)$ (even though we don't have a formula for g to "plug in").

Figure 1: A Calculus II Digital Exam Problem

The nature of the problems in both types of tasks was varied in order to expose students to different ways ICT tools might be useful and to highlight the various ways students chose to use them.

Preliminary Results

The full analysis of student work provided rich data on use of ICT tools. Below, a very brief overview of the initial results is given. One specific example is used to illustrate how the digital tasks highlighted the variety of uses of digital tools and how the data led to this preliminary characterization of mathematical digital literacy.

Notable Results from Surveys

Thirty-five students responded to the post-course survey. When asked to rate their comfort level with digital tools at the beginning and end of the semester on a scale of 1 (not comfortable at all) to 10 (very comfortable), every student reported the same or greater levels of confidence. The mean change in self-reported comfort level was 1.6 with a median change of 1. All students reported using some technology outside of the digital assignments and course requirements.

When students were asked to describe how they used digital tools in the class, the most common response was for visualization. In particular, students noted the value of Desmos for graphing equations and of Mathematica for graphing three-dimensional solids. They also valued the ability to quickly perform calculations and to check answers, though many noted that learning the syntax for Mathematica was difficult, at least initially.

An Example of Results From Student Work

Analysis of student work on the digital tasks illustrated the different ways in which students engaged with digital tools. For one instance, two examples of student work on the problem from Figure 1 are shown in Figure 2:

Student 1 submission of #3	Student 2 submission of #3

In[1]:= `(* Problem 3 *)`
`p15[x_] = Integrate[Normal[Series[Sin[x^2], {x, 0, 15}]], x]`
`N[p15[3/2]]`
Out[1]= $\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \frac{x^{15}}{75\,600}$
Out[2]= 0.777928
	`(*3*)`
	`Clear[f, a];`
	`f[x_] := Sin[x^2]`
	`a = 0; (*center*)`
	`Integrate[Normal[Series[f[x], {x, a, 15}]] , x]`
	`p8[x_] = f[a] + Integrate[Normal[Series[f[x], {x, a, 15}]] , x]`
	`(*we must add the constant term which is f(center)*)`
	`N[p8[3/2]] (*approximation for g(1)*)`
	$\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \frac{x^{15}}{75\,600}$
	$\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \frac{x^{15}}{75\,600}$
	0.777928

Figure 2: Samples of student work on Calculus II Digital Exam problem #3

Both students chose Mathematica for this particular problem. This is not surprising given the nature of the problem and the tools with which most students were comfortable. However, their processes differ. Student 1 submitted a concise and correct solution. Student 2 also submitted a correct solution but copied previously used code provided by the instructor to find the 8th Taylor polynomial. Note that Student 2 did not bother to change p8 to p15 even though the problem is to find the 15th Taylor polynomial. A comparison suggests that Student 1's solution exhibited greater digital literacy since they were comfortable enough with the content and syntax to simplify their code whereas Student 2 attempted to mimic a previous application of digital tools.

Toward an Understanding of Mathematical Digital Literacy

Students tended to use digital tools in the following major ways:

1. *Determine which tool should be used to solve a given problem*.
2. *Learn and apply syntax of technological tool (sometimes based on template)*.
3. *Decide how to translate mathematics into input in chosen tool*.
4. *Interpret technological results to find a proposed solution.*
5. *Use technology to justify that a proposed solution is correct*.
6. *Display and submit answer and supporting work digitally.*

Though there was much variation in the particular ways students engaged in these activities with ICT tools, they fell into these six main categories of use. Such a categorization permits some initial conjectures about components of mathematical digital literacy:

Component 1: Ability to assess and choose tools based on potential use along multiple proficiencies

Component 2: Translation between digital and mathematical contexts, including multiple representations (notational, graphical, syntactical) and digital and mathematical troubleshooting

Component 3: Using ICT tools to enhance or complement (rather than replace) mathematical understanding

Component 4: Using ICT tools to communicate mathematics

These components are related to the seven proficiencies with ICT tools described by ETS (2003), but are specific to mathematics. A more nuanced and detailed analysis is underway and will be described in greater detail in the proposed presentation.

Conclusion

Work remains to be done to fully characterize digital literacy for mathematics. However, this preliminary study supports the idea that a focus on learning and doing mathematics within digital environments increases student facility and comfort with ICT tools. The ways students utilized digital tools provides some initial indications of important components of digital literacy.

The Proposed Presentation

The proposed preliminary report would include the information summarized in this proposal in addition to more specific examples of student work and more careful and nuanced descriptions of components of digital literacy. As a preliminary report, the authors hope to use this as an opportunity for feedback from experienced and engaged mathematics educators to shape future research and analysis on this subject. In addition to welcoming critical assessment and feedback of this preliminary research, the authors propose the following questions to be considered by the audience:

1. How do we, as a research community, move toward a fuller understanding and description of what digital literacy means in mathematics? What research designs might be useful or beneficial?
2. How does such an understanding remain responsive to changes in availability and capability of digital tools?

3. How might we begin to understand the relationship between mathematical digital literacy, mathematical conceptual understanding, and proficiency with mathematical practices?

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Blended Processing: Mathematics in Chemical Kinetics

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This work investigates the following research question: How do non-major students understand and use mathematics to solve chemical kinetics problems involving integrated rate laws? Personal constructs, a blend of personal and social constructivism, serves as the theoretical framework for this study. Semi-structured interviews with 36 general chemistry students, 5 upper-level physical chemistry students, and 3 chemical engineering students were conducted using a think-aloud protocol. Audio and written data were collected using a Livescribe pen. The audio data were transcribed, and screenshots of students' written data were inserted into the transcripts; these transcripts were refashioned into problem-solving maps. Open coding of the problem-solving maps reveals initial themes regarding students' understanding and use of mathematics when solving chemical kinetics problems. Blended processing was used as a methodological framework to guide the coding process. Through this analysis, distinctive types of blended processing have emerged.

Key words: Rates, Problem Solving, Blended Processing, Chemistry, Kinetics

Understanding fundamental chemistry concepts is intrinsically tied to understanding mathematical symbolism and operations, as well as the ability to translate between equations and physical reality. This nature has led researchers, such as Becker and Towns (2012), to investigate students' understanding and use of mathematics in scientific contexts. These lines of inquiry are providing insight to researchers and practitioners on how to best enhance students' abilities to interpret and use mathematical expressions with conceptual reasoning.

Multiple studies have shown that students need a rich conceptual understanding of algebra and calculus in order to succeed in the physical chemistry classroom (Derrick and Derrick, 2002; Hahn and Polik 2004; Nicoll and Francisco, 2001). Thompson and colleagues showed that students struggled when writing mathematical expressions to describe a physical process (Bucy, Thompson, and Mountcastle, 2007; Thompson, Bucy, & Mountcastle, 2006). Conversely, students also demonstrated difficulty interpreting physical meaning from mathematical equations. In another study, students seemed to hold an isolated understanding of topics in physics and mathematics (Pollock, Thompson, and Mountcastle, 2007). This was studied further by Wemyss, Bajracharya, and Thompson (2011) who showed that when students are given analogous questions in the context of mathematics and physics, they perform better on the former. This finding is in line with other studies on students' understanding of mathematics in the context of science (Bassok & Holyoak, 1989; Beichner, 1994; Black & Wittmann, 2007; Christensen & Thompson, 2012; Cui, Rebello, & Bennett, 2005, 2007; Cui, Rebello, Fletcher, & Bennett, 2006; Orton, 1983a, 1983b; Shaffer & McDermott, 2005; Zandieh, 2000).

Problem solving is also a key component in the learning and practicing of chemistry (Bodner & Herron, 2002; Summerfield, Overton, & Belt, 2003). As Maloney (2011) found, there are many problem-solving strategies or steps required to solve quantitative problems. In quantitative problem solving, there is often an initial qualitative analysis to understand what the problem is asking (Reif, 1983). This conceptual reasoning step is not only important and beneficial to students, but also demonstrates problem-solving expertise (Hull, Kuo, Gupta, & Elby, 2013; Kuo, Hull, Gupta, & Elby, 2013; Reif, 1983). Such research has led to the development of problem-solving strategies to help students mirror expert-like behaviors (Heller, Keith, & Anderson, 1992; Huffmann, 1997; Reif, 2008; Van Heuvelen, 1991). However, examining how students use and understand the equations when problem solving in science contexts has rarely been studied.

Chemical kinetics was chosen as a rich context to study quantitative problem solving among chemistry students because of its highly quantitative nature. Furthermore, this work contributes to two key gaps in the literature. In their recent review, Bain and Towns (2016) revealed that research in the area of chemical kinetics at the undergraduate level are rare, unlike other chemistry content areas. Additionally, the National Research Council reported that research on upper-level students and courses were scarce in discipline-based education research (DBER) (Singer, Nielsen, & Schweingruber, 2012). This study targets these literature gaps by investigating how both introductory- and upper-level students use and understand mathematics in the context of chemical kinetics problems. Therefore, the following question serves as the guiding research question for this work: How do non-major students in a second-semester general chemistry course, a physical chemistry course, and a chemical engineering course understand and use mathematics to solve chemical kinetics problems involving integrated rate laws?

Theoretical Underpinnings

The theoretical framework that guides this study is Kelly's (1955) theory of personal constructs. This theory is a combination of personal and social constructivism that posits that while individual's knowledge constructions may differ, they can be similar to one another because of social interaction. The theory of personal constructs provides an appropriate theoretical lens to investigate how individuals understand and use mathematics to solve kinetics problems in that each participant has their own individually constructed understanding of the content.

Blended processing serves as the methodological framework for this study. This framework builds on the basic tenant of constructivism, knowledge as being constructed in the mind of the learner, by describing how an individual's different knowledge constructions interact, or "blend" (Bodner, 1986; Bodner, Klobuchar, & Gleelan, 2001). Blended processing is a framework stemming from the field of cognitive science that explores human information integration (Coulson & Oakley, 2000). It provides a way to describe and understand individuals' mental spaces (knowledge constructions) and their interactions (Bing & Redish, 2007; Hu & Rebello, 2013). When multiple mental spaces are activated by external stimulus, knowledge elements from each space interact and are organized in a "blended space", allowing an individual to make sense of cognitive input in an emergent fashion (Bing & Redish, 2007; Coulson & Oakley, 2000; Fauconnier & Turner, 1996, 1998, 2002; Hu & Rebello, 2013).

Methods

The methods outlined below were chosen as they are consistent with the theoretical underpinnings guiding this work and allow for the investigation of student problem solving in chemical kinetics.

Data Collection

Semi-structured individual interviews were selected as the primary mode of data collection. A stratified purposeful sampling technique was employed, providing a participant sample of second-semester general chemistry students and upper-level physical chemistry students (Patton, 2002). The upper-level students were sampled from two courses: a physical chemistry for biological and life science majors course and a chemical reactions engineering course. Non-major science, technology, engineering, and mathematics (STEM) students served as the target study sample because they are a larger population both at the introductory- and upper-level. Additionally, because the ability to integrate different knowledge domains when solving a problem demonstrates expert-like reasoning, this exploratory study aimed to explore how students in interdisciplinary fields (like engineering) solve problems.

A pilot study comprising of four second-semester general chemistry students was conducted initially to test the viability of the interview protocol (Table 1). Audio and written data were collected via a Livescribe pen. The pilot interviews were conducted by a team of two graduate-student researchers in order to develop a shared understanding of the interview environment, prompts, and probing style. After preliminary analysis and discussion by the research team, the full study interviews were conducted independently by one the two graduate-student researchers. This data collection occurred over two semesters, yielding 36 introductory-level and 8 upper-level interviews (Table 1).

Table 1

Interviews by population and semester

Study population	**Number of students interviewed**	
	Fall 2015	**Spring 2016**
Pilot study (second-semester general chemistry)	4	-
Second-semester general chemistry (non-chemistry STEM majors)	17	19
Physical chemistry for biological and life science majors	-	5
Chemical reaction engineering (chemical engineering majors)	-	3

The interview protocol consists of four prompts. Two prompts provide the participants with an integrated rate law equation and ask them to describe it. The other two prompts are chemical kinetics questions that are reminiscent of homework- or exam-style questions. These prompts provide students with data and information about a chemical reaction scenario, asking them to reason about a relevant chemical kinetics quantity.

Data Analysis

The data were transcribed verbatim. Participants' written work were inserted into each transcript where appropriate. In order to make the data more manageable for analysis, problem-solving (PS) maps were generated for each interview. Discrete problem-solving steps in student responses were identified. These steps and the corresponding transcript and written data were organized into tables chronologically. Table 2 includes an excerpt from one of the pilot study PS maps.

Table 2

Excerpt from Trip's PS map (he was prompted to explain the second-order integrated rate law)

Student's problem solving	**PS Steps**
"Okay, so, this is the second-order integrated rate law."	Recognizes equation
"And so, it starts with, you have your rate equation for a second order reaction. And then you, if you integrate both sides with respect to time. dt. Then you end up, and then you rearrange it and you get this." $\int rate\,dt = \int k[A]^2\,dt$	Recognizes origin of equation
"So, basically, the purpose of this is so you can have a function of concentration versus time. Instead of just concentration versus rate. That way it's easier to use in like the lab."	Highlights purpose of equation

Keeping the methodological framework of blended processing in mind, multiple rounds of open coding of the PS maps were conducted (Patton, 2002). Often times codes were assigned to excerpts of data as they were organized as steps in the map, meaning a single problem-solving step received one code. Other times multiple codes were assigned to the data in a single step; further, there were instances in which multiple, consecutive steps received a single code. This coding process is ongoing, where the code book is continually being refined via constant comparison methodology (Patton, 2002).

Preliminary Results

Preliminary analysis shows variation in how students integrate knowledge domains to make sense of concepts to solve kinetics problems. A non-blended understanding has the potential to limit students' problem-solving abilities, while a blended understanding serves to support more productive problem solving. Our data suggests that blended processing is not a binary phenomenon; rather, it is dynamic spectrum. For example, when discussing the purpose of a rate law, a general chemistry participant, Hazel, argued that the equation is useful for solving for a third variable whenever you are given two of the values. This type of understanding could limit a student's ability to understand and make sense a more authentic problem. In contrast, more blended understandings of concepts, such as those demonstrated by other general chemistry participants, Trip and Damien, where the rate law is interpreted in terms of relationships and changes between variables, are likely to be more supportive of successful problem solving.

Preliminary analysis has also revealed distinctive types of blending processing in students' problem solving: mathematics blending, chemistry to mathematics, and mathematics to chemistry. Mathematics blending describes student blending of conceptual and formal mathematical reasoning. The second type of blending describes when students take chemistry concepts/information and translate that to mathematical concepts/symbolism. Alternatively, there is also evidence of mathematics to chemistry blending. This is the most common type of blending demonstrated by the pilot study interview participants. While we have only done preliminary analysis on the general-chemistry-level interviews, we do see similar types of reasoning with upper-level physical chemistry students.

We have also noted a variation in problem solving approaches, varying from very simple approaches utilizing one method with little reasoning to more complex approaches trying multiple methods, often with conceptual predictions and justifications. The latter represents a more sophisticated approach to problem solving, as it draws on multiple types of understanding and explicitly incorporates justification for problem solving steps. We plan to explore the relation of 1) success in problem solving and 2) complexity of problem-solving approaches to blended processing.

Conclusions and Questions

While our preliminary results suggest that there is evidence of blending among our student participants, this evidence is sparse and irregular. Chemistry faculty members want mathematics to be connected to (blended with) the chemistry represented in these problems. Ultimately, meaningfully blending and integrating mathematical understandings to science and engineering concepts and problem solving will help students develop a deeper understanding of STEM disciplines. Therefore, practitioners must explicitly model the cognitive practices of blended processing. Furthermore, they should provide students the time and space to practice blending and assess blended processing on course assessments.

The following questions outline potential avenues for investigation.

1. How is student problem-solving success related to blended processing?
2. How is student problem-solving sophistication/complexity related to blended processing?
3. What is the nature of chemical kinetics assessment in the participants' courses?
4. How do we foster this blending across disciplines in our classrooms?

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Raising reasoning through revision: A case study of an inquiry-based college geometry course

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Geometry is the subject where U.S. students are weakest on international assessments, but college geometry is an area of proof that is understudied. Since geometry is secondary students' only exposure to proof, it is vital our secondary teachers can prove effectively in this content area. The purpose of this case study, drawn from a larger project, was to understand how, if at all, pre-service teachers' proof schemes became more axiomatic throughout a one-semester inquiry-based college geometry course. Participants in this study, Kayla and Lindsey, were pre-service teachers enrolled in an inquiry based college geometry course. Although Kayla had two prior proof courses and Lindsey had none, both participants were using a perceptual proof scheme at the beginning of the semester. However, by the end of the semester, the chance to revise their proofs and discuss problems with their peers helped both students advance to more axiomatic geometric thinking.

Key words: College geometry, inquiry based learning, proof

Compared to other nations, the students of the United States of America are floundering in geometry. The Trends in International Mathematics and Science Study (TIMSS) evinced that twenty-one educational systems, including China, Japan, Israel, and England, have higher geometry scores than the U.S. (Mullis, Martin, Foy, & Arora, 2012). This deficiency is because pre- and in-service teachers possess an inadequate understanding of the structure of geometry. Despite this difficulty for students and educators, college geometry remains severely under-researched (Speer & Kung, 2016).

Geometry arises from a set of undefined terms and axioms through which all other theorems and definitions are constructed. Hence, a thorough understanding of geometry involves a deep understanding of proof; yet, teachers possess a narrow understanding of proof. Studies indicate that pre- and in-service teachers believe proof only helps explain ideas used in mathematical concepts, and they do not recognize the ability of proof to systemize results (Mingus & Grassl, 1999; Knuth, 2002b). Teachers lack the geometry content knowledge required for geometry proofs, and they are convinced by empirical evidence as well (Jones, 1997; Knuth 2002a). Consequently, teachers with inadequate proof and geometry understanding cannot be expected to impart adequate proof and geometry knowledge to students.

Pre-service teachers in undergraduate proof courses do not thoroughly understand what arguments qualify as proof (Weber, 2001). They lack comprehension of the mathematical language and concepts necessary to proof (Selden, 2012), and they possess an incomplete understanding of definitions and theorems (Weber, 2001; Selden & Selden, 2008). In typical direct-instruction, lecture proof courses, students are expected to develop proficient proof skills with little guidance. Without recurrent feedback, students will likely cultivate ineffective strategies (Weber, 2001). These ineffective strategies are typically proof schemes dependent upon external and empirical convictions, such as the authoritarian, ritual, and perceptual proof schemes (Harel & Sowder, 2007). In order to successfully write a proof, students need to employ effective strategies or proof schemes with arguments based on axioms and logical deductions, such as the intuitive and structural axiomatic proof schemes.

A typical proof learning environment is dominated by the instructor with little student participation – an instructor-centered environment (Selden, 2012; Padraig & McLoughlin, 2009). This learning environment affects teachers' conceptions of proof, conceptions already determined to be limited (Mingus & Grassl, 1999; Knuth, 2002b). The typical instructor-centered learning environment, where students copy instructors' proofs, does not induce the logic and proof techniques needed to construct a proof in all students (Harel & Sowder, 1998). Alternatively, a proof course should consist primarily of student-student and student-teacher interactions (Selden & Selden, 2008). One potential approach is an inquiry-based learning pedagogy where students are active learners, and the instructor is responsible for facilitating students' learning (Padraig & McLoughlin, 2009).

The purpose of this study was to evaluate the effects of an inquiry-based learning environment on students' development of proof comprehension throughout an undergraduate geometry course. Through this study, we can determine whether or not this type of instructional environment is a potential solution to pre-service teachers' shallow understanding of proof and geometry. Mathematicians and teacher educators can then make more informed decisions on how to structure college geometry courses. We argue that the revisions present in an inquiry based classroom were vital to help students develop from a perceptual to a more axiomatic proof scheme.

Methods

The theoretical perspective used in this project was the reduced Toulmin model of argumentation. In mathematics education literature, there are two formats in which the model appears. I will distinguish them as the reduced Toulmin model (Figure 1) and the extended Toulmin model. The reduced model consists of three types of statements that represent different pieces of the argument, and the extended model consists of six statements. The three statements in the reduced Toulmin model are as follows: The data (D) is the foundation on which the argument is based. The conclusion (C) is the statement the arguer intends to convince. The warrant (W) justifies the relationship between the data and the conclusion. When students are asked to prove a claim, the conclusion is correct because it is given to the students. However, the data is typically a mixture of right and wrong. This amalgamation occurs because of the warrant the student applies – their reasoning from a piece of information to the conclusion. Although a warrant is specific to an argument, a warrant-type is a category of warrants with similar properties.

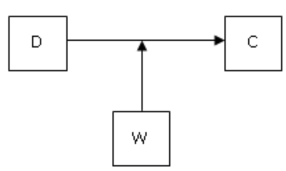


Figure 1. Reduced Toulmin model. Adopted from (Inglis, Meija-Ramos, & Simpson, 2007).

This study took place at a midsized, rural, research university in the South, and the students who participated were those enrolled in a college geometry course based upon Miller (2010) geometry course notes. The data collected was part of a larger study; this study is a case study of two students – Kayla and Lindsey. Both Kayla and Lindsey are Caucasian females whose majors were math education. Kayla was classified as a senior and had two prior proof courses, and Lindsey was classified as a freshman with no prior proof experience. Students in the course were provided with course notes that presented open-ended problems related to a specific learning goal. For each new assignment, students were assigned a specific problem from the provided course notes and a group. If a group appeared to be making little progress or moving in an unproductive direction, the teacher would use guided questioning to redirect students' thoughts. If multiple groups stopped progressing, the teacher would initiate a whole class discussion.

To determine students' proof comprehension, researchers examined the assignments students turned in. Students were allowed to revise and resubmit all assignments, and these were analyzed as well. Researchers also used observations to gain further understanding of students' proof comprehension. As students discussed their ideas, a researcher sat behind them listening and taking notes on their interactions.

The submissions were analyzed by assignment, and all the drafts from an individual participant were analyzed at the same time. After this initial reading of blinded assignments, researchers would journal their impressions of the coding and the trajectory exhibited in the multiple submissions. These journals were used to operationalize the proof schemes in Harel & Sowder (1998) and Harel (2007), and to construct the standards of evidence (Table 1).

Table 1

Standards of evidence

Proof Scheme Warrant	Identifiers
Authoritarian (1)	Argument produced after intentional scaffolding or direct instruction from the teacher/student did not participate in the group's reasoning, and/or writing process/The correct pieces of reasoning were discussed in class, and the argument does not arise from the students' data.
Ritual (2)	Student misapplies multiple axioms and theorems/Argument mirrors the format of a known correct argument, but the argument does not arise from the axiomatic system/Student restates the axiom or theorem as it is originally written despite the fact that the terminology does not relate to the context of the proof.
Perceptual (3)	Student refers to or provides only a diagram as justification for reasoning/The argument is driven by students' perceptual observation of the figure he or she drew and not by the implications of axioms and theorems.
Low Analytical (4)	The argument follows the correct deductive process, but the student does not establish the definitions, theorems, or axioms the process utilizes/The argument follows the correct deductive process, but there is one instance in which the student relies on another lower warrant/The argument follows the correct deductive process, but at least one clarification statement is necessary for validity/The argument follows the correct deductive process, but an axiom or theroem is misinterpreted and misapplied.
Intuitive Axiomatic (5)	The argument is built from undefined terms and axioms, definitions, and theorems of an axiomatic system that is intuitively grasped such as Euclidean geometry.
Structural Axiomatic (6)	The argument is logical and made up of systematic application of axioms and theorems/If any portion of the argument could be clarified, the clarification is not necessary for the argument's validity.

Findings

After analyzing the data, we mapped Kayla's and Lindsey's warrants throughout the course. Figure 2 presents a chronological summary of their proof scheme warrants. On all assignments, except Midterm Problem 3biii, students worked on the solutions with a group during class.

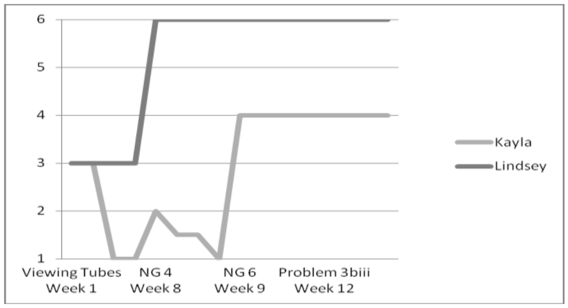


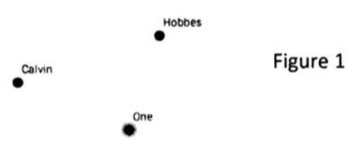
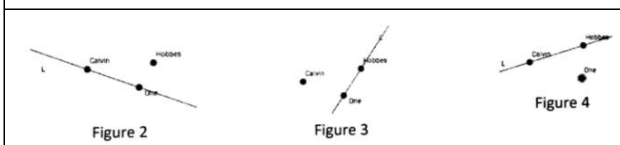
Figure 2. Proof schemes used throughout the semester

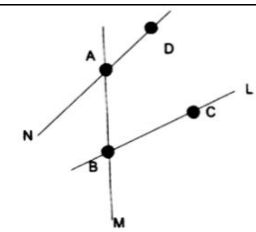
The Viewing Tubes assignment asked students to determine the formula to find the area of the wall that can be seen through a tube. To accomplish this, students needed to establish that the two triangles resulting from the situation are similar. Kayla and Lindsey both started this assignment utilizing the perceptual proof scheme. In their claim that the triangles are similar, the students produced and were convinced by a diagram. As they received feedback on their arguments, Lindsey maintained the perceptual proof scheme and Kayla's reasoning regressed to the authoritarian proof scheme.

Neutral Geometry Worksheet 4 (NG 4) required students to prove each point belongs to at least two different lines. Kayla's warrant on her first draft of NG 4 is the ritual proof scheme. This first draft mirrors another worksheet utilizing the Neutral Geometry (NG) axioms, and Kayla successfully completed this previous worksheet. Kayla's revisions to this assignment come from statements directly made by the teacher or a class discussion. Hence, as her drafts of NG 4 progress, Kayla's reasoning becomes solely authoritarian. Lindsey's exploration of the NG Axiom System, however, allowed her to garner a better understanding of how axioms develop and argument. The warrant on Lindsey's first NG 4 draft is then the structural axiomatic proof scheme because she applies the axioms to form a valid deductive argument.

Table 2

Kayla & Lindsey's initial NG proofs

Kayla NG 4 Draft 1	**Lindsey NG 4 Draft 1**
Let there exist 3 different points, (Calvin, Hobbes, One) (Figure 1).	Assumption: Let there be a point, A.
	By Axiom 3, there exists at least one line, L. CASE 1: If the point A lies on Line L, then at least point is not on L, point B, by Axiom 2.
If there exits at least one line (Calvin, One), (Hobbes, One), or (Calvin, Hobbes) by Axiom 3, there must also exist a pair of points (Calvin, One), (Hobbes, One), (Calvin, Hobbes) that are connected by L. (Figure, 2, 3, 4)(Axiom 1)	Since points A and B are two different points, then by Axiom 4, another line, Line M contains points A and B. Because B does not lie on L, line L and line M are two different lines. Therefore, point A belongs to two different lines, Line L and Line M.
	CASE 2: If point A does not line on Line L, then by Axiom 1, at least two points, points B and C belong to Line L. By Axiom 4, since A and B are two different points, there exists a line containing both A and B, line M.
By Axiom 2, there must also exist a point that does not belong to L. In Figure 2, (Calvin, One) belong to L while Hobbes does not exist on L. This similarly applies to Figure 3 and Figure 4 when L consists of (Calvin, Hobbes) or	By Axiom 2, there exists another point not on Line M, point D. Since Point A and Point D are two different points, then by Axiom 4, there exists a line

<table>
<tr><td>(One, Hobbes). Given Figure 4, Calvin also exists on a line consisting of (Calvin, Hobbes). Thus Calvin exists on 2 lines (Calvin, One) (Figure 2) and (Calvin, Hobbes) (Figure 4).</td><td>containing points A and D.</td></tr>
<tr><td rowspan="2">Figure 1, L containing (Calvin, One), there must exist a Hobbes not on L (Axiom 2). For all L containing two points there will be a third point not on L. By Axiom 4, any two points can be connected by a line. For the three points, Calvin, Hobbes, One, therefore there can be at least three lines (L, M, and N) (Figure 4)</td><td>
</td></tr>
<tr><td>Because points A and D are two different points, and point D does not belong to line M or Line L, then Line M, N, and L are different lines. Therefore, point A belongs to 2 different lines. Line N and Line M.</td></tr>
</table>

Neutral Geometry Worksheet Problem 3 (NG 6) assignment asked students to prove there exists a line not containing a given point on the NG Axiom System. Kayla, for the first time, exhibits analytical reasoning, and her warrant for her drafts is the low analytical proof scheme. Her argument follows a correct deductive process, but she does not utilize the axioms to establish truths. Also, she partially relies on the ritual proof scheme because she continues to use the complex notation from she mirrored in NG 4. Lindsey' warrant is the structural axiomatic proof scheme be she produces a valid deductive argument.

The last assignment analyzed in this study was Midterm Problem 3biii (Problem 3biii), and students were to prove, using the NG axiom system, that for any given point there exist two other distinct points such that the collection of points is non-collinear. Kayla maintains the reasoning she previous exhibited, and her warrant is the low analytical proof scheme. Again, she develops a correct deductive argument, and she is finally able to move past her ritualistic beliefs about arguments in the NG axiom system. For a portion of the argument, however, she still relies on the perceptual proof scheme. Although her reasoning exhibits an analytical characteristic, it is still hindered by preconceptions about proof. Lindsey maintained her reasoning level from NG 6. Her warrant on Problem 3biii is the structural axiomatic proof scheme – she produces a valid deductive argument.

Discussion

Overall, both students began the semester as perceptual provers, and admitted they focused more on trying to remember high school geometry than on engaging in the problems. Kayla's preconceived notions about proof inhibited her ability to move past a perceptual proof schema; whereas without any prior proof courses, Lindsey had no preconceived ideas to overcome and was quickly able to move her reasoning to an axiomatic level. The non-Euclidean geometries used in all assignments after Viewing Tubes forced Kayla and Lindsay to fully engage with the problem to understand its solution. Kayla reported that she had never revisited an assignment in her previous proof courses, and it appears that the revisions they completed helped them to begin to reason more axiomatically; this was Lindsey's first proofs class. Lindsey, who earned an A in the course, needed less time and fewer revisions to begin axiomatic reasoning than did Kayla, who earned a B. Although more inquiry is needed to determine if non-Euclidean is the most effective way to help pre-service teachers understand definitions and proof in their college geometry course, we recommend that revision of partially correct proofs be a incorporated into college geometry classes.

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Exploring Mathematics Graduate Teaching Assistants' Developmental Stages for Teaching

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Traditional training programs that address mathematics graduate teaching assistants' (MGTAs) teaching practices are offered when they first arrive to campus, when they have little, if any, teaching experience. However, not much research has investigated how MGTAs' thinking about and facility with teaching change over the course of their graduate programs and, consequently, how their need for training changes over time. The goal of this study is to understand MGTAs developmental stages for teaching and how understanding these stages can inform the creation of a multi-year training program. Eleven MGTAs from a large, doctoral granting institution were surveyed and interviewed over the course of an academic year. Survey and interview responses were examined using a specific model of teacher development. Preliminary analyses, suggestions for multi-year MGTA training programs, and questions for future research are discussed.

Keywords: Graduate Teaching Assistants, Professional Development, Training for Teaching

Background

Years of research have provided evidence that the form of instruction in science, technology, engineering, and mathematics (STEM) disciplines in post-secondary institutions is significantly problematic for undergraduate learners. In particular, Seymour and Hewitt (1997) found that undergraduates in STEM courses were most likely to point to poor pedagogy as the reason for dropping out of a STEM discipline. More recently, researchers have found that students in traditional lectures have higher failure rates than students in classrooms that support active learning and student engagement in mathematical work (Chen, 2013; Freeman et al., 2014; PCAST, 2012). Mathematics instructors' and professors' teaching practices have an impact on students, as do graduate teaching assistants, with undergraduate students' interest in a subject experiencing a greater decline when they are taught by a graduate teaching assistant (Bettinger & Long, 2004).

As a result of these findings and the understanding that mathematics graduate teaching assistants (MGTAs) represent the future instructors and professors of mathematics, researchers have investigated the teaching practices of MGTAs (Belnap, 2005; Latulippe, 2007; McGivney-Burrelle, DeFranco, Vinsonhaler, & Santucci, 2001; Speer, 2001). In particular, researchers have developed courses and other training programs with the goal of providing MGTAs with a new vision of instruction that would offer undergraduate learners more meaningful ways of engaging in mathematics that lecture-based, direct-instruction does not provide (DeFranco & McGivney-Burrelle, 2001; Harris, Froman, & Surles, 2009; Speer, 2001). However, these studies found that MGTAs' instructional practices did not change as a result of these programs. Despite investigations of the efficacy of training programs, the field of MGTA professional development has not yet reached a consensus on the breadth and scope of training programs for MGTAs, with training programs varying from a few hours, to an intensive week, to training incrementally spread over the course of an academic term (Deshler, Hauk, & Speer, 2015; Harris et al., 2009; McGivney-Burrelle et al., 2001).

Of particular note about MGTA training programs, however, is that most are implemented solely in MGTAs' first year of graduate school; more specifically, when MGTAs first arrive to their programs, when they know little about teaching and likely have not had experience teaching. This is of concern because, over the course of their graduate programs, MGTAs' knowledge of teaching and mathematics will change significantly. Yet, in general, MGTAs will not have explicit attention paid to the development of their teaching practices as their knowledge and beliefs change, and as they encounter particular obstacles to instructional change and innovation. Such obstacles include MGTAs' own histories as mathematics learners (Deshler et al., 2015), the social context of the department (DeFranco & McGivney-Burrelle, 2001), belief among faculty members that attention to teaching is a distraction from research (Harris et al., 2009), and the structures of teaching assistant work (Beisiegel & Simmt, 2012).

Little is known about how MGTAs develop their teaching practices over the course of their graduate programs, even though their teaching practices have a significant, negative impact on undergraduate learners (Bettinger & Long, 2004). Researchers have pointed to the development of MGTA training programs as needing to be informed by many factors and should take into account such needs of the graduate students at different stages of their development (Park, 2004). Additionally, DeFranco and McGivney-Burrelle (2001) note that such training programs should be "viewed as ongoing professional development experiences that support [MGTAs] through the long and complex process of changing their teaching practices" (p. 688). With this in mind, the purpose of the study is to understand MGTAs' transitions, possible stages and changes in their thinking about teaching and learning as they progress through their programs. The research questions that guide this study are:

1. Do MGTAs go through developmental stages as teachers over the course of their graduate programs?
2. What implications do these stages have for how MGTAs are trained and supported over the course of their graduate programs?

The long-term goal of this study is to use this new knowledge to create a well-informed, scaffolded training program that will attend to the stages MGTAs go through as their teaching and thoughts about teaching evolve.

Theoretical Framework

Besides what has been learned through the study of training programs that address MGTAs' teaching practices, little is known about what stages MGTAs go through and how they can be supported as their views of teaching and learning evolve over the two to six years they are in a graduate program. Looking to the K-12 literature, researchers have studied schoolteachers' experiences in order to gain an understanding of teachers' transitions over time. For instance, Katz (1972) described four developmental stages, which include: (1) *survival* of the first year of teaching, with particular focus on classroom management and the routines of classrooms and schools; (2) *consolidation*, in which teachers begin to understand which skills they have mastered, and what tasks they still need to master; (3) a period of *renewal*, when teachers become tired of their routines and start to think of how things might happen differently; and (4) reaching *maturity*, where teachers think more broadly about the contexts of schools and students' learning (p. 52-53). Importantly, Katz (1972) notes that the third stage of renewal, of beginning to question standard teaching practices, does not begin until the third or fourth year of teaching. If MGTAs experience similar developmental stages, then the current programs that address their teaching practices only at the beginning of their graduate programs are likely inadequate and more thought should be given to MGTAs' possible developmental stages and their needs at those stages.

Context of the Study

In order to understand the MGTAs' transitions in teaching over time, two beginning-of-the-academic-year surveys were developed, one for new and one for experienced MGTAs. Additionally, protocols for mid-year and end-of-year interviews were created. Surveys are used at the beginning of the year because of logistical issues, such as varied arrival times to campus, and to also capture baseline information. The surveys include open-ended questions that inquire about MGTAs' thoughts about teaching and learning mathematics, how they would describe a well-taught mathematics lesson, and what had influenced the way they think about teaching. Additionally, Likert-Scale items include those that address MGTAs' epistemic beliefs and self-efficacy. Mid- and end-of-year interviews allow a deeper view their teaching practices, MGTAs' most recent teaching experiences, whether they feel that they are receiving adequate support, and what other support they they feel they need. The intention of the study is to survey and interview participants for the duration of their graduate programs in order to capture, longitudinally, any changes to their views of teaching and their need of support for teaching.

In 2015, the MGTAs were recruited from a department of mathematics in a large, doctoral granting institution. Approximately 5,000 students enroll in a lower-division mathematics course (such as Pre-calculus, Differential, Integral or Vector Calculus, Business Calculus, or Differential Equations) each year at this university, with the structure of most courses having three hours of lecture with 150-250 students per class and taught by an instructor. MGTAs are generally assigned to run recitations (1 hour workshops) of smaller groups of students from the large lecture sections. MGTAs are not assigned to courses based on knowledge, skill, or experience. Rather, assignments to courses depend mostly upon scheduling, although MGTAs are asked what their preferences are. At the beginning of the academic year, newly arrived MGTAs receive 2 ½ days of training for their teaching assignment, with a primary focus on how to support active learning and student engagement in mathematical work during recitations and lectures. During the summer after their first year, MGTAs are offered the opportunity to teach their own course as the instructor of record. Only informal mentoring happens before and during the summer sessions.

Eleven of 60 MGTAs agreed to participate in the first year of the study. Participants include 3 first-year, 2 second-year, 4 third-year, 1 fourth-year, and 1 fifth-year MGTA. At the time of submitting this proposal, each of the participants had completed the beginning-of-the-year survey as well as the mid-year and end-of-year interviews. The participants who remain in the mathematics graduate program will complete surveys and interviews in the upcoming academic year and new participants will be recruited. Participants' responses to survey and interview questions were analyzed using thematic analysis (Braun & Clarke, 2006), with a deductive approach that looked for instances of the participants' experiences that could be elucidated with Katz's (1972) four-stage model of teacher development. At the end of the first year of this study, data has not yet been collected that would illustrate each participant's transitions during the course of their programs. However, comparisons will be made between the first- and later-year MGTAs in order to gain a preliminary understanding of their developmental stages.

Preliminary Findings

The first year MGTAs alluded to *surviving* the first year as teaching assistants. For example, one first year MGTA described her initial experience teaching in this way: "The first [term], I was completely on my own and I didn't know what I was doing." In this statement, I see that, despite the training the MGTA had received, she was still fairly uncertain about her teaching. The newness of teaching was a disorienting and isolating, and she did not seem to rely on what she had heard or learned during her training. As another example, a first year MGTA stated this about his first term:

> By that point the quarter was just getting really hectic and I wasn't able to plan as much as I usually like to plan for courses. Sometimes I was looking at the material for about two hours before I started that day whereas usually I like to look at it the day before or during the weekend or something. And so sometimes, though, the classes that I went to where I was kind of doing it on the fly, where I was literally looking at it like an hour or two before class. A lot of times it's just more like get the notes done, go in, and do it.

In this participant's statement, I observe that the busy life of a new mathematics graduate student was enough to counteract the training that he had received when he first arrived. He noted that he had intended to devote more time to preparing for teaching, even creating revised drafts of lessons, with the aim of posing several open-ended questions during class. However, under the time constraints of his life as a graduate student, he switched to survival mode, with a routine of going through notes and presenting material and not implementing teaching strategies that would promote active learning.

One first-year MGTA spoke about his experience of being assigned to teach a course during the summer term at the end of his first year in the graduate program:

> I've never taught a day in my life – and when I went to – I originally just asked [two instructors] three questions. I asked them, "What worked in your classroom and what didn't?", "What advice do you have?" and some other questions. And so they answered that and then they said, "Hey, if you want to sit down and talk, we can." And, if that hadn't happened – that was out of their own kindness – and, if that hadn't happened, I feel like I'd be drowning right now because I wouldn't know what to do.

This MGTA noted that he would not be surviving, but would instead be drowning, in his first teaching experience had he not independently sought out advice and course materials from instructors. Another interesting thing to note is his statement that he had "never taught a day in his life." He had led multiple recitations during the two terms immediately prior to teaching his own course, and yet neither that experience nor the initial training seemed to provide him with enough resources to feel comfortable in his role as an instructor.

Later-year MGTAs spoke differently about their teaching experiences. For example, one third-year MGTA summarized her transition from the *survival* stage to the *renewal* stage:

> I think previously, I was more focusing on, "I just want to survive my first teaching experiences." So, now that this is my fourth time teaching, I feel a little bit more comfortable trying to incorporate more active learning in my classroom, and trying non-traditional techniques whereas previously, when I taught, for example, my first time teaching my own class and I taught Calculus, I did mostly lecture because I just wanted to do what I felt most comfortable with – what I felt I could be successful at.

From her statement it is not clear that she had tired of her routine of lecturing, as suggested by Katz (1972), for the transition from survival to renewal. Yet, it is clear that she is now comfortable enough in the classroom that she is beginning to think of what she might do differently; in this case, incorporate active learning. This is interesting as the MGTA training she had received three years prior had promoted active learning and it was only at the end of her third year that she was ready to consider and enact active learning strategies.

The third- and fourth-year participants' statements illustrated their transition into *maturity*, as they were thinking more broadly of educational issues (Katz, 1972). For example, one MGTA noted that she had become more aware of different learning styles:

> I've learned to understand more deeply some of the things students have going on in order for me to be more conscious of. I've also been very aware of different learning styles. I'm making sure to – some people are visual. I am visual. So I make sure to have everything written down. But I also know people are auditory learners. So I also read everything.

Another later-year MGTA spoke of her awareness of multiple choice assessment questions and potential biases within multiple choice questions, noting her observation that questions are "biased for gender and ethnicity and culture." This particular MGTA had also begun to think about the intersection of teaching mathematics and social justice issues.

Discussion

Katz's (1972) developmental stages are useful for defining and understanding the stages that MGTAs go through. Through this framework, I observe that MGTAs think about different, and progressively more sophisticated, aspects of instruction as they progress through their graduate programs and developmental stages. These findings suggest that MGTAs are not necessarily ready to enact more demanding instruction (e.g., active learning) in their first experiences running recitations, when they are teaching for the first time and likely in survival mode. The findings of this study also suggest that MGTAs have more advanced thoughts about teaching in their third and fourth years, thoughts that go beyond what is typically offered in initial MGTA training programs.

One conclusion is that MGTA training programs should be multi-year training programs that address these stages and the MGTAs' needs in each of these stages. A third-year MGTA illustrates how the initial training did not impact her at first and that follow-up training in her third year might be useful. Here she refers to the teaching strategies she had learned in her initial training:

> I probably couldn't recall them all, right? They're just somewhere in the recess of my head. It might be good to hear them in the context of now having three years' experience. And certainly what I thought was important before I had ever taught is going to be different than hearing it after three years' experience. So maybe even if I heard the same thing, it would carry more weight now. I would have a better understanding for the context it would fit in.

Katz (1972) noted the stage of *renewal*, of thinking of how teaching might happen differently, does not being until a teachers' third or fourth year in the classroom. Thus, with Katz's finding and the participant's statement, I suggest that training programs that extend into third and later years might gain more traction in helping MGTAs enact more demanding forms of instruction, such as active learning. Once MGTAs have more experience teaching, the instructional practices espoused by training programs can become more relevant, relatable, and practicable.

Intended Questions for the Audience

For the presentation at RUME, questions that I intend to pose and discuss with the audience are:

- Does Katz's (1972) framework seem adequate for understanding and describing the developmental stages for MGTAs?
- How might the results of this study be useful to and inform training programs for MGTAs?
- What other questions and ideas should be answered in order to better inform the development of a multi-year MGTA training program?

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Flip vs. Fold: What is so important about the Rigidity of a Motion?

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This report will investigate the mathematical and pedagogical consequences of flipping versus folding in the identification of reflection symmetry. Preliminary results are presented from a teaching experiment aimed at exploring the development of one undergraduate student's understanding of symmetry. The analysis indicated that throughout the teaching experiment, the student held two distinct versions of reflection symmetry. One version was what most would identify as reflection, while the other was an iterative process based on the participant's ability to fold the figure. In this report, we share what this student identified as symmetries and how she justified her methods. In addition, we discuss why the definition of isometry necessitates a rigid motion and how the motion of folding is insufficient for identifying symmetries correctly. Lastly, we consider why understanding a rigid motion is advantageous for students who want to consider symmetries in more sophisticated mathematical contexts such as group theory.

Keywords: Symmetry, Abstract Algebra, Intellectual Need, Teaching Experiment

Introduction

Symmetry can be thought of in two different ways (Larsen, 2010). Consider the following definitions of symmetry from different textbooks, "an object or design has symmetry if part of it is repeated to create a balanced pattern" (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998, p. 206) and "a symmetry of a figure is a rigid motion which leaves the figure unchanged" (Farmer, 1991, p. 27). The first definition describes symmetry as a property of a figure, and the second as a motion preformed on a figure. Throughout their K-12 experiences, students are introduced to both conceptions of symmetry as part of their mathematics instruction (Common Core State Standard [CCSS], 2010). In particular, as early as forth grade, students are introduced to the concept of symmetry as a property of a figure and expected to, "recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts" (CCSS, 2010). In later grades, students are introduced to the concept of symmetry closer to that of a motion performed on a figure. For example, in 8th grade students are expected to "verify experimentally the properties of rotations, reflections, and translations" (CCSS, 2010). For more advanced courses in mathematics, it is often beneficial that students are able to consider symmetries as strictly isometries. This is especially true in the context of group theory, where students need to be able to think of the property of symmetry in terms of a transformation that can be combined (composed) in such a way that leaves the given figure unchanged (Larsen, 2010).

The context of geometric symmetry has proved to provide a 'rich and natural context for developing the concepts of group theory' (Larsen, 2009, p. 136), since the ideas of symmetry and equivalence are fundamental concepts in group theory (Burn, 1996). However, outside of APOS theory (Asiala, Kleiman, Brown, & Mathews, 1998), little research has been done into how students develop an isometry understanding of symmetry. We have recently begun to investigate how students may develop an understanding of symmetry and symmetry equivalence through a series of task-based interviews. In this preliminary report, we present our findings from a teaching experiment aimed at exploring the development of one undergraduate student's understanding of symmetry. In particular, we report on how this student identified symmetries of various figures and how she justified her methods.

Theoretical Perspective

In the context of our research we are considering symmetry as an isometry, or a rigid motion, and therefore it must be a motion that does not deform the shape. Yet since it is not uncommon for students to have been taught to fold a figure in the context of symmetry (CCSS, 2010), we have found ourselves faced with an interesting question, "If symmetry is a motion, what does it matter if one flips or folds?" We are interested in not only the mathematical consequences of the different motions, but also the pedagogical ones. If students are in fact more accustomed to a folding motion, as per Harel (2013), the issue we must address is that of how we might *necessitate* one a flip motion (rigid) over a fold (non-rigid). In other words, as we help to facilitate students through the Measuring Symmetries Task (Larsen & Bartlo, 2009), we must be cognizant of how we can make flip, instead of fold, a solution to what students recognize as a problem.

Methods

The first author conducted a teaching experiment (Steffe & Thompson, 2000), exploring how an undergraduate student, Birdie, developed an understanding of symmetry over a series of 5 task-based interviews. Birdie is a high achieving sophomore civil engineering student who had recently completed Calculus 3 at a large urban university in the western United States. Data collection consisted of all Birdie's written work as well as video recordings of each interview, which were later transcribed. Throughout the interviews, the student explored her ideas of symmetries by working through the Measuring Symmetry Task (Larsen & Bartlo, 2009). The task is designed to build on the student's aesthetic sense and intuition to help the development of formal ideas of symmetry, and later focuses on the idea of symmetry equivalence. The task begins by asking the student to consider the following figures, see Figure 1, and to rank them from least to most symmetric.

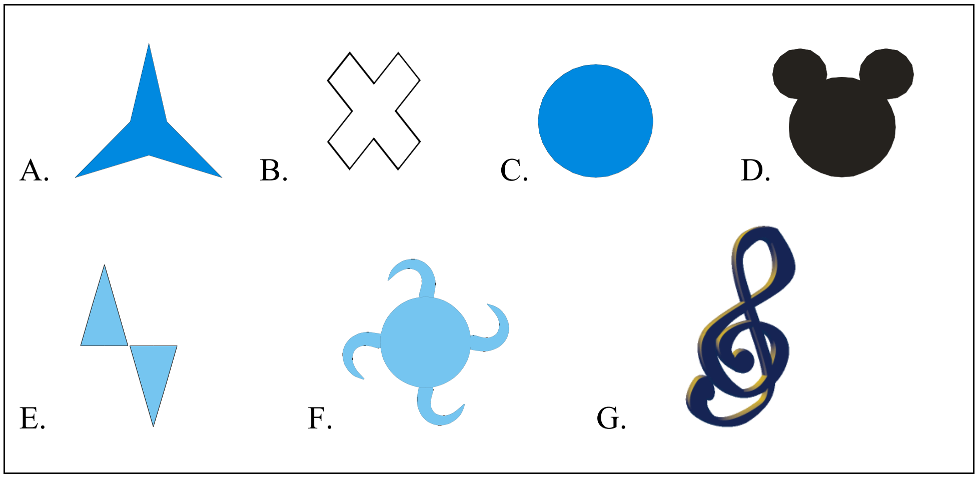


Figure 1. Figures (A – G) given in the Measuring Symmetry Task

Through the course of the teaching experiment the student was also asked to:

- Measure (i.e., assign a number to) the symmetry of a figure in a way that is consistent with the ranking
- Define the various types of symmetries they had counted
- Negotiate between their definition and counting system for inconsistencies
- Consider equivalence of symmetries

Ongoing thematic analysis (Braun & Clarke, 2006) is being conducted in an attempt to identify Birdie's various ideas of symmetry. As one of the main research goals of the teaching experiment is to discover how Birdie developed an isometry understanding of symmetry, we are particularly interested in what parts of the definition of symmetry she could recreate on her own. There did not seem to be a point in the Measuring Symmetry Task that necessitated a flipping motion instead of folding, which was the catalyst in trying to find the true distinction between the motions.

Initial Results

From the first time Birdie offered an explanation as to why she ranked the figures in the way that she did, she offered 2 distinct versions of what she eventually called, "line symmetry." When considering figure D (see Figure 1), Birdie assigned a ranking of 1. She explained:

> Birdie: Um, this (motions to figure D) I can at least cut once, and I guess this is asymmetry (in reference to figure G)...
> Interviewer: Can you describe what you mean by cut?
> Birdie: So if I were to fold it in the middle right here I can fold it once and it would be symmetric on that line.
> Interviewer: Can you use the transparency to show me what you mean by fold? ...(she folds the transparency in half) like actually fold it in half, ok. And then it lines up like that, ok.

This is precisely the reflection symmetry one would expect for figure D, and mathematically accurate, one reflection about a vertical axis down the middle of the figure.

As Birdie continued to describe her ranking for the figures, she quickly offered a different version of line symmetry when considering figure E (see Figure 1). In early interviews, Birdie gave figure E a rank of between 2-6 symmetries as she developed and negotiated her ideas of line symmetry. Notice that mathematically figure E has only a trivial and non-trivial rotation, and no reflections. The following transcripts show her initial thoughts on a second version of line symmetry.

> Birdie: Um, this one's (figure E) kinda the same situation where, if I fold it in the middle, it lines up to be something that's more symmetric in my mind.
> Interviewer: Ok, what parts are lining up when you fold it in the middle?
> Birdie: Um, actually it's not lining up, but it also has rotational symmetry.
> Interviewer: Ok.
> Birdie: Alright, well when I fold it, it lines up to be something that can be folded again into great symmetry.
> Interviewer: Ok so maybe by a compound fold?
> Birdie: Yeah, I don't know what the rules are here, but I'm just assuming it won't be...

Interviewer: We haven't established any so it's whatever you want to do at this point.

Birdie continued to develop and refine both versions of line symmetry as the experiment continued and was able to begin to articulate them in her written description of what she was counting (symmetries). Figure 2 presents Birdie's written description of symmetry.

Symmetry is broken up into two groups for me, line & rotational. Line symmetry being if you draw a straight line through the object from one side to the other & fold it on that line the two sides will match up perfectly (perfectly being without any one the reflected side not matching a half) & either create the same shape or a new one. Which then can then be done again with another 'fold' aka drawing another straight line and so forth until it will no longer match up with the reflected side no matter where the line is drawn.

Figure 2. Birdie's written description of her conceptions of symmetry.

Her description in Figure 2, suggests that Birdie recognized that while one version of symmetry considered the original shape, the other considered new shapes as the result of folding. This iterative folding was a sort of process in which she was considering the resulting shape from a fold, and if it had line symmetry in addition to the original figure. Birdie was always aware that one process deformed the original shape. The following transcript shows Birdie's distinction between these two methods when asked to clarify her assigning 4 folds to figure B (see Figure 1), which only has 2 reflection symmetries:

> Birdie: I guess the argument is more, are you counting it as the same shape or as a different shape, each time you fold it? I mean, technically it's a different shape just looking at itself, but considering the past of it, it's the deformed shape that you create from the original shape.

We explore the mathematics behind both of Birdie's motions for identifying line symmetry; the single folding motion that Birdie eventually identifies as equivalent to 'flipping' and the iterative process of 'folding.' Flipping can be described as a reflective symmetry. Reflective symmetry is absolutely a type of isometry, and therefore a distance preserving injection. The 'folding' motion however is not an injection, as it sends the figure to a proper subset of the original domain. The folding motion deforms the figure and so it is also not distance preserving.

By using an iterative folding process Birdie was able to identify line symmetries that were not mathematically accurate. The figure below (Figure 3) shows Birdie folding two different figures, each of which she counted more symmetries than the figure would have by the formal definition.

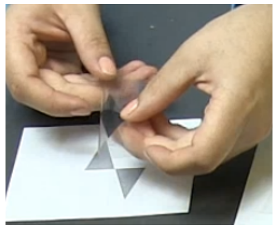
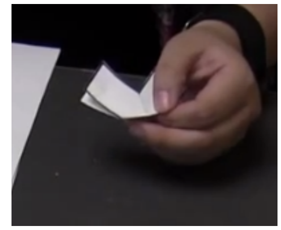
E. 		B. 	
This figure has 0 reflection symmetries	Birdie identified up to 4 line symmetries	This figure has 2 reflection symmetries	Birdie described 6 line symmetries through iterative folding

Figure 3. Examples of Birdie's additional line symmetries from iterative folding.

Conclusion

The word ‘symmetry’ is given multiple definitions, and while that makes both symmetry as the property of a figure and symmetry as a motion preformed on a figure valid, only the second will be advantageous when thinking of symmetry as a more abstract mathematical object in contexts such as group theory. Our work with Birdie has suggested that students beginning to work with symmetries as isometries may be (heavily) influenced by the notion of symmetry as a property and the motion of folding. There is no doubt that symmetry has been found to be a rich and natural context for developing group theory (Larsen, 2009), where students can rely on their intuition and aesthetic sense, that provides rich opportunity for student discourse (Larsen & Bartlo, 2009). Yet our research has shown that to fully develop an isometry understanding of symmetry required a student to negotiate between their pre-existing idea of symmetry and the folding motion/s that she had associated with it.

Questions for the audience / Related ideas we’re still exploring

- While we know that the deforming motion of the fold is not the same as the rigid motion of the flip and therefore will not create the same group structure under composition we are curious what kind of algebraic structure if any can be used to describe the folding motions, and what the operation might be.

- Under what circumstances does a fold motion accurately identity reflection symmetry? Birdie had two kinds of line symmetry one of which was a single folding motion and the other was a series of iterative folds. The single folding motion was successful at identifying reflective symmetries even though it was not a rigid motion. Would it always be? Why?

- What kinds of situations necessitate a flip over a fold that are experientially real for students that helps facilitate their recreation of a mathematically accurate definition of symmetry that includes a rigid motion?

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Examining Students' Procedural and Conceptual Understanding of Eigenvectors and Eigenvalues in the Context of Inquiry-Oriented Instruction

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This study examines students' procedural and conceptual understanding as evidenced by their written responses to two questions designed to assess aspects of their understanding of eigenvalues and eigenvectors. This analysis draws on data taken from 126 students whose instructors taught using a particular inquiry-oriented instructional approach and 129 comparable students whose instructors did not use this instructional approach. In this proposal, we offer examples of student responses that provide insight into their reasoning and summarize broad trends observed in our quantitative analysis. In general, students in both groups performed better on the procedural item than on the conceptual item. Additionally, the group of students who were taught with the inquiry-oriented approach outperformed the group of students who were taught using other approaches.

Key words: eigenvalues, eigenvectors, linear algebra, inquiry-oriented, student thinking

Linear algebra is a mandatory course for many science, technology, engineering, and mathematics (STEM) students. The theoretical nature of linear algebra makes it a difficult course for many students because it may be their first time to deal with this kind of abstract and conceptual content (Carlson, 1993). Carlson (1993) also posited that this difficulty arises from the prevalence of procedural and computational emphases in students' coursework prior to linear algebra, and that it might therefore be difficult for students to connect new linear algebra topics and their previous knowledge. To address this issue, researchers have developed inquiry-oriented instructional materials and strategies to help students develop more robust, conceptual ways of reasoning about core topics in introductory linear algebra (e.g. Wawro, Rasmussen, Zandieh, & Larson, 2013). In this proposal we examine assessment data to identify ways in which students reasoned about eigenvectors and eigenvalues. In particular, we identify differences in the performance of students whose instructors taught with a particular inquiry-oriented approach to teaching eigenvectors and eigenvalues and comparable students whose instructors did not use this approach.

In this work we draw on data from an assessment that was developed to align with four core introductory linear algebra concepts addressed in the IOLA instructional materials: linear independence and span; systems of linear equations; linear transformations; and eigenvalues and eigenvectors (Haider et al., 2016). The focus of this study is to identify the ways students understand and reason about eigenvalues and eigenvectors. In the assessment, there were two questions, question 8 and 9 that addressed eigenvalues and eigenvectors. Question 8 was a procedural item related to the eigenvalue of a given matrix and question 9 focused on the conceptual understanding of the eigenvectors. The research questions for this proposal are:

- How did students reason about eigenvectors and eigenvalues in the context of questions designed to assess aspects of student's procedural and conceptual understanding?
- How do the performance and ways of reasoning of students whose instructors adopted an inquiry-oriented approach to teaching linear algebra compare to the performance and ways of reasoning of other students?

Literature & Theoretical Framing

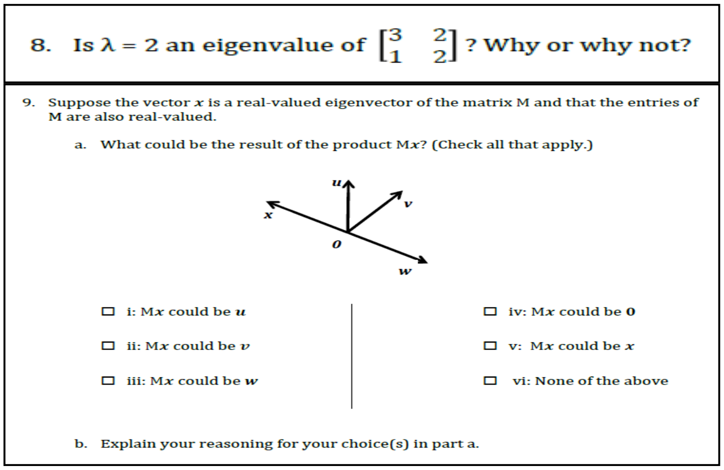
Many have argued that the shift from a predominantly procedural approach to mathematics many students experience before college to a more conceptual approach causes a lot of difficulties for students as they transition to university mathematics; linear algebra is a topic in which students struggle to develop a conceptual understanding (Carlson, 1993; Dorier & Sierpenska, 2001; Dorier, Robert, Robinet & Rogalski, 2000; Stewart & Thomas, 2009). Across the literature on the teaching and learning of eigenvalues and eigenvectors, procedural thought processes are featured prominently. For example, Stewart and Thomas (2006) highlighted the example of the conceptual processes and difficulties students find in learning about eigenvalues and eigenvectors, where a formal definition may be immediately linked to a symbolic presentation and its manipulation. Thomas & Stewart (2011) highlighted a difficulty students find when faced with formal definitions for eigenvalues and eigenvectors. Since these definitions contain an embedded symbolic form ($Ax = \lambda x$), students often move quickly into symbolic manipulations of algebraic and matrix representations such as transforming $Ax = \lambda x$ to $(A - \lambda I)x = 0$ without making sense of the reasons behind these symbolic shifts. Schonefeld (1995) used eigenpictures ("stroboscopic" pictures) to show x and Ax at the same time by using multiple line segments on the x-y-axis. He observed that graphical representations of eigenvalues and eigenvectors got little attention in the literature and that a picture may benefit more than algebraic presentations. It is also documented that students struggle to coordinate algebraic with geometric interpretations (e.g. Stewart & Thomas, 2010; Larson & Zandieh, 2013) and the students' understanding of eigenvectors is not always well connected to concepts of other topics of linear algebra (Lapp, Nyman, & Berry, 2010). To support students in developing a better understanding of the formal definition and the geometrical interpretations of the eigenvalues and eigenvectors, researchers have developed a variety of instructional interventions (e.g. Tabaghi & Sinclair, 2013; Zanidieh, Wawro, & Rasmusen, 2016).

Researchers often make reference to conceptual understanding and procedural understanding when discussing student's thinking about mathematical concepts (Hiebert, 1986). To operationalize our distinction between concepts and procedures, we draw on Vinner's (1997) distinction between conceptual and analytical behavior. According to Vinner (1997), students are in a conceptual mode of thinking if their behavior provides evidence that they are attending to concepts, their meanings and their interrelations. On the other hand, students are in an analytical mode of thinking when solving routine mathematical problems if they act in the way expected and certain analytical thought processes occur. In case they do not act in such ways but succeed in making the impression that they are analytically involved in problem solving then they are in pseudo-analytical mode of thinking. Vinner (1997) argued that in most problem solving situations when students are asked to solve a problem their focus usually is on which procedure should be chosen and not on why a certain procedure works. Based on this, we draw on Vinner's (1997) definition of conceptual versus analytical (procedural) as an analytic tool for interpreting student's answers to the questions involving eigenvalues and eigenvectors.

Data Sources

In our previous work, we have developed an assessment that covers the four focal topics of linear algebra mentioned earlier (Haider et al. 2016). This assessment was administered at the end of linear algebra course at different public and private institutions across the country. This was a paper-and-pencil assessment that includes 9 items and the students were allocated one hour to complete. It was designed to measure students' understanding of introductory linear algebra topics. Every item of the assessment contained a component of open-ended justification for students to elaborate their conceptual understanding of the topic.

Figure 1. Question # 8 and 9 (Eigenvalues/Eigenvectors) of the assessment



8. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$? Why or why not?

9. Suppose the vector x is a real-valued eigenvector of the matrix M and that the entries of M are also real-valued.

   a. What could be the result of the product Mx? (Check all that apply.)

- ☐ i: Mx could be u
- ☐ ii: Mx could be v
- ☐ iii: Mx could be w
- ☐ iv: Mx could be $\mathbf{0}$
- ☐ v: Mx could be x
- ☐ vi: None of the above

   b. Explain your reasoning for your choice(s) in part a.

We have collected assessment data from two groups of students: students whose instructors received instructional supports to teach linear algebra using a particular inquiry-oriented approach as part of the NSF-supported TIMES research project (who we will refer to as TIMES students), and students whose instructors did not receive these supports (who we will refer to as non-TIMES students. The instructors who participated in the TIMES project attended a 3-day summer workshop, participated online workgroup conversations for one hour per week for a semester, and implemented inquiry-oriented curricular material in their linear algebra class. We have collected the assessment data of 126 Times students across six TIMES instructors and 129 non-Times students across three non-TIMES instructors from different institutions in the US. Non-TIMES instructors were recruited from linear algebra instructors either at the same institutions as TIMES instructors or at other similar institutions (e.g. similar geographic area, similar size of student population, similar acceptance rate at institution) to collect assessment data for comparison of TIMES and non-TIMES students. In this study, we focused on in-depth analysis of students' reasoning on the assessment questions related to eigenvalues and eigenvectors. Both items are shown in the figures above.

The inquiry-oriented approach to learning eigenvalues and eigenvectors associated with this study is characterized in detail elsewhere (Zandieh, Wawro & Rasmussen, 2016). This approach supports students in coming to first learn about eigenvalues and eigenvectors as a set of "stretch" factors and directions that can be used to more easily characterize a geometric transformation. In this sequence of tasks, students first work to describe the image of a figure in a plane under a transformation that is easily described in a non-standard coordinate system. Students then work to label points using standard and non-standard coordinate systems corresponding to the previous task; they also find matrices that transform points from one coordinate system to the other. The instructor works to link this work to the matrix equation $A=PDP^{-1}$ and subsequent tasks aim to leverage this conceptual basis as students learn more traditional computational methods associated with computing eigenvalues and eigenvectors.

Methods of Analysis

To identify different types of students' approaches to eigenvectors and eigenvalues assessment items, we conducted our analysis in three stages: (I) developed a coding scheme for both the descriptive (open ended) part and the non-descriptive (multiple-choice) part of the items, (II) coded assessment data by following the coding scheme, (III) ran statistical analysis of coded data for descriptive statistics and t-test to compare the performance of TIMES and non-TIMES students.

At the first stage, we decide the coding scheme for non-descriptive part of the problems. Later, we looked into the descriptive parts of the problems. We identified different correct students' approaches for both problems, which helped us to refine our coding scheme. The coding scheme for question 8 and 9 is given in the table below:

	Points Awarded and Criteria	Comments
Question 8	3 Points: For any of the following answers. 1: $\det(A-\lambda I)=0$ implies $(x-1)(x-4)=0$ implies $x=1$ or $x=4$ implies $\lambda=2$ is not an eigenvalue for the matrix A. 2: $\det(A-2I)=-2\neq 0$ implies $\lambda=2$ is not an eigenvalue of A. 3: $(A-2I)\begin{bmatrix}x\\y\end{bmatrix}=0$ implies $x=0$ and $y=0$ which is the trivial vector, implies $\lambda=2$ is not an eigenvector for the matrix A.	Fully correct answer
	2 Points: if any answer with computational mistakes and with good justification and conclusion will be credited with 2 credits (computational mistaken attempt cannot be awarded full credit like a fully correct answer)	Partially correct answer
	1 Point: if any answer missing any of the three rules (correct answer, correct computation and correct explanation) will be missing credits depending on the number of rules missed.	Some procedural knowledge
Question 9	3 Points: Writing $Mx=\lambda x$ or mentioning Mx as a constant multiple of x/scalar multiple of x with or without mentioning what the values of this constant/scalar; 0, 1 and -1 and giving the options x, w, and 0.	Fully correct answer
	2 Points: If a student mentions $Mx=\lambda x$ and miss any of the possible values of constant/scalar; 0, 1, and -1 and/or miss one of the given options x w, and 0.	Partially correct answer
	1 Point: Writing $Mx=\lambda x$ and/or miss two of the possible values of constant/scalar; 0, 1, and -1 and/or miss two of the given options x, w and 0.	Some understanding

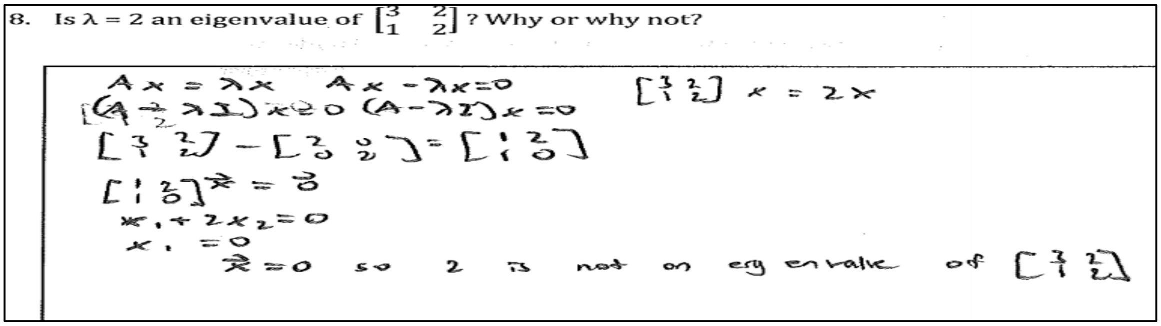
Table 1. Criteria for Awarding Points for question 8 and 9

At the second stage of analysis, we coded assessment data by following the above grading scheme. Two researchers looked at every students' attempt and decide a score independently and then matched with each other. If both researchers assigned different score to a particular student, then they discussed according to the grading scheme and agreed on a common score for that student. If both researchers have disagreement about a particular score, then a third researcher was consulted to make a consensus. At the third stage of statistical analysis, we checked the descriptive statistics to see the overall performance of students on the eigenvalue and eigenvectors questions. We also compared TIMES students with Non-TIMES students for both questions. We used t-test to compare the difference of means between questions 8 and 9 for both groups.

Initial Findings

In this section we summarize how students reasoned about the eigenvectors and eigenvalues items and how they med on both questions. We will provide some examples to show how they reasoned then point out using the quantitative data how the TIMES students performed compared to Non-Times students.

The following example shows that the student has a conceptual and procedural understanding in the sense that he/she by solving $(A - 2I)x = 0$, finding only the trivial solution and concluding that 2 is not an eigenvalue which is a nonstandard solution.


8. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$? Why or why not?

$Ax = \lambda x \quad Ax - \lambda x = 0 \quad \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} x = 2x$

$(A - \lambda I)x = 0 \quad (A - \lambda I)x = 0$

$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \vec{x} = \vec{0}$

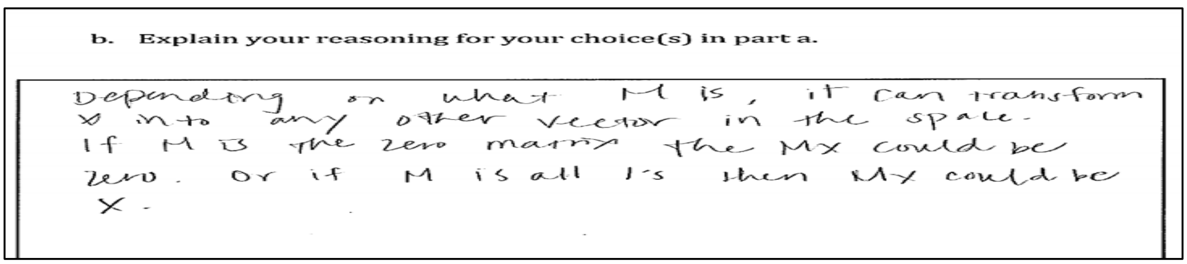
$x_1 + 2x_2 = 0$

$x_1 = 0$

$\vec{x} = 0$ so 2 is not an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

Figure 2: A student's example for Q8

Another example of a student who is interpreting the matrix M as something that can change in interpreting the eigenvector equation.


b. Explain your reasoning for your choice(s) in part a.

Depending on what M is, it can transform x into any other vector in the space. If M is the zero matrix the Mx could be zero. Or if M is all 1's then Mx could be x.

Figure 3: A student's example for Q9b

To see the difference in the students' performance we paid attention to the mean and standard deviation of the coded data. We also used t-test to compare the difference of means between both groups.

Question	All Students	TIMES Students	Non-TIMES Students
Q 8: Maximum Possible Points 3	Mean: 1.85 SD: 1.31	Mean: 2.0 SD: 1.23	Mean: 1.71 SD: 1.37
Q 9 (a & b): Maximum Possible Points 9	Mean: 1.50 SD: 0.82	Mean: 1.59 SD: 0.89	Mean: 1.42 SD: 0.74
Q 9 (b part only) Maximum Possible Points 3		Mean: 1.59	Mean: 0.54
		p-value = 0.0000273 < 0.05	

Table 2. Summary of TIMES and Non-TIMES Students' Performance

When examining the quantitative data to see how the two groups compared we noticed two things that seemed particularly noteworthy. First, both groups did better on the procedural item than the conceptual one. Second, TIMES students did better than non-TIMES on both items, but they outperformed non-TIMES on the conceptual item at a higher rate. In our session we will more deeply explore evidence of conceptual and procedural reasoning as it appeared in students' responses to the two items, and differences between the two groups of students.

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Supporting Students' Understanding of Calculus Concepts: Insights From Middle-grades Mathematics Education Research

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In this preliminary proposal, we report on results from a paired-student teaching experiment focused on college calculus students' developing notions of reversibility and reciprocity through compositions and transformations of linear relations. We anticipate fruitful discussions about relationships between numerical and quantitative reasoning and students' thinking about graphs.

Key words: Calculus, Derivative, Teaching Experiment

Introduction

The main topic of first-term calculus–the derivative–depends on deep, flexible understandings of rate and slope (Dawkins & Epperson, 2014; Zandieh, 2000) rooted in conceptions of rational numbers. Two explicit examples are learning to equivocate "increasing function" and "positive derivative" and learning to reason about reciprocal rates of change of an invertible function and its inverse. However, rational number conceptions are not usually the focus of pre-calculus coursework. In placement testing such as ALEKS, rational number topics are generally limited to arithmetic prerequisites for secondary school algebra. Research shows that development of numerical concepts in the elementary and middle grades can have a lasting effect on the ways students construct algebraic knowledge (Ellis, 2011; Hackenberg & Lee, 2015). We conjecture that differences in calculus students' reasoning about reversible and reciprocal relationships may extend from these differences. We report on our exploration of how differences in numerical conceptions affect college students' understandings of the derivative and how engaging in compositions and transformations of linear relations concurrent to calculus instruction may benefit students.

Theoretical Framework

We take a cognitive constructivist epistemologist perspective in viewing individuals' knowledge as a product of their organizing of their experiences interacting with the world to fit within or to construct mental *schemes* (von Glasersfeld, 1995). As others' thinking is fundamentally unknowable, schemes are researchers' constructs serving to fit their observations of others' activities. Our aim is to explore how ways students have constructed schemes for rational number concepts affect their learning of calculus topics.

A scheme consists of three parts: recognition of a situation, operations (mental actions), and an expected result. An example is the iterative fraction scheme, which is necessary for understanding improper fractions as numbers (Hackenberg & Lee, 2015). A student with an iterative fraction scheme understands the size of an improper fraction as the result of partitioning and iterating the size of '1'. If his or her scheme were *reversible*, the student also understands the size of 1 as the result of partitioning and iterating the size of an improper fraction. Reversible schemes are those for which the result of one sequence of mental operations is recognized as a situation for the reversed sequence of operations. Reversible schemes that are *interiorized* are those for which the recognition of a situation and an expected result of mental activity are experienced as synchronous. For a student with an interiorized iterative fraction scheme, the symbol '9/7' "contains" potential relations with numbers including '1', '1/7', '1/9' and '7/9', depending on the situation. These other numbers would be part of the student's concept image associated with '9/7' that could potentially be evoked (Tall & Vinner, 1981). The ways the associated units are coordinated depend on the student's *units coordinating structures*.

The construct of units coordinating structure has distinguished middle-grades students' reasoning with whole number multiplication, fractions integers, and linear algebraic expressions (Hackenberg & Lee, 2015; Ulrich, 2015). Research with pre-service and in-service teachers suggests that differences in units coordinating structures persist into adulthood (Busi, Lovin, Norton, Siegfried, Stevens, & Wilkins, 2015; Izsák, Jacobson, de Araujo, & Orrill, 2012). To our knowledge, units coordination has not previously been studied with calculus students. Of relevance for calculus students is whether students have constructed *two-level* or *three-level* units coordinating structures.

Students who have constructed only two-level units coordinating structures can assimilate composite units (units of units), but transformations of those units (e.g., forming a third level of units) are non-reversible. Such students can reason with three levels of units *in activity*, but they reason with two levels at a time. For example, a student assimilating fractions with a two-level units coordinating structure could form the size of 9/7 from a given size of '1' by partitioning the '1' into 7 parts and appending two more parts. Assimilating fractions with three levels of units is required for maintaining the relationships between the different units (seven 1/7 units within 1 and nine 1/7 units within 9/7) (Norton, Boyce, Phillips, Anwyll, & Ulrich, 2015). A construct related to units coordination that has been studied with calculus students and has been shown to correlate with success in calculus is *covariational reasoning.* As seen in Table 1, the description of mental actions in the Covariation Framework (Carlson, Jacobs, Coe, Larson, & Hsu, 2002) involve successively more structured coordinations of variables' values. Units coordinating structures constrain students' conceptions of (negative) integers and fractions, suggesting relation to Mental Actions 3 and 4.

Table 1
Stages 3-4 of the Covariation Framework (Carlson, Jacobs, Coe, Larson, & Hsu, 2002)

Mental action	Description of mental action	Behaviors
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable.	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input

Methods/Methodology

We assessed volunteer undergraduate students' backgrounds during the first week of their Summer 2016 8-week differential calculus course, using a units coordinating assessment (Norton, Boyce, Phillips, Anwyll, & Ulrich, 2015), and a pre-calculus concept assessment focusing particularly on functions concepts and covariational reasoning (PCA, Carlson, Oehrtman, & Engelke, 2010). We engaged two students in six weekly hour-long teaching sessions, which were video-recorded for ongoing and retrospective analysis by the researcher and a witness (Steffe & Thompson, 2000). The sessions were concurrent but separate from students' calculus classes, without interaction between the classroom instructors and researchers.

Preliminary Results

The two students (Susan and Kris, both pseudonyms) who participated in the teaching experiment were life sciences majors who had successfully completed pre-calculus coursework in college. Both were assessed as coordinating two levels of units: Susan used representations to form responses (coordinating units *in activity*) rather than creating representations to justify and explain interiorized reasoning, whereas Kris did not correctly coordinate three levels of units with fractions. Their responses on the PCA were similar, as Susan answered 16 of the 25 items correctly and Kris answered 14 items correctly. The eight items they each missed suggested interpreting graphs, function composition, domain and range, and covariational reasoning were pertinent topics for exploration. Following are descriptions of activities and reasoning exhibited during the teaching experiment.

Reasoning about linear relations

In the first session Kris and Susan were given Cartesian graphs depicting linear relations between time and volume, and volume and height, of water in a container (see Figure 1).

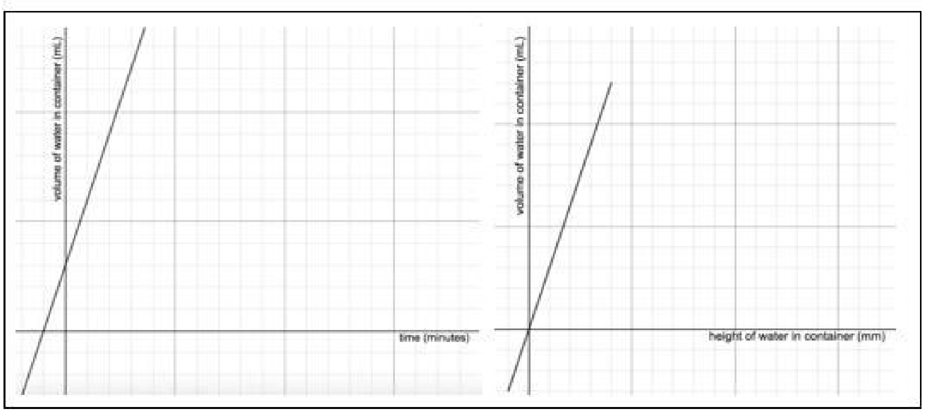

Figure 1. Graphs given to Kris (left) and Susan (right)

After establishing common interpretations of unit grid marks, the teacher asked for the height at 5 minutes. Susan used her time/volume graph to find the volume at 5 minutes and then found the height from Kris' graph. The teacher next asked how long it would take for the water in the container to reach a height of 27 mm, anticipating that they might represent and compose the two relations algebraically since they could not read the values directly from the graph. Susan instead focused on the graphs' slopes. "First we need to find out how much volume needs to be added, and then how much time that volume takes….From my graph we know there's 1 mm per 3 mL and from [Kris'] graph we know there's 3 mL per 1 minute." She was perturbed when multiplying $\frac{81\ mL}{27\ mm} \cdot \frac{1\ min}{3\ mL}$ resulted in 1 min/mm; it seemed she had anticipated a time corresponding with 27 mm. She did not consider the non-zero volume at time t=0 and insisted that knowing the slopes should be enough "since they were constant."

The sessions switched to the Desmos "Water Line" app, in which water is depicted flowing at a constant rate into different containers and students graph water height over time (https://teacher.desmos.com/waterline). The first container is a cylinder; the container is then modified in multiple ways (see Figure 2). In this context, Susan correctly explained why graph of the new cylinder's water line would be a positive vertical shift of the old cylinder's graph, despite Kris' conjecture of a negative shift and in contrast to her earlier reasoning.

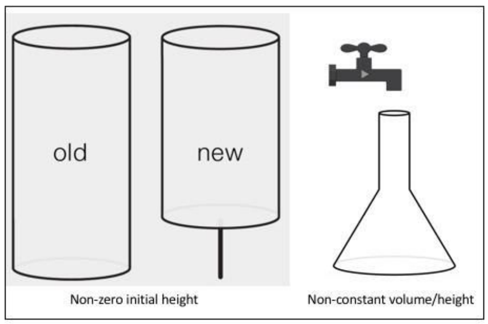

Figure 2. First three waterline containers

Reasoning about non-linear rates of change

Susan's water line graph for the third container indicated a linear component prefaced by an increasing, concave up, non-linear component, which she referred to as "exponential." When asked why she used the term exponential, she first exhibited shape-thinking (Moore & Thompson, 2015) regarding exponential growth , and then, referring to Figure 2, volunteered that "You can't really define this rate, because at each second it's filling at a different rate, because [the glass is] continually sloping inward. So it's going to be an exponential growth curve, because it's changing constantly." When asked what she meant by "it's changing constantly," she replied, "it's constantly getting quicker, and that is indicative of exponential growth." In response, the teacher introduced graphing calculator explorations with sliders to perturb the students' shape-thinking regarding exponential and polynomial functions, as well as activities relating the rate of change of a quadratic relation and its derivative, before returning to modeling water accumulation. *We are continuing to analyze relationships between Susan's and Kris' rational number concepts and calculus understandings. We will solicit feedback from the audience regarding our interpretations of the students' reasoning during the teaching sessions.*

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Signed Quantities: Mathematics Based Majors Struggle to Make Meaning

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Physics students struggle to make meaning of the negative sign in a variety of mathematical and physical contexts. This study is part of an ongoing concurrent mixed methods exploration of student understanding of negativity in physics. A set of multiple-choice items, modified from a prior study, was administered to over 500 calculus-based college students from diverse backgrounds. Results suggest that when the positive sign is an explicit part of a quantity, students struggle with positive quantity just as they do with negative quantities, and that the language that instructors use may inadvertently impute unintended meaning about signs.

Key words: signed numbers, negative, quantity, physics, vectors

Introduction

While notions of negativity have been part of mathematical thinking for well over a thousand years, many European mathematicians shunned negative numbers as recently as the advent of modern calculus. Negative numbers were seen to *"... darken the very whole of equations and make dark of thinking which are in their nature excessively obvious and simple."*(Maseres, 1758) More recently, mathematicians have isolated a variety of natures of negativity fundamental to algebraic reasoning that go beyond a "position on a number line" nature (Gallardo & Rojano, 1994; Thompson & Dreyfus, 1988; Nunes, 1993). College students are expected to reason with these various natures of negativity that form the foundation of scientific quantification, yet they experience the "darkening" of reasoning foreshadowed by 18th century mathematicians, struggling to separate the physical model from the rules associated with arithmetic. The work presented in this paper argues that the majority of engineering students in the U.S. struggle to make meaning of positive and negative quantities outside of the number line context learned in elementary school, in spite of successfully passing Calculus I and beyond in mathematics.

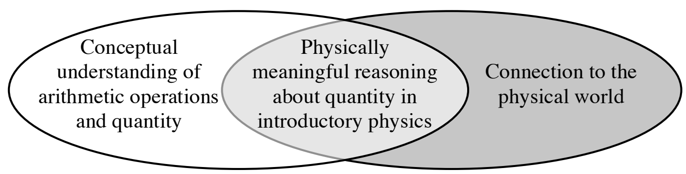
Developing flexibility with negative numbers is a known challenge in math education. In her study of algebra students, Vlassis found that full understanding of the concept of a negative number required that students develop flexibility with the various ways negative numbers are used in context (i.e., with the "negativity" of the number) (Vlassis, 2004).

Few published studies have focused on negativity in the context of the mathematics used in physics courses (Sherin, 2001; Bajracharya, Wemyss, and Thompson, 2012). In our own prior work on negativity (Brahmia & Boudreaux, 2016), we found that engineering students struggled to make sense of negative physics quantities, but struggled less with the unary, or "isolated number" nature of negativity, as compared with other natures. (The unary nature of negativity is embodied by position on a number line.) We found that students have particular difficulty with signed scalar product quantities, commonly associating the sign of the scalar with the direction of one of the factor vectors. We suspect that science contexts overwhelm some students' conceptual facility with negativity, and conclude that students' mathematical understanding is likely more fragile than what physics instructors may commonly assume based on the students' completed prerequisite math courses.

The current study extends this prior research. We wondered if there was something particularly "dark" about negative numbers *per se,* and if students would have a higher success rate had they been asked about signed positive numbers. Additionally, in open-ended versions of the assessment items we saw that students seemed to attribute meaning to particular phrases in the question statement that was not implied. We formulated the research questions below to focus the new study. The remainder of this paper describes our preliminary work in addressing these questions.

1. *Are difficulties with negative quantity associated with negativity per se, or do learners struggle in similar ways with positively signed quantity?*
2. *What assumptions regarding negativity do students make based on language commonly found in physics problems? Specifically, to what extent do students:*
   - *interpret "movement along the x-axis" to imply motion in the +x-direction?*
   - *interpret a negatively signed quantity to imply a motion in the negative direction?*

Figure 1: Double scope quantity reasoning blend



The cognitive blending theoretical framework (Fauconnier & Turner, 2008; Bing & Redish 2007) describes the interdependence of thinking about the mathematical and physical worlds that we feel is necessary for quantifying effectively with signed quantities in physics. Figure 1 illustrates a *double scope quantity reasoning blend*, in which two distinct domains of thinking are merged to form a new cognitive space optimally suited for productive work. Findings by Czocher support this view. She observed students enrolled in a differential equations course solving a variety of physics problems, and found that successful students functioned most of the time in a "mathematically structured real-world" in which they moved back and forth fluidly between physics ideas and mathematical concepts (Czocher, 2013).

Research Methods

Our work adopts a concurrent mixed methods strategy. We have used Vlassis's framework for negativity (Vlassis, 2004) to inform the design of six assessment items, three involving contexts from mechanics (ME), and three, contexts from electricity and magnetism (EM). (See Appendix.) Previously, we used multiple-choice (MC) versions of the items to reveal trends in large populations, and free-response versions to explore students' in-the-moment thinking (Brahmia & Boudreaux, 2016).

For the current study, the items were administered in a three semester, large enrollment, calculus-based physics course sequence at a large, diverse, public R1 university, and in an analogous three quarter sequence at a smaller, less diverse, public regional university. The items were ungraded, and were bundled with concept inventories routinely given as part of course assessment. At the R1 university, the course was composed almost entirely of engineering majors, while at the regional university, the course included not only engineers, but also students majoring in physics, chemistry, biology, and other STEM fields. In each case, students were concurrently enrolled in some level of calculus course.

To investigate the first research question, we administered MC versions of the ME items at the R1 institution in Spring 2016 to 551 students completing the second course in the physics sequence (which includes mechanics applications and thermodynamics). Half of these students (at random) received modified versions of the items, in which the negative quantities used in the original versions were replaced with positive quantities. To investigate the second research question, we extended the study described earlier (Brahmia & Boudreaux, 2016), in which the ME items were administered at the R1 institution in Fall 2015 at the end of both the first course in the sequence (which covers mechanics) and the third course (which covers EM). We examined responses on free-response versions (n=84, ME; n=138, EM), and made changes to the wording and the MC distractors of some items. These changes are described in the footnotes of the appendix. The modified versions of all items were then administered in Winter 2016 at the regional university, at both the start and end of the second course of the three-quarter physics sequence (which covers EM). Table I summarizes the administration of assessment items.

Table I: Administration of items in introductory, calc-based physics courses to assess student understanding of signed quantities. (FR=free-response, MC=multiple-choice)

Institution	**Administered at**	**Math Pre/Co Req.**	**Item context**	**Item format**
R1 University (2015/16)	End of 1st sem. course	PreCalc, Calc I	ME (negative quantities only)	MC and FR
	End of 2nd sem. course	Calc I	ME (neg. and pos. quantities)	MC only
	End of 3rd sem. course	Calc II, III	EM (neg. only)	MC and FR
Regional Univ. (2016)	Beginning and end of 2nd qtr. course	Calc I, II	ME and EM (mod. wording and choices; neg. only)	MC only

Findings

Figure 2 shows results from the positive and negative versions of items ME1-ME3. For all three item pairs, a chi-square test of significance yields p-values > 0.6. For items ME2 and ME3, effect sizes determined from log odds ratios are < 0.8 (Borenstein et al., 2009). The effect size for item ME1 is 0.13, a statistically small effect size. Prior research has found that physics students tend to inappropriately associate negative (positive) acceleration with decreasing (increasing) speed. We suspect that results on item ME1 are related to this difficulty, and thus do not interpret the finding as evidence that students experience inherent difficulty with negatively signed quantities (i.e., relative to positive quantities). Results overall suggest that students struggle to make sense of positive values just as they do with negative values, when the sign is an explicit part of the quantity.

Figure 2: Positive vs. Negative

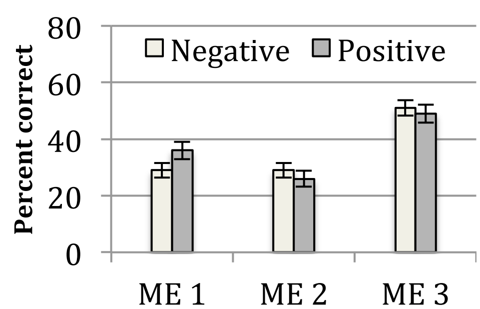
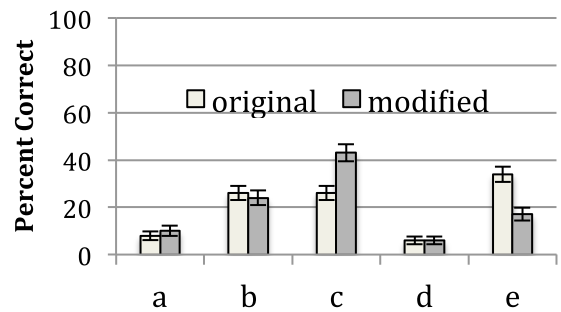


Figure 3 shows results from the original and modified versions of ME1 (see Appendix). We use these results to address the first part of the 2nd research question. Note that on the original version, choice "e" would be correct if the motion were assumed to be in the +x-direction. We note that on the modified version, given at the regional university, 17% fewer of the students selected e, with a corresponding 17% increase in the fraction selecting c, the correct answer. This result, statistically significant with a medium effect size, could indicate that students at the R1 institution tended to interpret "an object moves along the x-axis" to signify motion in the positive direction. An alternative explanation is that students at the regional university were not comparable to those at the R1, perhaps due to differences in course instruction. The statistically identical patterns of the other distractors, however, cast doubt on this explanation. We intend to replicate the experiment with samples drawn from the same population, and to present the findings as part of our talk.

Figure 3: Results from two versions of item ME1.



To address the second part of research question 2, we compared results on the original and modified versions of items EM1 and EM3 (see Appendix). No significant differences in performance were found. We conclude that although on free-response versions of the items, some students associated negatively signed quantities (i.e., quantities other than velocity) with motion in the negative direction, this association is not robust enough to distract students from other sensible choices in the MC format.

Implications for Mathematics Instruction

Based on prior research associated with negativity (Blaire et al. 2012), we suspect that understanding signed quantities increases students' cognitive load in science courses, which explains why, in the absence of a robust cognitive structure on which to hang the new signed quantities, students flounder to make meaning of the quantities and the language surrounding them.

We recognize the value of explicitly addressing mathematization with signed quantities in physics instruction. Here, however, we open a discussion of math instruction, in particular instruction between pre-algebra and calculus, with informal observations from an examination of sample math textbooks:

1. Sign and operation are often conflated using an equals sign (e.g., $5 + (-3) = 5 - 3$), and unsigned numbers are assumed positive. Adding a negative quantity and subtracting a positive one often have different meanings in science contexts. Although these operations yield the same arithmetic results, conflating them may lead students to struggle with the distinctions between sign and operation.

2. Orientation and sense are not always explicit in coordinate systems. Aligning the positive coordinate axis with the direction of motion eliminates the need for signed quantities. This, however, could be a missed opportunity to distinguish between orientation and sense. The opposite coordinate choice can prime students to understand signed numbers in science contexts.

Questions for the RUME community

1. What kinds of knowledge would math education researchers be interested in gaining from subsequent qualitative studies?

2. What audiences would be most interested in learning about this research?

Appendix

Multiple-choice versions of the assessment items and the percentage of students selecting each answer (Brahmia&Boudreaux, 2016). Correct answers are in bold.

	Unary structural signifier	Symmetrical operational sig.	Binary op. sig.
	Dir. of a vector component	Signifies work *by block* not *on block*	Position *rel. to* an origin
MECHANICS ITEMS	[a]***ME1:*** An object moves along the x-axis. The acceleration is measured to be $a_x = -8$ m/s^2. Consider the following statements about the "–" sign in "$a_x = -8$ m/s^2". Pick statement that best describes information the neg. sign conveys: a. The object moves in the negative direction. *(8%)* b. The object is slowing down*(26%)* c. **The object accelerates in the –x-direction (26%)** d. Both a and b *(6%)* e. Both b and c *(34%)*	[b]***ME2:*** A hand exerts a force on a block as block moves along a frictionless, horiz. surface. For a particular interval, the hand does W= –2.7 J of work. Consider the following statements about the "–" sign in "W = –2.7 J". *The neg. sign means:* I. the work done by hand is in neg. dir. II. the force exerted is in neg. dir. III. the work decreases the mechanical energy associated with the block Which are true? a. I only *(17%)* b. II only *(17%)* **c. III only (23%)** d. I and II only *(29%)* e. II and III only *(14%)*	***ME3:*** A cart is moving along the x-axis. At a specific instant the cart is at position x = –7 m. Consider the following statements about the "–" sign in "x = –7 m". Pick statement that best describes information the sign conveys. a. The cart moves in the negative direction *(6%)* b. **The cart is to the negative direction from the origin (67%)** c. The cart is slowing down *(6%)* d. Both a and b (*19%)* e. Both a and c *(2%)*

[a]Mod. version of ME1 at regional univ. replaced first sentence with *"An object moves in one dimension."*
[b]Mod. version of ME2 added "*IV. block moves in the neg. dir.,"* and replaced choice e with *"e. IV only"*

	Unary structural signifier	Symmetrical operational sig.	Binary op. sig.
	Dir. of a vector component	Signifies *opposite* type of charge	Potential *rel. to* a reference
E & M ITEMS	[c]***EM1:*** At a location along the x-axis, the E-field is $E_x = -10$ N/C. Consider the following statements about the "–" sign in "$E_x = -10$ N/C". Pick statement that best describes information the neg. sign conveys: a. The test charge is negative *(16%)* b. The field is being created by negative charge *(21%)* c. **The field points in the –*x*-direction (36%)** d. Both a and b *(12%)* e. Both b and c *(14%)*	***EM2:*** Valeria combs her hair and there is a transfer of charge such that $DQ_{comb}= -5$ mC. Consider the following statements about the "–" sign in "$DQ_{comb}= -5$ mC". *The neg. sign means:* I. neg. charge was added to the comb II. charge was removed from comb III. all of the charge in comb is neg. Which statements could be true? a. **I only (33%)** b. II only *(28%)* c. III only *(18%)* d. I and III only *(15%)* e. II and III only *(5%)*	[d]***EM3:*** A student uses a voltmeter to measure voltage across a battery. It reads –5V. Consider the following statements about the "–" sign in "– 5V". Pick statement that best describes information the neg. sign conveys: a. voltage is in opp. dir. as current (*32%)* b. there are 5V of neg. charge in battery *(14%)* c. voltage is in the neg. dir. *(18%)* d. **voltage at one term. is 5V less than the other (33%)** e. battery has neg. voltage *(3%)*

[c]Mod. version of EM1 at regional univ. replaced choice d with "*d. the test charge moves in the neg. dir.,"* and choice e with *"e. the electric field moves in the neg. dir."*
[d]Mod. version of EM3 replaced choice e with *"e. none of these is a valid interpretation"*

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Developing Student Understanding: The Case of Proof by Contradiction

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Proof is central to the curriculum for undergraduate mathematics majors. Despite transition-to-proof courses designed to facilitate the transition from computation-based mathematics to proof-based mathematics, students continue to struggle with mathematical proof. In particular, research suggests that proof by contradiction is a difficult proof methods for students to construct and comprehend. The purpose of this paper is to discuss preliminary results on student comprehension of proof by contradiction within a transition-to-proof course. Grounded in APOS Theory, this paper will illustrate that students' ability to negate quantification plays an early role in student comprehension of proof by contradiction.

Key words: Proof by Contradiction, Transition-to-proof course, Teaching Experiment

Proof is central to the curriculum for undergraduate mathematics majors. Despite transition-to-proof courses designed to facilitate the transition from computation-based mathematics to proof-based mathematics, students continue to struggle with mathematical proof (Samkoff & Weber, 2015). In particular, research suggests that proof by contradiction is a difficult proof methods for students to construct and comprehend (Antonini & Mariotti, 2008; Brown, 2011; Harel & Sowder, 1998). The purpose of this paper is to discuss preliminary results on student comprehension of proof by contradiction within a transition-to-proof course. In particular, this paper will address the following question: How does a student's conception of quantification affect their comprehension of proof by contradiction? Grounded in APOS Theory, a teaching experiment was designed to assess the development of student understanding of proof by contradiction. Results from a case study consisting of three teaching sessions with one student will be presented and discussed, concluding with a discussion of how quantification negation affects student comprehension of proof by contradiction.

APOS Theory and the ACE Teaching Cycle

APOS Theory is a cognitive framework that considers mathematical concepts to be composed of mental Actions, Processes, and Objects that are organized into Schemas. An *Action* is a transformation of Objects by the individual requiring memorized or external, step-by-step instructions on how to perform the operation. As an individual reflects on an Action, he/she can think of these Actions in his/her head without the need to actually perform them based on some memorized facts or external guide; this is referred to as a *Process*. As an individual reflects on a Process, they may think of the Process as a totality and can now perform transformations on the Process; this totality is referred to as an *Object*. Finally, a *Schema* is an individual's collection of Actions, Processes, Objects, and other Schemas that are linked by some general principals to form a coherent framework in the individual's mind (Dubinsky & McDonald, 2001). Utilizing the mental constructs of Actions, Processes, Objects, and Schemas, an outline of the hypothetical constructions students may need to make in order to understand a concept can be developed, referred to as a genetic decomposition. This genetic decomposition is used as a foundation to develop instructional materials.

One such pedagogical approach aligned with APOS Theory is the ACE teaching cycle; an instructional approach that consists of three phases: Activities, Classroom discussion, and Exercises. In the *Activities* phase, students work in groups to complete tasks designed to promote reflective abstraction rather than correct answers. These tasks should assist students in making the mental constructions suggested by a genetic decomposition. In the *Classroom discussion* phase, the instructor leads a discussion about mathematical concepts that the activities focused on. Students take a prominent role in this discussion while the instructor guides the conversation and presents an overview of what the students have discussed and introduces a formal, mathematical way of presenting the concept. In the *Exercises* phase, students work on standard problems designed to reinforce the Classroom discussion and support the continued development of the mental constructions suggested by the genetic decomposition. The Exercises also provide students with the opportunity to apply what they have learned in the Activities and Classroom discussion phases to related mathematical concepts (Arnon et al., 2014).

Methodology

This paper is situated in a larger research project on how students develop an understanding of proof by contradiction within a transition-to-proof course at a public R1 university in the southeastern United States. Data for this preliminary report consists of three sessions of a teaching experiment with a single student, Chandler, during summer 2016 from *Bridge to Higher Mathematics* - the first course in which students are formally introduced to mathematical proofs and their accompanying methods of proof. Chandler's understanding is similar to 5 other participants and can be considered representative of a general participant's understanding. Unlike a typical instructional sequence of ACE teaching cycle that, in a regular classroom, usually lasts for a week, this teaching experiment consisted of 5 shorter, consecutive teaching sessions each mimicking the ACE teaching cycle. That is, each session consisted of: students working on the Activity worksheet focusing on a particular component of the genetic decomposition for proof by contradiction (A); A discussion about the concepts from the worksheet (C); and a typical series of proof comprehension questions (Exercises, E) related to the content of the worksheet. A more detailed description for each session, including the role of the interviewer during each phase, is described below. In addition, Table 1 provides an overview of the content for the first three sessions with Chandler.

- Activity – Sessions began with a presented statement and proof. The interviewee would talk out how this statement and proof can be converted to propositional logic. During this phase, the interviewer acted as another student with incomplete knowledge.
- Classroom Discussion – After the Activity, the interviewer would ask the interviewee to summarize the structure of the proof. At this point, a formal structure of the proof would be given to the student, after which the interviewee would discuss how this structure compared to the one written during the Activity. During this phase, the interviewer acted as a knowledgeable agent who guided the student to make comparisons that would develop student understanding.
- Exercise – After the Classroom Discussion phase, the interviewee answered comprehension questions on their own, after which the interviewer prompted interviewee for their answers and their thinking behind the answers provided. Note that as textbooks

do not normally provide comprehension questions on proofs, the proof comprehension assessment model by Mejia-Ramos et al. (2012) was used to develop standard proof comprehension questions for the Exercises. During this phase, the interviewer acted as a knowledgeable agent to gain insight into the interviewee's thinking.

- Student's Questions – After the Exercise phase, the interviewee was encouraged to ask questions on any topic (not just those discussed in the interview).

Table 1. Overview of content per teaching experiment session

	Activity	Classroom Discussion	Exercise	Student's Questions
Session 1	Converting statements using propositional logic.	Discussion of these conversions and quantification in general.	Comprehension questions on a proof of the statement "The set of primes is infinite"	How do you negate the statement "$(\forall x)(\forall y)\big((P(x) \wedge P(y)\big) \rightarrow x = y)$"?
Session 2	Logical structure of the statement "If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes"	Discussion of procedure for proof by contradiction of implication statements.	Comprehension questions on the proof of the statement "If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes"	How do you write a proof by contradiction for the statement "$P \rightarrow Q_1 \wedge Q_2$" and similar statements?
Session 3	Logical structure of the statement "There is no odd integer than can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k."	Discussion of procedure for proof by contradiction of nonexistence statements.	Comprehension questions on a proof of the statement "There is no odd integer than can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k."	How do you write a proof for the Triangle Inequality?

Data Analysis

All sessions were video recorded and then transcribed by the interviewer. Transcripts of the three sessions went through multiple passes of analysis. First, sections of the transcripts were grouped by Activity, Classroom Discussion, Exercises, and Student's Questions for each session. Then, grouped sections were read together and general comments were made to find themes (if any) outside of the genetic decomposition. Two themes emerged from these general comments: (1) negating quantified statements and (2) procedures for proof writing. These themes were then coded throughout all three sessions. Finally, each interview was coded for evidence of developing an understanding of proof by contradiction. Due to space constraints, the rest of this section will briefly discuss how Chandler negated quantified statements and how this affected his comprehension of proof by contradiction.

Negating a statement is the first step toward writing a proof by contradiction and thus a student's conception of quantification negation directly affects proof comprehension of proofs by contradiction. On several occasions, Chandler expressed the need for explicit rules in order to negate quantified statements, indicating an Action conception of quantification negation. For example, during Session 2 Chandler discussed having a table with instructions on how to negate statements with single and multiple quantifiers, saying "Negation… I wish there was a table on how to negate things and what they look like, I don't know. There is this [points to page in textbook with rules for negating 'for all' and 'there exists' statements] for quantifiers." The interviewer then found a table online with negations of logical operators that Chandler confirmed was what he was looking for in addition to the page in the textbook.

In addition, Chandler had difficulties negating statements without first translating the statement into propositional logic and (sometimes with prompting) using rules to negate the propositional logic, again indicating an Action conception of quantification negation. For example, in Session 2 Chandler was asked to give the negation of the statement "Every even natural number greater than 2 is the sum of two primes." An excerpt of what transpired is provided below:

> CHANDLER: Every even natural number greater than 2 is not the sum of two primes? No, all? I don't know.
> [Interviewer prompts Chandler to write the statement in propositional logic]
> INTERVIEWER: Alright. So for every n here, that's how you said it to me. So parentheses for all n, in the natural numbers, if n is greater than 2, then n is p + q. So what would be the negation of this statement?
> CHANDLER: There is a natural number…
> INTERVIEWER: There is a natural number.
> CHANDLER: n greater than 2 that is not equal to p + q.

Note that at first, Chandler was unable to state the negation of the claim. However, once the claim was written in propositional logic, Chandler was able to recognize the negation of a "for all" statement. Chandler's need to convert the statement into propositional logic can be explained by his reliance on external, specific rules for negating quantifiers.

In terms of proof by contradiction comprehension, a student with an Action conception of quantification negation would need to have the logical structure of the proof explicitly written out. If the initial logical step is omitted (as they sometimes are in textbooks), he or she would not understand the proof as a whole. Chandler expressed a difficulty with an omitted negation in Session 1 with the following claim and first line of a proof:

> *Claim:* If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.
> *Proof:* Assume that every even natural number greater than 2 is the sum or two primes and that it is not the case that every odd natural number greater than 5 is the sum of three primes.

Chandler is initially unable to recognize the hidden negation step of the above proof, stating "I'm not sure when something contradicts, if you begin with 1, step 1. The assumption, or the given, was that P and Q. If you begin with P and not Q, it's already a contradiction, isn't it? No?" Once the intermediary step, "Assume it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes" and how this statement relates to the first statement of the proof "That is, assume that every even…" was presented, Chandler indicated he understood the first statement was a rewrite of the negation of the claim. This enabled Chandler to understand the logical underpinning of the proof.

Discussion

When Chandler was prompted to convert statements to propositional logic to negate and then convert back to mathematical language, he was successful in understanding (locally and globally) how statements of the proof logically followed. Converting statements to propositional logic for negation also aided Chandler in understanding the "logical leaps" of presented proofs, such as those that begin with the negated version of a statement and not with a phrase such as "Assume the statement is not true; that is …" that would indicate a proof by contradiction. While other authors have found difficulties negating quantification to affect student construction of proofs (Antonini & Mariotti, 2008; Lin, Lee, & Wu Yu, 2003), Chandler's comprehension after logical quantification suggests that a student's ability to negate quantification plays an early role in student comprehension of proof by contradiction.

Future Plans

Data collection and analysis will continue in spring 2017. A major goal for this semester will be to refine the preliminary genetic decomposition for proof by contradiction that guides the design of this study, provided below.

Preliminary Genetic Decomposition for Proof by Contradiction

1. Students outline the propositional logic of a given proof to develop specific step-by-step instructions to construct proofs by contradiction for the following types of statements: (i) implication, (ii) single-level quantification, and (iii) property claim.
2. Students interiorize each of the Actions in Step 1 individually by examining the purpose of statements of given proofs. These Processes become general steps to writing a proof by contradiction for statements of the form (i), (ii), and (iii).
3. Students coordinate the Processes from Step 2 by comparing and contrasting the general steps to determine the necessary steps for *any* proof by contradiction. This Process becomes general steps to writing any proof by contradiction and identifying a proof as a proof by contradiction.
4. Students encapsulate the Process in Step 3 as an Object by utilizing the law of excluded middles to show proof by contradiction is a valid proof method. Students can now comprehend proofs on a holistic level.
5. When necessary, students de-encapsulate the Object in Step 5 into a Process similar to Step 3 that then coordinates with a Process conception of quantification to prove multi-level quantified statements.

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Defining Functions: Choices That Affect Student Learning

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Textbook authors and instructors choose how to define the concept of function for students. This study examines the impact of definition choice on the mathematics work of graduate students, all of whom were mathematics majors and most of whom are in-service mathematics teachers. Data are student work on tasks requiring the application of different textbook definitions of functions. By drawing on ideas about action vs. object conceptions of function, it is hypothesized that certain linguistic features of definitions may affect students' abilities to use the definition and to build a robust concept image of function.

Key words: Definitions, Functions, Curriculum

Undergraduate mathematics programs must prepare future mathematics teachers to teach functions, a central topic in the secondary mathematics curriculum in the United States. Functions account for one of just six conceptual categories in the Common Core State Standards for Mathematics (CCSSM) for high school (Common Core Standards Initiative, 2010) and are a linchpin for much of the undergraduate mathematics curriculum. However, high school teachers often have a shaky understanding of the concept of function, exhibiting disconnected knowledge that is more procedural than conceptual (Doerr, 2004). Not surprisingly, their students are similarly challenged in their understanding of function (Kieran, 2007). This study investigates the impact of the choice definitions of functions in introductory algebra textbooks.

As shown below, different textbooks present definitions of functions that vary in their wording and are not always mathematically equivalent. These wording choices by textbook authors may have consequences in K-16 mathematics education. For example, the CCSSM's description of the standards for mathematical practice (i.e., what it means to *do* or *practice* mathematics) note that "mathematically proficient students understand and use stated assumptions, *definitions* [emphasis added], and previously established results in constructing arguments" (CCSI, 2010, p. 6). Accordingly, the big question which underlies this study is: How does the choice of definition of function impact learning and understanding? Unpacking this question, the study focuses on the questions:

1. Are some definitions more "usable" than others?
2. Does a definition respect what we know about students' knowledge development?

These questions are investigated by examining the work of in-service mathematics teachers, all of whom have undergraduate mathematics degrees, as they use different definitions of functions from high school textbooks. The preliminary results reported below indicate that, indeed, choice of definition matters.

Theoretical Framework

Understanding Functions

Much has been written about understanding of functions (for an overview see Kieran, 2007). For present purposes, the focus is to distinguish between (1) someone who sees a function as instructions to do something, and (2) one who sees a function as an object. There are several frameworks which explicate the broad idea behind this division. Sfard and Linchevski (1994), for example, described a theory of reification in which learners of mathematical concepts first hold an operational conception (process-oriented). They describe reification as "our mind's eye's ability to envision the result of processes as permanent entities [objects] in their own right" (p. 194). A learner's concept of function, through reification, progresses from operational to structural/object-oriented. APOS theory describes a similar trajectory for an individual's mental construction of a mathematical concept in terms actions, processes, and objects organized in schema (Dubinsky & McDonald, 2001). In the context of functions, an action conception may be to think of a function as a recipe (given by a mathematical expression) applied to some number. For someone with a process conception, the action is "interiorized" through repetition and reflection so that it can be thought about more abstractly (i.e., without evaluating the expression which determines the function by "plugging in" a number). Someone with an object conception can operate on functions as if they were indeed objects. Such a person may, for example, understand functions as a special type of mathematical relation, a set of ordered pairs. Thompson (1994) describes the object level as entailing "an image of functional process as defining a correspondence between two sets: a set of possible inputs to the process and a set of possible outputs from the process" (p. 27).

For the purposes of this study, the action/process vs. object distinction is sufficiently granular. In particular, for the present analysis, no distinction is necessary between action and process conceptions. The distinction between action/process and object conceptions is viewed as hierarchical with an action preceding object conceptions of function, as Thompson (1994) contended. Though, as acknowledged in APOS theory, an individual's progression through these stages may be messy and nonlinear. Furthermore, Sfard (1991) notes that "the ability of seeing a function or a number both as a process and as an object is indispensable for a deep understanding of mathematics, whatever the definition of 'understanding' is (p. 5)." In the spirit of Sfard, a definition of "understanding" is not attempted. Instead, the focus is the distinction between an action/process conception and an object conception. Paraphrasing Ponce (2007), there is a difference between a student who thinks of *a function does something* and one who sees *a function as a thing*?

U.S. High School Textbooks and Definitions of Functions

Herein, the action/process vs. object distinction will be related to linguistic features of textbooks' definitions of functions and students' work using the definitions. Textbooks are conceptualized as curricular materials in accordance with Remillard's (2005) definition, "printed, often published resources designed for use by teachers and students during instruction" (p. 213). The study is framed by Remillard's description of curriculum use as a participatory relationship that is influenced by characteristics of the individual (beliefs, pedagogical content knowledge, etc.) and by features of the curriculum (voice, representation of concepts, etc.). The curricular materials of relevance to this study are three U.S. high school introductory Algebra textbooks, specifically their treatment of functions. The textbooks were selected as a convenience sample; these were readily available books on the researcher's shelf. The main goal was to capture a variety of definitions, and these textbooks provided that. The following are the textbooks' definitions of function (emphasis added):

- Definition 1. (*CME Project Algebra 1)* "A function is a *rule that* assigns each element from the set of inputs to exactly one element from the set of outputs" (Center for Mathematics Education, 2009, p. 421).
- Definition 2. *(McDougal Little Algebra 1 California)* "A function consists of:
  A set called the domain containing numbers called inputs, and a set called the range containing numbers called outputs… A *pairing of* inputs with outputs such that each input is paired with exactly one output." (Larson, Boswell, & Kanold, 2006, p. 35).

- Definition 3. *(Holt Algebra 1)* "A relation is a set of ordered pairs. The domain of a relation is the set of the first coordinates. The range of a relation is the set of second coordinates. A *relation that* assigns to each value in the domain exactly one value in the range is called a function" (Burger, 2008, pp. 206–207).

According to these definitions, a function may be thought of as a rule, pairing, or relation; that is, a function can be thought of as three different *types of objects*. Other differences are apparent as well. Definition 3 describes first and second coordinates, the other two describe inputs and outputs. Definition 2 is not equivalent to the other definitions as it defines the inputs/outputs of functions to be numbers; the others are less restrictive. Thus, without much effort, it is easy to find differences in the way high school textbooks define this essential mathematical object. This study focuses on the *type of object* that a function is defined as and how that may help learners negotiate between action/process and object levels of understanding.

Methods

Mathematics graduate students were asked *to use* each of the three definitions of functions to do two tasks:

Task 1. Show that the equation $y = x + 1$ determines y as a function of x.

Task 2. Show that sequences are functions.

The research subjects were 23 graduate students in a master's program in mathematics (option in secondary mathematics education), all of whom had undergraduate degrees in mathematics or mathematics education and were credentialed secondary mathematics teachers. All but three were either currently teaching or had previously taught middle or high school mathematics. The research subjects' statuses as current or future teachers and mathematics majors are of relevance to the study's implications; participants may be viewed dually as students and teachers of mathematics.

Operating under the hypothesis that certain features may make a definition more "usable" for students and teachers, "using a definition" was operationalized by checking if the type of object was somehow referenced in a student's response. For example, to use Definition 1 a student would need to describe how the given equation or an arbitrary sequence determines a "rule". Of particular interest was whether the student used a verb form of the object. For example, in using Definition 2 (function as a pairing), did the student say something like, "A sequence *pairs* integers with terms." Data are being coded for whether the student used the definition, how s/he used it (e.g., using a verb form of the object), and whether s/he used it correctly.

This activity was assigned prior to any treatment of functions in the course. The nature of prior instruction about or experiences with functions was not probed. Though, the participants' undergraduate mathematics degrees were assumed to be preparation for answering the questions.

Preliminary Results

There is compelling preliminary evidence that students have more success if function is defined as an object which has an accessible verb form. This is exemplified by the work of the following student (a graduate student and high school mathematics teacher). In using Definition 1, she never explicitly stated how the equation in Task 1 determines a rule and instead focuses on actions (e.g., plugging in). In using Definition 2, she used the verb form of the noun/object "pairing" to describe how inputs are "paired with" outputs and provided a more complete answer.

- Using Definition 1 (function as rule): "The equation $y = x + 1$ determines y as a function of x because for each value that you plug into x, you will produce exactly one value for y.

For example, if you plug in 2 for x, you will get 3 for y."

- Using Definition 2 (function as pairing): "For each pairing of inputs and outputs, each input value is paired with exactly one output value. For example, if the domain is {–2, 0, 5} then the range is {–1, 1, 6} and the pairing of inputs and outputs would be {(–2, –1), (0, 1), (5, 6)}. Each input is paired with exactly one output."

This student's response for Definition 1 never addressed why or how the equation determines a rule. Yet her response for Definition 2 indicates that she does, in fact, understand the conventions for using a definition. By making use of the verb form of the noun "pairing," she explicitly connected her explanation to given definition.

In using Definition 2, participants had a preference for using the verb construction "is paired with" to explaining why the equation $y = x + 1$ is "a pairing." Furthermore, the definition itself makes use of both forms, perhaps modeling flexible language use for students. Though a verb form may be available for the noun/object "relation" in the form of "to relate," only one student used it one time whereas the phrase "paired with" appears in 30 out of the 46 student responses for Tasks 1 and 2 using Definition 2. It is hypothesized below that access to a verb form may help students coordinate between action/process (verb) and object (noun) conceptions of function.

There is indication that Definition 1 (definition as rule) is particularly challenging. Only 10 students correctly used the definition to answer Task 1, and two of those answers showed some confusion. Data are still being analyzed, but will be probed too see if the lack of a readily accessible verb form of the word *rule* may have played a role – the verb *to rule* certainly wouldn't have helped.

At least three other themes have emerged in the initial analysis that will be investigated through the data. First, students did very poorly in explaining why sequences are functions regardless of the choice of definition. Second, students' use of future tense may offer further evidence of a use of an action conception of function. For example many students wrote something similar to this student's explanation: "By substituting a value of x from the domain into this equation, one will find a value of y." That is, given an input, the function performs an *action* that will result in an output. In the first example of student work in this section, the future tense was used with Definition 1 (rule) but not Definition 2 (pairing). Third, many students displayed confusion about the difference between sets and elements of a set (e.g, "A sequence is a rule with each element in the input assigns exactly one element in the output."). Other students referred to x and y as sets. Such confusion could hinder use of definitions.

Discussion

This is a preliminary report, but it offers some evidence that the definitions we choose matter. It has implications for the undergraduate preparation of mathematics students in general and future mathematics teachers in particular. Students who have action/process conceptions of functions may be challenged by definitions which do not accommodate the use of action words. The connections between an action/process conception & verb and between an object conception & noun may be more than just a linguistic novelty. For example, there is not a convenient verb form for noun "rule" as there is for the noun "pairing." This subtle difference may make a difference in a student's ability to use a definition and negotiate the transition to an object conception of function. Perhaps making sense of Definition 1 (function as rule) is more likely to require a student to have an object conception of function. Though a verb form may be available for the noun/object "relation" in the form of "to relate," only one student used it one time whereas the phrase "paired with," wording used in the definition, was commonly used. At this point in the analysis, it seems like students have more success when using action verbs.

The abstract ideas of relations, sets, and rules may pose challenge for students who do not yet have object-level conceptions. It is noteworthy that these definitions came from an *introductory* high school algebra textbook, yet presented challenges for many mathematics majors. The study raises questions about choices of definition in introductory courses at all levels. Perhaps we should heed Thompson's (1994) call for "curricula that are mathematically sound, but nevertheless are constructed from the start with an eye to building students' understandings" (p. 40). It is not enough for a definition to merely be mathematically correct – it must also accommodate beginning students' action-level mathematical conceptions.

There are, of course, many other variables at play in this investigation. For instance, especially in the case of Task 2 (about sequences), many participants' lack of mathematical knowledge or fragile concept images obfuscated the role of the definitions' wording. Also, the use of language related to sets may have played some role in participants' abilities to use the definition. Ongoing data analysis will further reveal the extent to which word choice in definitions may help or hinder the transition from action/process to object-level conceptions of functions.

Analysis and interpretation of these data and directions for future research will be guided by audience discussion prompted, in part, by the following questions:

1. Regarding the emergent themes (language about sets, use of future tense, sequences as functions), which of these are worth further investigation? How do they fit in with the main idea of thinking about definition use by students with action-level conceptions?
2. In general, these participants (as math majors) knew the conventions of using a definition. What role is that playing? How does this impact the generalizability of the results to beginning students?
3. What are the implications for secondary mathematics teacher preparation programs? For definition choice in general?

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Students' Understanding of Test Statistics in Hypothesis Testing

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Hypothesis testing is a key concept included in many introductory statistics courses. Due to common misunderstandings of both scientists and students, the use of hypothesis testing to interpret experimental data has received criticism. This paper describes preliminary results obtained from a larger study designed to investigate introductory statistics students' understanding of one sample hypothesis testing. APOS theory is used as a guiding theoretical framework. Preliminary data analysis focused on two students' distinctions between test statistics when performing hypothesis tests on real world data. The results suggest a significant difference in these two students' understanding, one being identified having an action conception while the other had an object conception of hypothesis testing as situated in the study.

Key Words: Hypothesis Testing, Statistics, Test Statistics

Introduction

The use of statistics is crucial for numerous fields, such as business, medicine, education, and psychology. Due to its importance, statistics education has seen rapid growth over the past three decades (Vere-Jones, 1995). In the United States today, the Common Core State Standards for Mathematics calls for students to "understand statistics as a process for making inferences about population parameters based on a random sample from that population" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 81).

One method of making inferences (formulating a conclusion) about a population is hypothesis testing, which is widely used by researchers in the social sciences and is a key concept included in many introductory statistics courses. However, the use of hypothesis testing to interpret experimental data has received criticism (Nickerson, 2000) due to the common misunderstandings of both scientists and students when using this method (Batanero, 2000; Dolor & Noll, 2015; Vallecillos, 2000). LeMire (2010) defends the use of hypothesis testing and provides a framework that can be used to revise instructional content with the goal of further developing student understanding. This is an indication that on-going research should investigate students' understanding and curriculum effectiveness in light of the critiques surrounding methods of inference.

In this preliminary research report, we focus our attention on the following research question: What are students' understandings of hypothesis testing in two distinguished real world situations?

Theoretical Framework

The guiding theoretical framework for our larger study is APOS Theory (Asiala et al., 1996). APOS Theory is a framework which models an individual's mathematical conception using Actions, Processes, Objects, and Schema. An Action is an externally driven transformation of a mathematical object (or objects). An Action can be described as an individual needing an external cue to follow, such as a step-by-step example. Once Actions are repeated and reflected on, an individual can start to interiorize them to become a Process. A Process no longer requires step-by-step external cues. An individual is now able to internally imagine the steps in a transformation, without having to actually perform them. When an individual is then able to see the Process as a totality, is aware that transformations can be applied to it, and can construct these transformations, then the Process has been encapsulated into an Object. The collection of all mental constructions of Actions, Processes, and Objects forms an individual's Schema of a particular mathematical concept.

Methodology

The focus of our study is on university students who are enrolled in an introductory statistics course based on the emporium model. The emporium model, originated at Virginia Tech, includes key components of "interactive computer software, personalized on-demand assistance, and mandatory student participation" (Twig, 2011, p. 26). For this particular institution, each week students were required to spend three academic hours in a computerized mathematics lab, as well as attend one academic hour class each week with an instructor. The time in the mathematics lab was spent actively learning using the mathematical software MyStatLab by Pearson. Students were also engaged in activities such as reading and discussing about the subject content with their peers, graduate and undergraduate lab assistants, and instructors.

Elementary Statistics Using Excel was the textbook used in the course, written through Pearson and adapted specifically for the university (Triola, 2014). The textbook describes a test statistic as "a value used in making decisions about the null hypothesis," (p. 415). While assuming the null hypothesis is true, a test statistic is found by converting a sample statistic, whether that is a sample proportion or a sample mean, to a standardized score. As students for the course are required to calculate a p-value for most hypothesis tests, the text describes the p-value as the "probability of getting a value of the test statistic that is at least as extreme as the one representing the sample data, assuming that the null hypothesis is true," (Triola, 2014, p. 416). Although we discuss test statistics as one mathematical term or value, there is a distinction needed to be noted between test statistics calculated from sample proportions versus sample means. Specifically, in our study, the distinction needs to be made between the normal distribution and the Student's t distribution.

For proportions, students always assume a normal distribution, and thus calculate test statistics which are z-scores. When calculating a test statistic for a sample representing a population mean, we are referring to a standardized value that represents the extremeness of your sample in regards to what is expected. For means, students learn about test statistics in hypothesis testing for the normal distribution (z-scores) and the Student's t distribution (t-scores). Although these distributions appear similar, the distinction occurs depending on what we know about our sample. In particular, if we know the population standard deviation, then we know how the data is spread out and can use the normal distribution (z-scores). However, if we do not know the population standard deviation, but can estimate it with the sample standard deviation, we can use the Student's t distribution to estimate how the population is spread. For this reason, t-scores are greater than or equal to z-scores for the same value of n (equal as n approaches infinity) in order to overcompensate for the lack of knowledge of the distribution (see Figure 1).

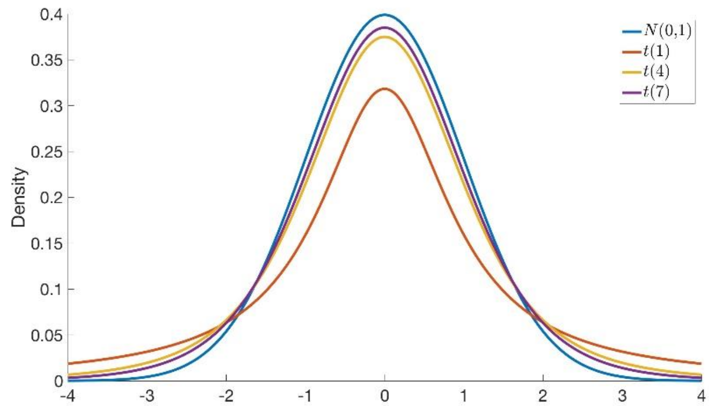

Figure 1. Graphical Representation of Normal Distribution and Student's *t* Distribution

Data was collected from these classes during the Fall 2014 and Spring 2015 semesters. Data consists of all students' work (more than 1,500 students) on hypothesis testing, including homework, quizzes, and tests. Semi-structured interviews took place with a targeted group of students of different capabilities. For this preliminary report, we will focus on two of the nine problem solving/interview sessions as the primary source of data. During the problem solving sessions, each participant worked alone on two hypothesis test questions. Following this, they participated in a semi-structured interview to further elaborate over the problems they solved. The first question asked the student to conduct and interpret a hypothesis test for a single population proportion. The second question asked the student to conduct and interpret a hypothesis test for a single population mean. The questions were as follows:

1. In a recent poll of 750 randomly selected adults, 588 said that it is morally wrong to not report all income on tax returns. Use a 0.05 significance level to test the claim that 70% of adults say that it is morally wrong to not report all income on tax returns. Use the *P*-value method. Use the normal distribution as an approximation of the binomial distribution.
2. Assume that a simple random sample has been selected from a normally distributed population and test the given claim. In a manual on how to have a number one song, it is stated that a song must be no longer than 210 seconds. A simple random sample of 40 current hit songs results in a mean length of 231.8 seconds and a standard deviation of 53.5 seconds. Use a 0.05 significance level to test the claim that the sample is from a population of songs with a mean greater than 210 seconds.

Students had seen these exact questions on their homework and quizzes when using the MyStatLab software. The only difference was that for the problem solving/interview sessions they did not have multiple choice/drop down menus as an option to answer the questions. Since there was active learning associated with these concepts in the mathematics lab and in class, students were expected to know how to conduct and interpret hypothesis tests for both questions, and in particular, they were expected to know to use the normal distribution to find the test statistic for Question 1 and the Student's *t* distribution to find the test statistic for Question 2.

Preliminary Results

Our preliminary results indicate that students are not always able to make distinctions between test statistics. Several students used the normal distribution to find the test statistic for not only Question 1, but also for Question 2. For those that did use the normal distribution when calculating the test statistic for Question 1 and the Student's t distribution for Question 2, some still could not articulate why they used two different test statistics or the relationship between the two test statistics. A common approach to finding the test statistic was to first look for key words, which suggests an Action conception of test statistic. This worked for some, however, others got confused with the language in the problems. For this preliminary report, we will describe the different approaches utilized by two students, Steve and Lana, and discuss the interpretations that helped them decide which test statistic to use.

Steve

For Question 1, Steve used the common approach of identifying key words to recognize which test statistic to use. He explained, "I immediately thought of these two formulas, and at first I wasn't sure which one to use, and then I was like, oh wait, there's no x-bar or mu or standard deviation. So that makes it pretty easy." He used a system of elimination to decide which formula not to use. Even though he correctly identified the test statistic, he went on to say that this is "just a formula that I've learned like any other" and that he "doesn't understand why we use that formula, other than we just use it". He concluded his explanation stating that Question 1 used a z-value because the problem was of proportions, what appears is likely a memorized rule applied to the problem.

For Question 2, Steve recognized that it was now a question of means, but mentioned that "it's basically the same problem" other than this discrepancy. Steve identified the distribution as normal because the question explicitly stated that "a simple random sample has been selected from a 'normally distributed' population". Based on the language in the problem, Steve associated the question with a normal distribution. He further explained that he would not know how to deal with a non-normally distributed population, that he could not recall learning anything other than a normal distribution, and that he did not know much about a Student's t distribution. Ironically, later in the interview, when asked if his answer was a z-value or t-value, he responded that it is "a t-value because you're testing means".

For Steve, it appeared as though he was basing his ideas of whether to use the test statistic associated with the normal distribution or the test statistic associated with the Student's t distribution off of key words found in the problems. When the problem for Question 2 included the phrase of "normally distributed", he used this to identify the question as a normal distribution. What could have been a result of memorized rules, led to a misconception of when to use the z-test and t-test. Steve appears to exhibit an Action conception of the test statistic as he relied on external cues, such as the key words in the problem, to identify which test statistic to use.

Lana

Lana, for Question 1, used the normal distribution to find the test statistic based on the fact that the problem was about proportions. To explain the test statistic, she initially explained how she was picturing the "big curve, the bell curve, and I'm picturing the test statistic is where the point that falls on there … so this is the mean right in the middle, and the test statistic is one side of it, saying this is how far away from what they are saying is the mean, this is what the mean of this, I guess that's what I'm thinking". Comparison of the mean and test statistic possibly indicated an

Object conception of test statistic. After her explanation, she then drew a picture to illustrate her thinking (Figure 2) of a graphical representation of the normal distribution.

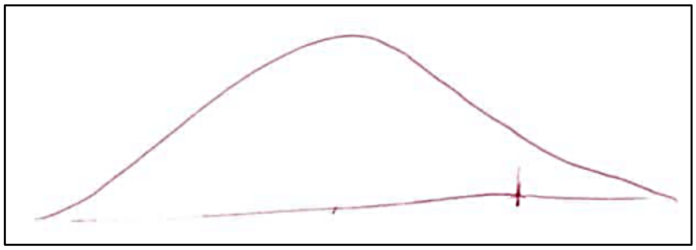

Figure 2. Lana's Graphical Representation of the Normal Distribution

For Question 2, initially, Lana in her written work used the normal distribution to find the test statistic. During the interview she became worried when asked if the question was a *z*-test. She mentioned that she remembered from high school to use a *t*-score if the sample is of 30 or less. After prompting, she realized she should have used a *t*-score, however, she also immediately recognized that her *z*-score would "probably not" be very different from the *t*-score. She knew that in the relationship between the two test statistics, "they are really close together".

Lana was able to picture in her head a graphical interpretation of the test statistic, and was also able to identify a relationship between the two test statistics, suggesting an Object conception. This was more than what most other students were capable of interpreting.

Concluding Remarks

Our data suggests that one area students have trouble with when conducting hypothesis tests on real world data is correctly identifying and interpreting which test statistic to use. Students with an Action level of understanding of hypothesis testing relied on memorizing facts and identifying key words. Without a deeper understanding, misconceptions emerged. Students who appear to base their understanding off of ideas and concepts, not memorized rules, seem to have a better grasp of the relationship between the two test statistics, and when and why to use each one. It is also suggested that not having the MyStatLab multiple choice/drop down solutions to choose from could have possibly influenced the students' responses in the analysis. The next step in analysis will be to further identify misconceptions and understanding related to hypothesis testing as a whole, not just the test statistic in particular.

Questions for the Audience

1. We have observed that the way students use certain mathematics software can influence students in developing a list of 'hints and shortcuts' for how to approach and solve certain types of problems. Does anyone in the audience have similar observations? How do we lower or eliminate this happening in our classes?
2. Does having multiple choice affect your students understanding of concepts? Do you observe your students trying to 'manipulate the system' to get correct answers instead of putting in the same amount of effort to understand the concept? How do you combat this issue?
3. Does the use of Excel versus other types of technology (calculators, statistical programs, etc) make a difference in the learning of students?

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Spatial Training and Calculus Ability: Investigating Impacts on Student Performance and Cognitive Learning Style

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Despite concerted efforts on the part of educational policy makers, women are still underrepresented in the STEM fields. Researchers have shown that calculus plays a major role in this gender disparity since it requires spatial skills to succeed: skills that women tend to lack compared to men. However, previous studies have shown that spatial ability is malleable and spatial skills can be improved with training. This pilot study employed spatial training in a third-term calculus course and measured the effects of this training on students' calculus ability, spatial rotation ability, and cognitive learning style. Associations between cognitive learning style and task performance were also measured. Preliminary results indicate that spatial training does not significantly impact student performance on a calculus skills assessment or a test of mental rotations, but effects on students' cognitive learning style are present.

Key words: spatial training, calculus skills, cognitive learning style

Introduction & Literature Review

While the call for more graduates in Science, Technology, Engineering and Mathematics (STEM) continues (Executive Office of the President, PCAST, 2012; Stieff & Uttal, 2015), women remain consistently underrepresented in these fields (Schlenker, 2015). The major cause of this gender gap may concern a sequence of courses in calculus. Calculus is universally required for STEM majors and, therefore, often acts as a gatekeeper to student success and continuation into STEM careers (Bressoud et al., 2015). However, women are 1.5 times more likely to change to a non-STEM career pathway after a calculus course than men (Ellis et al., 2016). Women may struggle with concepts in calculus because they rely heavily on visual representations: females to have been shown to have less developed spatial abilities (Ferrini-Mundy, 1987) and to underperform on spatial tasks compared to males, starting as young as four years of age (Levine et al., 1999; Voyer, Voyer, & Bryden, 1995). Other studies indicate that individuals with higher scores on spatial tests are more likely to enjoy STEM subjects and to choose careers in STEM (Wai, Lubinski, & Benbow, 2009).

While the body of literature seems to suggest that women may have a disadvantage in the STEM fields, encouraging evidence exists that practice with spatial concepts can improve spatial ability (Uttal, 2009; Stieff & Uttal, 2015) and that spatial training can decrease the gender gap in spatial thinking (Newcombe, 2010). We refer to this idea in the paper as *spatial training,* which we define as explicit instruction and practice on spatial skills such as rotation, planar views or unfolding of an object. At this time, knowledge about the benefits of spatial training is not conclusive concerning whether an increase in spatial ability has direct effects on performance in the STEM fields. A study conducted with engineering students found an association between spatial training and academic performance, as well as a closing of the gender gap (Sorby et al., 2013). However, very few studies exist that investigate the link between spatial training and mathematical skills. One study by Ferrini-Mundy (1987) required undergraduate students to complete spatial training exercises during a calculus course but did not find significant increases in calculus performance (although female students were better able to visualize solids of revolution than male students). Other research has found that while spatial training does reduce the gender gap in performance on spatial tasks, it fails to eliminate it (Uttal, 2009). Thus, a call has been made in the spatial cognition literature to extend this line of enquiry into the potential for spatial training to close the gender gap in the STEM fields by investigating new mediating variables and extending periods of spatial training (Casey, 2013).

One factor that may be associated with spatial and mathematical ability is the psychological construct of cognitive learning style. A cognitive learning style represents a coherence of a person's manner of cognitive function (i.e., information acquisition and processing) (Harvard Mental Imagery and Human-Computer Interaction Lab, 2013). Because the human visual system distinctly processes properties about objects (color, shape) and space (location and spatial relations), Kozhevnikov, Kosslyn, and Shephard (2005) have used neuropsychological evidence to propose the Object-Spatial-Verbal theoretical model of cognitive style. This model outlines three independent dimensions (object imagery, spatial imagery, and verbalization) to explain that object visualizers prefer to construct vivid, concrete, and detailed images of individual objects, spatial imagers schematically represent spatial relations of objects and spatial transformations, and verbalizers prefer to process and represent information verbally and rely on non-visual strategies (Kozhevnikov, Kosslyn, & Shephard, 2005).

It seems that preference for one of the three strategies has been shown to directly relate to one's performance on either mathematical, object imagery ability, or spatial ability tests (MM Virtual Design, 2016). Thus, cognitive learning styles may assist math educators to tailor material, assignments, and visualization media to students' individual differences in cognitive learning style and decision-making based on visual stimuli. Learning and performance based on visual information presented in a manner congruent to one's cognitive learning style may help close the gender gap in STEM. Indeed, Peters et al. (1995) found that women report using verbal strategies to solve mental rotation tasks more often than men and Casey (2013) points out that one reason why large gender differences are found for mental rotation tasks is because verbal strategies are often less effective than holistic mental rotation approaches used more often by men. Thus, measuring cognitive learning style in association with spatial ability and an understanding of calculus may afford information about whether those with predominant verbal cognitive learning styles are women, as well as whether their performance in calculus improves with spatial training.

This study targets the lack of literature investigating effects of spatial training on mathematical skills and seeks to answer the following research questions: (1) What are the impacts of spatial training on undergraduate students' performance in calculus? (2) Are differences present in the effects of spatial training between male and female students? And (3) What are the impacts of spatial training on students' cognitive learning style?

Methods

Context

The study took place at a mid-sized state-funded university in the northwestern United States in a summer 2016 quarter. All student participants were enrolled in a third-quarter calculus course covering the calculus of sequences and series, vectors equations, and multi-variable functions. The course was taught using an inquiry-based learning pedagogy; spatial training was incorporated during class time.

Spatial training was conducted for, on average, 10 minutes during every class meeting (twice per week for 10 weeks). Students completed exercises from a spatial training workbook developed by Sorby et al. (2013; and used with permission). Exercises in the workbook ranged from assessments of what a given shape would look like when rotated around a given axis, to asking students to draw an object from different angles using different cross-sections, to showing 2-D views of an object and asking students to draw the 3-D object. During the spatial training portion of class, students were asked to discuss the exercises in small groups and come to a consensus on the correct answer before answers were discussed with the whole class.

Participants

Participants had already completed two quarters of undergraduate calculus and spent an average of 12 hours a week (SD = 3.70) studying course material. Seventeen students (8 males, 9 females) attended class for both rounds of data collection and all but one took part in the study (n = 16; 8 males, 8 females, mean age = 21 years). Five (31%) participants reported enrollment in another math course during the quarter while 11 participants (69%) reported that they were not receiving other forms of math training at the time of the study.

Data Collection

Participants were given three separate instruments in a pre-post data collection model that measured spatial ability, calculus ability, and preferred cognitive learning style. The measurement of spatial ability was obtained using a 15-item version of the Purdue Spatial Visualization Test: Rotations (PSVT:R) (Guay, 1977). This multiple-choice instrument gives an example of an object and a rotation of the same shape and asks participants to choose (from 5 possibilities) the result of the same rotation on a new object. The PSVT:R was used to obtain a baseline measure of spatial ability and to quantify improvements from the spatial training. The Calculus Concept Inventory (CCI) was administered to measure calculus ability. This instrument, designed by Epstein (2013), measures students' understanding of the basic concepts of differential calculus. The CCI contains 22 questions about limits and derivatives, many of which require interpretation of a graph or visual aid. The Object-Spatial Imagery and Verbal Questionnaire (OSIVQ) developed by Blazhenkova and Kozhevnikov (2009) was used to measure students' predominant cognitive learning style. The OSIVQ consists of 45 questions to assess object imagers, spatial imagers, and verbalizers and takes approximately 10 minutes to complete. Each item asked students to rate their agreement on a 5-item Likert scale ranging from "totally disagree" (1) to "totally agree" (5) with statements of preference or ease of performing various tasks.

Additionally, basic demographic questions, such as age, gender, and the number of courses related to mathematics taken at the post-secondary level were asked. To gain an understanding of students' concurrent exposure to mathematical and spatial concepts outside of the training, and the course in general, students were asked to report whether they were receiving (or intended to receive) alternative instruction or tutoring during the term (e.g., enrollment in a different math course or spending time at the learning and teaching center), as well as the number of hours of this additional instruction per week.

All instruments were administered to participants during class time and no significant time pressure was placed on the students. Tasks were administered to participants in the following order: consent form; OSIVQ; Visualization and Rotation Purdue Spatial Visualization Test; Calculus Concept Inventory. Thus, the tasks were not completed simultaneously: only when a task was completed was the next task offered to a participant by one of the authors. No calculators or other electronic devices were used during task completion.

Results

The CCI and PSVT:R were scored by assigning 1 point for a correct answer and 0 points for an incorrect answer. Descriptive statistics for each inventory separated by gender are given in Table 1. To answer our first research question, students' pre- and post-test scores were computed and a paired-samples *t*-test was performed to determine significant improvements in calculus ability. No significant improvement was revealed over the term ($p > .05$). While male participants' average scores were higher on the CCI than women at the start of term ($M = 9.63$, $SD = 4.47$ and $M = 7.75$, $SD = 7.75$, respectively), they were not significantly higher ($p > .05$). This result was also borne out at the end of term whereby male students' average scores on the CCI were insignificantly higher than female students' scores ($M = 10.25$, $SD = 2.09$ and $M = 8.25$, $SD = 1.40$, respectively).

Table 1:
Descriptive Statistics for Test Variables Per Gender Type

Variable	Gender	Mean *Round 1*	Standard Deviation *Round 1*
CCI (scored out of 22)	Male	9.63	4.47
	Female	7.75	3.45
PSVT:R (scored out of 15)	Male	9.88	4.29
	Female	9.75	2.05
OSIVQ: Spatial (scored out of 75)	Male	49.75	2.74
	Female	46.13	3.44
OSIVQ: Object (scored out of 75)	Male	47.00	2.67
	Female	50.13	2.26
OSIVQ: Verbal (scored out of 75)	Male	42.75	1.40
	Female	39.50	1.58
		Round 2	*Round 2*
CCI (scored out of 22)	Male	10.25	2.09
	Female	8.25	1.40
PSVT:R (scored out of 15)	Male	10.5	1.45
	Female	9.25	1.24
OSIVQ: Spatial (scored out of 75)	Male	54.00	2.19
	Female	47.50	3.70
OSIVQ: Object (scored out of 75)	Male	52.38	3.42
	Female	49.25	2.04
OSIVQ: Verbal (scored out of 75)	Male	45.38	0.80
	Female	40.13	1.42

To determine an association between students' spatial ability and mathematical ability, a correlation coefficient for a student's final grade in Calculus III and their final PSVT:R score was calculated. This correlation ($r = 0.4283$) was not significant ($p > .05$). Additionally, a paired-samples *t*-test did not determine a significant improvement in students' rotational ability during the term ($p > .05$). Consistent with previous studies (Levine et al., 1999; Voyer, Voyer, & Bryden, 1995), male participants' average scores were higher on the PSVT:R than women at the start of term ($M = 9.87$, $SD = 4.29$ and $M = 9.75$, $SD = 2.05$, respectively), and at the end of the term ($M = 10.5$, $SD = 1.45$ and $M = 9.25$, $SD = 1.24$, respectively). However, these differences were not significant ($p > .05$).

On average, more students at the start of the term self-identified as object learners ($M = 48.56$, $SD = 7.00$) than they did as spatial learners ($M = 47.94$, $SD = 8.69$) or verbal learners (M

= 41.13, SD = 4.41). Indeed, scores on the spatial sub-scale were significantly higher than on the verbal sub-scale, $t(15) = 3.97$, $p = .001$. In addition, scores on the spatial sub-scale were significantly higher than those on the object learner sub-scale, $t(15) = -2.98$, $p < .01$.

After 10 weeks, the object cognitive learning style remained the predominant style for the class as a whole (M = 50.81, SD = 7.87) and, similar to the start of term, the second-most common learning style among the class was spatial (M = 50.75, SD = 8.96), followed by verbal (M = 42.75, SD = 4.16). However, unlike at the start of the term, these differences were significant: students sscored significantly higher on the object learning style sub-scale compared to the verbal learning style sub-scale, $t(15) = -3.57$, $p < .01$. They also self-scored significantly higher on the spatial sub-scale compared to the verbal sub-scale, $t(15) = -3.55$, $p < .01$.

Although mean scores on each of the three sub-scales increased over the term, paired-samples t-tests revealed only one significant difference between sub-scale scores over time. Students did not self-score significantly better or worse on the object or verbal subscales over time (all ps > .05). They did self-score significantly higher on the spatial sub-scale at the end of the term after receiving spatial training, $t(15) = -2.59$, $p < .05$.

At the start of the term, the men in the sample identified most, on average, as spatial learners (M = 49.75, SD = 2.74) and least as verbal learners (M = 42.75, SD = 1.40). This was also the case at the end of the term (see Table 1). In contrast, the highest average score among the three cognitive learning styles for women was on the object sub-scale (M = 50.13, SD = 2.26) while the lowest was on the verbal subscale (M = 39.50, SD = 1.58) and remained so at the end of term.

While participants' general perceptions of dominant learning styles remained stable over 10 weeks, each gender's scores increased on each sub-scale -- except that women's scores on the object learning style sub-scale decreased slightly (but insignificantly, $p > .05$) over time.

After the first round of data collection, independent samples t-tests revealed no significant differences between male and female participants' scores on the three OSIVQ sub-scales (all ps > .05). However, at the end of the term, a significant difference between men and women's scores on the verbal sub-scale of the OSIVQ was revealed, $t(14) = 3.22$, $p < .01$.

Finally, scores on the spatial sub-scale of the OSIVQ did not correlate significantly with high scores on the PSVT:R at the start of the term ($p > .05$) but did so at the end of term ($r = .62$, p =.01). No other significant correlations were revealed between scores on the PSVT:R and other sub-scales of the OSIVQ, nor were there any significant associations between scores on the OSIVQ sub-scales and the CCI at the start or end of the term.

Discussion & Future Work

This pilot study revealed that students who self-identified more as object or spatial learners than verbal learners did so significantly more strongly after spatial training. In particular, a strong identification as a spatial learner linearly associated with mental rotation ability after spatial training. However, spatial training alone did not significantly impact the calculus or mental rotation abilities of undergraduate students. The results of this research will inform an adjustment to the methods before a second study is undertaken with a second-term calculus course. Since the present study was done during a summer term, only a total of 18 classes occurred, with approximately 15 minutes of spatial training done per class. This may not be sufficient for students to be influenced by the training: indeed, less than half of the items in the workbook were completed during the term. Thus, the next iteration of this research will require students to complete some training modules independently to allow more time for an effect.

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Applying Variation Theory to Study Modeling Competencies

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This paper presents preliminary results of using variation theory to design modeling tasks in order to explore ways of strengthening undergraduate engineering students' modeling skills. The responses of two undergraduate engineering students enrolled in differential equations to a set of three versions of the same task are reported.

Key words: mathematical modeling, qualitative methods, differential equations, variation theory

Mathematical modeling has been identified as a critical component of students' mathematics education at all levels, but in particular for preparing them for successful STEM careers (Cai et al., 2014; PCAST, 2010). Mathematical modeling can be seen as defining a mathematical problem from a nonmathematical situation, a process Freudenthal referred to as *horizontal mathematizing* (Freudenthal, 1991). STEM students struggle to define mathematical problems because there can be an overwhelming number of real-world considerations that they might not be able to handle mathematically. The goal of this study was to explore how purposeful variations in the real world conditions explicitly given in modeling tasks might aid students in strengthening their abilities to connect real world conditions and constraints to mathematical properties, parameters, structures, and representations.

Theoretical Background

Mathematical Modeling

A mathematical model is a real-world system, a mathematical system, and a relationship between the two that maps the objects, properties and relationships from one system to the objects, properties, and relationships of the other system. Mathematical modeling is a nonlinear, iterative process that renders a real world situation as a mathematical problem that can be analyzed mathematically (e.g., Blum & Leiß, 2007). The results of mathematical analysis are then interpreted and validated against constraints from the real world situation. Two competencies have a reputation of being the most difficult: *simplifying/structuring* and *mathematizing* (Galbraith & Stillman, 2006; Stillman, Brown, & Galbraith, 2013). Simplifying/structuring consists of identifying conditions and assumptions of the real-world situation then selecting variables or constraints deemed to be important. Mathematizing introduces conventional representational systems (equations, graphs, tables) to present mathematical "properties and parameters that correspond to the situational conditions and assumptions that have been specified" (Zbiek & Conner, 2006, p. 99). Previous research has found that students' real-world reasoning is important to students' ability to define a mathematical problem (Stillman, 2000) and that the modeler's initial interpretation of a problem situation evolves in tandem with their construction of the mathematical model (Lesh, Doerr, Carmona, & Hjalmarson, 2003). Thus it is important to find ways to help students connect their real-world knowledge to their mathematical knowledge.

Variation Theory

In task design, variation theory focuses on what varies and what remains invariant in a sequence of tasks such that "desirable regularities might emerge from the learners' engagement with the task" (Watson & Mason, 2006, p. 93). Dimensions of variation occur across tasks (they operate like variables) and ranges of change are values those variables can take on. For the modeling tasks presented below, the dimensions of variation are the real world constraints for which assumptions are given explicitly. When the real world constraints are given explicitly as assumptions in the task statement, they can be immediately mathematized. For example, consider a liquid contaminated with 4 grams of chemical agent per liter entering a tank at a rate of 5 liters per minute. The amount of contaminant remaining after one minute is 20 grams. In contrast, not stating information about the liquid entering the tank could lead to the more general model $c_i(t)r_i(t)t$ of the amount of contaminant in the tank at time t, where c_i represents the concentration of contaminant entering and r_i represents the rate at which the liquid enters. The second mathematization requires the assumption that concentration and rate can vary in time.

One use of variation theory is to guide successive versions of tasks across design cycles. Here, I present students' responses to the first iteration of a set of tasks designed to direct students' attention to similarities and differences in real-world situations drawn from the same context – reflected by the information stated in the task. The students then reflected on the similarities and differences in mathematical structures and representations used to model them.

Methods

Participants and Interviews

Mance and Orys were sophomore male engineering undergraduates participating in a larger research study of undergraduate STEM majors' mathematical thinking during mathematical modeling (Czocher, 2016). Each was enrolled in a first course on differential equations. Mance's instructor focused on solution techniques. In contrast, Orys's instructor focused on applications and modeling with differential equations. I focus on the work of Mance and Orys here because they successfully engaged with all three versions of Tropical Fish Tank task (presented below). As part of the larger study, Mance and Orys participated in a series of one-on-one task-based interviews (Clement, 2000; Goldin, 2000). I refrained from teaching or correcting the students' work in order to minimize influence on their reasoning. My questions focused on making students' implicit assumptions explicit and on understanding how the participants were selecting mathematical representations and structure. The interviews were audio/video recorded and transcribed. Analysis examined the students' implicit and explicit assumptions and the ways they connected their real world knowledge to their mathematical knowledge.

The Task and Its Variations

The real-world situation presented in the Tropical Fish Tank task can be modeled with a first-order, inhomogeneous, differential equation. Three versions were used for this study:

Version 1. To regulate the pH balance in a 300L tropical fish tank, a buffering agent is dissolved in water and the solution is pumped into the tank. The strength of the buffering solution varies according to $1 - e^{-\frac{t}{60}}$ grams per liter. The buffering solution enters the tank at a rate of 5 liters per minute. How much buffering agent is in the tank at any point in time?

Version 2. To regulate the nutrients in a 300L tropical fish tank, a buffering agent is dissolved in water and the solution is pumped into the tank. Because the fish are more active during the night than during the day, the strength of the solution entering the tank varies. A book on pet care suggested that the strength might vary according to $\sin\left(\frac{t}{45}\right)$. The buffering solution enters the tank at a rate of $.25$* liters per minute. One morning, you notice that the pump removing soiled water is unreliable. How much of the nutrient is currently in the tank? [*Mance received a value of 5 liters per minute]. (Note: depending on the value of t, the function may take on negative values – this would violate real world constraints).

Version 3. To regulate the pH balance in a tropical fish tank, a buffering agent is dissolved in water and the solution is pumped into the tank. Another pump takes well-mixed solution from the tank. How much buffering agent is in the tank at any point in time?

Three dimensions varied for this set of tasks are real world constraints: the volume of liquid in the tank and the rates at which liquid flowed in and out of the tank (see *Table 1*). Because different information is given across the three versions, the students had to make assumptions (implicitly or explicitly) that would satisfy the constraints in each. The intention was that as a set the tasks would aid the students in realizing that the variations did not affect the mathematical structure used to model the Tropical Fish Tank.

Table 1 Dimensions of variation and range of permissible change for the Fish Tank Problem

Dimension of Variation	Possible Values	Reason Varied	Anticipated Student Responses
Rate in and rate out	5 (or .25) (Versions 1 & 2), Not Given	Draw attention to similarities among the models even if the concentrations were different; develop a more general model	Set rate out to be constant, give a specific value (e.g., 0); Set rate out to be constant, give a parameter; Set rate out so $\frac{dV}{dt} = 0$; Set rates to vary with time
Concentration	Exponential decay (Version 1), Sinusoidal (Version 2), Not Given (Versions 2 & 3)	Draw attention to similarities among the models even if the concentrations were different; develop a more general model	Consider real-world implications of the given concentrations; Assume concentration = 0; Set concentration to vary with time
Volume of the Tank	Given (Versions 1 & 2), Not Given (Version 3)	Dependent upon rates in and out	Assume constant, give a specific value; Assume constant, give a parameter; Assume varies with time

Borrowing Orys's notation, let $Q(t)$ represent the quantity of agent in the tank at time t, let $c(t)$ represent the concentration of the solution entering the tank, let V be the volume of the tank, and let $r_i(t), r_o(t)$ represent the rates at which the liquid flows in and out of the tank. In Version 1, $c, V,$ and r_i are given. The rate r_o must be assumed to be constant, left as a parameter, or left as an unknown function of time. In Version 2, c, V, r_i, are given while r_o must be inferred to be zero (note that depending on the situation imagined, this might force V to be a function of time). In Version 3, none of c, V, r_i, or r_o are given. They must be inferred as parameters or as functions of time. The differential equation that models the situation in Version 3 can be written as

$$\frac{dQ(t)}{dt} = r_i(t)c(t) - \frac{Q(t)}{V} r_o(t).$$

Results & Analysis

On all three versions, both students successfully obtained a differential equation to model the given situation, shown in *Table 2*. Both responded to the changed conditions in each version. For example, Orys observed that Version 3 seemed "to be missing a lot of stuff," "the usual things just aren't even mentioned like rates or volume of the tank…" He stated that he could make some assumptions but that he didn't know if it would be right. Orys's comments indicate his unease in engaging in *simplifying/structuring*.

Both students successfully handled the lack of information given about the rates in/out in Version 3. They also acknowledged that rate in/rate out need not be equal in any of the versions. Mance showed that $V(t)$ could be determined by integrating $\frac{dV}{dt} = r_i(t) - r_o(t)$. He indicated he was confident he could *mathematize* a nonconstant volume into the differential equation for the quantity of buffering agent in the tank, but that he was not sure if the differential equation with a non-constant coefficient of $Q(t)$ would "follow all the things we learned before." That is, though the model made sense to him, he was cautious in applying mathematics he knew to a novel mathematical context. In all 3 versions, both students eventually assumed that $\frac{dV}{dt} = 0$. Orys found an explicit expression for volume as a function of time (see *Table 2*), but did not solve the resulting differential equation. Mance's work illustrates a successful arc through *simplifying/structuring* and *mathematizing* regarding this constraint. He appropriately converted an assumption (volume of liquid does not change) to a mathematical property (rate of change of volume is zero) to a mathematical representation ($\frac{dV}{dt} = 0$), successfully *mathematizing* the assumption. He then explained the consequences of assuming that $\frac{dV}{dt} = 0$ and why it was reasonable to do so. Ultimately, both simplified the problem by assuming that the volume of liquid in the tank constant so that the tank wouldn't empty or overflow.

In Version 2, Mance did not notice that the function describing the concentration of the entering solution could turn negative. This is an example of how implicit assumptions which don't capture the real world situation be mathematized. On Version 2, Orys noted that the concentration should not be negative and decided to use $1 + \sin\left(\frac{t}{45}\right)$ to describe its oscillation around a unitized amount instead. Below are their differential equations for Version 3.

Table 2 The students' differential equations derived for Version 3

Mance: (in the notation from above)	Orys:
$\frac{dQ}{dt} = \big(r_i(t)c_i(t) - r_o(t)c_o(t)\big)V(t)$	$\frac{dQ}{dt}(t) + \frac{r_0(t)}{V(t)}Q(t) = r_i(t)c(t)$, with $V(t) = V_0 + r_i(t)t - r_o(t)t$ and V_0 as the initial volume of liquid in the tank

Mance stated that his equation on Version 3 was "the most general expression [he] could come up with." In this equation, the change in quantity of the agent does not depend on the amount of agent currently in the tank. This led the interviewer to probe Mance about the condition that the solution was *well-mixed* and why it was necessary, Mance indicated that if a solution was not well-mixed, the concentration of the tank would vary over space as well as over time. Indeed, without the assumption in place, the concentration of the solution leaving the tank would depend on where the overflow pump was placed. Thus Mance's model mathematizes one consequence of the well-mixed condition (concentration of agent in the tank is uniform) and not another (rate of change of quantity depends on current concentration).

Mance and Orys both reasoned using dimensional analysis. For example, on Version 2, Mance narrated his work to determine the rate at which agent entered the tank, "Your rate in ends up being like a number, it's 5 liters per minute and then you multiply that times your strength and you get some grams or whatever, your pH, per minute." On Version 3, Orys used dimensional analysis to set up a difference equation for the change of quantity of agent in the tank and let the limit $\Delta t \rightarrow 0$ in order to obtain a differential equation.

Discussion & Conclusions

Varying the assumptions and conditions relevant to the real world situation revealed that the students were reflecting on how the mathematics they knew could apply to the situations as they *simplified/structured* and *mathematized*. Mance commented that ideally, he would like to see worked examples with numbers given and more generalized representations of situations in order to increase his skill at deriving and solving equations. Constructing ever more general models may have helped these students explore the limitations of what the mathematical structure (or analytic technique) may be. Indeed, written and follow-up questions which varied parameters, changed assumptions, or otherwise focused on *simplifying/structuring* showed potential in helping Orys and Mance identify invariant mathematical properties and relationships that could be attached to real-world conditions and assumptions. This finding is consistent with research at the middle and high school level (Czocher & Maldonado, 2015).

Further, the preliminary results here seem to suggest that at least for these advanced engineering students in differential equations, the development of modeling skills was facilitated by familiarizing the student with certain mathematical conventions that effectively *mathematize horizontally* (Freudenthal, 1991). Two examples of such a convention are: *concentration is positive* or *time increases from zero*. The latter convention would have been well-assimilated by the time the students arrived in differential equations. Other examples of assumptions that became conventions were: *solutions are well-mixed* or *rate of change of a quantity can be expressed as a linear combination of base rates*. Other elements of the students' models were made up of more basic building blocks that could be assembled through the four basic operations and dimensional analysis. Using variance and invariance of real world conditions and drawing students' attention to how those variations impacted the mathematical properties, parameters, structures, and analytic techniques appropriate to successfully model the real world situation could aid students in seeing how mathematical choices connect to and depend on real world conditions. Such task sequences therefore seem to be a promising path for developing modeling skills, but of course require further study. With regard to the research setting, varying real world conditions provided multiple opportunities for the interviewer to explore and probe how the students were envisioning the real world situation (their situation models, Blum & Leiß, 2007) and how they chose to represent those conditions mathematically.

Questions for the Audience

1. When conceptualizing modeling competencies as learning objectives, how can task effectiveness be measured? 2. Would follow up questions, e.g., *what-if* scenarios posed by the interviewer, also qualify as variations in task dimensions? 3. In the context of modeling, it is important to nurture the skill of generating one's own assumptions. How can variations which target identifying real world assumptions and constraints and matching them with mathematical properties and parameters be implemented without overriding the dimensions the student spontaneously attends to?

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Do Students Really Know What a Function is?: Applying APOS Analysis to Student Small Group Presentations

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One of the fundamental concepts in mathematics is that of a function. This concept also appears to be a difficult concept to grasp for a large percentage of students. In order to assess the overall understanding of the concept of a function, we conducted an experiment with math majors at our university. In an upper division math problem solving course, the students were asked specific questions about the nature of functions. Students presented their understandings of function in groups of 2-3, which were recorded and then transcribed. Based on the IBL teaching methodology and the small group and classroom discussion data collected we have applied a sociocultural framework. The innovation we add is applying APOS, an individually oriented theory, to the collective level. Analyzing the video transcripts, we will discuss the overall trends in understanding as well as some of the common misconceptions that we have identified.

Keywords: Function, Problem Solving, Pre-service Secondary Teachers, APOS Theory

Purpose of the Study

The major goal of this study is to answer the question: Do students really know what a function is? This project was motivated by our experiences teaching upper and lower division math and science courses. We noticed an inability in a significant amount of students to apply the function concept to math and science problems. This in turn created an obstacle to success in a variety of undergraduate courses.

For example, the students in classes ranging from precalculus to linear algebra exhibited difficulty in evaluating functions of single or multiple variables. While learning to estimate areas under a curve with finite sums, students in calculus were unable to relate the height of a rectangle to the y-value of a given function. In introductory physics, students had difficulty applying formulas where certain letters represent constants and other represent variables (e.g. simple harmonic motion), as well as translating between representations of these functions in the graph form, and in the equation form.

Students appear to "get by" based on rote memorization and we wondered how much they actually understand. Without a comprehensive understanding and the ability to apply the function concept, we were concerned that higher order mathematical and scientific concepts would not be available to otherwise capable students. In addition to comprehension, the application of this knowledge can be found throughout a math or science degree. For example, the physical sciences require a basic understanding of the composition of functions. Differential and integral calculus requires deep understanding of how changes in the domain, even over an interval, affect the range of a function. Further, students with a degree in pure mathematics or math education require application of functions in a more abstract setting, specifically with regards to groups, vector spaces, Euclidean spaces, and more.

Theoretical Framework and Literature Review

One of the most pervasive big ideas (Schifter & Fosnot, 1993) in all of mathematics is the concept of a function. Michael Oehrtman and colleagues (Oehrtman, Carlson & Thompson, 2008) give an overview of the concepts necessary to understand the idea of function. Extensive studies that have been done on functions (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Carlson, 1998; Dubinsky & Harel, 1992; Thompson, 1994; Oehrtman et al., 2008) substantiate our experience that there is a lack of comprehensive understanding at the undergraduate level.

Our framework will be an integration of the APOS and sociocultural theories – where we take the unusual perspective of using the normally individual oriented APOS theory at the collective level.

In (Breidenbach et al., 1992), the following explanations of the terms action and process are given: "An action is a repeatable mental or physical manipulation of objects. Such a conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression)…A process conception of function involves a dynamic transformation of objects according to some repeatable means that, given the same original object, will always produce the same transformed object. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done." Building upon the work done in (Breidenbach et al., 1992) the faculty in (Cottrill et al., 1996) improve upon the theory adding, the concepts of:

…an object is defined as the result of "when the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations."

…a schema "as a coherent collection of actions, processes, objects, and other schemas that are linked in some way and brought to bear on a problem situation."

This rounds out the theory defining APOS as Action Process Object Schema. They further argue that the process view or "dynamic conception" must be preceded by the action or static view.

Although APOS is usually used to analyze individual conceptions it does tell us what to look for in how students talk about functions. That is, we categorize the group responses without saying that they capture any or all of the individual student conceptions. To be clear, we have identified our unit of analysis as the group of students not the individual students.

The sociocultural theory was initially introduced in (Vygotsky & Cole, 1981) and (Vygotsky, 1979). The sociocultural perspective explains the study design in terms of the class being taught with small group inquiry-based instruction, and synthesis of information occurring via whole class discussion, as well as the data collection of whole class video, and discourse analysis. The mathematical knowledge is constructed and transmitted via social interactions. In addition to this classroom culture, we keep in mind that that our students are pre-service teachers addressing mathematics through the lens of what they believe students should know. More specifically we can unpack the framework into the following three pieces:

Cultural	Social	APOS
Class culture IBL teaching method Type of students (pre-service teachers)	Group dynamics Classroom dynamics Data collection – video transcription	Analysis of the action vs. process understanding of functions

We apply the sociocultural perspective mainly to our study design, and the APOS theory to the data analysis. However, when analyzing the video transcript and seeing certain mathematical ideas arise, we interpret them in terms of the group dynamics that brought them to the surface. These theories fit together in a complementary way because we structure the instruction and data collection around a sociocultural perspective, and apply the APOS theory to the small group level to answer our question about whether students know what a function is, and more specifically whether the small groups collectively possess an action a process view.

Methods

This study took place during Spring 2013 and Spring 2014 semesters in a course designed to prepare future secondary educators with problem solving skills. There were 11 and 9 students enrolled in the courses, respectively. The students were geographically and ethnically diverse, as is common at our institution. The first author was responsible for teaching the course both semesters. This course is required for math education majors, and is a terminal math course, focused on problem solving and teaching. No new math content is introduced; rather students apply known concepts relating to high school math to solve novel problems at the college level.

Activity

One unit of the course is on the theme of Functions and Covariation (Carlson, Jacobs, Coe, Larsen & Hsu, 2002). The two authors taught one course period devoted to a "Teaching Functions" activity prior to the unit being taught. During the "Teaching Functions" activity, students were broken up into small groups of 2-3 people, for a total of 7 groups. They were given the following prompts and asked to produce a mini-lesson in response:

- What is a function?
- What is an example of a function?
- What is the role of the variable in a function?
- What are the characteristics of a function?
- What are the typical misconceptions about functions? (From the perspective of a high school student or person learning functions for the first time.)

Students had 30 minutes to plan the lesson. The written lesson plans were then collected. Each group gave a 5-10 minute presentation, which was followed with brief question and answer time. The presentations were video recorded and we transcribed the audio portion of the presentations for our analysis. We did a detailed analysis of the results of this activity and video. Similar research has been published recently with a video analysis of students investigating the Cartesian Connection (Moon, Brenner, Jacob & Okamoto, 2013). These authors argue that cognitive obstacles arise as a result of misunderstanding fundamental and pervasive big ideas in mathematics.

Analysis

The analysis was done with the help of RQDA software. This is a R-based Qualitative Data Analysis software package. RQDA serves as a way to highlight and sort different ideas within multiple files allowing for the compilation of information related to each specific idea. All of the code creation and text categorization was done collaboratively, and following best practices

(Dewar & Bennett, 2015), (Taylor-Powell & Renner, 2003). We used a thematic analysis on the transcription of each group's presentation.

Preliminary Results and Analysis

In order to summarize our initial findings, we will discuss the analysis as it pertains to the assigned questions. The first question the students were asked to address was based on the definition of a function. With regards to the definition of a function, we found that the students' responses could be summarized by four overall themes listed in order from inaccurate to accurate: 1) A function must have an equality or inequality with at least one unknown; 2) A function must be an equation with one variable; 3) For every x, there exists a unique y mapped to by x and the more sophisticated but less accurate definition: 4) A relationship between two quantities with a specific input/output role.

When asked to provide examples we note that most groups gave the linear and quadratic examples. This is perhaps not surprising, as these are some of the simplest examples of functions. We wonder whether our instructions to teach the lesson at the high school level influenced the types of examples that the students provided. We observed some other examples, such as the tangent, absolute value or other piecewise defined functions, as part of the later discussion that arose, for instance about one-to-one or onto functions or during a conversation about the vertical and horizontal line tests. The more sophisticated examples appear to result from classroom discussions evident of learning taking place through classroom dynamics.

The impetus for asking the students to identify the role of the variable came from our experience in the classroom. We had noticed many students making the same mistakes involving evaluating functions at expressions, such as believing that $f(x+h)=f(x)+h$, including errors forming composition of functions, as well as mistakes involving setting all functional expressions to 0 and trying to solve for x even when it is inappropriate to do so. We conjectured that these mistakes may be occurring due to a basic lack of understanding of the role of a variable. The responses from the students can be summarized in the following three concepts: 1) The relationship between unknowns; 2) A placeholder for a value; and 3) The concept of the independent variable as input vs. output. Although many groups did focus on plugging values into the variable, their responses overall appeared flexible enough that they might be less likely to fall into the mistakes we mentioned earlier.

Related to our earlier question about the role of a variable, we were also interested in how the students would relate the various characteristics of a function. One group specifically reflected on their experience working with functions in their upper division numerical methods course, and how in this course the characteristic of the graph allowed them to solidify the initial ideas presented in their precalculus courses. This addresses a question we had about students making connections between courses and synthesizing and applying the concept and characteristics of a function. We were also interested to note that one group said, "what they're going to learn in high school is to work with numbers but this could be vectors, or anything, it could be all kinds of numerical objects, all kinds of restrictions that you put in there and you get all things out." This shows that some students believe the domain could be a set other than the real numbers.

An analysis of the student presentations identified seven main themes in response to the characteristics question. They are 1) It has a 'number' of variables; 2) Has to have an equals sign; 3) Functions can be represented in a graph; 4) One-to-one, onto or Inverse; 5) Vertical line test; 6) Domain & Range; and 7) F(x) vs F(y). This last theme refers to the importance of identifying the independent variable; e.g. that it is possible for a single expression to define a function of x but not of y or vice-versa.

The final question concerning misconceptions was very rich. The students addressed common misconceptions, but also had a few of their own. The common misconceptions that they identified their potential students having were as follows: 1) All functions are one-to-one; 2) All functions have domain all real numbers; 3) The function is only dependent on the expression x or that y=f(x); 4) x as a function of y vs. y as a function of x; 5) Not necessarily a connection between the graph and equation; and 6) Not necessarily a connection between the notation and evaluation of expression. We found the variety of misconceptions identified by the students interesting. From the breadth of misconceptions we infer a high level of intuitive understanding of functions. Students appear to have reflected on and learned from their own mistakes as well as their experiences tutoring and observing others actual mistakes.

It should be noted that the two authors actively chose not to correct or comment upon the content of the students' presentations during the course of the exercise. Whilst we did not correct mistakes, we noticed that over the course of the conversation they as a group they came to consensus on the correct answer. The self-correction was seen in both years. For example, it was surprising to note that one group believed that an inequality is an example of a function. This error spawned a discussion within the class about whether an inequality is a function. Another student attempted to rationalize the example trying to help their colleagues. The group finally conceded that the inequality does not meet the definition of a function on real numbers. We asked ourselves what was the sociocultural dynamic in the group or classroom that resulted in this idea. Upon further viewings of the presentation it seemed that one member of the pair was more dominant. The less dominant student who began the presentation provided correct information inconsistent with the later incorrect ideas. We also noted that several groups were triples and perhaps this pairing of the dominant student with a wrong idea did not lend itself to be corrected in the small group.

While the APOS theory is typically used to analyze the individual, here we apply the theory to the groups, thereby defining the small groups as our unit of analysis. We identified three groups as possessing only an *Action View*. Predominantly these students described a function as something that you plug a number into and get another number out of. For example, one group described plugging numbers into functions to evaluate them and to create t-charts. They claim that the relationship between the variables is just a plug-in process. We categorized three groups as possessing a *Process View*. The group that went the furthest into process interestingly did not present the plug-in model. They presented two different instances of a process understanding. This group explained, "…and when we're introducing the function concept what's going to be more effective probably is that we're not talking about a letter with a hidden value. We're talking about a relation between some value that we don't know and another value that we don't know, but we can still say a relation between them." They do not view a function as an equation that needs to be solved, but instead describe a dynamic relationship. The second instance demonstrated an understanding of the relationship between input and output and their reversal as defining an inverse function, rather than algebraic procedures such as switching x and y and solving. The remaining group did not have their transcript coded as either "action" or "process."

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Teachers' Beliefs & Knowledge of the Everyday Value of High School Algebra & Geometry: Is One More Useful than the Other?

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Despite recent emphases on teaching that mathematics is useful, little is known about teachers' beliefs about the value of the topics they teach. While teachers have been found to believe mathematics overall is worthwhile, it may be that this belief varies among subjects. This study examines the beliefs and knowledge of students enrolled in teacher preparatory courses by employing a survey and algebra and geometry tasks related to each subject's usefulness and connection to real-world applications. Preliminary results show they value algebra above geometry in terms of future use by their students. However, while their confidence was equal in their abilities to produce real-world applications, they were more successful producing those related to geometry than algebra. Further survey and clinical interview data collection is planned with additional pre-service and in-service teachers to examine beliefs and knowledge expert (in-service) teachers bring to teaching to inform the preparation of secondary school teachers.

Key words: pre-service teacher education, algebra, geometry, teacher beliefs, pedagogical content knowledge, mathematical content knowledge

Introduction

The Common Core State Standards for Mathematics, as well as the Next Generation Science Standards, and the National Council for Teachers in Mathematics, have each expressed the need for students to value mathematics and to see it as a useful pursuit. The Common Core's Standards for Mathematical Practice includes in its description of mathematical proficiency, a "productive disposition (habitual inclination to see mathematics as sensible*, useful, and worthwhile*, [emphasis added] coupled with a belief in diligence and one's own efficacy)" (National Governors Association Center for Best Practices and the Council of Chief State School Officers, 2010, p. 6).

Research has shown that teachers see mathematics, as an entire discipline, as valuable. Yet little is known of mathematics teachers' beliefs regarding the usefulness and value of the individual subjects they teach, or whether their beliefs are aligned with their practice. This study was designed to examine whether pre-service teachers' beliefs towards algebra and geometry might differ, as well as their abilities to demonstrate the relevance and usefulness of the mathematics by producing real-world examples or stories.

Research on Teachers' Beliefs and Knowledge

Research into teacher beliefs has been popular for decades. Despite this, little has been done that narrows in on an area of mathematics, particularly in regards to its nature or value. Peterson et al. (1989) conducted a study of teachers' pedagogical content beliefs regarding addition and subtraction with first grade teachers. They write, "Unfortunately, research on teachers' beliefs has not been concerned with subject-matter content…Thus, these studies are limited because they report findings on teachers' beliefs across a wide range of curriculum areas and grade levels" (Peterson, Fennema, Carpenter, & Loef, 1989, p. 3). In other words, it is one thing for a teacher to believe that mathematics is a valuable and useful pursuit, and another thing to believe that algebra or geometry is a valuable and useful pursuit. Furthermore, believing in the importance of all topics within mathematics for all of students is yet another thing entirely. Hence, there is a lot yet unknown related to teachers' beliefs about the value of the subjects that they teach. This is despite the accepted view that teachers' beliefs shape instructional practices and student learning opportunities (see, e.g., Leder, Pehkonen, & Torner, 2002; Philipp, 2007).

On the other hand, research into teacher knowledge often focuses on a specific topic in order to categorize the knowledge they bring to bear, not only in solving mathematical problems, but teaching them as well. Work such as Ma's (1999) pinpoints topics within mathematics to examine teachers' mathematical knowledge in that small area. However, Ma's research was with elementary teachers. At the secondary level, less has been done to examine teachers' mathematical knowledge related to topics they teach within the wide field of mathematics. Therefore, less is known about variations in teachers' levels of expertise from topic to topic, beyond the elementary level. Hogan, Rabinowitz, and Craven (2003), stated that "Because the scope of the school curriculum in a given subject is sufficiently broad to preclude expertise in all topics within the curriculum, a teacher may feel more knowledgeable and confident within one specific content area" (p. 239). Therefore, it would appear necessary to examine teachers' knowledge on a variety of mathematics curriculum topics in order to gain a complete picture of secondary mathematics teachers' knowledge and abilities to apply that knowledge in practice.

While the U.S. lags behind a number of nations in mathematics, its weakest subject is certainly geometry. Results from the Third International Mathematics and Science Study (TIMSS) have repeatedly shown that of all the mathematical domains tested, the U.S. does best in algebra and worst in geometry (Clements, 2003; Provasnik et al., 2012). In fact, U.S. secondary students have been found to score near or at the bottom on every geometry item on the TIMSS (Clements, 2003). Thus, the focus of this study is teachers' beliefs and knowledge of algebra and geometry, as relevant to the learning standards for grades 8-12 and with the goal of providing insights into teachers' knowledge and beliefs that can inform the design of pre- and in-service professional development.

Research Design

Research Questions

In order to add to the existing work on teacher knowledge and beliefs, the specific research questions addressed in this study are,

1. How do pre-service and in-service teachers perceive the importance of learning algebra and geometry?
2. What kind of real-world algebra and geometry problems or models are pre-service and in-service teachers able to produce?

These questions were designed to explore both mathematics teachers' knowledge and beliefs at two different stages, during preparation and in practice. Studies have shown that teachers gain certain knowledge "on the job" that they may not have opportunities to acquire through teacher preparation programs, e.g., knowledge of common student difficulties, knowledge beyond understanding mathematics for oneself but for teaching mathematics, etc. (Borko & Livingston, 1989; Livingston & Borko, 1989, 1990; Ma, 1999). Whether that pertains to knowledge of and beliefs about the value and usefulness of the mathematics they teach is the subject of this study.

Theoretical Perspective

This study is being conducted with cognitive, expert-novice educational psychology theoretical framework. That is, it assumes that experts are characterized as having the ability to organize their knowledge in ways that reflect deep conceptual understanding of the area of study (Bransford, Brown, & Cocking, 1999). This organized knowledge allows them to observe meaning and patterns that would go unnoticed by novices. This also allows experts to contextualize their knowledge, making it flexible to retrieve and to apply. Novices, on the other hand, struggle to make connections between what they see as unrelated rules and facts, which prevents them from developing conceptual understanding.

While the research defining expert and novice knowledge is well-established and accepted, there is less research on characterizing expert beliefs. Wieman (2007) described experts in his work on science education as believing science has a coherent conceptual structure. Novices, on the other hand, see science as isolated pieces of information that are transmitted by authority. Since they believe science to be unorganized, they rely on memorization rather than deep conceptual learning. Their knowledge remains fragmented, lacking the structure they also believe the subject to be lacking.

In this study I am applying this theory to beliefs about mathematics and examining the extent to which mathematics teachers demonstrate expert beliefs about the nature of mathematics by conceiving of it as a coherent conceptual structure, interconnected and organized. In this framework, novice beliefs about mathematics would be characterized by assumptions that mathematics is made up of disparate facts and algorithms, meant to be transmitted from teachers, the authority, to their students. Their students then "learn" this material by repetition and rote memorization.

Methods

A two-part survey was administered to 35 pre-service teachers enrolled in an algebra course for elementary education majors near the end of the spring 2016 semester at a northeastern land-grant university. The first part of this survey, focused on beliefs, was modeled after the "Usefulness" portion of the Fennema-Sherman attitude survey for students of mathematics (Fennema & Sherman, 1976). The 12 Fennema-Sherman items were reduced to five, eliminating redundancies to increase participation and limit fatigue (e.g., see Figure 1 below).

1. I believe knowing algebra will help my students earn a living.

1	2	3	4	5
Strongly Disagree	Disagree	Undecided	Agree	Strongly Agree

Figure 1: Sample Beliefs Survey Item (modeled after Fennema-Sherman)

In addition, three items were created to gather data on teachers' attitudes towards the use of real-world problems in their teaching (e.g., see Figure 2 below). All eight items were written both in terms of algebra and geometry.

6. I believe it is important to show my students real-world applications of geometry.

1	2	3	4	5
Strongly Disagree	Disagree	Undecided	Agree	Strongly Agree

Figure 2: Sample Beliefs Survey Item (author-generated)

The second part of the survey consisted of two two-part tasks modeled after Ma's (1999) interviews with U.S. and Chinese teachers on the division of fractions, one each related to algebra and geometry. Teachers were asked to work through a problem and then to generate a scenario that could be modeled by the computations they just completed (see Figure 3 below).

People often adopt different approaches when solving a problem. How would you solve the following problem? Show your solution below.

1. Solve for x. $\frac{16}{1} = \frac{240}{x}$

Now imagine you are teaching an algebra course. To make this topic more meaningful to your students, you want to create real-world scenarios to show them the relevance of algebra in their daily lives. What would be a good story or model to demonstrate the application of this problem?

Figure 3: Sample Algebra Task

Three different tasks were designed for each subject to allow survey participants to have varied problems from their neighbors. Methods of analysis are discussed in conjunction with results presented below.

Preliminary Results

The beliefs survey items were scored using a Likert scale from one to five, one being "strongly disagree" and five being "strongly agree" with three being "undecided." Scores were then inversed for statements that were phrased negatively, so that a five became a one, a four became a two, etc. In this way, a score above a three meant a positive belief towards the subject, and scores below three meant a negative belief. This also allowed for measuring differences in beliefs by subtracting one from the other, between disciplines. Therefore, a positive difference when subtracting geometry from algebra meant the participant valued algebra more than geometry, no difference meant a belief in equal value, and a negative difference meant they valued algebra less than geometry.

For all but one of the beliefs survey questions, participants believed algebra to be equally as valuable or more valuable than geometry for their students' lives. Most felt real-word examples were important for both subjects (n=34 vs. n=33) but would be somewhat more often used in class for algebra (n=18 vs. n=14). However, despite the majority believing in the importance of real-world examples, the number who were actually confident in their ability to produce these examples was fewer (n=23 for both). In addition, about half of participants ranked the importance higher than their confidence (approximately 56% for algebra and 53% for geometry). The numbers here are close due to the small sample size however, they suggest a trend of pre-service teachers believing algebra to be more useful than geometry and, despite feeling that real-world examples are important in both subjects, having little confidence in their ability to demonstrate these for their students.

In the second half of the survey, participants were asked to solve a problem and then generate real-world scenarios that could be described by the solution they just produced. The first part (solving the problem) was scored "correct," "partially correct," or "incorrect." If a participant left the question blank, this was also scored as "incorrect." The second part was scored using a 0-3 point rubric. A detailed complete example that correctly reflected the problem given was scored a 3. A related example lacking detail was scored a 2. An unrelated or so incomplete as to be unclear example was scored a 1. If a participant left the question blank then this was scored a 0. Using these scoring guides on this small sample, it appears there is no visible relationship between correctness and ability to produce real-world examples (see Table 1 below). Despite nearly half of the participants answering the algebra task problem correctly, slightly more were unable or unwilling to produce a related real-world example. In fact, only 5 participants scored a 3. On the other hand, one-fifth of participants answered the geometry task problem correctly and again, only 5 participants wrote real-world examples that were scored a 3. Thus, even though participants had a higher success rate solving the algebra tasks than the geometry tasks, they had an equal success rate producing geometry-related real-world examples.

Algebra					Geometry				
	0	1	2	3		0	1	2	3
Incorrect	7	2	0	0	Incorrect	8	6	0	0
Partially Correct	5	2	1	2	Partially Correct	6	2	4	2
Correct	6	5	2	3	Correct	2	2	0	3

Table 1: Algebra and Geometry Task Scores

Conclusions and Implications

The goal of this study is to add to the body of knowledge available to instructors of pre-service teachers. Preliminary findings suggest that pre-service teachers generally believe algebra and geometry to be useful to their students' futures but they lack confidence in their ability to produce real-world applications and, in fact, are often unable to, given common mathematical tasks. A typical pre-service teacher in this study believes algebra to be somewhat more useful than geometry but is unable to generate rich real-world examples for either, despite a belief in the importance of doing so. This suggests that pre-service teacher preparatory coursework needs to include more instruction on the applications of the mathematics they will teach, particularly everyday use and career applications of geometry. This would both serve to better prepare them to teach mathematics as useful and also to see it as useful themselves. The Common Core Standards has identified the need for students to see the value of mathematics, therefore it is vital that our teachers going into the classroom be able to see the value of their subjects, and be able to present it clearly and purposefully to their students, including via real-world examples.

This survey will be administered to pre-service teachers enrolled in college mathematics courses again in the fall of 2016. From this, participants will be identified for clinical interviews to gain further insight into pre-service teachers' knowledge, beliefs and abilities to generate real-world examples. Additionally, clinical interviews will be conducted with in-service high school mathematics teachers from the surrounding area. This will allow for a deeper picture of what pre- and in-service teachers believe and both the pedagogical content knowledge and mathematical content knowledge they possess and how they differ with their level of teaching experience. Coupled with findings from pre-service teachers, findings about in-service teachers can then inform the design of instruction to enhance pre-service teachers' capacities to provide rich learning opportunities for their students.

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Second Semester Calculus Students and the Contrapositive of the Nth Term Test

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Little is known about the difficulties second semester calculus students have determining series convergence, and why students have such difficulty. This report seeks to add to the existing literature on series by analyzing second semester calculus student responses to a multiple choice item that involves the use of the contrapositive of the nth term test. We frame our discussion in terms of what these answer choices might say in terms of student concept images of series and sequences. We also analyze what prerequisite knowledge might help students be more successful in answering questions about series and sequences typically seen in a second semester calculus course.

Key words: Calculus, Nth Term Test, Series, Sequences, Contrapositive

Researchers have called for more research in the area of infinite series (González-Martín, Nardi, & Biza, 2011). Some of the existing literature on series includes the role infinity plays on students' understanding of series(Sierpińska, 1987), student understanding of definitions (Roh, 2008; Martínez-Planell, R., Gonzalez, A., DiCristina, G., & Acevedo, V., 2012), student's beliefs about their role as a learner and the relationship between these beliefs and approaches to solving convergence problems (Alcock & Simpson, 2004; Alcock & Simpson, 2005), how series are introduced to students (González-Martín, Nardi, & Biza, 2011), the difficulties students have accepting that comparison tests can be inconclusive (Nardi and Iannone, 2001), and the development of a framework used to analyze student errors determining the convergence of series (Earls & Demeke, 2016).

This preliminary report seeks to add to the existing literature on series convergence by analyzing second semester calculus student solutions to a problem that could be solved using the contrapositive of the nth term test. The report is part of a larger dissertation study that seeks to determine the difficulties students have finding the convergence of sequences and series in second semester calculus, and in what ways these difficulties are related to prerequisite knowledge students' instructors mightexpect their students to have from precalculus and first semester calculus prior to entering the course.

Conceptual Framework

Tall and Vinner (1981) use the term "…*concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). A student's concept image of a particular concept usually results from working with examples and non-examples (Vinner & Dreyfus, 1989). The concept image for concepts that do not have graphical components or have weak graphical components will include symbols, formulas, and associated properties (Vinner & Dreyfus, 1989).

Tall and Vinner (1981) define *concept definition* as "…a form of words used to specify that concept" (p. 152). They also differentiate between a student's *formal* concept definition and a *personal* concept definition. A formal concept definition is the definition of a concept that is agreed upon by the mathematical community. A personal concept definition, however, might be constructed by the student and could change over time.

Although distinct, concept images and concept definitions are intricately related. Tall and Vinner (1981) describe this relationship by saying, "for each individual a concept definition generates its *own* concept image" (p. 153). In other words, the words used to describe a particular concept generate a mental image associated with the concept. As an example of this relationship, consider the function concept. The formal concept definition of function can be described as a relation between two sets where each element of the first set is assigned exactly one element of the second set. However, a student studying functions might not remember this definition, and the concept image for the student might include the idea that a function must be given by a rule or formula.

In the discussion that follows, this paper describes what one multiple choice question might say about students' concept images of sequences and series in general and the nth term test in particular.

Research Methodology

The targeted population for this study is second semester calculus students enrolled at a large research university in the northeastern United States. One hundred seventy-nine students responded to an anonymous six question multiple choice survey on sequences and series with a cover sheet. The cover sheet asked students to list their previous three mathematical courses, whether or not they had experience with sequences and series prior to entering the course, age, year, gender, race, and expected grade in course.

The main research aims of the full dissertation study are to (1) determine what misconceptions of sequences and series are revealed when students solve problems on sequences and series typically seen in a second semester calculus course, (2) determine what ways, if at all, these misconceptions relate to the prerequisite knowledge students are expected to have prior to starting a second semester calculus course, and (3) determine what additional knowledge or conceptualization of sequences and series students might need to be successful in second semester calculus courses.

This preliminary report focuses on one problem on the multiple choice test, a problem that focused on students' understanding of the nth term test and its contrapositive. This question was chosen for this report because more students answered this problem incorrectly than any of the others:

Instructions: Please Circle The Letter of The Best Response.

Suppose that you know that

$$\sum_{n=1}^{\infty} b_n = 3$$

What can you say about $\lim_{n\to\infty} b_n$?

A. $\lim_{n\to\infty} b_n = \infty$
B. $\lim_{n\to\infty} b_n = 0$
C. $\lim_{n\to\infty} b_n = 3$
D. We can't say anything about $\lim_{n\to\infty} b_n$

Recall that the nth term test states that, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. Consequently, the contrapositive of this statement tells us that if the series converges, then the limit of the sequence must go to 0. Therefore, choice B is the correct answer.

This question was recommended by a mathematics professor with many years of experience as a second semester calculus instructor. Distractors were chosen based on responses given to this question during interviews in a pilot study. This choice was made based on Kehoe's (1995) recommendation that any multiple choice assessment should have three to four well written answer choices. All of the questions on the multiple choice test were reviewed for appropriateness (as in, typical problems that a second semester calculus student should be able to solve) and correctness by a mathematics graduate student, a mathematics education graduate student, and an experienced mathematics professor.

Data was analyzed using first exploratory data analysis (EDA) followed by confirmatory data analysis (CDA). Exploratory analysis involves looking at variables individually, then two at a time, and then multiple variables at a time. Behrens (1997) recommends starting with EDA before moving to CDA because EDA allows researchers, "…to find patterns in the data that allow researchers to build rich mental models of the phenomenon being examined" (p. 154).

During EDA, some hypotheses developed. For example, it appeared as though students that had experience with sequences and series coming into the course performed better on this question than those that lacked experience. The purpose of CDA is to test this and other hypotheses using statistical significance tests such as a chi-squared test, or Fisher's 2-tail test.These two statistical tests were chosen based on recommendations from the Institute for Digital Research and Education website (IDRE, n.d.). All statistical tests were performed using the JMP software.

Preliminary Results

Only 48 of the 179 students, or just over 26 percent, chose answer choice B, the correct answer to this question. Ninety-two students, which is over 50 percent, chose answer choice C, with seven students, under four percent, choosing answer choice A and 32,about 18 percent, choosing answer choice D.

Students that said they had experience with sequences and series in a previous course performed better on this question than those who said they had no experience. Sixty-eight students said they had experience with sequences and series prior to their second semester calculus course, and 111 said they had no experience. Unfortunately, the sample sizes were too small to determine if there was statistical significance within each answer choice. Consequently, I instead considered whether or not the students answered the question correctly.

Of the 68 students with prior experience, 25 students, or just over 36 percent, answered the question correctly. Of the 111 students with no prior experience, only 23, or just over 20 percent, answered the question correctly.

Fisher's two-tail test revealed a p-value of .0238, indicating that this difference is statistically significant. Likelihood ratio and Pearson chi squared tests were also performed and revealed p-values of .0198 and .0187, further indicating the likelihood that the difference in performance between students with prior experience and those without prior experience is statistically significant.

The full contingency table from JMP is shown below to give a visual representation and more detail on what was presented above. The no/yes on the left side of the table indicates whether or not students had experience with sequences and series prior to entering the course. The no/yes on the top of the table indicates whether or not students answered the question correctly:

Figure 1: Contingency Table – Question six correct by experience

Experience? Question 6 Correct?

Count Total % Col % Row %	no	yes	Total
no	88 49.16 67.18 79.28	23 12.85 47.92 20.72	111 62.01
yes	43 24.02 32.82 63.24	25 13.97 52.08 36.76	68 37.99
Total	131 73.18	48 26.82	179

Tests

N	DF	-LogLike	RSquare (U)
179	1	2.7146923	0.0261

Test	ChiSquare	Prob>ChiSq
Likelihood Ratio	5.429	0.0198*
Pearson	5.531	0.0187*

Fisher's Exact Test	Prob	Alternative Hypothesis
Left	0.9939	Prob(Question 6 Correct?=yes) is greater for Experience?=no than yes
Right	0.0153*	Prob(Question 6 Correct?=yes) is greater for Experience?=yes than no
2-Tail	0.0238*	Prob(Question 6 Correct?=yes) is different across Experience?

Discussion

There are several plausible explanations for answer choice C, which over 50 percent of the population thought was the correct answer. Note that the value for the sequence convergence in answer choice C is the same as the value of the series convergence. It is possible that students are having difficulty understanding the difference between sequence notation and series notation. In other words, their concept image of sequences and series may include the fact that the symbols are interchangeable. A review of existing literature on semiotics confirms that students have a difficult time reading and understanding symbols in mathematics (Marjoram, 1974; Chirume, 2012; Earle, 1977). However, more research is needed to understand the role of symbols on student concept images of sequences and series.

Another possible reason for answer choice C is that students don't see the difference between sequences and series because of their use in everyday conversation. That is to say, in the English language, the words “sequence” and “series” are interchangeable. As an example, consider the sentence, “A long sequence of events has led me to a career in mathematics education.” Replacing the word ‘sequence’ with the word ‘series’ leaves the meaning of this sentence unchanged. However, sequences and series are very different in mathematics. Perhaps further research can explore the effect of the English language on student concept images of sequences and series.

The results above indicate that students that had some prior experience with sequences and series performed significantly better than students whose first experience with sequences and series came in this particular second semester calculus course. This prior experience, however, varied greatly. Some students noted on the cover sheet that they had seen sequences and series in BC calculus in high school. Others said that they were familiar with finite series from high school precalculus courses. Still others noted that they recognized series from the definition of the integral in a first semester calculus course. Regardless of the level and depth of experience, some exposure to sequences and series prior to entering the course seemed to have helped. It is worth noting, however, that even among the students with prior experience, most students, about 64 percent, still answered the question incorrectly.

It is difficult to know exactly how the concept images of those who answered this question correctly differ from those that did not answer the question correctly. It does appear, however, that the concept images of those that answered the question correctly may include a proper understanding of the nth term test and its contrapositive. Those that did not answer the question correctly appear to have an incomplete concept image of the nth term test. In particular, it appears as though they do not recognize the contrapositive of the nth term test. The difficulties that students had in this problem with the contrapositive of the nth term test are consistent with existing literature describing student and teacher difficulties understanding the logical equivalence of a statement and its contrapositive (Gregg, 1997).

Questions

I plan to continue the statistical analysis of the other problems on the multiple choice assessment. As this is part of a larger dissertation study, I also intend to analyze the transcripts of students that solved this problem during interviews. To help with further analysis, I would like my presentation to receive feedback on the following questions:

(1) What other correlations do you think I should look for as I analyze the other multiple choice items?
(2) What other methods can I use to further analyze student concept images of the nth term test aside from the transcript analysis?
(3) Are there any other methodological suggestions you have that might help enhance this study and future research projects on student concept images of sequences and series?

Implications

Moving forward, further research can be done to examine the effect of semiotics and word meanings in the English language on student concept images of sequences and series. Future research can also examine the understandings of sequences and series that students have in an undergraduate real analysis course, and this can be compared to the misconceptions of second semester calculus students. Results from studies such as the larger dissertation study have potential implications for the development of curriculum materials and teaching strategies that can be used to strengthen student concept images.

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Let's Talk About Teaching: Investigating Instructors' Social Networks

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Researchers who evaluate efforts to improve STEM undergraduate education have recently begun to explore the importance of instructors' informal teaching discussion networks. These informal networks allow for the flow of knowledge between instructors that can include information about how to implement research-based instructional practices and creative perspectives that lead to innovative solutions to address localized classroom challenges. In this report, we reanalyze the network data from three pioneering studies in this area to explore the features of mathematics department networks as compared to other STEM department networks at multiple institutions. We plan to discuss implications of these features on the design and implementation of change efforts.

Key words: Social Network Analysis, Instructional Change, Academic Departments

Numerous STEM-focused calls for enhancement of undergraduate education have been focused on improving instruction. These calls have resulted in the creation of evidence-based instructional practices that have been shown to improve student learning outcomes. However, many instructors remain unaware of these practices or face challenges in adopting these techniques to their unique classroom environments and student population. Change efforts have recognized these challenges and the importance of the networks of instructors that provide expertise and creative ideas to successfully implement and coordinate instructional practices. For example, Kezar (2014) specifically identifies the need for social network analysis to harness the power of these networks to inform instructional change.

Social Network Analysis (SNA) is an investigation of social phenomena via the techniques of graph theory. In SNA, the vertices of a graph are individuals, the edges are relationships between two individuals, and the graph is called a sociogram. In this study, the individuals are STEM instructors and the relationships that connect the individuals are discussions about teaching. Each graph consists of all of the instructors in an academic department and the discussions among them. Figure 1 provides a sample discussion network in a STEM department.[1]

We use this preliminary report to begin to analyze datasets of networks from multiple departments. In particular, we are interested in investigating if mathematics departments have unique features when compared to other STEM networks. For example, mathematics departments often have comparatively large numbers of non-major students who take many lower and upper division courses. This results in large numbers of students per mathematics course and requires more instructors to support multiple sections. In other STEM departments, such as engineering, students are often majors and class sizes are smaller. Furthermore, mathematics departments are part of a larger disciplinary culture that influences how instructors are trained in graduate school and encultured into the discipline. We suspect that these disciplinary differences might manifest in varying department social network structures. Finally,

[1] All images and analysis were completed using UCINET 6 and NetDraw software (Borgatti, 2002; Borgatti, Everett, & Freeman, 2002).

as many academic departments do have similar policies and purposes, we may uncover similar social network features that reproduce themselves despite institutional and disciplinary differences. This finding in itself would be insightful and imply that change efforts in any STEM department would have similar network-based advantages and challenges. Our final report on this study will discuss these implications.

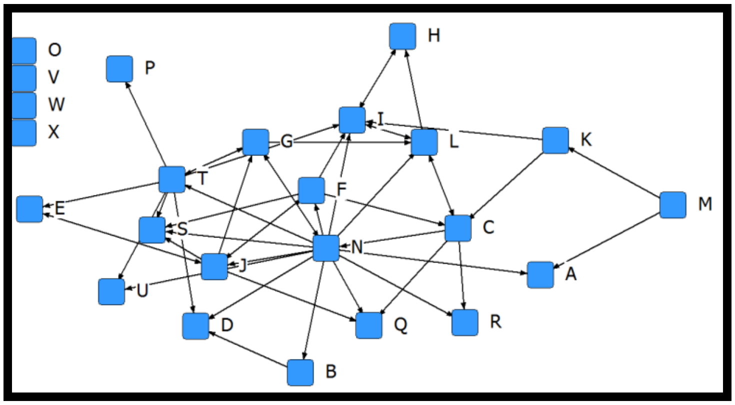

Figure 1: Discussion network sociogram where vertices are individuals and edges are discussions about teaching.

Theoretical Perspective

The goal of instructional change efforts is to improve the practice of instructors in the classroom. Changing an individual's practice requires addressing not only the individual's knowledge and abilities but also the contextual environment within which the individual works. Thus, the development of curriculum and practices by external experts that are meant to be adopted directly by instructors are not likely to be successful. This process does not provide flexibility or insight into the specific context of the individual. Networks may be especially important in remedying this challenge by allowing for adaption and adoption of evidence-based instructional practice. First, network members will be familiar with the challenges of the context and provide opportunities for collective sensemaking to articulate the purpose and features of the practice (Kezar & Eckel, 2000). Second, network structures that have infrequent but trustworthy relationships connecting subgroups of individuals promote innovation because of the ability for an individual to hear and create novel ideas (Levin & Cross, 2015). Finally, changing behavior at the department level can create supports and pressure for even more individuals to change their practice, which in turn will increase the impact of the change efforts (DeHaan, 2005).

Three Studies on Discussion Networks in Departments

This preliminary report represents the cooperative efforts of multiple investigators to reanalyze datasets in order to identify trends across multiple departments and institutions. The datasets represent three different studies, with different methods and different purposes. In our reanalysis, we are challenged to reconcile as many differences as possible across the datasets and to acknowledge those that could not be remedied in both the methods and results. In this section, we discuss the similarities and differences in the methods of the three studies.

The first study investigated social networks of mathematics departments across multiple institutions. In this study, members of mathematics departments at six different institutions were given an online survey to report discussion networks. This study included graduate students as members of the department. The survey listed all of the members of the department and asked respondents to mark each person "with whom they discussed instructional activities" during the last term in which they taught courses. Other items of the survey measured advice networks, friendship networks, and collective trust measures.

The second and third study shared similar survey designs. In both of these studies, respondents were given the opportunity to list up to seven individuals within the department with whom they "discussed teaching-related issues at least once a month." The members of the departments were provided in dropdown lists and the respondent also included the frequency with which the discussions occurred: nearly every day, weekly, monthly, and less than once a month. If the discussion occurred less than once a month, then no relationship was recorded between the two individuals. The second study included 15 STEM departments at a single institution. The third study was of six STEM departments at a single institution and included advice networks and data were collected twice (two years apart).

Reanalysis of Social Network Data

In order to analyze the data of the three studies, we made the following adjustments and choices. First, we added professional rank as an attribute for each individual. This allows us to remove graduate students from the sample, if necessary. Next, we chose to measure discussion relationships only if they occurred during a term in the first study, and at least once a month in the second and third study. We were unable to reconcile the difference in the nomination methods. In the first study, respondents could list ties with all individuals within the department. In the second and third study, this list was limited to seven people. We will need to discuss these differences when we are using measures that are likely to be greatly influenced by nomination approach, such as density of the network. Finally, we investigated if sample bias was influencing our results by comparing descriptive statistics of study one, study two, and study three before beginning our comparison of department-level network measures.

Preliminary Results and Lessons Learned

We begin by reporting descriptive statistics of each of the study's networks. We are interested in determining if institutional and/or study-based characteristics influence metrics, and if so, which metrics. Table 1 displays the average values and standard deviation for network degree centralization, average degree, density, average distance, and diameter. [2] Because of the limited number of departments, we do not make statistical claims.

From this table, we can see that the studies have the most impact on the degree centralization of the network, and relatively smaller impact on the other metrics. Because study one also was only mathematics departments, we may consider if this difference in degree centralization was partially due to the disciplinary culture. However, recall that study one had nomination methods that did not limit respondents to seven entries. This may also be the cause of the difference. In the future we hope to investigate this finding with even more mathematics departments' metrics and values. Future work will theorize what impact these metrics are likely to have on change efforts through sensemaking, creativity, and social norms.

Table 1: Average values and standard deviation for common network metrics of each study

	Degree Centralization	**Average degree**	**Density**	**Average Distance**	**Diameter**
Study 1 Average (Standard Deviation)	40.1 (9.7)	4.3 (0.9)	0.13 (.05)	2.2 (0.4)	4.5 (0.8)
Study 2 Average (Standard Deviation)	25.4 (7.3)	3.3 (0.8)	0.14 (0.06)	2.6 (0.3)	5.3 (0.9)
Study 3 Average (Standard Deviation)	20.4 (5.5)	3.6 (1.1)	0.13 (0.06)	2.4 (0.5)	4.9 (1.1)

These preliminary results give us confidence in pursuing our initial research questions regarding the differences or similarities among networks of various departments. We also stress the importance of coordinating research efforts in order to make large-scale impact on instructional change efforts. If each researcher has different survey designs and measurements with no standardization, then finding implications with broad application will be difficult. We therefore, call for a standard practice among network researchers in this area to produce studies that can build on the findings of one another and support the development of this area of research.

Audience Discussion Questions

The audience can help further this study be providing discussion around the following questions.

- Do audience members have anecdotal experience of large-scale differences of mathematics departments from other STEM disciplines?
- Do audience members believe that the data adjustments necessary for comparing networks are likely to challenge the validity of the study?

[2] The degree centralization is a measure of the degree to which ties are concentrated among a few key individuals. Average degree is average number of edges each individual has. Average distance is the average number of edges that span the distance between any two individuals in the network. The diameter of the network is the shortest distance between the two most distant individuals. For calculations of these metrics see Wasserman and Faust (1994).

- What other features of networks do audience members expect to be important for improving instructional practices?

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Instrumental Genesis and Generalization in Multivariable Calculus

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Raising Calculus to the Surface is a multi-year project designed to introduce important topics from multivariable calculus through the use of physical manipulatives. This report focuses on data collected through a series of task-based interviews with multivariable calculus students enrolled in a course featuring these manipulatives. To explain the students' activity, a two-dimensional framework was designed based upon characterizations of their interaction with the instruments and the generality of their mathematical activity. The report concludes by discussing the contributions to the field and possible future uses of the framework.

Key words: Multivariable Calculus, Instrumental Genesis

Literature Review

Multivariable calculus is a course that is important for the introduction of ideas that are complex and essential for STEM students (PCAST, 2012). Some research has been done on student conceptions in multivariable calculus of function (e.g. Martinez-Planell & Trigueros-Gaisman, 2012), derivative (Martinez-Planell, Trigueros-Gaisman & McGee, 2014), and integral (Jones & Dorko, 2015), yet the overall body of work remains thin. Further, little is understood about the impact of non-traditional instructional approaches on student learning of these key ideas.

Raising Calculus to the Surface is a multi-year project with an innovative curriculum designed to introduce important topics from multivariable calculus through student exploration with physical manipulatives. Students use, as representations of two-variable functions, surfaces which are molded from clear plastic and have a dry erase surface. Accompanying tools include an inclinometer (used to measure the slope at a point on the surface in a given direction), domain mats (dry erase sheets with coordinate lines or contour lines), and squares (made of clear dry erase plastic and used to simulate a tangent plane). Students in small groups complete activity sheets in-class, which emphasize collaborative learning, student inquiry, and measurement with quantitative reasoning (for further details see Wangberg & Johnson (2013)).

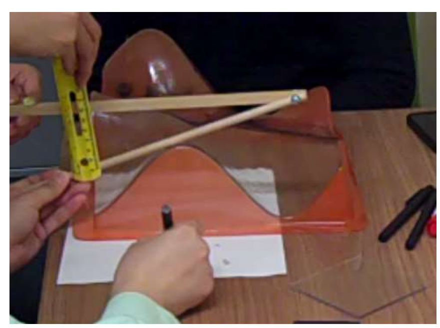

Figure 1

For students who use tools to complete tasks and answer questions in their activities on multivariable calculus, little is understood about how they enhance their understanding of calculus, or what role the tools play in that process.

Research Question

At the nexus of these issues lies the following research question:

What is the interplay between instrumental genesis and generalization as students develop conceptions of multivariable calculus during exploration involving the use of a physical manipulative?

Theoretical Perspective

Verillon & Rabardel (1995) presented Rabardel's theory of instrumental genesis to explain the complex process by which a person engaged in achieving a goal adopts the use of some assisting object. The material object when first introduced is an artifact. For it to be a productive tool, the user must attach to the artifact a role in solving the present task. Actions and behaviors cognitively organized by the user for a set of situations comprise a utilization scheme. Schemes can be constructed personally by the user, or received in a social context. The result of instrumental genesis is to have an instrument, an artifact endowed with a set of utilization schemes for tasks, a combination of material object and cognitive structures. During instrumental genesis, the artifact shapes the user through interactions which enhance the user's understanding of the subject matter, a process known as instrumentation. Additionally, the user shapes the artifact by developing schemes for interacting with the artifact, a process known as instrumentalization. Thus, as user and instrument develop their partnership, each one causes a transformation in the other. Subsequent to the development of the theory, instrumental genesis was applied in mathematics education to understand student use of graphing calculators, computer algebra systems (Artigue, 2002), and dynamic geometry software (Leung & Chan & Lopez-Real, 2006).

The curriculum used by students in this project has as its general pedagogical pattern exploration of a concrete scenario followed by mathematizing the observations into a rigorous formula. Thus the students' inductive reasoning is a natural target of inquiry. This form of reasoning is both one of the most important higher-level cognitive functions, and one of the most complex (Zhong et al., 2009). For this analysis, we use the lens of generalization as described by Tall (1991). Generalization is “an extension of familiar processes” in order to “operate on a broader range of examples” (Tall, 1991, p11-12).

Methodology

The data for this preliminary report were obtained from semi-structured task-based interviews with ten multivariable calculus students enrolled in a class using the Raising Calculus to the Surface materials. All students completed an activity sheet in class meant to introduce the concept of linear approximation for functions of two variables. In the interviews, which all took place within three days of that activity, students reflected on their experiences and responded to prompts further probing their ideas of linearization and differentiation in multivariable calculus. The data was analyzed using grounded theory (Strauss & Corbin, 1990). After an initial viewing of the data, categories naturally arose, and the data was reviewed and coded to confirm those categories.

Results

Analysis of the interviews revealed that students’ interactions with the artifacts depended both upon whether the student engaged in instrumentalization or instrumentation in regards to the artifact and whether the engagement occurred while the student was addressing a specific or general case of the mathematical phenomena. Therefore, we created a framework which classifies student interactions with the artifacts using those two dimensions (as illustrated in Table 1.)

Table 1

		Level of Generality	
		Specific	*General*
Relationship to instrument	*Instrumentalization*	Instrumentalization-Specific	Instrumentalization-General
	Instrumentation	Instrumentation-Specific	Instrumentation-General

Instrumentalization - Specific

Interactions were identified as instrumentalization-specific when the student brought prior conceptualizations of a specific case of a mathematical phenomenon to bear upon an artifact and as a result created schemes for utilizing the artifact meaningfully in this specific context.

Instrumentalization-Specific Example. During a previous conversation about the meaning of the term "partial derivative," R described the concept as a ratio of the change of the height of a function with respect to a change in either the x or y directions. The student was asked to determine the partial derivative at a specific point on the surface and given an inclinometer and a ruler as artifacts to use along with the surface. An inclinometer is a device that consists of two wooden rods connected at a hinge (see Figure 1). On one of the rods is a level which allows the user to confirm that one of the arms is horizontal while manipulating the other arm to represent the slope of the surface. Once introduced to the artifacts, R began experimenting with holding the inclinometer at different orientations until he determined which of those orientations best represented his understanding of partial derivatives.

> R: And this one [references the horizontal arm] we try to do it in such a way that this little ball is in the middle [references the bubble used for leveling the arm]. Then we will have something like this [holds the inclinometer such that it has a horizontal arm and the other arm reflects the slope of the surface] … Then you measure how high it is [holds a ruler vertically to the device]. So it will be just like a triangle.

The student then proceeded to explain how to use the ruler to measure the change in height and change in distance as indicated by the inclinometer in order to calculate the partial derivative as the ratio of these two measured quantities.

The student used his prior understanding of the partial derivative as a ratio of change in height over change in distance in order to create schemes for interacting with the surface and the inclinometer, therefore this was classified as instrumentalization-specific.

Instrumentalization - General

Interactions were identified as instrumentalization-general when the student brought prior conceptualizations of a general case of a mathematical phenomenon to bear upon an artifact and as a result created schemes for engaging with the artifact.

Instrumentalization-General Example. A group of three students were asked to determine whether the partial derivative of a two variable function at a point should be equal to the partial derivative of the tangent plane to that surface at that point. One student, J, began to argue that the two partial derivatives should be equal by relating back to his experiences in single variable calculus. He picked up a plastic square and used it along with the surface to reason about this relationship.

> J: If we take this and put it here. [Student picks up the plastic square and places it on the surface as a tangent plane.] … So you just cut it right here, [student makes a downward chopping motion with his hand across the plane and the surface] and you view it sideways. You would see a function with a line at that point.

The student proceeded to articulate that this representation should be viewed as similar to the single variable calculus relationship between a function and its tangent line.

It is important to note that this was the student's first experience encountering the plastic square. Once it was introduced, the student developed a scheme of placing the plastic square on the surface to represent a tangent plane along with a "cutting" scheme for viewing a cross-section of the surface. This development of new schemes for interacting with the surface led us to classify this action as instrumentalization. Additionally, the student was engaged in a conversation about the relationship between the tangent plane and the surface at any point in the domain, so the student's actions were not about a specific instance of this relationship, but rather about all such relationships across the surface, leading us to classify the activity as instrumentalization within a generalized context.

Instrumentation - Specific

Interactions were identified as instrumentation-specific when the student's interactions with the artifact led the student to develop new conceptions of a specific case of a mathematical phenomenon.

Instrumentation-Specific Example. A group of three students were asked to determine whether the partial derivative of a two variable function at a point should be equal to the partial derivative of the tangent plane to that surface at that point. The group first considered a single point, marked in blue on the surface, to determine whether they are the same at that point. During this conversation, one student, M, picked up the inclinometer and used a previously developed scheme to measure the partial derivative on both the surface and on the plastic square placed tangent to that surface.

> M: [while measuring the partial derivatives with the inclinometer] I guess it's the same. For either the plane or the surface, because this plane is tangent to the blue dot…

In this activity we see M used previously developed schemes, namely the scheme for measuring a partial derivative and the scheme for representing a tangent plane, to convince himself that in this case both calculations would yield the same result and thus the partial derivatives of the surface and of the tangent plane at that one point must be equal. Therefore this was classified as instrumentation-specific.

Instrumentation - General

Interactions were identified as instrumentation-general when the student's interactions with the artifact led the student to develop new conceptions of a general case of a mathematical phenomenon.

Instrumentation-General Example. A group of two students were challenged to use only the value of a multivariable function and its partial derivatives at a point to approximate a nearby value of the multivariable function. The students used the surface and the plastic square along with schemes for interacting with these artifacts developed during the classroom activity to reason about how to approximate the nearby value and why that value is an approximation and not the exact value of the multivariable function.

> [Student places the plastic square on the surface as a tangent plane and uses the dry-erase marker to draw lines parallel to the x and y-axes. The student then labels the nearby point in question on both the surface and the tangent plane.]
> P: We can get the value of this point on the plane, but we cannot get the value on the surface. It's an approximation, because the slope will change. In the x and y direction the slope will change at any point. On the plane it's not going to change.

In this activity the student evoked previously developed schemes for using the instruments to reason about linear approximation. It was through the use of those schemes that the student identified the reason why the value should not be exact: that the surface had a changing slope while the tangent plane had a constant slope. Earlier in the interview, the students were unable to explain why this result should not be exact. Because the students used their prior utilization schemes for the instruments about partial derivatives and tangent planes to develop this new understanding about approximation which holds at every point on the surface, we classified the activity as instrumentation within a generalized context.

Discussion

In this report, we have aimed to extend the knowledge of the field related to the crucial area of student understanding in multivariable calculus. Due to the complexity of the material, individual examples may be easier for students to grasp initially, and it can be a challenge to extend these to general principles. Further, an additional challenge is presented by the 3-dimensional nature of the mathematical objects. Students can use 3-dimensional tools to aid their study, but their use can be a complex process.

We have designed a framework to better understand student activity while using the Raising Calculus to the Surface materials. The research presented in this report extends the perspective of instrumental genesis in two important ways. First, it incorporates into the perspective a measure of the level of generality, allowing for analysis based both upon activity with the instrument and the type of mathematical activity in which the student is engaged. Second, the perspective of instrumental genesis has been primarily used to describe engagement with technological instruments; our research extends this framework to apply to the use of physical models and measurement tools.

When the categories described in this framework are coordinated temporally, they can create rich descriptions of student instrument use. Using this framework to describe mathematical activity, not only by categorization, but by moves between the different states of instrumental genesis and level of generality, may be a fruitful area of future study.

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A Continued Exploration of Self – Inquiry in the Context of Proof and Problem Solving

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Self-inquiry is the process of posing questions to oneself while solving a problem. The authors' previous work has explored the self-inquiry of undergraduate mathematics majors and a mathematics professor. Student self-inquiry was explored via structured interviews requiring the solution of both mathematical and non-mathematical problems. The professor's self-inquiry was explored through self-reporting of questions asked in an advanced problem-solving context. Using transcripts of the student interviews, a coding scheme for questions posed was developed and extended after coding the professor's self-inquiry. Previous results will again be highlighted here but will be followed by a discussion of self-inquiry in the context of an introduction to mathematical proof course. Data from the introduction to proof course is being collected and will be analyzed using the already developed coding scheme. This analysis will be compared and contrasted to previous self-inquiry results and we will present questions about possible future directions for exploring self-inquiry.

Key words: Problem Solving, Proof, Self-Inquiry, Logic, Questioning

Many teachers refuse to simply answer a student's question; instead, these teachers insist on responding to the student's misconceptions with other related questions that the student can answer, slowly scaffolding the student's responses until the student has answered (knowingly or unknowingly) their own question. This method, when done correctly, allows the student to recollect related knowledge, receive a confidence boost in their own knowledge of the subject, and receive a lesson in problem-solving strategies that could be utilized to solve future problems. This method of answering questions with other questions seems to work extremely well for student ownership of material, but the question remains as to why students don't ask themselves some or all of these leading questions. Since the student is capable of answering the posed questions that lead them to the solution, what is stopping the student from posing these questions themselves? Is effective self-inquiry a mark of a "good" student? What types of questions do these "good" students ask themselves while problem solving? More importantly, how can we foster pedagogical knowledge from these "good" students' questions so that teachers can guide all students toward productive self-inquiry?

We would expect that experts in mathematics do things differently than the masses. It therefore makes sense to also rigorously study exactly what characterizes expert mathematical thought, ultimately aiming to transfer this understanding to better educate undergraduates in mathematics. Indeed, much recent work in undergraduate mathematics education has explored this very idea. From how experts read proofs (Inglis & Alcock, 2012) and vet the work of their peers (Inglis, Mejia-Ramos, Weber & Alcock, 2013) to how they make conjectures (Belnap & Parrott, 2013) and use metaphors/perceptuo-motor activity (Soto-Johnson, H., Oehrtman, M., Noblet, K., Roberson, L., & Rozner, S., 2012), a clearer picture of expert mathematical practices is beginning to emerge. Adding expert self-inquiry to this developing picture of mathematical practices and comparing and contrasting it to undergraduate self-inquiry may shed further light on teaching practices that foster productive self-inquiry.

This preliminary report will summarize the authors' previous forays into self-inquiry, and motivate and describe the most recent data collection efforts in an introduction to mathematical proofs course. Finally we will pose questions and lead a discussion with the audience about data analysis, relevance of self-inquiry in RUME and possible future directions.

The Initial Exploration Into Self-Inquiry

A detailed initial exploration motivated by the questions posed above (Grundmeier, Retsek & Stepanek, 2013) suggests marked differences between the questioning profiles of "strong", "average" and "weak" students. This work led to a classification of the questions being posed by undergraduates. The following question tree was developed to exhaust the coding of all questions posed. The question tree will be explained in greater detail during the presentation, but essentially serves to group questions during problem solving into three main categories (and many subsequent subcategories) ranging from "static" to "dynamic" on a spectrum of action taken in the course of problem solving. See Figure 1.

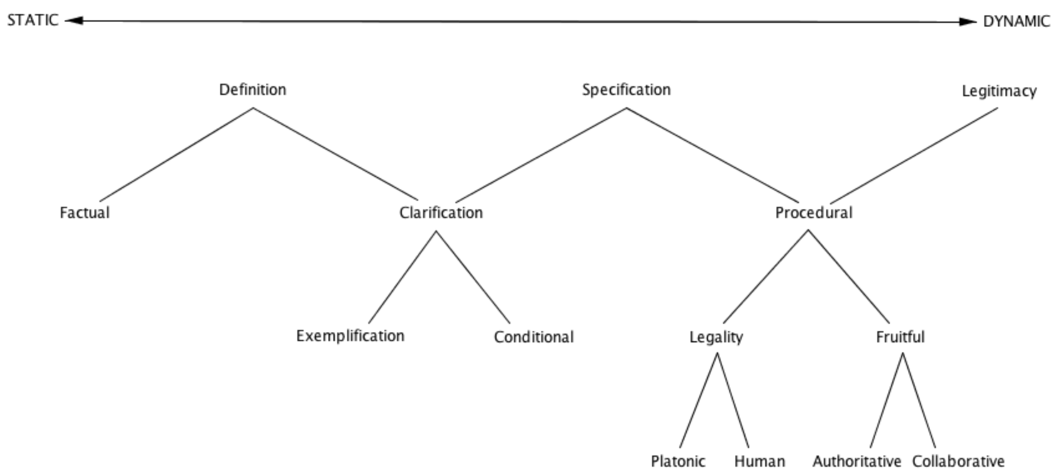

Figure 1: The question tree

In order to explore the self-inquiry of "good" students the authors defined the statistic RSQ (Relative Success Quotient) and calculated an RSQ for all students. To calculate the RSQ the authors focused on the 11 upper division courses that had been taken by at least 7 of the participants. For each course the average GPA and standard deviation of grades were calculated for the last 5 years of course offerings. A participant's RSQ is then calculated as the average number of standard deviations their grades are away from the mean for the courses they had completed from the 11 chosen. Participants clearly fell into three RSQ categories, deemed high, middle and low, and data has been organized accordingly. The table below highlights data from the mathematical task.

	Low	**Middle**	**High**
RSQ	*-0.167*	*0.479*	*.879*
Average # of Q's	13.75	16	9.2
Definition Q's	29.09%	18.75%	28.26%
Specification Q's	29.09%	46.875%	26.09%
Legitimacy Q's	41.82%	34.38%	45.65%

It is interesting to note that the students with a high RSQ were more efficient problem solvers who asked fewer questions. More interesting, though, is that these fewer questions focused on legitimizing their problem solving efforts and a smaller percentage of questions that served the purpose of specifying the problem-solving situation or related to definitions.

Comparison to an Experts Self – Inquiry

In order to shed further light on these questions, the authors designed and undertook a similar data collection process wherein a single "expert" recorded his own self-inquiry over an extended period of study on an advanced mathematical topic. The overarching goal of this study was to compare and contrast these expert questions to those of undergraduate students (Grundmeier, Retsek & Stepanek, 2013). Following an identical coding scheme, analysis of expert questions posed during the problem solving process shed further light on what makes for "good" self-inquiry and tested the adequacy and completeness of earlier question coding schemes.

As the main activity in a quarter long sabbatical the participant worked through the majority of the first two chapters of the text *Real Analysis: Modern Techniques and Their Applications* (Folland, 1984). This text was chosen because it is frequently used in graduate course work and would allow for a faculty mentor just in case mathematical questions needed to be referred to a colleague.

The participant's typical plan for working on the material was to carefully read each section of the text while noting questions that arose. He then attempted the problems that had been assigned in a recent Real Analysis course. While attempting to solve each problem the participant would document all questions that arose as well as the time between each question. Many problems required multiple attempts before a solution or proof was reached and questions during each attempt were recorded separately. For example, in organizing the data many headings such as "Section 2.3, Problem #22 attempt #2" appear. The choice to record these questions separately was made for a number of reasons. First there was often a significant amount of time between attempts, as the participant might have tried another problem in between or needed to sleep on the strategy he was using. Second, the participant assumed there would be overlap between the questions asked which might be important to analyze and discuss. Finally, it may be interesting to determine if the types of questions asked were different after some time subconsciously considering the problem. Working through this process for the first two chapters of the text led to the collection of 404 questions. A broad comparison of self-inquiry is presented in the table below.

	Low	**Middle**	**High**	**Expert**
RSQ	*-0.167*	*0.479*	*.879*	*NA*
Average # of Q's	13.75	16	9.2	10.54
Definition Q's	29.09%	18.75%	28.26%	25.7
Specification Q's	29.09%	46.875%	26.09%	42.9
Legitimacy Q's	41.82%	34.38%	45.65%	31.4

This initial analysis was followed by a more detailed analysis by question type that is presented in the tables below. All data represent percentage of questions within question type.

Definition Questions				
	Low	Middle	High	Expert
Factual - Definition	0	8.3	7.7	35.2
Clarification - Definition	81.3	75	53.8	44.4
Exemplification - Clarification - Definition	12.5	8.3	30.8	11.1
Conditional - Clarification - Definition	6.2	8.4	7.7	9.3

Specification Questions				
	Low	Middle	High	Expert
Clarification - Specification	56.2	33.3	41.6	21.1
Conditional - Clarification - Specification	18.8	33.3	16.7	22.2
Procedural - Specification	18.8	23.4	25	45.6
Human - Legality - Procedural - Specification	6.2	10	16.7	0
Exemplification - Clarification - Specification	0	0	0	11.1

Legitimizing Questions				
	Low	Middle	High	Expert
Procedural - Legitimacy	21.7	0	4.7	19.7
Platonic - Legality - Procedural - Legitimacy	60.9	77.3	76.2	10.6
Human - Legality - Procedural - Legitimacy	8.7	18.2	14.3	1.5
Collaborative - Fruitful - Procedural - Legitimacy	8.7	0	0	47.0
Authoritative - Fruitful - Procedural - Legitimacy	0	4.5	4.8	21.2

This analysis suggests the following points that will be discussed during this report.

- Expert may be more satisfied than undergraduates with precise and working senses of definitions with less need for exemplification.
- Undergraduates focus information gathering on clarification while the expert focuses information gathering on execution of procedures.
- Expert uses exemplification in the context of specification while undergraduates don't.
- Undergraduates focus legitimization on checking their work while the expert generates ideas for progress.

Further Data Collection and Analysis

With the goal of continued refinement of the question tree and to continue our foray into self-inquiry the authors will collect further data in an introduction to proof course during the fall 2016 quarter. While we have explored self-inquiry during the problem solving process we have decided that a potentially fruitful further direction is a focused exploration of students' self-inquiry at the beginning of the problem solving / proving process. During the proof course students will be required to state the first question that comes to mind after their initial consideration of the problem statement. This inquiry requirement will be pervasive throughout the course and be an expectation on all assignments. With approximately 50 students in two sections of the course and the requirement of student's to submit a portfolio with all problem solutions / proofs from the course the author's have the potential to collect at least 5000 questions and problem solutions to code.

Once data collection is complete the authors will code students' self-inquiry using the question tree as well as score their problem solutions with the goal of addressing the following questions.

1. Are there similarities and/or differences between self-inquiry in the context of mathematical proof and self-inquiry in the previous contexts explored?
2. Does the initial self-inquiry of "good" students in the context of this course differ from other students?
3. Is their any correlation between type of initial question and quality of the problem solution / proof?

Questions for the Audience

While related research has been conducted in secondary education and reading comprehension (Kramarski & Dudai, 2009; King, 1989) and in general mathematical thinking (Schoenfeld, 1992), it seems that the self-inquiry of undergraduates and experts has not been explored. Therefore another goal of this project is to continue this line of inquiry and add to the current mathematics education research related to problem solving and mathematical proof. The following questions for the audience may help shape our future research direction:

1. Is self-inquiry a relevant enough topic to the RUME community to deserve more attention and exploration?
2. Are there other potentially fruitful ways to analyze this data set and/or make comparisons of self-inquiry?
3. What other data collection tools or research design options would help explore self-inquiry?
4. Should we attempt further data collection of expert self-inquiry? If so, how might we do this is some authentic way.
5. How do we extend our work beyond the first question asked in the context of mathematical proof?

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Undergraduates' Reasoning about Integration of Complex Functions within Three Worlds of Mathematics

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Recent research illustrates the importance of studying students' nuanced mathematical argumentation, as well as students' tendency to invoke attributes of real numbers that no longer apply to situations in complex analysis. This preliminary report explicates a study exploring undergraduate student pairs' reasoning about integration of complex functions. I am particularly interested in students' attention to the idiosyncratic hypotheses of powerful integration theorems as they evaluate integrals. Here reasoning is treated as contributing to collective argumentation within one or more of Tall's (2013) three worlds of mathematics. Data were collected via task-based, semistructured interviews with pairs of undergraduates to elicit such reasoning, and classroom observations of the six class sessions devoted to integration prior to the interviews. All interviews have been transcribed and current analysis consists of conducting a Toulmin (2003) analysis, augmented by a three-world classification. Potential implications of this work and connections to the associated literature are also discussed.

Key words: collective argumentation, complex variables, integration, reasoning

Introduction and Literature Review

According to Tall (2013), "Mathematics is often considered to be a logical and coherent subject, but the successive developments in mathematical thinking may involve a particular manner of working that is supportive in one context but becomes problematic in another" (p. xv). The mathematics education literature on the teaching and learning of complex numbers reveals that such difficulties can arise when learning complex analysis. For instance, Danenhower (2000) identified a theme of "thinking real, doing complex" (p. 101) wherein individuals demonstrated a proclivity towards invoking attributes of real numbers that do not necessarily apply in the complex setting. Troup (2015) found further evidence of this phenomenon when undergraduates reasoned about derivatives of complex functions.

It is possible, then, that undergraduates might be tempted to initially reason about integration of complex functions as area under a curve, as this is one common interpretation in the setting of certain real-valued functions. This could be especially prevalent given that even within the context of real-valued functions, the literature reveals numerous examples of students' difficulties with integration (Grundmeier, Hansen, & Sousa, 2006; Judson & Nishimori, 2005; Mahir, 2009; Orton, 1983; Palmiter, 1991; Rasslan & Tall, 2002). However, many of these studies documented the *product* of students' deficiencies and misconceptions rather than the *process* of students' reasoning. As such, while students might end up with faulty conclusions about integration, their process of reasoning might actually be teeming with healthy connections to intuition or past experiences. Indeed, if nurtured properly, such connections between experientially-based intuition and formal mathematics could benefit students' reasoning in courses such as complex variables or analysis (Soto-Johnson, Hancock, & Oehrtman, 2016).

Moreover, by carefully documenting students' successful reasoning about undergraduate mathematics topics, we are able to gain insight into "what deep understanding and complex justifications are possible for students as they engage in mathematics" (Wawro, 2015, p. 355).

Students' reasoning within the subject of complex variables could particularly benefit from such an investigation, as the mathematical activity within this course is often situated somewhere between formal proof and symbolic calculation. In particular, students that integrate complex functions often invoke powerful theorems, which rely on idiosyncratic hypotheses and draw on ideas from topology and real analysis. While formal proof is typically not the focus of undergraduate courses in complex variables (Committee on the Undergraduate Program in Mathematics, 2015), application of such theorems requires that students at least recognize when these hypotheses apply. Hence it is possible that students might draw upon a combination of intuition, visualization, symbolic manipulation, and formal deduction when integrating complex functions. Accordingly, integration of complex functions serves as an appropriate topic to elicit the complex justifications that Wawro advocated for. Integration of complex functions is also an important topic for undergraduates with respect to practical applications. For instance, it is extensively used in physics and engineering to analyze and compute flux and potential. Moreover, one can apply techniques using integration of complex functions in order to drastically simplify or enable evaluation of certain real-valued integrals.

Despite the practical and theoretical assets inherent to integration of complex functions, there exists no educational research regarding undergraduates' reasoning in this mathematical domain. In particular, it is unclear as of yet how undergraduate students reason algebraically, geometrically, and formally with the notion of integration of complex functions. This study serves to ameliorate this gap in the literature and to inform the teaching and learning of complex variables by investigating undergraduates' multifaceted argumentation about integration of complex functions. Using Tall's (2013) Three Worlds of Mathematics framework, my ongoing research seeks to answer the following guiding questions:

1. How do pairs of undergraduate students attend to the idiosyncratic assumptions present in integration theorems, when evaluating specific integrals?
2. How do pairs of undergraduate students invoke the embodied, symbolic, and formal worlds during collective argumentation regarding integration of complex functions?

Clearly, this requires a careful consideration about what constitutes mathematical reasoning. According to the National Council of Teachers of Mathematics (NCTM), reasoning is characterized as "the process of drawing conclusions on the basis of evidence or stated assumptions" (NCTM, 2009; p. 4). Hence, because reasoning is not directly observable as a mental process, researchers can use individuals' argumentation, including the components mentioned by the NCTM, as a window into the mind. A common model used to document individuals' argumentation was formulated by Toulmin (2003) and consists of six components: data, warrant, backing, qualifier, rebuttal, and claim. According to Toulmin, any argument is based upon the arguer attempting to convince his or her audience of some *claim* (C), or asserted conclusion. This claim is necessarily grounded in foundational evidence, or *data* (D), on which the claim is based. The arguer can then supply a *warrant* (W) justifying the link between the given data and the purported claim. A *modal qualifier* (Q) is often necessary to explicitly reference "the degree of force which our data confer on our claim in virtue of our warrant" (p. 93). Depending on the warrant provided, there might also be circumstances in which the intended claim does not hold; in this case, conditions of *rebuttal* (R) are needed to indicate when the "general authority of the warrant would have to be set aside" (p. 94).

In the mathematics education literature, participants' mathematical argumentation has been analyzed with the aid of Toulmin's model in several different contexts. In the in-class setting, some researchers (Krummheuer, 1995; Krummheuer, 2007; Rasmussen et al., 2004; Stephan &

Rasmussen, 2002) felt that a reduced Toulmin model where the qualifier and rebuttal are omitted was appropriate, and rarely found evidence of explicit backing. Moreover, Krummheuer (2007) illuminated warrants invoked by the participants that did not even relate to the mathematical content directly, such as an appeal to the teacher's perceived authority. However, when more formal arguments such as proofs are concerned, researchers (Alcock & Weber, 2005; Inglis, Mejia-Ramos, & Simpson, 2007; Simpson, 2015) argued for the use of the full Toulmin model. They also mentioned that simply reading the finished product of a purported proof is inherently difficult because some components of the Toulmin model, such as backing and sometimes even the warrants, are implicit and cannot be elicited through real-time discourse with the proof author. Thus it would appear that an investigation into undergraduates' nuanced argumentation about integration of complex functions should adopt the full Toulmin model and incorporate opportunities for clarification, as in an interview setting.

Theoretical Perspective

In this study, I adopted Tall's (2013) Three Worlds of Mathematics as a way to theoretically orient my inquiry into undergraduates' reasoning pertaining to integration of complex functions. This perspective traces all mathematical knowledge back to three distinct but interrelated forms of thought: *conceptual-embodied*, *operational-symbolic,* and *axiomatic-formal.* According to Tall, conceptual embodiment begins with the study of objects and their properties, progressing towards mental visualization and eventually description through increasingly subtle language. The second world of operational symbolism grows out of actions on objects and is symbolized in potentially flexible ways via *procepts*, or symbols operating dually as process and concept (Tall, 2008). Tall's (2013) third world is that of *axiomatic formalism*, wherein individuals build "formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof" (p. 17). These three worlds can also combine to form, for example, *embodied symbolic* or *symbolic formal* reasoning.

As mentioned previously, Tall (2013) argued that our previous experiences with mathematics can either support or create conflict with new and abstracted mathematical notions. He referred to the knowledge structures predicated on these prior experiences as *met-befores*. Tall also argued that mathematical growth can be traced back to three innate *set-befores* of recognition, repetition, and language. These set-befores foster three forms of compression: categorization, encapsulation, and definition. Through this compression, individuals build so-called *crystalline* structures, which incorporate many equivalent formulations of a mathematical object and can be unpacked in various worlds.

These worlds can also lend additional specificity to a mathematical argument, in that "each world develops its own 'warrants for truth'" (Tall, 2004, p. 287). For instance, in the embodied world, truth is initially established based on what is *seen* to be true by the learner visually. In contrast, within the symbolic world, truth is established in arithmetic based on calculation. Finally, in the formal world, a statement is true either by assumption as an axiom, or because it can be proved formally from the axioms. Hence Tall's three-world perspective can complement the Toulmin analysis of a mathematical argument by adding specificity with regard to the types of backing and warrants used. As such, I classify participants' Toulmin components as embodied, symbolic, formal, or various mixtures of these, as viewed through Tall's three-world lens. Therefore, in the context of this study I define *reasoning* as mathematical argumentation within one or more of the three worlds.

Additional specificity with regard to backing exists due to Simpson (2015), who examined how earlier papers (e.g. Evens & Houssart, 2004; Inglis et al., 2007; Stephan & Rasmussen, 2002) reported its use. Simpson found that there were three distinct roles for backing of warrants within an argument. The first, *backing for the warrant's validity*, was invoked to explain why the warrant applies to a given argument. A second type of backing served to "highlight the logical field in which the warrants are acceptable," which Simpson characterized as *backing for the warrant's field* (p. 12). Finally, a third type, *backing for the warrant's correctness*, illustrated that a given warrant is actually correct.

Given that my study considers how pairs of students reason about integration tasks, it is additionally important that I consider how each individual contributes to an argument. According to Krummheuer (1995), *collective argumentation* takes place when multiple participants construct arguments through emergent social interaction. As part of this interaction, an individual invokes one of four speaker roles. As *author*, a speaker is both syntactically and semantically responsible for his or her statement. On the other hand, a speaker might claim responsibility for neither the semantic nor syntactic aspects of an utterance, in which case he or she acts as *relayer*. Alternatively, a speaker "uses the words of someone else to mean *something different* from the meaning ascribed to the utterance of the original speaker" (Krummheuer, 2007, p. 67, italics in original) as a *ghostee*. Finally, when a speaker revoices a previously mentioned idea using his or her own language, he or she is acting as *spokesman*.

Methods

In order to rigorously address my research questions, I enlisted the help of two pairs of undergraduate students to partake in a videotaped, semistructured (Merriam, 2009), task-based interview comprised of two 90-minute portions. Participants were selected from undergraduate students at a military academy in the United States, enrolled in the complex variables course during the spring 2015 semester. My first pair of participants consisted of Sean and Riley. Sean was a fourth-year physics and mathematics major and Riley was a second-year applied mathematics major with a cyberwarfare concentration. The second pair consisted of Dan, a third-year mathematics major, and Frank, a second-year applied mathematics major with an aero concentration. Table 1 summarizes my methods of analysis for the interview data; note that step one is now complete.

To obtain a rich understanding of the context in which these participants learned about integration of complex functions, I also observed and videotaped six class sessions at participants' undergraduate institution. These observations and ensuing field notes allowed me to document what mathematical content was introduced and emphasized during the integration unit in the complex variables course. They also allowed me to discern the nature of mathematical argumentation that was deemed appropriate for the complex variables course.

Discussion

Recently, Soto-Johnson et al. (2016) found that mathematicians drew upon a wealth of personal embodied experiences when discussing their conceptions of continuity of complex functions. Although their study pertained to the population of mathematicians, Soto-Johnson et al. hypothesized that meaningfully connecting experientially-based intuition and formal mathematics could also benefit students' reasoning in courses such as complex variables. In part, my research serves to reveal how undergraduates reconcile their met-befores with the formal idiosyncrasies present in integration theorems. Hence, my hope is that my inquiry into students' reasoning about integration might illuminate ways in which instructors can cultivate healthy connections between students' embodied intuition and rigorous, formal mathematics.

Additionally, I anticipate that my study will complement and extend the mathematics education literature regarding students' mathematical argumentation. Wawro (2015) found that her participant's argumentative successes were primarily due to the fact that he was "flexible in his use of symbolic representations, proficient in navigating the various interpretations of matrix equations, and explicit in referencing concept definitions within his justifications" (p. 336). Accordingly, this suggests a potentially strong connection between representational fluency and effective mathematical argumentation. I anticipate that the results of my study will serve to corroborate this finding by exploring how students' embodiment, symbolism, and formalism collectively inform their argumentation about integration.

Table 1
Interview Analysis Summary

Step	Description
1. Transcription	Document participants' exact verbiage; provide rich description of gesture and written inscriptions
2. Code Toulmin components	Classify participants' arguments for each task according to data, warrant, backing, rebuttal, qualifier, and claim as in Toulmin's (2003) model
3. Code for speaker roles	Categorize participants' speaking roles as that of author, relayer, ghostee, or spokesman (Levinson, 1988, as cited in Krummheuer, 2007)
4. Code for three worlds	Further classify participants' arguments from step 2 according to Tall's (2013) three worlds framework
5. Code for backing types	Refine coded arguments from step 3 by categorizing backing according to the types identified by Simpson (2015)
6. Thematic analysis	Reflect on the coded data from steps 1-5 to identify common themes within and across interviews

Potential Questions for the Audience

(1) How might I connect each of the analysis pieces to address my research questions? (2) How might one develop a teaching experiment where instructors draw students' attention to any implicit assumptions used when evaluating integrals in complex analysis?

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DNR-Based Professional Development: Factors that Afford or Constrain Implementation

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DNR-based professional development (DBPD) is a long-running program spanning seven years with multiple cohorts of in-service secondary mathematics teacher participants. This report investigates teacher change among five key variables: facilitating public debate, using holistic problems, attending to students' intellectual need, attending to meaning of quantities and use of students' contributions. Is there evidence that DBPD contributed to higher implementation among participants over time? What factors afford/constrain DNR implementation over time? Classroom observation data indicate the largest impact was found in teachers' attention to meaning of quantities and students' intellectual necessity while interview data provide insights to what affords and constrains DNR implementation.

Key words: Teacher change, DNR-based instruction, Intellectual Need, Professional Development, In-service

In their review of 106 articles reporting findings on mathematics professional development programs between 1985 and 2008, Goldsmith, Doerr and Lewis (2013, p. 21) point out that "existing research tends to focus on program effectiveness rather than on teachers' learning," while much less has been said about "*how* teachers develop knowledge, beliefs, or instructional practices". Indeed, DNR researchers have conducted, and continue to conduct, studies exploring the processes by which teachers develop knowledge of mathematics, pedagogy and student thinking while also describing the contexts in which this learning occurs. For example, Harel, Fuller and Soto (2014) described characteristics of the teaching practice of a DNR expert aimed at helping a group of teachers transition from result pattern generalizing (RPG), a form of the empirical proof scheme, to process pattern generalizing (PPG), a form of deductive proof scheme (RPG and PPG are described in Harel, 2001). Harel (2013b, 2014) describe two of the content areas covered by DBPD in summer institutes and follow-up sessions.

An extension of these studies – our current study – examines the extent to which the evidence that DNR-based professional development (DBPD) has contributed to the implementation of DNR principles by participating teachers and, in keeping with the recommendation of Goldsmith et al, seeks information about the factors that help or hinder implementation. This study reports such findings and contributes in areas by specifying a set of teaching practices we take as indicators of DNR implementation and combining direct classroom observation with participant interviews to yield both information about program effectiveness and provide insight into learning processes.

Theoretical Framework

DNR-based instruction in mathematics (Harel, 2008a; Harel, 2008b), is a theoretical framework that stipulates conditions for achieving critical goals such as provoking students' intellectual need to learn mathematics, helping them acquire mathematical ways of understanding and ways of thinking, and assuring that they internalize and retain the mathematics they learn. Though DNR is a system of premises, constructs and instructional principles, here we attend to only the necessity principle and the construct of teaching practices.

In DNR, the necessity principle stipulates that in order for students to learn what teachers intend they must have an *intellectual need* for that targeted piece of knowledge and the construction of this knowledge is brought about through a series of equilibrium and disequilibrium phases as learners engage in problematic situations (Harel, 2013a).

Harel (2008b) defines, "a *teaching action* is a curricular or instructional measure or decision a teacher carries out for the purpose of achieving a cognitive objective, establishing a new didactical contract (Brousseau, 1997), or implementing an existing one." Characteristics of teaching actions are called *teaching behaviors*. Teaching actions and teaching behaviors taken together are called teaching practices. The necessity principle, described above, is a foundational claim about the importance of teacher awareness of students' thinking and difficulties, the need to plan for and handle students' problem-solving approaches, as well as the importance of availing oneself of the opportunities that could arise if the right problems are posed and students are given the chance to make and express their own meaning for problems that are not broken down for them in advance.

Methods

DBPD consisted of two related support structures: (1) summer institutes and mid-year follow-up and sessions similar to what was described in Harel, Fuller and Soto (2014) and (2) on-site professional development. Both efforts targeted the teachers' knowledge of mathematics (in terms of ways of understanding and ways of thinking), knowledge of student learning, and knowledge of pedagogy. This report examined DNR implementation for 34 teachers, focusing on the following five teaching practices: *public debate, holistic problems, intellectual need, attention to meaning,* and *taking contributions seriously* as defined below.

- *Assigning Holistic Problems:* A holistic problem is one where a person must figure out, from the problem statement, the elements needed for its solution (Harel and Stevens, 2011). It does not contain hints or cues as to what is needed to solve it. In contrast, a non-holistic problem is broken down into small parts, each of which attends to one or two isolated elements. Often each of such parts is a one-step problem. (No/Yes)
- *Intellectual need:* Do students have a need for understanding the mathematics the teacher intends to teach? Does the teacher appeal to a problematic situation that puzzles students when introducing new mathematics? (No/Yes)
- *Attention to meaning*: When a problem has a context, unknown quantities have meaning with respect to that context (e.g. units related to quantities). Does the teacher attempt to attend to the meaning of quantities within the context of the problem? (No/Yes)
- *Public debate*: Is there evidence to believe that the whole class is following the discussion? Is the teacher making a successful effort to engage the whole class in debate through questioning and solicitation of contributions? Public debate also includes the need to evaluate mental images and their validity and efficiency. (No/Yes)
- *Taking student contributions seriously*: A student's contribution is considered to be taken seriously when it is allowed to live in the public space for discussion without immediate teacher evaluation. When taking contributions seriously, teachers solicit ideas and mental images from students, and facilitate public debate about these ideas to highlight and critique both underlying mathematics. (No/Yes)

Repeated classroom observations of teacher participants were conducted and used to evaluate participants' implementation of DNR and to chart changes in participants' teaching over time. Two forms of data were generated using these observations. First, researchers examined whether or not a particular teaching practice was demonstrated in each participant's classroom during an entire classroom observation across two later years of the program's existence. Second, researchers looked at interview data with participants conducted after classroom observations that could be used to give insight into factors that afford or constrain implementation. A summary of findings follows.

Findings

Three forms of findings constitute results of investigation into the program's impact on DNR implementation.

Classroom Observation Data (All classes)

The following tables show percentages of classrooms in which a particular teaching practice was present during year 3 and 4 of the program.

Attention to Meaning		
	Year 3	Year 4
Present	25.9 %	61.5 %

Intellectual Need		
	Year 3	Year 4
Present	63 %	80.8 %

Public Debate		
	Year 3	Year 4
Present	70.4 %	76.9 %

Taking student contributions seriously		
	Year 3	Year 4
Present	81.5%	84.6 %

Holistic Problems		
	Year 3	Year 4
Present	81.5%	80.8 %

The most dramatic results can be seen in the change of the percentage of classrooms in which teachers attended to meaning and to the intellectual need of students and raise questions about aspects of DBPD that may have contributed to these shifts. Similarly, one might ask what may have constrained increased implementation of holding public debate, taking student contributions seriously and using holistic problems. There are many possibilities, including the difficulties inherent in the teaching practices themselves, the students' comfort levels sharing ideas in class or even a potential ceiling effect. Note that each of these three teaching practices began with a relatively high rate of classrooms demonstrating them, 70%, 81.5% and 81.5% respectively. For this reason, we decided to investigate individual data.

Interview Data. During interviews with project staff, the following factors were reported as reasons participants felt they were able to implement DNR in their own classrooms.

- Resonance of the major principles of the DNR framework with their own values, including a strong dedication to problem-solving, belief in student-centered instruction, and a belief that current curriculum needs modifications to accomplish meaningful instruction.
- Community-driven programmatic elements such as a time, place, and leadership for sharing ideas and peer support building, one-on-one mentoring from more experienced teachers and regular contact with community members

- Academic freedom/autonomy to implement DNR-compatible teaching practices by administrators
- The program's on-site support
- Adoption of the Common Core State Standards in Mathematics and compatibility with DNR
- A strong sense of dedication to students

Factors reported to constrain DNR implementation included:
- Dissonance with the major principles of DNR, either throughout or growing with time
- Difficulty of Didactical Engineering/adapting DNR's instructional principles to curriculum
- Lack of academic freedom with respect to content and/or teaching practices (either perceived or embodied in administrators and/or parents including pressure to teach algorithmic proficiency at sites/desire to maintain consistent testing results with high SES from parents, students and administrators and large class sizes generating extraneous work)
- Self-efficacy and retention issues

Discussion

We began the report with insights from Goldsmith, Doerr and Lewis (2013) describing some of what has been done and what still needs attention in mathematics professional development over nearly three decades. On the surface, this report seems to do the opposite of what Goldsmith et al recommend, appearing, at least initially, to focus on "program effectiveness" rather than teacher learning. However, we argue that DNR researchers have been carrying out the work of investigating how teachers develop the forms of knowledge, beliefs and instructional practices needed in the common core era for quite some time. Evaluation of a PD program attempting to implement DNR is a necessary part of theory building. Indeed, knowledge about the kinds of hurdles teachers face when attempting to turn theory into practice can inform the theory itself as researchers attempt to make adjustments to their theoretical perspectives moving forward, starting a new cycle of theory building, implementation and evaluation.

Intended Questions for the Audience

1. We are aware that there lurking methodological questions we have yet to ask about classroom observation protocols with respect to each of the five teaching practices. What are some of them? Would duration help on some of these variables?
2. How can these findings inform future DBPD? DNR as a theoretical framework?
3. We have data, by individual participant, with respect to each of the five teaching practices discussed. I have bar graphs readily available with mean and standard deviations. How might this data be used to address our questions?

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An exploration of students' discourse using Sim2Bil within group work: A commognitive perspective

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This paper reports on critical aspects of three engineering students' discourse in group work using a digital tool called Sim2Bil while solving mathematical tasks. Applying a commognitive perspective, where mathematical discourse is characterized by words used, visual mediators applied, narratives developed and routines established, we investigate how these characteristics are influenced by the technological environment. It is found that all of the aspects of the students' discourse are influenced by Sim2Bil. For instance, a "trial and error" routine directly connected to the use of the tool is present in the students' discourse.

Key words: Commognition, Discourse, Engineering, Group work, Technological environment

Introduction

Working with mathematics involves using different resources. For instance, students use computers, calculators, and textbooks. These artifacts have been developed through history, and technological tools in particular have undergone major changes in the last decades. Nowadays, software for simulations, animations, graph plotting, dynamic geometry, CAS, and so forth, have emerged in mathematics education. These cultural tools change practices and give possibilities to work with mathematics in new ways.

Much research has been conducted on students' interaction with technological tools (Beatty & Geiger, 2010; Wijers, Jonker, & Drijvers, 2010), and an important aspect of future education may lay in expanding technology-supported collaborative work between students (Lowyck, 2014). The technological tools considered are, for instance, computer algebra systems (CAS) (e.g. Artigue, 2002) and tools for graph plotting (e.g. Swidan & Yerushalmy, 2014). Some studies have applied the commognitive framework to study how students are communicating using a digital tool (e.g. Ng, 2016) and the impact technological environments have on pupils' mathematical thinking (e.g. Sinclair & Yurita, 2008).

Previous studies conclude that technology can support students in their graphical approach to integrals (Berry & Nyman, 2003; Swidan & Yerushalmy, 2014). To supplement this research, in the study reported in the present paper we set up a kinematical context for integrals, and the students use integrals for a simulation of movement within a technological environment where students work with a digital tool called Sim2Bil. The students' interaction with words and visualizations and how they were used in their communication has been analyzed in a previous paper by Hogstad, Isabwe, and Vos (2016). In the present paper, we will view the data from a discursive perspective, analyzing the characteristics of the students' discourse with the aim of investigating how the digital tool influences the discourse.

Theoretical Framework

The commognitive theory (Sfard, 2008) takes a participationist perspective (Sfard, 1998) – learning is seen as the process of changing and individualizing discourse to become increasingly able to participate in a certain discourse community. The theory provides analytical tools for analyzing discourses, and especially mathematical discourse. From a commognitive perspective, discourses are different types of communication "set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). Thus, doing mathematics is engaging in mathematical discourse, and this mathematical discourse – indeed any discourse – can be distinguished by four characteristics: word use, visual mediators, narratives and routines (ibid, p. 133-134). *Word use* refers to words specific to the discourse or common words used in discourse-specific ways. *Visual mediators* are "the visible objects that are operated upon as a part of the process of communication" (Sfard, 2008, p.133). Every image of concretes, symbols and icons operated on in communication are visual mediators. *Narratives* are "any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to endorsement or rejection" within the discourse (ibid, p.134). Sfard includes mathematical definitions, theorems, axioms and proofs, as well as formulas and equations, under the term narrative. Other examples could be statements students make during a problem solving process, such as the statement they consider to be a solution to a given problem. Finally, *routines* are "repetitive patterns characteristic of the given discourse" (ibid, p. 135). Such repetitive patterns can be seen in almost any aspect of mathematical discourse: in forms of categorizing, modes of attending to the environment, in ways of viewing situations as "the same" or different and so on (ibid, p.135).

Routines are divided into three types: explorations, deeds and rituals. *Explorations* are routines where the goal is to produce endorsed narratives (ibid, p. 223). *Deeds* involve actions performed with the goal of achieving change in objects. A routine may either be an exploration or a deed, depending on what the performers (in our case the students) are trying to achieve. In particular, the difference can become unclear in those cases when "the objects on which the deed is performed are, in themselves, discursive rather than primary" (p.239). *Rituals* are those actions whose goal is neither to produce an endorsed narrative nor a change in objects but rather to gain the attention and approval of others and to become a part of a social group.

Methods

The overall aim of the study is to investigate how students' mathematical practice is influenced by working in a specially designed small-scale digital learning environment outside of regular lectures. The participants in the activity reported on in this paper were three engineering students. They were in the second semester of their first year at university. Their first year included courses in calculus, linear algebra and physics (including kinematics). Sim2Bil was not familiar to them.

Sim2Bil is an interactive digital tool, the interface of which is shown in Figure 1. At the top left there are two cars – one red, one green – which can drive from a starting line to a finish line. This is called the simulation window. Below, there are visualized two graphs, one for the velocity-time function of each car. This is called the graph window. At the bottom right, there are velocity functions for each of the cars. A user can insert parameters here to make 3rd degree polynomials. This is called the formula window. At the top right is a menu window, which is not used in the study and will not be further explained. As a default setting, there are two functions given in the formula window: v1 = 100 (for the green car); and v2 = 50t (for the red car). With these functions, the students can engage in the environment by pressing the start-button to see the cars run differently and finish together. The distance between the starting line and the finish line is 400 meters. The preset functions make the cars reach the finish line simultaneously (thus framing the forthcoming requirements), and they also make the graphical representations appear, thus showing how a user can operate with the tool. The shading of the areas under the graphs will appear gradually through an animation, increasing with time. These areas represent the distance travelled for the cars.

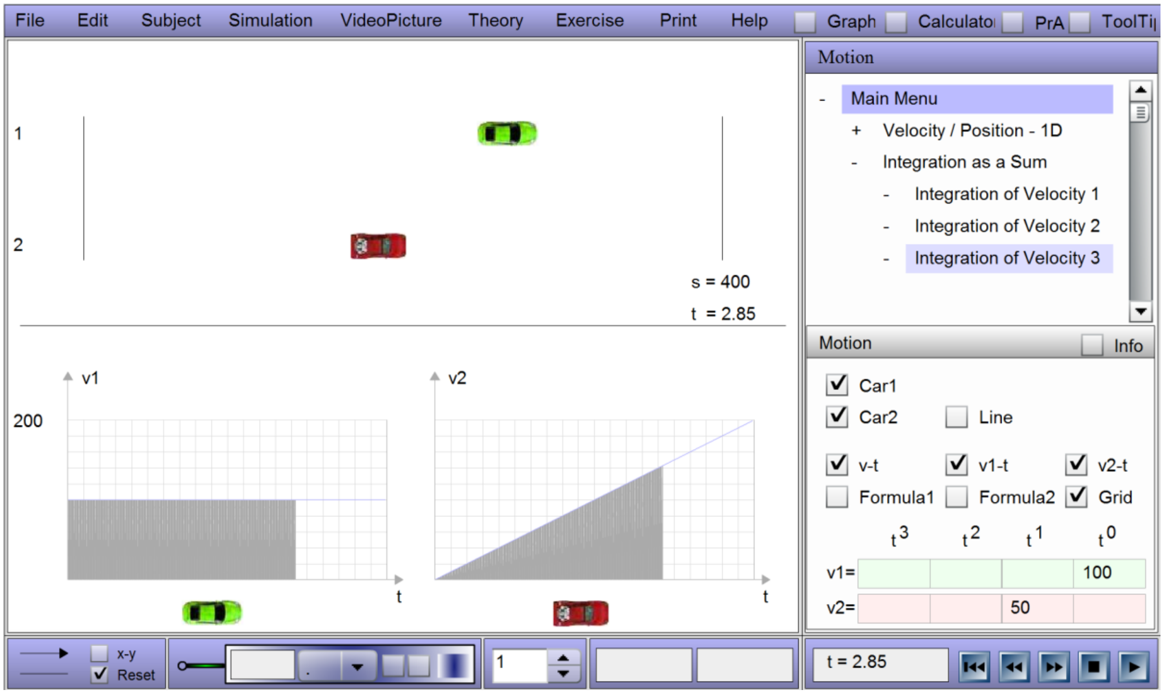

Figure 1. Interface of Sim2Bil.

The students received the following problem consisting of four tasks:

> Look around the screen. Find the start button (down at the right corner). The simulation runs for maximum 4 seconds.
>
> a) Press "Start" in the program, and explain to each other what happens. What do the shaded areas represent?
> b) Determine other numbers in the table, so that the cars run with different velocities, and arrive at the finish line at the same time.
> c) What can you do to make the green car be only half way when the red car reaches the finish line?
> d) Find the velocities of the green and the red car (v1 and v2), so that v2 is half of v1 when they reach the finish line simultaneously at 4 sec. Can you prove that your answer is correct?

The problem is designed for collaboration between the students since the tasks can be solved using different approaches. The formulation of the problem was presented on paper and Sim2Bil was set up on a laptop in front of the students. Other available resources were a calculator, book of formulae, and pen and paper for writing. It was anticipated that the students would see relations between distances travelled, shaded areas under the graphs, and the integral of the velocity functions. The formula window constrains possible solutions by allowing the students to set in parameters only up to 3rd degree functions.

The activity was video recorded using two cameras: one moving, directed at the students and their writing; and one fixed, directed at the computer screen to capture mouse movement, students' inputs within the interface and gestures like pointing towards the computer screen. One researcher was present during the activity and introduced the interface and tasks to the students. No specific time frame was given for the group work, and it turned out that the students spent in total of 45 minutes on the tasks. The video recordings were then transcribed and coded in terms of the different characteristics of discourse. In what follows, for reasons of space, attention is restricted to the first two subtasks.

Analysis

Briefly describing the students' work on the subtasks, they begun by reading the tasks aloud. They then pressed the start-button and watched the cars run before discussing what the shaded areas represent. Thereafter they discussed the degree of velocity functions in order to find new parameters which they in turn set in the formula window to see if the cars arrive together.

Looking at the discursive practice of the students, a number of characteristic features can be found. Starting with discourse-specific words, there are several words used to describe mathematical objects, such as: function, graph, area, integral, unknown, rectangle and triangle to mention a few. Still, although the students are mainly engaging in mathematical discourse, they also use terminology that might be considered to belong mainly to discourses of other disciplines. For instance, there are words used which belong to a discourse of physics, such as: position, distance, velocity and acceleration. Actually, one of the students recognized the discourse as being within physics by stating at the end of the session: "that is enough physics for today". There are also a number of words directly connected to the interface, for instance start (-button), start line and finish line.

Different visual mediators are present in the discourse of the students. Some are of types traditionally connected to mathematical discourse, for instance mathematical symbols and graphs. These are in some cases written on paper, and in some cases are aspects of the digital tool. There are also visual mediators directly connected to the tool, such as the moving cars and the representation of the two velocity functions by a number of boxes (see the bottom right corner of the interface depicted in Figure 1). In addition, gestures such as pointing or tracing the shape of a parabola with your hand are also used. Such gestural actions seem to be used partly for making sure that the students are talking about the same objects.

Concerning narratives, these are also of different types – those connected to the tool and those connected to the underlying mathematical content. Mathematical narratives include, for instance, formulas for the velocity functions as general third-degree polynomials, as well as expressions for distances in terms of integrals of velocity functions. Narratives more specifically connected to the digital tool include interpretations of the various features of the interface in terms of a discourse of mathematics or physics:

Student E: What do the shaded areas represent? It's the area under the graph.
Student S: Yes. The area, which then is the distance. It is a function.

Another example concerns formulating connections between different features of the interface: "It [pointing at the graph] changes when we push here [by inserting parameters].

The influence of the Sim2Bil tool can also be seen in the types of routines present in the students' discourse. Some routines are directly connected to the use of the tool. One of the most common is “running the simulation”, that is, inserting values for the parameters, pressing start, watching the cars run and then pressing reset. On some occasions, this routine had the characteristics of a deed, concentrating on the running of the cars, with the students’ attention focused on whether they arrived simultaneously at the finish line. On other occasions, the routine was more exploratory, with the focus on the graphs describing the velocity functions, and how these influence the movement of the cars. This routine is closely connected to a routine which we might call “trial and error”. Here, values for the parameters are for the most part picked seemingly at random or at least without any explicit reasoning on the students’ part. Then the resulting effect on the graphs and the running of the cars is noted, new values are picked, the effects are noted, and so on. This process is denoted by the students as “playing around”, and particularly one of the students repeatedly invokes this when the other students shift their attention to symbolic and numerical calculations away from the screen: “Can’t we just play around a little? Isn’t that good?”

On the other hand, there are also examples of routines that are more conventionally mathematical, such as “calculating values of expressions”. The calculations consist of writing algebraic expressions (mostly integrals of polynomial velocity functions) on paper and then using a calculator to calculate numerical values. These routines were mostly of an exploratory character. You might argue that there is an element of the deed about them, since the focus is on the numerical results, but these numbers are in turn going to be used to formulate narratives about the comparative movement of the cars in the digital environment.

Discussion

What we have presented above is a mostly descriptive account of the characteristics of the students’ discourse when working on the tasks. Still, looking at these characteristics it is clear that the Sim2Bil tool influences all aspects of the discourse. In particular, there is evidence of routines directly connected to the tool. This corresponds to the findings of Sinclair and Yurita (2008), where the introduction of a digital tool (Geometer’s Sketchpad) changes the routines engaged in the classroom.

The study is also in tune with other studies reporting on new forms of communication mobilized in dynamic environments. For example, Ng (2016) shows how students utilized a variety of resources in their communication, and developed routines for conjecturing and verifying calculus relationships. Such routines can also be found in the discourse of the students in the present study.

The environment described in this study enabled the students to connect algebraic expressions, graphical representations and movements of objects. As the students in the present paper see the growth of the shaded areas as an animation in the graphical window, the idea of area as a function is visually mediated. Although the distinction between dynamic and static visual mediators has so far not been explicitly addressed in the commognitive framework, the distinction is important because of the potential for dynamic visual mediators to evoke mathematical relations, as also stated by Ng (2016).

However, the findings presented in the present paper are only a first step, and in future publications we intend to deepen this analysis, looking at what opportunities for learning are offered by the tool, and how the tool (and the tasks) influences the mathematical thinking and reasoning of the students, as manifested in their discursive practice.

Question for discussions: How to interpret the complex interplay between technological aspects and mathematical aspects?

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The role of undergraduate mathematics faculty in the development of African American male mathematics majors

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Historically Black Colleges and Universities (HBCUs) have a longstanding legacy of supporting African American students in mathematics. The undergraduate mathematics faculty members play a unique role in supporting and developing astute mathematics students, especially African American male students. This preliminary research report highlights the experiences of a cohort of 16 African American male mathematics majors at an all-male, private HBCU by investigating the role of the mathematics faculty members. Using qualitative research methods grounded in critical race theory, preliminary data show these African American male mathematics majors benefited (mathematically and racially) by their supportive mathematics faculty members.

Keywords: Undergraduate mathematics education, HBCUs, African American men, Supportive faculty

Introduction

This preliminary research report analyzes the mathematics experiences of a cohort of 16 African American male mathematics majors at an all-male, private Historically Black College/University (HBCU) in the southeastern region in the United States. HBCUs have a historical legacy of developing mathematics majors/mathematicians, and this institution was recently recognized by the American Mathematical Society (AMS) as the Programs that Make a Difference Award for their commitment to producing African American male mathematics majors to increase diversity in the mathematical sciences (Borum, Hilton, & Walker, 2016; Jett, 2013). This research report hones in on the role of undergraduate mathematics faculty members in the development of these 16 African American male mathematics majors. The overarching research aim was to ascertain intrinsic and extrinsic factors that led to the undergraduate mathematical persistence for this cohort of students.

As it stands, this research study adds to the body of scholarship investigating the schooling experiences of African American male students (see, e.g., Duncan, 2002; Harper, 2013; Noguera, 2008; Strayhorn, 2015). In addition, this study adds to the thread of research highlighting the mathematical strengths of African American male mathematics students (Berry, 2008; Stinson, 2013). Given that a large number of these studies have been conducted at the K–12 level, it is important to gain insights from studying African American male students who are persisting in college mathematics. African American male students' stories of mathematical undergraduate persistence are largely absent in the research literature. Thus, this research project is designed to fill this void in the research literature and shift the discourse concerning the mathematics experiences of African American male mathematics collegians with respect to their mathematics faculty members.

Review of the Literature

There have been fruitful efforts designed to improve the mathematics achievement outcomes of African American students. One effort that has been successful in promoting high levels of undergraduate mathematics performance among African American students is the

Mathematics Workshop Program (MWP) at the University of California, Berkeley (Fullilove & Treisman, 1990). The MWP is cited as being successful for the following reasons: the workshops create environments that promote mathematics academic excellence among peers; the students spend more time on learning activities and learning tasks as opposed to just solving mathematics problems; and the students who participate in MWP are believed to continue in college longer than those students who do not participate in the workshop because they obtain social and study skills that can be used throughout their college matriculation.

A research team at the University of Maryland Baltimore County studied high-achieving African American men (Hrabowski, Maton, & Greif, 1998). At this institution, researchers became concerned about the status of African American male students in college science, mathematics, and engineering (SME; SME is synonymous with STEM) majors and decided to learn more about this group by studying the habits of the highest-achieving students who were enrolled in the Meyerhoff Program. Although the program now serves students from all racial and ethnic backgrounds who desire to pursue a doctorate in the sciences or engineering, the first year consisted of African American male students only. Hrabowski et al. (1998) hoped to identify attitudes, behaviors, habits, perspectives, and strategies of the highest-achieving African American male students in the program. According to Hrabowski et al. (1998), the following factors are critical for success in college among African Americans in mathematics and science: an adequate high school academic preparation, analytical skills, strong study skills, time management skills, advising, academic as well as social integration, and motivation and support.

Ellington and Frederick (2010) examined the experiences of eight high-achieving junior and senior mathematics majors to ascertain the factors that shaped their mathematical success and persistence. Their findings revealed the majors' success was informed by participation in academic programs at the K–12 level and college scholarship programs, access to advanced mathematics courses, and support from family, classmates, and teachers. In another study, McGee (2015) investigated the factors undergirding academic resilience among 23 high-achieving African American mathematics and engineering majors at the junior, senior, and graduate levels. Using the life-story interview process, she reported on a subset of two participants (one Black female and one Black male) from her larger study. McGee introduced the Fragile and Robust Mathematical Identity Framework to understand the interplay between mathematics and racial identity. Using this framework, she found that her two participants were able to thrive in these majors while grappling with various forms of racialization.

While the previously mentioned studies have moved the field forward concerning the experiences of African American students, we know comparatively little about the role of undergraduate mathematics faculty in the development of African American (male) students. This study, therefore, complements and expands existing research efforts in the field by examining the role of mathematics faculty in the development of mathematics majors for this population of students. All in all, this research report builds on scholarship from scholars who honor the mathematical talents and gifts in African American students (see, e.g. Berry, 2008; Cooper, 2004; Delpit, 2012; Ellington & Frederick, 2010; Jett, Stinson, & Williams, 2015; Leonard & Martin, 2013; McGee & Martin, 2011; Stinson, 2006; Thompson & Lewis, 2005; Walker, 2006, 2014). Moreover, this study also reveals how complexities about the constructs of race and/or gender may influence the mathematical development of African American male students given the theoretical frame.

Theoretical Framework

The experiences of African American students have been documented in the mathematics education research literature, and scholars in the field have paid attention to how their raciliazed experiences influence their mathematics learning (Larnell, 2016; McGee, 2015). As such, Critical Race Theory (CRT) was employed as the theoretical framework for this research project (see Bell, 1992; DuBois, 1903/2003 for comprehensive discussions concerning the country's racial history). Racism is an institutionalized force that has been used both historically and currently to dismiss and oppress people of African descent and other people of color. Solórzano and Yosso (2002) argue that "substantive discussions of racism are missing from critical discourse in education" (p. 37). As it stands, issues of race and racism have been underexplored in mathematics education research (Martin, 2009). There are, however, a contemporary group of mathematics education researchers foregrounding issues of race and racism in their analyses (see, Jett, 2016; Larnell, 2016; McGee, 2015; Stinson, 2013; Terry, 2011). In an attempt to extend analogous mathematics education research drawing from race-based frameworks, CRT was used to examine the role of undergraduate mathematics faculty members in the development of a cohort of 16 African American male mathematics majors.

With CRT, there are five foundational tenets, and these tenets are the hallmarks driving this theoretical perspective. These philosophical underpinnings include the following:

1) CRT asserts that "racism is normal, not aberrant, in American society" (Delgado & Stefancic, 2000, p. xvi).
2) CRT adheres to interest convergence, which advances that the dominant culture advances racial justice and other race based initiatives when it serves their interest (Delgado & Stefancic, 2001).
3) CRT asserts that race is orchestrated as a social construction (Ladson-Billings, 2013).
4) CRT explores the intersectionality of various constructs such as race, sex, class, gender, and sexual orientation to explore how these intersections make for broader understandings of these constructs (Delgado & Stefancic, 2001).
5) CRT utilizes voice to serve as a counter-narrative to the dominant discourse surrounding racial groups (Dixson & Rousseau, 2005).

These tenets of CRT were be used to frame the interview questions and to analyze the data.

Research Question

The overarching research questions (RQ) for this portion of study were as follows:

RQ1: How do undergraduate mathematics faculty members (at this particular HBCU) either help or hinder the mathematical development of African American male mathematics majors?

RQ2: What are the (student) identified strengths and weaknesses of the undergraduate mathematics faculty members as it pertains to the development of African American male mathematics majors?

Methodology

This research study employed qualitative research methods. More specifically, the qualitative research data collection methods included the following: 1) a pre-survey, 2) a semi-structured interview, and 3) a member checking prompt (Bogdan & Biklen, 2007; Patton, 2015).

1) The pre-survey was given to the participants prior to the first interview. This pre-survey solicited information from the participants pertaining to their demographics, family dynamics, and education. The information obtained from the pre-survey was used to inform the

first interview as well as to substantiate the data for coding and analysis. 2) Next, the participants completed a semi-structured interview (ranging from one to two hours in length). The interview amplified the participants' voices by honoring and using their own words to share their mathematics experiences. The utilization of "voice" as well as narratives aligns with qualitative research methods and CRT's fifth tenet. 3) The final method included allowing the participants to member check my findings. In other words, the member checking aspect allowed the participants to verify whether I reported their words, findings, and interpretations accurately.

With regard to data analysis, I employed coding to analyze the data. After analyzing the individual interviews, I searched for similar and dissimilar patterns in the data and articulated explanations for different phenomena (Glesne, 2006). In addition, I wrote reflective notes in my researcher's notebook, which aided during both the data collection and coding processes (Bogdan & Biklen, 2007). Tenets of CRT were used to code and assist with analyzing their experiences as racialized beings. All in all, this qualitative data analysis process offered me an opportunity to verify my findings with the participants and to address questions pertinent to the analysis of data on the role of undergraduate mathematics faculty members in the development of African American male mathematics majors.

Preliminary Findings

Qualitative data have been collected for this research project, but the data are in the early stages of data analysis. However, preliminary data indicate that these 16 African American male mathematics majors were affirmed at their HBCU by their supportive mathematics faculty members. More specifically, the mathematics faculty members were dedicated to providing the mathematics majors with a challenging undergraduate curriculum. The faculty members were supporting and caring, and the majors spoke deliberately about the effective guidance and mentorship offered by these departmental members, especially the department chair (an African American male alumnus of the department). Conclusively, these faculty members had social constructions about who could be mathematically astute, and these ideological orientations informed their mathematical and racial empowerment of these African American male mathematics majors (Ladson-Billings, 2013).

A more thorough discussion of the preliminary finding concerning the role of the undergraduate mathematics faculty members will be shared during this presentation. In doing so, I will also share some ways to improve the undergraduate mathematics space as articulated by the majors. The data will be shared in light of the research questions, with connections to the previously cited literature, and with theoretical linkage to CRT.

Discussion Questions

The following discussion questions and prompts will allow participants in this session to engage in dialogue, offer feedback for strengthening the work, reflect on their own undergraduate mathematics practices, and recommend suggestions for future areas of scholarly exploration for this line of research:

• Please share any suggestions or insights from your experiences working with African American male mathematics majors that have yielded successful mathematics outcomes.
• Please share any suggestions or insights from your experiences working with mathematics majors from other marginalized groups that have yielded successful mathematics outcomes for that particular population of students.
• What are some implications of this work for undergraduate mathematics instructors?

• What are some implications of this work for undergraduate mathematics education researchers?
• What are your thoughts and recommendations concerning extending and furthering this work?

<u>Goals</u>

One goal of this session is to highlight the critical role of the mathematics faculty in the development of African American male mathematics majors at an all-male, private HBCU. This particular institution has a legacy of producing many African American male mathematics majors as espoused by national data and reports. Another goal is to disseminate more stories of mathematical persistence to influence and develop more African American male students into the mathematics pipeline who have a desire to explore various mathematical and mathematics-related pursuits. Finally, a goal is to generate more conversations concerning the participation and underrepresentation of African American male students in undergraduate mathematics degree programs.

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Abstraction and Quantitative Reasoning in Construction of Fractions as Operators

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Drawing from task-based interviews, classroom observation, and participants' homework, the present study examines ten middle grades preservice teachers' understanding of the role of fractions as operators, with an eye toward exploring how fractional reasoning is constructed. The results point to the construction of the reversible distributive partitioning scheme as a requisite for understanding fractions as operators. Further discussion will suggest that school curricula and teacher education programs may need to be adjusted to reflect more current understanding of both early childhood cognitive development and future teachers' fractional knowledge.

Key words: Cognition, Teacher Education-Preservice, Teacher Knowledge

Writing about four decades ago, Kieran (1976) remarked that "most school curriculum materials simply treat rational numbers as objects of computation. Hence, child and adolescents miss many of the important interpretations of rational numbers" (p. 102). Since then, substantial research has further emphasized that there are multiple interpretations—or "subconstructs"—of fractions (e.g., Behr, Khoury, Harel, Post, & Lesh, 1997), and numerous books have been written reflecting this point of review in teaching rational numbers (e.g., Clarke, Fisher, Marks, Ross, 2010). The current study concerns itself primarily with the role of fractions as "operators," referring to fractions that link two quantities of the same kind or of different kinds (see Vergnaud, 1988).

Students are normally taught fractions from third to fifth grade, before reaching the expected age range identified by Lovell (1972) for developing the proportionality schema (i.e., ages 12-14). Two subsequent studies on adolescents' comprehension of fractions as operators (Kieren, 1976) add some empirical weight to Lovell's (1972) earlier findings, showing a strong relationship between age level and scores that is consistent with this suggested age range. In that regard, Kieren (1976) suggest an important relationship between students' development of the proportionality schema and their comprehension of fractions as operators.

Post, Behr, and Lesh (1982) noted that understanding proportional reasoning is only possible once students have reached Piaget's "formal operational state," which generally happens in the fifth grade. In that case, many students – perhaps a majority – are not ready to learn the operator role of fractions at the time when fractional foundations are being laid in school curricula. However, in the middle grades, if students cannot understand fractions both as quantities and operators, this can be a barrier to their comprehension of later topics in mathematics.

In light of these considerations, this study investigates preservice teachers' understanding of fractions as operators through task–based clinical interviews designed to reveal any areas of struggle, in order to further analyze the nature of these issues—particularly in regard to what they can reveal about how fractional reasoning is constructed—and to discuss possible solutions.

Theoretical Framework

Steffe (2003) conducted a study that can serve as an exemplar of the radical constructivist approach that forms an essential basis of my epistemological and analytical framework. Steffe

(2003) adopted as part of his analytical framework the notion that both social interaction and an individual's action are crucial in learning. He regarded social interaction as a means of " generating situation" for the construction of students' cognitive schemes, and learning as a product of the "auto-regulation" of constructs within the individual's understanding. The emphasis on the individual's self-regulation and auto-regulation is in line with von Glasersfeld basic principles of radical constructivism (1990, p.22). What we perceive as reality is thus regarded as a construct within the mind of each individual.

Hackenberg and Lee (2015) showed that only MC3 (able to coordinate three levels of units) students could use fractions as multipliers in the context of writing equations. Unit coordination is related to the ability to construct certain schemes, such as distributive sharing and reversible distributive partitioning (Steffe & Olive, 2010). To illustrate, students who can share two identical bars among three people have constructed the "distributive sharing" scheme. To carry out this operation, they would partition each of the two bars into three parts and pull out one part from each bar, understanding that those two parts make up 2/6 of the original two bars. However, unless they had also constructed the "reversible distributive partitioning scheme," they would not understand that 1/3 of two bars is identical to the two individual thirds of each bar. Thus, students who have constructed the reversible distributive partitioning scheme understand that taking one-third of each of two objects (not necessarily identical) is equivalent to taking one-third of both objects together.

While I make use of a constructivist theoretical framework, my conceptual framework includes both the constructivist notions of fractional schemes and operations and Vergnaud's (1988) explanation of fractions as quantities and operators. For the purposes of the current study, it seems appropriate to make use of a qualitative case study design that will allow for a context-rich and focused investigation using task-based clinical interviews and participant observation.

Research Design

The present research is designed to address the following questions: How do middle grades preservice teachers' reason with fractions as operators?

1. What can their reasoning with fractions as operators reveal about their construction of fractional knowledge?
2. What is the nature of their struggles in constructing fractions as operators, and what are some possible solutions to these issues in their understanding?

Building on the concept of the clinical interview developed by Piaget (1951), the "constructivist teaching experiment" (see Steffe & Thompson, 2000, p. 285; Steffe, 2003) is a method by which the researcher's ongoing analysis can interact with data in a fluid way. According to a description by Steffe & Ulrich (2014), "[a] teacher/researcher, through reviewing the records of one or more earlier teaching episodes, may formulate hypotheses to be tested in the next episode…" (p.104). While this research tracks students' learning trajectories across successive teaching episodes, the pedagogical interventions themselves are supplied by the participants' course instructor, rather than directly by the interviewer. The method employed in the current research, therefore, may be regarded as a modified teaching experiment method; rather than one person acting as the "teacher-researcher" in "teaching episodes" (Steffe & Thompson, 2000). I conduct interviews as the "researcher," while the course instructor acts as the "teacher." The interviews themselves—without the expressly pedagogical element characteristic of the teaching episodes of a teaching experiment—closely resemble "task-based clinical interviews" (Goldin, 2000).

This study makes use of a multiple case study design, examining the fractional knowledge of preservice middle grades math teachers in a Math Education department at a major public university located in the Southeastern United States. The primary data source is a series of clinical interviews conducted over the course of two semesters using ten volunteer participants from a math content course. More specifically, I make use of data from task-based clinical interviews, as well as from classroom observation notes and homework submissions.

From among the initial volunteers in the math content course, I divide the participants into groups according to whether or not they have constructed the distributive sharing scheme. These selections are based on analysis of an initial round of interviews, classroom observation notes and homework submissions.

Result and Significance

All clinical interviews were recorded on video, transcribed verbatim, coded and annotated for emerging themes, and analyzed. At the beginning of the interview all of participants were unable to make composite units for fractions acting as operators, which means that they had not constructed the reversible distributive partitioning scheme. In fraction multiplication and fraction equation problems, two of the participants either could not produce the drawn model of certain problems, or they could not provide a coherent interpretation of their own drawings. All participants showed signs of confusion as a result of incorrectly applying procedural knowledge. Moreover, the current progress of my data analysis indicates that the construction of the distributive sharing scheme is necessary but not sufficient to ensure a students' conceptual understanding of fractions as operators.

During each interview, variations of fraction multiplication problems were given, and students were asked to design word problems and represent the problem situation visually. For one of the problems, 4/5 times 1/3, several students drew a rectangle and partitioning it into 3 rows and 4 columns (or vice versa). They shaded 1 row and 4 columns and identified the 4 parts in the overlapping region as their answer. One of the word problems a student designed for this problem was: "You have a recipe that calls for one-third cups of flour, and you want to make four fifths of the recipe, so how much flower do you need to use?" When I asked her to show one cup and one recipe, she identified the largest box as both one cup and one recipe. This indicates that she used both fractions as extensive quantities (instead of 4/5 as the operator to show the relationship between the quantity 1/3 and the product 4/15). This shows her limitations in representing two different units in one drawing and in using one fraction as on operator.

Another identifiable limitation was in making composite units. Before classroom instruction, when asked to show 7/3 of two unit bars, two students responded that they must find one-third of one bar and then repeat it 7 times. Thus, by partitioning both unit bars into three parts and repeating one of these parts seven times, in fact they showed 7/3 of just one bar. Their difficulty was in understanding the two bars together as one composite unit (treating it as the "whole" for 7/3).

After the completion of the unit on fractions in the participants' math course, students already knew that the correct answer to the problem (finding 7/3 of two unit bars) could be given as 14 parts, each being 1/3 the size of one unit bar. However, in explaining the anser with a drawing, three out of ten students showed limitations in partitioning and iterating with composite units. All students gave the answer as 14/3 of one bar, but three students still answered that they needed to understand 7/3 as equivalent to 14/6 to determine the result relative to two bars. If they could form one composite unit out of the two bars, they were able to find 1/3 of the two bars by partitioning each bars into thirds and taking 1/3 from each bar, after which they could also make a composite unit of these two parts and iterate it seven times to make 7/3 of the two unit bars. In that case, the students do not need to consider 1/3 of one bar as 1/6 of two bars. This shows the limitation of these three students' reasoning with iterable composite units, corresponding to limitations in using fractions as operators.

Conclusion

In fraction multiplication contexts, the multiplicand represents an extensive or intensive quantity with one unit as a whole. However, the multiplier shows the relationship between the multiplicand and the product of multiplication. Thus the whole of the multiplier becomes the multiplicand, and students have to be able to use the multiplicand as one composite unit to perform the operation. This requires students' construction of the distributive sharing scheme.

These preliminary findings, although limited in scope, may direct further research that can case additional light on the construction of fractional reasoning, both in preservice teachers and in primary school students. Considering the foundational importance of being able to understand fractions as operators, in the very least we need to ensure that preservice teachers have a strong grasp of this fundamental fractional concept in order to teach middle grades students effectively. Moreover, if further research lends additional support to the notion that primary school is generally too early for students to understand the operator role of fractions, school curricula may need to be adjusted to reflect this.

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Students' Social Adaptation to Mathematical Tasks

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In this study, an advanced undergraduate geometry class taught in an inquiry-based learning setting was observed for social and socio-mathematical norms. Three pairs of students engaged in three task-based, semi-structured interviews: paired, individually, then paired again, solving the Seven Bridges of Königsberg and related tasks. A fourth stimulated-recall interview was performed using episodes from the last paired interview. Classroom observations and interview discourses were open coded for themes, structure, and function to analyze the norms developed within the classroom and by each pair as shaped by their social interactions. Tentative findings include: 1) norms of consensus, autonomy, and argumentation produced within the classroom, 2) varying metaphors across interview contexts, and 3) reliance on empirical strategies rather than structural reasoning. In this preliminary report, evidence from collected data is shared and a brief discussion how these results could help inform IBL teaching methods is included.

Keywords: Active Learning Strategies, Cooperative Learning, Inquiry-based Learning, Socio-Cultural Theory, Transfer

The White House has issued a call to action for incorporating active STEM learning strategies in K-12 and higher education (White House Office of Science and Technology Policy, 2016). Some studies have shown the effectiveness of inquiry-based learning as an active learning strategy for all mathematics students and especially minority groups (Kogan & Laursen, 2014; Laursen, Hassi, Cogan, & Weston, 2014). Almost no literature can be found, however, on the socio-cultural theoretical underpinnings of the transfer of inquiry-based learning to individual work.

The purpose of the dissertation study shared in this preliminary report was to explore the social and socio-mathematical norms of classroom using inquiry-based learning and the reproduction (or non-reproduction) of those norms in an interview setting. The research questions of this ongoing study are:

1. What were the social and socio-mathematical norms of the classroom?
2. How were these norms reproduced (or not) by the students in the interview settings?
3. In what ways did changes in the social context of the interview settings affect the mathematical practices produced by the students?

Methods

The setting of the study was at a four-year university in the Rocky Mountain region in an upper level mathematics course. In this inquiry-based learning geometry course students worked in small groups on projects and submitted written reports, either as a group or individually. The instructor of the course, Dr. Jackson (pseudonym), had taught the course using inquiry-based learning on more than twenty occasions. All class sessions were observed using written notes, video recording, and audio recording. During these observations, the researcher (first author) described the propositions used by students and the instructor, as well as the perceived sources of authority for those propositions. Descriptions of propositions and sources of authority were then open coded for themes.

The students in the course were a mixture of elementary education majors with an emphasis in mathematics, and mathematics majors with emphases in liberal arts, applied mathematics, or secondary education. Six students, Leo, Jemma, Skye, Melinda, Phil, and Grant (pseudonyms) agreed to participate in interviews, which were conducted as a mixture of individual and paired settings, the first three being task-based and the fourth being a stimulated recall interview (See Figure 1 for a full description of the interview structure). Interviews were then transcribed in full and each utterance was coded for structure. Open coding was used to describe the function of each utterance. Examples of functions included expressing beliefs, proffering hypotheses, and making conjectures. In this report, interview analysis from only one pair, Leo and Jemma, is shared.

Interview	Setting	Purpose
Interview 1	Paired, Task-Based	To investigate reproduction or non-reproduction of social and socio-mathematical norms of the classroom while performing a novel task as a pair.
Interview 2	Individual, Task-Based	To investigate influence of paired setting on individual cognition.
Interview 3	Paired, Task-Based	To investigate the influence (or non-influence) of individual cognition on discourse of the paired setting.
Interview 4	Individual, Stimulated-Recall	To gain insight into the covert thoughts of participants during Interview 3.

Figure 1. Stages of the interviews

The Interview 1 Task was the traditional Seven Bridges of Königsberg. Pairs were prompted with a map and tasked with providing a path that crossed every bridge once and only once, or provide an explanation as to why no such path existed. The Interview 2 Task was an amendment to the Interview 1 task, in that the individuals were instructed to: 1) construct a bridge such that one could create a path that crossed each bridge once and only once starting in the Red District and ending in the Gold District but it was not possible to construct such a path starting in the Blue District and ending in the Gold District (see Figure 2), 2) construct a second bridge such that one could create a path starting in the Blue District and ending in the Gold District, but no such path could be created starting in the Red District and ending in the Gold District, and 3) construct a third path such that one could create paths that crossed each bridge once and only once, starting and ending in both the Red and Blue Districts.

The Interview 3 Task was to create a continuous curve that crossed each edge of a figure (See Figure 3) once and only once or provide an explanation as to why such a curve was not possible.

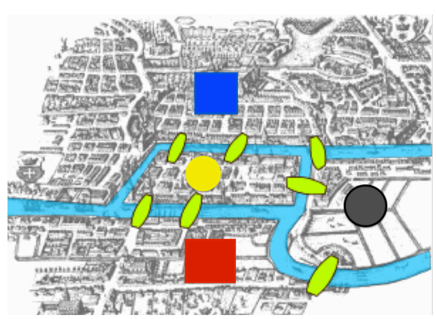

Figure 2. Seven Bridges of Königsberg with locations of Red District (southern square), Blue District (northern square), Gold District (central circle), and Grey District (eastern circle).

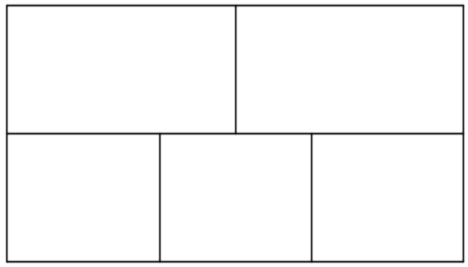

Figure 3. Figure of the Interview 3 Task

The Interview 4 was a stimulated recall interview using episodes video recorded from Interview 3. Episodes were chosen based on the propositions made during Interview 3. Participants were instructed to explain any thoughts or feelings they had during each episode, and could pause or rewind if they felt compelled to do so. After each episode, the interviewer (first author) would ask follow-up questions if the participant had not already addressed them.

Preliminary Results

Classroom Observation Findings

Three themes emerged in observing the classroom: *consensus*, *autonomy*, and *argumentation* (see Figure 4 for a summary of the themes and subthemes). Propositions were sorted into three categories: *beliefs*, *hypotheses*, and *conjectures*. Beliefs were defined as propositions that were made without prior evidence, hypotheses were defined as propositions made with unverified (by the participants) evidence, and conjectures were defined as propositions made with verified evidence. Categories of sources of authority were *group consensus*, *prior group consensus*, *authority figure*, *implicit*, *preference*, and *logic*. *Group consensus* was evidenced by a small group or the whole class reaching a sense of agreement. *Prior group consensus* was evidenced by a reference to consensuses that had been reached previously. *Authority figure* was evidenced by an appeal to either Dr. Jackson or the researcher (first author). *Implicit* was evidenced by no overt appeal to external authority, implying implicit appeal to the norms established within the classroom such as prior group consensus. *Preference* was evidenced by an appeal to personal preference.

In analyzing these themes and their interactions with one another, I could describe the evolution of the classroom could be described as follows: Students, in general, began the course lacking direction and authority to make and evaluate their own propositions. Dr. Jackson was able to source their authority in group consensus by holding whole-class discussions and building group consensus, as well as referencing these consensuses at later times. This afforded students with the opportunity to transition from lacking autonomy, to tentative autonomy, to free expression. With each new topic or discussion, however, this sense of autonomy could reset and students could revert back to lacking autonomy. By focusing on stylistic and structural aspects of argumentation, Dr. Jackson could then rebuild the students' autonomy by allowing them to express preferences and validate those preferences with logic. This cyclic process of building-referencing-losing authority continued throughout the course, even after the final project had been completed.

Theme	Subtheme	Propositions	Authority Sources	Student Quote
Consensus	*Building Consensus*	Beliefs, Hypotheses	Group Consensus	"Do you think that's a good direction? Looking at straight lines in terms of angles?"
	Referencing Consensus	None	Prior Group Consensus	"So we said that Axiom 4' is saying that there can't be two lines that share the same two points."
Autonomy	*Lacking*	None	None	"So where do we start?"
	Tentativity	Beliefs, Hypotheses, Conjectures	Authority Figure	"Does this make sense? Is it ok to do this?"
	Free Expression	Beliefs, Hypotheses, Conjectures	Implicit, Preference	"It's just saying that if we have two triangles that are the same, then the sides and the angles will be the same", "I guess we decide because it's our definition"
Argumentation	*Stylistic*	NA	Preference	"I don't think 'evenly on itself' is a very clear definition."
	Structural	NA	Logic	"If we define straight as being the shortest distance between two points, that doesn't work on the sphere."

Figure 4. Summary of observation themes of the classroom.

Interview Findings

Leo and Jemma began Interview 1 by recalling that they did something similar in Discrete Mathematics, a course they took before, albeit separately. Leo recalled the terms *Euler path* and *Euler circuit*, but was unable to use them on the task. Both recalled that the solution involved the number of even or odd vertices. Jemma then suggested that they "… know it, with like, doors?" They began using a *door-and-room* metaphor, implying that each section of the Task 1 map was a room and each bridge was a door between rooms (see Figure 5). The pair concluded eventually that there was no such path, because there could be at most two odd sections. They also claimed that if there were all even sections, the path would start and end in the same section.

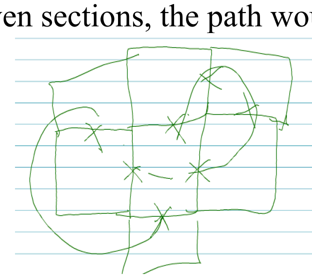

Figure 5. Leo and Jemma's picture depicting the *door-and-room* metaphor.

During Interview 2, Leo used a diagram that was more similar to a mathematical graph than a *door-and-room* diagram (see Figure 6). Leo eventually concluded, "With exactly two odds, you must start in one and end in the other." Jemma, however, used a diagram that more closely resembled the map given in the prompt (see Figure 6). Jemma eventually concluded, "If you start in an odd section, you can end elsewhere."

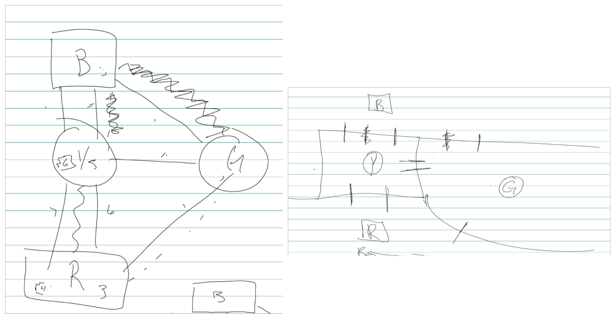

Figure 6. Leo's Interview 2 diagram (left) and Jemma's Interview 2 diagram (right).

For Interview 3, Leo initially asked if the "outside" needed to be considered. Jemma did not respond, and instead began path-tracing, claiming, "Each of these is like our space […] and there is a door on every single one of them." The pair then labeled the degree of each "space" with Leo accounting for three, five-degree sections on the "inside" and Jemma stating that, on the outside, "there would be nine." They then concluded that the curve was not possible, since they knew "three" parts of the figure had an "odd number of edges, and you can't have more than two."

In summary, Leo and Jemma built consensus around their prior understanding of the problem from Discrete Mathematics, including their use of the *door-and-room* metaphor. Individually, neither participant used the *door-and-room* metaphor, instead opting to use other diagrams. For Interview 3, the pair reverted back to using the *door-and-room* metaphor and successfully completed the task. During stimulated recall, Leo lamented that they relied so heavily on path-tracing and in-out strategies and not relying on their conjectures. He also said that he was uncertain initially as to how the previous tasks were related to the Interview 3 task. Jemma also lamented their use of path-tracing. She said that she viewed all the problems as *doors-and-rooms*, and that enabled her to solve the Interview 3 task.

Conclusion

The analysis of the classroom indicated the norms of the inquiry-based, geometry classroom as *consensus*, *autonomy*, and *argumentation*. Through the cyclic process of building and referencing consensus, Dr. Jackson was able to help the students develop a sense of autonomy in making choices regarding both stylistic and structural argumentation. During interviews, it seems that the pair of Leo and Jemma reproduced the norm of building consensus by agreeing upon their usage of the *doors-and-rooms* metaphor to solve the task. Both participants, however, did not use their agreed upon metaphor to complete the Interview 2 task. They later used the *doors-and-rooms* metaphor to complete the Interview 3 task, with Jemma feeling fully autonomous to apply the *door-and-room* metaphor, while Leo felt less sure.

These results indicated that while classroom norms may be reproduced in interview settings, they might not necessarily be reproduced when students work individually. As such, further study is required to assess the transfer of inquiry-based learning to an individual's own work.

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Instruction in Precalculus and Single-Variable Calculus: A Bird's Eye View

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Improvement of mathematics courses in the first two years of college has recently become a priority in the United States. This is evidenced by multiple calls to enhance undergraduate education in the mathematical sciences and by funding allocated to related research and instructional improvement projects. As stakeholders make decisions to invest in the improvement of these courses, it is critical that these decisions be informed by reliable information regarding how these courses are currently being taught. The work described here is an effort to lay this groundwork by painting a comprehensive portrait of instruction in precalculus and single variable calculus (P2C2) in the United States. In this report we address two research questions; 1) What instructional formats are currently in place in the P2C2 sequence? and 2) How common are these instructional formats nationally?

Key words: Precalculus, Calculus, Census Survey, Instructional Approaches

Improvement of mathematics courses in the first two years of college has recently become a national priority. This is evident from multiple calls to enhance undergraduate education in the mathematical sciences (e.g., *A Common Vision for Undergraduate Mathematical Science Programs in 2025,* Saxe et al., 2015) as well as from available funding allocated to related research by the NSF, through several programs. Given its key role in most STEM fields, the precalculus and single variable calculus courses (Precalculus to Calculus 2, P2C2) are of particularly importance and are the focus of this report.

The request for improvements of introductory post-secondary mathematics courses in general, and courses in the P2C2 sequence in particular, is warranted by the educational research in the area. It is well documented that within these courses, student learning outcomes lack rich conceptual understanding (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Tallman, Carlson, Bressoud, & Pearson, 2016; Thompson, 1994). In addition, research indicates that students in calculus have difficulties leveraging their knowledge to solve non-routine problems (Selden, Selden, & Hauk, 2000) and students' impoverished understanding of pre-calculus concepts can influence their understanding of concepts found later in the sequence (Thompson, 1994).

Perhaps most problematic is the retention rate of STEM-intending students; less than 40 percent of STEM-intending majors actually complete the degree (PCAST, 2012). Moreover, PCAST predicts there will be economic implications if this rate persists. Research indicates that many lose interest in STEM after taking courses in the calculus sequence (e.g., Bressoud, Mesa, & Rasmussen, 2015; Seymour & Hewitt, 1997). The literature repeatedly points to a relationship between retention and instruction. For instance, Ellis, Kelton, and Rasmussen (2014) found that some pedagogical activities (demonstrating how to work specific problems, preparing extra material to help students understand calculus concepts or procedures, holding a whole-class

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discussion, and requiring students to explain their thinking on exams) were significantly related to STEM intending students persistence. Additionally, Seymour and Hewitt (1997) found students who transfer from a STEM major often cite traditional and uninspiring instruction as reasons for their switch.

The AMS noted that "no single pedagogical method will be suitable for every classroom… Success in education is not achieved by simple formulas: there are many different successful ways of teaching mathematics, techniques adapted to the variation of talents of both students and teachers" (Friedlander et al., 2012, p. 1205). Thus, successful P2C2 sequences may come in a variety of forms involving a variety of instructional techniques. As we work towards the future of P2C2 instruction, it is important to understand how things stand now. This is important because the current situation represents the starting point of potential improvement efforts and likely will provide significant affordances and constraints as these efforts unfold. The work described here is an effort to lay part of the groundwork for improvement efforts by painting a comprehensive portrait of the instruction that is currently being implemented in the P2C2 sequence across the United States. In this report we address the following research questions:

1. What instructional formats are currently in place in the P2C2 sequence?
2. How common are these instructional formats nationally?

Methods

The data reported here comes from a census survey undertaken as part of a larger, multiphase project. The survey was administered to all institutions across the country whose mathematics departments offer a graduate degree in mathematics between April and August 2015. We focused on institutions with graduate degrees in mathematics because they produce the vast majority of STEM graduates and because such institutions typically have significant research missions resulting in a challenging need to balance the demands of research and teaching. In the United States, there are 330 departments which offer graduate degrees in mathematics; 178 of the 330 are PhD-granting institutions and 152 of the 330 offer Master's degrees. The overall response rate was 67.6% (223/330); 75% (134/178) of the PhD-granting institutions and 59% (89/152) of the Master's-granting institutions participated in the survey.

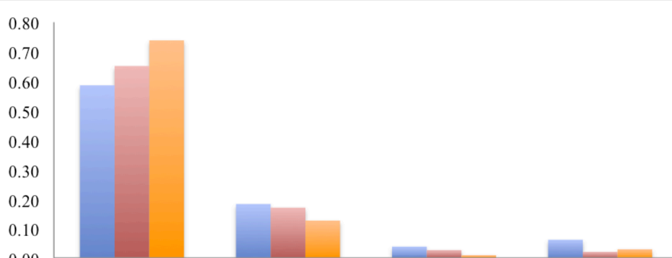
The survey was designed to gather information about each department's P2C2 program as well as the individual courses that comprise the sequence. In particular, the survey identified programs that are currently in place and their prevalence, revealed initiatives to improve current programs, and assessed degrees of implementation of the seven features of successful programs identified in the CSPCC project (Bressoud et al., 2015). The survey itself had three main parts. Part 1 asked the participants to name the courses that make up the mainstream P2C2 sequence in his or her department. Part II asked for information about departmental practices in regards to the P2C2 sequence. Part III asked for information about specific courses in the P2C2 sequence, including number of course offerings per term, total enrollment per term, DFW rates, contact hours, instructors, instructional approaches, recitations sections, and coordinated aspects across sections. This study reports on a subset of the data collected from Part III of the survey, where participants provided information about the individual courses that make up the P2C2 sequence.

The following section includes descriptive statistics (i.e., frequencies, proportions, etc.) gathered from questions related to instruction in the P2C2 sequence. Our aim is to identify instructional patterns of existing P2C2 programs by analyzing the aggregated data as well as zooming into the individual components by stratifying the data based on institutional type (PhD vs. MA/MS), course (precalculus vs. calculus vs. calculus 2), and course type (honors vs. regular).

Sample Results

The census survey collected information about the primary instructional format used in regular course meeting and that used in recitation sections (where applicable). In this report we focus on the format of the regular course meetings, and leave analysis of recitation sections for future work. We begin with a question where participants were asked, for each course, to identify whether the course is normally taught using 1) *mostly lecture*, 2) *some active learning alongside lecture*, 3) *mostly active learning*, or 4) *computer assisted instruction with lecture*. Participants were also provided with the opportunity to report that there was too much variation in the instructional format of that course across sections or allowed to indicate that a course was taught using an instructional format that was not listed. If a participant indicated either of these last two options they were asked to provide additional explanations. In this report we do not consider these written-in responses, but that information will be discussed in our presentation.

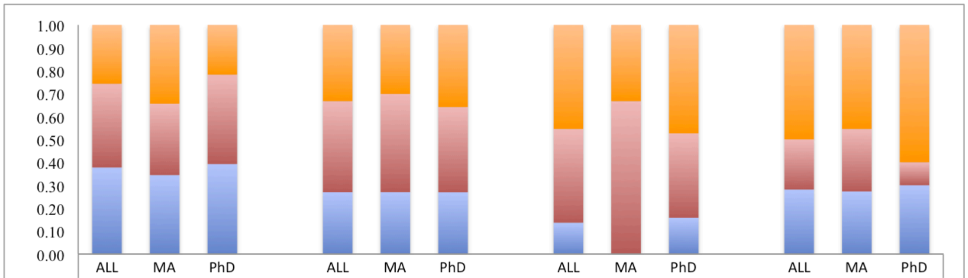
Two hundred institutions provided information about the instructional format for 881 courses. Of these, 66% were identified as being taught using mostly lecture, 16% used some active learning in tandem with lecture, 2.5% of courses were taught using mostly active learning, and 3.6% of these courses used computer-based instruction alongside lecture. When these data were stratified by course (PC, C1, and C2), we see that the use of lecture increases throughout the P2C2 sequence, while the use of active learning decreases. The data also indicates that computer-based instruction coupled with lecture is most prevalent in Precalculus courses. These results are shown in Figure 1.



	Lecture	Lecture + Some AL	Mostly AL	Lecture + Computer
PC	0.59	0.18	0.04	0.06
C1	0.65	0.17	0.03	0.02
C2	0.74	0.13	0.01	0.03

Figure 1. P2C2 courses with selected primary instructional format. N_{PC}=256, N_{C1}=327, N_{C2}=298.

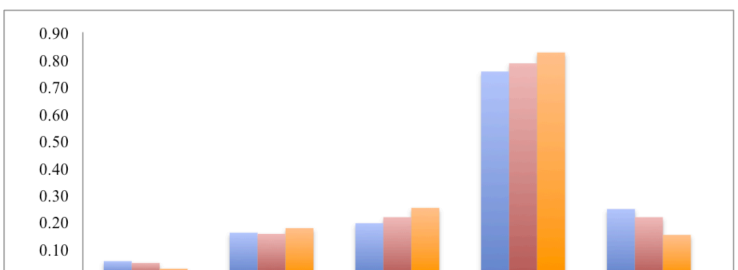
Figure 2 shows the prevalence of these instructional approaches across the P2C2 sequence for Master's-granting and PhD-granting institutions. These results indicated that Master's-granting institutions are more likely than PhD-granting institutions to teach Precalculus using mostly lecture, while PhD-granting institutions rely slightly more on lecture within the single-variable calculus courses. For both Master's- and PhD-granting institutions, courses taught using some active learning concurrently with lecture were more prevalent in Calculus 1, followed by Precalculus, with Calculus 2 employing this instructional format the least. Very few P2C2 courses are being taught using mostly active learning techniques in regular course meetings - 22 out of the 881 reported. The majority of these (19/22) come from PhD-granting institutions, and it was only at PhD-granting institutions that courses past Calculus 1 are taught in such a way. Furthermore, we note that the proportion of courses being taught using mostly active learning decrease along the P2C2 sequence at PhD-granting institutions. Additionally these results indicated that PhD-granting institutions are more likely than Master's-granting institutions to teach both Precalculus and Calculus 2 using computer-based instruction alongside lecture, while Master's-granting institutions rely slightly more on computer-based instruction coupled with lecture to teach Calculus 1.



	Lecture ALL	Lecture MA	Lecture PhD	Lecture + Some AL ALL	Lecture + Some AL MA	Lecture + Some AL PhD	Mostly AL ALL	Mostly AL MA	Mostly AL PhD	Lecture + Computer ALL	Lecture + Computer MA	Lecture + Computer PhD
PC	0.26	0.34	0.22	0.33	0.30	0.36	0.45	0.33	0.47	0.50	0.45	0.60
C1	0.37	0.31	0.39	0.40	0.43	0.37	0.41	0.67	0.37	0.22	0.27	0.10
C2	0.38	0.34	0.39	0.27	0.27	0.27	0.14	0.00	0.16	0.28	0.27	0.30

Figure 2. P2C2 courses with selected primary instructional format for MA- and PhD-granting institutions. Sample sizes for lecture: N_{ALL}=583, N_{MA}=183, N_{PhD}=400. For lecture incorporating some active learning: N_{ALL}=141, N_{MA}=63, N_{PhD}=78. For mostly active learning: N_{ALL}=22, N_{MA}=3, N_{PhD}=19. For lecture incorporating computer-based instruction: N_{ALL}=32, N_{MA}=22, N_{PhD}=10.

For courses that were identified as incorporating at least some active learning as part of their instructional format, participants were asked to provide additional information about the type of active learning strategies used during instruction. In particular, participants were asked whether these courses incorporated 1) *POGIL*, 2) *IBL,* 3) *clicker surveys*, 4) *group work*, or 5) *flipped instruction*. Participants were also provided with the opportunity to indicate that an active learning strategy other than those provided was used and prompted to provide additional information. As before, we will focus on courses that incorporated one of the provided responses for this question and discuss other active learning strategies identified by participants in our presentation.



	POGIL	IBL	Clicker Sureveys	Group Work	Flipped
PC	0.05	0.16	0.19	0.75	0.25
C1	0.05	0.15	0.22	0.78	0.22
C2	0.03	0.18	0.25	0.83	0.15

Figure 3. Selected types of active learning strategies within the P2C2 sequence. N_{PC}=57, N_{C1}=65, N_{C2}=40. Options are not mutually exclusive.

Of the 881 total courses, 163 were identified as incorporating at least some active learning. Of these, the most prevalent active learning technique was group work, reported in 78% of these courses. Additionally, 4% identified POGIL as a strategy used in these courses, 10% IBL, 21% clicker surveys, and 21% flipped (note that these options are not mutually exclusive). Figure 3 shows the prevalence of the various active learning strategies across the P2C2 sequence. These results indicate that the proportion of courses that use POGIL and flipped instruction decrease throughout the sequence. In contrast, both the use of group work and clicker surveys increase throughout the P2C2 sequence.

Conclusion

The results reported here provide some insight into the prevalence of instructional formats currently in place in the P2C2 sequence. In particular, we reported on the types of instructional approaches used in the regular course meetings. While the majority of courses are taught in a traditional lecture format, our analysis reveals that alternative instructional formats, such as active learning, are also currently present within the P2C2 sequence. Approximately 19% of courses within the P2C2 sequence are currently taught using at least some active learning, while 3.6% of courses are being taught using computer-based instruction alongside lecture. However, we note that the proportion of courses being taught in traditional lecture format increases through the sequence (from 59% to 74%), while all other formats decreased in frequency. Our analysis also reveals that courses being taught using at least some active learning employ a variety of active learning strategies. The most popular active learning strategy be used during regular class meetings is group work, followed by clicker surveys and flipped.

In addition to these results, we will present a similar analysis of the instructional formats being used in recitation sections accompanying P2C2 courses (when applicable). We will supplement the analyses presented here with information from the open-ended responses, explaining some of the variation in instructional format across instructors and identifying instructional approaches (e.g., emporium model) that were not included in our original set of options.

This preliminary report and subsequent analyses lay out the landscape of instructional format and course delivery for P2C2 courses across the country. We believe that the information in this report and presentation will be useful to those studying pedagogy at the undergraduate level, presenting a starting point and indicating the relative frequency of usage of different approaches. This information may be informative for stakeholders whose decisions affect improvement efforts in P2C2 courses.

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Students’ Inclination to Use Visual Images during Problem Solving

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This is a preliminary report on a study to investigate the inclination of Calculus III students to use visual reasoning in problem solving situations. One of our research hypotheses was that there is a correlation between students’ inclination towards visual representations when taking notes and their use of visual representations in problem solving situations. Surprisingly, preliminary analysis of the results suggests that there may not be a correlation, although work is ongoing.

Key words: Visual representation, visualization, problem solving, calculus

Introduction

Even though the development of computer graphics and educational software with its ubiquitous penetration into educational practice, may lead some to believe that the days of using free hand sketching in the teaching process are over, research demonstrates that the use of free hand sketching in classroom practice has significant impact on students’ learning. (Walker, Winner, Hetland, Simmons, & Goldsmith, 2011) The very process of sketching involves powerful hand-mind coordination; one reason for this is that the concretization of abstract mathematical objects as creations of our own hands is subject to easy manipulation or transformation.

Research into the role of imagery in the cognitive process has been intensified in the last two decades. The usefulness of visual representations of mathematical objects in the process of learning mathematics has been described as a vehicle that promotes abstraction and generalization. (Zimmerman & Cunningham, 1991; Arcavi, 1999; Presmeg, 2006).What do we mean by visualization? Phillips, Norris and Macnab (2010) published an overview of educational literature about visualization published between 1974 and 2009; they found 23 explicit definitions of visualization and related terms, such as “imagery” or “visual aid”, and point out that some of these are even contradictory. For our theoretical framework, we will utilize their definition of a “visualization object”:

> Visualization objects are “physical objects that are viewed and interpreted by a person for the purpose of understanding something other than the object itself. These objects can be pictures, 3D representations, schematic representations, animations, etc. Other sensory data such as sound can be integral parts of these objects and the objects may appear on many media such as paper, computer screens and slides. (Phillips et al.,2010, p.26)

Traditional calculus content is particularly suitable for implementation of visual pedagogy. In a Calculus III class taught by the first author, we strived to create a learning environment in which the main ideas about three dimensional coordinate systems, classification of quadratic surfaces, and min-max problems involving multivariable functions would first be taught focusing on a visual point of view and then from an algebraic or analytic point of view. We also followed Inglis and Mejia-Ramos’s (2009) suggestion that students are more inclined to accept figures as evidence for the claim when descriptive text explaining the claim accompanies the figure. In this study, we focused on students’ ability to reproduce the images during enacted lessons but did not evaluate students’ sketches for the basic requirements of perspective, visibility or line intersections.

The purpose of this research was to study students' desire and ability to reproduce three-dimensional images of the mathematical objects presented during instruction and how this desire and ability correlates with their use of images in solving test questions. The research questions for this work are:

1. How do students use sketches and other visual representations when taking notes in a Calculus III class?
2. What is the relationship between students' inclination and ability to hand sketch visual images and their use of visual imagery when problem solving?

Methods

Research Context and Participants

In a Calculus III class of 45 students, 30 students agreed to participate in this research project. The book used for this course was Stewart's *Essential Calculus, Early Transcendentals*. Designed as a one semester, four credit hours course, the class met three times a week with the first two class sessions for an hour and 15 minutes each and the last session meeting for 50 minutes. Students' class assessments consisted of a final exam, two one hour and 15 minutes tests, four 15 minutes quizzes, web-assigned homework, and handwritten homework each being adequately weighted. Students were encouraged to take notes for each class in a separate notebook and promised extra points for this activity. Their notebooks were examined twice during the semester and we collected copies of the notes for two particular class periods, which were also videotaped for research purposes.

The first recorded lecture was "Double integrals in polar coordinates" and the second recorded lecture was entitled "Triple integrals in spherical coordinates". We focus only on the videotaped lecture about triple integrals in spherical coordinates. The teaching style of the instructor was a traditional lecture, with occasional questions asked to students in the class in order to gage their understanding of the concepts discussed. The change of coordinate system and questions associated with this notion in the Calculus III curriculum is rich with concepts that can easily be illustrated on the whiteboard: segments, lines, cylinders or spheres in 3-D coordinate system most of the time do not present a challenging sketching task for the instructor.

Coding

Visual images from class. We first carefully watched the video recording of the lecture presented during the 50-minute class and identified all images that were presented on the whiteboard; these were typical visual representations of the Cartesian, cylindrical and spherical coordinate systems to visualization of the infinitesimal volume element in the spherical coordinate system.

The imagery presented during the enacted lesson, was subject to a natural classification. We distinguished three categories of images: primary image, secondary image, and second-layer image. We define *primary image* as an image on which the derivation of the analytical portion of the presentation, related to a justification (proof) of a proposition relies. This image may also play an essential role in the explanation of a new mathematical concept. Usually this image will stay on the board during an enacted lesson for a substantial amount of time. In Figure 1, we provide two examples of primary images that were used in the lecture about triple integrals in spherical coordinates.

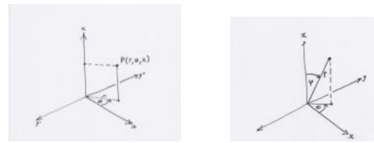

Figure 1. Primary Images

A *second-layer image* (SLI) is an image that has been superimposed on a primary image later in the exposition, bringing new aspects of the presented notion, or illustrating a portion of the proof of a proposition. Most of the time, during the enacted lesson, students are inclined to sketch a completely new illustration rather than revisiting a primary image and superimposing on this image a new one. Illustrative elements that have been superimposed on the primary images given in Figure 1 are indicated as shaded areas.

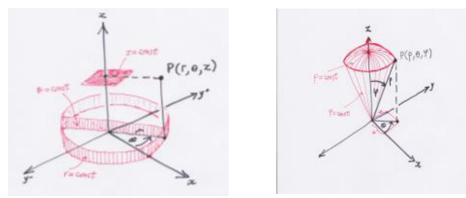

Figure 2. Second-Layer Image

Our third category of images is *secondary images*. These are images that have been used to clarify a particular argument related to the primary image, illustrate a particular point in the analytical portion of the argument, or used as a review of a specific notion used in the exposition. Usually these images will stay a short time on the board, serving its purpose and not interfering with the primary image. We illustrate this category in Figure 3.

Figure 3. Secondary Images

The left picture from Figure 3 helps students recall the definition of sine and cosine function and the right picture illustrates four graphs of the equation $\varphi = c$ (c is a constant) in the spherical coordinate system, for four different choices of c all between 0 and π.

There were 16 images presented on the whiteboard during the class period (Lecture entitled "Triple integrals in spherical coordinates"), with one of the images appearing twice in the lecture, at the beginning and towards the end, and being counted as two separate images. Out of these, 4 were counted as SLI and 5 were counted as secondary images as far as the instructor's drawing.

The researchers then coded each of the 30 notebooks by identifying which of the 16 images from the whiteboard were recorded by the students. This resulted in one tally for each of the students, which was the number of primary, SLI, and secondary images that the students had reproduced in their notebooks.

Coding visual images for problem solving. In order to investigate students' use of visual reasoning when problem solving, we chose to consider two problems from the exams given in the class. Each of these problems were designed so that visual representation of the problem would reveal a short path towards its solution. In the first problem (T1), we asked for unit vectors that make an angle of 30º with the vector < 2015, 2015 >. In the second problem (F), we ask students to evaluate an iterated integral, with its domain of integration given in cartesian coordinates by converting it to polar coordinates.

Table 1.

Code	**Description**
No code	No attempt at problem
1	No drawing used in solution
2	An attempt at sketch for some notions
3	Good sketch not used in solution
4	Sketch was essential and provided insight for student
Multiply code by 2	*Student was successful*

Students' use of images was coded as shown in Table 1. We were interested not only in the use of images when solving problems, but also in a successful use of visual information and so codes were multiplied by 2 if the solutions were correct. We are trying to quantify the instant when a logical sequence of analytical expressions is guided by the visual information.

In Table 2, we present a portion of the table with the students' full coding. This table was used to analyze quantitatively the data to answer the research questions. In the next section, we discuss some of the results.

Table 2.

Student	**Codes for notebooks**				**Codes for problem solving on exam**			
	Primary	SLI	Sec.	Primary + +SLI +Sec.	T1	F	T1+F	
7	7	1	1	9	-	1•1	1	
8	7	2	0	9	1•1	4•2	9	
9	9	1	0	10	1•1	1•1	2	
10	8	2	0	10	1•1	4•1	5	

Preliminary Results

Bold horizontal segments in Figure 4 show the total number of images that each student reproduced in their notebooks (Primary + SLI + Sec). The triangular diagrams show the total score (T1+F). Visual inspection shows that the relationship between the two is low. Due to space constraints, we have not shown graphs that break out the differences between image types.

Statistical analysis of the results reveal that the correlation index for the columns (T1+F), that measures how successful are students in producing visual information in problem solving situations and column (Primary+SLI+Sec) that indicates students' ability and inclination to hand-sketch is 0.05. In other words, there is no a correlation between these two variables. The comparison of other variables reveal similar trends. For example, 30% of the students successfully completed prblem (F), but there is not a correlation between their success in this problem with their tendency to give a visual representation of this problem (correlation index in this case is 0.334).

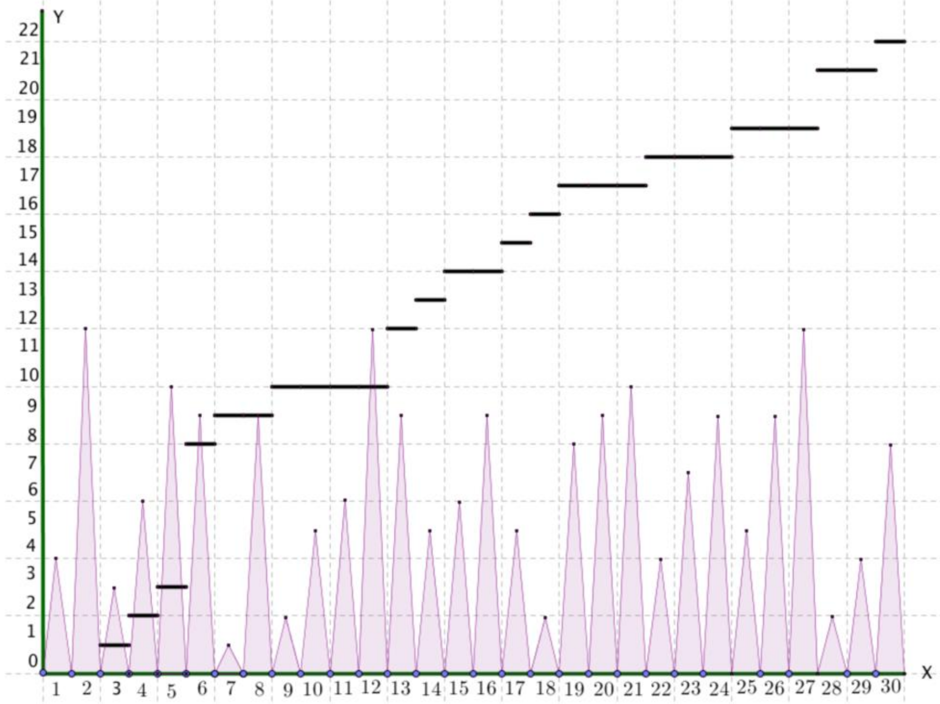

Figure 4.

Even though throughout the course the instructor exhibited a preference for visual explanations followed by analytic aspects of problem solving, it is surprising how little of this way of reasoning was accepted by students. Only four students (13%) successfully answered this question, all using the visual information in the most substantial way. A higher number of students using visual representation in the second problem (67%) may be explained by the way the polar coordinate system is introduced in all calculus books and in enacted lessons. A picture depicting how polar coordinates relate to the familiar Cartesian coordinates always accompanies it in the book.

Conclusion and Questions

These preliminary results indicate that just asking students to sketch visual representations in their notebooks is not sufficient for them to begin using more diagrams and graphs to problem solve in exams. Instructors need to consider additional ideas to encourage the use of visual methods. This might include homework that requires sketching, or using group work to have students talk and work together, with visualization as a focus.

Intended Questions for the Audience

1. What other coding ideas might help us move forward on connecting visualization and problem solving in a classroom setting?
2. Is it important to give value to getting a correct answer? How might we address this in other ways than doubling the scores?
3. What other data might be useful to collect in order to continue investigating visualization in problem solving?

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Students' Epistemological Frames and Their Interpretation of Lectures in Advanced Mathematics

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In this paper, we present a comparative case study of two students with different epistemological frames watching the same real analysis lectures. We show the general point that students with different epistemological frames can interpret the same lecture in radically different ways. We also identify epistemological frames that are useful or counterproductive for understanding a lecture on how the rational numbers are constructed from the integers. These results illustrate how different students interpretations of a lecture are not inherently tied to the lecture, but rather depend on the student and that student's perspective on mathematics. Thus, improving student learning may depend on more than improving the quality of the lectures, but also changing student's beliefs and orientations about mathematics and mathematics learning.

Key words: epistemic frames, student understanding of lecture, real analysis

In recent years, there has been a substantial increase in research into how proof-oriented university mathematics courses are traditionally taught. This research has led to important insights into what transpires in lectures in advanced mathematics (e.g., Fukawa- Connelly, 2012; Fukawa-Connelly & Newton, 2014; Hemmi, 2010; Lew et al., 2016; Mills, 2014; Pinto, 2013; Weber, 2004; Weinberg et al., 2016) and mathematics professors' motivation for their pedagogical behavior (e.g., Alcock, 2010; Lew et al., 2016; Nardi, 2007; Weber, 2012).

While the nature of lecturing in advanced mathematics is now better understood, there has been comparatively little research into how tertiary students interpret the advanced mathematics lectures that they observe. The goal of the current paper is to address this gap. In particular, we aim to shed light on the issue: When professors provide lectures in advanced mathematics that are clear to other mathematicians, why are these lectures often confusing to the students who attend them? Why do *some* students learn from a lecture while other students find the same lecture less clear? What dispositions prepare students to learn from lectures in advanced mathematics?

To address these broad issues, we adapt the notion of *epistemological frame*. We elaborate on the nature of epistemological frames shortly, but an epistemological frame can be thought of as a person's tacit answer to the question, "what sort of activity is this?" (c.f., Goffman, 1997; Redish, 2004). In this paper, we present a comparative case study of two students with different epistemological frames watching the same real analysis lectures. The first goal of this paper is to illustrate the general point that students with different epistemological frames can interpret the same lecture in radically different ways. The second goal of this paper is to identify epistemological frames that are useful or counterproductive for understanding a lecture on how the rational numbers are constructed from the integers.

Theoretical Perspective

In this paper, we adapt the notion of *epistemological frame*, as it is used in the physics education literature (e.g., Redish, 2004). Goffman (1997) introduced the notion of *frame* to describe how individuals develop expectations to help them make sense of the complex social spaces that they inhabit. For instance, most adults in the Western world have a "restaurant frame" (Schank, 1990). When individuals enter a restaurant, they carry with them numerous assumptions: for instance, the individual will eat a meal, the staff at the restaurant will prepare the meal, the items listed on the menu list are what the individual may order, the prices next to the item denote how much the food will cost, the individual will be expected to pay for their meal using cash or a credit card, and so on. Such expectations are valuable in that they enable us to cope with the enormous complexity of most situations that we regularly encounter. However, occasionally our frames can be problematic in cases where they are inaccurate or when two different individuals frame the same situation in different ways. For instance, a visitor to the United States might not be aware of the expectation to leave a 20% tip on the bill, which could lead her to inadvertently insult the waiter by not leaving the presumed gratuity.

In an educational situation, we refer to an individual's epistemic frame as consisting of their epistemological expectations. These consist of an individual's responses to questions such as "what do I expect to learn?", "how do I build new knowledge?", and "what counts as knowledge or an intellectual contribution in this environment?" (cf., Redish, 2004). Physics educators have illustrated how when a teacher and her students approach an activity with different epistemological frames, the students may not reap the educational benefits that the teacher intended. For instance, Redish (2004) described one student in a physics tutorial who was given a prompt to make a qualitative prediction using her conceptual understanding of physical principles. However, this student viewed intellectual contributions in a physics classroom as consisting of deriving numerical answers from textbook formulas. As a result, she participated in the activity by engaging in computations, thereby avoiding the conceptual considerations the activity was designed to elicit. Hammer (1995) illustrated how epistemological frames can explain how some students successfully learn physics in a traditional classroom while other students do not.

In this paper, we use epistemic frames to account for two students' interpretations of a lecture in advanced mathematics. Although the phrase "epistemic frame" is not often used in mathematics education, mathematics educators have used similar constructs to highlight students' classroom participation and identify differences in how teachers and students sometimes frame mathematical activity. For instance, Thompson (2013) described a secondary algebra teacher, Sheila, whose discourse with students was saturated with references to the conceptual meaning of the operations that were being discussed and performed, and a student, Mindi, who struggled in Sheila's course. Through his interviews with Mindi, Thompson found that Mindi believed that doing algebra consisted of applying rules and learning algebra consisted of memorizing rules; Mindi saw little value to having a conceptual understanding of these rules. As a result, she ignored the frequent references to meaning in Sheila's discourse.

In advanced mathematics, Solomon (2006) and Alcock and Simpson (2004, 2005) have explored the relationship between students' epistemologies and their learning in advanced mathematics classrooms, with both emphasizing that students who believe mathematical knowledge comes from external sources have a difficult time in these environments. Similarly, many mathematics educators have documented how the types of arguments that students found convincing influenced their proof-related behavior (e.g., Harel & Sowder, 2007; Stylianides, Stylianides, & Weber, in press). In this paper, we consider a more fine-grained epistemological distinction than the ones discussed above. The two students that we considered both showed evidence of having a desire to understand the material that they were learning, an internal loci of control and deductive proof schemes. Nonetheless, their different epistemological frames led them to interpret the same real analysis lecture in very different ways.

Methods

Rationale

The goal of this study was to understand individual student's interpretations of a real analysis lecture at a fine-grained level. Doing this in an authentic manner posed significant methodological difficulties. If we observed students in an *actual* real analysis lecture, it would be infeasible for us to know how a student was interpreting a professor's comments *in the moment* since we would be unable to interrupt the lecture to probe the student's thinking. We could follow Lew and colleagues (2016) in using cued recall by interviewing a student about a lecture they attended shortly after the lecture had occurred. However, by that time, much of the student's initial impressions would be forgotten. To manage this difficulty, we interviewed two students as they watched real analysis lectures that had previously been posted on YouTube. Here, the students can act as if they were attending an actual lecture yet the interviewer or student could pause the video to discuss their in-the-moment impressions of what was being discussed.

In this paper, we report a comparative case study (Yin, 2013) in which we attempt to illustrate how a particular phenomenon unfolds within a given context. There are some particular nuances of the study, such as the individual students' backgrounds or the content of the lecture, that may not generalize to other situations. What we contend is representative is the phenomenon of how an individual student's epistemological frame can influence his or her understanding of a mathematics lecture.

Participants

Two participants, Alan and Brittany (pseudonyms), agreed to participate in this study. Both participants were mathematics majors at a large state university in the northeast United States. Both students had completed a transition-to-proof course in the previous semester. At the university in which this study occurred, mathematics majors were required to complete a course in real analysis, for which the transition-to-proof course was a prerequisite. Some mathematics majors enrolled in real analysis in the semester after completing their transition-to-proof course while others, like Alan and Brittany, did not. Like many mathematics majors, Alan and Brittany elected to take an abstract algebra course immediately after their transition-to-proof course, choosing to postpone their real analysis requirement to a subsequent semester. Hence, while neither had, both participants could have taken a real analysis course in the semester of this study.

Procedure

The materials consisted of the first two 60 minute class meetings from a real analysis course taught by the president of the Mathematics Association of America and award winning professor Francis Su at Harvey Mudd University[1]. The lectures consisted of Professor Su beginning the real analysis course by first constructing the rational numbers and then the real numbers from the integers. Prior to conducting the study, the research team studied the lecture and parsed the lecture into five to ten minute segments in which coherent mathematical content was being

[1] Videos can be found at: http://analysisyawp.blogspot.com

presented. The research team also identified points in the lecture where important content was being conveyed.

Each participant met weekly with the first author for four weeks. Each of the four interviews was audio-recorded. The first interview was a one-hour interview in which the participant discussed the content from the transition-to-proof course to get a sense of their understanding of the relevant content (particularly with number theory, functions, and proof) as well as their learning strategies and dispositions. The next three interviews were two-hour interviews conducted in the style of a constructivist teaching experiment (Steffe & Thompson, 2000) in which the research team attempted to build and refine the mental schemes, which we termed epistemological frames, that each participant was using to interpret the real analysis lectures. During each interview, the participant watched the lecture and was instructed to stop the tape to discuss anything that they observed to be important, interesting, confusing, or otherwise noteworthy. The interviewer would also stop the tape to probe the participant's thinking when the professor had stated something that the research team had identified as important or at the end of a segment. The interviewer would then ask the participant to describe their impressions at that point.

After each interview, the members of the research team would engage in concurrent analysis by listening to the recordings of the interview and forming initial hypotheses about the schemes that the participants were using to interpret the lectures. They would then meet to synthesize these hypotheses and develop prompts to assess the viability of their hypotheses. These prompts were designed such that if a participant held a particular epistemological frame, we could expect them to respond in a particular way. We began the next interview segment by providing participants with these prompts, which was then followed by them resuming watching the lecture videos.

After all four interviewers were conducted, we transcribed all four interviews and clarified our initial hypotheses of participants' epistemological frames from the prior concurrent analysis. We then engaged in the following cyclic retrospective analysis: For each aspect of a participant's hypothesized epistemological frame, each member of the research team individually read the transcripts, identifying all instances that either supported or disconfirmed that the participant held this frame. The research team then met to determine how well the proposed epistemological frame was supported by the data. When there were aspects of the hypothesized frames that were not supported by the data (i.e., there were few confirming instances or significant disconfirming instances), we either removed this aspect from the epistemological frames that we attributed to the students or we revised the epistemological frame and repeated our analysis. The result of this retrospective analysis were epistemological frames for Alan and Brittany that were grounded in our data.

Results

Epistemological Frames

For the sake of brevity, we briefly discuss four aspects of Alan and Brittany's epistemological frames. In our presentation, we will elaborate on these and provide supporting evidence. We first note two aspects that Alan and Brittany had in common: (i) Both participants appreciated, understood, and enjoyed mathematical proof and (ii) both exhibited an internal locus of control. We also note two areas in which they differed: (iii) Alan valued definitions of terms that were unambiguous while Brittany preferred definitions that were comprehensible and relatable to her prior ways of knowing, and (iv) Alan viewed the purpose of these lectures as making his understanding of the rational numbers more reliable by basing it on a more secure footing (the reliability of the integers) while Brittany viewed the purpose of the lectures as providing a common base of knowledge shared by all students in the class from which real analysis can be built.

Differing interpretations of the lecture. We note that Brittany's epistemological frames, while sensible and productive in some situations, led her to interpret some aspects of the lecture in unproductive ways. For instance, when Professor Su deduced common properties of the real numbers (e.g., the integers were contained in the rationals), Brittany became frustrated because this was obvious and "everyone already knew this". By contrast, Alan claimed that the point of the lecture was to illustrate how Professor Su's construction of the rational numbers was sufficient to prove these important properties (cf., Weber, 2002). Brittany found many of Dr. Su's proofs of common facts about the rational numbers to be unnecessarily complicated, since she was aware of simpler proofs of these statements that were appropriate for middle school students. For instance, Professor Su defined rational numbers as equivalence classes of ordered pairs and defined addition as $(a, b) + (c, d) = (ad + bc, bd)$. Professor Su then proved this operation was well-defined. Brittany thought that the simple proof that $a/b + c/d = ad/bd + bc/bd = (ad+bc)/bd$ was superior to Professor Su's formal demonstration. In contrast, Alan abhorred the suggestion of using such a proof, since we had no right at this stage to say $a/b = ad/bd$. To Alan, the point of this exercise was to ignore everything we knew about the rational numbers. In general, Brittany wanted to apply her robust understanding of the rational numbers to the content that Professor Su was discussing and was annoyed that Professor Su did not do so. Alan continually reminded himself that he was only allowed to consider what he knew about the integers, the definitions Professor Su had provided, and the deductions made from them.

Discussion and significance

We use the general finding that students' epistemological frames can enable or prevent students from interpreting mathematical lectures in a productive manner to make two points. First, previous research on lectures in advanced mathematics has generally focused on what the professor says but did not consider student's interpretation of what was said. Our results illustrate how students' interpretations of a lecture are not inherent in the lecture itself but also depend on the student doing the interpreting. Consequently, we argue that ignoring students' interpretations of lectures is a significant shortcoming of most studies on lectures in advanced mathematics. Second, our results suggest that the key to improving students' learning from lectures does not consist only of improving the quality of the lectures. Rather, it is important to attend to their epistemological frames as well, a point that Solomon (2006) and Dawkins and Weber (in press) argue has received limited attention in the mathematics education literature.

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Mathematical Modelling and Mathematical Competencies: The case of Biology students.

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The research aims at introducing modelling tasks in order to engage students more actively into learning mathematics through tasks that are biologically 'colored'. My focus is on the individual progression (if there is any) of students' mathematical competencies during a sequence of modelling sessions that will be part of a regular course of their first year calculus. My ultimate goal is to construct a dynamic competence profile for every student that will participate in the project. Taking the above into consideration, my research suggests a number of interventions in a standard freshmen mathematics course for biology students, interventions that offer a fruitful didactical environment where students can sharpen their mathematical competencies.

Key words: mathematical modelling, mathematical competencies, tasks, progression

Introduction

There is an increasing amount of literature which provides documentation for the learning benefits associated with engaging students in mathematical modeling. There is a 'red thread' among many researchers who, through the description of mathematical modelling processes, displayed the variety of many opportunities for educational benefits (Kaiser et al., 2006). Students engaged in modeling may develop a deep understanding of the content and an ability to solve novel problems (e.g. Wynne et al. 2001, Lehrer & Schauble 2005). Other studies (Schwarz & White 2005; Windschitl et al. 2008) have shown that modeling curricula can bring students into alignment with the epistemic aims of science and help them develop more sophisticated ideas about the scientific enterprise as a whole. Sriraman et al. (2009) blended the notion of interdisciplinarity with modelling, highlighting the necessity for creativity and giftedness across disciplines. It comes as no surprise that

> Both the National Research Council (NRC) and the National Science Foundation (NSF) in the U.S is increasingly funding universities to initiate inter-disciplinary doctoral programs between mathematics and the other sciences with the goal of producing design scientists adept at using mathematical modeling in interdisciplinary fields (Sriraman & Lesh, 2006, p.247).

Theoretical Stance

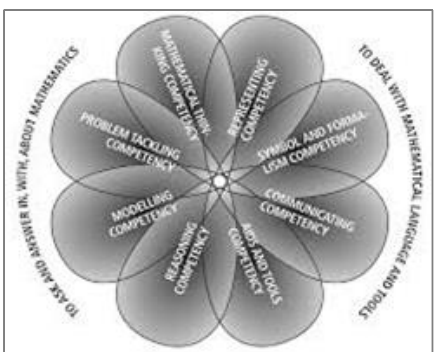

Figure 1. The KOM model (Blomhøj & Jensen, 2007)

A long-lasting and ongoing discussion among researchers and members of Educational Institutes centers on students' assessment and the need for a solid and valid evaluation (e.g. Galbraith 2007, Haines and Crouch 2007, Vos 2007). A different approach though occurred by an important number of researchers when they turned their view on students' competencies and mathematical competencies (e.g. Greer and Verschaffel 2007, Henning and Keune 2007, De Bock et al. 2007, Houston 2007, Blomhøj and Jensen 2007).

The Danish KOM (Kompetencer og matematiklæring) project (Fig. 1) Niss (2003) focused on basing the description of mathematics curricula primarily on the notion of mathematical competence. The framework they proposed could apply at any educational level. Niss and Højgaard (2011) managed to combine assessment and competence by introducing a three-dimensional model of progression of each competence, which I describe in the *Data Analysis* section. This model will be part of my set of tools for my data analysis since it focuses on the progression of students' development of a certain competence. It is possible that one mental, verbal or written action may describe two different competencies (overlapping); therefore it is important to locate discrete elements that characterize every competence even though in mathematical modelling activities students need to combine mental process in terms of combined competency profiles. We can find some attempts towards this direction from Andersen et al. (2001) and OECD (1999, 2001) focusing on the PISA investigations. These studies include an international comparison of secondary school students' competence profiles. The research reported here contributes to this research program by extending such analyses to university students.

The competence framework in my research will be based on the general term of mathematical literacy which combines the development of mathematical concepts and terms while dealing with real-world (realistic) tasks.

In my research I will use the notion of mathematical competence as something that students must bring into action in order to meet the challenges of the future. I consider this future-directedness rather important in educational terms because I am interested in the ways in which a student puts his or her mathematical knowledge to functional use. This will also give a strong connection to what Blum et al. (2002) considered as vital elements of modelling competences. He described a student, who is competent in modelling, as one who is able to structure, mathematize and solve problems. Furthermore in line with what Maaß (2004) considered as modelling competencies it is important to understand that knowledge alone is not sufficient for a student to develop his/her mathematical competencies. A student has to use and direct his knowledge with a suitable and specific way in order to be successful in modelling and this is where my study focuses on: observing, monitoring and analyzing that process.

Besides the competence theoretical framework I will adopt the theory of Didactical Transposition. Bosch and Gascón (2014) refer to four different bodies of knowledge (see Figure 2) where the transformations applied to a "content" or a body of knowledge since it is produced and put into use, until it is actually taught and learned in a given educational institution.

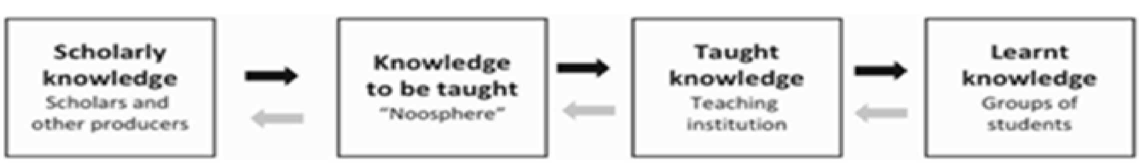

Figure 2. The didactical transposition (Bosch & Gascón)

Research Questions and Design

In this report I address the following two research questions:

1) What is the dynamic of a student's competence profile through the course of the mathematical modelling unit?
2) What competencies are deemed necessary for a student in a biology department?

By the term dynamic in the first research question I include the notion of progression (of a student's competence) from the very beginning of the project and also focus on identifying the initial set of competencies that a student brings when he or she enters the tertiary level.

In this study I make use of a design based research approach (Kelly & Lesh, 2000) in which an iterative process of design, implementation, and analysis takes place. More specifically, this study takes place in two phases. In Phase 1 (already completed), I investigated students' mathematical competencies during their engagement with a series of modelling tasks. These students were on their first year in a university's Department of Biology and the modelling session followed up a regular first year calculus course.

Phase 2 is ongoing and takes place with first year students in biology at the Norwegian university where this study takes place begin with a standard 10 week mathematics course on calculus. The modelling sessions occur weekly during their first semester. Approximately 100 students are organized in 3 separate classrooms where there is a 50 minute modelling session where students engage with modelling tasks. The students in each classroom are organized in small groups of three or four. One small group from every classroom is chosen to be monitored with audio and video devices. Every two sessions are considered to be a single modelling block where a new mathematical tool will be introduced. By the end of every session, a modelling task will be assigned to the students as part of their obligatory assignments for their mathematics course. Every modelling block will be designed in respect to the competence theoretical framework that I adopted. In addition the sessions will provide new knowledge that is also necessary for the successful engagement of students with the home assignments.

Methods for Data Generation

Data for Phase 1 and 2 consists of students' written work (tasks and assignments) and recordings (video and audio) which capture all kinds of discourses that are taking place during the sessions. In Phase 2 a questionnaire will also be given to the students at the beginning and at the end of the project. Discussions between the students and with the lecturer will be recorded both in video and in audio form. In every classroom separate cameras and audio devices will record the focusing on the group and to the whole classroom. In addition, the selected groups will be provided with a special device (LiveScribe 3 Smartpen) for more accurate and secured data collection. The same equipment will be used in all interviews with the task designer.

Data Analysis

Data from RQ_1 and RQ_2 will allow me to address the four bodies of knowledge proposed by the ATD, the first two from a detailed task-design analysis, the third from the above mentioned recordings and the last from a general assessment (formal exams and general performance in the classroom during the sessions). A task-design analysis, for example, can provide what the existing literature (mathematical biology) provides on population dynamics and exponential growth (scholarly knowledge) but also which task was finally decided to be presented (knowledge to be taught) and this will happen for every different modeling block.

The multi-dimensional model functions in such a way that whenever one or more dimensions may display a change (progression) then the volume (student's competence profile) of the cuboid changes. At this point a better analysis of these three dimensions is necessary.

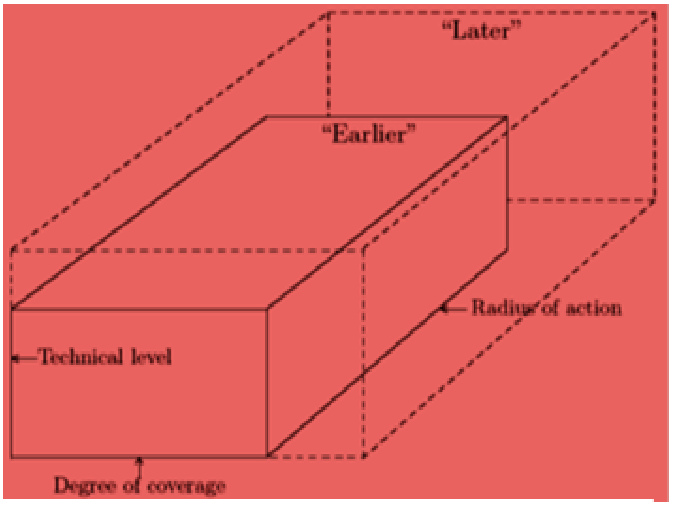

Figure 3.The three dimensional model (Niss & Højgaard, 2011)

- **Technical Level**: indicates how and to what degree (how advanced) a student may use his/her tools and mathematical entities which belong to his/her cognitive set of knowledge in order to activate a certain competence.
- **Radius of Action**: illustrates the range of action a student may take in terms of context and didactical situations. It shows where a student can activate a specific competence.
- **Degree of Coverage**: indicates to what extent a student is developing a competence in terms of its specific characteristics.

For the needs of this analysis I constructed a coding system which breaks down into smaller parts the verbal, mental and written actions of every single competence. This system, which is illustrated with data from Phase 1, functions as a decoding tool that assigns every student's discourse action to specific parts of a certain mathematical competence. In order to be successful in this attempt I need strong indicators that correspond to a specific competence and the frequency of appearance of these codes can be an indicator of progression (or stagnation) of a specific competence. It is in my intention to improve the reliability of this coding system by grading every code depending on the different tasks the students encountered during the modelling sessions.

Abstract from my coding system: the Reasoning Competence

- When a student is able to follow and assess a chain of arguments.
  Code: Flw. Arg.

- Knowing the difference between a formal mathematical proof and other kinds of mathematical reasoning. Code: Pr. ≠ Math. R.

- Separating main lines from details and ideas from technicalities during a line of arguments posed by anyone in the classroom. Code: Sep.

- When a student has the skill to devise formal and informal mathematical arguments. This may differ from a typical mathematical proof in our study therefore we could include the term: proving statements. Code: Pr. St.

Data from Phase 1

At the extract below we can see a discussion, between the members of Group2 in a university department of Biology about a modelling task. The students should come up with a solution in a time frame of 15 minutes and then present their possible solutions on a whiteboard in front of the other groups. The colored parts of the extract are based on the coding system above.

The task

Uncontrolled geometric growth of the bacteria Escherichia coli (E. coli) is the theme of the best-selling Michael Crichton's science fiction thriller, The Andromeda Strain. At some point the author claims that: ''**In a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth.**'' If a single cell of the bacterium E. coli divides every 20 minutes, how many E. coli would be there in 24 hours? The mass of an E. coli bacterium is 1.7×10^{-12} g while the mass of the Earth is 6.0×10^{27} g.

- Is Crichton's claim accurate?
- If not, how much time should be allowed for this statement to be correct?

The students are trying to find a way to mathematize the assumption: if a single cell of the bacterium E. coli divides every 20 minutes. They should come up with this expression: 2^{3x}

1. A: what about 24x8 (checking calculator) and then I get…oh we have to compare it with, we can take 10 to the power of…isn't that very close to Earth's mass?
2. B: close is not enough for mathematics.
3. A: yes but we have something to compare it with. Is 8 our ground number (meaning base) or is it 24? I don't know what power I should put.
4. B: we have 2, 4, 8 … (almost silent)
5. C: so it's always double.
6. B: so it goes 16, 32, 64…
7. A: it may be 2^x ? Since it's always changing.
8. C: But our ground number?
9. A: Our ground number is 2, when we have 4 it is 2^2 then 2^n=8
10. B: No you have to put 3 to get 2^3=8
11. C: You have 8, 16, 3, 64… so is there …? How is it called?
12. A: (writes 2^3, 2^4…) Is this the first line with 2^1? The starting point? Oh we can take just 2^{25} and then we have (a huge number appears at the calculator)

The Reasoning Competence is not the only one that appears in the text but for the interests and page restrictions of this report I included only this specific type of mathematical competence. It is quite possible that episodes of overlapping competencies may occur but this is not an obstacle when it comes to identifying the progression of a specific competence.

Goals & Addressed Questions

My main goal is to create a dynamic competence profile for every student and it is in my intention to redefine the term good student by that of competent and try to find a way to identify students' learning skills, which in this study are considered as mathematical competencies. I therefore consider that the didactical environment of mathematical modelling is a suitable one for my research interests.

It would be more than helpful for my dissertation, if I could have some feedback on the following questions that are closely related with my data analysis:

1. What statistical tool would be ideal for my code analysis?
2. Is there a solid connection between ATD, mathematical competencies and mathematical modelling tasks?

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A Preliminary Investigation of the Reification of "Choosing" in Counting Problems

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In a recent combinatorics-focused teaching experiment with two undergraduate students, the students developed a robust understanding of a three-stage counting process that provided a solution for problems involving combinations. So strong was the students' three-stage process, they did not seem to naturally conceive of the singular process of "choosing," which is an important aspect of understanding combinations. In this preliminary report, I question whether or not the students engaged in reification, which Sfard and Linchevski describe as "our mind's eye's ability to envision the result of processes as permanent entities in their own right"(1994, p. 194). I raise questions about what aspects of the student work might have fostered or hindered their ability to reify choosing, as well as what might be taken as evidence that reification has occurred in the context of combinatorics.

Key words: Combinatorics, Reification, Discrete Mathematics, Counting Problems

Combinatorial enumeration, or the solving of "counting problems," is increasingly relevant in our computerized age, and counting has applications in a variety of mathematical and scientific areas. Recent research in undergraduate mathematics education has investigated a variety of aspects of counting, including examining productive ways of thinking about counting in general (e.g., Halani, 2012; Lockwood, 2014) and student reasoning about various facets of counting (Lockwood, 2013; Lockwood & Gibson, 2016). Following such research, in this paper I explore one particular aspect of counting: two students' work toward the reification of the notion of "choosing." By examining this phenomenon, I hope to both bring to light insights about the teaching and learning of the essential combinatorial construct of choosing, and also to speak to the nature of the broader mathematical practice of reification.

In this preliminary report, I share results from a paired teaching experiment that was designed to examine the students' generalization within a combinatorial context. The students were quite successful in their generalizing activity and their combinatorial problem solving, but a surprising phenomenon emerged regarding their development of solutions to problems involving combinations. Specifically, I was surprised to find that conceiving of "choosing" as an action was not something that seemed natural to them, and, interestingly, it was difficult to motivate them to reify this choosing process. By describing their work and exploring potential reasons for their resistance to reifying, I seek to answer the following questions: *What factors affected (limited and facilitated) these two students' reification of the process of choosing? What does this say about the nature of reification in the context of combinatorics?*

Literature Review and Theoretical Perspective

Reification. Sfard and Linchevski (1994) describe reification as "our mind's eye's ability to envision the result of processes as permanent entities in their own right" (p. 194). In other words, reification occurs when someone can conceive of the result of a process as something that may be operated on or with. Sfard and Linchevski elaborate this phenomenon in algebra especially, as the following example suggests. They offer the algebraic expression of $3(x + 5) + 1$, and they notice that we could view this expression in a variety of different ways. First, it may be a computational process that is "seen as a sequence of instructions: Add 5 to the number at hand, multiply the result by three and add 1" (p. 191). It could, however, also be viewed as an object in its own right – as a function that could itself be operated on or treated as the input of another function. This dual way of thinking about an expression like $3(x + 5) + 1$ is in line with Sfard's (1991) distinction between structural and operational conceptions, in which "the same representation, the same mathematical concepts, may sometimes be interpreted as processes and at other times as objects" (Sfard & Linchevski, 1994, p. 193).

Sfard and Linchevski (1994) go on to argue that reification can help promote relational understanding (or conceptual knowledge). Generally, they convey that reification is difficult, saying that their data "provided sufficient evidence that reification is inherently very difficult. It is so difficult, in fact, that at a certain level and in certain contexts, a structural approach may remain practically out of reach for some students" (p. 220). The importance yet difficulty of reification is one motivating factor for this study, and by exploring the data in depth I hope to gain insight both into the nature of reification (especially within a combinatorial context) and also how reification might be fostered for students. Additionally, I have previously noted the ways in which Sfard's (1991) structural versus operational duality naturally applies to combinatorics (e.g., Lockwood, Reed, & Caughman, in press), and I believe that there is much more to be investigated about this dual relationship in combinatorics, especially regarding combinations. The results from this study provide but one preliminary perspective of a more complex and complete picture of structural and operational conceptions in combinatorics.

Combinations. I now offer a brief mathematical discussion of combinations. I use Thompson's (2008) notion of a conceptual analysis, which he describes as a method to "describe what students might understand when they know a particular idea in various ways" (p. 42). One reason that Thompson (referencing Glasersfeld, 1995) gives for developing a conceptual analysis is to "generate models of knowing that help us think about how others might know particular ideas" (p. 43). This reason aligns with the purposes of this paper, and I highlight a couple of key aspects of what it may entail to come to know combinations.

"Combination" problems have as outcomes combinations, or sets, of distinct objects (as opposed to permutations, in which outcomes are ordered sequences). To count combinations, one may go through a three-stage process (this three-stage process describes a typical way in which the formula for combinations is taught – start with permutations and then divide by duplicate outcomes). If we consider the number of ways to choose k people from a set of n distinct people, we can first arrange all n people in a line, say. We can then note that we only care about the first k people, and thus arrangements of the last n-k people all yield duplicate outcomes. Thus, b $\frac{n!}{(n-k)!}$ yields permutations of k people from n people. Then, to count combinations, because any k! arrangement of the k people are but one combination we wish to count, we can divide that product by k!. This three-stage counting process is reflected in the formula for combinations, which is given by $\frac{n!}{(n-k)!k!}$.

While it is important to be able to explain this formula, another aspect of conceiving of combinations is to understand that although this three-stage process can effectively explain the formula, it is sometimes useful to be able to think about choosing as one reified process that can itself be a stage in the counting process. In particular, the result of choosing can itself be viewed as something that can be acted upon. When I say the act of "choosing," then, I mean conceiving of a single, particular act in which I can choose some subset of objects from a set of distinct objects. The reification of choosing is an important feature of successful combinatorial enumeration because there are many problems in which choosing itself becomes a stage in the counting process. Two previous results point to this phenomenon and highlight the importance of being able to conceive of and use choosing as a stage in the counting process. In Lockwood, Swinyard, & Caughman (2015), we highlighted an episode in which students struggled on one particular problem that involved combinations as a stage in the counting process. In Lockwood, Wasserman, & McGuffey (2016) we also demonstrated that certain combinations are particularly difficult for students, who view such problems as qualitatively different than other combination problems. The results from that study also indicate that solving counting problems that involve combinations as a stage in the counting process are difficult. Both of these studies point to the fact that choosing can be a stage in the counting process is an important but difficult idea for students to understand.

Methods

The data in this study is part of a larger project intended to explore students' generalizing activity in a variety of mathematical domains. For the part of the project described in this paper, we conducted a 15-session teaching experiment with two calculus students (Rose and Sanjeev, pseudonyms). Each videotaped session was ~60 minutes in length, and we met approximately three times a week for five weeks. We covered a variety of material in the teaching experiment, including having students generalize formulas after solving counting problems, and initially explore combinatorial proof. In this paper I focus on a particular phenomenon regarding the students' conceptions of combinations. During the teaching experiment, I started to observe that the students were solving problems involving combinations in a way that suggested they were not reifying the process of choosing. In response, I gave students tasks with the aim of examining their conceptions of choosing and to attempt to foster the reification process.

For data analysis, I had the interviews transcribed. My research team and I then took a pass through the data in which we created enhanced transcripts, which involves adding screen shots and relevant nonverbal cues to the transcript. In reviewing the videotapes and creating enhanced transcripts, we flagged situations and episodes in which students were using the choosing formula, and we returned to those episodes. We tried to use those episodes to articulate a narrative (Auerbach & Silverstein, 2003) of the students' understanding and use of combinations.

Results

In the limited space provided, I describe a handful of key episodes that demonstrate the students' development and use of the formula for combinations. I argue that their initial conceptions likely affected the ways in which they later reasoned about combinations. I try to point out that the students had such a strong conception of a particular counting process that they did not naturally seem to view choosing as a single, reified counting process. I also raise questions about whether the approach I took in teaching these concepts (starting with permutations and deriving combinations) had some unintended consequences in terms of the students' ability to reify the process of choosing.

Students' development of the formulas for permutations and combinations. A noteworthy aspect of the students' initial work is that the students developed a surprisingly strong sense of equivalence, which they used to derive the formula for permutations. We asked the problem, *How many 3-letter sequences are there using the letters A, B, C, D, E, F if no repetition of letters is allowed?*, and they understood their formula of 6!/3! as dividing out by equivalent arrangements in the three elements of the "tail" – or the three positions they did not care about.

In the exchange below, Rose suggested an answer of 6!/3!, which Sanjeev had not previously considered. As he reasoned through and ultimately explains the process, this argument became his key way of reasoning about this kind of problem.

Rose: *Well, I divide it by 3 factorial, because the last 3 spots can be in any order, and they don't matter. It doesn't really –*

Sanjeev: *Are you saying hypothetically – if you were to continue adding spots – like if you were to go all the way up to 6.*

Rose: *Yeah, if you wanted combinations with all 6 letters that means when you get the last 3, and you'd get how many combinations there are with all 6 letters.*

Sanjeev: *Oh, I see. So if you have 6 letters then 6 factorial represents the total number of arrangements you can make with 6 letters. Because you're only looking for a sequence of 3. Then you can take out that last 3, because it doesn't matter. That order these last 3 would be in. And so when you divide that out, and this part this gets divided out here, you're left with (inaudible).*

In developing the formula for combinations, the students understood that if they considered combinations (as opposed to permutations) they needed to account for duplicates, and they realized that they needed to divide by an additional factor. Ultimately, they arrived at the correct formula for combinations.

The students had thus established a strong understanding of division and equivalence in determining the formulas for both permutations and combinations. We saw subsequently that this became their go-to way of solving combination problems. This can be from Session 4 with a problem about iPhones that says, *In a shipment of 1,000 iPhones, 25 are defective. How many ways can we select a set of 50 non-defective iPhones?* Figure 1 demonstrates exactly their three stage counting process – arrange all of the distinct elements, divide by the "tail" that you "do not care about," and then additionally divide (or multiply by 1 over the reciprocal) in order to account for duplicates again.

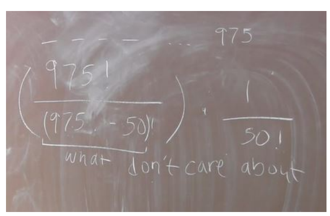

Figure 1 – The students' response to the iPhones problem

I want to emphasize that it was impressive that Rose and Sanjeev had such a solid understanding of the division and of the counting process, and especially that they had an understanding of how combinations and permutations are related. Almost more than any students I had worked with before, they had a particularly solid sense of division and equivalence and of how to make that work out in terms of the set of outcomes. However, at the same time I started to notice that every time they explained or thought about or talked through a combination problem, they always referred to that three-stage process. I began to wonder if they *necessarily* went through this process as an inherent part of their understanding of combinations.

As further evidence of their conception of this three-stage process for solving combination problems, and for the relationship between permutations and combinations, consider the students' description of four key problem types in Table 1. After the students solved some initial problems we had them characterize and group the problems according to type and to articulate a formula for each one. Their formulas and written responses for factorials, permutations, and combinations are given in Table 1. Note that they actually defined the definition of combinations as building on the formula for permutations, providing further evidence for the nature of the process for counting combinations. Subsequently, there were surprising episodes in which the students seemed to focus on their process rather than implement a single act of choosing.

Formula	Problem Type	Students' Written Description
$n!$	Arrangements	How many ways to arrange a given number of elements.
$\frac{n!}{(n-k)!}$	Permutations	How many ways to arrange a given number of elements into a given number of spots without reusing any elements
$\frac{n!}{(n-k)!k!}$	Combinations	[continue from permutations]…and divide by the factorial of the given spots to delete repeated sequences because any arrangement of the same given elements is considered the same combination.

Table 1 – The students' descriptions of what they were trying to count

Eventually, after more prompting and work with the students, they did perhaps demonstrate that they could consider the result of their three-stage process as an object in and of itself. For example, they were solving a problem in which they were making a committee with 3 men from 7 men. As they had the following exchange they wrote $\left(\frac{7!}{4!}\right)/3!$.

Rose: *So that means for this scenario, we have what was it, seven men. I'm going to write it over here. We have seven men, and it looks like we're going to isolate the three persons by 4 factorial.*

Sanjeev: *Which are the ones we don't care how they're arranged.*

Rose: *Yeah. And then, within this group of 3, there's 3 factorial ways to arrange them.*

This again emphasizes their attention to the three-stage process. However, they then did seem to be able to think about operating on the result of that expression, as Rose indicated .

Rose: *Because this will tell us how many ways you can arrange the men. And for every way you can arrange the men, it goes with one of the three arrangements of the women. So if there are three arrangements for women, then, that's three times this [the expression for the men].*

Discussion and Conclusion

Although I could provide only a limited number of examples due to space, the results presented in this preliminary report are intended to raise questions about what it takes to provide evidence that reification has taken place. There did seem to be evidence that the students could think of their combination expression as "how many ways you can arrange the men," which suggests being able to conceive of the results of their three-stage process. And yet, such utterances came only after the students clearly articulated this process. Some lingering questions, then, include, *What is evidence for reification in the context of combinatorics?* and *Do the utterances such as "this will tell us how many ways you can arrange the men" count as evidence that they conceive of "choosing" as a reified process?* In addition, I wonder what role symbolization plays in reification, as I had not prompted the students to express the formula for combinations using a single symbol. Also, I wonder if the way in which the tasks were sequenced hindered their reification process in any way. A final question is whether or not it is necessary for students to reify the process of choosing in order to be successful counters. I hope to facilitate discussion about such questions through the presentation of this preliminary report.

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Outcomes Beyond Success in a Problem Centered Developmental Mathematics Class

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Low success rates in the pre-college level, or developmental, curriculum at many community colleges has resulted in the creation of classes that use problem solving and group work to help students become more mathematically empowered. This preliminary report describes one such class at a Midwestern community college and then outlines the results from a pre- and post-survey of students taking the class, focusing on whether students' attitudes towards mathematics changed while enrolled in the class. Further analysis will examine how students evaluated the class and ranked the class structures. Generally, males, younger students, and Black students were less likely to complete the course. Students who came close to completing the class had an overall positive shift in their attitudes towards mathematics.

Keywords: developmental mathematics, community college, problem solving, attitudes

Each year over 1 million students invest substantial time and money to take pre-college level classes that do not count towards a degree (Parsad, Lewis & Greene, 2003). Often called "developmental," such classes provide *all* students with the chance to be *college ready* (McCabe, 2000). However, as many as 67% of students who start developmental classes do not finish (Bailey, Jeong & Cho, 2010), which is particularly concerning given that developmental students disproportionately come from minority and low socio-economic backgrounds (Attewell, Lavin, Domina, & Levey, 2006). Despite the volume of students in developmental programs, research has rarely examined individual classes and students. Given this, an acute need exists for research on developmental math that looks beyond success rates (Mesa, Wladis & Watkins, 2014).

Developmental math tends to be taught using lecture with an emphasis on procedural knowledge (Grubb et al., 1999; Mesa, Celis, & Lande, 2014). The low success rates, combined with the fact that community college students have high levels of math anxiety (Sprute & Beilock, 2016) and appreciate knowing how what they learn relates to their lives (Cox, 2009), suggests that the traditional ways of teaching developmental math do not provide sufficient opportunities for students to change their opinions about mathematics or themselves as doers of mathematics. This study examines the outcomes beyond success of one developmental math implementation that attempts to better meet the needs of the developmental population.

Mathematical Literacy: Streamlining the Curriculum

Low success rates in developmental mathematics has led community college educators to create curriculum pathways that accelerate students through their required pre-college-level course work while promoting more real world connections. *Mathematical Literacy* at Fields Community College (FCC; all names are pseudonyms) is one such initiative. Rather than learning through lecture, students engage in group work. The teacher primarily acts as a facilitator while the students work on real world problems. The course is designed to be completed in one semester, which fulfills the students' developmental needs in a shorter time frame than the traditional algebra sequence.

The *Mathematical Literacy* movement is fairly new, but early results from national initiatives suggest students are more successful than students taught in traditional classrooms (Strother & Sowers, 2014; Yamada, 2014). Less understood is how students' relationship to math changes after taking classes like this and students' experiences in these classes. Developmental classes offer one of the last opportunities for students to explore mathematics and challenge their ideas about math, so it is important to investigate whether and how students' attitudes towards math change in these new developmental classes. In particular, I investigate:

- Who finishes *Mathematical Literacy* at FCC?
- Do the attitudes of students who finish *Mathematical Literacy* change during the course of the semester? How do attitudes vary between students with different characteristics?
- Is there a relationship between the magnitude of students' attitudes towards mathematics and their background characteristics?
- How are students' evaluations of the class related to their attitudes towards mathematics?

Methods

Sample

In total, 150 *Mathematical Literacy* students from eight surveyed sections at FCC participated. All FCC *Mathematical Literacy* instructors were invited to participate. Sections were surveyed in all instances when the instructor agreed to participate. Three sections occurred in the in fall (n = 53) and five sections in the spring (n = 97). The number of students participating from a particular section ranged from 15 to 24 students.

Data Sources

Data for this study come from a pre- and post-survey given to students during the first and last week of the semester, respectively. Both the pre- and post-survey included the Attitudes Toward Mathematics Inventory (ATMI; Tapia & Marsh, 2004) to measure mathematics attitudes along four dimensions related to mathematics performance: enjoyment of mathematics (8 items), motivation (9 items), self-confidence (15 items), and value (8 items). Students answered items using a 5 point Likert scale on items. The pre- and post-surveys also include single item questions asking about the nature of mathematics (i.e., "learning math is mostly memorizing facts") and about students' level of comfort with certain types of class structures (i.e., "I learn mathematics best when I get to work in a group"). In addition, both surveys contained free-response items asking about students' mathematical backgrounds (pre-survey), demographics (pre-survey), educational plans (pre-survey), expected grades (post-survey), and course evaluation (post-survey).

Analytic Methods

To determine who completes *Mathematical Literacy* I ran descriptive statistics on the pre-survey sample and compared these to the descriptive statistics of the post-survey sample.

To examine whether or not the attitudes of students' towards mathematics changed over the semester, I conducted a confirmatory factor analysis on the ATMI scales. For brevity, in this proposal I report only the results from the original scales. Analysis using the attitude scales were run on both the original scale and the scales created after the confirmatory factor analysis. The general trends of the findings discussed in the results section are the same for both sets of scales.

Using the attitude scales, I performed paired sample t-tests to determine whether attitudes of those who come close to completing the class changed significantly over the 16 week class. To test for differences in attitudes growth between different sub-populations, I ran independent sample t-tests between different sub-populations of interest.

To examine the relationship between the changes in attitudes and students' background characteristics, I performed two-level Hierarchical Linear Models (HLM) with teachers as a level-two variable and students as level one. Separate models were run for each of the four attitudes, with the post-survey attitude scores as the dependent variable. Independent variables were students' pre-survey attitude score, gender, race, age, anticipated grade, whether or not they had taken previous developmental math classes, and the highest degree they intended to earn.

I will use descriptive statistics, basic qualitative coding, and HLM modeling to examine the attitude scales and class evaluation data and their relationship to students' backgrounds.

Results

Who Completes Mathematical Literacy

Although eight sections took the pre-survey, one class section did not take the post-survey because the survey schedule did not work out for the instructor. Within the students who had the opportunity to take both the pre- and post-survey resulted in a 63.3% retention rate between the pre- and post-survey. Students in this group who did not take the post-survey were, for the most part, no longer actively pursuing the course for personal or academic reasons. All of the students who took the post-survey reported that they expected to pass the class.

Descriptive statistics for participants who took only the pre-survey, compared to those who took both surveys are summarized in Table 1. Notably, the whole survey sample is more female than male and majority white, which is consistent with the developmental population at FCC. Those who only took the pre-survey were more likely to be male, more likely to be Black, and were a couple years younger than the students who took both surveys.

Table 1. Demographics of Survey Samples

	Whole sample		Pre-survey only		Took both surveys	
	n	%	*n*	%	*n*	%
Gender						
Male	65	45.14	34	52.31	31	47.69
Female	79	54.86	27	34.18	52	65.82
Race						
White	52	56.52	8	15.38	44	84.62
Black	29	31.52	21	72.41	8	27.59
Hispanic	5	5.43	1	20.00	4	80.00
Asian	5	5.43	0	0.00	5	100.00
Other	1	1.09	0	0.00	1	100.00
Taken prior developmental	93	35.86	44	47.31	49	52.69
No prior developmental	52	64.14	21	40.38	31	59.62
Age (years)	150	22.71 (7.88)	69	21.59 (7.31)	81	23.66 (8.26)

Note: Age reports the mean age of the participants in years followed by the sample standard deviation.

Changing Student Attitudes in *Mathematical Literacy*

Students who took both surveys had average positive gains on all four of the measured attitudes towards math. The shift in their enjoyment of math was significant. Students who were enrolled in their first developmental class had an overall positive increase in their value of mathematics, compared to those who had taken previous developmental math (Table 2).

Students were significantly more likely to disagree with the statement "there is only one way to solve a mathematics problem" at the end of the semester than they were at the beginning of the semester (Table 3). On other nature of math items there were no significant differences. They were also likely to shift whether or not they thought they learned mathematics best in a group or not, but these results were not significant. The high mean pre-post difference combined with the high standard deviation for these items ($\overline{x}$ = -0.185, sd = 1.246) suggests, however, that students were changing their mind about whether they agreed with these statements or not.

Table 2. Attitude Change Results for Selected Sub-populations

	Comparison of pre-post-survey attitude scores (complete data on both surveys)				Average growth in attitude scores of those with no prior developmental math compared to those that did				
					N		Pre-post difference		
Attitude	*N*	Pre-score	Post-score	*t*-stat	No prior devel. math	Prior devel. math	No prior devel. math	Prior devel. math	*t*-stat
Confidence	61	2.859	2.986	1.562	36	23	0.174	0.064	0.638
Enjoyment	74	2.941	3.093	2.406*	44	28	0.202	0.076	0.943
Motivation	73	2.830	2.925	1.637	44	27	0.073	0.136	-0.502
Value	68	3.756	3.813	1.064	40	26	0.144	-0.082	2.047*

*p<0.05

Note: Attitude scores were scaled to be on a 5 point scale with 1 corresponding to "Strongly agree" and 5 corresponding to "Strongly disagree."

Table 3. Attitudes Shifts towards Math and Classroom Structures

Item statement	N	Mean post-pre difference	Standard deviation
Learning mathematics is mostly memorizing facts.	77	-0.039	1.032
There is only one way to solve a mathematics problem.	80	-0.288*	1.171
I enjoy working in small groups in math class.	78	-0.077	1.066
I learn mathematics best when I get to work in a group.	81	-0.185	1.246
I learn mathematics best when I work by myself.	81	0.185	1.205
The math I learn in school rarely helps me when I use math in my daily life.	78	-0.115	1.032

*p<0.05

Note: Items were scored on a 5 point Likert scale, with 1 corresponding to "Strongly agree" and 5 corresponding to "Strongly disagree."

The Relationship between Attitude Change and Background Characteristics

For each of the measured attitudes, the biggest significant predictor of how attitudes changed was their pre-survey attitude score. However, intending to earn a Masters or higher significantly predicted an increase in the value of mathematics. Expecting to earn a grade of A in the class significantly predicted an increase in their mathematical confidence. The full results of the HLMs using the original scales on the ATMI are presented in Table 4.

Discussion & Conclusions

The results presented in the previous section highlight many positive outcomes beyond success rates in a *Mathematical Literacy* classroom. A 63% retention rate may seem low, but this number is higher than many developmental math instructors experience when teaching using traditional methods. That the students who completed the class experienced an average positive growth in their attitudes towards and views of mathematics also warrants excitement. Although only enjoyment showed a significant increase, several others were close to being significant. A larger sample could produce more robust results. Given that community college students do

Table 4. HLM Coefficients for Models Predicting Post-survey Attitude Scores

	Value	Confidence	Enjoyment	Motivation
Fixed effects				
Pre-attitude score	0.692***	0.560***	0.647***	0.738***
	(0.085)	(0.090)	(0.088)	(0.089)
Male	0.081	-0.040	0.169	0.179
	(0.099)	(0.150)	(0.129)	(0.124)
Race/Ethnicity				
Black	-0.020	-0.218	0.138	0.086
	(0.159)	(0.230)	(0.198)	(0.204)
Hispanic	-0.012	-0.404	0.029	0.095
	(0.235)	(0.363)	(0.294)	(0.310)
Asian	0.436*	-0.519+	0.118	-0.078
	(0.198)	(0.311)	(0.244)	(0.230)
Other	0.464	0.478	0.475	0.000
	(0.382)	(0.542)	(0.512)	(.)
Age (years)	-0.003	-0.001	0.009	0.005
	(0.006)	(0.010)	(0.009)	(0.008)
Prior developmental	-0.180+	0.047	-0.170	0.042
	(0.104)	(0.156)	(0.134)	(0.131)
Expected grade				
A	0.110	0.606**	0.319*	0.231
	(0.123)	(0.190)	(0.159)	(0.158)
B	0.076	0.367*	0.069	-0.052
	(0.108)	(0.157)	(0.136)	(0.133)
Anticipated degree				
Bachelors	0.023	-0.026	-0.070	-0.023
	(0.127)	(0.204)	(0.170)	(0.164)
Masters or higher	0.364**	0.166	-0.010	0.190
	(0.138)	(0.229)	(0.183)	(0.183)
Unknown	0.174	0.322	0.519	0.401
	(0.373)	(0.386)	(0.366)	(0.350)
Constant	1.046**	1.092***	1.070***	0.667*
	(0.335)	(0.300)	(0.297)	(0.302)
Random Effects				
Teacher	0.000**	0.000**	0.000*	0.040
Residual	0.338***	0.477***	0.464***	0.442***
Sample size	65	58	71	69

+ p<0.10, * p<0.05, ** p<0.01, *** p<0.001

experience a lot of mathematics anxiety, these results are certainly worth investigating further to determine which parts of *Mathematical Literacy* contribute to the students' shift in attitudes.

That said, some results suggest that *Mathematical Literacy* does not reach everyone equally. The overall rate of persistence in *Mathematical Literacy* is higher than for traditional classes, but the students who did not complete the class were more likely to be Black and male than the students who did. Community college mathematics classrooms are an important and under-examined area in mathematics education. Given that Black students disproportionally enroll in developmental math, understanding the issues that keep these students from succeeding in *Mathematical Literacy* deserves a closer look.

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Underrepresented Students Succeeding in Math: The Challenges and Coping Strategies of Mathematically Talented Post-secondary Students

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Retaining mathematically talented underrepresented students in mathematics programs requires understanding the challenges the face during their post-secondary mathematics education. Using Swail's framework (2003), this study investigates the self-identified challenges undergraduate and graduate mathematics students face and the coping mechanisms that helped them navigate and overcome those challenges. The vast majority of the challenges both groups of students encountered were cognitive in nature, suggesting that programs wishing to retain students should focus on providing social and institutional supports to provide balance.

Key words: Underrepresented students, mathematics, post-secondary programs

In the United States, completing a graduate program in mathematics was, until recently, primarily the domain of white American citizens and international students. In 2012−2013 the U.S. awarded 1,843 doctoral degrees in mathematical science, with 46.5% going to U.S. citizens. Of domestic students, only 24 graduates were African American and 25 Latino (AMS, 2013).

The reasons for low numbers of underrepresented students in math stem from many causes. Underrepresented undergraduate students generally have lower rates of degree attainment than the general college student population (Gladieux & Swail, 1998; Mortenson, 2005) and mathematics based fields (Bonous-Harmarth, 2000). At both the undergraduate and graduate level, underrepresented students pursuing STEM degrees face challenges related to their identity (Fries-Britt & Turner, 2002; Gildersleeve, Croom, & Vasquez, 2011; Malone & Barbarino 2008). At the graduate level academic challenges have been documented for students of color (Cooper, 2004; Herzig 2002). Cooper (2004) found that black graduate reported differences between graduate and undergraduate culture related to studying. Although they felt comfortable in the program, some reported hostile and competitive climates. Students of color who eventually earned a graduate degree in a mathematics based field reported frequently being judged by the color of their skin while earning their degree (MacLachlan, 2006).

Many students earning advanced degrees face adversity, but the unique and compounding challenges underrepresented students face means it is particularly important to understand their challenges and how they overcome them at both the graduate and undergraduates levels. Few studies have examined successful underrepresented students in mathematics. Expanding this knowledge base can help inform services that support students pursuing math. Towards this end, we compare the challenges to success and the subsequent coping strategies mathematically talented underrepresented undergraduate and graduate students identify. In particular we ask:

1. What are the self-reported challenges the mathematically talented underrepresented students face?
2. What are the strategies utilized by students to help them manage and persist through these challenges?
3. How do the challenges faced and coping strategies used vary between undergraduate and graduate students?

To help answer these questions, we focus on student participants in the National Alliance for Doctoral Studies in the Mathematical Sciences (henceforth referred to as "the Alliance"). The

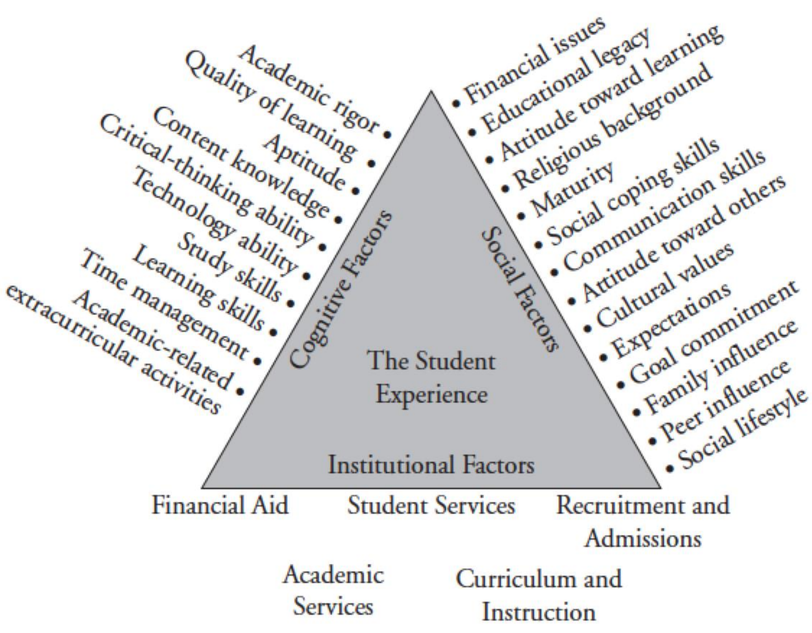
Alliance is a nationwide program seeking to facilitate the recruitment and retention of talented, underrepresented domestic students in mathematics disciplines through the doctorate.

Conceptual Framework

This study uses Swail's (2003) Framework for Retention of Minority Students, which captures the interplay of the many factors affecting student persistence and achievement by categorizing them into three main components: 1) Social; 2) Cognitive; and 3) Institutional. These constructs are presented around a triangle, with each component forming one edge (Figure 1). The graphical representation signifies that the three components impact one another while also influencing a students' persistence or achievement; persistence optimally occurs when equilibrium exists between the three factors. For example, a student facing large institutional challenges must have social and/or cognitive supports that compensate for the deficiency. Swail's framework, although underutilized in research, accounts for both students' personal attributes and the contextual factors related to an institution's role in persistence (Swail, 2003).

While Swail's (2003) retention framework is most commonly utilized by institutions to improve student retention, the Alliance's goal to retain students in mathematics is comparable. Alliance students are currently persisting towards a mathematics degree and have access to many of the factors the literature suggests cultivate success. This unique set of circumstances makes the Alliance population uniquely suited to understand the challenges of STEM pipeline persisters and the ways in which they overcome those challenges.

Figure 1. *Swail's Framework of Minority Student Persistence*



Source: Swail (2003), p. 77

Data and Methods

Sample

Participants at the 2014 Alliance conference were asked to complete an anonymous survey during lunch of the main day. Of the approximately 300 conference attendees, 210 individuals took the survey, which included students, faculty mentors, and university representatives. We focus on the responses of the participating undergraduate (*n*=127) and graduate students (*n*=25). Table 1 provides basic demographics on these students. The two page survey consisted of both free-response items asking about various dimensions of success and quantitative rankings of various influences on success.

In this study, we draw the data from four of the questions: two free-response and two ranking questions. The free response items asked:

1. *Among the various experiences you've had in your program, please describe one that you consider to be the most challenging.*
2. *If applicable, why did you think this was a challenging experience and how did you overcome this challenge?*

The first ranking question asked students to order the influence of various individuals on their success. The options were: faculty/teachers, advisors, mentors, peers, family, or others. The second ranking question asked students to assign a percentage to different people and structures that indicated how much of their success was attributable to that entity, so that the sum of the percentages assigned was 100%. Their choices were: mentors, financial aid, the Alliance, natural ability, hard work, peer relationships, and other.

Table 1. Demographics of Survey Sample

	Undergraduate		Graduate		
	Male	Female	Male	Female	Total
White	9	9	1	2	21
Black	9	25	5	1	40
Hispanic	34	23	6	4	67
Asian	0	2	0	0	2
Other/Multiracial	3	10	2	3	18
Total	55	69	14	10	148

Methods of Analysis

The analysis of the items from this survey used qualitative coding combined with statistical tests for the two free response items and statistical methods for the two ranking questions. We discuss each, in turn, below.

Free response items. The free response items asked students to identify the biggest challenge faced during their program and then describe how they overcame that challenge. We read the responses and identified the challenge and the coping strategy using Swail's (2003) Framework for Retention of Minority Students. The authors each coded all the responses separately. Discrepancies were discussed until consensus was achieved.

One hundred twelve undergraduates and 23 graduate students described the challenges they faced while in school. In comparison, only 74 undergraduates and 19 graduate students answered the follow-up question. For this preliminary analysis, each challenge and each coping strategy was coded as social, cognitive, or institutional. If a student mentioned more than one challenge or coping strategy, both codes were assigned. When the student identified a challenge but not their coping strategy, or the student indicated that the challenge had not been resolved, the coding for the coping strategy was left blank.

After coding we ran Pearson Chi-squared tests to determine whether the undergraduate or graduate students varied significantly in the types of challenges they faced. We ran similar tests to determine whether graduate students were more likely to resolve challenges with both social and cognitive coping strategies.

Quantitative items. The first quantitative item asks students to rank which individuals were critical to their success. The second asks students to assign a percentage indicating how much particular sources contributed to their success. Responses that did not sum to 100% were removed from the analysis. Data from these questions were analyzed using independent sample *t*-tests to determine if the rank or weight of each source varied significantly between the undergraduate and graduate students.

Preliminary Results

The results presented in this section represent the initial analysis done on the research questions discussed above. By the time of the conference we plan to have explored these findings in more detail, as discussed in each of the results sub-sections.

Challenges and Coping Strategies

Both undergraduate and graduate students experienced similar challenges and the majority of the challenges they faced were cognitive. For example, a female undergraduate stated that for her "balancing all aspects of life was hard. Finding the right balance between school, family, social, and sleep is hard to find so that none are ignored." A male graduate student noted that "…trying to learn anything when I wasn't prepared for the stuff" was his biggest challenge. Both of these statements received a cognitive code they mention need for balance and academic background respectively, both of which are cognitive factors in Swail's (2003) framework. The first statement also received a social code because she notes social obligations as a challenge, which falls under the domain of social in the framework.

Graduate students did use coping strategies from multiple domains significantly more often than undergraduate students. They were also more likely to draw on social coping strategies than undergraduates. Table 2 documents the relevant test statistics. In our next steps we will code the cognitive responses into the secondary cognitive categories of Swail's framework to see whether graduate and undergraduate students face different types of cognitive challenges.

Table 2. Nature of Challenges and Coping Strategies by Student Status

		Nature of Conflict						
	N	Social	Cognitive	Institutional	Social & Cognitive	Social & Institutional	Cognitive & Institutional	All three
Undergraduate	112	36	76	12	11	0	1	0
Graduate	23	5	16	3	1	0	0	0
Pearson Chi-squared		0.977	0.026	0.105[a]				
		Nature of Coping Strategy						
Undergraduate	74	24	44	11	2	1	2	0
Graduate	19	11	12	5	4	4	2	1
Pearson Chi-squared		4.176*	0.769	0.238				

*p<0.05

[a] Cell sizes in this analysis were smaller than is technically required for a reliable Person Chi-squared estimate.

Factors Students Attribute to Success

Undergraduate and graduate students agreed on the relative rank of various individuals on their success: both groups felt that the person most important to their success was themselves.

Families and peers were the least important. Graduate students did rank their family and peers a bit higher than undergraduate students, but the difference was not significant.

Of those surveyed, 23 graduate and 114 undergraduate students completed the question asking students to assign a percentage indicating how much various influences impacted their success. The only significant difference was that graduate students attributed less of their success to natural ability (9.78% compared to 16.52% for the undergraduates). Table 3 shows the means for each of the tested influences and the relevant test statistics.

Table 3. Average Percent of Influence on Student Success

Influence on Success	Student Level	*N*	Mean	Std. Deviation (s.e.)	*t*
Mentor	Graduate	23	19.13	14.05 (2.93)	.446
	Undergraduate	114	17.81	12.74 (1.19)	
Financial Aid	Graduate	23	15.61	11.54 (2.41)	1.068
	Undergraduate	114	12.96	10.74 (1.01)	
Natural Ability	Graduate	23	9.78	7.16 (1.49)	-2.764*
	Undergraduate	114	16.52	11.21 (1.05)	
Hard work	Graduate	23	30.00	17.58 (3.67)	-.269
	Undergraduate	114	30.98	15.49 (1.45)	
Peer relationships	Graduate	23	16.48	15.81 (3.30)	1.742
	Undergraduate	114	12.30	9.11 (0.85)	
Alliance	Graduate	23	7.96	7.97 (1.66)	.381
	Undergraduate	114	7.40	6.05 (0.57)	
Other	Graduate	23	1.09	5.21 (1.09)	-.655
	Undergraduate	114	2.05	6.63 (0.62)	

*$p<0.05$

Significance

This study provides insight into the types of challenges persistent underrepresented undergraduate and graduate students face in their mathematics programs. The fact that most of their challenges were cognitive is suggestive. Swail's (2003) framework argues that to persist students must have balance between the three domains. Because students in mathematics programs face overwhelming cognitive challenges at both the undergraduate and graduate level, programs focused on retaining underrepresented students should strive to balance the necessary cognitive challenges with rich, meaningful social and institutional supports.

The fact that graduate students were more likely to cope with the challenges they faced using structures from multiple domains of Swail's framework is also suggestive. Although undergraduate and graduate students face similar challenges, graduate students are more likely to resolve those challenges using multiple types of support. The quantitative data from the survey speak to this—graduate students believe their natural ability is not as important to their success as undergraduates, suggesting they look outside of themselves for support. Conversely, the key to promoting undergraduate retention in STEM may require encouraging underrepresented undergraduates to take advantage of the opportunities many diversity offices provide. More generally, the findings presented in this study provide support for previous research positing that an awareness of the environmental factors that exist for underrepresented students and how they perceive those factors is integral to their successful transition and persistence.

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Student Gesture Use When Explaining the Second-Derivative Test and Optimization: Mimicking the Instructor?

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The Second-Derivative Test and optimization can naturally evoke gestures from an instructor while he or she is teaching. We wanted to establish how student learning might be affected by an instructor's use of gesture. Students viewed either a gesture-rich or gesture-free video of an instructor solving an optimization problem, and were interviewed a week later to assess both their understandings of optimization and how they used gesture to support their explanations. Very few gestures were used when the students explained how they solved the optimization problem. However, when describing the second derivative test separate from optimization, students used a number of gestures. We conclude that further study should be undertaken, but such study should be focused on the Second-Derivative Test without the context of optimization problems.

Key words: Gesture, Calculus, Optimization, Instruction

Background

Gesture use in the classroom may be used to increase student interest and foster student success in calculus. Recent studies suggest that instructors' use of gesture promotes student learning, and that instructors can intentionally alter their gesture production during instruction (Alibali, et al., 2013a; Alibali, et al., 2013b; Cook, Duffy, & Fenn, 2013; Hostetter, Bieda, Alibali, Nathan, & Knuth, 2006). As calculus can be considered the study of motion, it is a natural place to examine gesture. There are several types of calculus problems that require students to visualize or imagine situations involving changing rates, with optimization being one example. LaRue (2016) studied student responses and approaches to optimization problems, while classifying each action in a student's problem-solving process using Vinner's (1997) conceptual, analytical, pseudo-conceptual and pseudo-analytical definitions. We use these definitions, along with Tall and Vinner's (1981) concept image, to describe student understanding.

We seek to answer the following questions: Will students who view a gesture-enhanced lesson demonstrate a greater understanding of the presented concept than students who view a gesture-free lesson? How do students use gesture when explaining their work? Do they mimic any gestures used by their instructor? We present a pilot study that attempts to answer these questions.

Methods

Data collection was completed in two phases. The first phase included eleven students, each of whom were currently enrolled in a second-, third-, or fourth-semester Calculus class. The average amount of time it had been since each student had taken first-semester Calculus was about three-and-a-half years. These students were asked to take a short pretest, watch one of two possible videos, and then complete a short posttest. The second phase, which occurred a week later, involved semi-structured interviews in which students were asked questions to assess their understanding and recall of the first phase's material.

The six-question pretest had students determine if points on a function were maxima, minima, neither, or “can’t tell,” using three of the standard function representations: algebraic, graphical, and verbal. The verbal problem was an optimization problem in which they were asked to maximize the area of a rectangle for a given perimeter. After completion of the pretest, students were shown one of two videos in which a similar perimeter/area optimization problem was presented, solved, and verified using the Second-Derivative Test. Both videos followed essentially the same script, but one video contained instruction that was given without any of the categories of gesture outlined in Engelke Infante (2016), while the second video contained instruction that utilized such gesture. After watching and taking notes on the video, the students were given a three-question posttest consisting of three optimization problems: the first was very similar to the one in the video, the second was an extension of the first, and the third was a different type of optimization problem. In the interest of space, we consider only the optimization problems that were similar to what was presented in the video. Six students viewed the gesture video, and the other five viewed the gesture-free video. Rubrics were created for each problem, and tests were graded and scored.

For the second phase, we requested interviews with all participants. Three students agreed to participate and were asked to 1) tell us what you know about the Second-Derivative Test, and 2) solve a simple perimeter/area optimization problem. As students reached three specific points in their solution process, they were asked “How do you know that the functions you have chosen represent area and perimeter?”, “How do you know to take the derivative at this step?”, and “How do you guarantee that the answer you obtained was a maximum?” Follow-up questions were asked to clarify responses and ascertain that we had gained as much student knowledge as possible. The interviews were videotaped and an analysis of subject gesture use was begun.

Data and Results

To assess the extent that the gesture-rich video influenced student understanding, we compared the student scores of the last item on the pre-test (the only optimization problem) with the average scores of the two similar optimization problems in the posttest. The initial results suggest that the students who viewed the gesture-rich video improved their scores on the optimization problems more than the students who watched the gesture-free video, as shown in Table 1.

Table 1
Scores of the Optimization Problems, Pre- and Post-video

Students	Pretest Problem	Posttest Problems	Improvement
All Students	26%	52%	26%
Gesture-rich video	22%	55%	33%
Gesture-free video	32%	48%	16%

During the interviews, we expected students to recall statements and solution methods that were shown to them in the video. However, of the three students interviewed, none could give an accurate description of the Second-Derivative Test, and while two students completed the initial optimization problem correctly, they did not verify their answers using the Second-Derivative Test, which was the method of verification in the video. The third student also obtained a correct answer to the initial optimization problem, but he did so without using calculus. When given an additional optimization problem to solve, he was unable to solve it. Further discussion of the interview responses is given below.

The Pretest and Posttest

The pretest suggests that the students had poor mastery and recall of first-semester calculus objectives that required the use of more abstract algebraic representations of functions. The first question (Pre #1) asked students to determine extrema of a function, f, given its graph. The next four problems (Pre #2-5) asked students to determine extrema of f given either the graph of f' or a list of algebraic information about f, f', and f'' (e.g., $f(1) = 8, f'(1) = 0, f''(1) = 2$.) The final pretest problem (Pre #6) asked students to solve an optimization problem requiring them to maximize the area of a rectangle for a given perimeter. Table 2 lists the scores for all problems on the pretest and posttest and shows this drop in scores after the pretest's first problem. We were not totally surprised by this score decrease, given that the literature indicates that the function concept takes a long time to develop, and that students have difficulty moving between different function representations (Carlson, 1998; Carlson, Oehrtman, & Engelke, 2010).

We found a similar score decrease across the posttest optimization problems. The posttest problems asked the students to: 1) solve a problem similar to the example problem in the video, 2) solve a problem with a slight extension of the example problem, and 3) solve an optimization problem with a different context. It was our goal to determine the extent to which they could make extensions from the example that had been presented. As seen in Table 2, students scored well on the most similar problem (Post #1) and scores decreased as extensions needed to be made.

Table 2
All Pre- and Posttest Scores

Students	Pre #1	Pre #2-5	Pre #6	Post #1	Post #2	Post #3
All Students	75%	24%	26%	67%	36%	20%
Gesture-Rich Video	87%	25%	22%	70%	40%	12%
Gesture-Free Video	60%	23%	32%	64%	32%	22%

The Interviews

During the interviews, no student accurately described the Second-Derivative Test. Each student gave responses that were a mixture of correct statements, pseudo-conceptual and pseudo-analytical ideas (Vinner, 1997), and statements that were incorrect or off-topic. Two of the three students were able to complete an optimization problem correctly, but neither of them used a version of the Second-Derivative Test to verify their answers, like had been done in the video example. We present case studies of each student's responses that focus on their gesture use and what they said about the Second-Derivative test.

Students Who Watched the Gesture Video

Student 105. Student 105 initially stated that the Second-Derivative Test was "concavity." His first use of gesture occurred when he began discussing a sign chart which was part of his concept image of the test. While stating that one must find critical points, he used his hands to signal a hierarchy of derivatives by moving them lower at each step while saying "… [given a] function, you find the first derivative, and then you find the second derivative." He seemed to show confusion between f, f', and f'', stating that one must "plug [critical number] back into the original function and that will tell you exactly where it changes … it will change concavity" and later compounding this error by drawing a sign chart that involved plugging critical values into f, claiming that the signs of those results determined decrease and increase, and stating incorrectly that "from decreasing to increasing is concave down, and … from increasing to decreasing is concave up." Thus, while his concept image includes this sign chart, his reasoning behind the creation of this chart is pseudo-conceptual. Nevertheless, he used many gestures during this explanation, indicating concavity by using five "U"-shaped hand positions and two "U"-shaped tracing motions within a span of about 40 seconds during the explanation of his sign chart.

He was able to solve the optimization problem correctly, using the method shown in the video minus the verification step. However, when asked about why we take the derivative of our area function, he noted because "The first derivative tells us the maximum and minimum, the second derivative tells you the concavity." While this statement is technically correct, it is pseudo-analytical as it is not an explicit reason as to why we take a derivative, and he did not include any of the "U"-shaped gestures he used before. When asked how he knew that his final answer led to a maximum, he did not invoke the Second-Derivative Test. Instead, he constructed a table by testing at least one value larger and one value smaller than his critical point; the areas resulting from those numbers were calculated and found to be less than the area resulting from his answer. Here, we see evidence of pseudo-analytical reasoning and the absence of gesture use. Aside from a few general hand motions given to emphasize certain words, he invoked minimal gesture during these particular explanations.

Student 109. Student 109 mentioned that the Second-Derivative Test was "to determine the point at which the (first) derivative changed direction," while positioning his index finger higher than his middle finger and then switching their positions to signify this change. When recalling the Second-Derivative Test, he invoked the image of a bell curve, and used gestures in his description, using two hand motions to trace, first upward, then downward, an imaginary bell curve in the space in front of him while saying, respectively, "climbing up the bell curve," and "it would not start to go negative, but start decreasing," and finishing with the same finger gesture as before while stating "it's the change of the first derivative."

Like Student 105, Student 109 was able to solve the optimization problem using the method shown in the video, and used a similar table instead of the Second-Derivative Test to show that his answer led to a maximum. Student 109 did not use much gesture during the explanation to his solution of the optimization problem; his gesture use consisted primarily of tracing elements of the many diagrams he drew to compliment his words. For example, after drawing a bell curve with a small horizontal tangent line at the top, he said "at this point, slope should equal zero, and the tangent line would be flat," while tracing over the horizontal tangent line.

Student Who Watched the Gesture-Free Video

Student 106. Student 106 initially said that the Second-Derivative test told us "rate of change" and "maximum and minimum." With that latter statement, he presented simultaneously a raised and lowered horizontal, flattened hand to delineate the ideas of maximum and minimum. He later said the test is "where the first derivative is equal to zero …," at which point he seemed to trace a small horizontal segment with his pencil, maybe to represent an image of a horizontal tangent line. He continued "… and it tells you the maximum and minimum, or maybe the local maximum and minimum of the first derivative," while making the same flattened-hand gesture during both times he said "maximum and minimum." Thus, it appears that Student 106 has at least a partial conceptual understanding of the second derivative.

He answered the perimeter/area optimization question by quickly stating that the shape had to be a square. When asked how he knew this to be true, he stated that "I know that circles have probably the largest area, then I think the square, in my mind is next … so I just figured that I'll just make a square out of the rectangle." Since this answer did not take up much time, the interviewer gave him a second optimization problem to consider. He talked through several possible ways to approach the problem, but finally settled on constructing a graph. He drew two increasing segments followed by three decreasing segments and did not have an explanation as to why he drew this shape. While he was describing how he was interpreting his graph, he made a downward pointing motion when describing a function as decreasing, moved his hand in a precise straight line when saying that a function was linear, and held up his pencil horizontally then moved a finger down at the appropriate time while saying that the first derivative equaling zero can give us a maximum because the "slope tells us where it levels off and starts to go back down."

Ongoing Work

The fact that students who viewed the gesture video showed more improvement when solving optimization problems on the posttest appears to be a promising area to explore. The interviews given a week later provided mixed results. While the students were describing their understanding of the Second-Derivative test in the interview, they used several gestures, some of which were similar to what they had seen in the video (Student 105) and some that were distinctly different (Student 106, who viewed the gesture-free video). However, students' performance on similar optimization problems was not as strong as indicated on the posttest. Students did not verify that their answers were extrema. When asked how they knew their answer was a maximum, they did not invoke the second derivative as had been presented in the video. This leads us to believe that we were potentially asking them to recall too much mathematics. Analysis of the interviews is ongoing.

Since the amount of mathematics in our script and tests seemed too much for these students, our next stage of research will be done on a smaller scale. We plan to devise a smaller, 4-5 minute script that only defines the Second-Derivative Test, eliminating the surrounding context of an optimization problem. This script will be filmed in four ways: with no gesture at all, with only tracing and pointing gestures at the board, with only hand gestures in the space between the instructor and the student, and with both of the preceding types of gesture. It is our hope that a simplified script, with more focused pre- and posttests, will allow for greater student success. With this success, we hope to not only have a greater understanding of how instructor use of no gesture, limited gestures, or many gestures might affect student learning, but also see students using more gesture to augment the explanation of their mathematical processes. Hence, we will be able to make suggestions for how instructors can incorporate gesture into their teaching to facilitate student understanding and success. As our work progresses over the next several months, we are certain several interesting questions will arise that we will bring to the conference.

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Experts' Varied Concept Images of the Symbol dx in Integrals and Differential Equations

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The mathematical symbol "dx" is a symbol for which there can exist different views about its characteristics, purposes, and roles. We wished to see how experts viewed the dx in a variety of settings. We chose four mathematical contexts and interviewed four mathematics professors in order to understand their various concept images of the dx. While there was little agreement among the experts' responses, most of them did have a strong concept image that remained consistent throughout their interviews, despite our attempts to create cognitive conflict between the different mathematical contexts. We conclude that the existence of a range in the experts' opinions is noteworthy, and that further study should be conducted in order to more fully explore this range and any implications for instruction that may result from it.

Key words: Calculus, Integrals, Differential Equations, Concept Image

In this preliminary report, we examine instructors' views of some of the various roles of the symbol dx. This symbol can be found in a variety of mathematical settings, including definite and indefinite integrals, the formal definition for differential of a function, Leibniz notation for derivatives, the process of integration by substitution, and several types of ordinary differential equations. We wanted to see whether instructors' concept images (Tall & Vinner, 1981) of this symbol would be consistent throughout all of these settings, or if different concept images would manifest given different settings.

Research has been done on how students perceive the dx in a definite integral. Sealey and Thompson (in press) summarized four such perspectives noted in the literature: a marker that points out the variable of integration (Artigue, Menigaux, & Viennot, 1990; Hu & Rebello, 2013; Jones, 2013; Nguyen & Rebello, 2011), a graphical width of a rectangle (Bajracharya, Wemyss, & Thompson, 2012; Wemyss, Bajracharya, Thompson, & Wagner, 2011), a small amount of a given physical quantity (Artigue et al., 1990; Hu & Rebello, 2013; Roundy, Manogue, Wagner, Weber, & Dray, 2015), or a difference or change in a quantity (Von Korff & Rebello, 2012). López-Gay, Martinez, & Martinez (2015) gave other concept images found in physics, including dx as an infinitesimal increment or linear estimate. While Hu and Rebello (2013) emphasized the importance of conceptualizing the dx as a width, Sealey and Thompson (in press) noted the importance of conceptualizing it as a difference in other mathematical contexts.

The purpose of our research is to explore further concept images of the dx in some additional mathematical contexts. Specifically, we chose to investigate experts' perspectives of the dx in definite and indefinite integration, the formal definition of the differential of a function, integration by substitution, and separable and exact ordinary differential equations. To create beginning points of reference, two surveys of textbooks were conducted. The first survey included nine books (Barnett & Ziegle, 1989; Breusch, 1969; Ellis & Gulick, 1988; Fisher & Ziebur, 1965; Hughes-Hallet, et. al., 2006; Mizrahi & Sullivan, 1982; Rees & Sparks, 1969; Stein, 1967; Stewart, 1987) which contained material found in traditional first- and second-semester calculus courses. We analyzed and compared any sections of these books in which definite integrals, indefinite integrals, differentials of functions, and integration by substitution were introduced and/or defined. The second survey included three books (Boyce & DiPrima, 2012; Stewart, 1987; Zill, 1997) that contained information on basic ordinary differential equations, with which we compared and contrasted the sections that introduced separable and exact differential equations.

We found that only Hughes-Hallet et al. (2006) presented the idea that the dx in the definite integral comes from the factor Δx found in a Riemann Sum, while the other books stated that the dx in a definite integral was merely notation with its only purpose to serve as a dummy variable that indicated the variable of integration. All of the books clearly stated the definition that, if y were a function of x, the differential of y is given by the formula $dy = y'(x)dx$, with all but one book stipulating that x and dx were independent variables, x is any number in the domain of y, and dx is any real number. Every book approached the evaluation of an integral that required substitution by the usual method of determining a $u(x)$ and using the relation $du = u'(x)dx$. However, there was no discussion as to the nature or roles of the various dxs that were seen throughout this process. Similarly, no matter the solution methods offered for separable or exact ODEs, no book explained the roles of the dx, nor discussed the permissibility of multiplying or dividing by dx throughout the solution process.

Theoretical Perspective and Methods

Tall and Vinner's (1981) concept image and definition were used to structure the design of the study and the analysis of the data. Because of the discrepancies and varieties of concept images and definitions found in the existing research and collections of textbooks, we wanted to interview experienced mathematics faculty to see if their concept images would be different, not only from the books' images, but also from each other's. We wanted to determine if their individual concept images would be more well-formed and align more closely to a formal concept definition than what the textbooks seemed to provide. Assuming the existence of well-formed concept images, we also wanted to see whether we might find some instances of potential cognitive conflict.

During a series of clinical interviews, four professors were shown a series of mathematical symbols, definitions, and situations in which the symbol dx was present. Faculty members were chosen so that there was some variety in their research areas. Participants Sonya, Johnny, and Jackson each had research and/or teaching experience in analysis and differential equations, while Kurtis' research areas included combinatorics and graph theory.

Prior to the interviews, we created an interview protocol, listing the order in which the aforementioned mathematical symbols, definitions, and situations would be presented to the subjects. Thirteen such symbols, definitions, and situations were divided into four categories, listed in Table 1. In addition to the question of how the subjects perceived the role of the dx in each category, follow-up questions were posed if the subject stated something that differed markedly from the surveyed textbooks or other subjects' responses. Johnny requested that his interview not be videotaped; thus his impressions have been taken from the authors' notes. All of the other interviews were videotaped and later transcribed.

Data and Results

Since we were interviewing experienced instructors, it was possible that their individual concept images might have converged to a formal concept definition. But, as the textbooks did not show one formal definition but a variety of ways in which to think about the dx, we anticipated that the professors might not all have the same concept image. Data analysis is ongoing, but

Table 1
A Summary of Our Categories and Uses of dx

Categories	Symbols, Definitions, or Situations Containing dx
Integrals	$\int f(x)\boxed{dx}$, $\int_a^b f(x)\boxed{dx}$, and $\int_b^a f(x)\boxed{dx}$
Definitions and Notation	If $y = y(x)$, the notation $\frac{dy}{\boxed{dx}}$ and definition $dy = y'(x)\boxed{dx}$ If $x = x(t)$, the notation $\frac{\boxed{dx}}{dt}$ and definition $\boxed{dx} = x'(t)dt$
Integration by Substitution	$\int_1^4 \frac{\cos\sqrt{x}}{2\sqrt{x}}\boxed{dx}$ versus $\int_1^2 \cos x\boxed{dx}$, after $\int_1^4 \frac{\cos\sqrt{t}}{2\sqrt{t}}dt$ used the substitution $dx = \frac{1}{2\sqrt{t}}dt$
Two ODEs	1) The separable equation $\frac{dy}{\boxed{dx}} = g(y)h(x)$, and the solution steps $\frac{1}{g(y)}dy = h(x)\boxed{dx}$ and $\int \frac{1}{g(y)}dy = \int h(x)\boxed{dx}$ 2) The exact equation $(2xy - 9x^2)\boxed{dx} + (x^2 + 2y + 1)dy = 0$

preliminary analysis indeed shows that while all four interview subjects were very consistent within their personal concept image, these images differed from one another, and did not align with a single formal concept image. Summaries of the subjects' responses for each of our four contexts and some of the subjects' personal concept images are given below.

The dx in Definite and Indefinite Integration

Sonya and Jackson stated that the dx in a definite integral comes from a limiting process applied to the width represented by the bases of Riemann Sum rectangles. Kurtis seemed to have a similar idea but was not as specific, saying that "it [the dx] comes from the Δx [in a Riemann Sum]" without mentioning the image of rectangle widths. Johnny initially described the dx as arising from the limit of "cuts in the interval between a and b on the *x*-axis," but later changed his answer to "dummy variable" after some thought. All subjects except Kurtis also claimed that they viewed the dx in an indefinite integral no differently than they viewed the dx in a definite integral. Kurtis, however, claimed that the indefinite integral's dx had no meaning beyond being half of a notation (the other half being the integral sign) which signaled antidifferentiation.

The dx in Definitions and Leibniz Notation

Kurtis said that $dy = y'(x)dx$ if and only if $\frac{dy}{dx} = y'(x)$, but did not feel that this meant that one could simply multiply or divide by dx to go from one form to the other. Sonya agreed that such multiplication or division was not possible, while Johnny and Jackson had no problem with multiplying or dividing by dx in this way. Another area of contention was that part of Johnny's initial response when presented with the four notations and definitions in this section was to state an idea also found in Rees and Sparks (1969): the dx in $dy = y'(x)dx$ could be either an independent variable or a function of two other variables, and that this definition of the differential of a function would still hold. Sonya and Jackson initially thought instead that dx was strictly a dependent or independent variable depending on its position in the definition, but Sonya came around to Johnny's view after that view was presented to her. Kurtis only went so far as to claim that the relationship between the dy and dx in $dy = y'(x)dx$ was the same as the relationship between the dx and dt in $dx = x'(t)dt$.

The dx in Integration by Substitution

All subjects except Sonya seemed to feel that even though the substitution process began with the "$dx = x'(t)dt$" definition of the differential of a function, once the substitution was made, the dx had transformed into a simple dummy variable, and was therefore now bereft of any deeper meaning. Jackson additionally mentioned that while the initial dummy variable (in $\int_1^4 \frac{\cos\sqrt{x}}{2\sqrt{x}}\,dx$) and the transformed dummy variable (in $\int_1^2 \cos x\,dx$) had similar roles as infinitesimal widths, we could still think of them as different, since one was the limit as *n* goes to infinity of $\frac{4-1}{n}$ while the other was the limit as *n* goes to infinity of $\frac{2-1}{n}$. This idea that the two versions of dx have different sizes was also expressed by Sonya, but her image included an idea that both dxs were on different levels; specifically "macroscopic/microscopic" levels.

The dx in Separable and Exact ODEs

Sonya felt that even though it may appear that we could multiply by dx in order to separate variables in the separable equation, what is really happening instead is that we are multiplying by Δx and then passing through the limit. Kurtis agreed with the idea that we are not really multiplying by dx, but seemed to think that it was always fine to proceed as if that is what were really happening. Johnny and Jackson, as before, had no problem with multiplying or dividing by dx. Similar responses occurred during the explanations of the exact ODE. Sonya was still uncomfortable with the idea of "moving the dx around" but admitted that it is how solving differential equations is usually taught. Johnny and Jackson did not have this discomfort, and Kurtis declined to answer, stating that he was not as familiar with exact differential equations.

Personal Concept Images

While analyzing all of the subjects' responses, it was found that Johnny, Kurtis, and Sonya seemed to have central images that ran throughout all of their answers. Johnny's overall view seemed to be summarized by his initial response when presented with the two notations and definitions: "in some ways, they are all the same." He noted that even though the traditional definition $dy = y'(x)dx$ came with the idea that *x* and dx were independent variables, there was nothing stopping us from assuming that dx could also be a function depending on other independent variables (for example, $dy = y'(x)dx$ while $dx = x'(s)ds$), a view shared by Rees and Sparks (1969). If one were to continue this chain down to the last link, then the last differential in this chain can also fit that definition, as in our example: $ds = 1 \cdot ds$, since the derivative of s with respect to s is 1. He repeatedly said that these relations between differentials were "meaningful only in their relation to one another." Thus, for example, one can multiply or divide by a dx while manipulating an ODE, since that ODE also contains a dy, but in the "integration by substitution" process, once the substitution has been made, the dx becomes a dummy variable, since we no longer have a second differential.

Johnny's central view of all dxs outside of integration having a numerical basis seems to run counter to Kurtis' central view, which was that every instance of dx or dy was merely a product of a "useful notation" and had no mathematical meaning as a numerical entity. Kurtis said many times that all of these manipulations were products of a "perfectly good notation," and thus easy for educators use when introducing concepts like the Chain Rule or integration by substitution, but while it may appear that mathematical operations with a dx might be implied, Kurtis was adamant that this was not the case.

An image of Sonya's was that she was uncomfortable multiplying or dividing by dx, since it was an infinitesimal quantity created by "passing Δx through the limit." To her, no matter the situation in which a dx was being used in algebraic manipulations, the "real story" was that we were instead manipulating Δx (a measureable quantity) and then passing through the limit, turning all Δxs into dxs. She noted that the convenience of simply saying "multiplying by dx" was helpful for instruction, but that we should be more careful about telling our students "we can multiply by dx." Yet when it was presented to her, she seemed to accept Johnny's and Rees and Sparks (1969) view that dx could be an independent variable no matter the presentation. This might seem to contradict her idea that dx was only some infinitesimal quantity unable to be manipulated. Further research will explore whether these two views are an example of cognitive conflict, or whether deeper questioning will lead to a more complete view of her total concept image.

Discussion

Many of the subjects' responses suggest possible areas for future research. It is possible to use differentials to develop implicit differentiation (Mizrahi & Sullivan, 1982; Rees & Sparks, 1969) or generate derivatives by using differentials instead of taking the limit of a difference quotient (Dray, 2013; Dray & Manogue, 2010). Additional data collection could tell us if any of the subjects' concept images allow or conflict with these developments. Sonya mentioned that the convenience of multiplying by dx would not be appropriate with higher-order differentials; additional data collection could tell us if any of the subjects' concept images agree or disagree. Several books gave differential rules that parallel derivative rules (an example being $d(uv) = u\,dv + v\,du$), and there are also proofs of the Chain Rule and various methods of finding the solutions to separable differential equations in which it appears that differentials are being multiplied, divided, or canceled. The belief in whether one could perform such manipulations with dx divided our subjects equally; additional data collection could tell us what percentage of other experts will feel that such manipulations are acceptable.

This preliminary report also suggests implications for teaching. Even though only four subjects were interviewed, their concept images had some variety to them. One could say that there was a continuum of answers, from Johnny's central thought that all differentials outside of integration were really the same and had analytic properties, to Kurtis's assertion that all differentials were only part of a really good notation, with Sonya's and Jackson's views falling somewhere in the middle. Further research might further define spaces on this continuum or perhaps show a greater concentration of images at one or both ends. Whatever the dispersion of concept images on this continuum, the fact that such a dispersion exists perhaps begs the question of how the existence of different views of differentials held by textbooks and instructors might affect student learning.

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Locating a Realistic Starting Point for the Guided Reinvention of Limit at Infinity With Community College Students Prior to Pre-Calculus

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In this paper, I describe a teaching experiment conducted with a pair of undergraduate students at a two-year community college. My primary goal was to explore a realistic starting point for the guided reinvention of the concept of limit at infinity for students who had not yet studied limits. The teaching experiment included 5 weekly hour-long sessions in which the two students were presented with tasks that involved describing the behavior of certain real-world phenomena. The initial analysis revealed that these students showed ways of thinking that anticipate the formal concept of limit at infinity. Further analysis will be used to develop an appropriate instructional sequence with a realistic starting point to be used in future teaching experiments in which students will be engaged in the guided reinvention of a formal definition of limit at infinity.

Key words: Calculus, Guided Reinvention, Limit at Infinity, Two-Year College

The limit concept is notoriously difficult in learning (and teaching) Calculus. Prior research has documented common student misconceptions and models of how students might understand the limit concept (Tall & Vinner, 1981; Williams, 1991). Most, if not all, research on students' understanding of the limit concept has involved participants who had previously studied Calculus or were enrolled in Calculus at the time of the study. If we aim to improve instruction on limits by understanding what can cause difficulties for students, there are two potential issues with this approach to participant selection. First, it is difficult to discern whether cognitive obstacles are imposed by traditional instruction on limits or are natural products of the mathematical concepts themselves. In other words, what if the difficulties students face when learning limits could be avoided by using alternative instructional or curricular approaches? Second, by nature of making it through a Calculus course (or even making it *to* a Calculus course), the students in such studies have already demonstrated a certain level of mathematical ability (or determination). A different approach is needed in order to examine how students think about limits, without them first having been taught explicitly about limits. In this paper, I aim to describe how two students reasoned about limits prior to formal instruction on limits by engaging them in the guided reinvention of the concept of limit at infinity using their informal understandings of realistic situations with asymptotic behavior as a starting point.

Background and Related Literature

Guided Reinvention

Guided reinvention has often been used in undergraduate mathematics education research in order to develop instructional sequences that aim to make advanced mathematics content more accessible to students (Cook, 2014; Larsen, 2009; Oehrtman, Swinyard, & Martin, 2014; Swinyard, 2011). Guided reinvention is a teaching approach that is aligned with Hans Freudenthal's Realistic Mathematics Education (RME) philosophy, which proposes that mathematics instruction should focus on the activities of doing mathematics rather than the products of those activities (Freudenthal, 1973; 1991). The starting point for instruction should be a context that is realistic relative to students' experiences so that it will be within their reach to develop ways of reasoning related to the mathematical concept(s) that are to be learned. From that starting point the instructor guides students to reinvent the desired piece of mathematics.

The Limit Concept

The limit concept has a fairly complicated structure, which includes many advanced concepts. In order to make the definition accessible to first-time Calculus students, it is often described informally. The most rigorous informal description (of limit at infinity) is usually given in textbooks, such as "$\lim_{x\to\infty} f(x) = L$ means that *f(x)* can be made *arbitrarily close* to L (or as close as desired) by making *x sufficiently large*," where the terms *arbitrarily close* and *sufficiently large* have technical mathematical meanings.

Through the APOS framework, Cottrill et al. (1996) described how students might learn the limit concept through a coordination of these two processes. Swinyard & Larsen (2012) added that student understanding of the formal limit concept could potentially be hindered by two obstacles: (1) a tendency to take a domain-first perspective in reasoning about limits and (2) the difficulty of thinking about an infinite process as being completed. Swinyard (2011) demonstrated that students can be guided to reinvent a formal definition of limit from the starting point of a strong informal understanding of limit. I am not aware of any previous attempts to examine how or if students can be supported to reinvent any aspect of the limit concept from a starting point that does not require prior understanding of the limit concept itself.

Research Question and Methods

The purpose of this study was to gain insights into how students – who have not had instruction on limits in Pre-calculus or Calculus – might come to understand the concept of limit at infinity in the context of guided reinvention of limit at infinity. In particular, the goal was to explore one potential starting point from which the concept of limit at infinity could be reinvented. I aimed to answer the following research question: Can instructional tasks in the context of describing functional behavior evoke ways of thinking that anticipate desired ways of reasoning for understanding the concept of limit at infinity?

This study involved a teaching experiment (Steffe & Thompson, 2000) with a pair of students (Zelda and Tetra, pseudonyms) who were recruited from a College Algebra course at a two-year community college. The participants were selected based on the fact that they had not previously taken a course in Precalculus or Calculus (verified by the student on a preliminary questionnaire) and on their ability and willingness to discuss their thoughts about mathematics with others (verified through a conversation between myself and the students' College Algebra instructor). The teaching experiment consisted of five 60-minute instructional sessions. All instructional sessions were video- and audio-recorded for analysis. Before each instructional session, I conducted a thought experiment to imagine a possible hypothetical learning trajectory (Simon, 1995) for the upcoming session. Between each pair of sessions, I conducted ongoing analysis (Strauss & Corbin, 1998), which involved comparing the proposed hypothetical learning trajectory to what actually took place during the sessions, and then planning instructional tasks for the next session. After the conclusion of the teaching experiment, I conducted an initial reflective analysis (Cobb, 2000) of all five sessions together. The focus of this initial analysis was to identify events during the teaching experiment in which the students showed signs of reasoning about the instructional tasks in ways that anticipate the concept of limit at infinity.

Overview of the Teaching Experiment

For this preliminary report, I will highlight a few key moments from the first three sessions of the teaching experiment related to the development of the concept of limit at infinity. In Session 1, participants described how the quantities changed in each of the realistic situations in Table 1 (among others that were dropped after Session 1). Then, during Session 2 the students looked for similarities and differences among the situations in Table 1. During this session the students indicated that they wanted to be able to use graphs and formulas to solve these problems. So, I gave them formulas (for Situations 1 and 5) and graphs (for Situations 2 and 3) to work with in Session 3. When students indicated that the limit (or "end value") would not be reached based on the formulas or graphs, I posed the question of how close the quantity would get to it. In Sessions 4 and 5 I asked students to predict a value of the quantity at some future time that was not displayed on the graph. I introduced two new situations (involving exponential growth and circular motion) in order to contrast the other examples in which their predictions were getting better as I asked them to predict values of the quantity after longer periods of time.

Situation 1	Imagine taking a pie out of the oven after it has been baking at 350 degrees. Describe how the temperature of the pie changes as time passes.
Situation 2	Imagine a person bungee jumping over the side of a bridge that is 200 ft. in the air. Describe how the height from the ground of the bungee jumper changes as time passes.
Situation 3	Imagine dropping a tennis ball from the roof of a tall building and watching it hit the sidewalk below. Describe how the height of the ball from the ground changes as time passes.
Situation 5	A group of scientists store 10 g of a radioactive substance. Every 30 days half of the substance decays (breaks down into other substances and energy). Does the substance ever completely decay?

Table 1. Realistic situation prompts from Session 1

Results

I organize the results to discuss 3 such examples from the first three sessions, where students' dialogue about these situations seemed to lead them to discuss 3 important aspects for reinventing limit: infinite process, closeness, and coordinating domain and range.

Dealing With an Infinite Process

When comparing the four situations in Session 2, one of the main observations the students focused on was how the quantities in Situations 1-3 eventually reached a constant value whereas in Situation 5 the quantity never reached zero. For Situations 1-3, the students were only concerned with a finite timeline, as demonstrated by Zelda's description of Situation 1: "…it's just like really hot… and then over time it will just cool off gradually and then eventually, I guess, turn into room temperature." Then, when the students were first describing Situation 5 in Session 1, they spontaneously began to write out a list of values, believing that they would determine when the mass reached zero. The students paused after calculating the first eight values and had the following exchange:

Tetra: *This is what I was trying to say about the infinite numbers between zero and one. That's what I was trying to say. It'll keep going.*
Zelda: *So, would it never decay?*
Tetra: *It's just not going to decay.*
Zelda: *Ever? [with a tone of surprise]*

At this point Zelda was struggling with the idea of an infinite process. In Session 2, she decided to use her calculator to continue the process that they had started in Session 1, saying "because I do feel like it comes to zero at a point." She ended up dividing by two over forty times before giving up and saying "It will just always have a number to divide in half, like always. Even if it's like zero-point-one or zero-point-something." Engaging in this task allowed the infinite process to become experientially realistic to her.

Anticipating Arbitrarily Close

Since the students had determined that the mass of the radioactive substance in Situation 5 would never completely decay, I posed the question: "How close would the mass get to zero?" The following dialogue transpired.

Tetra: *I can't comprehend that number because, like, every time you would divide it, the decimal just keeps getting longer and longer and longer and longer and longer and longer. So, it would be like some crazy, never-ending decimal.*
Zelda: *I don't know how to figure that out. Unless you give us a number that's like to you that's close enough to zero.*

The students' comments indicate that any comprehensible number is insufficient in describing how close the mass gets to zero. Tetra's comment about the decimal just getting longer demonstrates the students' struggle with seeing an infinite process as being completed. In addition, Zelda's comment "Unless you give us a number…" was connected to her knowing that for any given value, they could use the formula to identify the corresponding number of days needed to reach that number. In other words, there was a sense of "arbitrariness" to the "give us a number that's close enough to you, and we can figure it out" idea.

Coordinating Domain and Range

Another interesting interaction occurred near the end of Session 3. Prior to giving the students a graph for Situation 2, I asked them to recall their prior description of the situation from Session 1, and Tetra sketched a graph (Figure 1).

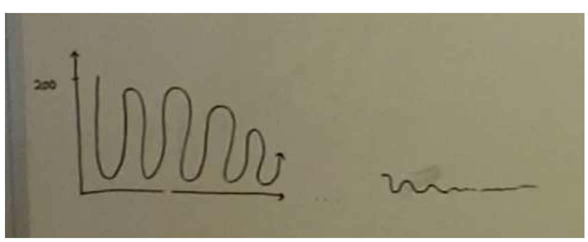

Figure 1. Students' graph of the height of the bungee jumper (Situation 2)

Based on our previous discussions, the students seemed to believe that the bungee jumper would eventually stop bouncing. So, I asked them how long it would take.

Tetra: *I feel like all of these are the same. They just never end.*
Zelda: *No, I think he'd stop bouncing. I think he'd just like stay. I don't think it would take that long to stop bouncing completely.*
Will: *So, on the graph, what would it look like?*
Tetra: *Just taper off.*
Will: *But, like, how would you draw it?*
Tetra: *So, let's say after a point it would be really, really small. Like, I could draw it bigger, but like, on the other bounce, it would keep getting smaller until it's kind of like… [draws the portion on the right in Figure 2]*
Zelda: *And then he's just still, or like swinging at the same height.*
Tetra: *And then just get smaller.*

Relying on her intuition that the bungee jumper would eventually stop bouncing, Zelda saw the stopping point as an end to the situation. On the other hand, Tetra was open to the idea that the bungee jumper would continue to bounce forever. As a result, she was able to focus on the quantity *approaching* a limit rather than simply *reaching* it. This perspective also fostered a shift in her language. Where she had previously used language like "at a point" to refer to some moment in the future when the limit was reached, she began to say "after a point" to indicate that she was also considering what happened beyond the moment when she considered the quantity to be close to the limit. Moreover, she coordinated her notion of "after a point" (similar to *sufficiently large*) with the bounces being really small and continuing to get smaller.

Discussion

The two students in this study were able to reason about these realistic situation contexts in ways that indicate that these contexts and tasks could potentially be used as a starting point for the guided reinvention of the concept of limit at infinity for students who have not previously learned about limits. In particular, describing a situation in which the students believed that the limit (or "end value") would not be reached could provide a context for discussing the completion of infinite processes. The task of determining how close the changing quantities could get to the limit (Task 3) supported discussions in which the notions of *arbitrarily small* and *sufficiently large* were present in the students' comments.

These findings come with several limitations. Although the students were able to reflect on an infinite process, it was not in the context of the infinite processes involved in the limit definition in which students think of values in the range becoming successively closer to the limit value and values in the domain becoming successively larger (Cottrill et al., 1996). Moreover, there was no indication that the students were able to resolve the idea of an infinite process being completed, which is one of the student difficulties with the limit concept described by Swinyard & Larsen (2012). Lastly, although Tetra showed signs of coordinating the notion of sufficiently large in the domain with the notion of small variations in the range, she seemed to be taking a domain-first perspective. Overall, important issues related to the concept of limit at infinity, including potential student difficulties, were evoked during these students' engagement in the instructional tasks; however, more insight is needed to help the students overcome and resolve these difficulties in future implementations of the teaching experiment.

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Exploring Student Conceptions of Binary Operation

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Binary Operations are essential to many undergraduate mathematics courses. However, little is known about student conceptions around binary operation. This report presents preliminary results from nine student surveys about the topic. The question set was developed in response to Group Concept Inventory (GCI) results. We look at three activities closely related to binary operation: identifying when an instantiation is a binary operation, identifying when two instantiation are the same binary operation, and generating an original binary operation instantiation. We use the lens of variation theory to make sense of student responses. We found that students' concept image of binary operation may be missing key attributes (such as requiring two inputs) and contain unnecessary attributes (requiring a general rule.)

Key words: Group theory, variation theory, abstract algebra, binary operation

Students begin working with binary operation in elementary mathematics classes. However, binary operation is not defined abstractly until advanced mathematics courses such as abstract algebra. Well-known operations such as addition and multiplication provide the examples of binary operations for much of mathematical education. Taking a variation theory approach, student conceptions of binary operation are built from the examples they encounter and particularly what variations exist across this space. This may lead to overgeneralizing properties such as operations always having a formal name or symbolic rule to describe them. Furthermore, critical attributes of binary operation may remain hidden such as the necessity for binary operations to be defined on two elements of a set. By giving nine modern algebra students a set of non-routine binary operation prompts, we identify critical aspects of binary operation that may be hidden in routine contexts.

Concept Understanding in Advanced Mathematics

There are a number of ways researchers have made sense of concept understanding in advanced mathematics. In this study, we use the concept image/concept definition framework (Tall & Vinner, 1981) and variation theory (Marton & Booth, 1997) to make sense of the snapshot data collected about students' conceptions of binary operation. Tall and Vinner's work on concept image and definition introduced a way of discussing all of the cognitive associations with a given mathematical concept. Concept definitions exist formally, but also personally for each student. Concept image captures all associations such as examples, applications, and representations of a concept. Tall and Vinner's work was especially powerful in its ability to capture a lack of coherence that often exists in novice's concept images and definitions. The definitions may not be cohesive with images, and different parts of a concept image may not align with itself.

Variation Theory (Marton & Booth, 1997, Watson & Mason, 2006) provides one lens for making sense of potential lack of coherence. The theory posits that learning occurs in contexts where variations occur. Experiences with examples of a concept, such as binary operation, may expose a learner to different attributes to eventually become part of their greater conception around a topic. Marton and Booth likened this process to a jigsaw puzzle where "the parts need to be found and then fitted into place" (p. 180) to understand the whole. Through exposure to varying aspects, the learner can then determine which aspects are parts of the whole or the structure of a mathematical concept and which are allowed to vary, *dimensions of possible variation* (Watson & Mason). A students' concept image reflects the attributes that they have been exposed to (and assimilated) in varying situations.

Prior Research on Student Conceptions of Binary Operation

Informally, a binary operation can be thought of as a rule for combining two elements of a set to produce a single element (from this same set.) Addition is a binary operation on the set of real numbers because any two real numbers can be added to another real number. Formally, binary operations are defined as a function from the Cartesian product of some set *S (SxS*) to *S*. In this way binary operation brings together two concepts that may have been seen as disjoint in prior situations: operations and functions. Research on the specialized case of arithmetic dominates the operation literature with framework such as Slavit's (1998) *operation sense* framework. This framework builds formal operations from standard processes (such as combining groups developing into addition), but does not extend to the abstract notion of binary operation. To adopt the language of variation, until university courses, variation in operations are likely limited to these well-developed arithmetic operations.

Because binary operations are a special case of functions, student conceptions around functions may play an additional role in their understanding. Students have been documented as struggling with functions across grade levels (Oehrtman, Carlson, & Thompson, 2008). For example, students may prefer a certain representation such as a written symbolic rule (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Vinner & Dreyfus, 1989). This may reflect students typically being exposed to functions that can be defined in that manner leaving students to overgeneralize that a written symbolic rule is a structural aspect of function.

Existing frameworks on binary operation have taken a number of approaches. Brown, DeVries, Dubinsky, and Thomas (1997) presented a genetic decomposition of binary operation where students may have an action conception (explicitly combining two inputs to arrive at an output), a process conception (a general process for combining inputs to arrive at outputs), or an object conception (seeing binary operations as things that can be acted on as objects themselves.) Novotná, Stehlíková, and Hoch (2006) provided a structure sense framework capturing the transition from familiar operations to unfamiliar. Ehmke, Pesonen, and Haapasalo (2011) presented a framework leveraging different representations to distinguish students with procedural and conceptual understandings. Each of these frameworks in some way captured going from concrete to abstract understanding. The variation theory approach in this paper aims to compliment these process frameworks with a more nuanced view of exactly what attributes of binary operation may be influencing student conceptions.

Some of these ideas have been broached in misconception literature in other subject areas. Mevarech (1983) found statistics students overgeneralized properties onto binary operations such as associativity. Zaslavsky and Peled (1996) had pre-service and in-service teachers generate examples of binary operations resulting in a number of issues including defining unary operations rather than binary operations.

The study presented below serves as a follow-up to initial results from the Group Concept Inventory (Melhuish & Fasteen, 2016). We conjectured that student conceptions of binary operation accounted for performance on questions targeting subgroups, the associative property, and groups themselves. For example, in results from the GCI, over half of students surveyed felt that the set {0,1,2} was a subgroup in $\mathbf{Z}_6$. Follow-up interviews reflected that the students might not think that the addition modulus 3 was different than addition modulus 6. These students' responses raised questions regarding student conceptualizations about what makes two binary operation instantiations the same or different.

Methods and the Question Set

The nine surveys analyzed herein come from two modern algebra classes at a large, public university. The students were surveyed at the end of their course. In order to make sense of student thinking around binary operation, we identified three activities that are tightly linked to understanding of mathematical concepts:

1. ***Is or is not***. Determining if a given instantiation is an example of a concept (Ehmke, Pesonen, and Haapasalo, 2011)
2. ***Same or different.*** Determining if two instantiations are mathematical the same
3. **Generating**. Creating an instantiation meeting some criteria (Zazkis, & Leiken, 2007)

The survey questions prompted students to engage in one of those activities. We looked at what attributes of binary definition students might attend to, how students determine if two binary operations are the same, and if students can generate a non-standard binary operation. For each question, students were also prompted to explain their reasoning. The surveys were then analyzed with the lens of variability: which attributes of binary operation seemed to guide student solutions.

Results and Analysis

The Binary Operation Concept Attributes

Each student was asked to determine if functions were binary operations including: addition mod 3 on {0, 1, 2}, division on the reals, x^2 on the reals, and a binary operation defined element-wise on the set {1, 2}. The responses to these prompts had two notable patterns: 1) students attended to closure over other attributes; 2) students did not always attend to binary requirement.

Closure seemed to the primary attribute attended to across all prompts. The eight students who elaborated on their responses either used the term closure or provided an explanation such as: "Yes. When you add any two numbers in the set it produces another in the set." However, this closure manifested differently depending on whether the student had integrated the attribute of binary. When evaluating addition mod 3, one approach was to take all six combinations of three elements, add together the pairs, and determine the result was still in the set. However, two of the students looked at the elements and appeared to treat the modulation as the operation. For example, one student stated: "Yes, 0 mod3=0, 1 mod3=1, 2 mod3=2. All the elements of the operation are in the set." Similarly, all eight students who responded to the prompt about x^2 claimed that this was in fact a binary operation. I conjecture this is for one of a two reasons 1) a unary approach, 2) restricting the domain of a known binary operation: multiplication. For example, see the students' response in Figure 1.

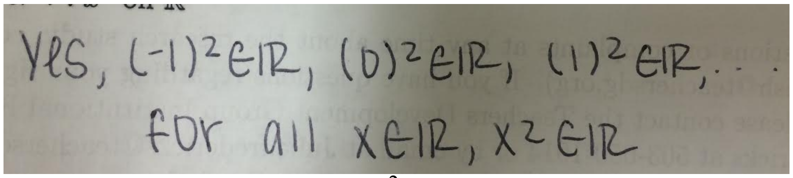

Figure 1. Unary approach to x^2

In this case, the student appeared to be inputting members of the reals rather than ordered pairs or two inputs. As in the modular addition case, this student illustrated that the attribute of binary may not be realized as structural to binary operation. Alternately, some students rewrote x^2 as $x*x$ then explained that it "gives you another number that is [sic] ***R***." In this case, the students seemed to unintentionally be restricting the domain from all sets of two elements, to only ordered pairs of the form: (x, x). These students may not be attending to the necessity of domain to include all combinations of two elements from the corresponding set.

Attributes of the Same Binary Operation

The students were also given four pairs of operations to determine if they were the same or different respectively. These prompts were intended to probe what attributes of the binary operation instantiations determine if they are unique. Of the nine students, five thought that division and multiplication were the same operation on ***R***. (One student provided the caveat that 0 had to be removed for division.) The arguments generally relied on the relationship between division and multiplication: they are inverse operations, or alternately, division can be rewritten as a multiplication statement. ($a/b = a*(1/b)$). There are similarities across multiplication and division; however, the commonalities are not sufficient attributes to be the same operation. Students also argued that addition modulo 3 and addition modulo 6 are the same. Half of the eight students felt these were the same operation with reasoning such as: "Yes, because op 1 is addition under mod 3, whereas, op 2 is also addition under mod 6." In the third of these tasks, students were given a binary operation defined on a table and a second defined element-wise. The elements were named the same, but had had a differing output for one combination. Of the seven students who addressed this prompt, all seven appropriately matched elements and used the rationale to declare the operations different. Yet, these same students used different rationale to declare multiplication and division the same operation and addition modulo 3 and 6 as the same operation. Table 1 contains a breakdown of attributes that students allowed to vary when determining if two binary operations are the same.

Table 1. Sameness Attributes.

Name	**Description**	**Permissible Variation**
Superficial Sameness	Students attend to attributes that provide similarities disconnected from the definition of binary operation.	element names, domain of operation, output for given pair of elements
Literal Sameness	Students attend to attributes connected to binary operation requiring a literal sameness	none
Isomorphic Sameness	Students attend to attributes up to isomorphism	names of elements

Attributes when Generating Binary Operations

The final question in the survey stemmed from a GCI question. Students were prompted to generate a binary operation such that the set {1,2,4} forms a group. Five students indicated it was impossible to construct such an operation, one student did not know how to approach the prompt, two students provided a familiar binary operation not meeting criteria, and one student provided no rationale for their response. One sample student response was, “No. Because the finite set does not hold enough elements to have the properties necessary when performing an operation on the set to form a group.” Another student listed the four arithmetic operation putting, “No?” next to them. These students’ approaches reflected student responses from the GCI (Melhuish & Fasteen, 2016). Through generating an unfamiliar example, students are limited to a set of examples with what they perceive as permissible variations. If students were to create a binary operation on this set, they would need to create one that had no pre-set name or explicit symbolic rule. This may not be a variation that students have experienced and aligns with other research on how students conceive of functions.

Discussion

Looking across the three activities highlights 1) binary operations are not trivial structures for students in introductory group theory, 2) variation theory provides a lens for making sense of what variations are permissible in students’ concept images. The students in this study seemed to include some form of closure as a structural attribute of binary operation. The desire for closure was found in the *is or is not* question set and the *creating* question. However, binary, the need for two inputs, appeared to be lacking across several of the surveys. To take a variation-based approach, it is unlikely these students are ever presented with a function that does not take two inputs as typical textbook treatments do not include operations that are not binary (Gallian, 2016; Fraleigh, 2003). Without varying this attribute to illustrate non-examples, students may not realize that two inputs are a structural aspect of the binary operation concept. Going a step further, we can look at what attributes students attend to when differentiating between members of the set of binary operations. In this case, students looked at a number of attributes ranging from superficial similarity to deep, structural similarity. The student responses reflected their understanding of binary operation on a whole. Particularly, the students often shifted from the defining attribute of a binary operation being the elements and the way they are operated on to other aspects such as both belonging to a similar family. We argue that the notion of identity (or sameness) amongst elements of a set is essential to understanding the attributes of greater concept. Furthermore, across first and second question sets, students often altered the domain such as restricting to (x,x) inputs in the x^2 question. The notion of binary operation as function (with a particular domain) may not be prioritized as an attribute when approaching such tasks.

Understanding binary operation is essential to appropriately addressing any number of tasks and concepts in group theory. This preliminary analysis shows that students’ concept image of binary operation may overlook key attributes (such as two inputs) and pick up unnecessary attributes (such as requiring a symbolic name.) The consequences of exploring student understanding using variation theory are largely pedagogical in nature. Studies have started looking at dimensions of variability available for students in abstract algebra courses in terms of groups and rings (Cook & Fukawa-Connelly, 2015; Fukawa-Connelly, 2014). Pairing such studies with student responses can provide a powerful impetus towards consciously varying examples and non-examples to best bring attention to structural attributes.

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The Lead TA Influence: Teaching Practices Focused on for an Active Learning Classroom

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Across the nation, there is increased national interest in improving the way mathematics departments prepare their GTAs. In particular, this research focuses on how the mentor GTAs in the graduate teaching assistant program under consideration share effective teaching practices and how this effects changes in the teaching practice of GTAs. I report preliminary results on how the focus of particular teaching practices of mentor GTAs (known as lead TAs) change over the period of one term through their participation in professional development. With an understanding of the differences and the similarities between the focuses of the lead TAs, an analysis of the differences between the Calculus I and II GTAs will become more apparent. The research presented here represents the start of an increased understanding of how GTAs form their own teaching practices.

Key words: Graduate teaching assistants, professional development, teaching practices

Across the nation, mathematics departments have begun to change the way they structure the teaching of the Calculus sequence. One of the seven recommendations that emerged as a result of the MAA sponsored study of successful Calculus programs (Bressoud, Mesa, & Rasmussen, 2015) was to improve the professional development offered to the Graduate Teaching Assistants (GTAs) involved in the teaching of Calculus. Though the departments have the common goal of improving the teaching practice of GTAs, the structure of the professional development programs for GTAs varies greatly among mathematics departments (Belnap & Allred, 2009). The research has been on the various structures of professional development programs or on the outcomes of the program. While it is important to know what the outcomes of the program are, it is equally important to understand how those changes occurred so as to improve our professional development programs. Little work has been done investigating the process GTAs go through in their evolving understanding of effective instructional practice and, ultimately, improving their practice.

At a large southwestern university, several changes were made to the calculus program, including to the structure of the professional development program for the GTAs. Before the changes were implemented, the GTAs had no training before they began to work. Currently, the GTAs participate in a three-day seminar before the semester begins. They also meet with a professor in mathematics education in a professional development course that meets throughout the academic year and with a mentor GTA, or lead TA. The focus of the seminar, the course, and the mentoring is on effective teaching practices and developing ways to reflect on one's instructional practice.

This paper discusses preliminary results of analysis on the nature of the discussions regarding effective teaching practices in the various debriefing sessions between the lead TA and their fellow GTAs. In addition to these debriefs, the GTAs participate in weekly meetings with their course coordinator to prepare the activity for the following week. When coupled with analysis of debriefs, the differences between the two lead TAs will help provide insight into the ways the lead TAs have possible influence over the ways the teaching practices of their fellow GTAs is shaped throughout the term.

Background

In many ways, professional development can feel like a complex game of telephone. The leaders and creators of the professional development have certain ideas of effective instructional practice that they are attempting to convey to the teachers or the facilitators with whom they are working. However, the facilitators and teachers are appropriately going to interpret it in their own way and share and use their transformed version of their ideas of effective practice. This particular phenomenon has been well documented in the K-12 literature. Research on the Standards movement reform of the 1980's and 1990's documented only a modest impact of the initiative on teachers' practice. What the research did document was that teachers selectively took up reform ideas and adopted only the surface-level features (Spillane & Zeuli, 1999). Researchers explained the adaptation in terms of teachers' learning processes and suggested that implementation varied because teachers drew on prior knowledge and practices when interpreting the message about the new standards and instructional practices (Coburn, Hill, & Spillane, 2016; Coburn, 2001; Cohen & Ball, 1990). Additionally, the interpretation created by the teachers sometimes did not align with the professional development and messages from school districts and school administrators and so resulted in inconsistent instructional guidance (Coburn, 2001).

Though the K-12 research focuses on how standards are adopted or polices are implemented at the state level and then interpreted by district and local administrators (and, in some cases, content coaches), the context is still similar to the process of changing teaching at the university level. I argue that when faculty and graduate students undertake reform teaching, all of those involved, including the department chair, course coordinators, faculty who take on professional development of teaching assistants, and the teaching assistants, co-construct the message of the reform. It begins with a small group of faculty with the goal to promote high-quality instruction and its success ultimately, in large part, depends upon the learning of the teaching assistants who interact with the college students most frequently. In the K-12 literature, focus has been placed on the ways in which the teachers themselves are interpreting instructional policy and the ways in which their communities affect their interpretations (Coburn, 2001; Stein & Coburn, 2008). Through these studies, researchers have been able to record the significant impact on understanding that discussion with other teachers has on an individual teacher's learning of the instructional policy. My particular study focuses on something similar at the undergraduate level, where I am interested in gaining a better understanding of how GTAs make sense of and interpret what they learn about how to lead a student-centered classroom.

Within the various studies done on the range of professional development programs available for GTAs, most studies can be described by three main themes: temporal, structural, and topical. In temporal, the duration of the professional development is discussed and how it varies across the nation (e.g. Belnap & Allred, 2009). In structural, the focus is on the various ways the programs for professional development of GTAs is structured across the nation (e.g., Ellis, 2015; Palmer, 2011). In topical studies there is an effort to create a list of standard topics focused on within each of the professional development programs described in previous studies (e.g. McDaniels, 2010). Finally, outside of the three topics described above, there are a group of studies on the efficacy of particular professional development programs (e.g. Griffith, O'Loughlin, Kearns, Braun, & Heacock, 2010).

There have been only a handful of studies done exclusively on the state of professional development of GTAs across the nation (Belnap & Allred, 2009; Ellis, 2015; Kalish et al., 2011; Palmer, 2011; Robinson, 2011). Outside of the national studies, there are also a handful of articles on particular programs at specific institutions, with a focus on the structure of the program or the efficacy of the program (e.g. Griffith, O'Loughlin, Kearns, Braun, & Heacock, 2010; Marbach-Ad, Shields, Kent, Higgins, & Thompson, 2010). So, while there have been studies that describe the various forms of professional development or that give an idea of what GTAs have learned from their experiences in professional development, little to no work has been done on the ways in which the GTAs are actually taking-up and implementing what they have gleaned from the professional development. In other words, the focus has been on the product and not the process. This research contributes to understanding how the GTAs are appropriating and transforming various teaching practices to fit their own needs over time.

Setting

At the large, public southwestern university, several significant changes have been made to the Calculus I and II courses, with professional development on teaching for GTAs who lead the break-out sections taking a central role. The professional development for the GTAs focuses on supporting the enactment of student-centered instruction so the GTAs are prepared to lead group work and whole class discussions around challenging problems. The GTAs are being asked to lead the Calculus homework and problem-solving sessions in a manner different than the instruction they likely experienced. Some have never taught before and among those who have, the paradigm is likely new to them.

In addition to a professional development program, the structure of the GTA program has been changed to include a lead TA. The lead TA position is filled by a more experienced GTA providing support to his or her fellow GTAs with a professional development aspect that occurs both before the term begins and throughout the term (Ellis, 2015). At the university in the study, the lead TAs for Calculus I and II were chosen based on their prior experience with teaching or participating in an active learning classroom. Throughout the semester, the lead TA visited the activity day sections of his fellow GTAs to observe the class and met with the GTAs after to debrief about how the class went. The lead TA attempted to make these visits about three times a semester for all of the other GTAs that he was able to observe. Finally, the lead TA served as a liaison for the other GTAs to the coordinator of the course, the mathematics education researchers involved in the professional development, and occasionally the head of the department.

With the lead TA holding such a centralized position within the program, capturing the ways they have influence over how the professional development on leading a student-centered classroom is appropriated and transformed is important to research. What role does the lead TA play in shaping the information provided through the various professional development meetings? To begin answering this question, the analysis compares the teaching practices discussed by each lead TA during the debriefings. In the next section, I go into more detail on how this was done.

Methods of Analysis and Initial Results

To contribute to the research base, I collected data for one term at a large, southwestern public university with two groups of GTAs, one for Calculus I and the other for Calculus II. The mentor GTAs who performed the observations and debriefs are known as the lead TAs and they have more experience with student-centered instruction than the other GTAs. The data used in this study includes the discourse in the various formal professional development settings in which they were involved and the debriefings between the lead TAs and the other GTAs for Calculus I and II.

I transcribed each of the videos and audio recordings, creating a new utterance (Bakhtin, Emerson, & Holquist, 1986) each time the topic changed within each speaker's turn. The utterances are open coded (Strauss & Corbin, 1994) for the various teaching practices discussed. With this list of teaching practices, the instructional practices the lead TAs deemed important enough or relevant to discuss with their fellow GTAs during the debrief sessions is evident for a comparison between the two lead TAs. With this analysis, I will have the basis for understanding how the lead TAs shaped the teaching practices of their fellow GTAs over the course of the term and how the communities of Calculus I and II GTAs differ in their focuses over the period of a term.

For the final analysis of how the lead TAs shaped the teaching practices of the GTAs over the term, I am be using a framework known as the Vygotsky space (Harré, 1983). In this framework, a particular teaching practice can be tracked as it is appropriated and transformed by the GTAs throughout the term. A representation of the Vygotsky space can be seen below in Figure 1. Since the analysis of the data in this preliminary report will not be using the Vygotsky space framework, a detailed explanation of the framework is beyond the scope of this proposal.

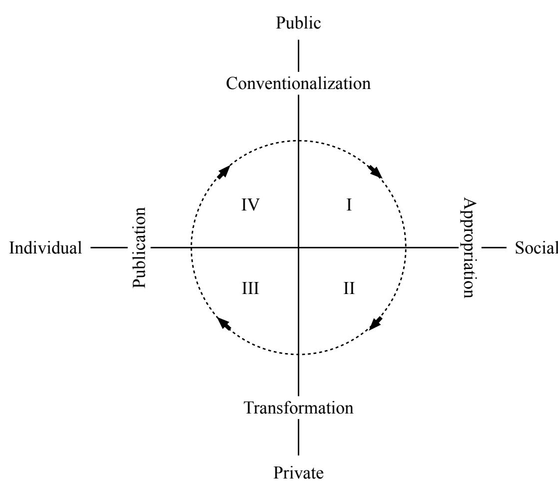

Figure 1: The Vygotsky Space

Preliminary Analysis

Each of the lead TAs met with their fellow GTAs approximately three times throughout the term. In these debriefings, the lead TA would discuss both things he found positive in the class and things he felt could be improved. The lead TA for Calculus I observed his fellow GTAs over the period of two weeks, twice during the term. He conducted his debriefings after the observation in his office and the meetings lasted between 20 and 40 minutes. In contrast, the lead TA for Calculus II conducted two observations a week and rotated the GTAs so that he observed each of them two times throughout the term. He conducted his debriefings directly after his observation so as to give the GTAs the information before they went to their next class and the meetings lasted between 5 and 10 minutes. Each of these debriefings were audio recorded and transcribed.

The teaching practices focused on in the debriefs by the two GTAs did differ in some ways but there were several main similarities. What differed the most between the two lead TAs were the ways they discussed particular teaching practices. For instance, both lead TAs focused on the ways to launch a task at the beginning of a class but there were distinct differences in the ways they talked about the launch. The lead TA for Calculus I tended to focus on the goal aspect of launching a task:

> So one thing that you did do that we want to stress is like at the beginning, you really want to outline their goal. What they should be doing. And I thought you did that well. You said the goal is to match the limits and then create a limit sentence so you can say it in English and you know exactly what it means. So that was really good. And you want to try and do that every activity, like, make sure they know what they're working towards because if they don't then they think that they're done really fast and you're like, what about this? They're like, oh I didn't know we had to do that. Just typical student stuff.

In contrast, the lead TA for Calculus II focused more on the wording and the tone of the launch:

> You did a really quick launch. You did a nice quick launch… Yeah. Um, so in your intro, sort of avoid saying things like the professor's says or the professors wanted me to say. It sort of exudes that you don't really care what you're doing. You're just doing it because you have to.

The difference between the two lead TAs could possibly come from the differences between the focuses of the course coordinators. For Calculus I, the weekly meetings with the course coordinator do involve a discussion about the activity for the coming week but the lead is mostly taken by the lead TA. In contrast, the comment from the Calculus II lead TA illustrates how the lead TA appropriated the use of the word 'launch.' The lead TA also adopted the professor's message that they were to convey that this was a most interesting mathematics problem. The lead TA chose to discuss the GTA's reference to the professor as deflection of autonomy, authority, and, consequently, conveying a lack of personal interest in the problem. With the focus on the way the activity was introduced to the students, the lead TA made sure to focus and comment on those particular practices during his first two observations. These differences between the interpretations and weekly meeting discussions have the potential to highly influence the lead TAs and, as a consequence, the other GTAs. With an analysis of the rest of the data, we can better understand the GTA's interpretation and whether or not the practices of the GTAs changed over the course of the term based on these suggestions.

Conclusion

The study discussed in this proposal is still within the preliminary stage of analysis. However, with a better understanding of the teaching practices focused on by the lead TAs, there will be a better understanding of what practices were deemed important by the community of GTAs and how those practices changed over the course of the term. With this information, the field can begin to understand how GTAs change their practice over time and improve the professional development offered to graduate students who are new to the practice of teaching.

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Professional Development Linking the Concept of Inverse in Abstract Algebra to Function Inverses in the High School Curriculum

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*Pre-service and in-service high school teachers often do not leverage their experience with abstract algebra when interpreting the notation of inverse functions. For this study, we have designed a professional development activity in which teachers can explore inverses in different sets with different binary operations to elicit pseudo-empirical abstraction of the relationship "element * inverse = identity." We used a scripting task found in previous literature to measure the impact of the activity on both the teacher's understanding of inverses and how the teacher would explain the inverse function notation to students. We claim that emphasizing the role of the identity element when discussing inverses can help pre-service teachers overcome misconceptions about inverse functions.*

Key words: teacher education, abstract algebra, horizon content knowledge

Introduction and Literature Review

Students in high school or college algebra classes commonly misinterpret the notation $f^{-1}(x)$ as $\frac{1}{f(x)}$. From an advanced mathematical perspective, we understand that the root of this issue is that the student is not attending to, or is not aware of, the fact that the notation $f^{-1}(x)$ generally refers to the inverse with respect to function composition and not function multiplication. Really, the fact that function composition is the binary operation at hand is merely a convention and the student's interpretation, while it does not conform to standard mathematical interpretations, is not unreasonable.

As instructors who have had experiences with students at this level, we realize that the student is probably not thinking about binary operations at all and is merely generalizing from their experiences with negative exponents of real numbers. But how do high school teachers, who have presumably had training in abstract algebra, think about such a situation? Zazkis & Mamolo (2012) report that they informally asked ten in-service teachers how they might respond to a student who has this confusion. Eight of the ten responded that the meaning of the $\square^{-1}$ is context-dependent. For example, a teacher may explain that the meaning of the symbol $\square^{-1}$ changes depending on what it is "next to." Only two of the ten teachers referenced an inverse with respect to an operation.

Continuing this line of research, Zazkis and Kontorovich (2016) crafted a scripting task for pre-service teachers to investigate how they would respond to a student's question about the symbol $\square^{-1}$ in a classroom setting. The pre-service teachers were given the beginning of a script between a teacher and a student (see Figure 1) with directions to finish writing the dialogue between the teacher and student (or several students). Their analysis divided the 22 scripts into two groups: those that explained the symbol $\square^{-1}$ as always meaning inverse with respect to an operation and those that explained that the symbol $\square^{-1}$ changes meanings (inverse or reciprocal) depending on the context. Fourteen of the 22 pre-service teachers fell into the latter category.

T: So today we will continue our exploration of how to find an inverse function for a given function. Consider for example $f(x) = 2x + 5$. Yes, Dina?
S: So, you said yesterday that f^{-1} stands for an inverse function.
T: This is correct.
S: But we learned that the power (-1) means 1 over, that is, $5^{-1} = \frac{1}{5}$, right?
T: Right.
S: So, is this the same symbol, or what?
T:

Figure 1: *Scripting Task (Zazkis & Kontorovich, 2016)*

We have developed a professional development activity designed to help pre-service and in-service teachers make connections between inverses as they appear in the school curriculum and inverses with respect to binary operations in a group theoretic context. To measure the impact of our professional development activity, we utilized the same scripting task (Figure 1) as a pre- and post-test. From our results from the pre-test we will identify two misconceptions held by pre-service teachers: an "opposite" scheme for inverse, and a "get to one" scheme for inverse. We present evidence that group theoretic activities designed to help teachers pseudo-empirically abstract the generalized property "element * inverse = identity" can help them move from a "same-symbol different meaning" understanding to a "same symbol same meaning" understanding. In particular, an emphasis on identifying the identity element with respect to an operation <u>before</u> discussing inverses can help teachers overcome a "get to one" scheme.

Theoretical Perspective

The data from this study will be analyzed through the lens of Piagetian genetic epistemology and von Glasersfeld's radical constructivism. According to these theories, "…what [people] are able to observe about the world is more dependent on what they already know – that is, on their own special system of thinking - than it is on what actually exists" (Gallagher & Ried, 1981, p. 1). These structures of knowing are referred to as schemes or "units of generalized behavior (or actions) that provide the basis for mental operations" (Driscoll, 2005, p.192). When a learner encounters something that does not fit into an existing scheme, the learner must accommodate this new object by expanding his existing scheme or by creating a new one (vonGlasersfeld, 1995). Piaget used theories of abstraction to describe how assimilation and accommodation can occur. Pseudo-empirical abstraction can be defined as, "abstraction based on the observation of perceptible results, with coordination drawn from activities exerted on objects, reflection on the products of activity" (Ellis, 2016). The process of acting on objects, reflecting on these activities, and coordination of those actions can cause perturbation, which can lead to the learner accommodating their current scheme to encompass this new element.

Horizon content knowledge is one part of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). While some view horizon content knowledge as a connection between the mathematics that students are doing and more advanced mathematics that the *students* will encounter (Ball, et. al, 2008; Fernandez & Figueiras, 2014), others perceive it as connected to the *teacher's* knowledge of advanced mathematics (Wasserman, 2013; Zazkis & Mamolo, 2011). Our working definition of HCK is the teacher's advanced mathematical knowledge and the threads that connect it to the students' mathematical understanding that guide a teacher's planning and in-action instruction.

Task Design

The professional development tasks were designed to give teachers experience with several different sets and operations. We chose three sets: the Real Numbers, the integers modulo 12 (the numbers on a clock), and the set of functions. We chose the first two sets because the teachers have experience doing calculations in both of those settings, and the latter set because it was relevant to the discussion about inverse functions. We used colloquial language whenever possible so that the activities are accessible to those who were not familiar with group theory.

For the real numbers, the participants were asked to begin with the binary operation of addition. We gave the following definition: "The *additive identity* element is the element that 'does nothing' when you add it to another element," and asked them to identify the additive identity. Then we gave them the definition: "An *additive inverse* for an element in the set is the element that you must add to get back to the additive identity," and asked them to find additive inverses of several different elements and then to write a statement describing additive inverses in the real numbers in general. We then asked corresponding questions regarding multiplication.

For the set of integers modulo 12, we chose to use only the numbers appearing on the face of a clock, because we wanted them to have experience with a set in which the additive identity is represented with something other than zero. Since they may not be as familiar with computing in this context, we first gave them some true statements (i.e. 8+9=5) and asked them to explain why each one is true and to generate some of their own statements. Then they were asked to identify the additive identity element, find additive inverses of several elements in the set, and write a general statement describing additive inverses in this set. The tasks for clock multiplication were similar, but before we talked about multiplicative inverses we gave them a completed multiplication table and asked them to circle all of the times that the multiplicative identity appears in the table. Then they were asked to identify some of the elements that had multiplicative inverses.

At this point, we had the participants make a table in which they described the different sets, operations, identity elements, and types of inverses that they had seen in the exercises. The goal of this activity was for them to begin to abstract the notion: element * inverse = identity.

The final set was the set of functions. We began with function addition. We gave them a few functions to work with and had them practice adding functions. We then asked them to identify the additive identity function, and asked them to graph it. We had a discussion about why this function was the additive identity, and asked them to find the additive inverses of the given functions. Then we moved on to function composition, asking them to compose some given functions and think about an identity function with respect to composition (most groups were able to figure it out). We then proposed the function $i(x) = x$, and asked them to compose the given functions with the function *i.* They then found inverses for the given (injective) functions with respect to composition and composed the given functions with their inverses to see that that composition results in the identity function $i(x) = x$.

Methods

Participants were recruited via email and word of mouth from a mathematics department at a large Midwestern public university and a local high school. Participant groups included two graduate students in mathematics who piloted the activities, three senior level pre-service secondary teachers who were doing their student teaching, three sophomore level pre-service secondary teachers, and three in-service teachers from a local school district. We implemented the professional development in groups of two or three. The activity took approximately two hours. First, we gave the participants the scripting task (Figure 1) and had them write a script individually. Then we collected the scripts and went directly into the professional development activities, which they worked on in groups. At the end, we gave their scripts back and had them individually write a reflection on their script, identifying things they might change in their script and areas in which their own understanding had changed as a result of the activities. All groups were video recorded and their written work was collected.

Results

The results reported here will focus on an analysis of the scripts that the participants wrote before they did the group theory activity, and the reflections on the scripting task that they wrote after the activity. We found, similar to Zazkis's previous work, that only a few of our participants (3 out of 11) explained the notation $\square^{-1}$ as referring to an inverse with respect to an operation: both of the graduate students and one in-service teacher. The scripts also revealed that several of the participants held misconceptions about inverses. We identified two schemes that displayed underdeveloped understandings: the *opposite scheme* and the *get-to-one* scheme.

The *opposite scheme* is characterized by the vague idea that "inverse means opposite." They pay no attention to the identity element at all in this scheme. When asked to find an inverse of an element, they will do something to make it an "opposite," whether that is changing the sign or finding the reciprocal, and often cannot move smoothly between the "two types of opposites." One participant fixated on the symmetry that additive inverses have on the number line. He then had difficulty thinking about multiplicative inverses because he couldn't construct a visual representation. Others referred to the inverse of an operation: "An inverse means opposite, so the opposite (if you're thinking about multiplication) is division. So 5^{-1} means you would divide by five. An inverse function is different, but it still essentially means opposite."

The *get-to-one* scheme is characterized by the idea that "the inverse is what gets you to one." Participants who displayed this scheme began to attend to the role played by the identity element, but incorrectly generalized that the identity is always 1. One participant, Ben, showed in his script that composition of a function and its inverse results in x. Ben seemed to be unsatisfied with the x, so he wrote that $x = 1x$ and drew a box around the coefficient 1. We interpret this action as his way to assimilate this result into his get-to-one scheme (see Figure 2).

T: So, the inverse of an object, be it a fraction or a function, is the thing that turns the thing into 1.
S: Okay?
T: Remember when we found the inverse of a fraction? We got the reciprocal, which was the flipped fraction and when we multiplied it with the original fraction we got one.
S: Yeah…
T: So, that is what the inverse fraction does. And I will show you exactly what I mean by this.

So, for $f(x)$ the inverse is $\frac{1}{2}x-\frac{5}{2}$. Now… plug it in $2(\frac{1}{2}x-\frac{5}{2})+5=x-5+5=x=1x$

Figure 2: *Ben's Scripting Task Exhibiting a Get-to-One Inverse Scheme*

Five of the six undergraduates and two of the three in-service teachers reported that they think differently about inverses after the group theoretic activity. All but two mentioned that their understanding about the identity changed. Dan said, "I had forgotten how to find the inverse of a function because I hadn't done them in so long and I was only taught the process, not the reasoning." Markus also reported, "Yes, this session was a good refresher for me on what inverse was." Sloan explained, "Before this session, I never really realized how many different types of inverses were possible depending upon the operations and sets. Seeing the relationships between the various inverses and correlating identities is enlightening…"

It is interesting that although they had found inverse functions in the past and had likely encountered the computation $f(f^{-1}(x))=f^{-1}(f(x))=x$, the pre-service teachers didn't know how to interpret this result. Jane said, "I now think of inverses as connected to the identity" and Kim notes "the composition of functions having the identity x is very interesting and I hadn't thought of the idea of the operation actually having an identity."

Dan, Sloan, and Ben all showed evidence of a get-to-one scheme in their first scripting task. After the exercises, both Sloan and Ben said that they had never thought about different types of identity elements and had expanded their understanding of inverses.

Six of the eleven participants said that they would change how they talked about inverse functions to students. Sloan said, "I would make an effort to clarify which operation [and] identity I would use as I explain." Kim said, "The word 'identity' would be used much more. However, the same concept regarding 'neutralizing' the function I believe I would continue to use. When the composition [gives] x, the composition is neutral."

Discussion

Previous research has shown that many pre-service teachers do not leverage their understanding of abstract algebra when interpreting the notation $\square^{-1}$. Because of teacher shortages, it is quite possible that many teachers have never had a course in abstract algebra at all. Thus, professional development activities designed to broaden teachers' mathematical experiences can be beneficial. Zazkis and Mamolo (2011) suggest that a teacher could make use of her horizon content knowledge by explaining the situation in terms of an inverse with respect to an operation. Our work contributes by incorporating professional development tasks aimed to help teachers make these connections. Our tasks emphasize the role of the identity element when thinking about inverses. This emphasis seems to help teachers overcome the misconception that an inverse is the element that "gets you back to 1." Having the teachers experience finding identity elements and inverses in different sets with different operations helped many of them to generalize the relationship "element * inverse = identity" which can lead them to interpret the notation $\square^{-1}$ as meaning an inverse in general. This study is limited by the small number of participants, however, the goal of this work was to design and conduct a professional development activity focusing on a particular connection between an advanced mathematical concept and secondary school mathematics in a way that teachers can leverage in their classrooms. Future research is needed to continue the development of such professional development activities.

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Textbook Formations of Independence

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The noun independence and adjective independent are applied in multiple mathematical contexts. In probability, independent events do not affect each other, but in algebra and regression, an independent variable has a non-symmetric effect on a dependent variable. Further complicating matters, independence in everyday language represents something in between. Prior research has shown that students and professors struggle to apply concepts of independence. As part of an investigation into curriculum about independence, textbook definitions about independence were examined. Across nine books, a mix of algebra and statistics texts, substantial variations existed in definitions of independent events and independent variables. Variations included the register of representation, verbal against algebraic, and the strength of the dependent effect. Little written guidance was provided to help learners navigate across the multiple formations.

Key words: Independence; Probability; Variable; Semiotics; Lexical Ambiguity

Independence is an old concept in probability. Over 250 years ago, De Moivre defined independent events in *The Doctrine of Chances*; two events are *independent* when "the happening of one neither forwards nor obstructs the happening of the other" and *dependent* if the "probability of either's happening is altered by the happening of the other" (1756, p. 6). The concept is symmetric; either event can serve as the "one" or the "other". In probability, the term maintains that definition today. Independent events are sufficiently common that the authors of the Common Core State Standards chose to include the definition in the high school standards (National Governors Association [NGA] Center for Best Practices & Council of Chief State School Officers [CCSSO], 2010, p. 82).

Despite 250 years of history—or perhaps because of 250 years of history—people have trouble determining if events are independent. Manage and Scariano (2010) surveyed 219 college students in US introductory statistics classes; only 23% correctly answered a multiple-choice question on the definition of independence. Molnar (2016) surveyed 25 US high school mathematics teachers; only 3 (12%) correctly solved a problem about two events in a table. Outside the USA, D'Amelio (2009) wrote about students' and professors' challenges in Argentina; about half the French pre-service teachers surveyed by Nabbout-Cheiban (2016) incorrectly solved problems about independent events.

One potential reason behind the trouble is that the colloquial non-probabilistic definition of independence differs from De Moivre's statement. According to the Oxford English Dictionary, the adjective *independent* refers to something "not depending on the authority of another, not in a position of subordination or subjection; not subject to external control or rule; self-governing, autonomous, free" ("Independent," 2015). In everyday language, independence and dependence are not necessarily symmetric. For instance, when choosing where to live, young children are usually dependent on their parents' decisions, but parents have more autonomy.

Another complication arises from the labeling of *independent* and *dependent* variables when describing algebraic functions. In Common Core standards, Grade 6 students should "write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable" (NGA Center for Best Practices & CCSSO, 2010, p. 44). With defined sides, this definition is never symmetric, although closer to the sometimes-symmetric everyday definition than the always-symmetric probability version. Incomplete textbook explanations do not assist students. Leatham (2012) gathered 10 school mathematics textbooks containing problems on independent and dependent variables. Of 73 total problems, 32 (44%) provided absolutely no information to determine the independent variable; others provided only partial information. Leatham concluded that most textbook problems sent mixed messages, "implicitly impeding students from developing a robust understanding of independent and dependent variables" (p. 357).

Some statistics textbooks tag on an additional non-symmetric definition of independent variables in regression models, where the independent variable controls the value of the dependent variable. Because the regression definition is similar to the algebraic one, and the definition does not appear in state standards, it is less crucial. Besides, between colloquial, algebraic, and probabilistic definitions, ample opportunities exist for confusion.

Our entire investigation will consider the curriculum around independence through multiple lenses as defined in Gerhke, Knapp, and Sirotnik (1992). In this preliminary report, we examine planned textbook curriculum in algebra and statistics textbooks. Later, we will interview teachers to ask about intended curriculum and analyze student artifacts of experienced curriculum.

Theoretical Framework

Independence is a non-physical mathematical concept. As Duval wrote in 2006, mathematical objects are never physically visible. Humans comprehend mathematics only through symbols and signs, "but the mathematical objects must never be confused with the semiotic representations that are used" (p. 107). Therefore, our framework for understanding mathematical confusions about objects labeled independent is semiotic.

For example, a mathematical object that can change between more than one value, dependent on circumstances, receives the representation *variable* in written English. In college algebra, the object receives a single letter representation such as X. In either setting, the object could have been expressed with another semiotic representation, such as changing-number or ϖ or 변수 (the Korean representation for the concept of variable).

In Duval's (2006) framework, systems of representations are called *registers*; earlier, variable was presented in the registers of written English, algebra, and Korean. Explanations for mathematical objects, known as *formations*, are designed to help students connect a concept with its representation in a register. Students are expected to learn formations. For instance, in the Common Core probability standards, students are asked to explain the formation of "independence in everyday language and everyday situations" (NGA Center for Best Practices & CCSSO, 2010, p. 82).

When authors write textbooks, they generate written formations for mathematical objects. Some authors include formations in other registers or ask students to convert between registers. Our textbook research uses the semiotic framework to ask the following questions.

1. For independent events, what formations and registers are used in definitions?
2. For independent variables, what formations and registers are used in definitions?

We had also proposed a third question, about explanations provided to distinguish applications of the semiotic symbol *independence*, but we did not find many of these explanations. We comment more about this in the section on questions for the audience.

Method

Encyclopedic search of pre-algebra, algebra, and probability textbooks would be very long, given the abundance of available textbooks. For this RUME report, we decided to concentrate on a small sample of books we knew had recent college-level use, similar to how Cook and Stewart (2014) examined recently-published textbooks on linear algebra. We selected three college introductory statistics textbooks (Bluman, 2014; Bowerman, O'Connell, & Murphee, 2014; Diez, Barr, & Cetinkaya-Rundel, 2015) and three college algebra textbooks (Bittinger, Beecher, Ellenbogen, & Penna, 2013; Crauder, Evans, & Noell, 2014; Miller, 2014). For comparative purposes, we added three US secondary school algebra textbooks (Benson, Dodge, Dodge, Hamberg, Milauskas, & Rukin, 1991; Bittinger, 1999; Brown, Dolciani, Sorgenfrey, and Kane, 1990). In each textbook, we recorded the initial formative definition involving *independent* or *independence* and examined problems in the text related to independence.

Results

Each introductory statistics textbook had a different definition of independent in regards to events. One college algebra textbook and one secondary school algebra textbook also contained formations because the books included sections on probability. The initial definitions are presented in Table 1, with statistics book definitions first.

Table 1
Initial Textbook Definitions Related to Independent Events in Probability

Textbook	Definition
Bluman (2014, p. 213)	Independent Events – Two events A and B are independent events if the fact that A occurs does not affect the probability of B occurring.
Bowerman et al. (2014, p. 171)	Independent Events – Two events A and B are independent if and only if 1.) $P(A\|B) = P(A)$ or, equivalently, 2.) $P(B\|A) = P(B)$ Here we assume that P(A) and P(B) are greater than 0.
Diez et al. (2015, p. 85)	Independent Process – Two processes are independent if knowing the outcome of one provides no useful information about the outcome of the other.
Brown et al. (1990, p. 756)	Independent Events – Two events A and B are independent if and only if: $P(A \cap B) = P(A)P(B)$.
Miller (2014, p. 779)	Independent Events – If events A and B are independent events, then probability that both A and B will occur is P(A and B) = P(A)*P(B).

The authors of a statistics textbook, Bowerman et al. (2014), presented the most challenging definition, using both the probability algebra register $P(A|B) = P(A)$ and conditional probability. The two algebra textbooks also contain formulations in the probability algebra register, varying slightly in the semiotic sign for and (and versus $\cap$), but do not require another mathematical concept. Relying on an additional concept complicates the structure. If a student cannot convert $P(A|B)$ into a mental concept, the student will not comprehend independence. Research results, summarized by Falk in 1986, have shown that conditional probability confuses many students.

Asking learners to construct a mental formulation of independence through another challenging concept, plus a conversion from the algebra register, is highly demanding.

On the other hand, Diez et al. (2015) and Bluman (2014) avoided conditional probability and the algebra register, relying only on written English. The computational formula appears later. Placing the written definition first reduces cognitive load by requiring less symbolic conversion. Nevertheless, despite language similarities, their definitions are not alike. A process is a larger concept than an event; events are sets of outcomes inside random processes. By defining independence on processes, not two events inside a process, Diez et al. (2015) have offered a different conception than the other authors. Interestingly, Diez et al. later refer to independent events, writing "if two events are independent, then knowing the outcome of one should provide no information about the other" (2015, p. 94). The shift between larger processes and smaller events may not appear notable, but for a concept with demonstrated problems, all shifts in formulation matter. Bluman's (2014) definition is the clearest.

Turning to variables, four algebra books and two statistics books contained a definition for independent variables. We do not know why the other two algebra books did not; perhaps the authors considered the concept a prerequisite. Initial definitions are presented in Table 2, with college algebra textbooks first, then secondary school algebra textbooks, then statistics books.

Table 2
Initial Textbook Definitions Related to Independent Variables

Textbook	Definition
Bittenger et al. (2013, p. 62)	Independent Variable – In the equation $y = \frac{3}{5}x + 2$, the value of y depends on the value chosen for x, so x is said to be the independent variable.
Miller (2014, p. 17)	Independent Variables – One type of mathematical model is a formula that approximates the value of one variable based on one or more independent variables.
Benson et al. (1991, p. 346)	Independent Variable – In a function, the variable whose value is subject to choice. The independent variable affects the value of the dependent variable.
Bittenger (1999, p. 156)	Independent Variables – Write an equation like $y = x^2 – 5$, which we have graphed in this section, it is understood that y is the dependent variable and x is the independent variable, since y is calculated after first choosing x and y is expressed in terms of x.
Bluman (2014, p. 19)	Independent Variable – In an experimental study, the one that is being manipulated by the researcher. … The resultant variable is called the dependent variable.
Bowerman et al. (2014, p. 487)	Independent Variable – Regression analysis is a statistical technique in which we use observed data to relate a variable of interest, which is called the dependent (or response) variable, to one or more independent (or predictor) variables.

As with the independent event definitions, we see multiple registers in Table 2. Both books with Bittenger (1999, 2013) as an author initially used the algebraic register. Although the letter x is a common algebraic formulation for an independent variable, writing in two registers complicates the concept. In other surveyed books, the symbolic x appears later.

Both Bittenger books (1999, 2013) introduce independence through the verb choose; independent variables have values selected, but dependent variables do not. The noun choice appears in Benson et al.'s (1991) definition; Bluman's (2014) definition of manipulation in experiments has synonymous language. The other two books do not mention choosing a value for independent variables. Both Miller (2014) and Bowerman et al. (2014) utilize a word related to prediction. Only Miller's 2014 mathematics book—not a probability and statistics text—describes the connection as approximate, although *affect* in Benson et al.'s (1991) definition is not as strictly causative as the other books or the Common Core formulation. Overall, free choice versus control appears to be the dominant formulation. The two statistics books that use the term independent variable do not vary much from algebra formulations, a slightly surprising result.

Questions for the Audience

The third probability and statistics textbook made a type of distinction we had hoped to see frequently. When discussing regression modeling, Diez et al. add in a footnote that applying the words independent and dependent "becomes confusing since a *pair* of variables might be independent or dependent, so we avoid this language" (2015, p. 18, emphasis in original). The other books do not make distinctions or connections between formations involving the word independent. One possibility for the paucity of connections we saw is our sample size. Although our recollections of other texts include few connecting and distinguishing statements, a larger search could identify more. Alternatively, we could interacting with more teachers and students.
1. Would a more comprehensive textbook search be fruitful?

The three concepts are unalike in the direction of the relationship. As described earlier, the probability relationship is symmetric; the algebra relationship is not symmetric; the colloquial relationship is sometimes symmetric. In an earlier discussion, one person suggested that relationships be defined in terms of causal direction, bi-directional versus uni-directional.
2. What research could be done to investigate this idea? Given the untested nature of any new causal definitions, the investigation would likely have to occur outside standard classroom flow.

This leads into another question, the primary one in the minds of teachers—and us.
3. What are other possible solutions to the misconceptions and lexical confusion?

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Connecting Secondary and Tertiary Mathematics: Abstract Algebra and Inverse

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This study explores how practicing teachers make connections between secondary and tertiary mathematics. Using three frameworks for teacher knowledge of mathematics, coupled with key developmental understandings (KDUs) (Simon, 2006) as related to teacher knowledge (Murray & Wasserman, 2016), we observe how a professional development workshop focused abstract algebra impacts teachers' understanding and teaching of secondary mathematics.

Key Words: mathematical connections, professional development, teacher knowledge

Within the mathematics and mathematics education communities, ongoing consideration has been given to the knowledge secondary mathematics teachers require to provide effective instruction. At the focus of this debate is what mathematical content knowledge secondary teachers must have in order to communicate mathematics to their students, evaluate student reasoning, and make informed curricular and instructional decisions. Many believe that mathematics teachers should have a solid base of mathematical knowledge and mindfulness of how tertiary mathematics is connected to secondary mathematics (Papick, 2011). But some have shown how more mathematics preparation does not necessarily improve instruction (Darling-Hammond, 2000; Monk, 1994). Thus, questions endure about what connections are between secondary and tertiary mathematics are important and how knowledge of these connections may impact classroom practice. While there are many who support the notion that mathematics preparation for secondary teachers must involve knowledge of vertical connections, less is known about how inclusion of courses such as abstract algebra in teacher preparation programs may bring this about.

Framework

In order to explore the nature of vertical connections in mathematics, we draw on three areas of teachers' knowledge of mathematics research: Mathematical Knowledge for Teaching (MKT) (e.g., Ball, Thames, & Phelps, 2008), Advanced Mathematical Thinking (AMT) (e.g., Zazkis & Leiken, 2010), and Knowledge of Algebra for Teaching (KAT) (e.g., McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012). These frameworks, along with the research on key developmental understandings (KDUs) (Simon, 2006) as related to teacher knowledge (Murray & Wasserman, 2016), provide a way to think about how understanding of tertiary content may impact teachers' understanding of the teaching and learning of secondary mathematics.

Building on Shulman's seminal work on teacher knowledge (1986, 1987), Ball and colleagues (2008) conceptualized the domains of MKT according to elementary mathematics. Although there exist many challenges in translating the definition of these domains in a secondary context (Baldinger), our current work utilizes MKT constructs to better understand the nature of knowledge needed for secondary teaching. In particular, we consider the development of horizon content knowledge (HCK) as it pertains to secondary mathematics through exposure to tertiary mathematics. According to Ball, Thames, & Phelps (2008) HCK is "an awareness of how mathematical topics are related over the span of mathematics in the curriculum" (p. 403), but this construct varies considerably from the elementary to secondary level (Howell, Lai, &

Phelps, 2008). For the purpose of the current work, we use the definition of HCK as rendered by Jacobson et. al., (2013), as "an orientation to and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory" (p. 4). We use this definition to consider how instruction in abstract algebra coupled with secondary tasks prompting teachers to apply this knowledge may develop an awareness of connections between abstract algebra and secondary mathematics. We believe that knowledge of specialized content such as abstract algebra may provide secondary teachers with a better understanding of the mathematical horizon as it pertains to their teaching of secondary algebra.

Even with more students taking algebra, the preparation of algebra teachers is still not well researched (Stein et. al., 2011 in McCrory et. al., 2012). This lack of research on teacher preparation and shortcomings of existing frameworks serves as motivation for the development of KAT. This framework contains three domains of knowledge believed to be essential to the teaching of secondary algebra: s*chool algebra, advanced mathematics,* and *algebra for teaching* (McCrory, et. al., 2012). Knowledge of advanced mathematics is pertinent as this is the knowledge that provides teachers with "some perspective on the trajectory and growth of mathematical ideas beyond school algebra" (McCrory et. al., 2012, p. 597), much like HCK. This knowledge is of special significance because many students experience difficulty transitioning from high school to college mathematics and many teachers perceive the undergraduate mathematics that they themselves learned as immaterial with regard to their teaching practice. For these reason, and because secondary teachers are typically required to graduate with an undergraduate degree in mathematics, we also draw upon AMT, defined as "knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college" (Zazkis & Leiken, 2010, p. 264).

Finally, in order to better understand how exposure to abstract algebra content may impact secondary teachers' knowledge of teaching mathematics, we consider mechanisms by which teachers develop awareness of connections and how this awareness may impact instruction. To accomplish this, we utilize the construct of KDUs (Simon, 2006). The essential characteristics of KDUs to teachers and teaching are that a KDU must involve a conceptual advance on the part of the teacher, and that without the knowledge, teachers must build their understanding through activities and reflection rather than explanation or demonstration. We posit that through awareness of connections between secondary and tertiary mathematics, it is possible for secondary teachers to develop KDUs, thereby furthering their understanding of and ability to teach secondary mathematics.

Our current work explores how in-service teachers make connections between secondary mathematics and abstract algebra. This work builds on the results of a smaller pilot study, which revealed an interesting change in participants' understanding of various mathematics concepts, including inverse. In the previous and current work, frame participants' understanding of inverse using the APOS framework (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1997). The APOS framework conceptualizes an individual's understanding of mathematical content according to four levels - *action*, *process*, *object*, and *schema*. At the action level, inverses are used algorithmically to perform mathematical tasks such as solving equations, e.g., multiply both sides of an equation by the multiplicative inverse. As a process, inverses are viewed as both an operation and mathematical property of equality. At the object level, inverses are understood as elements within a set defined with respect to a binary operation. Finally, at the schema level, a comprehensive understanding of inverse is attained and the operational/elemental duality of inverse is understood as well as its utility as a mathematical property of equality (Wright, Murray, & Basu, 2016).

In our pilot, we found that some participants initially discussed inverses through an action or process-level of understanding. After engagement in activities focused on the algebraic structures of groups, rings and fields, participants began to consider inverses as objects. We used this finding to further explore how an extended professional development workshop highlighting the connections between abstract algebra and secondary mathematics may not only change teachers' ideas about the inverse concept, but also influence their thinking about mathematics teaching and learning. The research questions for this study are: (1) How does understanding of tertiary mathematics change teachers' knowledge and teaching of secondary mathematics concepts? (2) How does exposure to and instruction in tertiary content, specifically abstract algebra, change the way teachers understand and teach the concept of inverse?

Methods

To answer these research questions, we conducted a four-day professional development (PD) workshop with four in-service teachers (Conor, Dylan, Orlaith, and Aidan) from an urban high school in northern New Jersey. The workshop consisted of four three-hour sessions held on consecutive days prior to the beginning of the school year.

Data Collection

During the workshop, participants worked through three packets of activities that included scripting tasks (Zazkis, 2013) (e.g., extend an imaginary interaction between a teacher and students in a form of dialogue, including explanations and/or examples), secondary content activities (e.g., describing the mathematical properties used to solve a multistep linear equation), and tertiary content activities. The tertiary activities focused on abstract algebra content including algebraic structures (groups, rings, and fields) and formal definitions of binary operations, inverse relation and function. The first two packets prompted participants to consider mathematical properties used when solving equations in a secondary classroom and how these properties relate to algebraic structures. The third packet focused on connecting the content on algebraic structures to functions. The purpose of these activities was to unpack how discussions about inverses through an abstract algebra lens might help participants reconsider functions as objects rather than actions or processes. Additionally, we sought to challenge conventional thinking about what a function is and common problems distinguishing between additive, multiplicative, and compositional inverses for functions.

Participants engaged in these tasks individually and as a group. One researcher engaged participants in group discussions, which were audio and video taped. We collected all written artifacts containing participants' responses and reflections for future analysis. Future data to be collected is classroom observations and participant interviews that will allow us to further explore how exposure to and instruction in abstract algebra content impacts instructional practice.

Data Analysis

In our initial analysis, each researchers independently isolated episodes from the video recordings that highlight teachers' understanding of inverse and identity. Once the significance of these episodes was mutually ratified, we transcribed and analyzed the audio using an initial or "open" coding method, searching for words or phrases that showed evidence of participants' making connections between secondary and tertiary mathematics. Open coding was the preferred method for our initial analysis as it allows for the development of tentative codes that may lead to further inquiry, thereby allowing the study to take direction naturally (Saldana, 2009).

Preliminary Results

We focus our results on the fourth day of the workshop to explore how abstract algebra impacts secondary teachers' understanding of inverse. Similar to our pilot, we found participants moved from an action/process level toward an object level of understanding of inverse. To highlight how the participants' understanding evolved, we divide our results into three sections: understandings of the concept of inverse and inverse function as action/process, the transition in understanding towards object, and impact on instructional practice.

Initial Understanding of Inverse

At the beginning of the session, participants read and responded to a scripting task capturing the conversation between a teacher and three students regarding the inverse of the function $f(x) = x^2$. Students in the task provided three answers for the inverse of this function: $1/x^2$, $-x^2$, and $\sqrt{x}$. After considering this scenario, Dylan, commented, "Only one of them [$1/x^2$, $-x^2$, or $\sqrt{x}$] is the inverse of the function $f(x) = x^2$, but they all have different thinking. For example, x^2 and $-x^2$. When you combine them it is 0 but does that mean that that's an inverse?"

Conor viewed the inconsistency of responses as a lack of "conciseness in mathematical vocabulary" rather than "just students' misconceptions". Based on this discussion, we see participants beginning to understand why there are inconsistencies in students' understandings of inverse functions. In trying to unpack this ambiguity, participants exhibited an understanding of inverse at the action/process level. For example, when participants were asked to communicate their understanding of inverse, Conor referred to multiplication and division as inverse operations, while Dylan asserted, "my definition of the inverse of a function is something that undoes another function." In providing this definition, Dylan overlooks the binary operation with respect to which the inverse is defined. He references an algorithmic approach to "undoing" without considering the operation that is being undone or the product of this undoing, arriving at the identity, thereby evidencing action-level understanding.

Transition in Inverse Concept Understanding

During the discussion of abstract algebra in connection with the inverse and identity function, participants transition from action/process to object level understanding. In the introductory scripting task described above where participants realize the ambiguity in their conception of inverse, they discuss the importance of clearly defining the term *inverse*. Orlaith states, "What does it mean that we are asking them for the inverse? So define the term inverse. I think that's where we would have to start." She also notes how inverse is different for functions, but expresses her inability to provide a specific definition for this concept. Through this reflection, we posit that Orlaith's HCK might not include the understanding of functional inverse. Specifically, she seems to be aware of how her inability to provide an accurate definition of a function's inverse is indicative of a gap in her knowledge of the broader mathematics territory of inverses in algebraic structures. We interpret this as evidence of a transition in her understanding of the concept of inverse.

We also observe transitioning through participants' written responses to reflection questions posed at the end of the four-day workshop. In particular, Conor writes, "You think about them in terms of other definitions/properties that contain the words inverse or identity." In addition, Aidan reflects on his transitioning knowledge by stating, "It is important to emphasize what operations we are working with." Through these reflections, we claim that participants have developed a deeper awareness of inverses as either a mathematical property (process level) or as an element within a set of objects that must be defined with respect to an explicit mathematical operation (object level). Furthermore, we posit that this advance in conceptual understanding may signify the development of a KDU with regard to the higher-level mathematical underpinnings of inverse.

Impact on Instructional Practice

At the conclusion of the workshop, participates seemed more aware of the connection between tertiary and secondary mathematics than they had been at the beginning. This awareness prompts teachers to reconsider their own practice in different ways. For example, during one conversation, Dylan mentions how he will integrate newfound understanding of inverse, particularly in regard to defining inverses with respect to their identity under composition, into this year's lessons. Upon realizing that the identity of a function under composition is the function "y=x" Dylan states, "I never taught it like that" and "I'm one-hundred percent using that this year!" When asked to reflect upon future instruction, he further states, "I will be lesson planning completely different. Our first unit is all about solving and inverse operations. I must change how I approach my introduction throughout this unit because I want my students to not have misconceptions as they approach higher levels." In so stating, Dylan is considering how this new knowledge impacts his teaching of secondary mathematics as it may help prepare his students for higher-level mathematics, showing an awareness of the mathematical horizon.

Applications and Implications

Based on our pilot findings in which participants' understanding of the mathematical concept of inverse evolved from action/process to object understanding, we created an extended PD workshop to push teachers' thinking about inverse, identity, and solving equations. This extended workshop allowed us to present classroom scenarios and scripting tasks and to delve more deeply into abstract algebra content.

In the current study, participants' understanding of inverse evolved in a similar way as our pilot participants. That is, participants began to think about inverses as objects within an algebraic structure, rather than as an action or process. The difference between the current work and the pilot came from the participant's connection to their classroom practice. Similar to the pilot, the participants reported a newfound appreciation for being precise and consistent with their language in the classroom, especially around definition of identity, inverse, and binary operations. In the current work, the participants went even further by connecting their experiences in undergraduate mathematics classes to their teaching. As Conor stated, "I left it all in the floor in college, but it's so important for Algebra II as well!"

As we continue our data collection and analysis, we will use classroom observations and interviews to verify that the teachers' reports of how this new understanding of content might impact instruction. The implications of this work are a first step in helping us understand how knowledge of mathematics is related to being an effective secondary mathematics teacher. We hope to engage mathematicians and mathematics teacher educators in discussions about how this data might provide confirming or disconfirming evidence for teachers' reports on instructional impact of the knowledge of connections between secondary and tertiary mathematics.

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Self-assessment Behaviors of College Mathematics Students: A Preliminary Report

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Research shows that low-achieving students are less able to accurately assess their own weaknesses. As a result, many might fail to see the need to explore the subject matter more deeply, in order to improve their conceptual understanding and procedural fluency. This study investigates undergraduate mathematics students' self-assessment behaviors. Students from a broad range of courses at three universities were asked to predict their expected grades on assignments, and these predictions were compared with the grades assessed by their instructors. They were also asked to justify their self-assessments if they did not give themselves full points. Preliminary results showed that students overall overestimate their grades. There was a significant difference between expected and actual grades. As test scores increased, the difference increased from negative to positive. Students in the B-range (between 80-89%) were the most accurate predictors.

Key words: [Self-assessment, Undergraduate Mathematics Teaching, Metacognition]

Studies suggest that higher-ability students also have better metacognitive skills (Chi et al, 1989; Recker & Pirolli, 1992; Shute & Gluck, 1996; Wood & Wood, 1999), and that low-achieving students are less accurate when assessing their own weaknesses (Langendyk, 2006). One implication of these findings is that many low-achievers may not see the need to explore the subject matter more deeply, in order to improve their conceptual understanding and procedural fluency. Students think that they are doing "just fine" even if their knowledge and performance are weak (Kruger & Dunning, 1999). This over-confidence could be a factor contributing to students' lack of success (Langendyk, 2006). Students often do not know when they need help or what form of support is appropriate (Aleven & Koedinger, 2002), and low-achieving students need external support in order to link assessment to learning (Langendyk, 2006).

Undergraduate mathematics courses have one of the highest dropped, failed or withdrawn (DFW) rates (Gardner Institute, 2013). However, little research has investigated mathematics students' self-assessment of performance, or the reasoning for their self-assessments. This study addresses these gaps in the literature by investigating the following research questions: 1) How accurately do undergraduate mathematics students self-assess their performance? 2) How do self-assessments of successful and unsuccessful performers compare? 3) What reasons do students give to justify their self-assessment of their performances? 4) How does students' self-assessment accuracy affect their self-regulated learning behaviors?

At this point, we only have information pertaining to the first two research questions.

Literature Review

This study is based on the theoretical framework of "meta-ignorance," also referred to as the Kruger-Dunning effect (Dunning & Kruger, 1999; Kruger & Dunning, 1999). This framework asserts that people's ignorance is often invisible to them, because "lack of expertise and knowledge often hides in the realm of the "unknown unknowns" or is disguised by erroneous beliefs and background knowledge" (Dunning, 2011, p. 248).

Research shows that good students have better metamemory (a conscious awareness of ones own processes with respect to memory) accuracy than do poor students, and are better able to predict what they know and do not know (Sinkavich, 1995). Dunning and Kruger (1999) found that when asked to rank their performances relative to peers, bottom-quartile students overestimated their performance, while top-quartile students underestimated. Self-assessments are more likely to be inaccurate on difficult tasks for which people lack requisite knowledge (Lichtenstein & Fischhoff, 1977). If the task is too difficult or the person is unskilled, there is a greater likelihood of overconfidence (Dunning & Kruger, 1999).

Dunning and Kruger (1999) assert that people unaware of their incompetence suffer a dual burden, as "not only do they reach erroneous conclusions and make unfortunate choices, but their incompetence robs them of the ability to realize it. Instead, they are left with the mistaken impression that they are doing just fine" (p. 1121). They further argue that such people are likely to get stuck and become unaware of their incompetence, because "the skills that engender competence in a particular domain are often the very same skills necessary to evaluate competence in that domain" (p. 1121). Incompetent individuals are unlikely to recognize correct judgment if they cannot produce correct judgment. In other words, they lack metacognition skills (Everson & Tobias, 1998).

Research shows that good students are more successful in the metacognitive task of evaluating their own performances, such as anticipating which test items they will get right or wrong (Austin, Gregory, & Galli, 2008; Sinkavich, 1995). A survey of 15-year old students across 34 countries showed that higher performance and more accurate self-perceptions of math skills are associated with each other (Chiu & Klassen, 2010). However, good students also underestimate their abilities when comparing themselves to peers (Dunning, 2011), a behavior known as the "false consensus effect" (Ross, Green, & House, 1977). Below-average students, on the other hand, falsely believe that they are above-average (Ferraro, 2010). When students are given their peers' work to grade, good students improve their ranking accuracy much more than their weaker peers (Dunning, 2011).

Dunning and Kruger (1999) found that students are willing to rate themselves more negatively if they are equipped with intellectual resources. They showed that even a short 20-minute lesson to solve a certain task improved students' logical reasoning and self-assessment accuracy. Research also shows a link between self-assessment and learning if students use their knowledge to formulate new strategies and learning goals (Boud & Falchikov, 2006). However, making people aware of their limitations does not necessarily induce them to overcome their limitations (Prasad et al., 2009). Many of them are unwilling to anticipate their incompetence even if they receive feedback on their work (Hacker, Bol, Horgan, & Rakow, 2000; Ferraro, 2010). Such unwillingness may be caused by self-esteem, self-defensiveness, or the difficulty they experience when trying to improve (Sheldon et al., 2014).

Little research has investigated undergraduate mathematics students' self-assessment behaviors, how they justify their self-assessments, and whether their self-assessment accuracy impacts their self-regulated learning behaviors. We hope that the results of this study would inform future mathematics instruction and the design of professional development activities for instructors to help students become better learners.

Methodology

Four faculty researchers collected data in their respective universities: a private university in central Georgia, and two public universities in north and southwest Georgia. Data were collected from 229 students in a broad range of undergraduate courses taught by the researchers: introduction to mathematical modeling, college algebra, elementary statistics, calculus I, II, and III, differential equations, and mathematical probability and statistics.

Students in these courses were given an initial survey asking their self-reported readiness to take the course and their expected end-of-semester grades. Students were asked to write their expected grades for all in-class quizzes and exams, which were graded based on the instructors' grading rubrics. Students were also asked to justify their self-assessment if they did not give themselves full points in those problems. Since they were asked to write their expected grades at the bottom of each assignment, they were able to compare the two scores after the assignments were returned to them.

Students' self-evaluation of their performances was also measured through a short survey after each exam. Toward the end of the semester, a purposeful sample of students (those who consistently overestimated, underestimated, or made almost accurate predictions of their scores) was asked to voluntarily participate in a few semi-structured interviews. They were reminded of the differences between the two scores and asked to explain their perceived reasons for the inaccuracy (or accuracy) of their assessment. This report is based on data from the spring 2016 semester, but does not include qualitative data from the interviews. Data will also be collected in the fall 2016 semester and a comprehensive study of the combined data will be made.

Based on existing research and our personal experiences, we hypothesized that top performers would be more accurate predictors of their scores, bottom performers would overestimate themselves and be less accurate predictors, and students would become better predictors as the semester progressed.

Results

This paper reports preliminary results from quantitative analysis pertaining to only the first two research questions. We have used both descriptive and inferential statistics. An independent t-test was used to determine the statistical significance of the average difference between the students' expected grades and the grades assigned by the instructor. Pearson's correlation test was used to determine the significance of the relationship between the students' predicted and actual grades. We found that students overall overestimated their scores. The test shows a statistically significant difference between their expected and actual grades ($t\ (1799)=-6.89$, $p < 0.01$), with a mean difference of 5.85 on a 100-point scale. For A students (scoring 90-100%), the difference was still statistically significant ($t\ (381)= 7.84$, $p < 0.01$), but they underestimated their performances by an average of 5.22 points on the 100-point scale. B students (scoring 80-89%) slightly overestimated themselves, but the difference between predicted and actual grades was not statistically significant. The mean difference was only 1.42 points.

For C students only (70-79%), the difference was statistically significant ($t\ (150)=-6.58$, $p < 0.01$). They overestimated themselves on average by 8.43 points. Students in the D range and below (69% and less) were the most miscalibrated, overestimating their performance ($t\ (500) = -14.4$, $p < .001$) by 20.25 points on a 100-point scale. The line graph (see Figure 1) shows how the difference between instructors' average grade and students' expected grade changes in relation to the average instructors' grade. This shows that bottom performers tend to overestimate themselves, and top performers tend to underestimate. The graph suggests that students in the B-range (80-89%) were more accurate predictors of their performances.

Preliminary test results did not support our initial hypothesis that top performers are more accurate predictors, as B students actually proved to be the most accurate. But the results supported the hypothesis that bottom performers are less accurate predictors than others.

The regression line in Figure 2 shows the relation between average grades vs. average differences. The two variables are strongly correlated, $r\ (89) = 0.67$, $p < 0.01$. This shows that as the test scores increase, the difference between the instructor's grade and students' predicted grade increases from negative to positive.

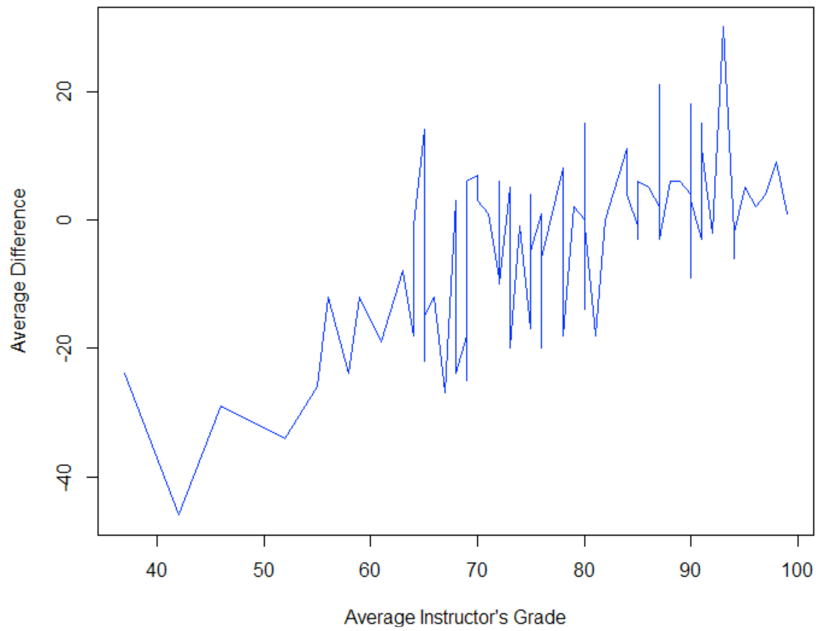

Figure 1. Average Instructors' Grades vs. Average Differences

We also looked at how the differences between predicted and actual grades changed as the semester progressed. Students overestimated their performance by 5.58 points on the first quiz, then underestimated by a mere 0.74 points on the second. Interestingly, they then overestimated by 5.7 and 6.3 points on the third and the last assignments (quizzes or tests). This did not validate our initial assumption that students would be more accurate predictors of their performances as the semester progressed.

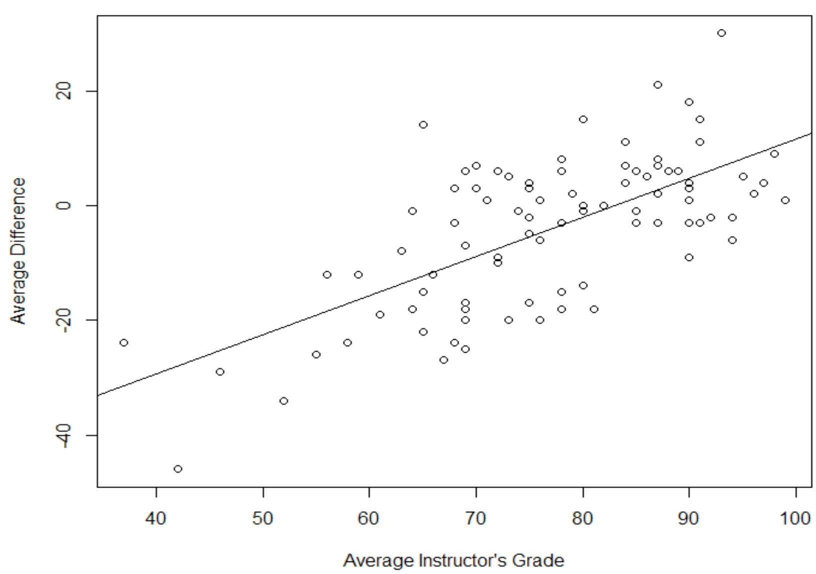

Figure 2. Average instructors' grades vs. average differences

Discussion

We had little information about college math students' self-assessment behaviors prior to this study. The results show that they are likely to be more miscalibrated when they either perform well or do not perform well, as also found in existing research (Kruger & Dunning, 1999; Langendyk, 2006). Both top (A students) and bottom performers (C and below) were inaccurate predictors. Top performers slightly underestimated their scores, while bottom performers overestimated by a huge margin. More interestingly, B-range students turned out to be the most accurate predictors, as the difference between their expected and actual grades was not statistically significant. This was in agreement with previous findings that students in between the top and bottom performers are more accurate predictors (Langendyk, 2006). We do not know why this group was able to better predict their performances, but we will have better insight once we finish analyzing our qualitative data.

Existing research shows that top performers in general underestimate their performance when comparing themselves to their peers (Dunning, 2011). Our study showed that top-achievers in college math courses still underestimate themselves, even when they are not comparing themselves with their peers. Since we haven't analyzed our qualitative data yet, we do not know why this group of students underestimate themselves. Either they are not completely confident that their solutions are correct, or they think they have not met instructors' expectations. The false consensus effect (Ross, Green, & House, 1977) in this group of students might be encouraging them to always work harder.

Lack of knowledge may prevent poor performers from knowing what they have done wrong (Dunning, 2011; Kruger & Dunning, 1999). If they had the ability to recognize right or wrong solutions, they probably would have been able to better predict their scores. Since they did not know that their solutions were incorrect, they might have been overly optimistic about their performances because they were hoping for a passing grade. Because of their mistaken belief that they know the subject matter, they might not even realize that they need to work harder to gain deeper understanding of the subject matter and gain procedural fluency.

We did not find studies that investigated college math students' self-assessment behaviors as we did in this study. One implication of our findings is that both low- and high-achievers need instructor support. Low-achievers need help figuring out conceptual and procedural gaps in their knowledge. They often do not know when they need help and what kind of support they need (Aleven & Koendinger, 2002). High-achievers, on the other hand, need help figuring out that what they know is actually correct. Preliminary findings show that they know the content but still lack confidence in their knowledge, as evident from their predicted scores. Since the students in between these two groups know, in general, whether they did right or wrong, it shows that these in-between students (mostly B students) have potential to improve. We will have better understanding about students' self-assessment behaviors once we finish analyzing their justifications for their self-assessments.

Results from this study showed that our initial assumption that students would become more accurate predictors of their performances as the semester progresses was not necessarily true. Even though they made almost accurate predictions of their scores in the second assignment (a quiz or a test), they then overestimated their performances on the next two quizzes or tests. Seeing the difference between their expected and actual grades on the first assignment might have made them more cautious about expecting higher scores on the second assignment. But why the students then became overly optimistic on the next assignments is unknown. We need to analyze interviews and collect more data to make any conclusions about this student behavior. This can be a good topic for further investigation.

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Identification Matters: Effects of Female Peer Role Models Differ By Gender Between High and Low Mathematically Identified Students

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We investigated a peer role model intervention designed to alleviate underrepresentation of women in STEM. Half of the Calculus break-out sections at a large university were visited by a peer role model and half served as controls. The female peer role models were expected to increase the sense of belonging and mathematical self-efficacy of women highly identified with mathematics. Our results show that peer role models have the intended effect on women highly identified with mathematics, but also have a positive effect on men with low mathematical identification.

Key words: Calculus, Gender, Experimental, Self-Efficacy, Mathematical Belonging

In spite of the need for more workers with training in science, technology, engineering, and mathematics (STEM) (PCAST, 2012), retention in STEM majors and programs of study is a persistent problem in the United States (e.g., Bressoud, Mesa, & Rasmussen, 2015; Chen, 2013; Seymour & Hewitt, 1997). It is particularly troubling that under-represented groups such as women and ethnic minorities are disproportionately affected (Ellis, Fosdick, Rasmussen, 2016; Hill, Corbett, & St. Rose, 2010; Lewis, Stout, Pollock, Finkelstein, & Ito, 2016). The Calculus sequence is critical to student success in a STEM major, and students' experience in Calculus has been shown to dissuade students from continuing. At the site of our study, a large, urban, southwestern university, women are almost twice as likely as men to opt out of taking Calculus II after completing Calculus I. This is similar to national data showing that even after controlling for academic preparation, career intention, and instruction, women are one and a half times more likely than men to leave after Calculus I (Ellis et al., 2016). We need to better understand how different populations are differentially affected by their Calculus experiences.

Several factors have been suggested to be potential causes for the retention disparities between men and women. One factor that researchers in the laboratory have identified as important in the performance of women highly-identified with mathematics is stereotype threat (Steele & Aronson, 1995). Stereotype threat is a concern stereotyped individuals have about confirming a negative ability stereotype that exists about their group. For instance, women routinely contend with the stereotype that they are inferior in math and this concern lowers their math performance below their actual math ability (Spencer, Steele, & Quinn, 1999). Among these women, exposure to female peer role models enabled positive performance and psychosocial outcomes, such as greater self-efficacy (Marx & Roman, 2002). Peer role models are inspiring in-group members who defy negative ability stereotypes about their group (i.e., mathematically-capable female students) (Marx & Roman, 2002). Researchers have suggested that the lack of mathematical confidence, or self-efficacy, is a potential deterrent to women's persistence (Ellis et al, 2016). Furthermore, stereotypes contribute to the perception that gender and ethnic minority students do not fit with being a STEM major leading to the minority students' feeling of not belonging, greater insecurity, and greater expectation of dropping out of STEM (London, Rosenthal, Levy, & Lobel, 2011). The absence of female peer role models in the classroom contributes to feelings of not belonging (Lockwood, 2006). For under-represented students, including women, belonging is crucial (Good, Rattan, & Dweck, 2012;

Hill, Corbett, & Rose, 2010; Walton & Cohen, 2007).

Wilson et al. (2015) conducted a 5-institution study linking belonging with academic engagement among STEM majors. Students filled out surveys containing measures of belonging at multiple levels (class, major, and university) as well as measures of behavioral and emotional engagement in their academic pursuits. The results were that class and major belonging correlated with greater behavioral and positive emotional engagement and lower negative emotional engagement. Belonging remained a significant predictor even when other measures such as self-efficacy were included in the models. Furthermore, while belonging and self-efficacy are correlated, they also have distinct aspects that contribute separately to positive outcomes for students. While Wilson et al. (2015) looked at correlates for a single point in time, Walton and Cohen (2011) demonstrated that a short intervention can result in long term stabilization of feelings of belonging and raise the GPA of minority students who participated.

In Calculus classrooms, women are aware that their gender is in the minority and this contributes to a feeling they do not belong and this can feel threatening for STEM-intending students. In numerous past laboratory studies, researchers have found that exposure to similar others who represent success in STEM (i.e., math-talented peer female role models) can alleviate the negative effect on women's math performance (e.g., Marx & Ko, 2012; Marx & Roman, 2002; Marx, Stapel, & Muller, 2005). Because female peer role models defy the stereotype and highlight women's presence in STEM, more exposure to these peer role models may also has the potential to foster women's belief that they do belong in STEM and that they have the ability to succeed in math. The results presented here are part of a larger study that seeks to examine the effects of peer role models on sense of belonging and self-efficacy in the setting of university mathematics classrooms, as well as long-term effects such as a higher percentage of women continuing in higher level mathematics courses.

Setting & Participants

Participants were undergraduates of all racial/ethnic groups in Calculus I at a large university in the southwestern U.S. All were eligible to participate. The Calculus I classes are conducted in large lecture halls with graduate teaching assistants (GTAs) leading two 50-minute break-out sections each week. There were 16 break-out sections that were typically about 35-40 students. We recruited and trained 4 female upper-division STEM majors to serve as near peer role models (2 Hispanic and 2 non-Hispanic white STEM majors). These near peers made 2 in-class role model presentations in half of the breakout sections of the Calculus I classes. The other half of the Calculus I class served as the control. The role model presentations closely followed the structure that we have found through laboratory studies to be key components of a role model. Half of the break-out sections were visited by one of four female near-peer role models twice during the semester. These smaller sections supported the opportunity for students to form a more personal connection with the peer role model. The focus of past work on role models has been conducted among under-represented minorities who were pre-selected to be highly STEM identified. Those who are most likely to experience threat are those who are most vested or identified with the domain (i.e. STEM) (Schmader, Johns, & Forbes, 2008). In a Calculus classroom with high and low math identified students, we have the opportunity to test whether role models can be beneficial to minority students at all levels of math identification.

The role model presentations consisted of:

1) an introduction establishing the role model's similarity to students and aptitude in

mathematics

2) a presentation of a mathematical topic, each topic related to helping people/the environment and was directly tied to mathematics being taught in the class but presented within some mathematical setting that was mathematically unfamiliar

3) explicit encouragement to persist in mathematics/STEM in order to understand the unfamiliar mathematics

The particular topics were chosen in effort to combat the notion that STEM careers are less likely than careers in other fields to fulfill communal goals (e.g., working with or helping other people) (Diekman, Brown, Johnston, & Clark, 2010).

Methodology

Following the second in-class role model presentation, all students (both intervention and control groups) filled out a questionnaire containing the *Mathematics Identification Questionnaire* (Brown & Josephs, 1999) and a modified version of Walton and Cohen's (2007) *Social Fit Measure* to assess sense of belonging in STEM. This measure contains questions such as, "I would feel comfortable in a math field" and "I feel that I would belong in a math field." We also combined Marx and Roman's (2002) Self-Appraised Math Ability Scale and a modified version of Schwarzer and Jerusalem's (1995) General Self-Efficacy Scale (GSE) to measure self-efficacy. The self-appraised math ability scale contains items such as "I deal poorly with challenges in math". All items in the GSE were worded to reflect mathematical self-efficacy (e.g., "I can always manage to solve difficult math problems if I try hard enough"). Responses were recorded on 1 (strongly disagree) to 7 (strongly agree) scale. Students then provided basic demographic information. In our analysis the sense of belongingness and self-efficacy dependent variables were regressed separately onto role model (yes vs. no), student sex (female vs. male), and the continuous variable math identification (centered).

Results

Sense of Belonging

Sense of belonging yielded a main effect of math identification, F (1, 120) = 41.73, $p <$ .001, showing a positive relation between math identified and sense of belonging in math. Of more interest was the significant 3-way interaction between role model, student sex, and math identification, F (1, 120) = 14.06, $p < .001$. Among low identified female students, those exposed to role models indicated feeling just as low a sense of belonging in math ($M = 4.12$) as those in the control ($M = 4.42$), $F < 1$. Whereas among high identified female students, those exposed to role models indicated a higher sense of belonging in math ($M = 5.60$) than those in the control ($M = 4.93$). The benefit of role model exposure among high math identified female students was further confirmed by comparisons with male students. Specifically, among high math identified students in the control condition, we observed the classic lower sense of belonging for female compared to male students ($Mfemale = 4.93$ vs. $Mmale = 5.59$). In contrast, among high math identified students in the role model condition, female students had a higher sense of belonging in math than male students ($Mfemale = 5.60$ vs. $Mmale = 4.95$). These results suggest that role models are not as beneficial for low as compared to high STEM identified female students. Interestingly, however, low math identified *male* students seem to derive benefit from female role model exposure such that those who were exposed to the role model had higher sense of belonging than those in the control (*M* = 4.02 vs. *M* = 4.62).

Self-Efficacy

Self efficacy also yielded a positive relation with math identification, F (1, 120) = 33.29, $p <$ .001. There was also a role model by student sex interaction, F (1, 120) = 4.75, $p < .05$. This demonstrated that in the control condition female students had lower self-efficacy than male students (*Mfemale* = 4.47 vs. *Mmale* = 5.17). In contrast, this gender difference in self-efficacy closed in the role model condition (*Mfemale* = 4.97 vs. *Mmale* = 4.95).

This 2-way interaction was subsumed under a 3-way interaction between role model, student sex, and math identification, F (1, 120) = 10.82, $p < .01$. A closer look at this interaction showed a remarkably similar pattern of effects as that found for sense of belonging. Specifically, low identified female students, those exposed to role models indicated about the same level of self-efficacy (*M* = 4.18) as those in the control (*M* = 4.28), $F < 1$. Whereas among high identified female students, those exposed to role models indicated higher self-efficacy (*M* = 5.76) than those in the control (*M* = 4.66). Comparisons between female and male students within the high math identified students further confirmed the beneficial effect of role model exposure on female students. Specifically, in the control condition, we observed the classic gender gap in self-efficacy (*Mfemale* = 4.66 vs. *Mmale* = 5.80). In contrast, in the role model condition, female students had self-efficacy than male students (*Mfemale* = 5.76 vs. *Mmale* = 5.14). Again, mirroring what we found for sense of belonging, role models seem to have an opposite impact among those in with low math identification. In particular, whereas role models did not seem to benefit female students (*Mrole model* = 4.18 vs. *Mcontrol* = 4.28), role models seem to have some benefit for male students (*Mrole model* = 4.75 vs. *Mcontrol* = 4.54).

Discussion

We posited that female students' career trajectories would benefit from exposure to female peer role models who, through their own success in mathematics, illustrate that women do belong in STEM. The female peer role models were expected to increase the sense of belonging and mathematical self-efficacy of women highly identified with mathematics. Our results show that peer role models have the intended effect on women highly identified with mathematics, but also have a positive effect on men with low mathematical identification. We will discuss possible reasons for the positive effect in men. The results are being analyzed for a second semester to see whether role models can alleviate female students' negative mathematics experiences in order to increase their interest and persistence in mathematics.

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Considerations for Explicit and Reflective Teaching of the Roles of Proof

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In a previous study we sought to understand the classroom activities that provided students the opportunity to engage in the five roles of proof described by Michael de Villiers (1990). In conducting the analysis for that study, we noticed that students' views of proof were sometimes not aligned with de Villiers' views. This led us to the current investigation, where we explore alignment between undergraduate students' views of the nature of proof and de Villiers'. We hypothesize that an explicit and reflective (ER) approach to instruction may be important if students are to learn about the nature of mathematics (in general) and the nature of proof (more specifically). We offer implications for both research and practice, with respect to the explicit and reflective instruction on roles of proof.

Key words: Roles of Proof, Nature of Mathematics, Nature of Proof, Transition-to-Proof

Researchers in mathematics education have made several efforts to understand the disciplinary practices of mathematicians (e.g., Burton, 1999; Nardi, 2008; Weber, 2008; Weber & Mejia-Ramos, 2011). As a result of such studies, the field benefits by gaining a deeper understanding of the nature of mathematical knowledge and inquiry. Proponents of situated learning theory (ourselves included) would argue that in order to learn mathematics, students must engage in authentic mathematical practices (Greeno, 1997; Lave & Wenger, 1991). From this perspective, understanding the work of experts in a field (such as mathematicians in mathematics) is an important aspect of instructor knowledge, allowing instructors to design learning environments that engage students in authentic disciplinary practices and thus aid in their learning of mathematics. But, do students need to go beyond engagement in legitimate mathematical practices within the classroom, and actually hold an understanding of the nature of mathematician's practice and the nature of mathematical knowledge? It is widely acknowledged that most students know very little about what mathematicians do (Hersh, 1997). Yet little research has been conducted regarding students' understanding of the nature of mathematics as a discipline (Jankvist, 2015). Perhaps an understanding of the nature of mathematical knowledge and inquiry may lead undergraduates to have a greater appreciation of mathematics or even lead to greater learning gains. But until systematic research is conducted into this area, these remain untested hypotheses.

Within science education, researchers have studied how students learn about the nature of scientific inquiry and scientific knowledge and the benefits of such knowledge (Lederman & Lederman, 2014). One of the main findings of that work is that engagement in authentic scientific practice alone is not sufficient for students (or teachers) to learn about the nature of science (Bell, Blair, Crawford, & Lederman, 2003). Although teachers often perceive that their students will *implicitly* learn the nature of science through engaging in scientific practice, research shows that students need *explicit and reflective* (ER) instruction on the nature of science in order to develop a sophisticated understanding (Bell et al., 2003). To teach the nature of science explicitly and reflectively means that students engage in authentic scientific practice, have that practice brought to their attention explicitly (e.g., by the instructor), and have the opportunity to reflect on the ideas that have been explicitly addressed (Lederman et al., 2014). Researchers in mathematics education sometimes claim that students have developed "desirable beliefs about the nature of mathematics" (Rasmussen & Kwon, 2007, p. 192) after participating in inquiry-oriented courses. However, these claims about student beliefs are made in regards to what it means to do mathematics within the particular classroom settings, and generalizations are not made about what students may believe about the nature of mathematics as a discipline (e.g., Fukawa-Connelly, 2012; Yackel & Rasmussen, 2002).

We believe there may be an implicit assumption possessed by some scholars that if students participate in inquiry-oriented classrooms and engage in mathematical practices similar to those of research mathematicians, then the students may come away from such classes with informed conceptions of what it means to know and do mathematics in the discipline. The authors of this paper admit to being guilty of this assumption in the past. We believed that undergraduates' understanding of the nature of proof in the discipline could be developed implicitly by engaging students in the five roles of proof described by de Villiers (1990). One of the learning objectives in our transition-to-proof course, taken from the course syllabus, was that students "Gain an appreciation of the many roles of proof and reasoning in the discipline of mathematics (e.g., verification, explanation, systemization, discovery, communication)." In a previous study, we identified the (classroom) activities that engaged students in those five roles of proof. Our implicit assumption was that by engaging in the five roles of proof, the students would gain a sophisticated understanding of those roles in the discipline. However, we found that even when students were engaged in a role of proof, this did not always lead to their understanding of the role in the broader discipline. Even after reading de Villiers' (1990) paper on the roles of proof, their written summaries of these roles suggested they had naïve conceptions that did not align with de Villiers'. This leads us to the hypothesis, based on the findings in science education, that an explicit and reflective (ER) approach to instruction may be important if we want students to learn about the nature of mathematics (in general) and the nature of proof (more specifically).

In this preliminary report, we intend to explore the following questions:

- What do undergraduate students in a transition-to-proof course understand about the nature of proof in the discipline? What do they not understand?
- Which roles of proof are in need of further explicit and reflective (ER) instruction?

By exploring these questions, we hope to gain a better understanding of the considerations necessary when teaching the nature of proof in a more explicit and reflective (ER) manner.

Conceptual Framework

A key construct in our research is the notion of the nature of proof as it is experienced by research mathematicians. We conceptualize the nature of proof using de Villiers' (1990) five roles of proof: verification, explanation, systematization, discovery, and communication. A mathematician engages in *verification* when a proof convinces the mathematician of the truth of a mathematical statement. The reason why the mathematical statement is true may be illuminated as a mathematician engages in the *explanation* role of proof. *Systematization* refers to proof's role in organizing and creating a deductive system of axioms, definitions, and theorems. A mathematician engaged in *discovery* may deduce an unanticipated result during the completion of a proof. Proof also provides a means for *communication* among mathematicians as they transmit mathematical knowledge and negotiate meaning and validity. We frame our work from the following assumption: Students have a sophisticated understanding of the nature of proof if their ideas surrounding the roles of proof align with those of de Villiers (1990).

Methodology

The data for this study were taken from an undergraduate transition-to-proof course at a southeastern university in the United States. The instructor (also one of the authors on this paper) designed the course with an aim toward broadening students' understanding of proof and providing experiences for students to engage with proof in ways similar to mathematicians in the discipline. There were thirteen students in the course: nine mathematics majors (seven of whom were prospective secondary mathematics teachers), and four mathematics minors. At the end of the semester the students took a two-part final exam. The second part (20% of the exam grade) of the final required students to read de Villiers (1990) paper, describe in their own words each of the five roles of proof, recall an instance during the course in which they engaged in one of the five roles (or describe an activity that might be used in the future for engaging students in such a role) and rank order their engagement in each role of proof throughout the semester.

The researchers used open process coding (Saldaña, 2009) to analyze the written descriptions of the 65 student recollections (five roles and thirteen students) of the instances in which they recalled being engaged in the roles of proof during the course. During subsequent analysis the researchers discussed the following questions:

1) How do students perceive that a certain activity engages them in a specific role of proof?
2) How do student perceptions align with the role of proof as articulated by de Villiers?
3) What are some implications for teaching such a course in the future?

When completing initial work related to the project, we focused on the first question. Here we concentrate our efforts on the second question related to comparing student perceptions to de Villiers' perceptions. After completing the introductory analysis, the first author went back to the data and reviewed each students' summary of the five roles of proof as well as their description of an activity which engaged them in the role of proof. He noted if the students' ideas were aligned with de Villiers, identified evidence for this determination, and noted any implications. He then reviewed these notes and identified several preliminary discussion points regarding student understanding of the roles of proof and the nature of mathematical knowledge/inquiry.

Results and Discussion

Verification

When describing the verification role of proof, de Villiers' (1990) challenged the naïve view that proof provides mathematicians with absolute certainty. While acknowledging that proof does indeed provide conviction, especially in the case of non-intuitive claims, lack of quasi-empirical falsification also plays a role in conviction. Our data suggest that reading de Villiers' arguments was not enough for students in the course to understand this position. Several students summarized de Villiers' description of verification using the language of "absolute certainty." David wrote, "Verification by proofs is to show the absolute certainty of a mathematical computation by proving it in every case." Similarly, Tina claimed,

> The verification of a proof is most often a way of knowing what you already know. By that I mean the things that we are most sure are true the verification makes us absolutely certain. If mathematicians had not verified the proofs of some of the greats from ancient times we still would not know that some of them were incorrect.

Students such as Tina and David may benefit from classroom discussions regarding the nature of proof and absolute certainty. Perhaps reading *and discussing* relevant philosophy of mathematics

(Hersh, 1993) or mathematics education literature (Weber & Mejia-Ramos, 2015) may prove useful in challenging students' absolutist conceptions of proof.

Explanation

We have noticed that the term "explanation" may be confusing for students. While the occasional student understood this role, many students interpreted explanation in a colloquial sense, e.g. explaining why one step follows from another within a proof. Jared claimed,

> Explanation is the reasoning behind the proof. It shows how steps are logically taken to get to the conclusion of the proof and why we take them. Overall, it is saying why we can go from the statement to the conclusion.

Similarly, Stephanie wrote,

> Proofs may provide more or less explanation and still be a valid proof. Explanation is just how much detail the writer of the proof goes into. If your audience doesn't know as much about mathematics you may want to give a full explanation of your proof.

Perhaps it would be beneficial for an instructor to specifically draw attention to how proving can provide *insight* or *understanding.* Hersh (1993) claimed that in teaching, the primary role of proof is for explanation. However, without explicit and reflective instruction, students ultimately may fail to understand that proofs serve this important function. Instructors may ask students to compare and contrast proofs in regards to the insight or understanding they provide.

Systematization

In general, the students in the course under study did not understand the systematization role, that proof may play a role in the organization of a theory. They viewed systematization as the use of established theorems, axioms, and definitions within a proof. Jeb wrote,

> Systemization is taking several smaller true statements and arranging them into one large true statement that is the proof. It is organizing or ordering the smaller sections of a proof to flow in such a way that they look like a single statement.

De Villiers (1990) noted that the systematization role is only understandable at an advanced stage of mathematics. It is the first author's opinion that in a course in which systematization has already been conducted by the organization of course materials, it will be difficult for students to understand systematization as a role of proof. Perhaps in a course in which the organization of materials is not predetermined (e.g. Fawcett, 1938), students may better understand this role.

Discovery

De Villiers (1990), in describing the discovery role of proof, described how mathematicians may prove an unexpected result when they realize, through deduction, that a proof generalizes from a specific case to a larger class of mathematical objects. Students in the course recalled times when they experienced discovery in this manner, but also wrote about additional ways they came to discovery during the course. For instance, Millie recalled a problem that asked her to "Prove, or disprove and salvage" a given mathematical statement:

> I decided [for all integers a, b, and c, if bc is divisible by a, then either b is divisible by a or c is divisible by a] was a false statement, and I provided a counter example. Then, I made a conjecture that the converse was true: for all integers a, b, and c, if either b is divisible by a or c is divisible by a, then bc is divisible by a. I proved my conjecture, and that conjecture was my own personal discovery.

Although not aligned with de Villiers' description of the discovery role of proof, we believe that Millie's learning experience was valuable. For instance, Susan explicitly mentions how classroom discoveries led her to develop a new conception of the nature of mathematics.

> Discovery came to me when working on a truth table to test possible outcomes of a proof. This truth table led to "Susan's conjecture." I also think this class, in general, led to a broader "discovery" that mathematics is a living, changing, developing thing. Unlike my former perspective that it had all been discovered many years ago and we are just reviewing and learning those truths.

We contend, that although these two students did not describe an instance in which they discovered a new result through deduction (as de Villiers described), their classroom discoveries were valuable. It is important that students come to realize that mathematics is a dynamic field in which new discoveries are made rather than a static body of knowledge. Explicit and reflective instruction related to the discovery role of proof should enable students to be aware of the variety of ways discovery may occur in mathematics (in addition to discovery as de Villiers described).

Communication

De Villiers describes the communication role of proof as being related to the negotiation of meaning and validity amongst mathematicians. We are encouraged that students seemed to understand this role. Krissy wrote,

> Our inability to come to a consensus among three people when evaluating a particular argument also demonstrated how difficult it might be for the global community of mathematicians to achieve agreement when it comes to proof style and validity.

The instructor designed several course activities with the goal of helping students understand the social nature of proof. For instance, students designed a course rubric that outlined what makes a valid mathematical argument and were constantly asked to critique the arguments of others. The instructor frequently made the social nature of proof an explicit part of classroom discussions and we believe this contributed to students' understanding of the communication role of proof.

Conclusions

Reading de Villiers' (1990) article may be a first step towards explicit and reflective (ER) instruction on the nature of proof, but it is not enough. The students submitted their reflections as part of a final assignment, and there was no subsequent class discussion related to the assignment. Our findings may have been different if students had the opportunity for discussion. We judge that if any role of proof was discussed explicitly and reflectively most often in class, it was the communication role. Students seemed to possess a clear understanding of how validity is negotiated within the community of mathematicians through proof.

In moving forward with this work, we would like to gain feedback from RUME conference participants on the following questions:

1. How important is it that students understand the nature of mathematics (in general) and the nature of proof (more specifically)? Why?
2. How can what we know from science education research (about explicit and reflective teaching of the nature of a discipline) transfer to mathematics education?

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Perturbing practice: The effects of virtual manipulatives as novel didactic objects on instruction

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Abstract: *The advancement of technology has significantly changed the practices of numerous professions, including teaching. When a school first adopts a new technology, established classroom practices are perturbed. These perturbations can have both positive and negative effects on teachers' abilities to teach mathematical concepts with the new technology. Therefore, before new technology can be introduced into mathematics classrooms, we need to better understand how technology affects instruction. Using interviews and classroom observations, I explored perturbations in mathematical classroom practice as an instructor implemented novel didactic objects. In particular, the instructor was using didactic objects designed to lay the foundation for developing a conceptual understanding of rational functions through the coordination of relative magnitudes of the numerator and denominator. The results are organized according to a framework that captures leader actions, communication, expectations of technology, roles, timing, student engagement, and mathematical conceptions.*

Key words: Virtual manipulative, Didactic objects, Rational functions

The advancement of technology over the past twenty years has significantly altered the practices and routines found in numerous professions. When school districts adopt a new technology, teachers experience immediate changes, or perturbations, in their existing practices. These perturbations in practice can have small or large, short- or long-lived, and positive or negative effects on teachers' ability to accomplish the work with the new technology. In mathematics education, new technology is regularly being introduced into instruction (Pope, 2013). This technology comes in many forms such as hardware (e.g., computers or graphing calculators), software (e.g., Geometer's Sketchpad), or educational website licenses (e.g., Nearpod).

The goal associated with the implementation of new technology in instruction is to facilitate instruction and improve student achievement and understanding. However, in order to achieve this goal, we need to better understand the process of adopting new technology in instruction. In particular, we need to account for teachers' current mathematical meanings of concepts, the perturbations experienced by teachers when implementing a new technology, and the effect these perturbations have on the instruction of mathematical concepts.

As a step along this path, this paper identifies perturbations that occur in mathematical classroom practices (MCPs) when an instructor uses novel virtual manipulatives to teach a concept for which there are already established instructional practices. In order to connect with the goal of introducing technology to improve student understanding, virtual manipulatives designed to support reflective mathematical discourse were chosen for the study. The observed perturbations in MCPs are organized in a framework based on perturbations from industry when a new technology was adopted (Edmondson et al., 2001; Pickering, 1995). Additionally, assimilation and accommodation (Piaget, 1967), cognitive conflict (Lee, Kwon, Park, Kim, Kwon, & Park, 2003), and covariational reasoning (Carlson et al., 2002), were used to tailor the framework to a mathematics classroom.

Virtual manipulatives as didactic objects. Manipulatives are physical objects or concrete models that can be touched and moved around by the learner (Durmus & Karakirik, 2006). In mathematics instruction, manipulatives afford opportunities for learners to interact with abstract mathematical concepts and procedures through visualization and movement. However, we now recognize that the benefits of using manipulatives do not necessarily require the sense of touch, e.g., moving around physical objects. Now, a new class of computer-based manipulatives has been created (Durmus & Karakirik, 2006; Moyer-Packenham, Salkind, & Bolyard, 2008), where a virtual manipulative is defined as a "web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer, Bolyard, & Spikell, 2002,p.373).

Research on the use of objects (physical or virtual) in mathematics instruction has traditionally focused solely on how the tool itself supports student learning and understanding in terms of cognition (Lee et al., 2003). However, there has been a shift to expand the focus beyond the object itself to include the accompanying discussion (Thompson, 2002). Accordingly, Thompson (2002) defines *didactic objects* (DOs) as tools or objects that are created with the intent of supporting reflective discourse (p.198) and considers them to have two components: first, the object itself, and, second, the classroom discussion that engages students in constructing mathematical understandings. This study explores DOs designed to scaffold a conceptual understanding of rational functions.

Practices in and out of the classroom. Practices or routines are ways of doing things that are known and shared by a group of people as they engage in some activity. They are established over time and emerge as a group works together repeatedly to accomplish an activity. Changing the tools that are used in an activity, therefore, changes the associated practices, both in the long and short term. In the long term, tools can transform practices and significantly change the very nature of an activity. In the short term, the introduction of a new tool or technology can perturb established practices and lead to the adoption of new practices. For example, Pickering (1995) noted multiple disruptions in established practices on the labor floor and within management due to the adoption of numerically controlled machine tools by General Electric's (GE) Aero Engine Group in the early 1960's. Similarly, Edmondson, Bohmer, & Pisano (2001) discovered disruptions in routines that occurred when minimally invasive cardiac surgery equipment was introduced to cardiac surgery in an emergency room. Table 1 contains a framework categorizing, summarizing, and providing examples of the perturbations in practice informed by research in industry.

Table 1. *Framework summarizing perturbations in practice in industrial contexts*

Aspects of practice	Description	Example
Leader Actions	Leader's interpretation of the technology and how the leader implements the technology	Edmondson et al. (2001) demonstrated how the surgeon's beliefs in the technology correlated with how the ER team adapted to the technology.
Communication	The discourse and environment	In Edmondson et al. (2001), the discourse in the ER changed from the surgeon being the only speaker to every member of the team needing to communicate.
Expectations of Technology	Predicted outcomes for the implementation process	In Pickering (1995), prior to implementation GE management expected the technology to increase production.
Roles and Responsibilities	The individual's original responsibilities are altered during the implementation process	In Pickering (1995), the role of workers evolved from button pushers to integral members in the success of the machines.

If we consider a mathematics classroom, then the teacher and students together represent a team of individuals with a shared, collective goal of learning, and with the teacher as the team leader. The teaching practices that have been established over time in the context of the classroom by the teacher and her or his students in the course of their ongoing interactions (Cobb, Stephan, McClain, & Gravemeijer, 2001) and that may be disrupted by the implementation of new technology include pacing, student engagement, communicative norms, etc. For mathematics classrooms, such practices also include the emergent mathematical conceptions of the students as well as the mathematical understandings the teacher plans to cultivate within students (Thompson, 2013). MCPs, once established, are maintained through reflection and consistency. When a teacher reflects on the effectiveness of a practice, he or she is assessing whether the practice is effective in helping attain the goal of learning. Maintaining established MCPs relies on the consistency of the teacher. Teachers continually reinforce MCPs (for example through selecting tasks and activities) that engage students in productive ways and help them to build mathematical understandings.

Establishing and maintaining classroom practices is arduous work and takes a fair amount of time, effort, and coordination to accomplish, which makes changing practices difficult as well (Thomson, 2004). If a practice is disrupted or forced to change, there may be an accompanying experience of discomfort. Teachers, like every working professional, easily fall into a comfort zone that is made up of carrying out established practices. When these established classroom mathematical practices are perturbed by a new technology, such as a virtual manipulative, the teacher might well experience disequilibrium or discomfort (Schwartz, 1999). Furthermore, the teacher beliefs of how the mathematics classroom should function, what mathematics concepts are important, and what resources are to be used for instruction can affect how the established practices are changed (Cobb & Yackel, 1996).

Rational Functions. Students are first introduced to rational functions in algebra courses, usually taken in high school. Traditionally, rational function instruction centers on finding the asymptotes algebraically. However, this calculational orientation does not provide students with a conceptual understanding of how rational functions behave. In particular, simply setting the denominator equal to zero does not capture the covariational relationship that exists between the two polynomials that make up the rational function. It does not support students' ability to see the relative magnitude of the numerator in terms of the denominator as a single quantity. This issue sets the stage for the adoption of a virtual manipulative that allows students to explore rational functions more dynamically and to construct a covariational understanding of how the functions behave near vertical asymptotes.

Methods

I focused on a single instructor (Elaine) and a pair of novel DOs for teaching rational functions. Elaine was a graduate student teaching Pathways Pre-calculus (Carlson et al., 2013) and had taken a technology and visualization course, but was unfamiliar with teaching rational functions with the DOs used in this study. Therefore the DOs are considered novel to Elaine. The DOs (Rat Bar and Rat Graph) were accompanied by teacher guides, containing display setting and questions to ask students to foster a discussion around four phases, as summarized in Table 2.

Table 2. *Phases of didactic objects*

Phase	Didactic Object	Purpose of the Activity	See Figure
1	Rat Bar	Assist students in conceptualizing and representing relative magnitude as a quotient of functions by comparing the relative lengths of the red and blue bars.	1a
2	Rat Bar	Assist students in internalizing relative magnitude as a quantity by no longer just seeing *two separate magnitudes* but instead seeing the relative magnitude of these magnitudes *as its own quantity*.	1b
3	Rat Bar	Assist students in coordinating the change in parameter values with the change in the relative magnitude.	1c
4	Rat Graph	Assist students in graphing the rational function by attending to changes in the relative magnitude of its numerator and denominator.	1d

In order to study the perturbations caused by the novel DOs, I used two pre-interviews, classroom observations, and a post-interview with the instructor. In the first pre-interview, which took place one week prior to the observations, I explored the participant's current understandings of rational functions and gathered descriptions of the participant's instructional practices prior to the introduction of the novel DO. In the second pre-interview, which took place two days prior to the observations, I guided the participant through an exploration of the DOs. The instructor was provided with a journal to record instructional preparations made following the second pre-interview. Two days after the pre-interviews, I conducted three real-time classroom observations covering the instruction on rational functions to identify moments of perturbations from my perspective. Using stimulated recall methodology (Stough, 2001), video clips of the moments I identified during the observations were then shown to Elaine during a post interview two days later so she could give a retrospective analysis of the instances that I had flagged as perturbations.

Figure 1 depicts the implementation of the DOs for each phase shown in Table 2. In Phase 1, the teacher displays various lengths of two bars and asks students to provide a numerical guess of their relative magnitude (Figure 1a); in Phase 2, the teacher changes the length of the two bars and asks students to use the distance between their fingers to represent the changing magnitude (Figure 1b); in Phase 3, the teacher changes the length of the two bars and asks students to use their fingers to coordinate the change of one magnitude relative to the other (Figure 1c); in Phase 4, the teacher shows students a graph of the numerator and denominator of a rational function and asks students to graph the resulting rational function (Figure 1d).

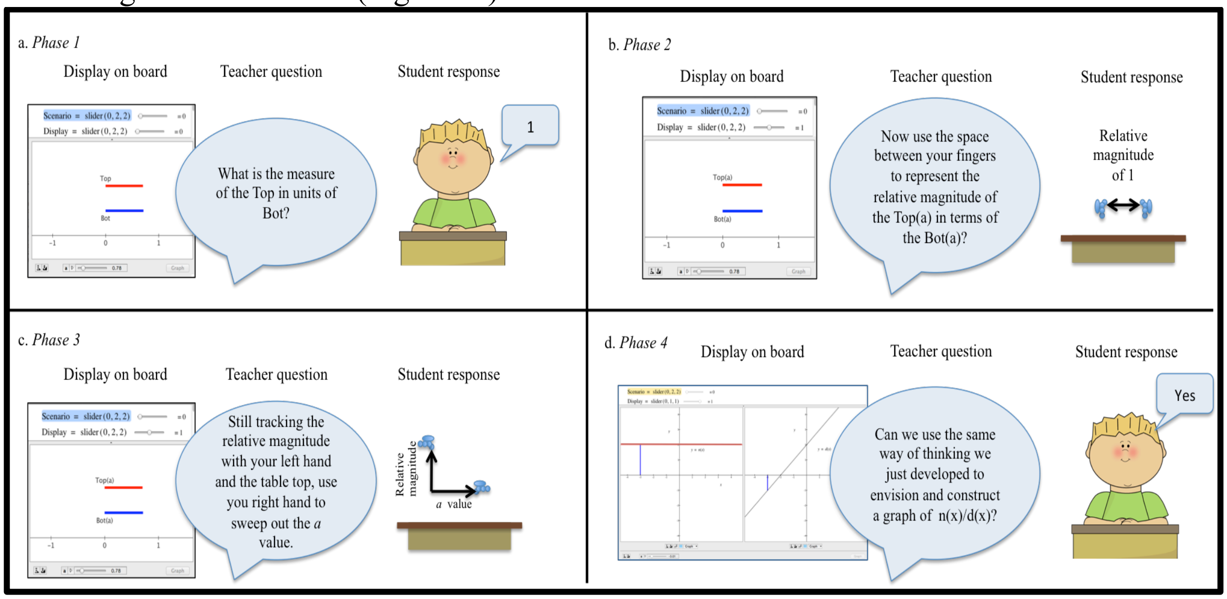

Figure 1. Four phases of using didactic objects for teaching rational functions

Preliminary Results

The preliminary results of the study provided converging evidence for the aspects of practice that are perturbed when novel technology is introduced in the context of industry, e.g. leader actions, communication, expectations of technology, and roles/responsibilities (Table 1). However, there were also ways in which the novel DOs perturbed practices in the classroom that were not observed in industry. These included student engagement and mathematical conceptions, as shown in Table 3 which describes and provides examples of the aspects of practice that were perturbed as a result of the introduction of novel DOs.

Table 3. *Framework summarizing perturbations in practice in mathematics classroom*

Aspects of practice	Description	Example
Leader Actions	How instructor perceives novel DO and how the instructor uses the technology in the classroom	Elaine's introduction to the DO demonstrated her uneasy feeling toward trying something new.
Communication	Classroom discourse surrounding the novel DO	Elaine's students no longer relayed exact answers but instead they explained the behavior of the function.
Expectations of Technology	What understandings the teacher expects students to develop	Elaine had expected the novel DO to take the exact amount of time as her previous lesson.
Roles and Responsibilities	Responsibility for assimilating conceptual and procedural	Elaine's role was altered from lecturer to discussion facilitator.
Student Engagement	Student participation while the DO is being used	Elaine's students became more active in the lesson through the activities that accompanied the virtual manipulatives.
Mathematical Conception	How students perceive the mathematics addressed by the novel DO	The novel DO change the emphasis of rational functions to behaviors rather than symbolic manipulation.

A possible reason for these additional perturbations in classroom practice (student engagement, mathematical conceptions) stems from differing expectations of technology. Industry adopts technology with the intent of increasing productivity and efficiency. In contrast, the purpose of using technology in a mathematics classroom is reorganizational (Sherman, 2014) and supports the development of deeper understandings (Thompson, 2002). All sources of data collected in this study point to the difficulties of accomplishing this task. Figure 2 displays two examples drawn from the classroom observations showing how two of the six aspects of practice, namely leader actions and mathematical conceptions, were affected by the implementation of the novel DOs.

Leader actions. In industry, it was noted that the introduction of new technology impacts the actions of the leader, which in turn affects how the team operates. This was also true in the observed classroom environment. In this case, the novel DOs caused Elaine to adopt a hesitant, foreboding approach to the upcoming lesson on rational functions. As seen in Figure 2, her introduction of the DOs to the students sounded much like a parent trying to explain to a child that vegetables may not taste good but that they are good for your health. Thus, as the leader, she gave students plenty of reason to be wary of the upcoming lesson and mathematics, instead of exuding confidence and a belief in the value of conceptual understanding. This is noteworthy because teachers, as classroom leaders, profoundly influence student beliefs in both the short and long term and thus ultimately shape the perception of what it means to understand and do mathematics (Thompson, 2013).

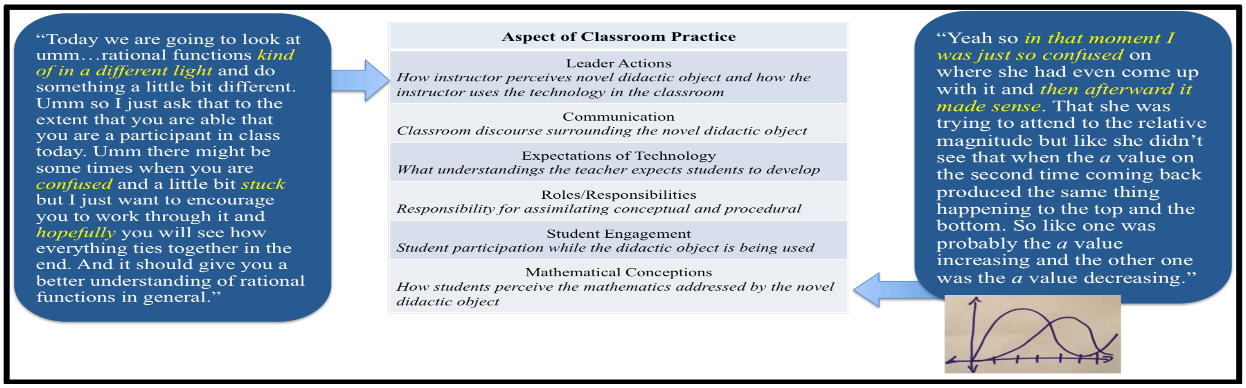

Figure 2. Examples of perturbations in mathematical classroom practices

Mathematical conceptions. However, unlike what was observed in industrial contexts, in the classroom the introduction of novel DOs also perturbed the conceptions of those involved, causing new conceptions to emerge and unexpected conceptions to surface. Thus, as seen in Figure 2 (right) Elaine was baffled by the mathematical conception of one of her students when working with the class through Phase 3. In this phase, the students are asked to construct a graph of the relative magnitude of the numerator in terms of the denominator. Elaine admitted to being stumped in the moment when the student drew a graph on the board of two functions on the board. After the class session ended, Elaine figured out the student's conception and how this was reflected in the presentation on the board. This is an example of how perturbations can lead to additional practices, in this case the practice of anticipating student responses (Stein, Engle, Smith, & Hughes, 2008).

Discussion and Conclusions

This preliminary work found converging evidence of perturbations found in industry (Edmondson et al., 2001; Pickering, 1995). Evidence from the study confirmed that novel technology caused perturbations in classroom practice with regard to leader action, communication, expectations of technology, and roles and responsibilities. Additional perturbations in student engagement and mathematical conception were found supporting the tailoring of the framework.

Although this study has limitations in scope, but sets the stage for delving more deeply into the ways in which novel DOs perturb classroom practices so that we can find ways to foster productive perturbations (e.g., supporting cognitive conflict and conceptual understanding) and mitigate the effects of less productive perturbations (e.g., transmitting a lack of confidence to students through the instructor's actions). The framework can guide development of interventions to smooth the path of using technology in classrooms. If teachers are more comfortable introducing and making use of new technology, we will be one step closer to improving student achievement and understanding.

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Comparing Graph Use in STEM Textbooks and Practitioner Journals

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In this study we focus on the use of graphical representations to find similarities and differences regarding how graphs are used in mathematics textbooks and how they are used in STE textbooks and journals. After highlighting the need for our study and summarizing the results of related studies, we present our methods. We then present key preliminary findings comparing how a selected pre-calculus textbook and certain textbooks and journals in various STE fields use graphical representations. We conclude with preliminary implications and questions.

Keywords: Graphs and graph use; STEM; Precalculus

The changing nature of the global market has highlighted the need for improving STEM education in the U.S. In order for the U.S. to compete with other nations, U.S. students need to enter STEM fields. However, currently U.S. STEM education is surpassed by other nations at the elementary and secondary levels (Holdren, Lander, & Varmus, 2011). Further, mathematics often plays the role of a "gatekeeper" for students' continued study and future success in STEM fields (Crisp, Nora, & Taggart, 2009; Gasiewski, Eagan, Garcia, Hurtado, & Chang, 2012). Given this role, we focus on college level mathematics, particularly precalculus and calculus. These levels are particularly important because more than one third of students intending to pursue a STEM major in the U.S. enroll in mathematics remediation (e.g., precalculus) (Radford, Pearson, Ho, Chambers, & Ferlazzo, 2012) and students interested in STEM majors are more likely to declare a non-STEM major after introductory calculus (Bressoud, Carlson, Mesa, & Rasmussen, 2013).

In the larger study we aim to explore the mathematics presented at these levels as it connects to the demands of science, technology and engineering (STE) fields. Our research question is, "How are graphical representations used in precalculus and calculus textbooks similar to and different from the graphical representations used in STE textbooks and practitioner journals?" Specifically, we focus on nuances of graphical representations of two covarying quantities in mathematics textbooks and in STE textbooks and journals. We note that at the time of submission we have examined one precalculus text and have not had the opportunity to examine any calculus textbooks, although this is our intention.

Literature Review

Rybarczyk (2011) and Roth, Bowen, and McGinn (1999) analyzed several textbooks and research journals in biology and ecology respectively, examining every visual representation in these sources (e.g., diagrams, photographs, graphs, tables, etc.). The researchers identified a mismatch in the types of visual representations used in science textbooks compared to journals. For instance, journals used graphical models to represent statistical data more frequently than textbooks. Because these researchers focused broadly on the different visual representations used across science textbooks and journals, they did not address nuances in how these sources represent two covarying quantities. Such nuances can impact students' interpretation of graphs (e.g., if graphs follow conventions commonly maintained in school mathematics) (Moore, Paoletti, Stevens, & Hobson, 2016; Moore, Silverman, Paoletti, & LaForest, 2014). Hence, in this study, we attempt to close this gap by producing a fine-grained analysis of how STE textbooks and journals use graphs to represent two covarying quantities and compare these to introductory college level mathematics textbooks.

Methods

To address our question, we have begun to gather data from textbooks and journals in STEM fields to explore how graphical representations are used in these different sources. To date, we have analyzed the five sources described in Table 1. We used the Open Syllabus Project (OSP, opensyllabusproject.org) to determine which textbooks were frequently used in STE courses.

Table 1

The Source Title, Author or sub-journals and reasons for including the source, by source.

Source Title (Short name)	Author or sub-journals	Reason for including
Glencoe Precalculus (Precalculus text)	*Glencoe Precalculus* 2014	Precalculus textbook from a major publisher.
Chemistry: The Central Science (Chemistry text)	Brown, LeMay, Bursten, Murphy, and Woodward (2012)	3rd ranked textbook in OSP under search for “chemistry”.
Engineering Mechanics: Statics (Statics text)	Hibbeler (2013)	1st ranked textbook in OSP under search for “statics”.
IEEE Journals and Physics Today (IEEE/Physics)	IEEE Electron Device Letters IEEE Network IEEE Communications Magazine IEEE Photonics Journal IEEE software IEEE Journal of Selected Topics in Quantum Electronics Ten IEEE “Transactions” journals Physics Today	IEEE is the world’s largest technical professional organization for engineers and scientists. They publish a variety of journals and magazines aimed at providing a venue for these professionals to share their knowledge. Physics Today, with a circulation over 100,000 is the membership magazine of the American Institute of Physics.
Journal of the American Medical Association (JAMA)		The most widely circulated medical journal in the world, JAMA publishes original research, editorials and reviews within the biomedical sciences.

With respect to the three textbooks, we focused on the graphical representations that the authors emphasized. As such, we analyzed all graphical representations that were included in the body of the text and did not analyze the graphs in the problem sets as it is up to instructor to assign these problems. With respect to the journals, for JAMA, we started with the most recent issue available through our university library (July 5, 2016) and backtracked through May 2016, identifying all articles with at least one graphical representation. For IEEE/Physics, we identified journals whose stated purpose seemed to align with informing practitioners. We then scoured these journals for articles with at least one graphical representation. For every article we found, we coded any graphs we observed in the article. As our coding of the graphs is both a method and result, we elaborate on how we coded each graph in the results section.

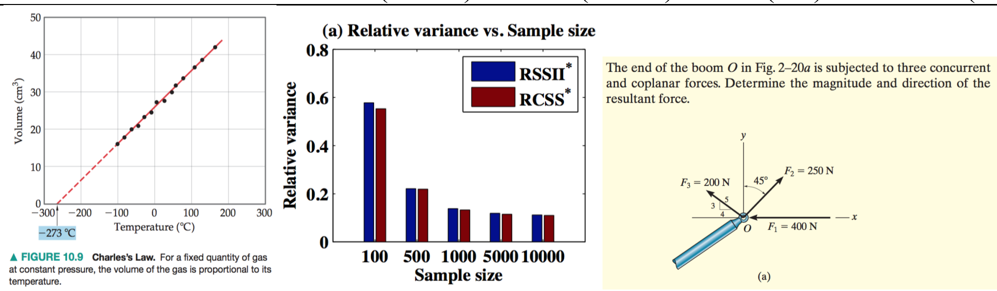
Results

We coded a total of 850 graphs across the five sources (Table 2). Our initial goal was to code the extent to which these sources used graphs to represent relationships between covarying quantities (Covarying Quantities in Table 2, Figure 1a). However, because certain sources used graphs for other purposes, we also coded the number of graphs in each source that were used for these purposes. For instance, and as reported by Rybarzyk (2011) and Roth, Bowen, and McGinn (1999), several sources often presented statistical graphs (Statistical Graphs in Table 2, Figure 1b). Other sources (specifically the Statics text) overlaid a coordinate system over an object or phenomena to help mathematize the situation (Imposing Axes in Table 2, Figure 1c). Table 2 presents the number of each type of graph we observed in each source. We highlight the prevalence of graphs representing two quantities in the precalculus text, chemistry text and IEEE/Physics journals. Statistical graphs played a smaller but still significant role in journal articles when compared to textbooks. Further, we note that the Engineering Statics text almost exclusively used graphs to impose axes on a given object or phenomena.

Table 2

The total number of graphs (N) as well as the number of graphs representing two quantities, statistical graphs, imposing axes, and imaginary planes versus the source.

Source	N	Covarying Quantities	Statistical Graphs	Imposing Axes	Imaginary Plane
Precalculus text	299	273 (91.3%)	23 (7.7%)	0 (0%)	3 (1.0%)
Chemistry text	74	74 (100%)	0 (0%)	0 (0%)	0 (0%)
Statics text	255	3 (1.2%)	0 (0%)	252 (98.8%)	0 (0%)
IEEE/Physics	166	144 (100%)	22 (13.3%)	0 (0%)	0 (0%)
JAMA	56	35 (62.5%)	21 (37.5%)	0 (0%)	0 (0%)



(a) (b) (c)

Figure 1. An example of a graph representing (a) two covarying quantities from Brown et al. (2012, p. 390), (b) statistical information from Li, Yu, Mao, and Jin (2016, p. 13) and (c) imposing axes on a situation from Hibbeler (2013, p. 37).

Because our main focus was on how these sources use graphs to represent the relationship between covarying quantities, all further analysis focused on graphs that fit this subcategory. We were interested in how frequently graphs represent two decontextualized quantities (typically x and y), one contextualized and one decontextualized quantity, or two contextualized quantities (e.g., Figure 1a) (Table 3). We note the significant differences across these sources in regards to using graphs to represent contextualized quantities. Unsurprisingly, STE textbooks and journals almost exclusively used graphs to represent two contextualized covarying quantities. In contrast, graphs in the precalculus textbook rarely represented two contextualized quantities. We found the lack of contextualized examples in the precalculus textbook surprising, and conjecture that we may obtain a different result when we examine calculus textbooks and possibly precalculus textbooks from other publishers.

Table 3

The total number of graphs representing two covarying quantities (CQ), the number of graphs representing two decontextualized, one contextualized one decontextualized, and two contextualized quantities versus the source.

Source	CQ	Two decontextualized	One contextualized one decontextualized	Two contextualized
Precalculus text	273	270 (98.9%)	1 (0.4%)	2 (0.7%)
Chemistry text	74	2 (2.7%)	1 (1.4%)	71 (95.9%)
Statics text	3	0 (0%)	0 (0%)	3 (100%)
IEEE/Physics	144	0 (0%)	0 (0%)	144 (100%)
JAMA	35	0 (0%)	0 (0%)	35 (100%)

Another aspect of graphs representing two quantities that we examined across these sources was the frequency with which conventions with respect to the location of the intersection of the coordinate axes was maintained (Table 4). We coded graphs representing two quantities as either having axes intersect at (0,0), as having axes intersecting at a value other than (0, 0) (e.g., Figure 1a), or if the graph had no scale and we were unable to infer the coordinate values of the intersection of the axes. We note that all graphs in the precalculus and chemistry textbooks had axes that intersected at (0, 0). However, practitioner journals followed this convention with less frequency; it was typical for the intersection of the axes in these sources to not be at (0, 0).

Table 4

The total number of graphs representing two covarying quantities (CQ), the number of these graphs with the intersection of the axes at (0, 0), not at (0, 0) and no scale versus the source.

Source	CQ	Axes intersect at (0, 0)	Axes do not intersect at (0, 0)	No Scale
Precalculus text	273	273 (100%)	0 (0%)	0 (0%)
Chemistry text	74	74 (100%)	0 (0%)	0 (0%)
Statics text	3	0 (0%)	0 (0%)	3 (100%)
IEEE/Physics	144	40 (27.8%)	89 (61.8%)	15 (10.4%)
JAMA	35	19 (54.3%)	16 (45.7%)	0 (0%)

We conjectured that mathematics textbooks used time as a quantity under consideration frequently and wanted to compare how frequently other sources used time. Hence, for all graphs representing covarying quantities with at least one contextualized quantity, we coded if time was represented on the graph (Table 5). Articles in JAMA represented time in a majority of their graphs representing contextualized quantities. The chemistry text and IEEE/Physics journals had significantly more graphs in which time was not a quantity under a consideration.

Table 5

The total number of contextualized graphs, graphs in which time is a quantity under consideration, and time is not a quantity under consideration by source.

Source	Contextualized Graphs	Time is a quantity under consideration	Time is not a quantity under consideration
Precalculus text	3	1 (33.3%)	2 (66.7%)
Chemistry text	72	10 (13.9%)	62 (86.1%)
Statics text	3	0 (0%)	3 (100%)
IEEE/Physics	144	35 (24.3%)	109 (75.7%)
JAMA	35	25 (71.4%)	10 (28.6%)

Preliminary Discussion, Implications, and Future Research

One important preliminary implication of our results is the importance of preparing students who may enter STE fields to use coordinate systems both to represent two covarying quantities and to mathematize a situation or phenomena. Although researchers have focused on students' understandings of representing relationships between covarying quantities (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu 2002; Kozhevnikov, Motes, & Hegarty, 2007; Thompson, 2011) there has been less focus on students' use of coordinate systems to help mathematize an object or phenomena. There have been some efforts to examine the mental operations that students' use when imposing axes onto a situation or phenomena (Lee, 2016; Lee & Hardison, 2016; Piaget & Inhelder, 1967), however, the extent to which the Statics textbook uses coordinate systems to mathematize a phenomena or situation reinforces the need to continue to examine students' understandings of this use of coordinate systems.

A second implication relates to previous researchers' findings in regards to students' interpretations of graphs. There is a large body of research examining individuals' struggles interpreting graphs in both mathematics and the sciences (e.g., Glazer, 2011; Leinhardt, Zaslavsky, & Stein, 1990; Shah & Hoeffner, 2002). For instance, several researchers have indicated students make iconic translations, interpreting graphs intended to represent two covarying quantities as an image of the situation (Carlson et.al., 2002; Leinhardt et al., 1990; Monk, 1992). We conjecture the lack of contextualized graphs in precalculus curriculum may help explain some of the observed struggles students encounter when interpreting contextualized graphs. Consistent with Shah and Hoeffner's (2002) argument, if graphs are presented only abstractly, students are likely to struggle translating this knowledge to graphs in contextualized situations; future research is needed to examine the validity of this conjecture.

A third implication relates to graphing conventions. Researchers (Gattis & Holyoak, 1996; Moore et al., 2016; Moore et al., 2014) have indicated that students often maintain what are conventions to teachers and researchers as inherent aspects of their mathematics. These findings reflect what students experience with their textbooks. If students repeatedly experience graphs in which these conventions are maintained, they may develop mathematical understandings that are constrained by such conventions. Hence, it may be unsurprising when students struggle to make sense of situations in which these conventions are not maintained.

A final implication relates to the extent to which different sources represent time as a quantity under consideration. Entering the study, we conjectured time would be a predominant quantity used in mathematics textbooks but found this was not the case in the precalculus textbook. In JAMA time was a quantity under consideration in a majority of graphs but in the chemistry textbook and IEEE/Physics journals time was a quantity under consideration in less than a quarter of the graphs. The extent to which time is used as a quantity under consideration in various STEM fields requires further examination.

Intended Questions

We intend to examine more precalculus and calculus textbooks. What other sources (both mathematics or sciences textbooks or journals) would be good to consider in this study? Why? What other data would be worth analyzing within the graphical representations? Why?
What do you see as some other possible implications of a study like this? What could we do to make our implications stronger?

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Student Conceptions of Three-Dimensional Solids

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In the study presented in this paper, the authors aim to construct a model for the processes by which students in a multivariable calculus class conceptualize solid regions in three dimensions. We designed and recorded student work from several tasks in which students must decode a description of a solid figure and answer questions assessing the strength of their conception of the figure. Presented here are findings from the analysis interviews and group work on one of these tasks in which students are asked to build a clay model of the solid region described by a set of inequalities in three variables.

Key words: multivariable calculus, spatial reasoning, three-dimensional solid

Introduction

As motivation for our study, we consider the following problem, variations of which appear in many multivariable calculus textbooks.

Let E be the solid region in $\mathbb{R}^3$ defined by the following set of inequalities.

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 1-z \\ z^2 & \leq & y & \leq & 1 \\ 0 & \leq & z & & \end{array} \tag{1}$$

Find the volume of E.

In anecdotal observations of student work, the authors found that students would attempt to solve this problem, apparently without attempting to conceptualize the solid figure E as a subset of $\mathbf{R}^3$. While this observation calls into question the validity of such problems in building and assessing students' understanding of volume, it also raised the question, through what process would students decode sets of inequalities such as (1) as a solid figure?

In the present study, we use data from 76 undergraduate multivariable calculus students (split into 29 response groups) collected in individual interviews and recorded group work, to identify common strategies and obstacles in decoding, processing, and communicating representations of solid figures.

Theoretical Background

There is a rich history of studying student conceptions of three-dimensional solids in the context of spatial reasoning. Much of the work in this area relies on methodology in which subjects must interpret a two-dimensional drawing of a solid figure and then perform some spatial task such as rotation (Bodner, Guay, 1997) or identification of cross-sections (Cohen, Hagerty, 2007). However in studies where a two-dimensional drawing is used, Gorgorió (1998) warns, "individuals' demonstration of their spatial orientation ability depends also on their abilities for interpreting and communicating spatial information." Indeed it has been shown that student misconceptions can lead to errors both during interpretation (Parzysz, 1988; Hallowell, et al, 2015), and communication (Ben-Chaim, Lappan, & Houang, 1989; Hershkowitz, 1990). Thus, building on the paradigm set forth by Parzysz (1991), we model student responses to the tasks as information which has passed through three phases: decoding, processing, and communication.

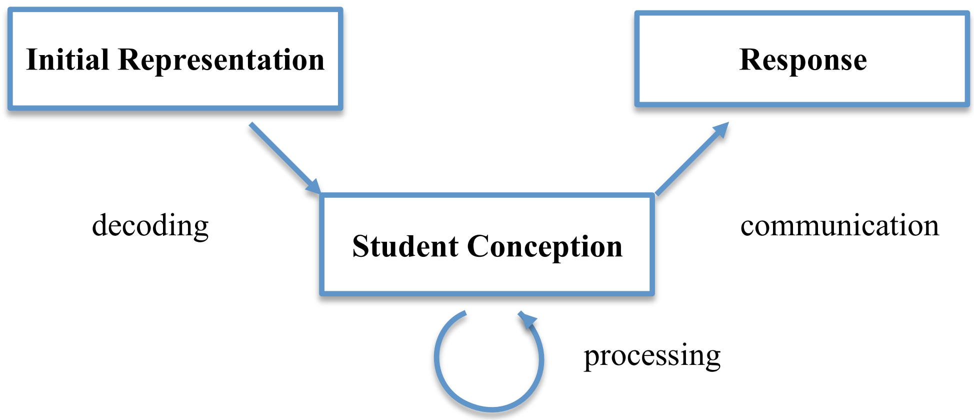


Figure 1: Model for student responses. This figure illustrates the flow of information resulting from students' response to a task. Errors or loss of information may occur in each phase.

The *decoding* phase encompasses interpretation of the given representation. The *processing* phase includes any mathematical or visuospatial treatments the subject performs, as well as conversions between registers. Finally, in the *communication* phase, subjects encode their responses to the task. Included in the communication phases are written work, drawings, conversation with other students, and building physical models.

In a multivariable calculus class, students are asked to become fluent in interpreting descriptions of figures in $\mathbf{R}^3$ which come in a variety of forms: written descriptions, equations or inequalities in three variables, two-dimensional drawings, and combinations of these. Viewed from the theoretical framework of semiotic representation theory, this requires students to coordinate representations in several registers (Duval, 1993), and perform conversions from one register to another (Duval, 2006). Trigueros and Martínez-Planell (2010) investigate how students use this coordination of registers to perform tasks related to surfaces and equations in multivariable calculus. Our study builds on this work by applying the same theoretical framework to solid figures in three dimensions.

Methodology

We designed several tasks meant to elicit responses which would give insight into the strategies students use and the obstacles they face in decoding and processing representations of solid figures. For this preliminary report, we will discuss results from Task 2 below.

Task 2: Let G be the solid region described by the inequalities

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 1 \\ 0 & \leq & y & \leq & 1 \\ 0 & \leq & z & \leq & x+y \end{array}$$

a Build the region G out of clay

b Write 2–4 sentences explaining how you determined the shape of the solid region from the inequalities.

Like many of the tasks in our study, Task 2 pairs a *close* representation of the solid (Parzysz, 1988)—a three-dimensional model—with the representation type we wish to study—in this case, a set of inequalities. Underlying the rationale behind this pairing is the hypothesis that students will face little difficulty in decoding the close representation, nor in communicating a robust enough conception of the figure into the medium of a clay model. Under this assumption, errors in student work on Task 2 could be attributed to misunderstandings or miscalculations related to the decoding and processing of the sets of inequalities[1].

We used a grounded theory approach to collect and analyze the data from participants in the study. Data collection involved video-taping students working in groups on the tasks during nine class sections from two different instructors in the spring and summer of 2016. We also collected students' written work and photographed clay models produced by the students. Additionally, we conducted individual interviews with four students. Through axial coding (Glaser & Strauss, 1967), we established categories for responses to the task, and identified several distinct strategies used by the students.

Preliminary Results

Here we present preliminary findings from analysis of Task 2 above. Four categories of clay models emerged from the coding of student responses: rectangular prism (P), tetrahedron/pyramid (T), conflicted (C), and accurate (A); examples of each are shown in Figure 2 below.

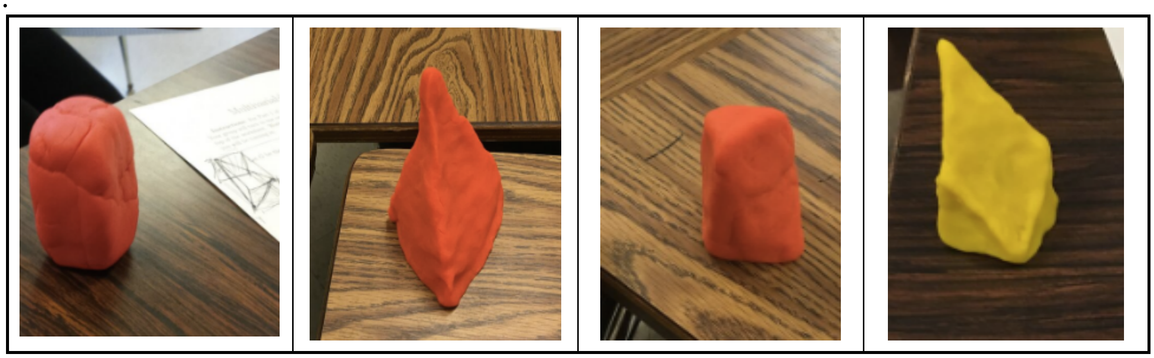


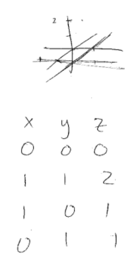
Figure 2: Examples of the four categories of responses. From left to right: rectangular prism (P), tetrahedron/pyramid (T), conflicted (C), and accurate (A).

[1] This assumption was not entirely accurate since some students who demonstrated a robust conception of the solid made errors in the model the produced related to the relative scaling in the *x*, *y*, and *z* variables.

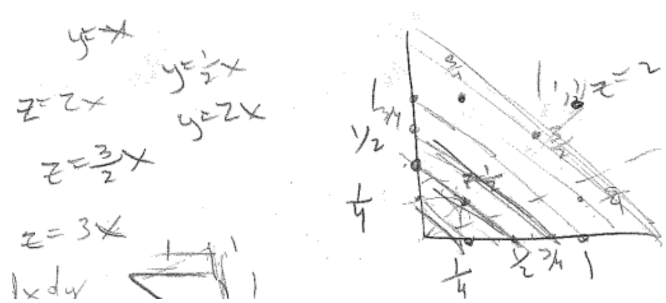
The categories of (P) and (T) are self-explanatory. A clay model was classified as accurate (A) if it reflected all the correct faces and edges of figure described by the given inequalities; models in which the x, y, and z variables were scaled differently could still be considered accurate. Models which fell into none of the other three categories were classified as conflicted (C); in all cases, these models were some hybrid between a pyramid and an accurate model.

Additionally, we identified several distinct strategies that were used by students while completing the tasks, which are described below.

1. Maximum/Minimum: Identifying the maximum and minimum values for each variable—in this context, this amounts to identifying that $z \leq 2$. Example: "The first two inequalities are the same, so we concluded that the base of the region is a square. The third inequality is dependent on x and y, and the largest value is 2, so the region can range from a rectangular prism to a 2d square."
2. Covariation/Path: Using language that invokes an image of two variables changing at the same time, often to describe an edge of the figure or a path along a face. Example: "the z starts at zero and the origin corner at the square then goes to 2 at the (1,1) corner."
3. Finding Vertices: Locating extremal points of the figure by converting some subset of the inequalities to equalities and solving. Example: see Figure 3.
4. Level Curves: Sketching level curves of $z = x + y$ in order to understand the top face of the figure. Example: see Figure 3.



Finding Vertices



Level Curves

Figure 3: Examples of Finding Vertices and Level Curves.

We organized strategies by response category, as shown in Table 1. The most notable trend is that response groups which produced an accurate model seemed to use a wider range of strategies than those which produced other models.

Applying our model for interpreting student responses from Figure 1, we can attempt to infer in which phase information was lost or mishandled from both the response type and the strategies used. For example, we hypothesize that for students who responded model type (P), most of the information never made it past the decoding phase—that students did not correctly interpret the meaning of the inequalities taken as a set. For students who responded with model type (T), it seemed that some information was lost in the decoding phase, and that only minimal processing had taken place. For model type (C), a hybrid between (T) and (A), the mishandling of information seemed to occur in the processing phase; these students had a conception of the figure in one representation register, but failed to accurately convert it to another.

Table 1: Strategies used, organized by response category. Responses submitted by a group of students were treated as a single response. In many cases, groups reported using more than one strategy, which is reflected in the table.

Student's Strategies	Rectangular Prism (P)	Tetrahed-ron/Pyramid (T)	Conflicted (C)	Accurate (A)
Responses in this category	5	2	7	15
Maximum/Minimum	5 (100%)	2 (100%)	4 (57%)	7 (47%)
Covariation/Path			1 (14%)	3 (20%)
Finding Vertices	1 (20%)	1 (50%)		7 (47%)
Level Curves			4 (57%)	5 (33%)

Conclusions

We plan on analyzing the rest of our collected data in a similar vein. We remain interested in using this information as a base line to flesh out the three phases of decoding, processing, and communication in further analysis of student's thinking on solid regions in $\mathbf{R}^3$.

Questions for Audience

1. For those who teach multivariable calculus: How do you communicate solid figures to students in the context of triple integrals and volume? What big ideas do you hope your students will get out of this part of your course?
2. The use of three-dimensional models of unfamiliar three-dimensional solids, constructed using 3D printing or clay, is somewhat novel to our research. What other lines of inquiry could we investigate with this medium?

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The Effects of the *Epsilon-N* Relationship on Convergence of Functions

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Much work has been done in recent years to study students' formulations of formal limiting processes. One of the most common goals is to foster a productive understanding of the relationship between the error bound epsilon and the domain of the convergence; what is called a range-first perspective. My study examined an advanced calculus student's understanding of the relationships involved in convergence of functions, and how his prior experience with limits influenced his concept image. I unpack his cognitive organization of the dependence relationships between epsilon, N, and x in functional convergence. This case study demonstrates the effects of a persistent understanding that ϵ depend on N in the convergence of sequences.

Key words: Sequences of Functions, Formal Definition of Convergent Sequences, Advanced Calculus

Mathematics students study convergent sequences throughout their college career. Formalizing sequential convergence provides an important yet difficult stage in the development of students' reasoning about advanced mathematics. A vital component of the formal definition of sequential convergence is the relationship between ϵ (*epsilon*) and the critical index N. I call this the *epsilon-N* dependence relationship, which mirrors the *epsilon-delta* dependence relationship within the formal definition of continuity of a function.

In advanced analysis courses, sequences take on forms beyond just real numbers. In particular, advanced calculus students encounter sequences of continuous functions towards the end of their instruction. The convergence of such functions is defined in a manner that is structurally similar to the familiar convergence of real numbers, but it has the added complexity of also accounting for variation within the domain of the function. Thus, keeping track of the definition's quantifiers becomes vitally important to understanding the different types of convergence and their implications on the functions themselves.

While conducting a larger study involving student conceptions of metric spaces, I interviewed two students to examine their concept images and definitions (Tall & Vinner, 1981) pertaining to convergence of function sequences. A case study of one student's interview demonstrates how reversal of the *epsilon-N* relationship may persist beyond convergence of real numbers to affect the convergence of functions. In this preliminary report, I will discuss this case to demonstrate the effect of this reversal on convergence of functions, and I will go into detail about some cognitive conflicts that seemed to arise within the student's concept image. I seek to answer the following research questions: 1) How does the addition of the domain value x into the definition for functional convergence affect student understanding of convergence?, and 2) Does reversal of the *epsilon-N* dependence relationship in real number convergence affect the dependence relationships for functional convergence?

Literature Review

There is an abundance of research on student initial understanding of limits, the potential difficulties that arise and initial interpretations of limit definitions (Bezuidenhout, 2001; Cornu, 1991; Cottrill et al., 1996, Davis & Vinner, 1986; Roh, 2008; Roh, 2009; Tall, 1992; Tall &

Vinner, 1981; Williams, 1991). There have also been investigations into students' intuitions about the nature of limits (Oehrtman 2003; Oehrtman 2009). Much of these investigations, however, pertained to students' intuition and reasoning with informal limits rather than formal understanding.

Targeted glimpses of student understanding of formal limit definitions began with Swinyard (2011). In his teaching experiment, two students successfully reinvented the formal definition of the limit of a function. Swinyard and Larsen (2012) also suggested some strategies for fostering productive generation of formal limit definitions such as a focus on the error bounds. This strategy is called adopting a range-first perspective as opposed to a domain-first perspective where the control of the convergence is given to the variation in x values. This was adapted to sequences by Oehrtman et al. (2014) to also include the index n in the domain-first perspective for purposes of sequential convergence study where students reinvented the definition of formal sequence convergence.

Other efforts have also been made to investigate student understanding of the formal definition of limits with special attention paid to the relationships between the variables controlling the limiting process (Adiredja, 2013; Adiredja & James 2013; Adiredja & James 2014).

This study will continue the literature on student understanding of formal limit definitions with specific attention to the relationships of the variables within convergence of sequences in advanced settings.

Theoretical Perspectives

Concept Image and Concept Definition

Formal mathematics often entails understanding a particular concept via its definition. Accompanied with each concept, however is also a collection of cognitive organizations beyond just the definition. This calls for a distinction to be made between the collection of cognitive objects that are brought to bear upon recollection of the concept, and the words that compose the definition of the concept. This is the distinction between a thinker's concept image and concept definition (Tall & Vinner, 1981).

Tall and Vinner define the concept image as the "total cognitive structure that is associated with the concept". This includes examples, graphs, images, relationships, and even the concept definition. The concept definition is the formal language that the student uses to identify the concept. This definition may be a personal definition given by the student, or it might be the formal definition accepted by the mathematical community. In this report, I use the notions of concept image and concept definition to characterize the student's conceptions.

Domain-First and Range First Perspectives

Informal limit notation is usually written in a manner such as $\lim_{n\to\infty} a_n$ or $a_n \to a$. In accordance with this notation, students transitioning from informal to formal notions of the limit may focus first on how the variation of the index affects the sequence entries. This is called a domain first perspective. Swinyard and Oehrtman (2014) suggest that focusing on the behavior of the range values before focusing on the domain values will foster correct interpretation and generation of the quantifiers within the formal limit definition.

Mathematical Discussion

I now briefly offer a discussion of the mathematics involved in this study to situate the subsequent discussion of the data. The convergence of a sequence of functions is an advanced concept that is encountered by mathematics students towards the end of the advanced calculus sequence. There are two types of convergence, however we will only be discussing point-wise convergence.

Because of the nature of convergence in metric spaces, the formal structure of functional convergence is very similar to that of the familiar convergence of real number sequences. A sequence of real numbers (a_n) converges to a real number a if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|a_n - a| < \epsilon$. For this type of convergence, the error bound ϵ controls the behavior of the sequence, since as ϵ decreases, a critical index N must be found.

A sequence of functions(f_n) converges point-wise to a function f on a domain D if $\forall \epsilon > 0$ and $\forall x \in D$ $\exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$. Note that at each x value in the domain of convergence we have the sequence of functions forms a convergent sequence of real numbers, thus the critical index N is controlled by both x and ϵ.

Methods

The episode reported on was part of a larger study exploring student's reasoning about metric spaces. This involved conducting teaching experiments with two students, each teaching experiment consisting of six hour-long sessions (Steffe & Thompson, 2000) with a single student. The particular episodes being explored in this report were semi-structured and task based (Hunting, 1997) so that student activity could be observed.

The participants in the study were both mathematics majors at a large northwestern university who had completed the advanced calculus instructional sequence and were studying real analysis applied to finite dimensional vector spaces at the time of the interviews. Both students demonstrated proficiency in their advanced calculus courses and had seen functional convergence before.

Each interview was video recorded, and the sessions have been reviewed multiple times highlighting episodes illuminating aspects of the students' concept images as well as sources of cognitive conflict and their resolution. The particular episodes reported on are transcribed along with time stamped pictures highlighting diagrams or writing done by the students.

Results

The goal of this interview was to explore Kyle's concept images and definitions pertaining to different types of convergent function sequences. This set the stage for a new metric space to be explored, and I was able to examine Kyle's concept image for convergence of functions.

The first step of the interview was to establish Kyle's concept definition for point-wise convergence of functions. This type of convergence would have been Kyle's first experience with functional convergence of any kind in the classroom. Kyle's first concept definition, the formulation I claim that is most relevant to his understanding, is given here: A sequence of functions $\{f_n\}$ converges pointwise to f if $\forall x \in D$, $\lim_{n\to\infty} f_n = f$. Kyle soon corrected this to add in $f_n(x)$ and $f(x)$.

This is significant because Kyle first chose to use the conventional limit notation given in a calculus setting instead of incorporating the ϵ-N definition. Kyle demonstrated a domain-first focus throughout the majority of the interview, and this focus on N controlling the limiting process was first displayed in his personal concept definition.

After a brief discussion about what the elements of his definition meant, I prompted Kyle to write a formal definition. When asked to discuss what the definition means, he said:

Kyle: When we are trying to show point-wise convergence we are trying to find some ϵ that will always be greater than the difference between our sequence and our function, however because we are selecting our ϵ and then selecting our x then this ϵ will actually be a function of x and the N from the naturals.

As he said this he wrote out $\epsilon(x, N)$. As we began to unpack the nature of the dependence of ϵ on N and x, he made the following statement:

Kyle: N plays the same role that it always plays when we're talking about convergence. It's the last index such that every index past this you're going to have this property $[|f_n(x) - f(x)|]$ be true, so in so much as that ϵ is pretty much always dependent on that natural because you want to say that past this natural N, you know this ϵ is always going to be true.

As we continued discussing the nature of the ϵ dependence on N and x, we discussed some typical examples of functions that converge in a point-wise manner such as x^n and x/n. In each of these examples Kyle would draw an ϵ window around the values of the functions at a particular point x, reinforcing that a different ϵ window was necessary as N increased without bound and as x varied across the domain. Kyle continually demonstrated that the index increasing without bound was the driving force of the convergence of the functions, verifying that he had adopted a domain-first perspective.

This domain-first perspective, which I infer had carried over from his experience with convergence of real numbers, now had an added complexity of incorporating the domain of the function as well into his scheme for convergence. As an example of this, Kyle described the following *x-N-epsilon* relationship as we examined the sequence of functions x/n:

Kyle: but if we think about what n is, like if we think about the Archimedian property. If you are thinking about $1/n$ as being less than ϵ, then that means that $2/n$ would imply that $1/n$ is less than $\epsilon/2$. So if you're increasing the values of these x's, your ϵ is going to get smaller and smaller.

This of course is not the case as there is no inherent relationship between ϵ and x. As we further discussed the effect of fixing an x value, Kyle had a moment of cognitive conflict where he commented that he didn't really mean ϵ was a function of x, just that if you changed the x you would change the ϵ, but then since we do this for all positive $epsilon$ maybe "$\epsilon/2$ is just as good as ϵ". Here I think he has a moment where he is considering the universal quantification of ϵ, but his reversal scheme is still his dominant stance.

His reversal was evident later when Kyle wrote out that to show x^n converges at $x = 1/2$ we needed to bound $(.5^n)$ by ϵ, and so we take the "nth root of *epsilon*" . This would happen if we had already found a critical index N and needed to find an error bound that worked for that N. This task then illuminated Kyle's potential sources of cognitive conflict which allowed a resolution to be found. When he could not verify the convergence in the manner he wanted, I suggested using the fact that 2^n is greater than n for all n, and so then he tried bounding $1/2^n$ by $1/n$ which could then be bounded by ϵ. When trying to prove the convergence of $1/n$, he recalled the Archimedean property, and after trying to find the ϵ using the property, I reminded him that the Archimedean property works for a fixed ϵ and showing that there exists some N that bounds $1/N$ by ϵ. This served to reorient Kyle's stance on the dependence.

By resolving that the N is found for a fixed ϵ, he immediately restructured his construct on not only the convergence of $1/2^n$ but also for all of his previous claims of dependence. This was then reinforced by showing x^n converged for different values of x, which altered the ciritcal index even with a fixed ϵ. After realizing that varying x and ϵ could potentially change the critical index, he claimed that N was doing all the work of convergence, thus reorienting his understanding of the dependence relationships.

Discussion and Conclusion

This episode is meant to serve as an example of the persistence of ϵ dependence within more advanced mathematical settings and the effect it can have on the understanding of more complicated processes than just convergence of real numbers. I argue that because the informal perspective on a limit was so ingrained in Kyle's concept image, it caused him to project the dependence of ϵ on the domain values x as well, even though no relationship should exist at all between ϵ and x. While Kyle could readily recall the formal concept definition of functional convergence, and even reason through a few examples, his personal concept definition dictated the contradictory *epsilon-N-x* relationship that seemed to permeate his concept image.

I contend that the main source of his interpretation of the dependence relationship was his previously held notion that ϵ depended on N for convergent real-number sequences. This demonstrates a lack of attention to the use of quantifiers since he correctly understood the need to fix an x value in evaluating the convergence. His proficiency of fixing x played a vital role in his cognitive re-organization when he observed the convergence of x^n at different fixed values of x and ϵ after the discussion about the Archimedian Property. This suggests a potentially strong instructional point to make is that point-wise convergence of functions is the convergence of a continuum of real-valued sequences all at once.

One final observation is that his understanding of the limiting process was robust enough to adjust upon integrating the Archimedean Property into his concept image. Kyle was secure in the convergence of a few key examples of function sequences so that he could experiment with his process of proving the functions converged to the limit he knew them to have. This case study highlights the complex relationships that exist between the different variables involved in functional convergence and the potential compounding of the *epsilon-N* reversal. An instructional takeaway from this is the importance of original instruction of formal convergence, as well as the need for attention to quantification in the interpretations of formal definitions. This study also serves as a first look into student understanding of convergence beyond that of real numbers.

Questions and further directions

This of course was a first glance at student reasoning with function convergence, and so there is still much to explore in this area. Clearly there is a natural relationship between understanding of convergence of real numbers and convergence of functions. Thus an extension of this study is to introduce function convergence to students with a robust understanding of real number convergence.

The following are for further discussion: 1) Are there ways in which we can investigate the relationship between the types of convergences? 2) Are there ways to specifically target universal quantification that might elicit useful understanding of the need to vary x and *epsilon* in functional convergence? 3) As pointwise convergence is the evaluation of a continuum of real-valued sequences, is there a way to leverage this natural connection to sequential convergence of real numbers in future investigations?

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The Effect of Attending Peer Tutoring on Course Grades in Calculus I

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Tutoring centers are common in universities in the United States, but the effects of tutoring on student success are often not examined statistically. This study utilizes multiple regression analysis to model the effect of tutoring attendance on final course grades in Calculus I. Our model predicted that every three visits to the tutoring center would increase a students' course grade by one percent, after controlling for prior academic ability. We also found that for lower achieving students attending tutoring had a greater impact on final grades.

Keywords: Calculus I, multiple regression, peer tutoring, undergraduate mathematics

Introduction and Literature Review

Although many institutions across the nation offer free tutoring to students (Johnson & Hansen, 2015) with the goal of improving lower division instruction, there have been few published studies investigating the impact of attending tutoring on performance (Xu, Hartman, Uribe & Mencke, 2001). A common approach is to simply look at success rates for students who attended peer tutoring versus success rates for students that did not attend tutoring (e.g. Garcia, Morales & Rivera, 2014; Jimenez, Acuna, Quiero, Lopez & Zahn, 2015). While this may provide some evidence that tutoring had a positive impact, we argue that this type of analysis is over-simplified. Unfortunately, reporting on student performance and retention in an in-depth manner requires resources in terms of staff time and collaboration with other disciplines that many tutoring centers simply do not have (MacGillivray, 2009).

Quantitative measures of the impact of tutoring on grades can be difficult because students tend to self-select and students of different mathematical abilities may attend tutoring for different reasons (Topping, 1996). One way to account for these factors is to use multiple regression. Regression models have been used to show that attending Peer Assisted Learning sessions can improve the grades of mathematics majors (Duah, Croft & Inglis, 2013) and that attending optional tutoring can improve the grades of college algebra students while controlling for students' mathematical abilities (Xu, et. al, 2001). This study will add to the literature by using multiple regression to measure the impact of attending optional tutoring sessions at the Mathematics Learning Success Center on course grades for Calculus I students at Oklahoma State University.

The research questions for the study are:

1. What is the effect of attending optional drop-in tutoring offered by the MLSC at Oklahoma State University on Calculus I students' course grades, after controlling for their high school math GPA and ACT math sub-score?
2. Do we see that attending optional drop-in tutoring at the MLSC benefits students of lower mathematical ability more than stronger students?

Method and Data

To evaluate the effectiveness of mathematics tutoring in calculus, student academic and tutoring data were collected from a public 4-year research university in the Midwest with an enrollment of approximately 25,000 students. Study participants include all 640 students enrolled in Calculus I for the fall 2015 semester. Since attending tutoring sessions is voluntary and students receive no credit for participation in tutoring, self-selection bias is acknowledged as a limitation of this study.

Data collected for the study includes: student course grade (percentage) in Calculus I, high school math grade point average, ACT math score, number of visits to the tutoring center, and duration of visits at the tutoring center. For visits with a missing duration, the average of the student's other visit durations is used as an estimate. Of the 640 students enrolled in Calculus I, there were 390 students who visited the tutoring center a total of 5193 times.

Table 1. Descriptive Statistics

	Mean	Standard Deviation	Minimum	Maximum
Final Course Grade (Percent)	73.93	22.49	3.44	101.46
High School Math GPA	3.51	0.58	1.63	4.75
ACT Math	26.55	3.60	14	36
Visits	8.11	13.75	0	102
Estimated Total Time (Minutes)	738.45	1435.43	0.00	11549.03

Of the 640 students in the study, 390 (60.9%) visited the tutoring center at least one time. For the students who did visit the tutoring center, the average number of visits was 13.3 per student with an average visit length of 78 minutes. These students had slightly higher prior academic achievement scores with an average high school math GPA of 3.58 and math ACT of 26.8 compared to 3.38 and 26.2 respectively for those who did not visit the tutoring center. The average course grade earned for students who attended tutoring was a B (80.4%) while the average for those who did not attend tutoring was a D (62.0%). There were 534 students who completed the final exam and 78 (12.2%) withdrew from the course.

Overall, as the number of times a student visits the tutoring center increases, so does the course grade. However, on average, students with a higher frequency of visits also had a higher high school math GPA, so it is unclear at this point whether the increase in exam score is a result of increased tutoring visits or prior math ability (see Table 2).

Table 2. Scores by Tutoring Visit Category

	Number of Students	Average Final Exam Score	Average High School Math GPA
0	250	62.0	3.38
1-5	169	78.2	3.52
6-10	64	80.3	3.54
11-20	72	82.3	3.65
21+	85	83.3	3.68
Overall	640	73.9	3.51

Results

To investigate the relationship between the variables, Pearson r correlations were computed (see Table 3). The correlation between visits and estimated total time is close to 1 which indicates possible multicollinearity and that likely only one of these variables will be needed in the final model.

Table 3. Correlation Matrix

	Final	High School Math GPA	ACT Math	Visits
High School Math GPA	0.50*			
ACT Math	0.42*	0.42*		
Visits	0.26*	0.15*	-0.03	
Estimated Total Time	0.23*	0.12*	-0.05	0.94*

Note. * significant at $p < 0.05$

Simple linear regressions using each independent variable separately as a predictor of the dependent variable indicate that each independent variable individually is a significant predictor of course grades. The initial multiple regression model will therefore include all four independent variables. Analysis of this model indicates significant overall statistical predictive ability, $F(4, 455)=60.63$, $p<0.0001$. The R^2 of this model is 0.348, which indicates that approximately 34.8% of the variance in course grades can be explained by the predictor variables. In this model, Estimated Total Time has a large p-value of 0.9016 indicating it is unlikely to be a meaningful predictor. Removing this variable and analyzing the subsequent model with predictor variables of high school math GPA, math ACT, and visits results in a significant overall model, $F(3, 485) =85.27$, $p<0.0001$, and an R^2 of 0.345. Each independent variable was found to have a p-value of less than 0.0001 indicating a they are all likely meaningful predictors. The R^2 values indicate the proportion of variance in course grades that can be uniquely accounted for by that predictor variable. Coefficients, correlations and collinearity statistics are found in Table 4.

Table 4. Multiple Linear Regression Results

	Parameter Estimate	P-Value	Partial R^2	Tolerance
Constant	-21.677			
High School Math GPA	12.993	<0.0001	0.13	0.81
ACT Math	1.782	<0.0001	0.10	0.83
Visits	0.333	<0.0001	0.06	0.97

The parameter estimates in Table 4 indicate the change in predicted course grade for each unit change in the predictor variable. As such, the parameter estimate of 0.333 for visits indicates that a student's course grade is predicted to be approximately one percentage point higher for every three visits to the tutoring center. The prediction equation for this regression is: Final Course Grade = -21.677 + 12.993(High School Math GPA) + 1.782(ACT Math)+0.333(Visits).

Interaction

It is also of interest to determine if students of lower mathematical ability, determined by high school math GPA, benefit more from tutoring than students with higher mathematical ability. In Table 5, mean course grades are compared between those who did attend tutoring and those who did not at different categories of high school GPA. From this table, it does appear that lower mathematical ability students benefit more from tutoring with a decreasing difference in mean course grades with increasing mathematical ability.

Table 5. Mean Final Exam Score by HS GPA and Tutoring Participation

HS GPA Category	No Tutoring	1+ Tutoring Visits	Mean Difference
2.00-2.49	25.9	66.6	40.7
2.50-2.99	56.4	72.7	16.3
3.00-3.49	58.1	74.2	16.1
3.50-3.99	71.6	86.4	14.8
4.00+	78.5	87.2	8.7

To investigate this analytically, a regression with an interaction between high school GPA and number of visits is analyzed. The results from this analysis are found below in Table 6. It should be noted that the independent variables in this analysis have been centered to mitigate multicollinearity due to the inclusion of both GPA and Visits and the interaction term between those variables.

Table 6. Multiple Linear Regression Results

	Parameter Estimate	P-Value	Partial R^2	Tolerance
Constant	74.105	<0.0001		
High School Math GPA	12.929	<0.0001	0.13	0.83
ACT Math	1.798	<0.0001	0.10	0.81
Visits	0.365	<0.0001	0.07	0.89
High School Math GPA/Visits Interaction	-0.203	0.0555	0.01	0.93

In this analysis the predictors of high school math GPA, ACT math, Visits and the interaction term all have very low p-values indicating they are likely meaningful predictors of the final course grade. However, it is worth noting that the partial R^2 of the interaction term of 0.01 indicates that the interaction is only able to account for an additional 1% of the variation in course grades over that of the other dependent variables. The overall R^2 of this model is .350 compared to .345 for the model without interaction which indicates that the predictor variables are able to account for 35.0% of the variance in course grades.

The sign of the coefficient of the interaction term suggests that visits for students with lower high school math GPA result in a larger increase in course grade than for students with a higher high school math GPA. The predicted increase in course grade per visit to the tutoring center of a student with a high school math GPA one standard deviation above the mean is 0.25 points. This increase is 0.37 points for students with an average high school math GPA and 0.48 points for students with a high school math GPA one standard deviation below the mean. In other words, students with lower prior math academic achievement see a larger increase in course grade per tutoring visit than students of higher prior math academic achievement. The prediction equation for this regression is: Final Course Grade = -27.752+ 14.576(High School Math GPA) + 1.798(ACT Math)+1.077(Visits)-0.203(Visits*High School Math GPA). The coefficients of this equation have been simplified from the centered form of the equation and therefore differ from those in Table 6.

Discussion

We attempted to control for students' prior mathematical ability by using high school GPA and ACT math sub-score as variables in the multiple regression model. We found that high school math GPA, ACT math sub-score, and the number of tutoring visits were all significant factors in predicting course grades. The model predicts that each visit to the tutoring center raises the student's grade by 0.33%. A student with the mean high school math GPA and the mean ACT math sub-score who does not attend tutoring is predicted to make a 60% in the course. If that same student attends tutoring twice a week for the whole semester (30 visits), the predicted course grade is raised to 70%.

To determine if tutoring attendance benefits low achieving students more than high achieving students, we developed a new regression model that includes the interaction between high school math GPA and tutoring visits. This interaction model has slightly better predictive power than the previous multiple regression model. The model predicts that a low achieving student (with high school math GPA and ACT math score each one standard deviation below the mean) would score 56% without tutoring, and would need to attend tutoring 28 times to raise their score to a passing grade (70%). In contrast, a high achieving student (with high school math GPA and ACT math score each one standard deviation above the mean) would score 86% without tutoring, and 28 visits to the tutoring center would only raise the student's grade 7 percentage points. Thus, for the lower achieving student, each visit to the tutoring center has more of an impact on his or her course grade.

There are several limitations to this study. We acknowledge that the students had the option to attend tutoring, so there is a self-selection bias that we attempted to control by using prior mathematical ability scores in the multiple regression model. We also have no data about whether or not students made use of other support services, such as office hours or independent study groups.

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Mental Constructions Involved in Differentiating a Function to a Function Power

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Functions of the form $f(x) = (g(x))^{h(x)}$*, including constant functions, power functions, and exponential functions, are fundamental examples of functions that differential calculus students should be able to differentiate. Yet students often struggle to distinguish between these forms. Drawing on APOS (Action-Process-Object-Schema) theory as well as Piaget and Garcia's triad of schema development, this paper offers a genetic decomposition of the schemas students build for determining the derivative of a function to a function power. In particular, we analyze how students determine which differentiation rules to use with different function structures of a function to a function power and how students construct a conception of logarithmic differentiation. An initial genetic decomposition informed by existing literature was refined using the results of a series of three clinical interviews with each of two calculus students. Findings include the necessity of a strong background in functions, logarithms, and other differentiation rules.*

Key words: Logarithms, Differentiation, Function, APOS, Schema

Purpose and Background

Logarithmic differentiation is useful when differentiating non-constant functions to non-constant powers. For example, consider the function $y = x^x$. To find the derivative of this function, a student would need to realize that neither the power rule nor the exponential rule applies so the function equation must be transformed into an equation to which standard differentiation rules apply. Taking the natural logarithm of both sides of the equation and implementing a property of logarithms yields $\ln y = \ln x^x = x \ln x$. This function can then be differentiated using standard differentiation rules, resulting in $\frac{dy}{dx} = x^x(\ln x + 1)$. When either the base or exponential function in a function to a function power (FFP), $f(x) = (g(x))^{h(x)}$, is constant you can, in contrast, use the appropriate rule for differentiating polynomial or exponential functions.

Therefore, appropriately carrying out logarithmic differentiation requires students to coordinate all other differentiation rules and properties of logarithms. Additionally, students who know when to apply the technique demonstrate recognition of differences between types of FFPs. Because of this, we felt that differentiating FFPs provided a rich context for studying how students construct and utilize their differentiation rules. In this paper, we examine the necessary mental constructions for students to differentiate FFP expressions and to distinguish between situations where the constant rule, power rule, exponential rule, and logarithmic differentiation are appropriate.

Theoretical Framework

We use both APOS theory (Dubinsky, 1991) and Piaget and Garcia's (1989) triad of schema development to analyze the mental constructions necessary to address problems like the one above. APOS theory is an extension of Piaget's work with reflective abstraction in which student concepts are categorized as Actions, Processes, Objects, or Schemas. An *action* is an external transformation of objects (Asiala, et al., 1996); for example, a student with an action concept of the derivative could directly apply the power rule to find the derivative of $f(x) = x^4$. A *process* is an interiorized action; for example, a student with a process conception of the derivative could find the derivative of $f(x) = (x^3 + 2)^2$ by expanding the binomial and then taking the derivative by using the power rule or by applying the chain rule and power rule. An *object* is an encapsulated process; for example, a student can think about the result of applying the power rule to an appropriate, arbitrary function as producing another function, not just as a series of steps to find a solution. A *schema* coordinates a student's objects and processes; for example, a student could recognize that finding the derivative of $f(x) = (x^3 + 2)^2$ could be accomplished by expanding the binomial and then taking the derivative by using the power rule and could be accomplished by applying the chain rule and power rule, but then decide which was simpler and differentiate accordingly.

Researchers such as Clark et al. (1997) have found categorization using APOS theory difficult in cases where students will be at different levels of abstraction for different elements of study: action concepts of some composite functions and process for others when studying the chain rule. In these cases, the triad of schema development can help look in more detail at how schemas develop (Dubinsky & McDonald, 2001). Three stages, Intra, Inter, and Trans, describe the development and coherence of connections made within a student's schema. At the Intra stage, a student focuses on individual objects rather than looking for connections between them. A student at the Inter stage recognizes some relationships between different actions, processes, objects, and schemas, but cannot explicitly connect all of the relationships. A student at the Trans stage has created a coherent structure that connects appropriate relationships and recognizes what is and is not within the scope of a schema.

Preliminary Genetic Decomposition

Students with only an action conception of function need to substitute specific values into a function and receive outputs to make sense of a function. Students with a process conception of function recognize that a function receives inputs and gives outputs without explicitly needing values with which to calculate, but still consider a function in terms of a dynamic activity (Arnon et al., 2014). A student with an object conception of function recognizes the set of outputs of a function as an entity, recognizes the relationship of the inputs to the outputs, and can distinguish between different types of functions. We hypothesize that an object conception is necessary when differentiating FFPs in order to distinguish between types of functions and choose applicable differentiation rules and techniques.

An action conception of logarithms requires logarithms to be calculated with specific values to have any meaning. At the process level, students can recognize logarithms are functions that obey specific properties and use those properties of logarithms appropriately because they recognize the process that has been applied to compute solutions. For many students, it takes a long time to attain a process understanding of logarithms, as they tend to overgeneralize algebraic rules as they "factor out" logarithms (i.e., $\ln x + \ln y = \ln(x + y)$) or "cancel" logarithms (i.e., $\frac{\ln x}{\ln y} = \frac{x}{y}$), (Liang and Wood, 2005). This process knowledge is necessary for logarithmic differentiation because complex equations involving the chain rule, such as $f(x) = (x^3 + 2)^{4x^3}$, require awareness of what can and cannot be separated using logarithmic properties.

A differentiation scheme at any level of development, from action conception to full schema development, includes many subschemes including a graphing scheme and separate schemes for determining derivatives according to different differentiation rules. In order to address finding the derivative of an FFP, all differentiation rules must be at least at a process level for derivatives requiring more than one rule to be coordinated in a new process, much as Maharaj's 2013 study concluded. However, each differentiation rule need not be viewed as a static object, because coordinating the rules and completing the procedure for finding the derivative could be viewed as a few more steps in a dynamic process.

Finally, students must develop a logarithmic differentiation scheme as a subscheme of their differentiation schemes. The logarithmic differentiation scheme coordinates their function scheme, logarithm scheme, and various differentiation rule subschemes in order to determine a derivative of an FFP when the functions in both the base and exponent are non-constant. Because this scheme must coordinate so many other schemes, decomposing students' mental constructions with the triad of schema development is useful.

Students at the Intra level of logarithmic differentiation would struggle with coordinating the constituent schemes described above, either because they had not developed all of them to a sufficient degree or because they would not recognize that their other differentiation schemes could not be applied to a situation. For example, students may not recall their logarithmic properties and therefore take the natural logarithm of the equation, but then use the power rule to differentiate the resulting logarithm rather than applying properties of logarithms, which was the point of performing logarithmic differentiation (i.e. $y = x^x$ yields $\ln y = \ln x^x$, but then the student incorrectly concludes $\frac{1}{y}\frac{dy}{dx} = \frac{x^{x-1}}{x^x}$). Alternatively, students might be able to perform the steps of logarithmic differentiation but would need to be told to apply it. Another possibility is that they might indiscriminately perform the process to find the derivative of any function, whether useful or not (i.e. determining the derivative of $y = \sin^{-1} x$ through logarithmic differentiation because it appears to contain a power).

Students at the Inter level would have organized the constituent differentiation rule subschemes sufficiently to recognize which functions require logarithmic differentiation. However, they would not recognize why logarithmic differentiation works or why functions with constant base and non-constant power and functions with non-constant base and constant power require different differentiation rules or why logarithmic differentiation might be applied to equations like $f(x) = x\sin(2x^3)\cos(3x^2 + 2)$.

Students at the Trans level would recognize that logarithmic differentiation is useful because it allows previously inaccessible functions to be transformed into functions that can be addressed with other differentiation rules. A secondary use of logarithmic differentiation, simplifying functions requiring multi-part product rule application like $f(x) = x\sin(2x^3)\cos(3x^2 + 2)$, would also be assimilated at this time, though a student may not choose to use logarithmic differentiation in this way. Moreover, students might recognize that changing the position of the constant in FFP situations (i.e. $y = e^x$ versus $y = x^{\mathrm{e}}$) alters the set of outputs of the function, so it also alters the rate of change of the function.

Methods

The participants for this study were two entering freshmen taking a differential calculus course in a six-week summer session. Both students had previously seen differential calculus in high school, participant A through AP Calculus AB and participant B through a pre-calculus course that also included calculus topics. Participants were recruited from the same section.

Each participant engaged in three semi-structured interviews (Fylan, 2005) lasting 30-50 minutes. The interview questions are listed in Figure 1. The first interview occurred after both students had been taught how to find derivatives of constants, power functions, e^x, and to use the product, quotient, and chain rules. The second interview occurred after students had been taught logarithmic differentiation. The last interview occurred within a day of the final exam for the course, approximately three weeks after the second interview. The interviews were video-recorded and participants' written work was collected. To analyze the data, participants' interviews were examined for evidence of the construction of processes, objects, and schemas relevant to developing a schema for logarithmic differentiation.

Interview 1	Interview 2	Interview 3
$f(x) = e^x + 3x$	$f(x) = 5\sin(x^2) + xe^x$	Sketch the graph of the derivative of the given function (which was $f(x) = 2.82x + 1.37$).
$f(x) = 4x^5 - 2e^x$	$f(x) = \ln(2x^2 - 1)$	
$f(x) = x^x$	$f(x) = 4^{2x} - \cos^3 x$	Sketch the graph of the derivative of the given function ($f(x) = 2.9(x + 2.6)^2 - 2.1$).
$f(x) = (x^3 + 1)^2$	$f(x) = x^x$	
$f(x) = \dfrac{7e^{2x}}{e^x}$	$f(x) = \dfrac{\cos x}{x}$	Sketch the graph of the derivative of the given function ($f(x) = e^{-x} + 2$).
$f(x) = e^\pi$		Sketch the graph of the derivative of the given function ($f(x) = 3x^3 + 2x^2 + 1$).
$f(x) = xe^x$	$f(x) = x\sin(2x^3)\cos(3x^2 + 2)$	
$f(x) = \pi x^2$		Sketch the graph of $f(x) = x^x$ and connect to the derivative of the function. (The derivative is $f'(x) = (1 + \ln x)x^x$.)
$f(x) = (x^3 + 2)^{4x^3}$	$f(x) = (x^3 + 2)^{4x^3}$	
$f(x) = (x^3 + 4)\sin x$	$f(x) = \sin^{-1}\left(e^{(1-x)^3}\right) + e^{2.4}$	Differentiate: $f(x) = \pi^2 + 2^x + x^2 + x^{1/x}$

Figure 1: Interview questions

Results

Both participants demonstrated at least an object conception of function; they could fluently distinguish between constant, x^n, and n^x forms when asked to identify what type of function each was. Participant A was also able to identify graphs of quadratic and cubic polynomials and exponential functions, though Participant B struggled to identify the graph of an exponential function. Additionally, when Participant A was asked to find the derivative of $f(x) = x^x$ in the first interview, before learning logarithmic differentiation, he noted similarities to $f(x) = e^x$, but also noted the base x, "was not a constant, or a number. It's just weird."

Participant B displayed a process conception of logarithms. He applied logarithm properties appropriately in all but one problem and actively avoided the error he had made in the second interview in the last interview. Specifically, in the second interview he chose to use logarithmic differentiation on $f(x) = 4^{2x} - \cos^3 x$, which led him to say that $\ln y = 2x \ln 4 - 3\ln(\cos x)$. However, in the last interview, he recognized directly applying logarithmic differentiation to $f(x) = \pi^2 + 2^x + x^2 + x^{1/x}$ "wouldn't help because it's not multiply or divide; it's the sum. They are different [functions] so that's not going to work."

Participant A was less comfortable with the use of logarithms, displaying process level conceptions intermittently. In interview 2, when asked to differentiate $f(x) = \ln(2x^2 - 1)$, he questioned aloud whether or not $f(x) = \ln(2x^2 - 1) = \ln(2x^2) - \ln(1)$. When asked if he could determine if this was true or not, the only method he could determine was to compare their derivatives. While this method worked, he could not construct an argument based on properties of exponents. In interview 3, when asked to differentiate $f(x) = \pi^2 + 2^x + x^2 + x^{1/x}$, he took the natural logarithm of both sides, but wrote each piece separately as $\ln y = \ln \pi^2 + x \ln 2 + 2 \ln x + \frac{1}{x} \ln x$. While he knew some procedures for working with logarithms, he did not display understanding of why the procedures worked.

Both participants had full schema conception for all of their differentiation rule subschemas. They were both very comfortable with differentiating using the product, quotient, chain, and basic differentiation rules. In many instances, they demonstrated multiple ways of approaching problems, including recognizing the derivative of $f(x) = (x^3 + 1)^2$ could be found by using the chain rule directly or by expanding the binomial and then applying the power rule.

Bringing together these constituent subschemas, Participant A had an Inter stage and Participant B had a Trans stage of logarithmic differentiation schema development. By the end of the course, neither participant tried to apply his logarithmic differentiation schema to functions where it was not helpful (like $f(x) = \sin^{-1}\left(e^{(1-x)^3}\right) + e^{2.4}$) and both recognized when it was necessary ($f(x) = x^x$) or helpful ($f(x) = x \sin(2x^3) \cos(3x^2 + 2)$). However, in interview 3, Participant A could not successfully differentiate $f(x) = \pi^2 + 2^x + x^2 + x^{1/x}$ because his logarithm schema was not sufficiently developed, whereas Participant B could and realized why his previous "factoring" of the logarithm was inaccurate.

Discussion

Because both participants struggled at times to process how a logarithm works and what properties it possesses, having a process conception of logarithms does appear to be necessary to address logarithmic differentiation problems. Despite expecting development of a Trans stage schema for logarithmic differentiation to be necessary for students to consider apply logarithmic differentiation to three-part product rule situations, Participant A, who only attained Inter stage, applied this ably. Thus we revise the genetic decomposition, claiming students recognize all possible opportunities to use logarithmic differentiation at the Inter stage, even if they do not understand exactly why it works.

The present genetic decomposition of a logarithmic differentiation schema requires an object level conception of function and a process level conception of logarithms and each differentiation rule. A student's logarithmic differentiation schema then coordinates these subschemas. The level to which a student is able to coordinate these subschemas is characterized as being at the Intra level if the student cannot coordinate them and does not realize when logarithmic differentiation should be performed, at the Inter level if the student recognizes when to apply logarithmic differentiation but does not understand why, and at the Trans level if the student recognizes the underlying structure of the functions and can therefore answer why logarithmic differentiation should be used.

By describing the mental constructions necessary to perform and know when to apply logarithmic differentiation through APOS theory and the triad of schema development, instructors may be better equipped to teach in a manner that encourages the development of students' mental constructions. Specifically, encouraging students to distinguish between different types of functions may assist students in strengthening their function schemas and differentiation rule schemas. This will then enable them to construct a logarithmic differentiation schema and a richer understanding of calculus.

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Pedagogical Practices for Fostering Mathematical Creativity in Proof-Based Courses: Three Case Studies

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Some mathematics education publications highlight the importance of fostering students' mathematical creativity in the undergraduate classroom. However, not many describe explicit instructional methodologies to accomplish this task. The authors attempted to address this gap using a formative assessment tool named the Creativity-in-Progress Rubric (CPR) on Proving. This tool was developed to encourage students to engage in practices that research studies, mathematicians, and students themselves suggest may promote creativity in processes of proving. Three instructors in different institutions used a variety of tasks, assignments, and in-class discussions in their proof-based courses centered around the CPR on Proving to explicitly discuss and foster mathematical creativity. These instructors' actions are explored using Levenson's four teacher roles of fostering mathematical creativity. In this report, preliminary results indicate that each of the three instructors assumed at least three of the four roles.

Keywords: Mathematical creativity, teaching practices, proof-based courses

Mathematical creativity has been emphasized by the MAA's Committee on the Undergraduate Program in Mathematics in its latest guidelines (Schumacher & Siegel, 2015). The guidelines state that "[a] successful major offers a program of courses to gradually and intentionally leads students from basic to advanced levels of critical and analytical thinking, while encouraging creativity and excitement about mathematics" (p. 9). Under *Cognitive Goals and Recommendations*, the guidelines also state "major programs should include activities designed to promote students' progress in learning to approach mathematical problems with curiosity and creativity and persist in the face of difficulties" (p. 10). In support, Nadjafikhah, Yaftian, and Bakhshalizadeh (2012) claim that one of the goals of any educational system should be to foster mathematical creativity. Mathematical creativity is discussed as an important aspect in undergraduate mathematics (e.g., Zazkis & Holton, 2009), but pedagogical actions that support its explicit fostering in classrooms are rarely mentioned or studied. Ervynck (1991) stated, "[W]e therefore see mathematical creativity, so totally neglected in current undergraduate mathematics courses, as a worthy focus of more attention in the teaching of advanced mathematics in the future" (p. 53). In this project, the authors attempt to explore this issue further by attempting to address the research question: ***What teacher actions or practices in the proof-based undergraduate classroom might foster students' perceptions of mathematical creativity?***

To explicate mathematical creativity with undergraduate students, three instructors from different universities in the U.S. implemented various practices such as designing assignments, creating tasks, and structuring class discussions in their courses. One common feature of these practices was that all three instructors centered their implementations around a tool created to enhance research-based actions for mathematical creativity in the proving process, the Creativity-in-Progress Rubric (CPR) on Proving (Savic et al., 2016; Karakok et al., 2016). This rubric was developed considering certain theoretical aspects of mathematical creativity, which is discussed in the following section.

Theoretical Perspective and Background Literature

There are over 100 different definitions of mathematical creativity (Mann, 2006) and multiple theoretical perspectives (Kozbelt, Beghetto, & Runco, 2010). In developing the CPR on Proving, the authors considered mathematical creativity as a *process* that involves different modes of thinking (Balka, 1974) rather than looking at the creative end-product (Runco & Jaeger, 2012). Mathematical creativity in the classroom often considers a *relative perspective*, similar to Liljedahl and Sriraman's (2006) description of mathematical creativity at the K-12 level: a process of offering new solutions or insights that are unexpected for the student, with respect to his/her mathematics background or the problems s/he has seen before. This particular definition acknowledges students' potential for creativity (both in process and product) in the mathematics classroom. Often literature cites this as “little-c” creativity (Beghetto, Kaufman, & Baxter, 2011), as opposed to “big-C” or an absolute perspective (Feldman, Csikszentmihalyi, & Gardner, 1994). Finally, the authors focus on creativity in the domain of mathematics, instead of exploring creative endeavors in general (Torrance, 1966). Many researchers (e.g., Baer, 1998; Milgram, Livne, Kaufman, & Baer, 2005) also stressed this distinction and the importance of domain-specific creativity: “creativity is not only domain-specific, but that it is necessary to define specific ability differences within domains” (Plucker & Zabelina, 2009, p. 6).

Creativity-in-Progress Rubric (CPR) on Proving

The CPR on Proving was rigorously constructed through triangulating research-based rubrics (Rhodes, 2008; Leikin, 2009), existing theoretical frameworks and related studies (Silver, 1997), conducting studies exploring mathematicians’ and students’ views on mathematical creativity (Tang et al., 2015), and investigating students' proving attempts (Savic et al., 2016).

There are two categories of actions that may help a student foster mathematical creativity: *Making Connections* and *Taking Risks*[1]. Making Connections is defined as the ability to connect the proving task with definitions, theorems, multiple representations, or examples from the current course that a student is in or possible prior course experiences. Taking Risks is defined as the ability to actively attempt a proof, demonstrate flexibility in using multiple approaches or techniques, posing questions about reasoning within the attempts, and evaluating those attempts. Making connections has three subcategories (between definitions/theorems, between examples, and between representations), and Taking Risks has four subcategories (tools and tricks, flexibility, posing questions, and evaluation of a proof attempt) that are designed to have students explicitly think about ways to develop aspects of their own mathematically creative processes.

Teaching for Development of Creativity

The literature for teachers’ actions to develop mathematical creativity at the undergraduate level is scarce. Zazkis and Holton (2009) cite a few implicit instances or strategies for encouraging mathematical creativity, including learner-generated examples (Watson & Mason,

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[1] For a final version of the CPR on Proving, see Karakok et al. (2016).

2005) and counterexamples (Koichu, 2008), multiple solutions/proofs (Leikin, 2007), and changing parameters of a mathematical situation (Brown & Walter, 1983). From the K-12 literature, there are quite a number of articles of fostering mathematical creativity through problem posing (e.g., Silver, 1997; Knuth, 2002) or open-ended problems (e.g., Kwon, Park, & Park, 2006). However, there is still a need to understand what teacher actions in the classroom could foster mathematical creativity, especially in undergraduate mathematics courses.

Methods

Participants

Three different instructors, Drs. Eme, X and Omar, from three different institutions (two located in Western US and one located in Northeastern US) participated in this study. Each instructor used the CPR in Proving; Drs. Eme and X removed the word "creativity" from the rubric in an attempt to minimize the explicit influence of the rubric on students' development of ideas on mathematical creativity. They also both implemented IBL teaching pedagogies, whereas Dr. Omar also included lectures. Dr. Eme introduced the CPR on Proving mid-semester in her Transition to Advanced Mathematics course in Spring 2016. Dr. X's implementation was in a seminar on Elementary Number Theory during Fall 2015 where he introduced the rubric in the third week. Dr. Omar implemented the CPR in his Combinatorics course in Spring 2016 and used it in Portfolio Assignments.

Data

Prior to the start of the semester, all instructors discussed their course goals with the researchers and shared their CPR on Proving implementation plans with the researchers. Drs. Eme and X were involved in the development of the CPR on Proving where as Dr. Omar approached the authors to utilize the CPR on Proving in a course that he was designing. All three instructors met with the authors regularly to discuss the process of their utilization of the CPR on Proving throughout the semester. All three instructors collected their students' work and utilization of the CPR on tasks. Dr. Eme also audio-recorded in-class sessions. Students in the three courses were invited to participate in interviews at the end of the semester. In this preliminary report, we share our analysis of notes from instructors' self-reported actions in class and implementation plans, along with recorded implementations of the CPR in their courses.

Three instructors had different ways to introduce the idea of mathematical creativity and implementation of the CPR on Proving. Two months into the course, Dr. Eme started a class period showing the class their own exam solutions: "...That's the exam 2 'solutions' and I say solutions in quotes because they're not all 100% correct, okay, but it doesn't matter. You know there are still really good ideas in there and that's what I want you to see." Dr. Eme had students practice using the CPR on a former student's scratch work in the same class period. The scratch work was for theorem: "If $3|n$, then n is a trapezoidal number (a number that can be decomposed into a sum of two or more consecutive integers)" The discussion below ensues:

1 Dr. Eme: What did you guys get for the first one?
2 Stephanie: Advancing
3 Dr. Eme: Advancing? Why?
4 Stephanie: Because they were able to utilize multiple theorems and
5 definitions…Definition Q, the consecutive integers, Definition test 3.
6 Dr. Eme: Good. Good. Other people agree? Disagree?
7 Tony: Agree.

Dr. Eme: Agree? Ok. How about "between representations?"
…
Cargo: That "The Between Representations" still confuses because I'm not sure exactly what it means? Is it supposed to be like using the notation or what?
Dr. Eme: Yeah, that's a good question. Does anybody have an answer?
Stephanie: I would say it's anyway you can rewrite it, or draw a picture, or anything you can do to represent that same concept but in a different way.
Cargo: OK.
Dr. Eme: That helps?
Cargo: Yeah.
Penny: That "concept" being.... the if $3|n$, let be trapezoidal number or any like any definition?

At another institution, Dr. X asked the students to use the CPR on five occasions throughout the semester, during homework as an evaluative tool, and during the final exam as extra credit. For example, in a homework problem, a student provided some scratch-work in his proof, bracketed the scratch-work, and wrote the reason why this scratch-work was not leading to a correct proof (in the student's words, "a mistake"). The student then promptly proved the theorem, utilizing the evaluation subcategory of the CPR on Proving.

Finally, Dr. Omar utilized the rubric while handing out problems labelled as "portfolio problems," which are, quoting from the syllabus, "much more involved, and the intention is to allow freedom to roam with it in any direction you wish." The students were required to use the rubric in a minimum three-page write up summarizing the proving processes they used. Unbeknownst to the students, many of these portfolio problems were open in mathematics, and the one portfolio problem had the same weight as the other three problems in the assignment which Dr. Omar viewed as "exercises". Dr. Omar stated that these three additional problems could be done by directly implementing ideas from class lectures or discussions.

Analytical Framework

To explore these three instructors' teaching actions, the authors adapted the work of Levenson (2011, 2013), who investigated fifth- and sixth-grade classes with the intention of explicating collective creativity and its effects on an individual's mathematical creativity. She described four teacher roles in fostering mathematical creativity:

1. choosing appropriate tasks,
2. fostering a safe environment where students can challenge norms without fear of repercussion,
3. playing the role of expert participant by providing a breakdown of the mathematics behind a process, and
4. setting the pace, allowing for incubation periods. (Levenson, 2013, p. 273)

The authors conducted preliminary analysis on instructors' actions during implementation of the CPR on Proving in their courses using these four roles. In particular, this preliminary analysis focused on how the CPR on Proving was utilized to foster mathematical creativity in their classroom, and using student interview data to support that fostering occurred.

Preliminary Analysis Results

The authors observed that all three instructors reported utilizing tasks that would encourage the mathematical creativity (property 1). For example, all three instructors choose tasks that were not all “Type 1” tasks, i.e. “proofs…can depend on a previous result in the notes” (Selden & Selden, 2013, p. 320), but rather either “Type 2:” “require formulating and proving a lemma not in the notes, but one that is relatively easy to notice, formulate, and prove” (p. 320) or “Type 3,” which is hard to notice a lemma needed. Moreover, Dr. Omar assigned open-ended tasks, and Drs. Eme and X had their students analyze solutions or proof processes of others as part of classroom discussions, which we view as an action related to choosing appropriate tasks since students were required to think analytically about both a solution and proving process.

The authors noticed from Dr. Eme’s classroom data that this instructor explicitly tried to foster a safe environment (property 2). As seen in the above episode, she carefully stated that the “solutions” the instructor was sharing out were not correct, but contained ideas that were “still really good.” This teacher action challenged the common norm of “only correct solutions should or would be valued” in this course. In addition, this action may also allow the students to challenge norms themselves without repercussion, since their instructor modeled such an action.

In the dialogue above, the students looked at another student’s proof of the trapezoidal number theorem. The authors claim that both the theorem and the evaluation of a student’s proof using the rubric are two examples of choosing appropriate tasks (property 1) for fostering students’ perceptions of creativity. Also, Dr. Eme is setting the pace to allow for incubation periods (property 4) in the course by letting students wrestle with how to interpret the CPR on Proving by analyzing the student’s scratch work (see lines 8-18).

Property 4 is more apparent in the “appropriate tasks” that Dr. Omar assigned during class, since he knew those tasks were open, and therefore necessitated incubation time. The tasks themselves are a form of property 1, since they had the necessary elements for mathematical creativity to occur (through the lens of CPR). That is not to say that mathematical creativity can only occur in open math problems; tasks that both Dr. Eme and X provided can also elicit relative mathematical creativity.

Finally, each of the four properties of teachers fostering mathematical creativity (Levenson, 2013), seemed to appear in different forms in these instructors’ implementations of the CPR on Proving. For example, even though Dr. Omar did not have regular in-class discussions about the CPR on Proving in class, his approach to assignments encouraged the students to experience his “role of expert participant” (property 3).

Discussion/Conclusion

Creativity in mathematics is important for both mathematicians and students' development of mathematical actions. Three instructors of this study used the CPR on Proving differently. However, all three had a shared goal: a recognition or awareness of students' own proving processes and the actions that could lead to the development and enhancement of students’ perceptions of mathematical creativity. Drs. Eme and X had explicit discussions related to rubric categories, which are important to “increase student learning, motivate students, support teachers in understanding and assessing student thinking, shift the mathematical authority from teacher (or textbook) to community” (Cirillo, 2013, p. 1). Discussions can lead to student reflections of their own work. Dr. Omar asked students to utilize the CPR on Proving to reflect on their portfolio problem assignments. The CPR on Proving seemed to facilitate the explicit valuing of such meta-cognitive practice. As Katz and Stupel (2015) stated, “Creative actions might benefit from meta-cognitive skills and vice versa, regarding the knowledge of one’s own cognition and the regulation of the creative process” (p. 69).

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The Saga of Alice Continues: Her Progress With Proof Frameworks Evaporates When She Encounters Unfamiliar Concepts, but Eventually Returns

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This case study continues the story of the development of Alice's proof-writing skills into the second semester. We analyzed the videotapes of her one-on-one sessions working through our inquiry-based transition-to-proof course notes. Our theoretical perspective informed our work and includes the view that proof construction is a sequence of mental, as well as physical, actions. It also includes the use of proof frameworks as a means of initiating a written proof. Previously, we documented Alice's early reluctance to use proof frameworks, followed by her subsequent seeming acceptance of, and proficiency with, them by the end of the first semester (Benkhalti, Selden, & Selden, 2016). However, upon first encountering semigroups, with which she had no prior experience, during the second semester, her proof writing deteriorated, as she coped with understanding the new concepts. But later, she began using proof frameworks again and seemed to regain a sense of self-efficacy.

Keywords: Transition-to-proof, Proof construction, Proof frameworks, Coping, Self-efficacy

This case study reports how Alice, in one-on-one sessions, after adopting the technique of proof frameworks (Selden, Benkhalti, & Selden, 2014) and completing several real analysis proofs, apparently hit a "brick wall" when she encountered a new concept, semigroups, in the second semester. However, after some time spent acquainting herself with semigroup concepts, such as ideals, idempotents, and homomorphisms, she "regained her footing" and started once again to use the technique of proof frameworks. Amongst other things, this case study illustrates the fragility of newly acquired proving skills, in the context of the acquisition of new mathematical concepts. It appears that Alice's proof-writing knowledge was initially, but not permanently, context bound.

Theoretical Perspective

We consider proof construction to be a sequence of mental and physical actions, some of which do not appear in the final written proof text. Such a sequence of actions is related to, and extends, what has been called a "possible construction path" for a proof (Selden & Selden, 2009). For example, suppose that in a partly completed proof, there is an "or" in the hypothesis of a statement yet to be proved: "If A or B, then C." Here, the situation is having to prove this statement. The interpretation is realizing the need to prove C by cases. The action is constructing two independent sub-proofs; one in which one supposes A and proves C, the other in which one supposes B and proves C.

When several similar situations in proof construction are followed by similar actions, an *automated link* may be established between such situations and actions. Subsequently, such a situation can be followed by the corresponding action, without the need for any conscious processing between the two (Selden, McKee, & Selden, 2010). When students are first learning proof construction, many actions, such as the construction of proof frameworks (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995), can become automated. A *proof framework* is determined by just the logical structure of the theorem statement and associated definitions. (For some examples, see Selden, Benkhalti, & Selden, 2014).

Related Research

Proof Frameworks

The idea of proof frameworks was introduced by Selden and Selden (1995), who stated:

> By a *proof framework* we mean a representation of the "top-level" logical structure of a proof, which does not depend on detailed knowledge of the relevant mathematical concepts, but which is rich enough to allow the reconstruction of the statement to be proved or one equivalent to it. A written representation of a proof framework might be a sequence of statements, interspersed with blank spaces, with the potential for being expanded into a proof by additional argument. (p. 129).

Selden and Selden (1995) went on to connect the ability to unpack the logical structure of mathematical statements with the ability to construct proof frameworks and with proof validation. They also pointed out that mental skills were involved (p. 132). The learning and mastering of such mental skills can involve much mental energy and considerable working memory. While Selden and Selden (1995) did not state this explicitly, in their sample validation in the Appendix, they did note that sometimes checking a sufficiently complex part of a proof might overload working memory and potentially lead to error. (p. 146).

Working Memory

It has been said that the "two major components of our cognitive architecture that are critical to [thinking and] learning are long-term memory and working memory" (Kalyuga, 2014). *Working memory* makes cognition possible but has a limited capacity that varies across individuals. It is associated with the conscious processing of information within one's focus of attention. However, working memory can only deal with several units, or chunks, of information at a time, especially when working with novel information (Cowan, 2001; Miller,1956). In contrast, *long-term memory* can be thought of as a learner-organized knowledge base that has essentially unlimited capacity and can be used to help alleviate the limited capacity of working memory (Ericsson & Kintsch, 1995). However, when working memory capacity is overloaded, errors and oversights are likely to occur.

Coping with Mathematical Abstraction and Formality

While the mathematics education research literature does not seem to have considered working memory overload during learning per se, there are a few studies of coping with abstractions. These could be reinterpreted as related to working memory overload causing confusion. For example, Hazzan (1999) investigated how Israeli freshman computer science students, taking their first course in abstract algebra in a "theorem-proof format", coped by "reducing the level of abstraction". Specifically, she found that they tended "to work on a lower level of abstraction than the one in which the concepts are introduced in class." (p. 75).

Leron and Hazzan (1997) pointed out that students in mathematical problem-solving situations “often experience confusion and loss of meaning.” (p. 265), and that students attempt to make sense of the problem situation “in order to better *cope* with it.” (p. 267). While this coping perspective occurs at all levels, they stated that “the phenomena of confusion and loss of meaning are even more pronounced in college mathematics courses.” (p. 282). They also suggested that more work on the coping perspective in mathematics education is needed. Pinto and Tall (1999) also considered university students’ coping mechanisms when confronted with formal definitions and proofs in real analysis. However not many university level mathematics education studies have specifically considered students’ coping perspectives.

Methodology: Conduct of the Study

In the second semester, as in the first, we met regularly for individual 75-minute sessions with Alice, a mature working professional, who wanted to learn how to construct proofs. Alice followed the same course notes previously written for an inquiry-based course used with beginning mathematics graduate students. As previously reported (Benkhalti, Selden, & Selden, 2016), Alice had a good undergraduate background in mathematics from some time ago and also had prior teaching experience. Further, she only worked on proofs, at her own pace, in front of us during the actual times we met. Because of this, we gained greater than normal insight into Alice’s mode of working and its development. Altogether, we met with Alice for 39 sessions.

We met in a small seminar room with blackboards on three sides, and Alice constructed original proofs at the blackboard, eventually using the middle blackboard almost exclusively for her evolving proofs. We videotaped every session and took field notes on what Alice wrote on the three boards, along with her interactions with us. For this study, we reviewed the second semester videos and field notes several times, looking for signs of Alice’s progress. However, somewhat to our surprise, after continued progress with her proof writing when dealing with real analysis proofs, she seemed to hit a “brick wall” upon beginning the abstract algebra (semigroups) section of the notes.

Alice’s Progression Through the Second Semester

At the end of the first semester, we (Benkhalti, Selden, & Selden, 2016) left our 19th meeting with Alice, feeling that she was making great progress, both with writing proof frameworks and with the problem-centered parts of proofs, and was developing a sense of self-efficacy (Bandura, 1994, 1995). She had just completed the proof of the theorem that the sum of continuous functions is continuous. This proof has a rather complicated proof framework that necessitates leaving three blanks spaces -- one for using the hypothesis appropriately, one for specifying a □, and one for showing that the chosen □ “works”, by showing the relevant distance is less than □.

Upon resuming in the second semester, Alice continued proving real analysis theorems, first attempting to prove that the product of two continuous functions, f and g, is continuous[1] in our first three meetings (i.e., our 20th-22nd meetings). She set up the proof framework correctly and explored the situation in scratch work. During this proving process, Alice made some astute

[1] Due to space limitations, we do not provide Alice’s proof.

observations, for example, having gotten to $|fg(x) - fg(a)| = |f(x)g(x) - f(a)g(a)| \leq |f(x)|\ |g(x) - g(a)| + |g(a)|\ |f(x) - f(a)|$, and having dealt with term involving $|g(a)|$, she noted that the former term was the "hard part" because $|f(x)|$, unlike $|g(a)|$, is not a constant. Somewhat later, Alice exhibited some self-monitoring, noting that she needed to move her sentence about the bound on $|f(x)|$, prior to setting δ equal to the "minimum of [the] three" deltas she had found. She also noted, in the 22nd meeting, that it seemed "weird" to write the restrictions on $|f(x) - f(a)|$ and $|g(x) - g(a)|$ without immediately explaining why she had chosen the bounds $\frac{\varepsilon}{2|g(a)|}$ and $\frac{\varepsilon}{2M_f}$, respectively, when applying the definition of continuity to f and g. Next, Alice proved polynomials are continuous with only a small amount of help from us.

By our 25th meeting, Alice had completed the real analysis section of the notes, and was ready to begin the abstract algebra (semigroups) section that starts with the definitions of binary operation and semigroup, followed by requests for examples. She provided only the most obvious of examples, such as the integers under addition or multiplication, and when asked for something "stranger", she said she could use the real numbers. When asked for another "strange" example "with no numbers at all", she suggested union as the binary operation, and with help, wrote up the example of the power set of a set of three elements. Next, when it came to providing examples of semigroups, she suggested the natural numbers with subtraction, but had to be prodded to check associativity; for this she considered (3 - 7) - 2 versus 3 - (7 - 2) and correctly inferred this was not a semigroup. To provide examples of left and right ideals, Alice needed to come up with a noncommutative semigroup, but she drew a blank. We suggested the semigroup of 2×2 real matrices under multiplication, and for an ideal, the subset of matrices of the form $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$. After some calculation, Alice concluded the subset is a right ideal, but not a left ideal.

Alice continued considering examples for the first 35 minutes into the next (26th) meeting, after which she came to the first semigroup theorem to prove: "Let S be a semigroup. Let L be a left ideal of S and R be a right ideal of S. Then $L \cap R \neq \emptyset$." She first wrote the definitions of semigroup, left ideal, right ideal, and ideal on the left-hand board, as she had done many times before. Then she wrote the first-level framework on the middle board, after which she went to the right-hand board and began doing some scratch work, which included drawing a Venn diagram of two overlapping circles, L and R, with an arrow pointing to the intersection. She wrote in her scratch work "$L \cap R = \emptyset$" and "there exists an element $a \in L \cap R$". It seemed that Alice was trying to clarify the theorem statement for herself, however, she had not yet attended to the second-level framework. We pointed this out. During the rest of her proving attempt, we seemed to need to remind Alice of relevant actions, such as considering what she knew about ideals (i.e., that they are nonempty), and hence, concluding that each of L and R contains an element, which she labeled l and r respectively, and using those, to try to "explore" to find an element in $L \cap R$, in order to conclude it was not empty. With our guidance, Alice finished the proof, but her sense of self-efficacy seemed shaken. Indeed, at the next (27th) meeting, Alice wanted to reprove the theorem before continuing. We now feel that she had been somewhat overwhelmed, or confused, by the new content, perhaps causing working memory overload. She had tried to cope as best she could by concentrating on the new concepts, while “forgetting” her prior proof writing skills. Alice’s hesitant behavior continued for eight more sessions.

Then, during the 35th meeting, Alice considered the theorem, “If S and T are semigroups and f: $S \rightarrow T$ is an onto homomorphism and I is an ideal of S, then $f(I)$ is an ideal of T”. She wrote the definition of homomorphism on the left board, wrote “What I know” on the right board, constructed the first-level proof framework, unpacked the conclusion, wrote the second-level proof framework, and decided to do a two-part proof (for left and right ideal). She seemed to have regained “her footing”. At the next meeting, she finished the proof, with some help from us.

Discussion

Alice began the second semester with the construction of additional real analysis proofs and seemed to be making very considerable progress, both with writing proof frameworks and with the harder problem-centered parts of proofs. By the end of the real analysis section of the course notes, we felt that she had developed greatly in her proving ability and had developed a sense of self-efficacy (Bandura, 1994, 1995) about proving. However, the subsequent introduction of unfamiliar, abstract content in the form of several definitions and theorems about semigroups at the 25th meeting seemed to cause her confusion, and she constructed the most obvious examples somewhat hesitantly. Also, when asked to prove the first theorem about semigroups, she did not begin by producing a proof framework, as she had previously done with the real analysis proofs, but rather began writing what she knew or could find in the notes, on the right-hand blackboard. Her proof construction, while not top-down, seemed to consist of first trying to gather as many semigroup ideas as she could, followed by trying to arrange them into a final proof. We feel that concentrating on understanding the unfamiliar abstract content was Alice’s initial way of coping. It was not until the 35th meeting, almost at the end of the second semester, that Alice seemed to have regained her sense of self-efficacy, and she again constructed proofs using the technique of proof frameworks that she had learned and perfected previously.

Implications

It seems that coping with newly introduced abstract concepts is not easy, even for someone as experienced as Alice. It also seems that one cannot expect, having learned the skill of constructing proof frameworks in more familiar settings, that this skill will be easily invoked while new abstract content is being learned, perhaps due to working memory overload. Amongst other things, this case study illustrates the fragility of recently acquired proving skills, in the context of the acquisition of new abstract mathematical concepts. It also suggests the difficulty due to such fragility can be overcome. Our own experiences as mathematicians suggests that one can (tacitly) learn not to be greatly disturbed by the introduction of several new abstract ideas. However, some school curricula avoid certain introductions of concepts, such as the Bourbaki definition of function, because they are considered too abstract (Tabach & Nachlieli, 2015). Further, as Hazzan (1999) found, students sometimes cope by “reducing the level of abstraction.” Yet Alice’s case suggests that, with time and effort, students can learn to cope with abstraction.

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Studying Pre-Service Teachers' Selection of Representations in a Technologically Enhanced Environment

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This paper reports the results of four groups of three pre-service teachers working on a task that had them investigate the fairness of dice. The teachers used an online collaborative environment to sample from six different dice and the environment provided them with various representations, which they used to support their arguments. All four groups preferred the frequency table over bar and pie charts representations. After working on the task, they evaluated the work of students on the same problem. They viewed students' work that used other representations as not convincing regardless of the correctness of their solution and showed preference to the students only using one representation. Implications for pre-service teacher training in statistics and how to promote the use of multiple representations are discussed at the end.

Keywords: statistics education, probability, representations, technology

Statistics and probability are important to understand everyday phenomena. Technologies allow us to efficiently collect, organize, and present our data. Other technologies allow us to simulate phenomena and explore the likelihood of different events. Starting in early grades, students are encouraged to learn concepts of statistics and probability. The National Council of Teachers of Mathematics (NCTM) suggests that students in elementary grades learn how to collect and present some basic data and make decisions using them (NCTM, 2000). Similarly, the Common Core State Standards (CCSS) recommends that students starting from sixth grade to learn about the likelihood of events and to use simulations to generate frequencies of events (CCSS Initiative, 2010). This indicates that mathematics teachers in all levels need to acquire a deep understanding of statistic and probability in order to teach the topic effectively. Though groups such as NCTM have identified the importance of teachers preparing their students to be statistically literate, little is understood about teachers' knowledge of the subject (Jones, Langrall, & Mooney, 2007). Externally provided visual representations such as bar graphs have been shown to aid in the development of student understanding by providing a model of the situation (Schwartz & Black, 1996). By studying the external representations teachers use as evidence in problem solving may provide insight into their knowledge. In this study, we report on the work of four groups of pre-service teachers who collaborated online to solve a probability task and evaluate middle school students' solutions of the same task. The main goal of study is to understand how PSTs use and analyze multiple representations while investigating the fairness of dice and how they evaluate middle school students' justifications for their solutions.

Overview of the Literature

Teaching statistics and probability are disciplines that require different knowledge to that needed to teach mathematics, such as non-mathematical activities like building meaning in data and choosing appropriate study designs (Groth, 2007; Cobb & Moore, 1997). Groth (2007, 2013) developed a hypothetical framework for the knowledge of teaching statistics, acknowledging in the framework the differences that statistics and mathematical knowledge. Due to the difference between the disciplines, investigating teachers' knowledge of statistics and probability is relatively a new direction in mathematics education (Garfield & Ben-Zvi, 2008; Garfield & Ben-Zvi, 2007). Nonetheless, multiple studies investigated the challenges students and teachers face when learning statistics and probability and how to deal with some of these challenges (Garfield & Ben-Zvi, 2007; Groth and Bergner, 2006). Other studies showed that incorporating technological tools in learning and teaching probability can support collaborative learning of probabilistic concepts effectively (Garfield et al., 2008; Lee & Hollebrands, 2011).

Researchers have shown the benefits of eliciting multiple representations when engaging in problem solving tasks (e.g., Shaughnessy, 2007). Computer programs have become useful in statistics education since they can quickly sample from a data set multiple times and provide multiple representations of the data. Beihler (2013) emphasized the benefits of technological environments as a pedagogical tool for learning elementary probability. Ben-Zvi and Arcavi (2001) used virtual environments to study students' use and construction of data representations to understand their way of thinking and found that in these environments students were able to learn in unique ways that promoted understanding compared to traditional environments. While multiple representations are beneficial and computer environments can quickly construct them, a study by Schnell and Prediger (2014) concluded that connecting different representations for students is complex, and if taught properly can foster deep understanding for patterns and variability.

Methodology and Data Source

Twelve pre-service teachers (PSTs) participated in the study. All of the PSTs were in their last semester of a two-year teacher education program at a large Northeastern university. The study consisted of the PSTs working in groups of three to solve three open-ended mathematics problems in a collaborative online environment, Virtual Math Teams with Geogebra Tool (VMTwG). The VMTwG tool integrates Geogebra with a white board and a chat box for synchronous discussion.

During each problem-solving session, the PSTs would meet in their group and work together to solve a task. The goal of the tasks was to have the PSTs not only come to a solution, but to form arguments to justify their solutions. After the PSTs worked on and discussed the task, they then watched together a video of children working on the same task as them. Discussion prompts for the video were provided that had the PSTs contrast and compare their problem-solving experience with those of the children in the video. Finally, individually, the PSTs would look at samples of work of students who worked on the same problem as them. The teachers were asked to review the student's representations and work and specifically address: (1) the correctness of the solution provided, (2) description of the strategy used, (3) the validity of the reasoning, and (4) whether or not they find the solution convincing and, if so, why. If they did not find the solution convincing, they were asked to indicate from studying the student work what pedagogical moves they might take to help the student develop a convincing argument. The data for this paper focus on the probability task which had the PSTs investigate the fairness of different dice using interactive simulations of each die. This module was chosen for analysis since the VMTwG software provided a variety of representations for the data and the PSTs were to choose which representations they wanted to support their argument whereas other modules had them construct their own representations. We were interested in studying which representations they chose for their argument and how it influenced their analysis of student work.

The PSTs were provided with simulations for six dice that were weighted differently. They could roll each dice between 1 and 1,000 times to determine whether the dice are fair. The

VMTwG environment provided them with three different representations of the data; a frequency table, a pie chart, and a bar graph (Figure 1). As a group, the PSTs were to determine which data representations they wanted to use as evidence to make an argument about the fairness of the dice. The video they watched contained a group of eight seventh graders engaged in a debate about how many times one would have to roll the dice to determine if the dice were fair or not. The student work, that was analyzed individually, was a series of charts produced by the same group of students. The charts contained screenshots of the tool, which the students used as evidence to support their conjectures about which set of dice were fair.

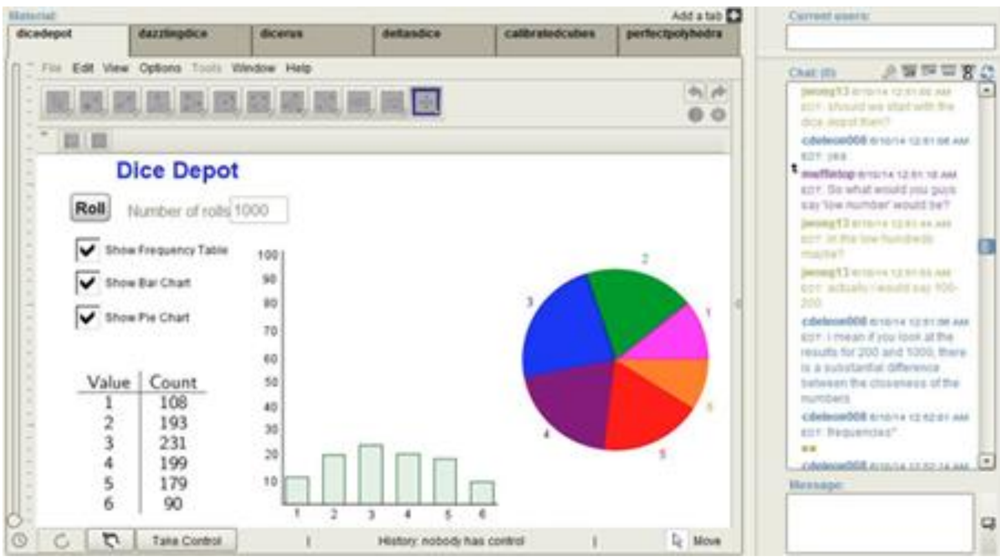

Figure 1. The VMT environment the PSTs worked in.

The VMTwG software contains a replay tool that allows a user to replay all actions that occurred in the tool. For analysis, two researchers openly coded the actions of the teachers inside of the tool, their chat logs, their group write up about which dice company they thought was fair, and their individual analysis of student work. Altogether, four groups of three teachers were analyzed.

Preliminary Results

Analyzing the chat logs and reports submitted by the PSTs revealed that while the groups saw benefits in using the bar graph and pie chart, all four groups chose to use the frequency table as evidence for their selection of dice in their reports. During the problem-solving session, all of the groups chose to use the frequency table to calculate the range of the distribution and used that to determine the fairness of the dice regardless of the sample size they used. The groups discussed selecting the dice that had “less range in the numbers” and dismissed dice with a “much wider range”. One teacher elaborated on her use of the frequency table. She reported that she “used this fraction [one sixth] with the frequency table to compare numbers.” She used the numbers provided in the frequency table to test whether the outcome for each side of the dice is more or less that one sixth of the sample size.

Furthermore, when responding individually to the student work produced by the middle school students, the PSTs indicated that they were not as convinced by the students who used bar graphs and pie charts as their evidence even though the students’ arguments were correct. Several of the PSTs also expressed that the children should only use one type of representations to make their argument and that multiple representations were “not appropriate”, “confusing”, and “made it difficult to decipher [their argument]”. In responding to a student who used both a bar chart and a pie graph, one of the teachers remarked, “The pie graph did not represent this argument as well as the other two types of data because the visual contrast is not as present in the pie as it is in the bar graph”.

Discussion and Questions

Though all four groups of PSTs had multiple representations available to them, all four groups chose to focus on the frequency table to support their own arguments. Their chat logs and write ups of the problem indicate that they found that representation as more convincing. When analyzing students’ work, the PSTs were dismissive of arguments by students who used the other representations. The preference for one type of representation by the PSTs may be problematic as researchers (e.g., Shaughnessy, 2007) have shown that multiple representations are a learning goal for statistics education and that various representations can help aid students in different ways. When the students whose work was evaluated by the PSTs used multiple representations, the PSTs suggested that they only use one to make their argument less confusing. While environments like the VMTwG have the ability to construct multiple representations of the same data, just providing the representations are not enough since it seems that learners tend to prefer and focus on one representation. Further work is needed to understand how to engage PSTs in environments like the VMTwG where they have to consider the affordances and constraints of different representations so they can foster a classroom environment that invites and uses multiple representations to support arguments.

Questions for the audience:

1) Do you see any benefits in having students construct their own representations instead of having them generated by the software? Dynamic representations have shown to be useful in statistics education, but the process of creating a representation seems important.
2) Any feedback or suggestions about the structuring of the study (participants engaging in the task, viewing videos, evaluating student work).

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Mathematicians' Interplay of the Three Worlds of the Derivative and Integral of Complex-valued Functions

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We engaged five research mathematicians in describing their images of differentiation and integration for functions of complex variables. Analyzing the data in terms of Tall's three worlds, we explore the connections between the physical embodiments of the mathematicians' reasoning, their descriptions for students, and their formalizations of these interpretations. For differentiation, the mathematicians relied heavily on direct application of concepts and analogies from differentiation of real-valued functions and employed rotation and stretching as a local linear description of the action of the function, corresponding to repeated mental imagery and physical gestures. For integrals, the mathematicians employed reasoning about line real-valued integrals, but acknowledged that they struggled to conceptually interpret what was being accumulated in the complex case. Instead, they all developed more personal meanings through a process of reconciling various aspects across their own conceptual-embodiment, operational-symbolic, and axiomatic-formal reasoning.

Key words: complex-valued function, derivative, integral, reasoning

Introduction and Literature Review

Authors of complex analysis texts introduce the definition of the derivative of a complex-valued function f at the point z_0 as the complex limit: $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ if it exists. Although this algebraic inscription is introduced as early as chapter 2 of many texts (Brown & Churchill, 2009; Paliouras & Meadows, 1990; Saff & Snider, 2003) authors claim, "geometric interpretations of derivatives of functions of a complex variable are not as immediate as they are for derivatives of functions of a real variable" (Brown & Churchill, 2009, p. 59). Such interpretations are then delayed until after conformal mappings are introduced which generally appear towards the end of the text. Similarly, integration of complex-valued functions is generally introduced using algebraic inscriptions similar to those employed in a multi-variable calculus course. If $w: [a, b] \to \mathbf{C}$ is a complex-valued function of a real variable such that $w(t) = u(t) + iv(t)$, where u and v are are real-valued, then $\int_a^b w(t)dt = \int_a^b u(t)dt + \ i\int_a^b v(t)dt$, provided the integrals exists. A contour integral over a path $\gamma: [a, b] \to \mathbf{C}$ of a function $f: \mathbf{C} \to \mathbf{C}$ is further defined as $\int_\gamma \ f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$. If any geometric representations are provided for these integrals, they are often for the integrals of *u* and *v* in Euclidean space rather than leveraging any complex geometry to visualize *w* or *f* and γ. Given complex variables textbooks do not offer introductory geometric interpretations of either differentiation or integration on the complex plane, it is possible that students enrolled in complex analysis courses may not have opportunities to develop rich visualization of either of these notions. As such, we felt it worthy to explore the research question: What geometric interpretations do mathematicians use to reason about differentiation and integration of complex-valued functions? Such research has the potential to contribute to the creation of design experiments and new curricula as well as to extend the literature on teaching and learning of complex variables which is scarce.

Much of the literature on the teaching and learning of complex variables is limited to exploring undergraduates' and inservice teachers' geometric reasoning of arithmetic and algebraic concepts of complex numbers (Danenhower, 2000, 2006; Harel, 2013; Karakok, Soto-Johnson, & Anderson-Dyben, 2015; Nemirovky, Rasmussen, Sweeney, & Wawro, 2013; Panaoura, Elia, Gagatsis, & Giatlilis, 2006; Soto-Johnson, 2014; Soto-Johnson & Troup, 2015). Much of this work illustrates undergraduates' and inservice teachers' tendency to prefer algebraic reasoning even when geometric reasoning simplifies the task. On the other hand, Nemirovsky et al. demonstrated how undergraduates' geometric reasoning of the arithmetic of complex numbers could be developed through the use of physical activities. These activities entailed using a tiled floor, string, and stick-on dots representing the complex plane, vectors, and points respectively. As a result of these activities the research participants were able to make connections between algebraic and geometric reasoning. Soto-Johnson (2014) and Soto-Johnson and Troup (2015) found similar results with high school students and undergraduates, who used a dynamic geometric environment (DGE) to explore the geometric meaning of the arithmetic of complex numbers. The dynamic movement of the string or dragging elements in the DGE allowed the research participants to reconcile their algebraic and geometric reasoning and to view the arithmetic of complex numbers as transformations on the complex plane.

Some researchers have begun to explore analytical concepts related to complex variables (Soto-Johnson, Hancock, & Oehrtman, 2016; Troup, 2015) but the work in this area remains scant. Troup (2015) showed how DGEs can assist in developing undergraduates' geometric reasoning of the derivative of complex numbers. Specifically, with the aid of technology the research participants abandoned their belief, which stemmed from their calculus experiences, that the derivative of a complex function represents the slope of the tangent line. They also came to recognize the derivative of a complex function as the dynamic notion of an *amplitwist* (Needham, 1997). While Troup focused his work on undergraduates, Soto-Johnson et al. focused their research on mathematicians' conception of continuity of complex-valued functions. Using Schiralli and Sinclair's (2003) framework of *conceptual* and *ideational mathematics*, the authors found that the participating mathematicians' ideational mathematics manifested itself via metaphors which appeared to stem from their prior physical embodied experiences. Furthermore, these metaphors did not tend to fully capture the mathematicians' conceptual mathematics. For example, although the concept of continuity accounts for the co-domain first, several of the metaphors accounted for the domain first and thus had a *domain-first* quality. Given the role that metaphor and embodiment played in the participants' responses, we decided to employ Tall's (2013) Three World of Mathematics Framework (discussed below) in this research project.

Although there is abundant literature on both students' understanding of derivatives and definite integrals in introductory calculus of real variables, little of it touches on interpretations that are likely foundations for generalization to geometric interpretations of the corresponding concepts in complex variables. Neither standard curricula nor research tend to address the derivative of a function from $\mathbf{R} \rightarrow \mathbf{R}$ as a linear map (multiplication by the value of the derivative at a point) in any representation as a potential precursor to the amplitwist idea. Introductory multivariable calculus also does not typically provide such a foundation and instead frames the derivative in terms of partial derivatives or equations of tangent planes. Even more, most texts (quite reasonably) rely on the independence of codomain functions for real-variable derivatives to simplify the situation to multidimensionality only in the domain. Although line integrals provide a possible foundation for contour integrals, little education research has focused on the related concepts. Multiplication in the integrand of a contour integral by a complex-valued $\gamma'(t)$, however, introduces a rotation of the real and complex components of *f* that does not have a convenient real-valued analog.

Theoretical Perspective

Tall (2013), coined the term *met-before* after the word metaphor and defined it as “a structure we have in our brains *now* as a result of experiences we have met before” (p. 23). These met-befores can be supportive or problematic in a new situation, such as in the case of Troup’s (2015) research participants’ initial notion of the complex-valued derivative as the slope of a tangent line. A natural counterpart to met-befores are *set-befores* which consist of “a mental structure that we are born with, which may take a little time to mature as our brains make connections in early life” (p. 23). Tall identified *recognition*, *repetition*, and *language* as the primary set-befores, which promote categorization, encapsulation, and definition which in turn foster mathematical knowledge through three worlds of mathematics.

The three worlds of mathematics are: *conceptual-embodiment*, *operational-symbolic*, and *axiomatic-formal* and though distinct, they are very much interrelated. Tall (2013) explained how the world of conceptual-embodiment is a result of our perceptions and actions which create mental imagery which are perfected via verbal communication. The second world, operational-symbolism, stems from embodied human actions which transform into symbolic procedures that in turn are compressed into procepts (Tall, 2008). The third world, axiomatic-formal builds on formal axiomatic systems whose properties are established through mathematical proof. It is important to note knowledge expressed in the axiomatic-formal world can be conveyed via the conceptual-embodiment world. This is especially evident in teaching abstract notions and this was evident in the work of Soto-Johnson et al. (2016). Given that Tall (2013) clearly states that these three worlds are intertwined, we will seek to document mathematicians’ interplay between the three worlds as they convey their geometric reasoning of the derivative and integral of a complex-valued function.

Methods

This research is part of a larger study where we explored mathematicians’ geometric reasoning of the arithmetic of complex numbers and analytic concepts of complex-valued functions. Five Ph.D. mathematicians from three different institutions, within the same state, participated in a 90-minute, video-taped, structured interview. Becky, Judy and Rafael’s area of expertise was complex analysis, Luke’s area of expertise was differential geometry, and Andrew’s background was in differential equations (names are pseudonyms). All five of the mathematicians have taught complex analysis on a regular basis and four of the mathematicians have published in this area. In this preliminary report, we focus on the participants’ responses to the questions about their interpretation of differentiation and integration. As part of the interview, the participants had access to a black or white board and were encouraged to use it as they saw fit. We also probed in instances where the response seemed unclear or in instances where we knew that the response was not a complete geometric representation of the concept.

Our data analysis is similar to the analysis conducted by Soto-Johnson et al. (2016), including time-lines which depict the interplay of the three worlds. At this time, we have transcribed all the interviews and began using a scheme similar to Soto-Johnson et al., while attempting to tease out conceptual mathematics into either operational-symbolic or axiomatic formal. We also provide rich descriptions of gesture as this is a prominent source of evidence for the conceptual-embodiment world. Below we provide a glimpse of the mathematicians' responses.

Preliminary Results and Discussion

We categorized segments of the mathematicians' responses to the differentiation questions into comparisons to differentiation of real-valued functions, amplitwist models, and a variety of idiosyncratic images. While interpreting the complex derivative, all of the mathematicians relied heavily on met-before comparisons to differentiation of functions from $\mathbf{R} \to \mathbf{R}$, $\mathbf{R}^2 \to \mathbf{R}$, and $\mathbf{R}^2 \to \mathbf{R}^2$. Especially when describing how they explain the ideas to students, they emphasized operational-symbolic reasoning as they stressed the need to write out the complex function in coordinates $f(x+iy) = u(x,y) + iv(x,y)$ and examine the limit definition of the derivative, paths along coordinate axes, partial derivatives, level curves, matrix representations of linear maps, and single and multi-variable Taylor series expansions. Judy, for example, acknowledged the need for real differentiability of f as a function from $\mathbf{R}^2 \to \mathbf{R}^2$, but asserted she believed most people do not understand the geometry of this well. Instead, she consistently pursued approaches involving the separate tangent planes of u and v in her arguments, ultimately attempting to work out the geometric relationship between the tangent planes of the component functions u and v. Luke and Rafael agreed that there was no significant geometric insight to be gained from interpretations of continuous partials implying real differentiability, but both connected images of real-valued linear functions to the complex case. Andrew and Becky resisted drawing geometric interpretations of derivatives when talking about them apart from specific examples.

Four of the five mathematicians relied heavily on geometric interpretations consistent with Needham's description of an amplitwist, drawing several pictures and making consistent and repeated physical gestures of the action. As such, they integrated conceptual-embodied reasoning. All of the participants emphasized aspects of dependence on the domain point, local behavior of the function, linear approximation of this behavior, errors approaching zero "more quickly" than $|z|$, relating to multiplication by $f'(z)$, and decomposing the action into a rotation and a stretch. All of the mathematicians represented these actions algebraically and graphically in Cartesian coordinates, but several either commented that polar representations are more appropriate or deliberately switched to using polar coordinates in their descriptions. They all drew coordinate grids in a domain plane and their images in a codomain plane to illustrate the local stretch and rotation in the mapping. These decisions reflected conceptual-embodied and operational-symbolic reasoning. Repeatedly they would either move their hands from one to the other on the board or an imagined version in space spreading their fingers apart to indicate stretching and twisting their wrist to indicate rotation, often doing so seemingly subconsciously as they were thinking about other tasks. When focusing directly on conveying these ideas, their gestures became more exaggerated, often involving both arms or their whole body.

The responses to the integration task tended to combine various met-befores from real-valued calculus, but for the most part, the mathematicians were uncertain about a clear geometric interpretation for integration of complex-valued functions. For example, Andrew, Luke, and Judy applied the met-before notion of line integrals from multi-variable calculus. Andrew immediately responded that he would not think of integration geometrically and followed up by saying that he would have to think about it. After some pause he started using operational-symbolic reasoning as he wrote $\int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx + udy)$ and indicated that one could easily determine an answer but that he didn't know how to make sense of it. Although, hesitant to interpret such symbolism, he attempted to reconcile operational-symbolic reasoning with conceptual-embodiment of contexts such as work and flux but concluded, "It just means if I would compute the integral this is what I would get. I don't think I have any deep insight. Is there deep insight?" Luke's initial response to the integration question was similar to Andrews, as he stated, "Wow, what does that mean? (long pause). Okay, so this is a hard question. If you're integrating (long pause). This is hard." Although starting with conceptual-embodiment to reason about a path in the plane, Luke shifted to reconciling operational-symbolic reasoning with axiomatic-formal reasoning as he wrote mathematical inscriptions and cited several theorems. He uttered that these theorems would allow him to "get some number," but did not "know what that number means," and concluded, "I don't have a good feel for it."

In contrast, Judy immediately remarked that she teaches integration of complex functions as integrating along the x-axis for the function $f(z) = u(z) + iv(z)$. Thus, she employed operational-symbolic reasoning but transitioned to conceptual-embodiment as she mentioned that "it's geometrically more helpful for students to think about an application like work done along the curve … imagine there's a force field and I'm traveling through it." As she uttered these these words she traced a curve in the air with her hands together. One aspect of Judy's response that was quite different from the other responses was that she discussed tangent planes in conjunction with the Cauchy-Riemann equations and as such appeared to be combining conceptual-embodiment reasoning with axiomatic-formal reasoning.

Rafael's response mimicked Needham's (1997) description the most, but through a creative and well-developed story regarding an early explorer on an ocean ship, who has a route on a map that is incorrect. Rafael mentioned that the route represents the path of integration and in order to determine the correct route one has to break up the incorrect route "into little line segments … [and] each one of these [segments] is a displacement vector." Each of these displacement vectors are then rotated and dilated and all summed in order to determine the true path. Rafael's response combines symbolic-operational and conceptual-embodied reasoning via diagrams that come to life with his gestures.

Discussion and Questions for the Audience

Besides contributing to the literature on the teaching and learning of complex numbers we hope to contribute to Tall's framework. For example, he provides rich theoretical examples that illustrate these three worlds for the derivative and integral of real-valued functions. Our work will provide further empirical evidence but for complex-valued functions. We anticipate this can be adopted for real-valued functions, which might create non-problematic met-befores. We anticipate posing questions to the audience about ways to analyze and present the evolution of the interplay between the mathematicians "three worlds" in time. We will also seek critical feedback on our coding scheme, especially from the community whose research adopts embodied cognition or Tall's Three Worlds of Mathematics.

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Evaluation of Graduate Student Professional Development and Instruction by Mathematics Departments: Results from a National Survey

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Findings from a recent national survey indicate that two thirds of graduate-degree-granting mathematics departments provide some form of teaching-related professional development to their graduate students. Despite the prevalence of such programs, little is known about how departments evaluate the quality of the graduate students' instruction or the efficacy of their professional development. We present a mixed-method analysis of data to shed light on both of these topics. We found that graduate students and their professional development are most often evaluated based on student evaluations. Other research indicates the ineffectiveness of student evaluations as measures of teaching, and so this finding indicates a need for research-guided evaluation tools for graduate student professional development.

Key words: Graduate Student Teaching Assistants, Professional Development, Institutional Change, Evaluation

In the United States (U.S.), graduate student teaching assistants (GTAs) play a large role in undergraduate mathematics education (Belnap & Allred, 2009; Ellis, 2014), though typically have little to no prior teaching experience and receive minimal teaching preparation. It is well documented that more rigorous teaching preparation can result in expert-like beliefs, knowledge, and practices (Alvine, Judson, Schein, & Yoshida, 2007; Hauk et al., 2009; Kung & Speer, 2009; Luft, Kurdziel, Roehrig & Turner, 2004), making up for the lack of teaching experience of graduate student instructors compared to other types of instructors. In particular, a recent national study found the presence of a robust GTA professional development (PD) program to be characteristic of departments with successful undergraduate calculus programs (Ellis, 2015). Given the need for effective preparation of GTAs in teaching, there is also a growing need to understand how these PD programs are, and can be, evaluated.

A primary goal of GTA PD is to help ensure that high quality instruction is provided to undergraduate students. Consequently, it stands to reason that departments would (or could) utilize some measures of teaching effectiveness in the evaluation of their programs. Unfortunately, assessing teaching practices presents many challenges. As noted recently:

> "…even with widespread national investments, education researchers, administrators, and faculty do not yet have shared and accepted ways to describe and measure important aspects of teaching. Developing the language and tools necessary to describe teaching practices in undergraduate education is crucial to achieving productive discussions about improving those practices" (American Association for the Advancement of Science (AAAS), 2013, p. 1).

The work reported on in this Preliminary Report is part of a longer-term effort to understand and support department change related to GTA PD. To help departments improve their GTA PD programs first we need to understand their current evaluation practices and what they take as evidence of effective instruction. Here we present findings from a U.S. national survey of graduate-degree granting (Master's and Ph.D) institutions to answer the following questions: *How are mathematics departments currently evaluating the success of their GTA PD programs?*

What data do departments gather related to teaching effectiveness? Answers to these questions provide insights into the current context. The findings presented here are the first phase of the larger, longer-term effort. These findings and the discussions that will occur as part of this Preliminary Report presentation will inform the design of the next phase that will include case studies and interviews. These sources will provide detailed data to further inform answers to the research questions that are the focus of our current presentation of findings from survey data.

Theoretical Background

As noted by others, "Documenting an existing practice is often the first step in improving it" (AAAS, 2013, p. 4). With our longer-term goals of supporting change to GTA PD programs, our initial efforts are focused on obtaining and analyzing baseline data about current practices of GTA PD program evaluation and assessment of teaching practices.

Typically a PD program for teachers is marked as successful based on a positive change in teachers' knowledge, beliefs, instructional practices, or their students' success (Sowder, 1997), and thus evaluating a PD program often involves evaluating at least one of these measures. A number of researchers have (separately) assessed multiple K-12 PD programs and determined common characteristics of successful ones (Elmore, 2002; Garet, Porter, Desimone, Birman, & Yoon, 2001; Hawley & Valli, 1999; Kilpatrick, Swafford, & Findell, 2001). These characteristics include, but are not limited to, programs occurring over long periods of time, a focus on content-specific understanding and student thinking and an opportunity for enactment of practices through teaching activities. Currently, there exists no comparable set of characteristics identified as common to successful GTA PD programs. We anticipate that frameworks used in these other studies of PD programs can inform our efforts and thus our survey question design was informed by the common characteristics identified by these researchers.

With the long-term goal of analyzing factors that influence how and why departments change (and supporting such efforts), we approach this work with an eye towards *change strategies*. Henderson, Beach, and Finklestein (2010) conducted a large-scale meta-analysis of research on facilitating change in undergraduate science, technology, engineering, and mathematics (STEM) instruction. They found that the least successful change strategies were developing and testing "best practice" curricular materials and then making these materials available to other faculty as well as other "top-down" policy-making meant to influence practices. Successful strategies involved shifting the focus from approaches with exact intended outcomes before implementation to those that acknowledge that the final outcomes will be shaped by the individuals and/or environment involved in the system. To leverage these findings in the context of mathematics departments' GTA PD program change we need rich data and insights into the "system" in which program and teaching practice evaluation occur. That information can then be used to help support departments to pursue what Henderson et al. (2010) characterized as one of the most effective change strategies: seeking to understand the system that one wishes to change and designing a strategy that is compatible with that system.

Methods

A survey was sent to department chairs at all graduate-degree granting mathematics departments in the U.S. (n = 330). The survey was designed to document a variety of features of current departmental efforts including their calculus program, planned changes related to this program, and their GTA PD program.

Department chairs were encouraged to have local departmental experts answer components of the survey with which they were most knowledgeable. For instance, in the case of questions about GTA PD programs, facilitators of the programs would be ideal for answering that section of the survey. The survey was administered using Qualtrics and distributed by the Mathematical Association of America (MAA) with follow up emails and phone calls to encourage participation. Response rate was 68% (n=223) of all institutions, 75% (n=134) of Ph.D.-granting and 59% (n=89) of Master's-granting institutions. For this report we present combined data from Ph.D.- and Master's-granting institutions. Results presented here come from analysis of survey data from the 148 departments (two-thirds of all responding institutions) that reported having a department-specific GTA PD program.

Here we discuss responses to a subset of questions related to GTA PD, as shown in Table 1.

Table 1: Overview of questions used for analysis.

#	Question (and multiple choice options)
3.	Which of the following activities, related to **evaluating** GTAs' teaching, does your program FORMALLY include? Mark all that apply. • GTAs are observed by a faculty member while teaching in the classroom • Student evaluations required by the university or department • Student evaluations are gathered specifically for the purpose of evaluating GTAs (in addition to or separate from the student evaluations required by the university or department)
3e.	• Other (please explain):
4.	How well does your teaching preparation program prepare new GTAs for their roles in the precalculus/calculus sequence? Very well Well Adequately Poorly Very poorly
4e.	• Please elaborate on your answer above.
5.	• Is the department generally satisfied with the effectiveness of the GTA teaching preparation programs currently in place?
5e.	• Yes • The programs are adequate, but could be improved (please explain) • No (please explain)
6.	What best characterizes the current status of your GTA teaching preparation programs? Mark all that apply. • No significant changes are planned • Changes have recently been implemented or are currently being implemented • Possible changes are being discussed

These questions focus on how departments evaluate graduate students in their roles as GTAs, how they assess success of their GTA PD program, and what data they use as evidence in their program assessment. This subset of questions includes both multiple choice questions and open-ended responses questions, asking responders to explain or elaborate their choices to the main questions. In elaborating their selections to the multiple-choice questions, many institutions pointed to specific aspects of their GTA PD program as evidence for their statements. From these responses we are able to gain insight into how departments currently evaluate their GTA PD programs. We conducted basic descriptive analyses on the multiple choice question data and thematic analyses on open-ended responses (Braun & Clarke, 2006). Thematic analysis is a bottom-up qualitative approach, where themes are data-driven.

Results

Analysis of data from Question 3 provides insights into the various ways departments evaluate GTA's teaching. Over 90% of departments with a GTA PD program use university/department-required student evaluations to evaluate their GTAs' teaching, while about three-quarters use teaching observations by faculty members and one-quarter use additional student evaluations that are specific to GTAs. Note that these percentages do not add up to 100% because responders could indicate the use of multiple evaluation methods.

Questions 4-6 provided information on the evaluation of the departments' PD programs. Findings show that 57% of respondents report that their program prepares graduate students well or very well for their roles, 66% of departments are satisfied with their programs, and there are no changes underway at 63% of the schools. This indicates that roughly 40% of graduate degree granting mathematics departments in the U.S. are less than happy with the current state of their GTA PD programs. It is these programs that will be especially in need of good evaluation tools as they move forward with changes to their programs.

Ninety-six respondents provided elaborations for responses to Question 4 (regarding how well the GTA PD program prepares GTAs). Thematic analysis revealed 11 themes in these responses related to what departments use to evaluate their programs. These themes are named and described in Table 2, along with their frequencies. Each department's response was coded with as many themes as were present and appropriate.

Table 2: Description of themes from open-ended responses and their frequencies.

Theme	Description	Frequency
Student evaluations	Department or university student evaluations used as data to rate GTA PD.	7
Prevented from teaching	GTAs are prevented from teaching if they are not already determined to be prepared. This may be based on performance in teaching a lower level class, being a recitation leader, through an interview, or through practice teaching.	6
Compared to others	The GTA PD program is evaluated in comparison to other departments in the same university or other university, the program is	4
Common Exams	GTAs' students' performance on common exams is used as data to rate GTA PD.	4
Student Grades	GTAs' students' course grades (or pass/fail rates) are used as data to rate GTA PD.	4
Observations	GTAs are observed teaching or leading recitation and these observations are used as data to rate GTA PD.	3
Complaints	The amount of complaints about the GTA is used as data to rate GTA PD.	2
Teaching Award	GTAs' receiving Department or University teaching awards is used as data to rate GTA PD.	2
Other	This included alumni surveys, listening to the advice experienced GTAs give to new GTAs, reviews from faculty, retention of students, and student performance in subsequent courses.	4
Too vague	Response included an evaluation of the program with no reference to data, with answers such as "could be improved" or "is a well oiled machine."	48
Description of program only	The response included a description of the program without mention of evaluation or other kinds of data.	28

As shown in Table 2, 76 of the 96 responses were coded as being either a description of the program without information about evaluation, or an indication of evaluation but with no description of the data related to the evaluation. This finding suggests that either program evaluation is not a prominent part of these programs, or the design of the survey question did not provide us access to this information, or both. This indicates that further investigation is necessary to obtain richer and more definitive answers to this question. Of the remaining responses, the most often used data were student evaluations, followed by student performance on common exams, student grades, comparison to other known programs, teaching observations, teaching awards, and complaints. Four responses involved "other" data, including alumni surveys, listening to advice experienced GTAs give to new GTAs, faculty reviews, student retention, and student performance in subsequent courses.

Discussion and Next Steps

Our findings indicate that although the majority of mathematics departments run a PD program for their GTAs, evaluation of these efforts is limited. This is not especially surprising given that it would be somewhat unusual for most mathematics faculty to possess the specialized knowledge and skills needed to engage in program and other types of evaluation. This points to a significant need for our field if we wish to leverage GTA PD to improve the teaching and learning of undergraduates and increase enrollment and retention rates in STEM majors. Of course, this situation is not unique to GTA PD programs and other researchers have noted the unique skill set required for this work: "Institutional and departmental policies affect everyone, yet most investigators researching undergraduate STEM teaching practice lack the tools and expertise to document institutional change" (AAAS, 2013, p. 47).

In addition, the above analysis shows that a primary means of evaluation of graduate students as teachers and of their professional development is student evaluations. An abundance of research indicates issues with using student evaluations as an evaluative tool (Basow, 1995; Centra & Gaubatz, 2000; Krautmann, & Sander, 1999). Influences on student evaluations are diverse and include gender, perceived enthusiasm and other factors. Therefore, it is problematic that student evaluation data is the primary source used to evaluate novice instructors. Relying exclusively on such data may provide incomplete and/or inaccurate information that, in turn, may provide inaccurate information about the GTA PD program's impact on the effectiveness of the instruction being provided to undergraduate students.

The results presented here reveal a number of additional measures that mathematics departments are or could use to evaluate GTA's teaching and GTA PD. These include measures of student performance, such as course grades, grades on common exams, and grades in subsequent courses, as well as more direct measures of teaching performance, such as observations and teaching awards. This finding is an initial insight into what approaches are being pursued in the particular context of a mathematics departments and as such, this information suggests practices to leverage when encouraging and supporting change in other departments. Our planned interview and case study investigations can augment these findings and provide additional insights into what mathematics faculty see as relevant and meaningful ways of examining programs and what they take as evidence of the effectiveness of those programs.

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An Explicit Method for Teaching Generalization to Pre-Service Teachers Using Computer Programming

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As colleagues in a Mathematics/Computer Science department, we found that many of our undergraduates were not able to participate successfully in the full range of STEM course offerings. In response to this need, we developed a strategy for explicit instruction in mathematical generalization. Our instructional design is grounded in a theory of mathematical learning that uses computer programming to induce students to build the mental frameworks needed for understanding a math concept. The design includes writing mini programs to explore a mathematical concept, finding general expressions in the code, making conjectures about the relationships among general expressions, and writing logical arguments for the conjectures. We share results from a study of 18 undergraduate math/secondary education majors. Our results indicate most pre-service teachers showed improvement in their level of abstraction over the concept of direct variation.

Key words: Generalization, Computer programming, Pre-Service teachers

Introduction

Abstraction and generalization are critical skills for navigation through the computer science and mathematics curriculum. Any effort to improve instruction should take into consideration how students learn. It is widely believed that teaching proof writing as a solitary activity may not provide students the building blocks to become proficient in reasoning about mathematical concepts. Stylianides claims that the four essential components of 'reasoning-and-proving' are 'identifying patterns, making conjectures, providing non-proof arguments and providing proofs' (Stylianides, 2008). Jenkins, et al., developed an explicit approach to teaching abstraction and generalization that considers the mental processes by which abstract concepts are acquired and utilized (2012). In developing the instructional treatment, close attention was paid to the theory of learning called APOS theory (Dubinsky, 1984). APOS is an acronym that stands for Action, Process, Object, and Schema. Each level denotes a cognitive classification of the learner's conception. APOS theory is an outgrowth of Piaget's theory of Reflective Abstraction (Piaget, 1971). As a constructivist theory, the basic tenet of APOS theory is that an individual's understanding of a mathematical topic develops through reflecting on problems and their solutions in a social context and constructing or reconstructing certain mental structures, then organizing these mental structures into schemas to use in dealing with problem situations. Specifically, in APOS theory, the process of reflective abstraction is the key to cognitive construction of logico-mathematical concepts (Dubinsky & McDonald, 2001, Beth & Piaget, 1966).

Programming as a Vehicle for Building Abstraction in the Mind of the Learner

Researchers in APOS theory have long employed computer programming as a means to teach undergraduate mathematics. In numerous studies, spanning several countries, and applied to a spectrum of mathematical topics, APOS theory has been applied to the use of computer experiences to encourage the construction of mental processes that lead to mathematical concepts (Asiala et al., 1998, Weller et al., 2008). The computer treatments have consistently yielded an increase in likelihood that students acquired the desired concepts. It is a commonly held belief among this substantial group of researchers that computer constructions are an intermediary between concrete objects and abstract entities (Dubinksy, 1997, Asiala et al., 1998).

Instructional Treatment: Building Mental Structures with Computer Programming

Applying this theoretical framework, an instructional treatment was developed using computer programs to push students to build the mental frameworks for abstraction and generalization. This instructional treatment consists of four stages as shown in Figure 1. In the first stage essential characteristics (ESS) of a problem are identified. Next, mini programs (PROG) are written to explore the essential characteristics. General expressions (GEN) are found in the programs. Participants are taught to write these generalizations as mathematical statements. Further exploration with computer programs (PROG) leads to more generalizations, and general expressions (GEN) are collected as participants conjecture about relationships between concepts. Participants are taught to write convincing arguments (CA) for some of the conjectures, using the general expressions (GEN).

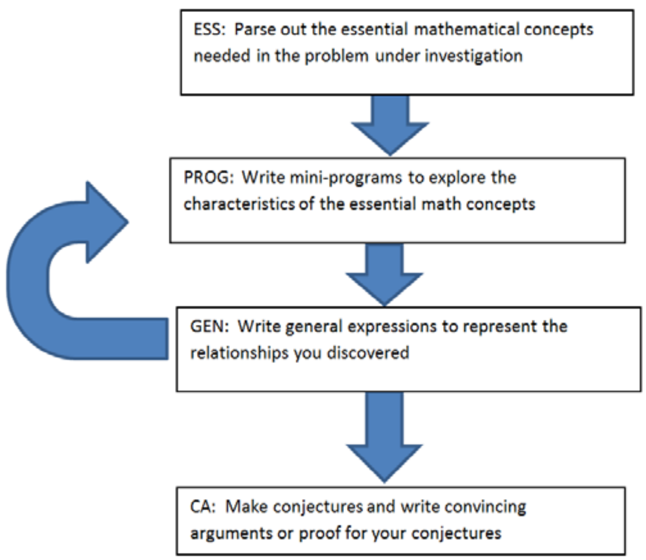


Figure 1. The four stages of the instructional treatment

Methodology

The participants were 18 undergraduate math/secondary education majors who were enrolled in a mathematical methods class. At this regional state university preservice teachers earn a bachelor of science in education and, in addition, complete all of the coursework for a mathematics major. This course is offered the fall semester before their internship.

Procedure Description

The instruction took place over a three day period. Each class session was 75 minutes. The format for the lessons included writing mini-programs using Python to explore concepts and make conjectures about relationships among concepts. For example, participants were asked to produce a table with columns consisting of distance, rate and time, then insert additional columns to show what happens when the rate is doubled while time is unchanged. From program output, they observe that distance doubled as rate doubled. They were taught to find the general expression for this relationship in their code and then to write it in mathematical language. After sufficient time exploring the relationship between increasing rate and resulting distance, they were led to make conjectures about relationships between distance, rate and time, in general. For example, they might conjecture that when the rate was multiplied by *k*, then the resulting distance was *k* times the original distance or "If $r_2=kr_1$, then $d_2=kd_1$." This was followed by instruction constructing convincing arguments for some of the conjectures. The proof writing activity was designed to push students to progress to the next level of cognition by affording them the opportunity to apply their conceptual knowledge in a different setting.

Data Collection/Analysis

All participants were pre-tested and post-tested to determine their level of abstraction for the concept being explored. Throughout the lessons, response sheets were also collected. This allowed the conceptual knowledge to be evaluated at multiple points to determine how mental frameworks were being built. Based on this data, each participant was assigned scores representing their level of abstraction over the concept before and after each lesson.

APOS analysis was used to assign level of abstraction demonstrated by a participant. Scores ranged from 0 to 3 representing No abstraction, Action, Process, or Object. Each entry was scored by at least three trained data analysts with a rubric developed for that particular concept and based on responses elicited on the pre- and post-tests and the response worksheets. Triangulated scores were tested for inter-rater reliability using Randolph's Kappa and assigned to each participant for each lesson, before and after instruction.

Results

Table 1 lists the item text associated with each scored response on the pre-test, response sheet, and post-test. Table 2 shows student scores for responses on the pre-test, participant response sheets, and post-test. Participant U0065 was not viable because they did not participate on the second day of the study. Twelve of the eighteen participants improved at least one level of abstraction in the direct variation lesson.

Response Item	Item Text
Pre-test 2,3	What happens to distance when time is fixed and you triple the rate? Write a convincing argument.
Computer 3,4	What happens to distance when time is fixed and you double the rate? Write a convincing argument.
Math 7,8	What happens to distance when time is fixed and rate is cut in half? Write a convincing argument.
Math 9,10	What happens to the distance when time is fixed and rate decreases? Write a convincing argument.
Post-test 2,3	What happens to distance when time is fixed and you triple the rate? Write a convincing argument.

Table 1. Question text on pre-test, response sheet, and post-test

Participant Code	Pre-test 2,3	Computer 3,4	Math 7,8	Math 9,10	Post-test 2,3	Change
U0018	1	1	3	3	3	2
U0064	0	1	3	3	2	2
U0065	0	1	N/A	N/A	1	1
U0066	0	1	3	3	3	3
U0067	1	1	3	3	3	2
U0068	1	1	3	3	3	2
U0069	2	1	3	3	2	0
U0070	1	1	3	3	2	1
U0071	1	1	3	3	1	0
U0033	0	0	2	3	0	0
U0038	0	0	3	3	2	2
U0041	0	0	2	3	2	2
U0042	1	1	2	3	3	2
U0045	0	0	2	3	2	2
U0047	0	1	2	3	2	2
U0048	0	0	2	3	3	3
U0049	0	1	2	3	2	2
U0051	2	1	2	3	2	0

Table 2. Scored participant responses on pre-test, response sheet, and post-test

The following snips from participant U0068 are representative of how the students' convincing arguments improved over the course of the instruction.

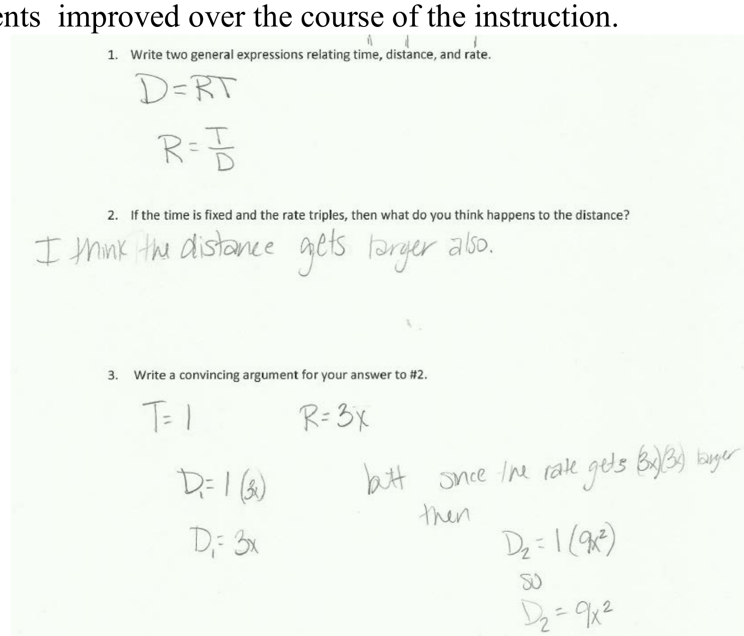
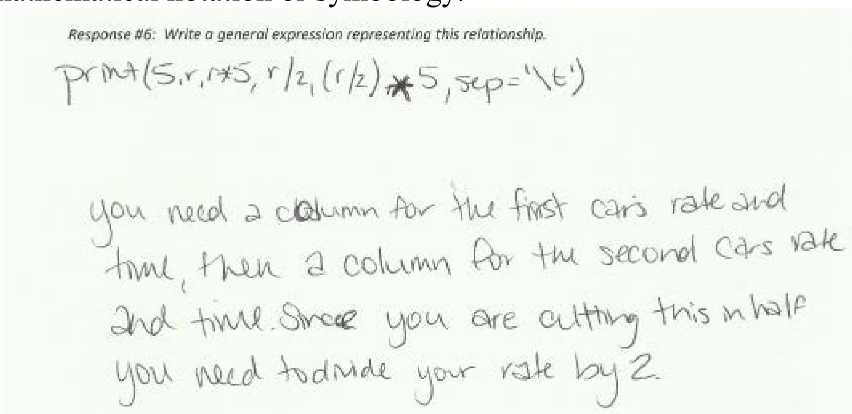


Figure 2. Pretest for participant U0068

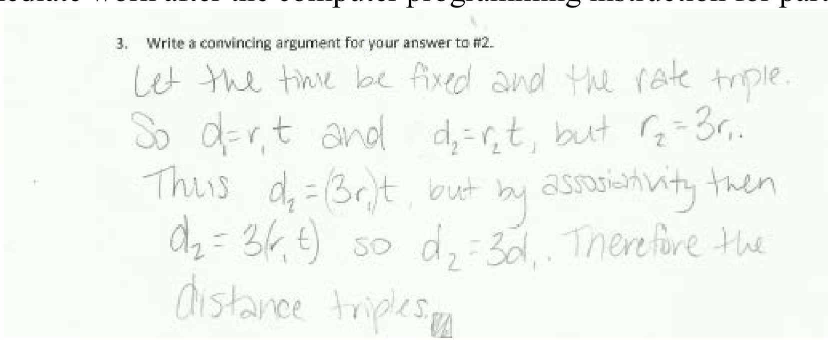
In the pretest, participant U0068 was able to recall the correct formulas to express the relationship between distance, rate and time. They had a correct "intuition" concerning the result, "distance gets larger also" (Figure 2). After the computer programming instruction, participant U0068 could state the effect of the increase specifically, rather than saying "the distance gets larger". When asked to give a general expression for the relationship, they referenced their computer code. In addition, they wrote a description of the output of the code and described the relationship in terms of the columns of data produced as output (Figure 3). The mathematics portion of the lesson taught students to find general expressions in their code and write them in mathematical notation or symbology.


Response #6: Write a general expression representing this relationship.

```
print(5,r,r*5, r/2,(r/2)*5, sep='\t')
```

you need a column for the first cars rate and time, then a column for the second cars rate and time. Since you are cutting this in half you need to divide your rate by 2.

Figure 3. Intermediate work after the computer programming instruction for participant U0068


3. Write a convincing argument for your answer to #2.

Let the time be fixed and the rate triple. So $d_1=r_1t$ and $d_2=r_2t$, but $r_2=3r_1$. Thus $d_2=(3r_1)t$, but by associativity then $d_2=3(r_1t)$ so $d_2=3d_1$. Therefore the distance triples.

Figure 4. Post test participant U0068

In the post-test, participant U0068 gave a convincing argument that used general expressions to express the relationship between distance, rate and time (Figure 4).

Conclusion

In this study we have described an explicit method for teaching generalization. We have reported results from a pre-service teachers in a mathematical methods course. We have shown that most participants' level of abstraction or ability to apply generalizations for direct variation increased. This is a strong indication that generalization can be taught explicitly. It suggests that further research into computer programming as an effective tool for teaching mathematical thinking is warranted.

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The Use of NCTM Articles as Reading Assignments to Motivate Prospective Elementary Teacher Engagement in Mathematics Courses

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In this study, we examine the use of assigning articles published in NCTM's practitioner journals as readings in mathematics content courses for prospective elementary teachers (PTs). In particular, we study the articles' roles in motivating PTs to engage in their content courses. As a conceptual foundation, we characterize NCTM articles as having potential to (1) increase PTs' "buy-in" of pedagogical approaches used in content courses, (2) challenge PTs' unproductive beliefs about mathematics, and (3) address mathematics content via children's thinking. We plan to analyze an existing dataset of PTs' online typed responses to assigned NCTM articles to identify whether and how their responses reflect increased motivation to engage in their content courses. We anticipate that our results will lead to an increased understanding of PTs' actual experiences related to the assigned article readings.

Key words: Prospective elementary teachers, subject matter knowledge, student motivation

Elementary teachers must have deep *subject matter knowledge* (Hill, et al., 2008) in order to teach mathematics well (Ball, 1990; Conference Board of the Mathematical Sciences, 2001; Kilpatrick, Swafford, & Findell, 2001; Ma, 1999). Thus it is of critical importance that PTs engage fully in their preparatory mathematics content courses in order to maximize their learning. This is particularly true in light of the fact that prospective elementary teachers (PTs) in the U.S. typically enter their content courses with limited subject matter knowledge (Ball, 1990; Ma 1999; Thanheiser, 2009; Thanheiser et al., 2014). And yet, it has been shown that PTs do not enter their content courses motivated and ready to learn (Kurz & Kokic, 2011; Moyer & Husman, 2006; Philipp et al., 2007; Thanheiser, 2009; Thanheiser et al., 2013).

Accordingly, mathematics teacher educators have developed pedagogical techniques and interventions that work to motivate PTs to engage in their content courses (Philipp et al., 2007; Thanheiser, 2009; Thanheiser et al., 2013; Thanheiser & Jansen, 2016). A recent example of one such intervention is the use of one-on-one content-based interviews that instructors can implement with their PT students (Thanheiser et al., 2013). Thanheiser and her colleagues showed that interviews can motivate PTs to learn by changing their beliefs about mathematics, and by increasing their awareness of their own unpreparedness to teach mathematics.

In this study, we investigate ways in which having PTs read and reflect on articles from NCTM's practitioner journals (e.g., *Teaching Children Mathematics* and *Mathematics Teaching in the Middle School*) might also motivate PTs to learn mathematics, according to the perspectives of the PTs themselves. As instructors of mathematics content courses for PTs, we find these reading assignments to be indispensable. Prior to this study, though, we know of no research specifically examining the use of NCTM articles in PT content courses and how such article assignments might support PTs in engaging in their content courses, promoting their development of subject matter knowledge for teaching mathematics. Our work addresses this, and moreover, our work illuminates PTs' own experiences and reactions to NCTM articles.

Conceptual Framework: Three Types of NCTM Articles to Motivate PTs' Learning

Below, we introduce three types of NCTM articles according to how we, as instructors of PT content courses, believe the articles boost PTs' motivation to learn mathematics. Further, we point to previous research literature to support these notions. Note that our discussion of each type of article is not intended to imply that every NCTM article can be categorized as being only one type. In other words, we see these three article types are interconnected, and indeed, many articles are two or three types simultaneously.

Type 1: Potential to increase PTs' "buy-in" of our pedagogical approaches

In our content courses for PTs, we strive to develop a collaborative, discussion-heavy learning environment that is built upon PTs' own ideas (Thanheiser, Browning, Moss, Watanabe, & Garza-Kling, 2010), and that fosters an orientation towards mathematics as a sense-making and problem-solving activity (Lampert, 2001). We do so not only to model for PTs how we hope they will someday conduct their own classrooms, but also because we believe this type of environment is necessary for PTs' own learning of mathematics. However, such an environment can differ drastically from the types of environments PTs experienced as K-12 students (Beswick, 2005; Comiti & Ball, 1996). For example, we ask PTs to share their thinking often in class, yet one study recently found that 85% of PTs had little to no experience being asked to share their thinking during any previous mathematics class prior to taking their first PT content course (Thanheiser & Jansen, 2016).

Because PTs hold preconceived notions about the teaching and learning of mathematics based on their own prior experiences as learners (Anderson, White, & Sullivan, 2005; Charalambous, Philippou, & Kyriakides, 2008), they are often skeptical about the unfamiliar style of teaching they are experiencing in their content courses. We believe NCTM articles that discuss the potential effectiveness of certain pedagogical approaches can help PTs make the transition to participating in class in ways that are new to them.

An example of this type of article is "Techniques for Small Group Discourse" (Kilic, et al., 2010). The authors open this article with a brief discussion of the importance of having children engage in discourse in a mathematics class. Then, the authors present two scenarios of teachers engaging elementary students in discourse, and they examine how the teachers' facilitation of student discourse promoted students' reasoning about the content. Although the intended audience of this article is practicing elementary teachers, we believe that having our PT students read about a classroom of children who deepened their understanding of a mathematical concept through dialogue might motivate them to similarly engage in dialogue during their content courses. We believe articles like this one can help PTs understand the rationale behind the teaching methods that we use in class, potentially decreasing their skepticism and increasing their willingness to engage with them.

Type 2: Potential to challenge PTs' beliefs about mathematics

A second type of article serves to challenge unproductive beliefs that PTs commonly have about mathematics (Philipp et al., 2007). For example, many PTs believe that mathematics consists entirely set of procedures or rules to be memorized (e.g., Ball, 1990; Thanheiser, 2009). Such PTs might not see the value in understanding why the procedures make sense, or that there is more than one correct way to solve a problem, thus potentially shutting down their motivation to learn the content, that they believe they already know, more deeply. Articles that push against PTs' commonly-held beliefs about mathematics might help them be more open to learning.

An example of this type of article is "Multicultural mathematics and alternative algorithms" (Philipp, 1996). This article opens with a child's invented algorithm for long division, introducing the idea that there is more than one way to solve a mathematics problem and that even young children can invent effective mathematical procedures. Then, the article presents and discusses examples of algorithms for multi-digit computation used around the world. None of the algorithms are the same as the standard algorithms traditionally taught in the United States. We believe that having our PT students read this article might help them recognize and re-think their beliefs centered on the idea that mathematics consists of one fixed set of procedures developed by somebody else. Additionally, this article might help challenge PTs' belief that the procedure that they are most familiar with is the only one, is used by everyone worldwide, and is the easiest. Our intention is that articles of this type help PTs think through beliefs that might restrict their motivation to learn mathematics that is new to them.

Type 3: Potential to address mathematics content via children's thinking

Prior research has pointed to the fact that PTs' analysis of children's mathematical thinking can be a motivator for PTs' own learning of content. For example, the use of artifacts of children's thinking (e.g., videos of children doing mathematics, or children's written work) has been linked to gains in PTs' mathematical understanding (Jacobs, Lamb, & Philipp, 2010). Our use of NCTM articles that discuss the mathematical details of children's ideas and strategies follows directly from this work. We believe that PTs are motivated to understand the mathematics more deeply themselves when they are confronted with artifacts of real children's mathematical thinking (e.g., via an article discussing a child's invented algorithm for long division).

An example of this type of article is "Tuheen's thinking about place value" (Wickett, 2009). This short article gives an account of a third-grader, Tuheen, who suggests that the regrouped "1"s in the multi-digit addition problem 59 + 67 be written as "10" and "100" instead of "1"s, to show their true value. The author then discusses how Tuheen's insight seemed to help strengthen his peers' understandings of place value in multi-digit addition. Because many PTs do not know the true value of the regrouped "1"s when they enter their content courses themselves (Thanheiser, 2009), we assign this article with the intention of helping them deepen their own understanding of place value and see that children can and do make sense of mathematics and come up with their own algorithms. As PTs work to follow the account of Tuheen's mathematical reasoning in the article, we believe they strengthen their own mathematical reasoning as well.

Research Questions

The above potential benefits of assigning NCTM articles are derived from our perceptions as teacher educators and may or may not be actualized, according to the perspectives and experiences of our students. And so we wonder whether there is evidence within PTs' written reactions to articles to support that they are, in fact, serving to motivate PTs' learning of mathematics. Further, we wonder what PTs' article responses tell us about the ways in which the ideas in the articles support their learning of mathematics. Therefore, in this study, we specifically ask the questions:

(1) Do PTs experience increased motivation to engage with the course material in mathematics content courses as a result of reading NCTM articles, according to PTs' responses to the articles?

(2) If so, how do PTs experience increased motivation to engage with the course material in mathematics content courses by reading NCTM articles, according to their responses?

Methods

Participants

Participants in this study were 42 PTs enrolled in the first course in a sequence of three quarter-long mathematics content courses for undergraduate PTs at a large, urban state university in the Pacific Northwest of the United States. The content of the course focused on whole numbers and operations. Thirteen of the participants were enrolled in the course in the Summer, and the class met for 140 minutes every weekday for three weeks (equivalent to four credits in one quarter). Twenty-nine of the participants were enrolled in the course in the Fall, and the class met for 110 minutes twice per week for ten weeks, which is also equivalent to four credits in one quarter.

Data Collection

As part of the homework assignments for the course, students were asked to read NCTM articles and respond online via a discussion forum in Desire2Learn (a course platform similar to Blackboard). Specifically, students were asked to: (1) type and post their initial summary and reaction to the article, and then (2) respond to at least one other person's post. There was no specific requirement for post lengths, yet the students were asked to make their responses substantive enough to convince the instructor that they read the article.

The dataset for the analyses below consists of all students' first summary/reaction discussion forum posts for each of three articles (one article representative of each type introduced above). Specifically the three articles are: "Techniques for Small Group Discourse" (Kilic et al., 2010, representative of a Type 1 article); Beliefs About Mathematics for "Multicultural Mathematics and Alternative Algorithms (Philipp, 1996, representative of a Type 2 article); and Content via Children's Thinking "Tuheen's Thinking About Place Value" (Wickett, 2009, representative of a Type 3 article).

Data Analysis Plan

This study is ongoing. At the time of this preliminary report, all data has been collected and analyses are in beginning stages, according to the plan outlined below.

In our first research question, we wonder whether the intended reasons for assigning NCTM articles appear within PTs' reactions to the articles. For example, for a Type 1 article, we ask, "Is there evidence to suggest that PTs actually experience increased buy-in to our pedagogical approaches, in response to reading this article?" Accordingly, all PTs' initial responses to each of three articles (one representative of each type) will be coded "yes" or "no", according to whether they made a statement relating to the type of the article. The percentage of "yes" and "no" responses will be reported. Further, if a response is coded "yes," the section(s) of the PTs' response that addressed the purpose of the article will be pulled out for analysis pertaining to the second research question. Figure 1 shows sample excerpts from our dataset that would be coded "yes" for each type of article.

Type of Article	Sample "Yes" Excerpt from a PTs' Response
Type 1: Potential to increase PTs' "buy-in" of our	"While the article primarily addresses the teacher's role in small-group facilitation, it provided me with an opportunity to reflect on my own experiences of working in small groups. The math classes that I have

pedagogical approaches	taken in high school were primarily lecture format, so working and collaborating with my peers is a new experience. Although I was tentative at first, I enjoy the collaborative atmosphere.”
Type 2: Potential to challenge PTs’ beliefs about mathematics	“Before reading the authors final comments I was thinking the alternative methods introduced in the article seemed much harder than the methods I was taught as a youngster but then the author recognized the fact that “people have the tendency to believe that the algorithm they use is easiest, regardless of what it is." I was shaking my head yes and then revisited the alternative methods with a more open mind.”
Type 3: Potential to address mathematics content via children’s thinking	“Wow! I never would have thought to do math that way, how very interesting. I think that that is a good way to do math in the beginning to show them what it means to carry and that the number (lets say 1) is actually a ten or hundred when carried over to that column.”

Figure 1. Sample PT responses that will be coded “yes” as evidence of the PT experiencing increased motivation, for each type of article.

In our second research question, we wonder how each type of article might motivate PTs to learn mathematical content, according to their own perspectives (as opposed to our own notions about how this might happen). For example, for a Type 1 article, we ask “In what ways (if any) do PTs’ responses suggest they are experiencing increased buy-in to our pedagogical approaches?” Here our goal will be to identify patterns in PTs’ open-ended responses to articles, and so we will use thematic analysis (Braun & Clarke, 2006) on the marked, relevant sections identified in our analyses for the first research question. Specifically, we will identify themes that illuminate the PTs’ perspectives on how an article’s intended purpose connects to their own motivation to learn. In this way, our use of thematic analysis will be focused on developing a deeper understanding of PTs’ perspectives pertaining to themes, using a narrow lens. This is in contrast to a more broad application of thematic analysis that would seek to capture all themes within the entirety of the PTs’ responses.

Discussion Questions for the RUME 2017 Audience

Because this is a preliminary report, we encourage feedback from the RUME 2017 audience to help shape the future directions of our work, as well as clarifying the community-wide implications of our findings. We specifically plan to pose the following questions to inspire discussion:

1) In our report, we introduce three types of NCTM articles according to the way in which they motivate PTs’ learning of mathematics content. What other “types” of NCTM articles might we consider including in future research?

2) For teachers of elementary mathematics content courses for PTs: What implications (if any) do the themes within our PTs’ responses have with respect to your own teaching?

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Analysis of Teachers' Conceptions of Variation

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The CCSSM emphasize statistical concepts for grades 6-12. A key factor in thinking statistically is to reason about variation and variability. This paper will present the analysis of survey questions and tasks given to in-service middle school teachers. The paper will attempt to answer the following question: "To what extent do middle school math teachers consider variation and variability when thinking about statistics and reasoning through statistical tasks?"

Key words: Statistics, Professional Development, Teacher Education

Organizations such as the American Statistical Association [ASA] (GAISE, 2005) and researchers (Gal, 2003; Gould, 2004; Davidian & Louis, 2012) have discussed the need for and importance of citizens' statistical literacy. Since 2010, many states across the United States have adopted the Common Core State Standards for Mathematics [CCSSM] (National Governors Association, 2010). These math standards explicitly contain statistics[1] standards for each grade band starting with the sixth grade. This emphasis on statistics in CCSSM is much greater than in previous standards (Tran, Teuscher, Dingman, & Reys, 2014). As a result, math teachers in the middle grades must now attend to statistical concepts as well as solely mathematical concepts.

However, peoples' prior experiences with statistics in formal educational settings are usually limited to ideas of center where ideas of variation are not emphasized until later grades, if at all (Pereira-Mendoza, 1986; Shaughnessy and Pfankuch, 2002; Noll & Shaughnessy, 2012). In order for teachers to teach statistics in a manner their students might find useful, the teachers themselves must first have productive beliefs, meanings, and ways of thinking about statistics. These productive meanings, beliefs, and ways of thinking about statistics must, by necessity, include productive ideas about variation and variability (Garfield & Ben-Zvi, 2005).

In this paper the researcher will answer the following question: *To what extent do middle school math teachers consider variation and variability when thinking about statistics and reasoning through statistical tasks?* This paper will include a preliminary analysis on the beliefs and conceptions that middle school math teachers participating in a professional development program hold about ideas of variation and variability. The beliefs and conceptions will be analyzed through the use of open-ended survey questions, and free-response statistical content questions.

Literature Review

Salisbury (1996) discusses variability and variation in data as the differences among people, among environments, and among things. Cobb and Moore (1997) describe statistics as a methodological discipline that arises from the omnipresence of variability. According to the GAISE (2005) framework, statistical thinking "must deal with this omnipresence of variability; statistical problem solving and decision making depend on understanding, explaining, and quantifying the variability in the data" (p. 6). Though the CCSSM (National Governors Association, 2010) does not formally define statistics, there is an implicit theme of variability

[1] Statistics used in this paper will mean the discipline of statistics, unless otherwise stated.

that is essential to this idea of statistics. The first 6th grade statistics standard in the CCSSM is for students to develop an understanding of statistical variability. A person who understands statistical variability will be able to recognize that statistical questions anticipate variability in data and considers its role in the answers (National Governors Association, 2010).

Several researchers have studied students' and teachers' conceptions of statistics as they relate to variability and variation (Shaughnessy et al, 1999; Torok & Watson, 2000; Saldanha & Thompson, 2002; Liu, 2005; Noll & Shaughnessy, 2012). For example, Saldanha and Thompson (2002), Liu (2005), and Noll and Shaughnessy (2012) discussed productive meanings for individuals to possess about sampling that involved individuals being able to envision variability of sample attributes between outcomes from repeatedly sampling from a population. Thus, the first 6th grade statistics standard is a large part of the foundation for a much more sophisticated statistical concept.

Methodology and Framework

The data collected for this study were gathered through the efforts of a large-scale professional development and research program. This program focused on middle school teachers in a Southwestern state in the United States. Each teacher in the program was asked to participate in professional development activities for two years. The project focused on promoting the mathematical and pedagogical development of its participants.

At the beginning of the second year, the researcher gave two assessment instruments to 50 teachers prior to formal professional development on statistical ideas. The first instrument was a set of mathematical tasks, both multiple choice and free-response, that related to statistical content the teachers were expected to teach in middle school (grades 6-8). The second instrument was an open-ended beliefs survey with seven questions about statistics and statistics teaching.

The researcher performed an initial examination of the teachers' survey responses by conducting several passes through the data. During each pass, the researcher examined the response for each question from each teacher before moving on to the responses for subsequent questions. The researcher created themes in teacher responses while using the lens of the GAISE (2005) framework in conjunction with CCSSM (2010) for statistics. Using this lens, the researcher examined the two survey questions and two content tasks where teachers had the most opportunity to consider variability as they thought about statistics and reasoned through statistical tasks.

Questions and Task Description

The focus of this paper is on the analysis of two of the beliefs survey questions and two of the tasks. The researcher analyzed the following two questions. Q1: Briefly give a definition for *statistics*. What do you take this to mean? What comes to mind when you see or hear the word *statistics*? Q2: To you, what are the differences, if any, between *statistics* and *mathematics*? What are the similarities, if any, between *statistics* and *mathematics*?

In addition to these survey questions, two tasks were analyzed: The Sampling Task and The Calorie Intake Task. The researcher designed The Sampling Task (Figure 1) to determine what teachers believed to be important aspects of sampling, as well as to determine which aspects of sampling that teachers would give evidence of noticing in their arguments for or against one of the choices.

You give your class a task to determine the average height of all the students in your school. Due to various time and logistic restrictions, your students are unable to measure each student at the school and instead must use sampling to determine the average height. Three of your students each come up with their own way to sample the school's population.

Chi's method: Assign the students in each class at her school a number. Randomly select one number of the numbers from each class. Record the height of the student that corresponds with that number.

Kendra's method: Wait until after-school clubs and sports start. Select one boy and one girl of each grade level (6th, 7th, and 8th) for each club or sport. Record their heights.

Diego's method: Pick an entire class from each grade (6th, 7th, and 8th) and record the heights for each student in each class.

Based only on the sampling method, whose conclusions would you be most willing to accept? Why would you be willing to accept this student's conclusions more than the other students' conclusions?

Figure 1: The Sampling Task

Each of the choices in The Sampling Task has a designed strength and weakness pertaining to sampling methodology. Chi's method is the only sampling method that includes random sampling. However, the number of classes in the school are not given in the prompt, thus the size of Chi's sample may be too inadequate to accept her conclusions. Kendra's method is the only sampling method that accounts for gender[2] in the sample. However, Kendra's sample may not be representative of the students at the school due to the existence or composition of certain sports. Diego's method has the potential for having the largest sample. However, Diego's sample, while large, may not be representative of the population at the school due to how he picked the classes. From the prompt, the teacher had to determine that Diego's method is not a census.

The histogram presented in The Caloric Intake Task (Figure 2) depicts a collection of calorie (kcal) counts.

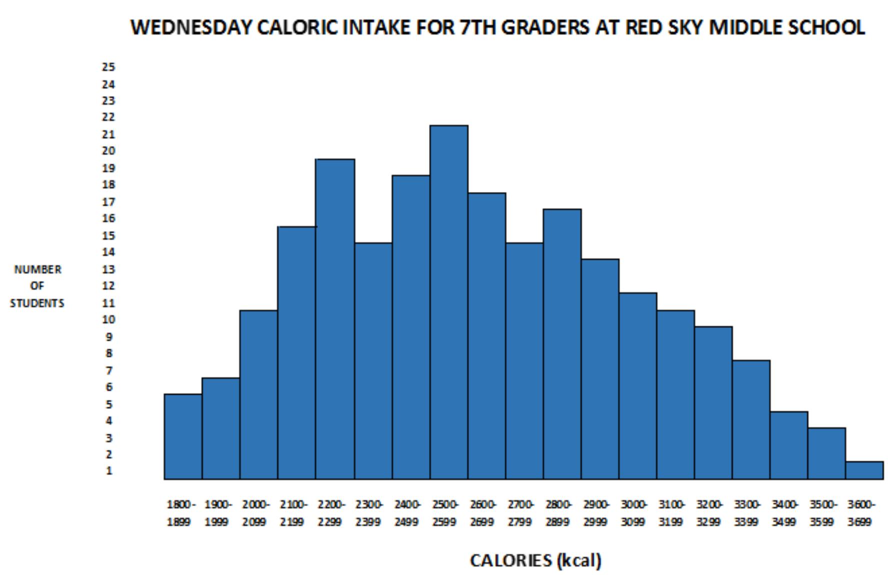

Figure 2: Histogram from The Caloric Intake Task

[2] Assuming gender as being dichotomous.

Using the histogram in Figure 2, the teachers were tasked with describing the data, developing questions to ask their students about the data, and explaining the statistical concepts underlying these questions. The researcher designed this task to determine what teachers believed to be important enough aspects of a distribution of data to ask their students. The researcher wanted to examine the ideas and language relating to the data's shape, center, and variability that the teachers would (or would not) use in posing questions to their hypothetical students.

Preliminary Results and Discussion

The new standards unambiguously state variability as a necessary condition for statistical thinking. However, the preliminary data from the teachers in this study suggest that teachers rarely consider the notion of variability when thinking about statistics or engaging in statistical tasks. In the beliefs survey, only one of the 50 teachers responded to Q1 with anything pertaining to variability.

An important difference between statistics and mathematics is that people who do statistics are focused on the variability of data. Thus, Q2 is a natural question to probe teachers' thinking about variability. Several teachers responded to the question with statements about utility, subjectivity, or practicality of statistics when related to mathematics. Several teachers also responded with the notion of statistics as a subset of mathematics. None of the teachers responded to Q2 by mentioning anything about variability being a key difference between the two, not even the teacher who had mentioned variability in her response to Q1.

Q1 and Q2 provided the best opportunity for teachers to discuss variability's role in statistics in the survey format. With only 1% of the total responses mentioning variability, the teachers seem to indicate that they do not consider variability when thinking about statistics. To strengthen this argument, analysis of the statistical tasks is presented below.

The Sampling Task and The Caloric Intake Task provided the teachers with opportunities to reason about statistics both personally and pedagogically. Of the 44 respondents to The Sampling Task, 31 selected only Deigo's method, 12 selected only Chi's method, and one selected Kendra's method or Diego's method. None of the teachers selected only Kendra's method, in fact, the teachers overwhelmingly selected against Kendra's method. As Table 1 shows, many teachers picked out the potential for sampling bias in Kendra's method as a weakness. None of the teachers (including the lone teacher who picked Diego and Kendra) mentioned the designed strength of Kendra's method.

Table 1: Teacher Responses to Kendra's Method

Sample of Teacher Responses to Kendra's Method	Kendra's method will undoubtedly be skewed because of clubs and especially sports.	Kendra is only athletes which could be biased to tall people.	Kendra's only samples a certain demographic and is not representative of "normal"	Kendra or Diego because their samples would be random vs biased. You must take into account not only the sample space but the direction the question at hand is leading you to a conclusion.

The variability in middle school students' heights due to gender[3] is familiar to middle school teachers. However, based on the responses, it seems to have gone unnoticed by the teachers that

[3]The height differences due to gender in middle school are obvious, but these claims can be corroborated by The Center for Disease Control. http://www.cdc.gov/nchs/data/series/sr_11/sr11_252.pdf

Kendra deliberately accounted for gender in her sampling. If some teachers did notice it, accounting for gender did not seem to be important enough for the teachers to give comment. Given this context that is near to the teachers, the researcher speculates that teachers who attend to the necessity of variability for statistics would have had no problem mentioning this strength of Kendra's design, even if the teacher did not choose Kendra's method.

The Caloric Intake Task was more open-ended than The Sampling Task. As stated previously, it was designed to allow the teachers to share what they felt might be salient statistical concepts for their students. A surface-level analysis revealed that 19 of the 31 teachers responded to the task with language or calculations that were related to variability. A deeper analysis of the teacher responses showed that only four of the 31 teachers asked meaningful questions about variability. Of these four teachers, two asked questions of their students about why the caloric intake between students could potentially vary. The other two teachers gave examples of how the caloric intake could potentially vary due to students' socioeconomic statuses, students' athlete statuses, and students' health statuses (Table 2).

Table 2: Teacher Responses to The Caloric Intake Task

Sample of Variability Language Responses	"Range of intake, mean, median, mode, MAD..."	"Students are between 1800 and 3699 calories which is a range of 1899 calories."	"What's the mean? What's the range?"	"I would ask my students about the spread and shape of the data as well as the skew."
Meaningful Variability Responses	"Why do you think the data varies?"	"What was the variation of the number of calories consumed?"	"Which groups of students might be in different categories? (athletes, poor who wouldn't get dinner at home, etc"	"There is a large range is calorie intake. I wonder how many students play sports - this could affect the needed calorie intake.

Based on the fact that the variability response rate for Q1 and Q2 was so low, it is safe to assume that the four teachers are atypical of the teachers in the program at-large. Most of the teachers do not seem to consider variability when thinking about statistics or reasoning through statistical tasks.

Future Work

Teachers responded to Q1 and The Caloric Intake Task with language that relates to variability. However, the teachers' responses to all four assessment items do not indicate that they are considering how variability influences statistical thinking. In the future, the researcher plans to 1) collect task-based interview data to assess the meanings that teachers have when they use variability language and 2) interact with teachers during lesson-planning activities to focus their attention on attending to variability while planning tasks and activities for their students.

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Implementation of Pre and Post Class Readings in Calculus

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Active learning practices highly depend on students' preparation for class in advance. However, reading Calculus can be a challenging task to students. We address this concern by assigning targeted pre-class readings and reading quizzes in two Calculus II classes. To study the effectiveness of these, we also provided them as post-class readings in two other classes. We report on our implementation and we discuss students' feedback about the readings and quizzes.

Key words: pre and post class readings, reading quizzes, exit quizzes, Calculus

Introduction

In recent years, Flipped Classroom and Inquiry-Based Learning pedagogies have emerged as methods of active learning. These active learning practices highly depend on students' preparation for class in advance. To prepare for class, professors often ask their students to read the textbook before they come to class. However, most faculty report that students rarely do this (Felder & Brent, 1996). We believe that this concern is even more evident in mathematics education. Reading mathematics texts before the class can be a challenging task to students especially in introductory classes such as Calculus. In this proposal we attempt to address this issue by implementing active learning practices (Prince, 2004) in four classes of Calculus II.

Our study is divided into two themes: students in two sections had access to short typed targeted notes before class combined with reading quizzes at the beginning of the class period. In two other sections students had access to the same notes after the class and the same reading quizzes were given to them as exit quizzes at the end of the class. After presenting a literature review, we describe the methods of this study. In particular, we describe the structure of these classes, the nature of the readings and quizzes, and the collected data. We report preliminary quantitative and qualitative results on students' usage of textbooks and these notes. We provide some of their feedback about the pre/post class readings and the respective reading/exit quizzes.

Literature Review

Many science education articles discuss reading assignments and reading quizzes. Hodges, Anderson, Carpenter, Cui, Gierasch, Leupen, and Wagner (2015) discussed different formats of reading quizzes in different STEM areas. We found in (Heiner, Banet, & Wieman, 2014) an implementation of targeted pre-reading assignments with an associated online quiz in two science classes, physics and biology. In an introductory physics course, Henderson and Rosenthal (2006) used reading questions instead of reading quizzes where students posed questions on the assigned readings to the instructor via email before class.

On the other hand, there have been numerous research articles demonstrating the difficulties students have with reading mathematics texts. For example, Shepherd, Selden and Selden (2009) believe that "many, perhaps most, first-year university students do not read large parts of their mathematics textbooks effectively, that is, they cannot work straightforward tasks based on their reading. Whether this is because they cannot read effectively, or choose not to do so, seems not to have been established" (p. 1). Shepherd (2010) also states that "the textbooks for many first-year university courses, such as college algebra, precalculus, and calculus seem to be written with the assumption that they will be read thoroughly and precisely" (p. 3). She also refers to a brief survey (JMM presentation, Exner & Shepherd, 2008) of Calculus I students where they found that few read the textbook at all. The authors of this presentation (as cited in Shepherd, 2010) provide a typical student comment: "When I think there is a formula I need, I'll go back and look if there is a formula, otherwise… there is very little chance that I'm going to read any of it. "

In addition, we refer the reader to Weinberg, Wiesner, Benesh, and Boester's study (2012) in which they surveyed 1156 undergraduate students in introductory mathematics classes about their textbook usage. Students reported that they used the examples instead of the expository text. Weinberg et al. results show that "instructors may play a role in students' textbook use. When students perceive that they are asked to use their textbook, they report that they are more likely to do so…Although the conclusions that can be drawn from these results are limited, they suggest that instructors may be able to increase students' use of their textbooks by asking their students to use their textbooks" (p. 23).

The aforementioned student's comment from (Exner & Shepherd, 2008) and Weinberg et al.'s results (2012) probably describe the attitude of the majority of Calculus students towards their Calculus textbooks. This motivated us to provide students with targeted summarized typed notes that can serve as a tangible resource for students' learning of Calculus concepts. By going through a literature search, we have found little evidence on the usage of targeted reading assignments in Calculus courses. In this preliminary report, we discuss the implementation of pre and post class targeted readings in Calculus classes. We also report on using quizzes that served as both "reading" (beginning of class) and "exit" (end of class) quizzes. Preliminary results from an end of the semester survey are presented in which we address the questions: *How much time do students spend on pre-class readings; how often do they use post class readings? How do they perceive these readings? How do they perceive reading and exit quizzes?*

Study Design

The study was conducted in four sections (Sections 2, 8 am; 4, 10.50 am; 6, 4.30 pm; 8, 8 am) of Calculus II classes at a comprehensive Northeastern university that has emphasis on sciences and engineering. At this institution, Calculus II covers integral Calculus and Series. The classes had a total of 90 students where a total of 69 students consented to participate in the study (26 females and 43 males). The majority of them were Engineering and Forensic Sciences majors. The classes met in Spring 2016 semester, three times a week for one hour and fifteen minutes.

To answer the questions above, we designed the study to have two sections (4 & 6) with pre-class readings where students took a quiz on new concepts and techniques at the beginning of the class period. Two other sections (2 & 8) had access to the same readings after the material was discussed in class and the same quizzes were taken as exit quizzes at the end of the class.

The readings were targeted in the sense that they were brief and prepared (typed) by the instructors. A typical note is a 1 page (at most 1.5 pages) long that has a short discussion about a concept followed by two or three examples. Fig. 1 is a sample note on Integration by Parts. We omit the second part of this note that included the two examples $\int \boldsymbol{xe^x dx}$ and $\int \boldsymbol{xlnx\, dx}$ followed by a brief generalization to any power of x. We designed the quizzes to be very similar to the examples in the readings that included any new formulas and multiple steps to guide the student's answers. In Fig. 2 we see a sample quiz question on Integration by Parts. It is worth noting that the quizzes constituted only 5 % of the final grade and hence did not have a huge impact on students' final grade.

Calculus-II
Integration by Parts

We have seen so far how to use the substitution rule. It is worth noting that the substitution rule for integrals is the opposite of the chain rule for derivatives:

- Chain Rule: $\big(f(g(x))\big)' = f'(g(x))g'(x)$,
- Substitution Rule: $\int f'(g(x))g'(x)\,dx = f(g(x)) + C$ using $u = g(x)$.

Now we will find the opposite of the product rule of derivatives. It is called **integration by parts**.

- Product Rule: $\big(uv\big)' = u'v + uv'$,
- Integration by Parts: $\int u\,dv = uv - \int v\,du$.

To get this formula, we rewrite the product rule as $\frac{d}{dx}(uv) = \frac{du}{dx}.v + u.\frac{dv}{dx}$, then we integrate both sides to get

$$uv = \int v\,du + \int u\,dv,$$

which can be written as $\int u\,dv = uv - \int v\,du$.

Figure 1: A sample note

To find $\int x^2 \ln x\,dx$ we need to use the formula $\int u\,dv = uv - \int v\,du$.

use $u = \ln x$ then $du = \ldots\ldots\ldots\ldots\ldots\, dx$

use $dv = x^2\,dx$ then integrate to get $v = \ldots\ldots\ldots\ldots$

Thus $\int x^2 \ln x\,dx = (\ldots\ldots\ldots)(\ldots\ldots\ldots) - \int \ldots\ldots\ldots\ldots\, dx =$

Figure 2: A sample quiz

The collected data for our study include two surveys, the first was given in the second week of classes and the second was given in the last week of classes. Our data also include the reading/exit quizzes, an early quiz on the first day of classes and a retention quiz at the end of the semester. The grades for these two quizzes were not counted towards students' grades but the students were not informed of this till after the quizzes were taken. For this report, we only present data from the two surveys.

Preliminary Results

In the early survey we asked students about their study habits and, in particular, how often they read the textbook prior to the next class. We found that about 65 % of the students who took the survey reported that they never or seldom read the textbook before coming to class, about 22 % of them said they read the textbook about half the time, and about 13 % said they read it usually or always. This shows that most students did not spend time outside the classroom to reinforce the learning of the concepts that were discussed in class before they come to the next class period. These results served as a motivation for assigning pre-class readings or providing targeted readings for an after-class reinforcement of concepts.

We now present some preliminary results from the end-of-semester survey. We analyzed two questions from the survey. In sections 4 and 6, we asked the students: How much time (on average) did you spend on each pre-class reading assignment while in sections 2 and 8, we asked the students: Did you use the typed notes when you worked on your homework or studied for tests (did not use them, sometimes, frequently, all the time). In all sections, students were asked to rate the usefulness/effectiveness of the typed notes. In sections 4 and 6, 59 % of students spent between 20 and 40 minutes, 16 % spent more than 40 minutes, while 25 % of students spent less than 20 minutes on the typed notes. The survey analysis showed that 69 % of students in these two sections found the notes either effective or very effective. In sections 2 and 8, 76 % of students either used them sometimes or frequently when they worked on homework assignments or studied for tests and 68 % of students found them either effective or very effective.

In the following, we support these positive students' perceptions by providing some of their feedback on the notes and the quizzes**.** Even though the quizzes constituted only 5 % of the final grade, they had a relatively high impact on students' pre-class reading efforts in sections 4 & 6, and on students' attention span in sections 2 & 8. We will support this claim via students' qualitative data from the end-of-semester survey.

We start with comments from students in sections 4 and 6 who had pre-class readings. We refer to students from these sections as S-number. Student S1 reported:

> The notes and quizzes constantly forced us to actively learn outside of class which I thought was very effective.

Student S2 found that:

> The typed notes and the quizzes are great additions to the class is probably why this class has not been as stressful as previous math courses.

Student S3 pinpointed the main purpose of having pre-class readings:

> The typed notes and quizzes made us have an understanding of the topic before class but left room for further understanding during class discussions.

Student S4 had a concern about the grades when s/he said:

> The notes are helpful but don't give the same understanding like doing it in class does which effects the grades since we do the quizzes in the beginning.

Student S5 commented:

> The typed notes were very helpful because they helped me have some understanding of the material before it was taught to me. There were only a few that were hard to understand but then the lecture was able to help me get a better grasp of the material. The quizzes were also very helpful because they pushed me to actually read the notes.

We also provide some feedback from students who had access to the typed notes after class and we present their feedback on the exit quizzes. We refer to students from these sections as T-number. The main theme of students' feedback was that the typed notes were helpful in doing the homework and that the quizzes kept the students' attention at a high level. For instance, student T1 said:

> The provided typed notes were a great help in doing the HW. They should of however, been uploaded before class instead of after. The quizzes were good to make sure that you are actually understanding the lesson that day.

Student T2's feedback was:

> The typed notes helped with the typed homework and the quizzes helped reinforce what I learned that day.

Student T3 found that:

> The notes, quizzes, & the assigned typed HW helped me comprehend the content even more than what I had expected.

While Student T4 commented:

> I liked the typed notes, but for the last few classes of the semester the typed notes did not cover everything we did in class. I like the way the quizzes are (right after the lesson)….I think that quizzes sometimes took away from the lecture when we could have used that time to finish the lecture.

Student T5 gave the following comment:

> Typed notes: I find it hard to understand math written out on a piece of paper. Quizzes: a nice grade booster and opportunity to show what I learned.

Discussion/Future Research

Our preliminary data analysis shows that the majority of the students reported that the typed notes and quizzes were helpful and conducive to learning. Some students went even further and recommended some changes. For example, Student T1, from a post-reading section with a final C+ grade in the course, expressed her/his preference to having access to the typed notes before class rather than after. Her/his feedback basically encapsulates instructors' hope of increased exposure to concepts outside the classroom. It is indeed our goal to assign these readings before class in every course, but for the purpose of this study, this was not the case in Spring 2016.

As pointed out by some students, there were a few notes that were technical and harder to read. We definitely agree and we will surely modify these notes as we implement these readings in the future. However, we do not plan to increase their size as we want to intentionally keep them short and concise. The purpose of the notes is not to replace the classroom mini-lectures or discussions, and consequently they will not cover everything that is discussed in class.

To address student T4's concern about the time taken by the quizzes (typically about 5 minutes), we looked into other alternatives such as online quizzes but we believe that in class quizzes give a better hands-on learning experience for students. We also claim that they provide a better assessment method of students' reading efforts and attention span.

For future research we hope to have a more in depth analysis of our data. In particular, we would like to have:

- A comparative analysis of the grades on quizzes and/or exams of the 4 sections through the lens of the targeted notes' usage,
- A study of the effect of these notes on students' content retention using the retention quiz,
- A study of the effect of pre-class readings on the classroom environment as reported by the instructors' observations, and
- An exploration of the shortcomings of this study such as the different meeting times and lack of instructions on how to read mathematics.

Preliminary Report Questions

1. Do the different meeting times of classes impact students' learning and performance?
2. What does existing research suggest for instructions to read mathematics?
3. What is the correlation between students' GPA and their motivation to read mathematics textbooks/notes?

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Opportunities to Learn from Teaching: A Case Study of Two Graduate Teaching Assistants

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Graduate students teach many first year undergraduate mathematics courses, such as College Algebra and Calculus. In this report, we focus on the opportunities to learn to teach that graduate student teaching assistants (GTAs) construct from reflecting on their teaching experiences. Research in professional development suggests that although reflection is absolutely essential to improving one's teaching, teachers have the greatest opportunity to learn from their teaching when they can mobilize their interpretations of teaching to inform specific and nuanced future actions. Yet, there are few studies addressing the ways in which GTAs develop opportunities to learn from reflection. In this case study, we examine how two graduate students developed the ability to link observations of student work to hypotheses about student thinking and then connect these hypotheses about student thinking to future teaching actions. These reflections were generated as a part of a professional development program for GTAs.

Key words: Post-Secondary Professional Development, Graduate Teaching Assistants, Student Work, Student Thinking, Teaching Actions

Providing and studying professional development programs for teachers has become a norm in K-12 education. Since instruction is the primary responsibility of K-12 teachers, the need for professional development is commonly accepted. In higher education, the need for professional development is gaining acceptance. While the primary function of colleges and universities is to provide students with an education, the instructors and professors who teach the classes often have responsibilities that extend beyond instruction. This is especially true at research universities, where a professor's appointment is often split between teaching and research. In a similar way, a graduate student's responsibilities are often split between the two. While it may be true that many graduate students are training to become researchers, their secondary role as an instructor is important. It is often the case that graduate students are the primary instructor of a course for the first time as a graduate teaching assistant (GTA), which is an assignment that mathematics departments should not take lightly. As providers of education, one of our primary goals should be to help our teachers provide high quality instruction. It is for this reason that we believe it is necessary and important to study not only *how* to provide professional development to graduate students, but also the *impact* that professional development has on their development as teachers and their future teaching actions.

Purpose and Relationship to Research Literature

While there are many important differences between teaching in K-12 and higher education (Speer, King, & Howell, 2015), there is also much that we can take from K-12 professional development when designing and studying professional development programs for GTAs. One thing that research has shown is that an instructor has more opportunities to learn when they plan future teaching actions by reflecting on previous instructional experience (Thompson, 1984). In the K-12 literature, studies have shown these opportunities to learn are impacted by both the clarity of the teacher's reflections on the nature of student's understandings and the amount of time available for the teacher to construct a response to student thinking (Horn, Kane, & Wilson, 2015). If the model from K-12 holds, then one would expect that graduate teaching assistants' opportunities to learn will also be more meaningful when they are given time to reflect.

However, providing time for reflection alone is not enough. The other important variable in the process is the clarity with which the teacher has the ability to reflect. While clarity could be measured in various ways, one potential means of measuring the clarity with which a teacher reflects is by studying the arguments the teacher makes. Since our study involved analyzing several reflective essays on teaching in which the GTAs made arguments to explain student thinking and propose future teaching actions, this measurement seemed best suited to our purposes. Toulmin (1958) developed an argumentation model for legal arguments using the components of grounds, claims, and warrants. In this model, grounds are the evidence underlying a claim and a warrant justifies the relationship between the grounds and the claim. More recently, Toulmin's model has been applied to other fields, including mathematics education (Inglis et al., 2007).

In Lai, Smith, Wakefield, Miller, St. Goar, Groothius, Wells (2016), a modified version of Toulmin's model was used to analyze the connections GTAs made between what they observed their students doing and their future plans for teaching. Lai et al. performed a qualitative analysis of 16 final papers written by mathematics GTAs and developed a coding scheme that categorized papers as low connectivity, medium connectivity, high connectivity with low coherence, and high connectivity. The researchers concluded that even when GTAs are teaching the same course, participating in the same professional development, and completing the same task, the clarity of their reflections on the nature of student thinking varies widely. Further, there is a wide variation in the GTAs apparent ability to connect student thinking to future actions. However, in this previous study, no attempt was made to look at the growth GTAs experienced. For example, while the authors found examples of GTAs with high connectivity in their final papers, there was no indication of what growth may have occurred over the course of the professional development program. An important question that resulted from this is whether or not these cases of high connectivity are representative cases of individuals who entered the professional development program with these reflective skills.

The purpose of this intrinsic case study (Stake, 1995) is to better understand the growth experienced by two mathematics GTAs over the course of their involvement in a professional development seminar. These two GTAs were selected because they have been identified as strong teachers and pedagogical leaders in the department. At this point in the research, growth will be generally defined as increase in coherence over time. The central question that is guiding our inquiry is: How did the GTAs grow over the course of the professional development seminar? We decompose the central question as follows:

1. How might the opportunities to learn afforded over the course of the professional development seminar help the GTAs to make changes in their teaching?
2. How might the GTAs' ability to clearly reflect on the nature of student thinking change over the course of the professional development seminar?

Our intention is to use Lai et al.'s research methods on a set of papers collected from GTAs over the course of an entire semester of professional development. Conducting this type of longitudinal case study will allow us to examine the growth of the GTAs, as opposed to just examining the final paper. Although the context of this case study makes it hard to generalize, this analysis will help identify what GTA growth looks like and provide one possible measure by which professional development programs for GTAs could be evaluated.

Theoretical Frameworks

This paper utilizes the coding framework discussed in Lai et al. (2016), which examines the connections between student data, student thinking, hypothesis, and future teaching actions. *Data* is defined to be the written student work collected by the instructor with the addition of any memory-recalled communication as observed by the instructor and recorded in their writing. After collecting data, the instructor interprets this data in the form of *student thinking*. That is, student thinking is the instructor's expression of how he or she believes the data should be interpreted as a reflection of the student's work. Instructors may hypothesize likely reasons for a student to think in the way in which the instructor has argued. This *hypothesis* depends upon the underlying origin of student thinking. Finally, an instructor may plan a *future teaching action* based upon any or all of the previous elements. These four elements, and the connections between them, form the framework used to analyze GTA work.

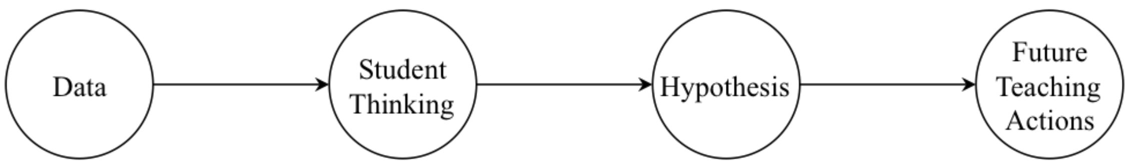


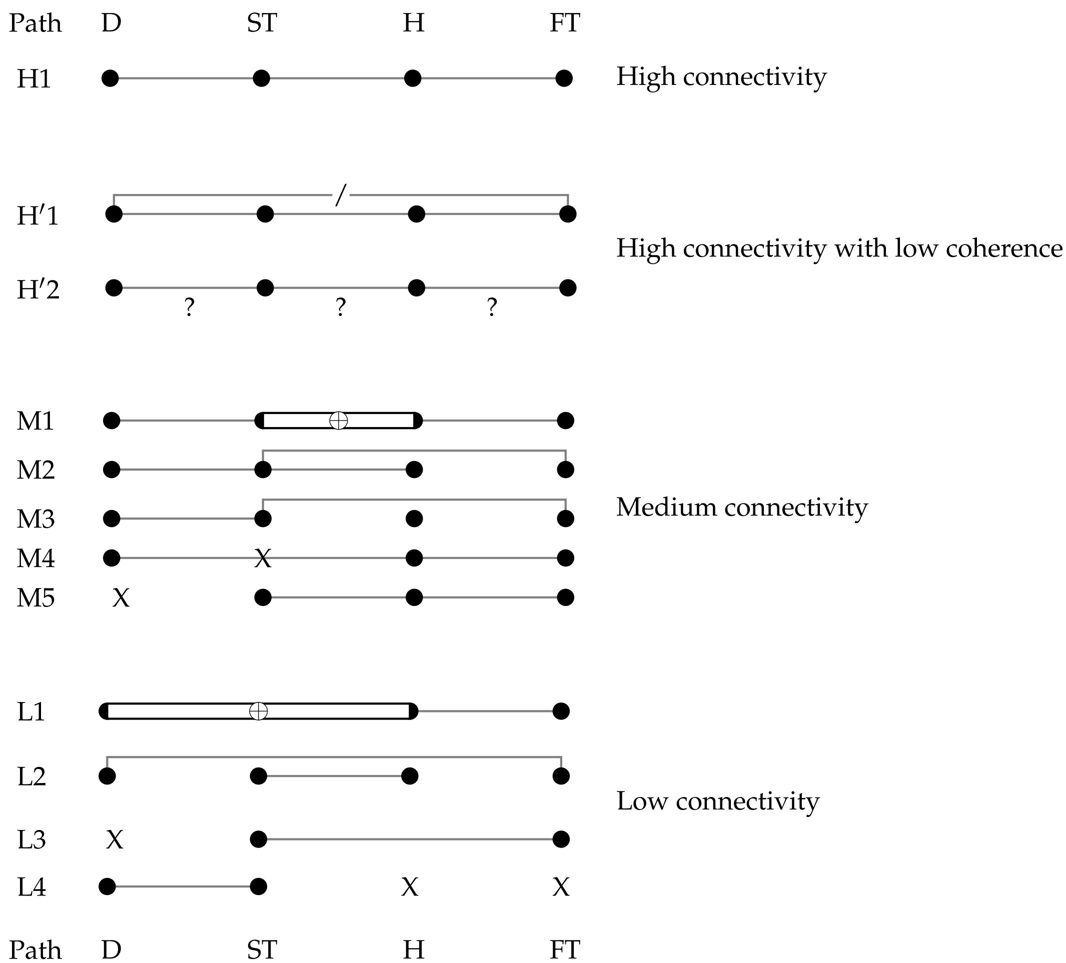
Figure 1. Model for GTA arguments

Using this framework, a GTA's work may be coded by looking for the presence or absence of any of the four elements and the connections between them. When looking at connections, coding captures not just the presence or absence of a connection but also the plausibility of a connection. Using this technique, a work may be categorized as having *low connectivity* if explicit reference to two or more of the four elements was missing, or the connections between these elements are not present. On the other hand, a work is said to have *high connectivity* if the GTA clearly articulates all four elements and makes explicit links between these elements. The tag *low coherence* is added when the four elements are articulated, but weak, implausible, or implicit links are present. A final category of *medium connectivity* is marked by the presence of at least three of the four elements, and attempted links between all of the components. A visual representation of these various types of connectivity is included in Figure 2.

Data and Methods

Using this framework, we plan to study the growth that we observed in the works of two particularly strong GTAs over the course of the fall 2015 semester when they were enrolled in the department's pedagogy seminar. At the university in the study, every graduate student who is assigned to be the primary instructor of a course for the first time is required to enroll in a seminar titled "Teaching and Learning Mathematics at the Post Secondary Level." This seminar meets for two hours a week in the fall semester and one hour a week in the spring semester. To help develop their ability to reflect, the graduate students in the course read educational literature, including articles ranging from Erlwanger's (1973) discussion of Benny to expository articles like Tsay and Hauk's (2013) explanation of constructivism.

While enrolled in the seminar, the graduate students enrolled in the course are primary instructors for their own mathematics course (either intermediate or college algebra). The typical enrollment in these courses is between 34 and 40 students. Graduate students are the sole instructor of record and, for most, this is their first experience teaching. Their experience teaching together with the articles they read provide the backbone of weekly discussions in which graduate students are given the opportunity to reflect on their own teaching, utilizing the vocabulary of mathematics education.



Key: D = data, ST = student thinking, H = hypothesis, FT = future teaching actions, X = component absent, ⊕ = conflated components, •—•= link was attempted and satisfied criteria, •$\overset{?}{—}$•= no consensus from research team on whether link is plausible, •-/-•= future teaching actions do not plausibly address student thinking identified in data.

Figure 2. Categories of connectivity of GTA arguments

The data for this study, which has already been collected, includes three assignments completed by the GTAs throughout the semester course. All three assignments asked the GTAs to write papers in which they analyze student performance on a quiz or exam question and provide both an interpretation of what went wrong for a student or a group of students and evidence supporting their interpretation. GTAs were also asked to form a hypothesis about what underlying experiences and beliefs may have contributed to this performance and ultimately develop an “intervention” to try and help the student overcome these experiences and beliefs in order to move into a more productive way of handling this type of problem. Data from 16 GTAs was collected and analyzed using the above-mentioned argumentation framework. The results of the final paper were previously analyzed and reported in Lai et al. (2016). This purpose of this paper is to discuss the growth observed in two GTAs who have been identified by the math department as particularly strong teachers. By comparing their three papers to each other, this study will allow us to analyze how these GTA's teaching may or may not have changed throughout the semester. In other words, while Lai et al. (2016) looked broadly at the work of all GTAs on a single assignment with the purpose of characterizing opportunities to learn, this study will look narrowly at the work of two GTAs across multiple assignments, so as to understand better how these opportunities to learn were utilized.

Significance

Even though providing professional development for GTAs is a rather new phenomenon, exemplary models of GTA professional development programs do exist (Bressoud, Mesa, & Rasmussen, 2015) and some studies of GTA professional development have been conducted (Hauk et al., 2009; Kung 2010; Kung & Speer, 2009; Speer, Gutmann, & Murphy, 2005). Yet, there is much that we have to learn. Ellis (2014) suggested that one means of better understanding how to prepare post-secondary mathematics instructors is to look at methods employed in the K-12 environment and use those methods to study the post-secondary environment. This study takes that approach by focusing on the clarity of reflections, as suggested by the K-12 literature. Also, this work provides an opportunity for further testing and refining of the framework proposed by Lai et al. (2016). Since this work is in its early stages, we believe that a case study is the right methodological approach. In order for researchers to better study GTA training, researchers need to better understand what is happening to the individuals in GTA training programs. Since the two subjects of this study were identified by as being particularly strong teachers, it would be useful to understand what this means and whether or not the framework suggested by Lai et al. (2016) identifies any features of their strength.

Questions for the Audience

To refine our ideas, we pose the following questions to our audience:

1. Are their areas of our framework that you think would be hard for you to utilize in analysis of GTAs in your program?
2. Is there something missing from our framework that would add a significant contribution to our understanding of the reflective process a GTA uses?

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Factors Influencing Instructor Use of Student Ideas in the Multivariable Calculus Classroom

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Despite overwhelming evidence of the effectiveness of student engagement in instruction, practicing mathematics instructors often use instructor-centric practices even if they value student engagement. Answering a call by Henderson and Dancy (2007) to study the implementations of researched-based curriculum in the classroom, this paper looks at the change in practices and values of instructors utilizing active-engagement activities in multivariable calculus classes. This curriculum incorporates context and multiple representations, and we look for evidence that addresses whether these features facilitate instructor use of student ideas in instruction.

Key words: Curriculum adoption, Student-centered Instruction

Despite an abundance of research highlighting the benefits of actively engaging students in the classroom, many practicing undergraduate mathematics instructors still utilize lecture and a host of non-student-centric practices. This often occurs even if instructors value student engagement.

We examine how two features of a research-based curriculum – contextualized problems and an emphasis on multiple representations (MR) – may or may not support instructors in attending to and using student ideas in instruction. In this study, we look at two guiding questions:

- What features of instructional materials support instructors in attending to and incorporating student ideas in instruction?
- Do contextualized problems and an emphasis on MR support instructors in valuing student contributions and incorporating them in their instruction?

Theoretical Framework

Activity theory (Engeström, 1987; Cole, 1996) notes that many factors influence change in any new activity. Instructors adopting a new curriculum have to navigate many new factors with the existing factors in the classroom. This might involve new roles of both students and instructors in the classroom, new agreements about how these populations interact and new rules for each of the new groups. In particular, instructors adopting a student-centered curriculum have to re-evaluate their expectations for how students and the instructor contribute toward the course. Instructors often have to settle into new roles while still being familiar with their prior roles.

There is reason to believe that context and MR might support an increase in the use of student ideas. For context, students may find contextualized problems more relevant to their interests and be more engaged with the task than for more abstract problems. Additionally, students can contribute conceptual and intuitive knowledge connected to the problem context. The use of MR could increase the possibility of finding multiple solution paths. Different groups can use different representations, providing opportunities for students to identify and discuss the connections and differences between them.

The instructional materials in this study incorporate context and MR so that students are able to touch, point at, and work with mathematical objects and explore properties even before these concepts have been introduced formally by the instructor. The contextualized activities present students with a meaningful scenario, effectively changing how they interact with math concepts:

Instead of asking how the mathematics they've learned is related to the real world, they're instead solving real-world problems and learning how the mathematics they discovered is connected to multivariable calculus. Lastly, the use of MR allows students to choose which way they want to solve a problem. They may choose to use a familiar representation, even if it is not the best choice for that problem. Natural questions exploring the connections between representations arise when their peers present similar or contradictory solutions using a different representation.

Methods

Instructional Context

Raising Calculus to the Surface (RC), an NSF (DUE-1246094) project funded in 2013, utilizes small group activities and open-ended questions designed to help students *discover* important multivariable calculus concepts prior formal introduction by their instructor. The *Raising Calculus* materials highlight four main features: physical manipulatives, open-ended prompts, contextualized problem situations, and an emphasis on MR. We focus on the relationship between the instructor's attention and use of student ideas in instruction and (1) how much they value the contextualized nature of the problems and (2) the extent to which they emphasized the use of MR.

Instructors rarely approached the *RC* project because the materials utilized inquiry. To facilitate adoption, instructors could modify the activities and choose which activities to incorporate into their course (Henderson and Dancy, 2007). All 11 activities utilized MR; Nine made meaningful use of context.

Data

Data for this paper comes from 16 of the 38 instructors who used the *RC* materials during one or two terms of the 2014-2015 academic year. The chosen 16 instructors completed anonymous open-response pre/post surveys. The paired surveys focused on the instructor's practices as well as their attitudes and beliefs about student learning in multivariable calculus. The pre-survey was completed prior to a professional development workshop in Summer 2014, and the post-survey was completed after each term of using the materials. Instructors completing the post-survey were shown their pre-survey responses, thus helping them identify and address changes in their course.

Categorization Process

The first two authors independently read and identified over 160 categories contained in all 23 instructor's survey responses. After comparing notes, the authors revised the categories, independently re-categorized instructor responses then met again and collapsed categories into four main themes. These themes were the instructor's instructional method, their use of students' ideas, their value and use of MR, and their value of context. We now describe how instructors were classified within each theme.

Instructor's use of instructional methods. No one question asked about their specific practices, but instructors often reported their instructional practices on the pre-survey or changes to their typical practices on the post-survey questions. An instructor was categorized as *Lecture* if they made no mention of having students engaged in *Small Group Skill Activities*, *open-ended Small Group activities*, or *Inquiry Activities*. Instructors could be categorized under the last three categories even if they still utilized lecture as the primary instructional method.

Instructor's use of students' ideas. Pre-survey questions asked instructors to describe the contributions an average student made to their classroom under the ideal setting (and to describe the actual contributions, if it differed from the ideal). A paired post-survey question allowed instructors to describe any changes in the contribution of the average student to the course. The post-survey also asked if (and how) students were more likely to explore their ideas with each other than in the instructor's previous course.

Instructors use student ideas in courses in a multitude of ways, making any one-dimensional representation of that data difficult. Nevertheless, subcategories chosen progress from no use of student ideas all the way to expecting or noticing students are contributing new content to the course. Responses such as using Think/Pair/Share or activities involving practice of previously introduced skills are sub-categorized under *Peers*, while responses involving group work with presentations to the class or whole-class discussion are sub-categorized as *Whole Class*. Instructors are categorized under the *Add New Content* category if they report their students contribute new ideas to the course.

Instructor's value and use of multiple representations. Survey questions asked instructors about the role various representations played in their course (pre-survey) and how this role changed after using the materials in the course (post-survey). Instructors often reported representations played a minimal role or a significant role in their course on the pre-survey, and frequently indicated it increased on the post-survey. The category Low indicates the instructor primarily emphasized symbolic and/or visual (height) features of graphs. The High category indicates the instructor utilized many representations, including contour plots, throughout the course. In the graphs that follow, pre-survey data points are marked to the left of the Low and High categories; the vector points toward the right of Low or High if the instructor noted that their use increased. The final category (Connecting Representations) indicates the instructor noted they value that students are making connections between different representations.

Instructor's Value of Context. No survey questions asked about context, yet many instructors mentioned their use and value of context or units increased on the post-survey question focused on MR[1]. For this category, Low indicates an instructor pays attention to units and quantities. High indicates an instructor is describing how context helps students understand mathematical concepts. Several instructors are categorized as No Context, indicating that their surveys contained no references to the value of context.

Limitations and Representation of the Data

The categories listed above contain multiple factors and dimensions; we have deliberately collapsed and ordered the data to assist us in visualizing changes contained within the data. Although the data is not ordinal, we present the data using graphs and vectors. Each of the 16 instructors are represented by a vector, with the tail (round circle) and tip (arrow) representing data from the pre-survey and post-survey, respectively. An instructor using the materials for two terms has a multi-segmented arrow. All sub-categories described in the previous section are included on the graphs, although not every sub-category is labeled. Small perturbations were added to the data points to keep nearby data points from overlapping.

[1] Context can be an important representation of a mathematical problem, but the survey questions directed instructors toward contour plots, graphs, symbolic, and verbal representations of functions.

Data Analysis

More Inquiry-Like Instructional Methods Corresponds to Increased Use of Students' Ideas

Figure 1 compares the change in the instructor's use of student ideas to their change in instructional methods. The data shows instructors utilizing *RC* materials incorporated open-ended small group activities into their course, although only 8 of the 16 instructors (those in group A) described using them in ways which let students discover concepts prior to formal introduction. Six instructors in group B changed from using primarily lecture-based practices where students engaged in instructor-led questions to using practices in which students shared their results with the whole class. One instructor in this group noted that he "liked the interactive atmosphere that took place in the classroom. There was a lot of student-led activity, including making conjectures, measurements, etc. that lead to sometimes heated debates about concepts.". He further stated that "using the surfaces made [him] re-think what [he] emphasize[d] in the course as an instructor." Although he wasn't expecting students to contribute new ideas as a result of adopting the materials, he used the experience to reflect upon his teaching practices.

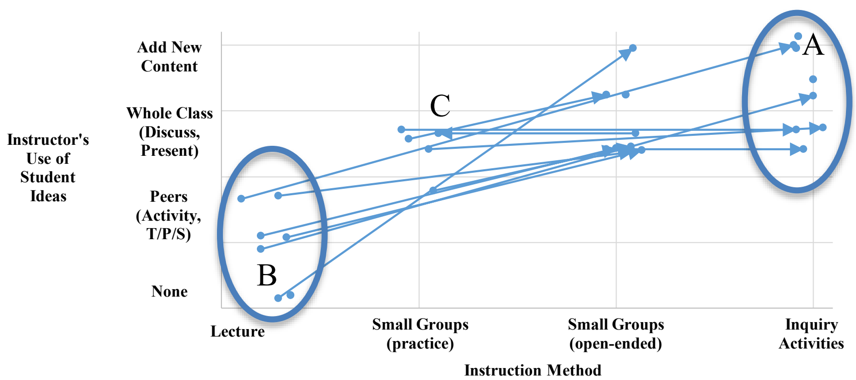

Figure 1: Instructor's value of context compared to their value of MR

These results are not surprising, as the *RC* materials include small group activities and use open-ended questions. Instructors could change and modify the activities, and those which landed outside Group A often indicated their changes made the materials more clear or less confusing for their students. Earlier work (Wangberg et al., 2016) suggests such modifications can make the activities easier to implement but could also unintentionally limit the conversations within the small groups. In the most extreme case, an instructor (just below C) typically taught his course using small group practice activities with whole-class discussion. He used the *RC* open-ended activities in small groups for one term, but modified the materials in the subsequent term so that his students could use them to practice previously learned concepts. Henderson and Dancy suggest that providing instructors with the freedom to modify materials can improve adoption. Instructors sometimes modified materials in order to align them with their more familiar instructional practices. Never the less, instructors tended to increase the use of student ideas as they incorporated open-ended activities into their course.

Valuing Multiple Representations Corresponds to Valuing Problem Context

Figure 2 compares changes in the instructor's value of context to their reported value and use of MR. It suggests instructors using *RC* materials report increased use of context or representations in their course, with 11 of the 16 instructors increasing one or both categories. We note four instructors never mentioned the value of context on their surveys[2].

Valuing Multiple Representations Corresponds to Increase in the Use of Student Ideas

Figure 3 compares changes in the instructor's use of student ideas in the classroom to the change in the instructor's value and use of MR[3]. Nine of the 16 instructors reported an increase in the use of student ideas, and six of these nine reported that the use or value of MR increased in their course. Two other instructors reported they had previously expected students to contribute ideas in the form of new content to the course.

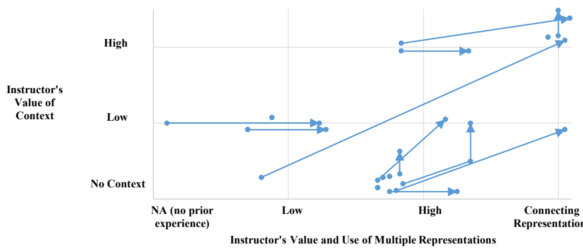

Figure 2: Instructor's value of context compared to their value of MR

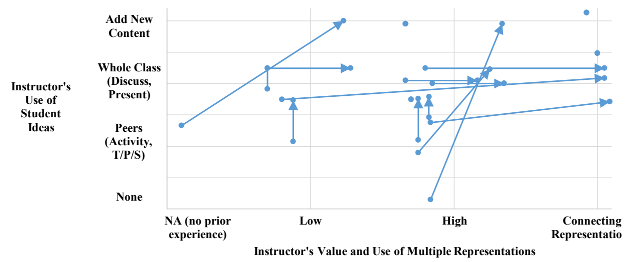

Figure 3: Instructor's expectation of student engagement and value of MR

Conclusion and Implications

Instructors rarely approached the *RC* project because the materials utilize inquiry, yet half of the instructors increased the use of student contributions in their course. While instructors could modify the materials to eliminate features like open-ended questions, which promote discussion, or utilize the activities after introducing the mathematical ideas, we note that is was much more difficult, given the nature of the activities, for instructors to remove the context from the activities or to limit an activity to a single representation of a function.

This paper investigates the connection between the instructor's use of student ideas in a course and the use of a research-based curriculum carefully incorporating MR in context-rich activities. Overall, we see increases in the use of students' ideas, the degree to which instructors value and use contextualized problems, and the degree to which instructors value and use MR. These corresponding increases support a hypothesized relationship between these aspects of instruction. Curriculum developers might consider incorporating contextualized problems and emphasize MR to support instructors in using and valuing student contributions in class.

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[2] Other data collected by the project may be able to fill gaps in the data.

[3] We chose to plot the change related to multiple representations, instead of context, as the data is more complete.

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Knowledge About Student Understanding of Eigentheory: Information Gained from Multiple Choice Extended Assessment

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Eigentheory is a conceptually complex idea whose application is widespread in mathematics and beyond. Herein we describe the development and use of an extended multiple choice assessment that gives us further insight into the ways students think about and understand eigenvectors, eigenvalues, and their related concepts.

Key words: linear algebra, eigenvector, eigenvalue, student understanding, assessment

The purpose of this preliminary report is to share results regarding student understanding of eigentheory that were gained from a multiple choice extended assessment instrument. We chose to focus on eigentheory because (a) it is a conceptually complex idea that builds from and relies upon student understanding of multiple key ideas in mathematics, and (b) its application is widespread in mathematics and beyond. Our aim to create an assessment instrument that captures nuances of students' conceptual understanding of eigentheory exists in tandem with our pursuit to frame what it might mean to have a deep understanding of eigentheory. As such, in this report we offer results both about student thinking and about possible affordances and constraints of various assessment instrument question formats.

Background and Literature

Research into people's understanding of eigenvectors and eigenvalues has had several different focuses**,** such as identifying the various processes and objects students need to understand in eigentheory (Stewart & Thomas, 2006; Thomas & Stewart, 2011), studying how mathematicians use gesture, time and motion to describe the concepts of eigenvector and eigenvalue (Sinclair & Gol Tabaghi, 2010), examining how dynamic geometry software can encourage students to think geometrically about eigentheory (Gol Tabaghi & Sinclair, 2013), and investigating the use of modeling problems and APOS theory to teach students the concepts of eigenvectors and eigenvalues (Salgado & Trigueros, 2015). Our current research into students' understanding of eigenvectors and eigenvalues has been influenced by the above work, but we endeavor to extend this growing body of knowledge in two ways. First, we hope to share further insights into how students think about and understand eigenvectors and eigenvalues that has not been reported on previously. Second, we are working towards the development of a framework for student understanding of eigentheory that ties together the work others have done in this area of research. We further discuss this framework within the next section.

Theoretical Framework

There exists a small collection of previous research into students' understanding of eigentheory towards developing a theoretical framework for what it means to have a deep understanding of eigenvectors and eigenvalues (Salgado & Trigueros, 2015; Thomas & Stewart, 2011). Although this research has merit, specifically pointing out the processes, objects, and coordinations necessary for understanding eigentheory, we feel there is more to understanding eigenvectors and eigenvalues than is conveyed therein. Our framework for student understanding of eigentheory aims to include the following ideas: (a) A distinction between the equations $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ and how these constitute two different ways to think about eigenvectors and eigenvalues; (b) the importance of eigenspaces, diagonalization, and their connection to the concepts of eigenvectors and eigenvalues; (c) how the concepts of eigentheory can be thought of within different modes of thinking (Sierpinska, 2000), modes of description (Hillel, 2000), or contexts (Zandieh, 2000); and (d) the various processes (e.g., matrix multiplication, scalar multiplication), entities (e.g., matrices, vectors), and theorems (e.g., invertible matrix theorem) needed to understand eigentheory and the calculations involved therein. While this framework is still under development, it informed our decisions about the creation and refinement of the assessment instrument, and, cyclically, the results of the assessment continue to inform the development of the framework. We describe the assessment more fully in the following section.

Methods

In this section, we describe the development and format of our multiple-choice extended (MCE) assessment instrument. We then describe the data collection and participants for the portion of the study presented in this proposal, followed by a description of our analysis.

Instrument Development

The MCE assessment instrument for eigentheory development grows from our prior work in student understanding of span and linear independence (Zandieh, Plaxco, Wawro, Rasmussen, Milbourne, & Czeranko, 2015) in which we developed the MCE-style question format. During this development, we considered literature on conceptually oriented assessment instruments in undergraduate mathematics and physics (Bradshaw, Izsak, Templin, J. & Jacobson, 2013; Carlson, Oehrtman, & Engelke, 2010; Epstein, 2013; Hestenes, Wells, & Swackhamer, 1992; Wilcox & Pollock, 2014). Questions written in a MCE style begin with a multiple-choice element and then prompt students to justify their answer by selecting all statements that could support their choice, a format based on a concept inventory in Upper-division Electrostatics created by Wilcox and Pollock (2014).

To develop the assessment instrument questions, we compiled a database of questions about eigenvectors, eigenvalues, and related concepts from literature on student understanding of eigenvectors and eigenvalues, online resources for clicker and classroom voting on linear algebra (Cline & Zullo, 2016), and previous linear algebra homework assignments, exams, and interview protocols used by research team members (e.g., Henderson, Rasmussen, Sweeney, Wawro, & Zandieh, 2010). The most promising questions that collectively addressed various aspects of our working framework were edited into the MCE format. The instrument has been administered in student interviews and as written homework twice and subsequently refined. This current proposal relies on a third administration of the assessment, described in further detail below.

Data Collection

In this proposal we present data from written assessments collected from three linear algebra classes taught by the same instructor at a large, research-intensive public university in the mid-Atlantic United States. Each class worked on one version of the MCE assessment for 20-25 minutes during the last day of class. All three versions consisted of the same six multiple choice question elements, with varying justification sections. Class 1 received a version in which students indicated if each of six given justifications were true and relevant, true but not relevant, or false (see Figure 1a). Class 2 received a version in which students only selected justification choices that were true and relevant (see Figure 1b). Lastly, Class 3 received an open-ended version in which they wrote their own justification for their choice.

The matrix $A = \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix}$ has $\lambda = 6$ as one of its eigenvalues. Which of the following vectors is an eigenvector of A with corresponding eigenvalue $\lambda = 6$?

(a) $x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ (b) $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Because … (indicate if the choice is true and relevant, true but not relevant, or false)

True & relevant	True not relevant	False	Statement
			(i) This vector $\boldsymbol{x}$ makes $A\boldsymbol{x} = 6\boldsymbol{x}$ a true statement.
			(ii) This vector $\boldsymbol{x}$ is the only vector in $\mathbb{R}^2$ for which $A\boldsymbol{x} = 6\boldsymbol{x}$.
			(iii) This vector $\boldsymbol{x}$ makes $(A - 6I)\boldsymbol{x} = \boldsymbol{0}$ a true statement
			(iv) Subtracting 6 from the diagonal of A yields this vector $\boldsymbol{x}$ as a column vector of the resulting matrix.
			(v) The vector $A\boldsymbol{x}$ is 6 times the magnitude and in the same direction as this vector $\boldsymbol{x}$.
			(vi) The matrix A also has $\lambda = -3$ as an eigenvalue.

(a)

The matrix $A = \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix}$ has $\lambda = 6$ as one of its eigenvalues. Which of the following vectors is an eigenvector of A with corresponding eigenvalue $\lambda = 6$?

(a) $x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ (b) $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Because … (select ALL that could justify your choice)

(i) This vector $\boldsymbol{x}$ makes $A\boldsymbol{x} = 6\boldsymbol{x}$ a true statement.

(ii) This vector $\boldsymbol{x}$ is the only vector in $\mathbb{R}^2$ for which $A\boldsymbol{x} = 6\boldsymbol{x}$.

(iii) This vector $\boldsymbol{x}$ makes $(A - 6I)\boldsymbol{x} = \boldsymbol{0}$ a true statement

(iv) Subtracting 6 from the diagonal of A yields this vector $\boldsymbol{x}$ as a column vector of the resulting matrix.

(v) The vector $A\boldsymbol{x}$ is 6 times the magnitude and in the same direction as this vector $\boldsymbol{x}$.

(vi) The matrix A also has $\lambda = -3$ as an eigenvalue.

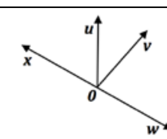
(b)

Figure 1. Comparison of (a) Class 1 MCE and (b) Class 2 MCE.

Analysis

After each class's written assessments were digitally scanned and grouped by question, spread sheets were designed to enter the data from Class 1 and Class 2. From the spreadsheets, the research group examined trends within each class before looking for trends across formats. Some basic percentages were calculated for the justification choices for Classes 1 and 2, as well as students' multiple-choice answers for all three classes.

2. Suppose the vector $\boldsymbol{x} \in \mathbb{R}^2$, in the two-dimensional sketch below, is an eigenvector of a 2x2 matrix M with real-valued eigenvalues. Which of the vectors $\boldsymbol{u}, \boldsymbol{v}$ or $\boldsymbol{w}$ illustrated below could be the result of the product $M\boldsymbol{x}$?



(a) $\boldsymbol{u}$ (b) $\boldsymbol{v}$ (c) $\boldsymbol{w}$
(d) Not enough information is given to know a possible result of the product $M\boldsymbol{x}$

3. Suppose A is a $n \times n$ matrix, and $\boldsymbol{y}$ and $\boldsymbol{z}$ are linearly independent eigenvectors of A with corresponding eigenvalue 2. Let $\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$. Is $\boldsymbol{v}$ an eigenvector of A?

(a) Yes, $\boldsymbol{v}$ is an eigenvector of A with eigenvalue 2.
(b) Yes, $\boldsymbol{v}$ is an eigenvector of A with eigenvalue 5.
(c) No, $\boldsymbol{v}$ is not an eigenvector of A.

Figure 2. Multiple Choice Stem to Questions 2-3 of the Eigentheory MCE.

The open-ended data was analyzed through multiple iterations of open coding. First, each team member individually summarized the key aspects of each student's justification process. Next, the team came together to develop a coding scheme for the student work to be used in the second iteration of coding. Each team member then individually coded the student responses with the new coding scheme before collectively determining a set of codes for each student. Lastly, the larger themes from the finalized coding were identified by examining common patterns across multiple students' solutions and justifications.

The focus then shifted to an analysis of identifying patterns in how students selected or supported their work in their justification for each question and what insight each format provided us about the students' reasoning. Currently the research team is focusing on the first three questions from the assessment (see Figures 1 and 2), with plans to analyze all six later.

Results

Table 1 shows the general performance of the three classes on the multiple-choice portion of the three questions. We make a few observations. First, classes performed similarly on all three questions, so comparing students' justifications across the three classes is reasonable. Second, Question 1 and Question 2 may seem somewhat uninformative because a majority of the students chose the correct answer. However, in the results and discussion to follow, we share how the MCE gives more information about student thinking through the justification portion of each question. Third, linear combinations of eigenvectors with the same eigenvalue, in other words, elements of an eigenspace, is a difficult topic. For sake of space, we focus on the student understanding we discover through the justifications of Question 1 in this proposal.

Table 1
Class Percentages for Answers Chosen on the Multiple Choice Portion

		Class 1 Judgment MCE (27 Students)	Class 2 Original MCE (29 Students)	Class 3 Open-Ended (28 Students)
Question 1	*Choice (a)*	14.8	24.1	7.1
	Choice (b)	81.5	75.9	92.9
	No Answer	3.7	0.0	0.0
Question 2	*Choice (a)*	0.0	0.0	0.0
	Choice (b)	0.0	6.9	0.0
	Choice (c)	92.6	82.8	96.4
	Choice (d)	3.7	10.3	3.6
	No Answer	3.7	0.0	0.0
Question 3	*Choice (a)*	25.9	48.3	21.4
	Choice (b)	3.7	6.9	21.4
	Choice (c)	55.6	31.0	53.6
	No Answer	14.8	13.8	3.6

NOTE: Correct answers are shaded.

Evidence of a Distinction Between the Equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}$ and $(\boldsymbol{A} - \boldsymbol{\lambda}\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$

Justification choices (i) and (iii) on Question 1 (see Figure 1) could be seen as the same statement (to an expert), with a rearrangement of the terms within the equations and an important use of the identity matrix. One might expect that students who selected choice (i) would also select choice (iii), and vice versa. However, four of the 27 students in Class 2 who selected justification choice (iii) did not select justification choice (i), and of the 26 students in Class 1 who said justification choice (i) was true and relevant, three said that justification choice (iii) was true but not relevant. Furthermore, 24 of the 28 students in Class 3, when writing down their justification, focused on some form of only one equation (10 used $A\boldsymbol{x} = \lambda\boldsymbol{x}$, and 14 used $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$), as opposed to the remaining 4 who wrote or used both equations. This gives us evidence that there are two distinct ways to think about eigenvectors and eigenvalues, encapsulated within the two equations, as we had conjectured in our framework development.

Geometric Reasoning on Question 1

Although all 27 students of Class 1 and 28 of the 29 students in Class 2 indicated that at least one of the symbolic justification choices (i) and (iii) were true and relevant, only 12 of the 27 students in Class 1 and 14 of the 29 students in Class 2 indicated that the geometric justification choice (v) was true and relevant. With only the Original MCE, we cannot know why the other 15 students in Class 2 did not select justification choice (v). However, with the MCE given to Class 1, we get more information, as 9 of the 15 students said this choice was true but not relevant, and 5 said this choice was false. We feel this either indicates that students tend to think about eigenvectors and eigenvalues symbolically more than they do geometrically, or that students see symbolic justifications as more acceptable to their teacher or the larger mathematics community. This is further corroborated by the data from Class 3, where none of the 28 students gave any geometric argument when justifying their answer.

Open-Ended Results

Results from the open-ended data consisted of identifying common strategies in students' justifications and possible refinements to the MCE justification choices based on these. From the open coding, two main strategies emerged in how the students approached the problem, namely based on the equation the student chose to focus on, as mentioned above in the section on the $A\boldsymbol{x} = \lambda\boldsymbol{x}$ or $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ equations. Within these two larger strategies, several sub-strategies presented themselves in student work. For example, of the 10 students that used the $A\boldsymbol{x} = \lambda\boldsymbol{x}$ approach, 8 also used a "plugging in" strategy to find the eigenvector, by simply multiplying each vector choice by the given matrix, and seeing which satisfied the $A\boldsymbol{x} = \lambda\boldsymbol{x}$ equation. We classified this approach as the "Eigenvector Definition Check" (EDC) approach.

These classified student solution strategies are also being considered in future refinement to the MCE justification choices. For example, 16 of the students used some form of solving a system of equations in their justification work; however, currently there is no justification choice that deals with this idea. Additionally, in previous work on the MCE, we had identified a solution strategy wherein students calculated the coefficient matrix $[A - 6I]$ and concluded incorrectly that a resulting column vector from the matrix was an eigenvector. Justification choice (iv) was added to the MCE to capture this solution method. However, disconcertingly, only two students in Class 3 wrote down this strategy, whereas in Class 1, 10 students selected (iv) as true and relevant and three students selected it as true and not relevant, and in Class 2, six of the 29 students selected it as true and relevant. Thus, further modification of this justification is needed to only tempt students who think of this strategy as the only way to find eigenvectors.

Discussion

One might posit we would learn more about student understanding of eigentheory through open-ended questions than other forms of written assessment. However, we have shown how the MCE format can provide rich information regarding how students think about and understand eigenvector and eigenvalue problems; in particular, we can gain more information about the aspects and contexts of eigentheory students see as connected and relevant in justifying their solutions. As we pursue research in this area, we ask for feedback on the following questions:

1. What nuances of student understanding of eigentheory do we learn from the various MCE formats, and which format do you think would be most promising in future use?
2. With the MCE data, what ideas for analysis (statistical tests or otherwise) do you believe would be useful and informative?

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Meta-Representational Competence with Linear Algebra in Quantum Mechanics

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In this report we shared our preliminary analysis of one student's meta-representational competence as he engages in solving a quantum mechanics problem involving linear algebra concepts, namely basis, eigenvectors, and eigenvalues. We provide detail on student A25, who serves as a paradigmatic example of a student's power and flexibility in thinking in and using different notation systems. This preliminary work lends credence to and inspires our conjecture that strong meta-representational competence (MRC) is necessary not only to be fluent and proficient in the mathematics involved in solving quantum mechanics problems but also to develop a robust understanding of the quantum mechanics content.

Key words: *linear algebra, physics, matrix notation, Dirac notation, symbolizing*

The National Research Council's (2012) report, which charges the U.S. to improve its undergraduate STEM education, specifically recommends "interdisciplinary studies of cross-cutting concepts and cognitive processes" (p. 3) in undergraduate STEM courses. It further states that "gaps remain in the understanding of student learning in upper division courses" (p. 199), and that interdisciplinary studies "could help to increase the coherence of students' learning experience across disciplines … and could facilitate an understanding of how to promote the transfer of knowledge from one setting to another" (p. 202). Our work contributes towards this national need for basic research by investigating students' understanding, symbolization, and interpretation of eigentheory and related key ideas from linear algebra in quantum physics.

In this preliminary report, we focus on one student's reflection on explicit symbolization choices he makes while solving quantum mechanics problems that involve linear algebra. In particular, we inspect data of one student solving an expectation value problem and his reasons for how and why he chooses a specific symbol system – either Dirac notation or matrix notation – for that particular situation. We align our analysis with the framework of meta-representational competence (diSessa, Hammer, Sherin, and Kolpakowski (1991), as well as the delineation of structural features of algebraic quantum notations offered by Gire and Price (2015). We conjecture that strong meta-representational competence (MRC) is necessary not only to be fluent and proficient in the mathematics involved in solving quantum mechanics problems but also to develop a robust understanding of the quantum mechanics content.

Background and Theoretical Framework

In this section, we give an overview of research conducted on student understanding of symbols and representations in mathematics and physics, as well as our theoretical orientation. We conclude with a brief introduction to eigentheory in Quantum Mechanics and Dirac notation.

Student Understanding of Symbols and Representations

The recognition of the importance of students having an understanding of the symbols used in mathematics and physics has grown over the past few decades. Arcavi (1994, 2005) coined this as "symbol sense," which includes (a) being "friendly" with symbols, (b) reading through symbols, (c) engineering symbolic expressions, (d) understanding different meanings based on equivalent expressions, (e) choosing which aspects of a mathematical situation to symbolize, (f) using symbolic manipulations flexibly, (g) recognizing meaning within symbols at any step in the solution process, and (h) sensing the different roles symbols can play in various contexts. Many researchers along this vein have examined the ways students learn, make sense of, and use mathematical symbols and notations (e.g., Kaput, 1998; Meira, 2002; Van Oers, 2002).

Research into students' competence with symbols, inscriptions, and representations is not limited to K-12 studies. For example, Harel and Kaput (2002) describe how mathematical notations play a key role in forming conceptual entities in higher mathematics. Additionally, in linear algebra research, Hillel (2000) described three modes of description (abstract, algebraic, and geometric) of the basic objects and operations in linear algebra and pointed out that "the ability to understand how vectors and transformation in one mode are differently represented, either within the same mode, or across modes is essential in coping with linear algebra" (p. 199).

Extending even further, research into students' understanding in quantum mechanics has looked at how students make sense of and use a novel notation, called Dirac notation (explained in the subsequent section). Singh and Marshman (2013) showed that even after graduate level instruction in quantum mechanics, students still struggle with Dirac notation, showing inconsistencies in its use among contexts and problems. More closely related with this current study, Gire and Price (2015) looked at four structural features of three different notation systems used in quantum mechanics (Dirac, matrix, and wave function) and how students' reasoning interacts with these features. The features identified by the authors are: (a) individuation, or "the degree to which important features are represented as separate and elemental" (p. 5); (b) externalization, or "the degree to which elements and features are externalized with markings included in the representation" (p. 7); (c) compactness; and (d) symbolic support for computation. Using problem-solving interviews with students as insight into these features, Gire and Price found that students readily used Dirac notation, and that the structural features vary across the different notations as well as among several contexts within quantum mechanics.

Relatedly, diSessa et al. (1991) importantly discovered that students have a great deal of knowledge about what good representations are and are able to critique and refine them, which the authors defined as Meta-Representational Competence (MRC). diSessa and Sherin (2000) explained that MRC includes inventing and designing new representations, judging and comparing the quality of representations, understanding the general and specific functions of representations, and quickly learning to use and understand new representations. Furthermore, diSessa (2002, 2004) offered a list of critical resources students possess as part of their MRC for judging the strength of representations, such as compactness, parsimony, and conventionality. Iszák and his colleagues (Iszák, 2003, 2004; Iszák, Çağlayan, & Olive, 2009) extended this research by looking at students' MRC with the creation and critique of algebraic representations, and demonstrated students and teachers have criteria for good algebraic representations, though the criteria may not align between teacher and student.

Brief Introduction to Eigentheory in Quantum Mechanics and Dirac Notation

In quantum mechanics, certain physical systems are modeled and made sense of using eigentheory. To a physical system we associate a Hilbert space (such as $\mathbb{C}^2$), to every possible state of the physical system we associate a vector in the Hilbert space, and to every possible observable we associate a Hermitian operator (usually given in its matrix form). The only possible result of a measurement is an eigenvalue of the operator, and after the measurement the system will be found in the corresponding eigenstate.

Dirac notation, also known as bra-ket or just ket notation, is a common notational system in quantum mechanics. A vector representing a possible state is symbolized with a ket, such as $|\psi\rangle$. Mathematically, kets behave like column vectors, such as $|\psi\rangle \doteq \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $a_1, a_2 \in \mathbb{C}$, and are usually normalized. To each ket is associated a bra, such as $\langle\psi| \doteq [a_1^* \quad a_2^*]$, its complex conjugate transpose that behaves like a row vector. The eigenvalue equations for observables are central to many calculations. For instance, the eigenvalue equations for S_x (the operator measuring the intrinsic angular momentum along the x-direction of a spin-½ particle) are $S_x|+\rangle = \frac{\hbar}{2}|+\rangle$ and $S_x|-\rangle = -\frac{\hbar}{2}|-\rangle$, where $|+\rangle$ and $|-\rangle$ represent the orthogonal eigenbasis of S_x. Because of the orthonormality of eigenbases of Hermitian operators, the inner products yield $\langle+|+\rangle = 1, \langle-|-\rangle = 1, \langle+|-\rangle = 0$, and $\langle-|+\rangle = 0$. An elegant use of Dirac notation involves change of basis; because the eigenbasis of one operator is often well known in terms of another, such as along the z-direction of a spin-½ particle, Dirac notation is seen to make basis change calculations efficient.

Methods

Participants for this study were junior physics majors at a large, public, research-intensive university in the Pacific Northwestern United States. They were drawn on a volunteer basis from a class of 35 students in a Spin and Quantum Measurements course; this course met for 7 class-hours per week for three weeks and involved many student-centered activities and discussions. Data for this report were collected during individual, semi-structured interviews (Bernard, 1988) conducted with 8 students at the end of the course.

To begin our analysis, we viewed the video and observed how students navigated the interview problems, with no set ideas ahead of time of what we would analyze, while keeping in mind the overarching research focus of students' understanding, symbolization, and interpretation of eigentheory in quantum physics. We looked for correct, incorrect, and unique reasoning approaches. Throughout our viewing, we noticed two students were particularly fluent in how they talked about and worked with both matrix and ket notations. This compelled us to investigate the literature about student use of symbols and notations, the most relevant of which were discussed above. As such, we began an analysis drawing on the work of diSessa and colleagues, Iszák and colleagues, and Gire and Price (2015). We reviewed the video data through this lens, coding for instances of students mentioning structural features of the mathematics or students making explicit meta-commentary on the representations they were using.

In this preliminary report, we focus on one student: A25, a double major in physics and nuclear engineering who had completed two quarters of linear algebra. The purpose for focusing on this participant is his demonstrated ability to articulate his thinking. During the interview, he demonstrated flexibility in reasoning about the concepts we were probing, and through his explanation a great deal of MRC was visible and analyzable.

Preliminary Results

In the beginning of the interview, student A25 volunteered that he sometimes explicitly chooses between doing calculations in matrix notation or in Dirac notation:

I: So how do you feel like, using eigenvectors and eigenvalues, in spins has been similar to and different from how you've experienced those in other classes?

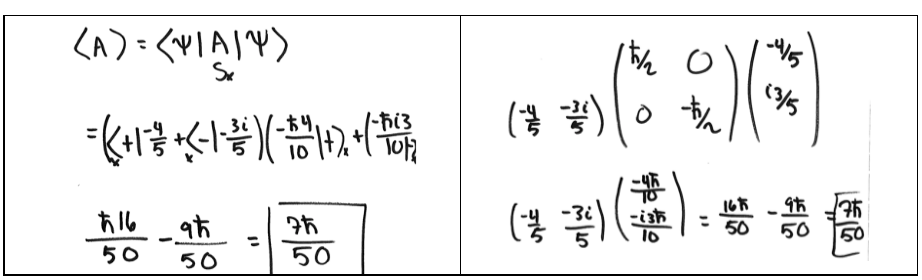
A25: Uh, well, it's very similar because you're doing a lot of the same math …the difference especially in physics, you're looking at kets. In, in at first I was kind of jarring, like to- to

> try to do the math in kets. But now, it's kinda- it's kinda easier, there's problems, there certain problems…where there's two ways to do them, they're kind of parallel, you can do it and you can expand the- the- the state in- in like as a- and expand them as- as kets in a different basis, or you can write that state as a- as a, as a vector, in that basis, and you can either do the matrix math for the like expectation values for example, you can do the matrix math or you can do the ket math, and sometimes it's, I'm finding that I, rather expand something in the ket.

From the transcript we see that A25 was aware that there exist multiple legitimate ways to solve the problem, seemingly understanding the various mathematical nuances and implications of his notational choices. His brief explanation highlights sentiments that are consistent with Arcavi's characteristics of symbol sense, such as being "friendly" with symbols and using them flexibly. We add, however, a metacognitive aspect of symbol sense here, noting that A25 was engaged in self-reflection rather than a researcher analyzing A25's engagement with symbols.

Because A25 volunteered expectation value problems as a situation in which he could use either notation, the interviewer had him work on such a problem right away, even though it was prepared to be at the end of the interview: "Consider the state $|\psi\rangle = -\frac{4}{5}|+\rangle_x + i\frac{3}{5}|-\rangle_x$ in a spin-1/2 system. Calculate the expectation value for the measurement of S_x." A25 immediately worked on the problem within Dirac notation, saying, "basically to find the expectation value… it's like denoted that way [writes $\langle A\rangle$] but really what you're doing is you're taking the, the bra of the state, and then you're putting the operator [writes $= \langle\psi|A|\psi\rangle$] in the middle of the inner product…" He continued to explain his work as he went, arriving at the correct answer of $7\hbar/50$ (see Figure 1a). Note that his work in Figure 1a involved the state's expansion and use of eigenvector equations for S_x in ket notation. In addition, this notation was first introduced to the students during this course; as such, A25 was clearly quick to use and understand this representation (a quality of MRC, diSessa & Sherin, 2000).

After discussing his work and solution, the interviewer asked: "Before you were telling about bra-ket versus matrix notation, you brought up an expectation value as an example of like, either or both, so can you, now that you had this problem, kinda revisit that?" A25 immediately solved the problem completely within matrix notation (see Figure 1b), explaining his steps. For instance, in line 1 in Figure 1b, he wrote the complex conjugate transpose of the vector representation of the state in the x basis, the matrix representation of the S_x operator in the x basis, and the vector representation of the state in the x basis. He also stated his process for computing the matrix times the column vector before he did the computation, and again in line 2 he explained "then I do it again, so, um, this time you're gonna get a number out," meaning he anticipated that a row vector times a column vector would be a number. We see this as flexibly using symbolic manipulations (Arcavi, 1995) and an anticipation of results.



(a) (b)

Figure 1. A25's expectation value problem, in ket notation (a) and matrix notation (b).

The interviewer then asked A25 to reflect on his preference between the two notations:

A25: Uh...To be honest, I don't really, I don't really know why I prefer this [Figure 1a], I think it's just because, um, I like this notation. This- this specific notation [Figure 1a line 1] like this to me is like a cleaner way of writing that [Figure 1b line 1] because that- I mean this and that [touching lines 1 in both figures] I feel like are your starting points, so you, you start here with this nice, like, looking thing [traces his finger under $\langle\psi|A|\psi\rangle$], or you start here with this big array of numbers [puts open hands around Figure 1b], and I prefer this [Figure 1a line 1], even though you have to expand this into basically the same amount of information [Figure 1a line 2]. And also, the nice thing about, about this [Figure 1a line 1], is it—actually this is really why it's better—is because you can, you can say ok S_x works- acts directly on these kets, you can just get rid of the matrix altogether...

We see his use of "nice looking thing" and "big array of numbers" in comparison to one another are an example of compactness. He also compares line 1 in 1(a) and line 2 in 1(b) regarding the "amount" of information they convey, which involves reflection on the physical and mathematical content expressed in the compared notations. Finally, acting directly on the expansion in terms of the eigenstates of the operator allow him to forego the matrix calculation entirely, which speaks to A25's view of compactness, parsimony, and symbolic support for using ket notation for this problem.

When asked about his notation preferences if the basis expansion of a given state vector and the operator "didn't match," A25 recalled a problem from their last homework that was "actually easier…to do the matrix multiplication," stating "you don't want to have to change these kets into different bases all over the place 'cause they're already all written in the same basis and you know what the operator is in that basis so you might as well just, do the matrix multiplication." This speaks to his awareness of symbolic support as well as using symbols flexibly. Finally, when asked if the notions of basis or eigenvectors/eigenvalues come up more in one notation than the other, A25 stated, "certainly…every time you write down a ket you're, you're very conscious of what basis you're in. In this one [points to Figure 1(b)] it's just kinda implied…all this [is] in the same basis, so you're just, you're just writing out numbers, an arrays of numbers, but here [in Figure 1(a)] you're thinking ok, this is the S_x operator, this is the x plus ket, this is the x minus bra…so I think that you're definitely more aware of what basis you're in when you're using this, because you have to be." This explanation is consistent with Gire and Price's (2015) notion of externalization, in that the ket notation allows features of the problem, namely basis, to be externalized in a way that the matrix notation did not provide for A25.

Conclusion

In this report we shared our preliminary analysis of one student's meta-representational competence as he engaged in solving a quantum mechanics problem involving linear algebra. This is a paradigmatic example of a student's power and flexibility in thinking in and using different notation systems. During our presentation, we will provide additional data and analysis on A25 and other students regarding observed MRC during their interviews. We would benefit from discussion with the audience regarding the following: (1) what aspects of MRC seem most important to success in solving quantum mechanics problems involving linear algebra, and (2) how tied to a robust understanding of the quantum mechanics content might MRC be, and how could that be explored through analysis?

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An Investigation of the Development of Partitive Meanings for Division with Fractions: What Does It Mean to *Split Something into 9/4 Groups*?

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In this paper we describe a study involving twelve preservice elementary teachers who were attending a community college. The design and implementation of this study were guided by the research question: In what ways do students reason through a sequence of tasks which progressively become more abstract, and which challenge primitive intuitions regarding partitive division? We highlight students' ways of thinking involved with division that are not easily generalizable, that favor numerical procedures over quantitative reasoning, and which are obstacles to the development of more robust meanings for division.

Keywords: Fraction Division; Partitive Division; Preservice Elementary Teachers

Discussion of the Literature

Researchers (Fischbein, Deri, Nello, & Marino, 1985; Harel, Behr, Lesh, & Post, 1994; Rizvi & Lawson, 2007; Simon, 1993; Thompson & Saldanha, 2003; Tirosh, 2000) have long acknowledged some primitive ways of thinking about partitive and quotitive division. Partitive division is when $a \div b$ is interpreted as the amount of the quantity referenced by a per one unit of the quantity referenced by b, given that the quantities are proportionally related. One primitive model for partitive division identified by researchers is *fair sharing*, which is characterized by thinking about splitting a into b equal parts. This primitive model requires the divisor to be a whole number, and thus the value of the quotient should be less than the value of the dividend; in other words, division makes smaller. Quotitive division is when $a \div b$ is interpreted as the number of copies of b that make a, which can also be interpreted as the measurement of a in units of b. Concerning quotitive division, the primitive model of *repeated subtraction* requires the divisor to be smaller than the dividend. These primitive models for division are rooted in reasoning with whole numbers and they continue to influence the reasoning of students and teachers, even after they are exposed to more sophisticated models (Fischbein et al., 1985). In particular, these primitive models obstruct sensible reasoning pertaining to division involving fractional values.

In a study of preservice elementary teachers, Simon (1993) noticed that the subjects could accurately execute procedures for long division of whole numbers, but that these procedures were not well connected to the subjects' meanings for division. He stated that "their lack of conceptual understanding given their algorithmic competence seems to challenge the idea that procedural practice eventually leads to understanding (Simon, 1993, p.249)." In a study of preservice elementary teachers, Tirosh (2000) investigated the impact of primitive models on division involving fractional values. She observed that the subjects heavily relied on procedures, such as *flip and multiply*, instead of relying on meanings for division. She introduced the subjects to formal justifications for the flip and multiply algorithm, but these formal arguments were based largely on symbolic manipulation of non-contextualized variables, and it was unclear whether these justifications would be accessible to elementary students. Rizvi and Lawson (2007) noticed that none of the 17 preservice teachers from their study could explain the flip and multiply algorithm, nor could any of the subjects initially pose a word problem that required division by a fractional value. The researchers attributed these deficiencies to a reliance on the primitive fair sharing and repeated subtraction models for division.

We have found little research on partitive meanings for division when the divisor is a fractional value. A quick survey of textbooks and online resources is likely to reveal that many attempts to connect numerical division of fractional values to a meaning for division are based on the quotitive division model. However, some researchers and educators (Beckmann, 2011; Gregg & Underwood Gregg, 2007; Kribs-Zaleta, 2008; Ott, Snook, & Gibson, 1991) have illustrated partitive models for division with fractions, but they do not discuss the development of these meanings. This paper reports on a study designed to investigate the development of partitive meanings for division with fractions.

The Partitive Model: What Does it Mean to *Split Something into 9/4 Groups*?

The design of this study assumed the subjects had existing meanings for fair sharing with whole numbers of groups, as well as meanings for fractions as operators. As such, we intended to build on these meanings by introducing the subjects to situations that require partitive division with whole divisors, followed by situations with non-whole divisors. Let's consider a learning progression that begins with primitive meanings for partitive division with whole divisors, and ends with robust meanings that accommodate non-whole divisors. Consider 6 cups of water fitting perfectly in 3 equally-sized whole containers. The relative size of one container's capacity to the total amount of water is critical – if 6 cups fit into 3 equally-sized containers, then one container holds 1/3 of the 6 cups. This way of thinking forms a meaningful foundation for the numerical equivalence of the expressions $6\div3$ and $1/3\times6$. Next, suppose 6 cups of water fit perfectly in 9/4 containers. Interpreting 9/4 containers as 9 quarter-containers allows one to reason that 1/9 of 6 cups will be in each quarter container with four copies constituting the capacity of one whole container (see Figure 1). Thus a whole container's capacity would be $4\times1/9\times6$, or $4/9\times6$ cups. This idea yields a numerical equivalence between $6\div9/4$ and $4/9\times6$. As another example, consider a situation where 6 cups of water fit perfectly in 2/3 of a container. Thus, 1/3 of the container holds 1/2 of 6 cups, and the whole container holds 3 times as much as 1/2 of 6 cups (see Figure 2). This way of thinking yields a numerical equivalence between $6\div2/3$ and $3/2\times6$. The study described in this paper investigates the development of these schemes for partitive division involving fractions.

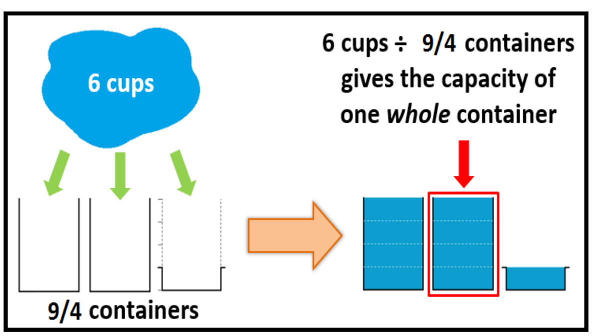

***Figure 1*. 6 cups in 9/4 containers.**

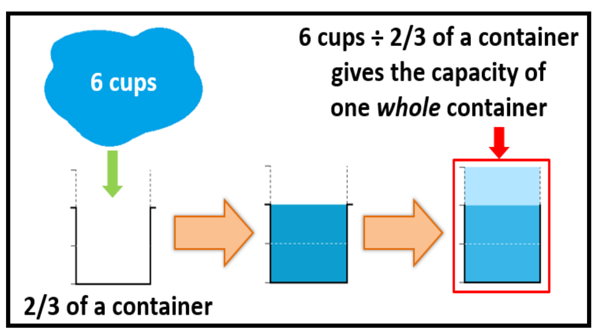

***Figure 2*. 6 cups in 2/3 of a container.**

Methodology

This study focused on twelve elementary education students in a *Mathematics for Elementary Teachers* course at a community college. An instructional unit, focused on the meanings of division, was implemented in the course and spanned three class sessions. Of the five learning objectives for this unit, two are the focus of this paper: (1) Use partitive and quotitive meanings for division (instead of algorithms) to divide by rational numbers and (2) Make sense of the *flip and multiply* procedure by using the partitive meaning for division. We issued a total of four assignments to each of the twelve students in the class. The first assignment preceded any formal class discussion on division and the remaining three assignments were given out after each class session. Two students, one higher-performing and one lower-performing, were selected to participate in videotaped interviews while they worked through the assignments. Leveraging Goldin's (2000) principles, the interviews with these two students were semi-structured and task-based, with the purpose of investigating their thinking. We asked the other ten students to each work on their assignments alone and without resources, to do the tasks from each assignment in order, and to clearly present their solutions in writing. The design of this investigation and subsequent data analysis were guided by the research question: *In what ways do students reason through a sequence of tasks which progressively become more abstract, and which challenge primitive intuitions regarding partitive division?*

Discussion of the Data

During this study, the subjects participated in a variety of tasks. In this paper, we narrow our discussion to the data from the following three tasks:

Task 1: Divide 27 gallons of water into 9/4 containers. How much water is in one whole container?
Task 2: Suppose an unknown amount of water is divided into 9/4 containers. What could you say about how much water is in one whole container?
Task 3: Explain why it is that when you divide by a fraction, you can multiply by the reciprocal of the fraction instead. In other words, explain the following: $a \div \frac{b}{c} = a \cdot \frac{c}{b}$

We designed these tasks to be successively more abstract. For Task 1, we considered a response to be correct if it was *12 gallons of water in one whole container*. For Task 2, we considered a response to be correct if it was conceptually equivalent to saying *4/9 of the water is in one whole container*. For Task 3, we considered a response to be correct if it was a generalization of valid thinking from Tasks 1 and 2, or some other valid explanation. Table 1 summarizes our analysis of the subjects' responses to these three tasks, and it reveals that as the tasks become more abstract, the students became less successful overall. We now discuss the thinking of the two students who were videotaped. We will refer to them as Adam and Sue.

Table 1. Summary of Results of Tasks 1 through 3.

	Task 1	Task 2	Task 3
Number of students with a correct response	8	4	1
Number of students with an incorrect response	3	5	4
Number who said "I don't know" or gave no response	1	3	7

The Case of Adam

Adam was the only student out of the twelve subjects who demonstrated valid reasoning during all three tasks. For Task 1, he drew 9/4 containers, then decided to procedurally compute 27÷9/4 by inverting and multiplying to get 108/9, which he reduced to 12 through procedural division. He admitted to using the numerical operation of division because the word *divide* was in the prompt. Once he calculated that the answer should be 12, he decided to find the amount in each quarter-container, by dividing 12 by 4 to get 3. He did not indicate that he could have also divided 27 by 9 to get 3. In fact, his calculation of 27×4/9 by first multiplying 27 by 4 is indicative that he was operating numerically and not thinking quantitatively about partitioning and iterating. He then confirmed that a total of 27 gallons was in the 9/4 containers by saying that 12 gallons were in each of the two whole containers and 3 more gallons were in the remaining quarter-container (see Figure 3).

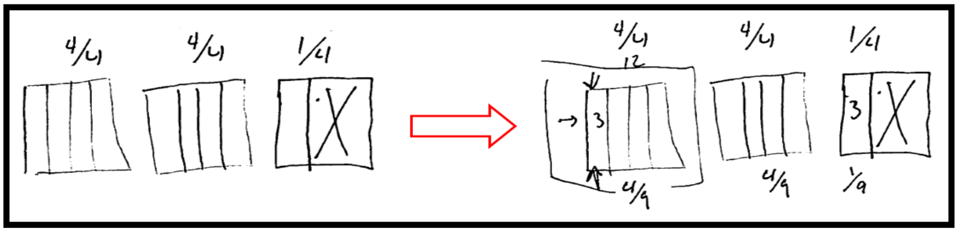

***Figure 3*. Adam's response to Task 1.**

During the interview, Adam's language was not consistent. He sometimes referred to each quarter-container as a whole container, as well as misspeaking about other referents. However, we suspect that his language was simply misrepresenting his valid thinking. Figure 3 also reveals that Adam labeled one container as "4/9", and he said that "4 out of the 9" pieces make one whole container. As such, and not surprisingly, for Task 2 he wrote that "4/9 of the whole" corresponded to one whole container. For Task 3, Adam demonstrated that he was beginning to generalize his thinking. He attempted to explain the algorithm by describing division by 7/2, as depicted in Figure 4. When pressed to speak in terms of *a*, *b*, and *c*, Adam responded by describing partitive thinking, but then gave a quotitive example. He said "The whole thing is the *a*...and I'm cutting it up to a certain amount of pieces. I don't know how many… abstract… but let's say it's one third. And I want to know how many one thirds fit into it [referring to the whole]." Ultimately, Adam was unable to generalize the partitive division models that he had earlier demonstrated using specific fractional values for the divisor.

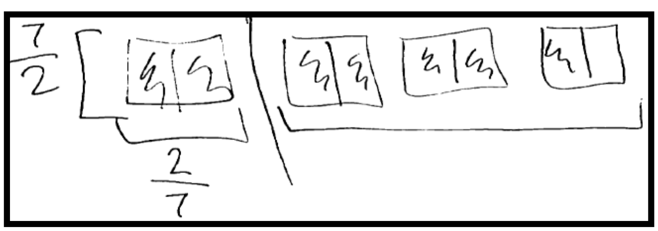

***Figure 4*. Adam uses division by 7/2 to illustrate the invert and multiply procedure.**

The Case of Sue

Sue appeared to have no trouble with Task 1. She drew nine contiguous boxes, marked "3" on each, indicated that four such boxes made a whole container, and concluded that 12 gallons were in one whole container. This is depicted in Figure 5.

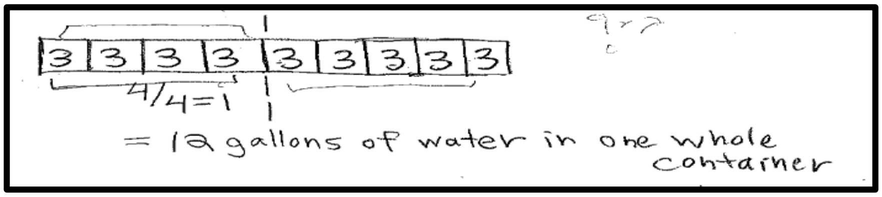

***Figure 5.* Sue's work on Task 1.**

On Task 2, Sue was immediately perturbed. She said she didn't know what to do, pointing out that the amount of water was not provided like it was in Task 1. The researcher asked what she would do if the amount of water was 18 cups. She proceeded to answer this question in the same way that she did for Task 1. The researcher then asked her to consider what she would do if "*x* cups" were the total amount of water, but she was ultimately unable to respond. Since Sue was stumped, the researcher returned to Task 1 and asked her how she knew to put a "3" in each box. She explained that she knew the total had to be 27 gallons and that there were nine boxes. She mentioned that each of the nine boxes had to have the same amount of water and mentioned a guess and check strategy. Two gallons per box was too little ("nine times two is only 18"), but nine times three gives the correct 27 gallons. The researcher then asked her why she answered that 12 gallons were in one whole container. She said that each box was one fourth of a container, so she added the four copies of three gallons to get 12 gallons. The researcher then drew nine contiguous boxes, shaded in four of them, and asked Sue how much of the entire collection was shaded. Sue promptly answered "four ninths". The fact that Sue answered "four ninths" so quickly in the latter situation indicates that Sue's meanings for fractions are likely limited to the *out of* model and that she does not have developed meanings for fractions as operators. As such, her schemes for solving Task 1 were not generalizable to the point where she could sensibly talk about Task 2. For Task 3, Sue did not know how to respond.

Conclusion

Primitive ways of thinking about division continue to be pervasive in mathematical thinking. This research explored the development of more robust meanings for partitive division, which are not thwarted by non-whole divisors. The initial data reveals that underdeveloped meanings for fractions are impediments to the maturation of robust meanings for division. For example, Sue mentioned "four ninths" when she saw 4 out of 9 boxes shaded, but she did not appear to think of four ninths as an operator on the total amount of water. The data also suggests that quantitative operations, such as partitioning and iterating, are often neglected in favor of procedural approaches to division. This was illustrated by Adam, in Task 1, when he procedurally calculated that 12 gallons were in each whole container; yet, there was no indication that he partitioned the total amount of water into 9 equal pieces. Additionally, we see examples of schemes for division that are not generalizable to more abstract levels of meaning. For example, Sue's guess and check scheme in Task 1 was dependent on knowing the total amount of water. It is evident that the development of partitive meanings for division with fractions depends on more robust meanings for both fractions and division. Additional research is needed to better investigate this claim.

Acknowledgement

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Corequisite Remediation and Math Pathways in Oklahoma

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We examine the current progress of implementing both corequisite remediation and math pathways in the state of Oklahoma. In this paper, we discuss the details of these effort and the underlying needs while providing a national perspective about the reforms. We present preliminary data from pilot sections of a corequisite College Algebra course and a new math pathway for degrees that require significant quantitative literacy but do not require engineering calculus. We also present statewide data on student course-taking patterns, degree requirements, and existing institutional efforts that will inform state-level decisions.

Key words: corequisite remediation, mathematics pathways, reform

Introduction

The state of Oklahoma is currently in the process of reforming introductory post-secondary mathematics options and curriculum across all 27 public higher education institutions with the goal of increasing success in college mathematics courses and therefore increasing degree attainment across the state. To accomplish these goals the Oklahoma State Regents for Higher Education, the governing body for all public higher education institutions in Oklahoma, adopted the Complete College America agenda (Complete College America, 2013). The main focus of the state reforms are 1) supplementing the current system of remedial courses with a corequisite model and 2) creating multiple introductory mathematics pathways better aligned to diverse degree programs.

Traditionally, to ensure preparation of entering students, colleges assess students using various criteria (e.g. SAT scores, ALEKS, etc.) then place them into college courses using these measures. Under-prepared students are placed in a remedial course sequence designed to fill in deficiencies from secondary mathematics and prepare them for college-level courses. Once a student completes the remedial course sequence, they are allowed to take credit-bearing math courses required for a degree. However, this remedial system often fails in its ultimate goal. In 2010, Bailey et al found that only 31% of community college students referred to a remedial courses sequence in mathematics completed it often due to a failure to enroll. Even more troubling, only half of the students completing the remedial course sequence enrolled in the gatekeeper college-level course passed (2010). Meaning only 15% of students referred to remediation passed the subsequent college credit-bearing course, which is significant as 58% of entering students community college students enrolled in a remedial mathematics course (Attewell et al., 2006). Overall, 28% of college students from two or four-year colleges enroll in remediation (Attewell et al., 2006). Furthermore, delaying enrollment in college-level courses in favor of remedial courses has the consequence of extending students' time to degree, meaning both an increased cost and decreased persistence toward a degree (Complete College America, 2011).

Alternatively, in corequisite remediation, underprepared students are placed directly in college-level courses with targeted assistance. The aim of the model is to allow under-prepared students to earn college credit upon entering while still providing the students with necessary perquisite material, thereby eliminating often multiple semesters of remedial courses and enabling students to progress to through their degree programs. Corequisite remediation has been successful in several pilot programs across the country, which will be briefly discussed in the next section.

The second focus of the reforms are creating mathematics pathways beyond the standard College Algebra/Calculus sequence. Creating additionally pathways provides students in non-STEM disciplines basic mathematics courses more relevant to their interests and needs. The reforms seek to increase collaboration between math departments, other academic degree programs, and employers as to the necessary mathematical knowledge and skills for students in their chosen field. For many, courses in statistics, quantitative literacy, or mathematical modeling are more applicable to future collegiate and career needs. The need for increasing such diverse mathematical competencies has been highlighted by several reports from professional associations including: the *Common Vision 2025* report by the Mathematical Association and America (Saxe, 2015), the *Guidelines for Assessment and Instruction in Statistics Education (GAISE): College Report* endorsed by the America Statistical Association (Aliaga, 2005), and the *Beyond Crossroads* Report of the American Mathematical Association of Two-Year Colleges (Foley, 2007). Additionally, by creating clearly defined mathematics pathways at the state level, Oklahoma is aiming to increase transferability of mathematics courses between public institutions.

In this paper, we address the following research questions:

1. What are the primary national trends and lessons in corequisite remediation and math pathways relevant to the goals and structures of the Oklahoma higher education system?
2. What are the primary obstacles in implementing corequisite remediation and math pathways in Oklahoma, and what factors can local and state leaders influence to address these challenges?

A National Perspective of the Reforms

Several states have either implemented or are in the process of implementing the reforms outlined above. We briefly describe progress in two of the states to lead these reform efforts, Georgia and Tennessee.

Currently, Georgia has two pathways: the traditional College Algebra/Calculus pathway and a non-algebra pathway which focuses on either quantitative reasoning or modeling. In Fall of 2014, Georgia piloted corequisite remediation for both pathways. 67% of the 2919 students in the non-corequisite sections passed the gateway course. In the corequisite sections, there was a total 1,132 students 64% of whom passed. Comparably, only 21% of students referred to remedial education in 2010 passed their gateway course within two years (Complete College America, 2015).

Beginning in Fall of 2014, Tennessee conducted a pilot corequisite program for an introductory statistics courses with 1,019 students at 9 different campuses. Tennessee saw similar results to Georgia, 63.3% of students assessed as being underprepared pass introductory statistics whereas under the previous remediation model only 12.3% had pass the introductory statistics course.

Similar results can been see in other states (Complete College America, 2016). We will continue to examine the progress and challenges of the reforms across the country and how their efforts can inform the reforms in Oklahoma.

Progress in Oklahoma

Corequisite Remediation

In fall of 2015, Oklahoma State University began piloting corequisite courses. The ALKES test is used to assess entering student college readiness. Normally, to place into College Algebra a score of 45 out of 100 is required. OSU ran three pilot sections consisting of 87 students scoring between 30 and 44 on the ALEKS test. The students in the corequisite sections attended class five days a week. Three days were dedicated to regular instruction, similar to the standard three-hour College Algebra class. During the other two days, an undergraduate learning assistant engaged students in active learning sessions designed by an experienced course coordinator to improve students' prerequisite knowledge. Students in the corequisite sections completed the same homework and took the same exams as students in regular College Algebra sections. Table 1 shows the percentage of the students in these courses earning a D and F, or withdrawing (D/F/W rate).

Table 1
Data from pilot sections.

	Enrollment	Proportion of first-generation students	Overall D/F/W rate	First-generation D/F/W rate
All sections	820	20.7%	31.6%	35.9%
Regular sections	733	18.8%	31.2%	37.0%
Pilot sections	87	36.8%	34.5%	31.3%

While the D/F/W rate overall for the pilot corequisite sections was slightly higher than for regular sections, the D/F/W rate for first generation was significantly lower. Traditionally only a small fraction of students enrolling in remedial courses succeed in that course, continue to College Algebra, and further succeed in College Algebra. Equally important is student persistence to and success in subsequent mathematics or statistics courses which is shown in Figure 1.

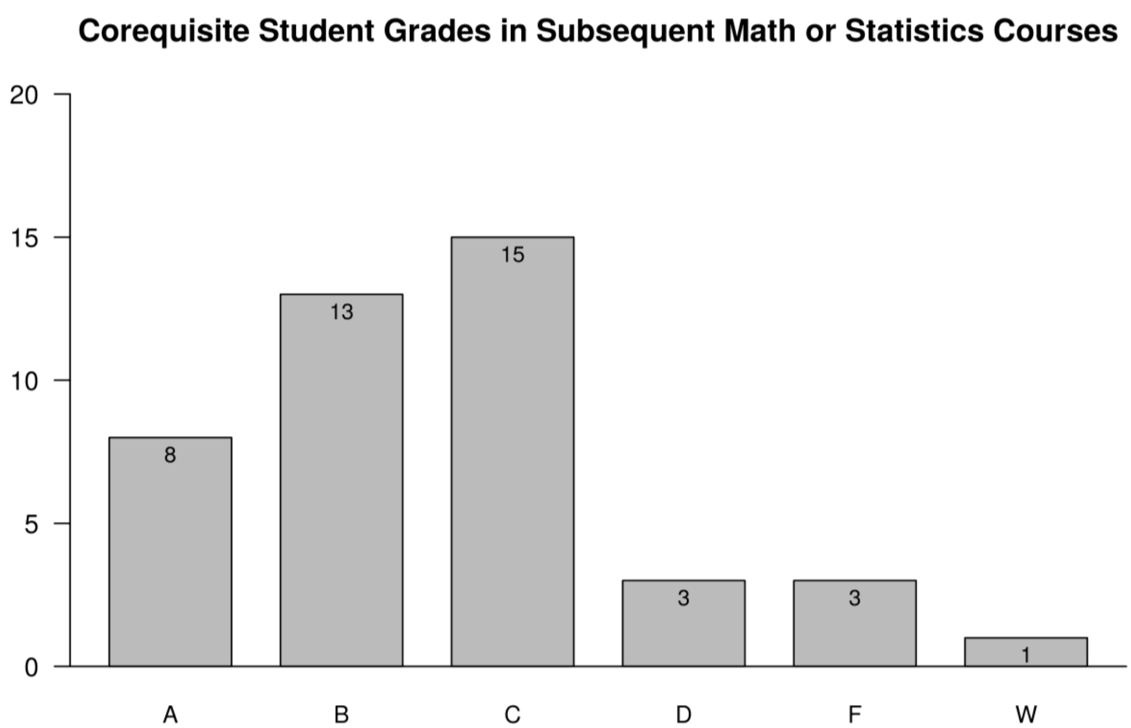

Figure 1. Corequisite students' grades in subsequent mathematics or statistics courses.

With regard to the state wide reforms, we have collected surveys from all public institutions in the state on their implementation of corequisite courses. We are in the process of analyzing this data, which will provide a useful baseline for comparison as the state reforms unfold.

Math Pathways

Oklahoma State University has also created a course *Mathematical Functions and Their Uses* (MATH 1483) which is being offered as an alternative to College Algebra for many business, social science, health, and agriculture degrees that do not require an engineering calculus course. The course emphasizes quantitative reasoning by modeling data with a calculator and/or excel and serves as an equally successful preparation for Business Calculus as College Algebra. The D/F/W rates are typically 15% and 25% for the fall and spring respectively. As degree requirements shift away from college algebra, we are in the process of analyzing D/F/W rates from subsequent gateway courses. Moreover, the shift to functions has led to an increased success in College Algebra. Additionally, in Fall 2016, the university began piloting a corequisite section of the functions course allowing traditionally underprepared students another pathway to earn college credit upon entering.

Statewide, we collected detailed information on the mathematics requirements for every major at 26 of the 27 state public institutions (we are still seeking information from the last institution) and complete statewide data for student enrollment and success in math courses by major. The aim is to better understand the current math pathway options available and clusters of majors that might benefit from shifts to new math pathways. As we continue our data analysis, we will be conducting interviews with individuals involved in the reforms in Oklahoma and individuals involved in the national reforms to better understand the data and process.

Discussion Questions

During our presentation, we will present a more complete analysis of the data described in this proposal with implications for state-level reform decisions. We seek a discussion on the primary challenges to a successful implementation of corequisite remediation and math pathways at-scale and productive theoretical perspectives on institutional and systemic change to guide ongoing research and evaluation.

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Instructor-Generated Concepts Framework for Elementary Algebra in the College Context

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The long-term aim of this study is to develop a conceptual framework outlining the concepts necessary for college students to be able to successfully complete fundamental tasks of elementary algebra. This paper is a preliminary report of one part of this research, which focuses on instructor perceptions of what concepts are fundamental to successful completion of elementary algebra tasks. The framework presented here is the result of an action research project conducted by five college instructors in the U.S. who teach elementary algebra.

Keywords: Elementary algebra, conceptual understanding, algebra concepts, tertiary education.

Elementary algebra and other developmental courses have consistently been shown to be barriers to student degree progress and completion in the U.S.. There is evidence that only as few as 20% of students who are placed into developmental mathematics ever successfully complete a credit-bearing math course (see e.g. (Bailey, Jeong, & Cho, 2010). At the same time, elementary algebra has higher enrollments than any other mathematics courses at U.S. community colleges (Blair, Kirkman, & Maxwell, 2010).

Significant research has been done in the primary and secondary context to explore which types of student thinking lead to more successful or less successful outcomes in student algebraic problem-solving, but little research has been conducted with students enrolled in elementary algebra courses in the tertiary context, despite the fact that there is significant evidence to suggest that mathematics learning is likely somewhat different in this context (Mesa, Wladis, & Watkins, 2014). One approach to investigate this setting is to conduct participatory action research in the tertiary context in order to explore how instructor experiences, including cyclical investigations of their own practice, can shed light on some ways that tertiary students learn elementary algebra concepts, and on which types of student understandings are important for successful completion of elementary algebra tasks.

Conceptualizations of algebra and fundamental algebraic concepts

There are a number of different conceptualizations of algebra that have been explored in the research literature. Usiskin (1988) laid out four conceptualizations of algebra: generalized arithmetic; the set of all procedures used for solving certain types of problems; a study of relationships among quantities; and a study of structure. Kaput (1995) in contrast identified five conceptualizations of algebra, the first four of which mirror somewhat closely those of Usiskin: generalization and formalization; syntactically-guided manipulations; the study of functions, relations, and joint variation; the study of structure; and a modeling language.

A number of different important algebraic concepts have been studied previously, typically in the primary or secondary context. A complete review of the literature is not possible due to space constraints, but we outline here briefly some of the major categories of research on algebra that are relevant to elementary algebra in the tertiary context and cite one or two key references for each:

- Variables and symbolic representation (Dubinsky, 1991; Kuchemann, 1978; Sfard, 1991)

- Functions and covariation (Blanton & Kaput, 2005; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Goldenberg, Lewis, & O'Keefe, 1992)
- Equivalence (Kieran, Boileau, Tanguay, & Drijvers, 2013; Knuth, Stephens, McNeil, & Alibali, 2006)
- Algebraic structure sense (Hoch, 2003; Hoch & Dreyfus, 2006; Linchevski & Livneh, 1999)

For an excellent and systematic review of historical developments in the conceptualization, teaching, and learning of algebra, see (Kieran, 2007). Some researchers have developed frameworks for organizing algebra as a subject, typically in the primary and secondary context (see e.g. (Nathan & Koedinger, 2000; Sfard & Linchevski, 1994). In addition, various national standards regarding the teaching of algebra exist, such as the Mathematical Association of America college algebra standards (Mathematical Association of America, 2011), the National Council of Teachers of Mathematics Standards (National Council of Teachers of Mathematics (NCTM), 2000), and the American Mathematical Association of Two-Year Colleges Beyond Crossroads standards (Wood, Bragg, Mahler, & Blair, 2006). However, while these standards stress the importance of conceptual understanding, their detailed explication of what students should learn tends to focus on computational tasks (e.g. being able to perform function composition) rather than on the specific conceptual ways of thinking that underpin those tasks (e.g. having a process view of function).

Teacher beliefs and expertise

The relationship between teacher beliefs and practice is complex; for example, teachers do not always employ teaching practices that strongly reflect their professed beliefs about how students learn. However, despite this complexity, there is significant research suggesting that teacher beliefs are often strongly related to the teaching practices that teachers implement in the classroom, and therefore are also related to student beliefs and learning experiences (see e.g. (Fang, 1996; Maggioni & Parkinson, 2008). So understanding teacher beliefs is one critical component of understanding instructor practice and its impact on student learning.

On the other hand, teacher expertise also has the potential to benefit the research community by contributing important information about what teachers have learned while teaching; this knowledge can then be used by researchers to generate and test new theories about how students learn and about what is effective in the classroom. As Schulman (1987) explains, "One of the more important tasks for the research community is to work with practitioners to develop codified representations of the practical pedagogical wisdom of able teachers" (p. 11).

This study uses a *teacher-as-researcher* interpretation of *action research*, as originally coined by Stenhouse (1975) and later expanded conceptually by Elliot (1991) and then Cochran-Smith and Lytle (1993; 2009). In this framework, teacher-practitioners investigate research questions not only to improve their own practice, but also to add to a larger body of knowledge than can be implemented by other teachers in similar contexts. This is a more inclusive view that includes practitioner experiences as a valid foundation for knowledge production.

Theoretical Framework

This study uses Vygotsky's (1986) theory of concept formation as a framework for investigating student understandings in elementary algebra. According to Vygotsky, algebraic symbols, graphs and other representations of mathematical objects and concepts mediate two interconnected processes: 1) the development of a mathematical concept in the individual; and 2) the individual's interaction with an external mathematical world where these representations are rigorously codified. Learners begin to use these representations before they have “full” understanding of their meaning, and it is through this experimentation and attempts at communication with “more knowledgeable” others over time that they internalize more formal and correct meanings for the objects that the representations symbolize.

Methodology

Five elementary algebra instructors collaborated on this action research project, some of whom are also active educational researchers. The group included faculty with doctorates in both mathematics and mathematics education, who have taught at the high school, community college, and university levels. Faculty came from varied backgrounds, and included both men and women, several different racial/ethnic, national, and immigrant backgrounds, and reflected a variety of different teaching styles.

Conceptual Framework

This study used the *Action Research Spiral* Framework (Kemmis & Wilkinson, 1998) to guide the process of collaborative exploration into student thinking about elementary algebra concepts. This framework outlines a cyclical practice in which practitioners cycle through the following steps repeatedly in a spiral: 1) plan; 2) act and observe; 3) reflect; 4) revised plan, etc.

First each instructor independently created a list of concepts that they saw as fundamental to elementary algebra. After all instructors had created their own list, all lists were combined. Then a series of discussions ensued, during which various topics and sub-topics on the original master list were combined, rephrased, removed, added, and otherwise revised. Instructors used the framework to inform the creation of assessments and classroom activities, used these in their classes, and then used their experiences to inform revisions in a cyclical process over four separate semesters.

In deciding on what concepts to explore, instructors were asked to think not just about the current elementary algebra course that they were teaching, but about tertiary elementary algebra in general, including variations in what might be included on the syllabus at different colleges. The syllabi of elementary algebra courses at a number of different colleges in the U.S. were consulted to give instructors an idea of the range. Several framing questions were used both during initial independent selection of topics and the subsequent discussions:

- What concepts would we want students to still understand a year after they have completed an elementary algebra course?
- What fundamental algebra ideas are necessary for future mathematics courses (e.g. college algebra, pre-calculus, calculus)?
- What fundamental ideas are of significant value in other liberal arts math courses (e.g. statistics), or necessary in order to be able to apply algebraic thinking in “real” life (e.g. financial calculations, risk calculations)?
- For which particular concepts might you conclude that students had missed the “whole point” of elementary algebra if they were to finish the course without understanding them?

Results

In the process of identifying a conceptual framework for elementary algebra, the group first identified a list of four broad types of tasks that they felt all successful elementary algebra students should be able to complete, whatever the differences across elementary algebra curricula (see Table 1). Then, using these tasks as an initial frame, the group developed a list of concepts that, based on their cyclical experience interacting with students in elementary algebra classes, they felt to be fundamental in order for students to successfully complete these tasks (see Table 2 for a partial reporting of that framework).

Table 1. Fundamental Elementary Algebra Tasks
1. Expressing and correctly interpreting relationships, patterns, and properties in expression or equation form, through correct use of algebraic symbols
2. Simplifying expressions by replacing them with equivalent expressions
3. Solving an equation/inequality or system of equations and correctly interpreting the solution set
4. Relating an equation, or properties of an equation, to a graph, and vice versa
This list of tasks was not intended by the instructors to include all of those that might be relevant to any elementary algebra class given in any context.

The instructors theorized that the fundamental elementary algebra tasks in Table 1 could only be completed if students understand the following concepts in the Framework given in Table 2.

Table 2. Elementary Algebra Concept Framework (selected topics presented in detail)
1. **Algebraic Symbolism:** Understands how to express relationships and patterns in expression or equation form, and can explain in words the pattern or relationship expressed by a given expression or equation *(The sub-concepts in this concept are not outlined here because of space constraints.)*
2. **Algebraic Structure:** Recognizes algebraic structure, with respect to the relevant context of a particular problem-solving goal a. Understands the role of a variable: i. That it can take on a wide variety of values; ii. That its value can vary or that it can represented a fixed unknown; iii. That any expression can be substituted in for a variable; iv. That a variable functions as a set of parentheses around whatever is substituted into its place; v. That every instance of a variable stands in for the same value; vi. That any repeating expression can be replaced by a variable, as long as that variable is defined to take on the original value that it replaced. vii. That during substitution, the structure of the expression outside the part being replaced remains independent of and unchanged by whatever is being substituted into it b. Is able to view expressions or equations with respect to a particular context, and in this context identify the relevant properties *(The sub-concepts for 2.b. are not outlined here because of space constraints.)*
3. **Properties/Generalizing Arithmetic Operations:** Understands the definitions of basic arithmetic operations and can use those definitions to describe general patterns and properties (in words and using equations). For example, understands basic definitions of addition, subtraction, multiplication, division/fractions, positive whole exponents and square roots and can use these definitions to determine when these operations (or combinations of these operations) have certain properties. *(The sub-concepts in this concept are not outlined here because of space constraints.)*
4. **Equality/Equivalence:** Understands equality/equivalence a. Understands what it means for two expressions to be equal i. Understands that two expressions are equal if and only if they are equal for all possible (combination of) values of (each of) the variable(s) ii. Understands that if two expressions are equal, one expression may be substituted for the other in any other expression or equation 1. Understands that simplifying (or otherwise rewriting) expressions is a process by which one expression is replaced by another equivalent expression b. Understands what it means for two equations to be equivalent *(The sub-concepts in 4.b. are not outlined here because of space constraints.)*
5. **Equations as Relationships between Variables:** Understands equations with two variables as something that describes the relationship between two variables, describing how one variable varies with respect to changes in the other variable. *(The sub-concepts in this concept are not outlined here because of space constraints.)*
6. **Thinking Graphically:** Understands how one and two-dimensional graphs describe the relationship depicted in

a particular equation or inequality and vice versa, and can describe how different operations will impact a graph. *(The sub-concepts in this concept are not outlined here because of space constraints.)*

In a cyclical process of experimentation, instructors developed assignments and assessment questions intended to either 1) assess the extent to which students understood one or more concepts listed on the framework; or 2) confront students with tasks that would require them to directly engage with a common misconception or a type of productive struggle that might better reveal their current understandings related to one of the concepts on the framework, or how they use those understandings to complete specific elementary algebra tasks.

An example of one question in the first category is below:

Assume that $a \neq 0$. Dale simplifies the expression a^3a^{-2} and gets the correct expression a. Which of the following must be true? There may be **more than one** correct answer—select **ALL** that are true.

a. $a^3a^{-2} = a$
b. If Dale lets $a = 10$ in both the expressions a^3a^{-2} and a, he will get two different answers.
c. Dale can substitute a for a^3a^{-2} anywhere it appears.
d. If Dale lets $a = 20$ in both expressions, he will get the same value for each expression.
e. Dale needs to know the value of a before he can say whether a^3a^{-2} and a are equal.

This question was designed to test the extent to which students understand the items listed under 4.a. in the framework in Figure 2. Based on the answers that students gave, instructors could then engage with students about their understanding of specific components of item 4.a. in order to better understand what those are and how they relate to one another. Based on conversations with students as a result of this question, the framework was revised: The first framework draft contained only item 4.a.ii.1; after repeated cycles of the research process the additional items under 4.a. were added and structured hierarchically.

An example of a task that falls into the second category is the following question, which one of the instructors used for an in-class activity:

Suppose there is a new algebraic operation called the bow tie, defined this way: $\bowtie a = \frac{1}{a} - a^2$
Use this definition to rewrite the following expressions (no need to simplify afterwards!) so that they no longer contain a bow tie symbol:

a) $\bowtie (-2)$
b) $\bowtie (x^2)$
c) $\bowtie \left(\frac{1}{y}\right)$

Instructors expected students to make the following mistakes somewhat frequently: $\bowtie (-2) = \frac{1}{-2} - 2^2$, $\bowtie (x^2) = \frac{1}{x^2} - x^2$, $\bowtie \left(\frac{1}{y}\right) = \frac{1}{y} - \left(\frac{1}{y}\right)^2$. The expectation was that students who made these mistakes did so through oversight—for example, they might forget to write both negative signs when substituting in -2 for $-a^2$. However, in one-on-one discussions with students, it became clear that many students did not forget to write the negative sign in this case; rather, they believed that because the -2 already had a negative sign, that this negative sign took the place of the one that was already in the expression $\frac{1}{a} - a^2$. This led to a revision of Framework item 2.a. to include sub-item vii.

Discussion and Plans for Next Steps

This framework reflects only the experience of one group of elementary algebra instructors at the college level and may not reflect the experiences of all tertiary elementary algebra instructors. This is also just one step of data collection in our larger goal of developing a conceptual framework for elementary algebra at the college level. Another ongoing study that is a part of this larger project is an extensive literature review synthesis whose goal is to develop a comprehensive research-based conceptual framework for elementary algebra that is based on existing literature from primary and secondary settings. The next step will then be to compare that framework to the instructor-generated framework in order to identify areas where they overlap and where they do not. Areas where the frameworks differ will then be used as a starting point for future research projects investigating how the algebraic understandings and learning processes of tertiary students in elementary algebra may differ from the experiences of primary and secondary students learning similar content.

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Career Decision Making Strategies of Calculus and Developmental Mathematics Students

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Abstract: Students who have persisted in mathematics coursework long enough to be present in calculus or who enter mathematics at the level of calculus would be expected have more robust notions concerning their career choices than those who enter developmental mathematics. In the current work, we give a preliminary comparison of data generated by a career decision making survey administered to students in a developmental mathematics course and to students in a first semester calculus course at a large research university during the fall 2015 semester. We consider some initial results for students who switch majors after a semester of mathematics coursework.

Keywords: Career Choices, Persistence, Calculus, Developmental Mathematics

Introduction

Students at most institutions of higher education will begin their studies with a mathematics course of some kind, and students who choose science, technology, engineering and mathematics (STEM) careers usually take mathematics courses during their first term. In addition, STEM intending students will typically eventually take calculus. These students have a variety of motivating factors for their choice of major and their career goals. During their mathematics course experiences, they may or may not make changes to their plans. In the current work, we present a preliminary study of career and major choices of students in developmental mathematics courses and first term calculus and provide some initial data from a study of how those intentions change over the course of a semester in those classes.

Theoretical Perspective

In this project, we surveyed students about a number of factors related to career exploration and career identity theory as part of a larger project to build models of developmental mathematics and calculus students who have declared a STEM major and career intentions. This work builds on early work on self-identity situated in disciplines (Marcia, 1966; Nosek, Banaji & Greenwald, 2002; Du, 2006; Hazari, Sadler & Sonnert, 2013) as well as some affective factors such as anxiety (Alexander & Martray, 1989) and personality constructs identified by the Big Five Inventory (John, Naumann & Soto 2008). This effort explores relationships between mathematics anxiety, personality, coursework and both persistence and identity. The data generated from a collection of five surveys is used to analyze when and why developmental mathematics and calculus students change majors. In this preliminary report we summarize the results from one of these surveys.

Methodology

In the fall semester of 2015 a series of surveys were administered to a cohort of students enrolled in both types of courses (developmental and calculus) to assess their anxiety levels, personality traits and career decision-making strategies. The survey consisted of a series of Likert-response questions combined with open answer questions intended to assess a student's methodology and reasons for choosing a major and career. This data and student demographic information were analyzed in an effort to begin to identify differences between the responses of these two groups. Identifying patterns in responses will help us begin to develop profiles of students in each cohort, but particularly in developmental mathematics, who will be successful in STEM majors. The survey was given in the second week of class and again in the final week of the semester. There were 458 calculus students and 80 developmental mathematics students who completed both pre- and post-surveys. Both surveys asked students about their thoughts on changing majors, therefore we have data about intentions to change majors both before and after a semester of mathematics coursework.

Results

Demographic data for each population are presented (Table 1). Developmental mathematics students were primarily female, non STEM intending. Calculus I students were more mostly male and STEM intending. We asked students their reasons for selecting a major and about the attractiveness of their field in a pre- and post-survey by providing the following two open-ended questions: "What seems attractive about your current major or career goals?" and "How did you come to decide on your current major?" Responses (pre and post) to these questions from both the developmental mathematics and calculus groups were coded for themes by two members of the research team. Codes were compared until there was total agreement among the researchers and each response was assigned up to three codes. Surprisingly, we do not note changes on these two statements from a pre and post comparison for both developmental mathematics and calculus groups. Even more unexpected, the summarized responses to the questions (Figure 1) show a parallel between factors influencing both groups of students. The majority of students chose their major based on personal interest and future career expectation. Many students also indicated that they were influenced by some external factors such as family and friends when making their decisions.

Table 1. Demographic information for Developmental Mathematics and Calculus I groups.

	Developmental Mathematics			Calculus I Groups		
	STEM	Non-STEM	Total	STEM	Non-STEM	Total
Female	8	50	58	102	48	150
Male	10	12	22	283	25	308
Total	18	62	80	385	73	458

In addition to the open ended questions that were coded, students were asked eight additional questions to be rated on a Likert scale. The questions are given in Table 2 and student response rates are given in Figure 2. We observe that most students in the two groups are confident in choosing their major, feel satisfaction with their selected major and are seriously committed to their current occupation. These characteristics of decision-making are consistent with the reasons for choosing their major. Moreover, the response rates for each category in the Likert scale are remarkably similar for both groups, indicating that student thinking about major and career choices is similar for both developmental mathematics students and calculus level students. This result contradicts the intuitive notion that students at the calculus level would have different reasoning about their career and major choices than those not ready for college level mathematics.

Figure 1. Student responses to open ended questions about factors affecting choice of major and attractiveness of major or career goals.

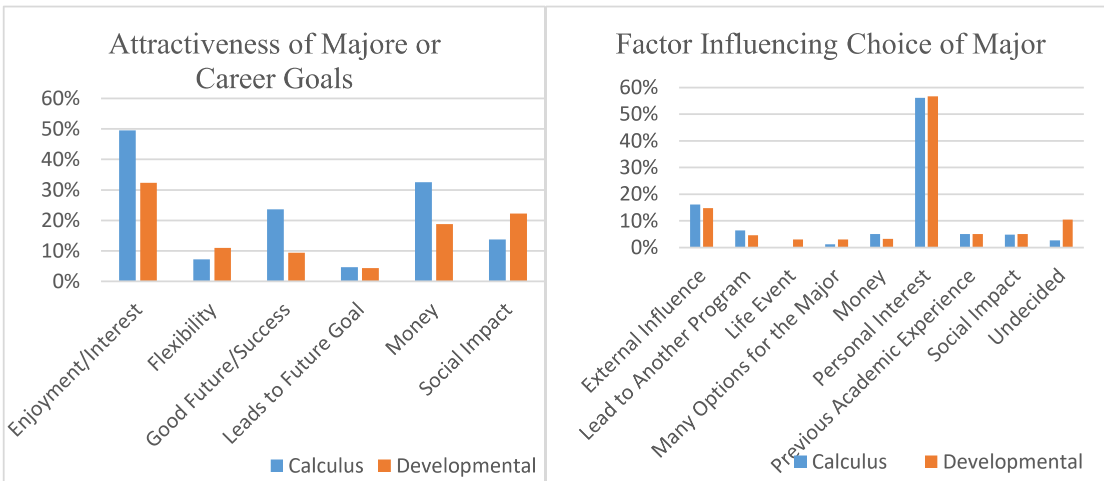
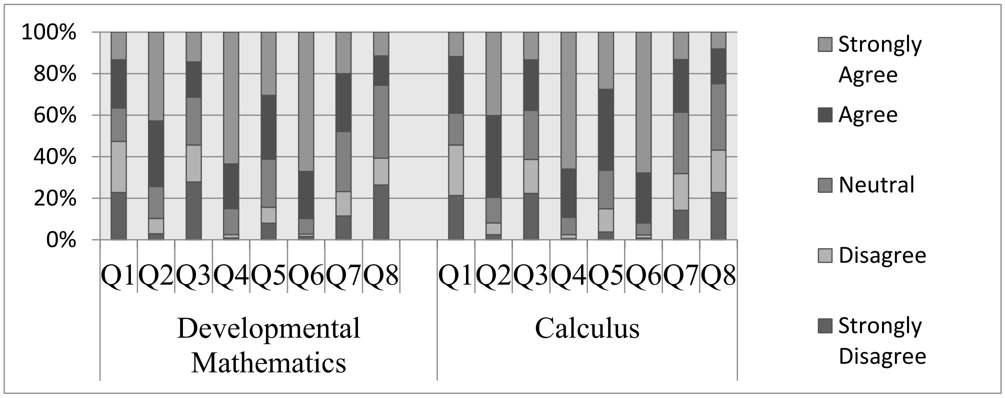


Table 2. Likert-scale questions asked in career decision making survey.

Q1	I had difficulty choosing a college major and a future career path.
Q2	I feel good about my current position with respect to my major.
Q3	Most parents have plans for their children, things they'd like them to do or go into. My parents have plans like that for me.
Q4	My parents feel good about what I'm doing now.
Q5	I would be willing to change my current plans if something better came along.
Q6	I see my occupation as being important to me in my life.
Q7	My family's opinion was very important in helping me choose my current major.
Q8	If my family were supportive, I would be likely to change my major.

Comparisons between pre- and post-survey responses, however, within the groups as well as between groups show that a substantially higher percentage of developmental mathematics students are deciding to change their major (18.12%) after a semester in a mathematics course than those in calculus (12.81%). We designate these students as “switchers” (Bressoud, Carlson, Pearson, Rasmussen, 2012). In addition, we asked students if they planned to change their major both at the beginning of the semester and at the end of the semester. At the beginning of the semester, 13.92% of developmental mathematics students indicated they planned to change majors, while 25.32% indicated that they planned to at the end of the semester. For calculus students, 21.27% planned to change majors at the beginning of the semester while 19.74% indicated they planned to change at the end of the semester.

Figure 2. Characteristics of career decision making processes



Students from the both courses who indicated an intent to change majors at the beginning of the term described different reasons for doing so (Table 3). Developmental mathematics students experienced a significantly lower level of confidence coming into the course, affecting their decision to switch, but few calculus students stated such a reason. On the other hand, the critical reason for change for calculus students at the beginning of the term was a loss of interest in their current major. Both groups of students also indicated their new major would be more interesting and be a better fit for their talents.

Table 3. Reasons for change-population indicating major change intention during pre-survey.

Statements	Developmental Mathematics	Calculus
I don't find my current major very interesting anymore	35.00%	26.67%
I found another major that is more interesting	30.00%	47.78%
I found another major that better fits my talents	35.00%	30.00%
My current major really wasn't what I expected	25.00%	22.22%
I just don't feel I belong when I am with other students in my major	15.00%	16.67%
My current department does not seem interested in me	10.00%	4.44%
I am not sure I can be successful in my current major	50.00%	16.67%

Specifically, examining the reasons for changing majors at the end of the term for both groups, we discovered a similar but more prominent pattern along with one exception (Table 4). A decrease in confidence appeared as an equally important reason for switchers in the calculus population (as well as in developmental mathematics). More importantly, some of the reasons that are not emphasized by students in the pre-semester survey results became critical for switchers in the calculus group at the end of the semester.

Table 4. Reasons for change-population indicating major change intention during post-survey.

Statements	Developmental Mathematics	Calculus
I don't find my current major very interesting anymore	30.77%	19.57%
I found another major that is more interesting	23.08%	47.83%
I found another major that better fits my talents	23.08%	56.52%
My current major really wasn't what I expected	30.77%	41.30%
I just don't feel I belong when I am with other students in my major	0.00%	17.39%
My current department does not seem interested in me	7.69%	8.70%
I am not sure I can be successful in my current major	46.15%	45.65%

Considering students in both groups by gender and by major (STEM or non-STEM), we found that in developmental mathematics courses there is an increase in number of students changing their major in all categories from the beginning to the end of the semester: male, female, STEM intending and non-STEM intending. The increase is greater for females (~+14%) than for males (~+5.5%) and greater for non-STEM intending (~+12%) than for STEM intending (~+5.5%). Conversely, for the calculus population, there was an increase in the number of non-STEM intending and female switchers, while the numbers of STEM intending (~-2%) and male (~-4%) switchers decreased. Rates for female and non-STEM intending students switching increased by approximately 2.5% and 4% respectively.

Conclusions

Despite the fact that there are several parallel patterns and no pre- and post-test differences in students' reasons for choosing their major, beliefs towards the attractiveness of their major, and the characteristics of how they chose their major between students in the developmental mathematics and calculus courses, we discovered differences in the reasons for switching between the course populations as well as differences in the switch rate between gender and the classification of STEM intention.

By comparing the overall rate of students' changes in their majors, we found that there are more students from the developmental mathematics courses changing their majors. This may be a preliminary indication that students who are taking the developmental mathematics course are comparatively more negative affected by mathematical experiences. In addition, the developmental course delivery format is significantly different than the calculus course delivery. The effect of the format of the course on the specific factors measured in this study has not yet been investigated. Losing interest and confidence are the two most critical explanations for changing their major in both groups, especially for the switchers. On the other hand, the developmental mathematics course and calculus courses perform similarly in keeping STEM intending students, since the decline in STEM intending students are significantly lower than the decline in non-STEM; the Calculus courses are doing comparatively good job retaining STEM intending students. This could inform the instruction in both courses so that we can retain more and increase the number of students who pursue a STEM degree.

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What Constitutes a Proof?
Complementary Voices of a Mathematician and a Mathematics Educator in a Co-Taught Undergraduate Course on Mathematical Proof and Proving

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The work reported in this paper is part of a study aimed at characterizing the processes and identifying the ways in which different kinds of expertise (mathematics vs. mathematics education) unfolded in the planning and teaching of an undergraduate course on Mathematical Proof and Proving (MPP), which was co-taught by a professor of mathematics and a professor of mathematics education. More specifically, the study aimed at unpacking the affordances and drawbacks of this collaboration. The collected data includes all 13 videotaped lessons in the 2012 semester, the second time the course was taught. The content of the course consisted of topics that were familiar/accessible to the students, e.g., high school level algebra, geometry, and basic number theory. In this paper we focus on how the views held by each instructor regarding what constitutes an acceptable proof and how it should be presented, are reflected in his/her teaching.

Key words: Mathematical Proof and Proving; Undergraduate Course; Problem-based Learning; Community of Practice

Introduction

The study reported herein was carried out in the context of an undergraduate course on Mathematical Proof and Proving (MPP)[1] that was designed in collaboration between mathematicians and mathematics educators, and was co-taught by a professor of mathematics (Jim) and a professor of mathematics education (Olga)[2]. A small portion of the study appeared in Zaslavsky, Sabouri, & Thoms (2013). The present paper is a preliminary report on work in progress - a qualitative analysis of the actual teaching of both instructors, viewed through a variety of lenses and guiding questions. The overarching goal of our study is to characterize the processes and to identify the ways in which the different kinds of expertise (mathematics vs. mathematics education) unfolded in the actual planning and teaching of the MPP course; in particular, we are looking for instances that could help understand how each expertise contributed to the course and complemented the other. In terms of student learning, the course was assessed by a special evaluation team. Their report indicated a gain in students' understanding of the notion of proof and their ability to prove. Thus, for the purpose of the study reported herein, we do not address questions regarding student learning; rather, we look at the teaching of both instructors and the interactions between them as they taught.

The MPP Course

Two assumptions led to initiating this MPP course: 1. Mathematical proof and proving are fundamental to mathematics; and 2. Many undergraduate students struggle with the notion of formal proof and the activity of mathematically proving (e.g., Harel and Sowder, 2007).

The notion of proof is often incorporated in mathematical content courses, and typically does not constitute the focal topic of a single undergraduate course. There are many transitional courses in mathematics, most of which combine learning about proof with learning unfamiliar fundamental topics in mathematics. Consequently, the cognitive load on students is high and they encounter more difficulty than necessary, since they need to deal with many issues

[1] This material is based upon work supported by the National Science Foundation under Grant No. 1044809.
[2] The names in this paper are pseudonyms.

concurrently: advanced mathematical ideas as well as proof and proving. The intention of the MPP course was to build on students' existing mathematical knowledge, and to draw on learning activities that involve familiar topics such as high school level algebra and geometry and basic number theory (e.g., familiar properties of integers such as divisibility).

The challenge of attending to students' learning difficulties, and at the same time maintaining an appropriate level of sound mathematics, led to a collaboration between mathematicians and mathematics educators. Moreover, the MPP course was designed and co-taught by two instructors – a full professor of mathematics and a full professor of mathematics education. The goal of this collaboration was to capitalize on each of the instructors' expertise. The mathematics educator brought her expertise on teaching mathematics, and in particular the teaching of proof and proving. She also brought expertise on students' difficulties in learning to prove. The mathematician brought his expertise in the discipline of mathematics and the knowledge and understanding of MPP that students will need for successful participation in more advanced undergraduate mathematics content courses. This collaboration was grounded in a mutual respect for each other's role and potential contribution, and on the recognition that the instructors had much to learn from each other.

While sharing the same concerns and long-term goals for the course, each instructor brought a different perspective on how students should be learning MPP and on how to attend to their difficulties. From the outset it became clear that although the structure and syllabus of the course were pre-determined in full agreement between the two instructors, each instructor had his/her own views and interpretations, and that the joint efforts to produce an MPP course to address the above concerns would require ongoing dialogue and reflection. The challenge was to bridge the different perspectives and use them as a springboard to enhance the course.

Conceptual Framework

The following perspectives on learning and teaching guided the design of the MPP course: 1. Students' interactions and classroom discourse contribute to learning [to prove] (Yackel, Rasmussen, and King, 2000; Zaslavsky and Shir, 2005; Smith, Nicholas, Yoo, and Oehler, 2009); 2. Tasks play a significant role in learning (Henningsen and Stein, 1997); 3. Uncertainty could promote the need to prove (Dewey, 1933; Fischbein, 1987; Harel and Sowder, 2007, 2009; Zaslavsky, 2005; Zaslavsky et al, 2012); 4. Class discussions and activities should address students' anticipated/manifested preconceptions and difficulties (Harel and Sowder, 2007, 2009; Weber, 2001; Reid, 2002; Buchbinder and Zaslavsky, 2009).

The decision to design and co-teach the course collaboratively, assigning two full professors as the MPP course instructors, is in a way a response to issues raised by Harel and Sowder (2009). Their study indicates that while mathematicians who teach undergraduate courses in mathematics have a broad and deep mathematical knowledge/understanding, many are not necessarily fully aware of students' difficulties in learning to prove, or of effective ways to scaffold their learning. In addition to the instructors, three mathematicians were involved in the design of the course, two doctoral students served as TAs and one served as a research assistant. The team members varied with respect to their expertise and experience, as well as their roles, which is one of the characteristics that Roth (1998) considers essential for a community. Theories of communities of practices provide us with tools for analyzing the various kinds of learning of the members of the community as well as the contribution of each member to the shared goals of the community (Rogoff, 1990; Roth, 1998; Lave & Wenger, 1991). These theories consider knowledge as developing socially within communities of practice.

An integral characteristic of our community of practice is associated with the notion of reflective practice (Dewey, 1933; Schön, 1983). The notions of reflection on-action and reflection in-action have been recognized as effective components that can contribute to the growth of teachers' knowledge about their practice. In our study, reflection was a key issue for the development of the instructors' awareness and understandings related to teaching and learning to prove.

Data Sources and Analysis

The data for the large scale study consists of video-tapes and field notes of all the classes in the semester (13 sessions), audio-records and field notes of weekly meetings held a day after each class, and email conversations between the team members. In addition, students' written homework and TA's comments and grades were scanned and documented.

The methodology employed in the analysis followed a qualitative research paradigm in which the researcher is part of the community under investigation. It borrows from Strauss and Corbin's (1998) Grounded Theory, according to which the researcher's perspective crystallizes as the evidence, documents, and pieces of information accumulate in an inductive process from which a theory emerges. The researcher acts as a reflective practitioner (Schön, 1983) whose ongoing reflectiveness and interpretativeness are essential components (Erickson, 1986). In our case, the researchers were members of the community of practice that they investigated.

For the purpose of this paper, we identified excerpts that reflected the instructors' views on what constitutes an acceptable proof, and analyzed how these views were exhibited.

Views on what is an acceptable proof

One of the main goals of the MPP course was to establish what constitutes an acceptable proof and how to produce and present a proof that is acceptable. At this point we restrict the discussion to what the instructors considered acceptable in the context of the course (we acknowledge that for different communities, the requirements of a proof could vary; e.g., mathematicians who present their proofs to colleagues, experts in the field, may not be expected to go into much detail from one step to another, as it is clear that the presenter as well as the audience can easily follow and fill in the gaps).

While Jim repeatedly insisted on the need to explicitly state the Given and the RTP (Required to Prove), Olga insisted on the need to back each step in the proof with a valid explanation for the inference that is made. Jim explained his stance mainly as a helpful strategy for constructing a proof, when this is not straightforward for the student. Olga supported her stance mainly from a metacognitive perspective. She distinguished between the way we think, e.g., in a draft version of a proof, and the way the proof should be presented formally.

How to Present It?

We illustrate the above views through an excerpt from the 6th session (of 13), that aimed at implementing the equivalence between a claim and its contrapositive for proving.

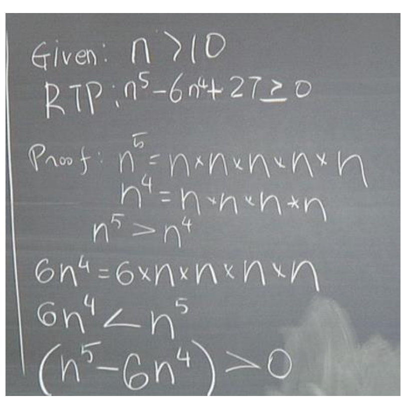

Figure 1. A student's proof of the contrapositive

Olga began the lesson with a review of some homework assignments, which included the following: "Prove that for any rational number n, if $n^5 - 6n^4 + 27 < 0$, then $n \leq 10$." Most of the students had proven the equivalent contrapositive claim, i.e.: "if $n > 10$, then $n^5 - 6n^4 + 27 \geq 0$. " The episode begins as a student is writing her proof on the board, explaining her reasoning as she goes. Her proof, as presented on the board, is shown in Figure 1.

Student: So if we know that n is greater than 10 and this number [expression] can be written... Since 6 is less than n, because we know that n is greater than 10, then we can say that... and then if we subtract ... we know that that would be a positive number. [writes on the board].. if we add 27 it would be greater than 0 also.

Olga: [turning to the class] What do you say? I see a lot of nodding. You agree with the proof? With the arguments, building the proof? How about the presentation?

Jim: I want to write that last line. She said it verbally without writing.

Olga: Ya, there are a few problems.

Jim: Just write the last line you said.

[Student adds the following line at the end: "$n^5 - 6n^4 + 27 > 0$"]

In Olga's feedback to the student she says:

> "This maybe makes it clearer, but still in order to present the proof very logically, it's important to write how we move…. in terms of presentation of the proof, it's still lacking. We look at it, it's not clear what follows from what and why. This was said orally. But the proof has to stand alone in writing. So all these missing parts should be written... Everything you write you have to account for…".

To make her point, Olga asks: "is n^5 always larger than n^4?"; this question pushed the students to realize that this holds only because of the Given, that is, because we know that $n > 10$." She uses this as an example of a missing justification in the proof.

Olga distinguishes between the way we think about a proof and the way we present it, as follows:

> "… we often think in less organized ways, but once you write a proof, we really want to make it flow…. So this, I think, is a good example of the slight difference between how we think and how the proof in the end has to look…"

In a different lesson, following Jim's emphasis on the need to clearly state the Given and the RTP, Olga says:

> "… It's more than writing what's given and what you have to prove. It's also accounting in each line you write, what is the status of what you wrote. Is it given, is it a known fact that you bring from some other place, which is fine. You have to annotate and write where it comes from, how you got to there... We need these words to make sense of what the status of each line is. … It's also a means of communication, but it's also a means of sort of control of what you're doing."

Interestingly, Jim was less worried about accountability for each line of the proof. He was more concerned with stating and using the Given and the RTP to come up with a valid proof. He repeatedly made utterances like the following:

> "…What's missing here completely is what's given and what do you have to prove. One thing I told you in the very first lecture is to do what? Write the Given and write RTP. …So notice here that this line is the implication, is assuming this is what's given and this is your conclusion.... This is a mistake that is commonly made because people confuse what is given and what you need to prove. If the people who wrote this proof started by GIVEN... then there will be no confusion of what should the last line be, verifying this.... When you cannot organize your thoughts... and tell me what's given, and what do you have to find, then there is

a serious issue…."

In response to an impasse a student expressed regarding a claim he could not think of a way to prove, Jim said:

> "… You construct the proof by playing. ok? ... You just start to play... How do you play intellectually? You take the object that is not recognizable to you and you start to play with this object. … You work with the "required to prove" to make it look more and more like the given. That should be your strategy. That's the only strategy that you have. There is no other strategy to follow. Ok? You should always think of playing with what you have to prove …".

The status of examples

Much has been written about the roles and status of examples for proving (e.g., Buchbinder & Zaslavsky, 2009; Ellis et al., 2013). Views vary from reluctance to rely on examples for proving to acknowledging the merits of example use for proving, particular the use of generic examples in proving (Leron & Zaslavsky, 2013).

Both Jim and Olga frequently used examples in their teaching of how to prove. Moreover, they accepted some proofs that relied on examples, as demonstrated in Figure 2 - a proof provided by one of the students.

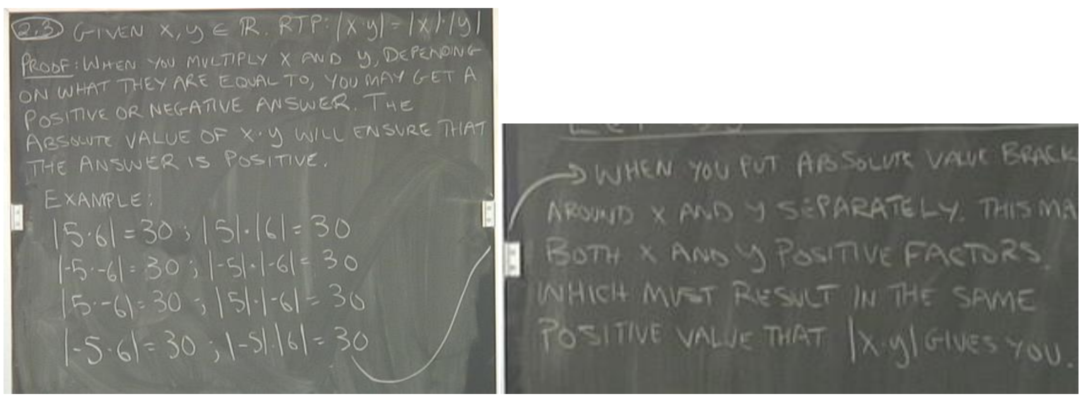

Figure 2. A student's proof by cases, with numerical examples

This proof can be seen as a generic proof, as each numerical case represents a general case. Both instructors accepted it as a valid proof, but Jim criticized its form of presentation, saying: "It's a badly written proof, but it is a proof. "

Concluding Remarks

This preliminary report captures some of the views that were expressed throughout the MPP course by the instructors regarding what a proof should consist of and how it should be presented. It highlights common views as well as differences in emphases between the two instructors, and suggests reasons for these differences. What we have presented is work-in-progress. In the oral presentation of our work we will provide a deeper and more comprehensive analysis, and will discuss implications of our findings for teacher preparation.

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STUDENTS' THINKING IN AN INQUIRY-BASED LINEAR ALGEBRA COURSE

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Elizabeth Thoren
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William Jacob
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This evaluation study compared the mathematical thinking of undergraduate students (as they responded to class work and interview prompts) who participated in an inquiry-based linear algebra course to a comparison group of students who participated in a traditional course.

Keywords: Linear Algebra, Inquiry-Based-Learning

The research literature shows that in traditional linear algebra classrooms students often memorize algorithmic methods that "work" even when not properly understood. Students develop understandings of matrix algebra and solving systems of linear equations using the Gaussian elimination, yet have problems with the more abstract notions of spanning set, linear independence and linear transformation (Stewart & Thomas, 2007). In the language of Sierpinska's modes of thinking (2000), the student dependent upon reducing a matrix to echelon form to determine whether vectors are linearly independent are thinking in *arithmetic mode*, whereas a student who is able to think more generally about objects in a system, by applying a definition or theorem when appropriate, is thinking in *structural mode.*

In this poster session we will share results from an evaluation study conducted in a large public research university, that examined how an inquiry-based linear algebra course supported the development of student mathematical reasoning from actions in the embodied world to formal structural thinking (See Figure1). Our framework, used to guide analysis of student work/interview responses, is adapted from Stewart & Thomas (2009, 2010), incorporates three mathematical worlds (embodied, symbolic and formal) and depicts a progression in the development of mathematical thinking (Tall, 2004).

	Embodied world/Synthetic Geometric Thinking	Symbolic World/ Arithmetic Thinking	Formal World/ Structural Thinking
Action	Adds multiples of two given vectors in R^2 or R^3 to visually determine whether a third is a linear combination of the given vectors.	Tests if a set of vectors is LI by constructing a matrix with the vectors as columns and row reducing it.	
Process	Generalizes this visualization process to two arbitrary vectors v_1, v_2, v_3 in R^2 or R^3.	Thinks about the action above without actually carrying it out.	
Object	Operates on this visualization of LI vectors (eg. Transforming them via reflection, rotation, etc)	Understands process as above and can operate on the resulting matrix (eg. knows that if matrix has a pivot in every column the original set of vectors is LI.	Shows set of given vectors is LI by definition by considering the vectors space that the vectors are in (eg. Gives a dimensional argument)

Figure 1. Framework of Progression of Mathematical Thinking (with Linear Algebra examples)

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The Influences of a Teacher's Mathematical Meaning of Constant Rate of Change on His Classroom Practices

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We investigated how a teacher's meaning of constant rate of change influenced his teaching practices. Findings revealed that a teacher with a strong mathematical meaning of constant rate of change was able to provide conceptually coherent explanations and pose questions that are based in his understanding of constant rate of change and his models of students' thinking.

Key words: Teacher Knowledge, Rate of Change, Teacher Questioning, Decentering

Introduction and Theoretical Framework

Many researchers have identified links between a teacher's mathematical knowledge and her teaching practice (Ball, Thames, & Phelps, 2008). However, few researchers have characterized how a teacher's meaning of a specific mathematical idea is expressed when teaching.

Steffe and Thompson (2000) described decentering in the context of teacher-student interactions as the mental actions involved in building a model of students' mathematical thinking. The teacher is acting in a decentered way when she adjusts her explanations and questions, based on her model of a student's thinking.

An individual's meaning for constant rate of change (*CROC*) might be a calculation for the slope of a line, without realizing that CROC describes how two quantities change together. An individual possessing a more robust meaning of CROC sees x and y changing together with the restriction that: i) the ratio of the changes in the two quantities remains constant (e.g., $m = \frac{\Delta y}{\Delta x}$) and ii) the changes in the two quantities are related by a constant multiple (e.g., $\Delta y = m\Delta x$).

Select Results

In the below excerpt the teacher is discussing the meaning of the statement $\Delta d = 6\Delta t$ with a student. The teacher began by prompting the student to explain the statement. The student responded by saying that the change in y is always 6 (lines 1 and 3). The teacher then poses a question (line 4) that is rooted in his meaning of CROC and his assessment of his student's expressed meaning. This short except reveals how the teacher's strong meaning for CROC enabled him to react productively (lines 4, 6, 8) to the thinking the student expressed. We claim that the teacher's ability to decenter was possible because of his strong meaning for CROC.

1 **Student:** *As the values of x and y change together the change in y is always 6.*
2 **Teacher:** *Say that again?*
3 **Student:** *Basically, this whole thing with 6.*
4 **Teacher:** *Is it? What if the change in t is ½? Then, what would the change in d be? 6? Or, what?*
5 **Student:** *Then it would be ½ times 6.*
6 **Teacher:** *Yeah but then that says that the change in d is 3, which isn't 6.*
7 **Student:** *So it's something times 6?*
8 **Teacher:** *What's that something?*
9 **Student:** *The change in t times 6?*

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Supporting Math Emporium Students' Learning Through Short Instructional Opportunities

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This study focuses on the concept of including traditional math classroom experiences in a math emporium course. The aim of the study is to gain an insight into the opinions of students about which emporium structure they prefer as well as which they believe they can be more successful in. Also, this study will analyze emporium students' academic success in both scenarios. To accomplish these goals, two sections of Algebra II in the math emporium were offered the option to attend short instructional opportunities led by the instructor.

Key Words: Math Emporium, Classroom Research, Student Opinions, Algebra 2

A math emporium is a self-paced math course that is completed with the aid of an instructional software package, which includes computational exercises, videos, practice exercises, and online quizzes (Twigg, 2011). Students work on math problems and ask for help when needed. Virginia Tech pioneered the first math emporium and studies have shown positive effects in terms of attendance, pass rates, and performance on end-of-course exams (Taylor, 2008; Twigg, 2011). In general, math emporiums follow the Virginia Tech model with three major places of variation; attendance, weekly traditional meetings, and size of facility (Twigg, 2011). The idea of having weekly traditional classroom meetings is not one that all schools adopt. As previously stated Virginia Tech does not offer traditional classroom meetings, while other schools, such as the University of Idaho, have weekly focus meetings (The Polya, 2016). This study focuses on the differences between a fully computer based instruction emporium and an emporium offering traditional classroom experiences. The U.S. Department of Education reported that there are advantages for blended learning and that "it was the additional opportunities for collaboration that produced the observed learning advantages" (U.S. 2010). An emporium that offers traditional classroom experiences is a type of blended learning and this study may be able to further support the claim that blended learning gives students an advantage.

The purpose of this study is to gain an insight into the opinions of emporium students at a public university in the Midwest as to which emporium structure they prefer, as well as which they believe they can be more successful in. In addition, this study will analyze the emporium students' academic success when they become involved in traditional classroom experiences, called short instructional opportunities, versus those who continue to use the fully computer based emporium system. The research questions for this study are:

1. What is the nature of students' perspectives of the benefits of incorporating short instructional opportunities into their emporium experiences?
2. How does attendance at the short instructional opportunities change over the course of the study?
3. What differences, if any, in student achievement were there between and among students who participated in short instructional opportunities and their peers?

Although the study is in the data collection stage, preliminary analysis suggests that at least sixty percent of students reported that having the opportunity to attend the hear the course material in a short instructional opportunity would aid in their success in the emporium.

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Mathematics Affirmations

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This poster reports on an action research project set in a developmental mathematics classroom in a community college. Students at the beginning of the semester expressed mathematics anxiety and did not believe they could succeed in the course. To help support students' learning, the instructor and students co-created a list of ten statements that became the prevailing philosophy in the class. These statements helped students alter their view of their own mathematics learning.

Key words: Developmental mathematics, student beliefs, mathematics anxiety

The first day teaching a developmental mathematics course, I distributed a self-efficacy survey and questionnaire to give me insight into my students' previous experiences learning mathematics and the challenges they foresaw which could affect their success. Several students discussed their responses personally and that based on their prior experiences, they did not believe they even had the *capability* to succeed in my class. Several mentioned mathematics anxiety. It was clear that if I wanted to help these students see themselves as mathematical beings (Nasir, 2002), I would have to find a creative solution to alter their perspective on their potential for success. These conversations had a sense of urgency; my impression was that their negative self-talk needed to be addressed before students would be open to learning mathematics.

Inspired by The Definite Dozen (Duncan-Andrade, 2010), my students and I co-created a list of ten *Math Affirmations* to increase students' self-efficacy (Bandura, 2008; Zientek, Yetkiner Ozel, Fong, & Griffin, 2013), reduce their anxiety (Dowker, Sarkar, & Looi, 2016), and develop a growth mindset (Dweck, 2006). The *Affirmations* included statements such as "I am capable of learning and doing math," "Hard work is often mistaken for luck or natural ability," and "Success comes from not being afraid to ask questions," and were recited every day. The norms of the course centered on reinforcing the *Affirmations* so that they were not an empty set of statements read aloud, but the foundation for the pedagogical structure of the course.

When the *Affirmations* were initially presented, many students felt optimistic that the *Affirmations* would help them be successful in this class. One student wrote, "I believe that *Math Affirmations* will help me to sharpen my math skills and also help me a be a confident student" Another supported this sentiment; "I think they [the *Affirmations*] can all benefit us in many ways". One student supported reading the *Affirmations* only once per week. Overall, student feedback at the beginning of the semester and during mid-semester feedback indicated a belief in the potential of the *Affirmations* to help improve students' mathematics performance or attitude.

These students were successful compared to other developmental mathematics classes, (Bahr, 2010; Bonham & Boylan, 2011; Boylan & Nolting, 2011; Waycaster, 2001). Twenty-nine students enrolled in the course. Eighty one percent of the students completed the course by taking the final exam; 57% passed the common final exam. Students were also highly satisfied with the course with 95% of students stating they would recommend the course and instructor to others and felt the instructor motivated them to do their best.

The *Math Affirmations* could be introduced to reduce any trepidation students may have about their ability to succeed. Future research will explore whether the *Affirmations* reduce anxiety, increase self-efficacy, and how they may contribute to student success.

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DEVELOPING AND TESTING A KNOWLEDGE SCALE AROUND THE NATURE OF MATHEMATICAL MODELING

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This study explored efforts to design and empirically test measure teachers' knowledge of the nature of mathematical modeling. The author begins by reviewing the literature on teachers' content knowledge and mathematical modeling, noting the effect of content knowledge on teachers' professional competence. Next, the author discusses the items used on the questionnaire to represent knowledge of the nature of mathematical modeling. Results from reliability, factor analysis, and scaling work with the items showed teachers' knowledge of the nature of mathematical modeling was unidimensional. The construct indicated by factor analysis formed psychometrically acceptable scale for measuring teachers' knowledge of the nature of mathematical modeling.

Key words: [modeling, mathematical knowledge for teaching, teacher knowledge, and teacher education]

Mathematical modeling strongly influences what mathematics students learn and how they learn it. Likewise, mathematical modeling has gained increased focus in standards and assessments for school mathematics—both nationally and internationally (NGA & Center, 2010; OECD, 2003). Consequently, there is much emphasis on the need to address the skills and understanding of mathematical modeling in school mathematics (Blum & Borromeo Ferri, 2009; NCTM, 2000; NGA & Center, 2010). However, most teachers have misconceptions about mathematical modeling and the modeling process (Gould, 2013) and lack knowledge about the nature of mathematical modeling (Blum & Borromeo Ferri, 2009). Therefore, this study investigated teachers' professional knowledge about the nature of mathematical modeling.

Significance and Related Literature

Ball, Thames, and Phelps (2008) emphasized the importance of teacher's content knowledge and explained that teachers of mathematics need certain knowledge domains to teach mathematics effectively. The practice of mathematical modeling is consistent with the Common Core standard: model with mathematics, and echoes the effective teaching practices and productive dispositions as explained in NCTM's principles to action (NCTM, 2014). Therefore, teachers' knowledge of the nature of mathematical modeling is worth investigating.

Methodology and Results

To achieve this, the author developed a Mathematical Modeling Knowledge Scale (MMKS) and pilot tested it with 71 practicing teachers of mathematics. The MMKS comprised 12 true or false items and an open-ended question. The scale development phases included item writing, experts' reviews, cognitive interviews, item analysis, and factor analysis. The overall internal consistency reliability of the MMKS for this sample was .80, indicating a relatively good reliable scale. Analysis on the open-ended item revealed that most teachers in this sample have misconceptions about the phrases *mathematical modeling* and *modeling process*.

Conclusion and Implications

Overall, the psychometric properties of the MMKS show promise in mathematics education research. The MMKS has the potential to benefit teacher education on mathematical modeling education. Results from this study and other published materials indicate a need for mathematical modeling education with both practicing and pre-service teachers of mathematics.

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Exploratory Activities with Dynamic Geometry Environment in Axiomatic Geometry

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In this study, I designed and implemented an instructional sequence of exploratory activities using a Dynamic Geometry Environment (DGE) in an axiomatic geometry course. The tasks in the sequence aimed at providing students with opportunities to encounter cognitive conflicts between their prior knowledge on Euclidean geometry and new observations on non-Euclidean geometry. However, some did not appear, some did appear and students recognized them, but could not resolve or just passed by. The conflict between what I intended in designing tasks and what I found in student responses seems to result from several aspects of design and implementation of the tasks.

Key words: Dynamic Geometry Environment, Axiomatic Geometry, Non-Euclidean Geometry

Researchers have addressed that exploration on non-Euclidean geometry can foster students' development of axiomatic reasoning by providing with informal experience of resolving cognitive conflicts toward their prior knowledge of Euclidean geometry (e.g., Hollebrands, Conner, & Smith, 2010). The purpose of the study is to investigate an effect of students' interactions with a Dynamic Geometry Environment (DGE) on developing axiomatic reasoning in terms of how they manipulate the geometric construction given in the DGE and how the corresponding responses from the DGE affect students' discussions.

I designed and implemented an instructional sequence of exploratory activities using Geometry Explorer (GEX) in an axiomatic geometry course at a large public university. For each of the tasks in the sequence, the students were asked to record their exploration using a screen-casting software. This form of data can capture the students' verbalized observations and arguments in the discussion synchronized with the visual representation. I analyzed the data to determine overall student performance and success or failure in the task design. The last activity of the sequence was coded to describe student use of GEX with analytic framework for student use of drag feature in relation to theories of variation (Leung, 2008).

In this study, I found several issues that resulted in success or failure of the task design as follows; (1) understanding of valid and robust geometric construction in DGE; (2) limitations of mathematical models in DGE; (3) concurrent multiple conflicts; (4) limitations of task design; (5) lack of strategic use of drag feature of DGE. The early experience of exploration brought into the issues on validity and robustness of geometric constructions in GEX. It embarked on discussing what it means to generalize figures by dragging activities in DGE and what they are expected to produce in the following investigations. On the other hands, the task designs in the activities have limitations in facilitating student's productive use of drag feature of GEX that resulted from several aspects involved in the design and implementation of the tasks. The spherical model of GEX constrained the students' wandering dragging across the entire surface of the sphere so that they could not successfully recognize the conflicts as planned in task design. Also, the geometric constructions given in the tasks was not robust enough to allow student to involve in guided dragging to find a particular example, the counter example of the exterior angle theorem in spherical geometry. The analysis on the students' use of drag feature indicated that the task design for exploratory activities need to take into consideration appropriate intervention to support developing systematic dragging strategies.

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STEM Major Mindset Changes During Their First Undergraduate Mathematics Course

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One of the reasons for the exodus in STEM majors is students' experiences in their first undergraduate mathematics course, usually introductory calculus. However, students with a growth mindset are more likely to persist past these initial courses. Although there is evidence that curricula like CLEAR calculus promoted gains in students' growth mindset, it is unclear how this curriculum compares to traditionally. The purpose of this quasi-experimental study was to investigate to what extent students enroll in CLEAR calculus become more growth mindset orientated than those that are enrolled in traditionally taught courses. The Patterns of Adaptive Learning Scale was used to measure the mindset of students in pre-calculus, calculus I, and calculus II. The analysis of the pilot data indicated CLEAR calculus students experience a small positive shift towards a growth mindset, while students in traditionally taught courses have a significantly more fixed mindset by the end of the semester.

Keywords: Calculus, mindset, transition to college

The initial transition course for STEM majors is calculus; most students who take this gateway course overestimate their preparation for the experience (Bressoud, Carlson, Mesa, & Rasmussen, 2013). One potential challenge in the post-No Child Left Behind era in the United States is that although students believe they are well prepared, the emphasis on standardized testing has placed a significant amount of emphasis on surface learning. This focus on surface learning leaves students unready to make connections between concepts in their initial undergraduate mathematics courses (Gueudet, 2008; Selden, 2005; Selden & Selden, 2002).One possible solution to help students transition smoothly to undergraduate mathematics is the use of formative assessments such as exit tickets; such assignments show promise in helping students to perceive their instructor as more approachable and caring about their success (Black & Wiliam 1998, 2009; Dibbs & Patterson, 2014).

However, the number formative assessments completed are a far stronger predictor of students' success than their weight in the course grade would indicate (Dibbs, 2015). One possible explanation for this effect was that students who completed more post-labs had different mindsets about learning mathematics than those that did not. Mindsets play a significant role in the overall success of calculus students. Dweck (2006) defines mindset in two different ways: fixed mindset and growth mindset. Students classified under the fixed mindset, if not immediately successful in introductory calculus often leave the STEM field. However, growth mindset students can persist and succeed, even after failures as severe as failing a course (Dweck, 2006). The hypothesis for this study was:There is no significant difference in the mindset changes between students enrolled in CLEAR Calculus and students enrolled in traditional calculus.

Participants were recruited from STEM majors enrolled in first-semester calculus (150 students enrolled) or second-semester calculus (100 students enrolled) courses at a rural Hispanic-serving research university in the South during the Fall 2016 semester. A modified version of the Patterns of Adaptive Learning Scale (PALS) was used to assess students' mindset. Data was analyzed via ANOVA and Tukey tests.

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Building a Cognitive Model for Symmetry: How Well Does an Existing Framework Fit?

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Symmetry has been found to be a rich and natural context for developing group theory (Larsen, 2009), yet the existing literature offers little insight on the complex cognitive processes in this domain. This poster will describe an attempt to use a pre-existing cognitive model of a student's understanding of symmetry, to help analyze the data from a recent teaching experiment aimed at exploring the development of one undergraduate student's understanding of symmetry. We share the ways in which the existing model accurately describes the student's cognitive processes associated with symmetry and also the places in which the model fell short in capturing the complexity of the student's thinking.

Keywords: Symmetry, Abstract Algebra, Cognitive Model, Teaching Experiment

Abstract algebra is an essential part of undergraduate mathematical learning and yet this subject is also known for its high level of difficulty at the collegiate level. Larsen has found the context of geometric symmetry proves to provide a 'rich and natural context for developing the concepts of group theory' (2009, p. 136), since the ideas of symmetry and equivalence are fundamental concepts in group theory (Burn, 1996). A recent teaching experiment (Steffe & Thompson, 2000) was conducted to identify a student's cognitive processes related to symmetry as they develop a mathematically robust definition of symmetry. This experiment explored how one undergraduate student, Birdie, developed an understanding of symmetry over a series of 5 task-based interviews. The student worked through the Measuring Symmetry Task (Larsen & Bartlo, 2009), which is designed to build on the student's aesthetic sense and intuition to help the development of formal ideas of symmetry. Asiala et al. (1998) offer a framework 'useful for understanding the mental constructions made by students learning about permutations and symmetries' (1998, p. 13), which utilizes the APOS perspective (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). This framework includes a genetic decomposition of an individual's understanding of symmetries. While the authors admit it is a 'very simple genetic decomposition' (1998, p. 18), they still found it to be very useful describing student conceptions of symmetry; when considering only rotation and reflection symmetries of fairly simple geometric objects.

In an initial attempt to use the APOS framework for symmetries as offered by Asiala et al. (1998) to describe Birdie's developing understanding of symmetry, we have found that it is fairly easy to illustrate that very often in the interviews Birdie is working with a "process" conception as described by Asiala et al., "A process conception of symmetry might be indicated by the subject's ability to imagine performing the symmetry without actually doing it." (1998, p. 18). However during our experiment, we also found that Birdie was able to articulate multiple processes for finding/identifying symmetries, some of which were mathematically accurate and some of which were not. For example she offered 2 distinct versions of what she called "line symmetry" one as a true reflection and the other as an iterative deformation of the figure based on Birdie's ability to fold, where Birdie performed both of these processes mentally. In this poster we challenge the ability of a simple genetic decomposition to accurately capture the complex ideas students associate with symmetry, especially in contexts when students are encouraged to rely on their intuition and aesthetic sense.

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Student Understanding of the Product Layer of the Integral in Volume Problems

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Research has shown that students have difficulties attending to the underlying product and summation structure of the integral when solving application problems. This study examines student conceptions of the product layer when solving volume problems. Participants were second-semester calculus students enrolled in a large, public university. Task-based interviews consisted of students working through and discussing volume problems. Preliminary results show that a majority of students' volume integral setups are highly formulaic and linked to memorized patterns and methods seen in class, as opposed to having a true understanding of the underlying structure of the problem. We plan to conduct more interviews of this type with additional volume problems and investigate other aspects such as visualization and gesture.

Key words: Calculus, definite integral, solid of revolution, volume

Some of the first types of definite integral application problems that students encounter are volume problems. Previous research has found that when solving definite integral application problems, students often rely on formulas, patterns, and previously encountered methods for setting up integrals (Yeatts & Hundhausen, 1992; Grundmeier, Hansen, & Sousa, 2006; Huang, 2010). Other studies have shown that students have very little idea of the dissecting, summing, and limiting processes involved in integration (Orton, 1983; Sealey, 2006, 2014; Jones, 2015). The goal of this study is to explore and analyze how students conceptualize volume of solid of revolution problems, and in particular, how they attend to the product layer (integrand and differential) during integral setup.

Our research is built on the foundation of the constructivist learning theory (Piaget, 1970), and the framework guiding our analysis of student understanding of definite integral concepts is based on Sealey's (2014) integral framework. We will also be discussing student's responses using Vinner's (1997) conceptual framework that describes and analyzes mathematical behaviors of students as conceptual/analytical or pseudo-conceptual/pseudo-analytical.

Clinical interviews were conducted with four students who were enrolled in second-semester calculus during Summer 2016. The video-taped interviews involved the participants working through three routine volume problems and talking aloud about their thought processes and problem-solving strategies. The interviewer asked several questions throughout the interview in order to determine if the students could unpack their methods to explain why they worked. The videos were transcribed and data analysis is ongoing.

Preliminary data analysis suggests that students have a very superficial understanding of integral volume problems. After some probing, it becomes clear that participants' responses are based on memorized formulas and not linked to the underlying structure of the definite integral. Almost all of the participants exhibited a strong connection between the axis of rotation and the variable of integration, but could not explain that connection.

In future studies, we plan to create volume problems that better expose the inconsistencies between students' integral setup and their conceptual understanding of integration, and develop activities that discourage students' pseudo-actions and foster more meaningful learning. Other avenues we would like to explore are how students use their visualizations when solving volume problems, and how gesture is incorporated into their understanding of these types of problems.

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Mathematics Education as a Research Field: Reflections from ICME-13

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Abstract: In an effort to broaden knowledge within the United States, the National Council of Teachers of Mathematics, with support from the National Science Foundation, funded multiple scholars' participation in the 13th International Congress of Mathematics Education. Working in NCTM theme groups these scholars met, discussed, and provided reports to various American educational organizations, so as to bring back findings related to a variety of ICME Topic Study Groups. The purpose of this poster is share findings from the "Mathematics Education as a Research Field" NCTM theme group.

For several decades mathematics education has been referred to as a rapidly expanding field of scholarly inquiry (Sierpinska & Kilpatrick, 1997; Artigue, 1999; Schoenfeld, 2000; English & Kirshner, 2015). Evidence of this growth can be seen in the significant increase in scholarly journals focused on mathematics education research and in the increasing number of areas of scholarly inquiry. As areas of inquiry have grown so has the diversity of the theories and research methods employed. With increasing diversity comes both opportunity and challenge. New theories, methods, and areas of inquiry offer not only opportunities to pursue novel phenomena but also the promise of fresh perspectives on persistent problems. However, diversification can also hinder communication and understanding among researchers – a point prophetically made in Thompson's (1982) article, *Were lions to speak, we wouldn't understand*.

To foster professional community among American researchers during this rapidly expanding period of growth, it is important (perhaps more than ever before) to seek out and understand diverse perspectives. Indeed, there is a need not to only to seek out those whose work and professional lives differ from our own but also to seek out and understand the ways in which our work is understood by others. It was with these aims in mind that a cadre of American scholars was sent to the *13th International Congress of Mathematics Education* (ICME 13) by the National Council of Teachers of Mathematics (NCTM), with funding from the National Science Foundation. The purpose of this poster is share findings from the reports of those scholars, who as part of the NCTM envoy participated in the *"Mathematics Education as a Research Field"* NCTM theme group: Hortensia Soto (group leader), Spencer Bagley, Stacy Brown, Jinfa Cai, Molly Keaton, Debra Junk, Robert Ronau, and Brandy Wiegers. Specifically, the poster will share questions and insights from a variety of ICME 13 topic study groups and do so with a focus on understanding distinct perspectives, new horizons, and shared domains of inquiry that may inform research on undergraduate mathematics education.

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Connecting Reading Comprehension Research and College Mathematics Instruction

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The poster will have three parts. The first is a literature review of reading comprehension research. Connections will be drawn between different research studies that apply to college mathematics instruction. In addition, common themes from research will be used to motivate the need for reading comprehension instruction in college mathematics. The second part will give results of a new research study involving interviews about reading comprehension strategies and instruction with interdisciplinary faculty from several institutions. The third part will describe proposed methodology of future research on reading comprehension in college mathematics instruction. It is hoped that during the poster session, participants will have a chance to brainstorm about possible future directions and methodologies for research in this area.

Keywords: reading in mathematics, clinical interviews

Faculty comment that students don't use the textbook. Students complain that the textbook wasn't useful. Courses have shifted from pure lecture to flipped classrooms, the Moore method, and other student-centered, constructivist approaches. In addition, there are more online courses. Often people don't know how to read newspapers or other articles that contain technical information and evaluate the accuracy. What is the missing piece that weaves through all of this? The ability to read and comprehend mathematics.

Several years ago I was given the opportunity to teach a freshman symposium course, taught by instructors from different disciplines, but with a focus on reading and writing. Instructors for the course undergo intensive summer training. I soon realized that the techniques from faculty across campus would be valuable when modified for my mathematics courses, and I began to research reading comprehension literature.

The first part of this poster will be an overview of reading research literature on literacy, on content area literacy, on disciplinary literacy, and on teaching reading in mathematics. Unfortunately, there are not a lot of resources for teaching reading in mathematics at the college level. Taking advantage of the poster format, this section will draw connections between different reading comprehension research studies that pertain to college mathematics instruction. Through this literature review, the need for college mathematics reading instruction will also be illustrated.

Due to the need for increased study of reading comprehension in college mathematics instruction, a new exploratory research study was completed that included open-ended, clinical interviews with ten faculty from mathematics, foreign language, philosophy, education, English, history, theology, and K-12. The second part of the poster will give details on the interpretive analysis of the interviews.

There is a need for additional research on this subject. A research study is planned, and the third part of the poster will give details on the proposed methodology of this study. It is hoped that participants during the poster session will have the opportunity to discuss future directions and methodologies for this research.

Diagrams for the Reasoning and Proof of Amortization Formula

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Thomas Mgonja, Utah Valley University

Many liberal arts or humanities students who are required to take quantitative reasoning in college have mathematics anxiety. One cause is a lack of symbolic skills for reasoning. Learning style theories suggest that different people learn in different ways. This study constructs a diagrammatic reasoning model for the concept of amortization to help students learn quantitative reasoning. The model also connects the basic concepts to the proof of the mortgage payment formula.

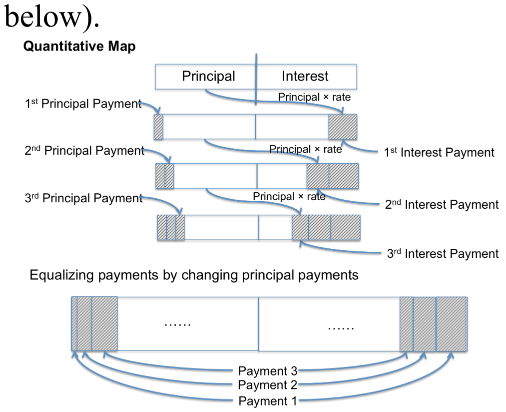
Key words: Diagrammatic Reasoning, Quantitative Reasoning, Amortization, Proof, non-STEM

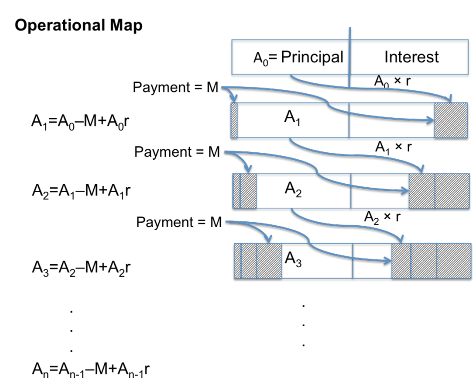
Many non-STEM major students (e.g., liberal arts and humanities) have negative attitudes towards and fear college mathematics classes such as quantitative reasoning (Henrich, 2011). This math anxiety is due in part to a lack of skills in symbolic and propositional calculations or manipulations of mathematics (Ashcraft, 2002; Hembree, 1990). The question that must be asked is, "Are there other alternative ways to help the students learn quantitative reasoning and appreciate mathematics?".

Learning style theories suggest that different people learn in different ways (Coffield et al., 2004; Pashler et al., 2009). In particular, diagrammatic reasoning has been tested as an alternative method for mathematical inference and proof (Jamnik, 2001; Kulpa, 2009). Diagrams allow richer properties and relations among elements in presenting mathematical structures and their meanings. They may significantly reduce the encoding process and make it easier for students to understanding and reason.

The purpose of the study is to construct a diagrammatic model for helping non-STEM major students perceive and process mathematical information visually while improving their math attitudes, and, consequently, become more effective at performing quantitative reasoning. In addition, we hope to assess the potential of the diagrammatic model in quantitative reasoning and in facilitating symbolic manipulations and proof.

We chose the concept of amortization for model construction because (1) it is introduced in the quantitative reasoning course, which is required of most liberal arts and humanities students, and (2) it involves the idea of complicated exponential growth (compounding process) and algebraic reasoning (symbolic manipulations on the mortgage payment formula $M = \frac{A\times r(1+r)^t}{(1+r)^t-1}$). We constructed two diagrammatic maps, the quantitative map and the operational map (See Figures below).





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A Discussion of The Use of Excel in Statistics Teaching and the Role of Technology in Improving Teaching and Learning Statistics with a Special Focus on the 'knitr' R-package.

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We cannot imagine teaching statistics today without using some form of technology. Teaching statistics courses in the past was very challenging due to time consumption in calculation. The computation is now done by computer and other software packages but the challenge now is that understanding the results and staying mistrustful to the results. In this discussion, we will discuss how we have been teaching introductory statistics courses with or without computer and provide some typical examples in excel spreadsheet. On the other hand, due to the recent development of the power of computing, we present a dynamic documentation of computational outputs from a statistical programming language using R markdown (included in the package "knitr") which is a simple formatting syntax for authoring HTML, PDF, and MS Word documents. Hence the main goal of this presentation is to give an outline of the method used in the past, its challenges as mentioned by Christine Duller (2008) and a demonstration of an R package which recently brought a great attention to the teachers of statistics and researchers. The usefulness of the package will be presented using some data analysis and graphs using R programming language. Since R Markdown supports dozens of static and dynamic output formats including HTML, Pdf, MS Word, Beamer, HTML slides, Tufte-style handouts, books, dashboards, shiny applications, scientific articles, websites, and more, it is more popular to the researcher, teacher of statistics and collaborators.
Key words: Teaching and Learning Statistics, Excel spreadsheet, "knitr" package, R Markdown

Literature Review

After we had an access of excel spreadsheet for teaching statistics classes, a number of research work have been published about teaching statistics. Christie, D. (2004) mentioned in his paper about "Resampling with excel-teaching statistics". Similarly Doane, D. (2004) talked about "Using Simulation to Teach Distributions". In (2008), Christie D. raised some issues on his paper "Teaching Statistics with Excel: A Big Challenge for Students and Lecturers". We can find many institutions use excels spread sheet for teaching and learning statistics in classroom.

After the development of the programming language R in 1993 and the availability of personal computer and other technologies, the programming language R (an open source) and many other programming languages have been widely used. The package "**knitr"** is an engine for dynamic report generation with R. It is a package in the statistical programming language R that enables integration of R code into LaTeX, LyX, HTML, Markdown, AsciiDoc, and re-structured text documents. We can see the popularity of R markdown and its usefulness for statistic teaching and research. Markdown is a lightweight markup language with plain text formatting syntax designed so that it can be converted to HTML and many other formats using a tool by the same name. Markdown is often used to format readme files, for writing messages in online discussion forums, and to create rich text using a plain text editor.

In this discussion, we attempt to address the usefulness of excel spreadsheet in teaching and learning statistics and also will discuss the latest development and show how efficient it is to use the package R Markdown for teaching and research. The main goal of this discussion is to provide high school and college introductory statistics instructors with a scenario of teaching statistics using excel and the recent development of the power of computing.

We have the following research question for this study:

1. What is the targeted group of people who use excel sheet for teaching and learning statistics?
2. What is the label of statistics and computation knowledge we need for using the programming language R and the package "knitr"?
3. What is the cost and availability of the programming language and computers etc.?

Theoretical Framework and Methodology

We will briefly discuss the heavy use of excel in teaching and research in different disciplines. Further, we discuss the statistical software R and its usefulness in teaching and learning statistics courses.

Results of the Research

We compare the usefulness of teaching statistics using excel for one group of audience and also look at the effectiveness and popularity of the statistical programming language R with a special focus on the package "knitr".

Discussion and Conclusions

Even though the use of excel in teaching in the past was very effective, due to its capacity and challenge for special data, statistical programming language R is more popular. Especially, for research and teaching statistics courses, it looks like people are more tending to use R.

Since R Markdown supports dozens of static and dynamic output formats including HTML, Pdf, MS Word, Beamer, HTML slides, Tufte-style handouts, books, dashboards, shiny applications, scientific articles, websites, and more, it is more popular to the researcher, teacher of statistics and collaborators.

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To Factorize or Not To Factorize: Novice Teachers' Struggles

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Focusing on the unique factorization domain (UFD) in college mathematics and polynomial factorization in school mathematics, this study examined how teachers' factorization concepts occur in the teaching context. We conducted semi-structured interviews with eight novice teachers. The result of this study can serve as a resource for teacher educators when teaching UFD in abstract algebra in the future.

Key words: Mathematical Knowledge for Teaching, Polynomial Factorization

What role does college mathematics knowledge play as teacher knowledge for teaching mathematics? To answer this question, much research has been devoted to mathematics teacher knowledge (see e.g. Ball, Hill & Bass, 2005; Evens & Ball 2009; Buchholtz et al., 2013). However, there is a lack of specific research on how college mathematical knowledge contribute to teaching school mathematics. With regard to the concepts that intersect between college mathematics and school mathematics, we focus on the concept of UFD in college mathematics and polynomial factorization in school mathematics. The purpose of this study is to examine how teachers are utilizing their college mathematics knowledge in the context of school mathematics.

Eight novice teachers were selected as our research participants because we considered teachers with the most recent college mathematics knowledge to be the most suitable participants for understanding how related college mathematics concepts are being used to understand school mathematics concepts. Using middle school-level polynomial factorization task, we conducted semi-structured interviews. All interviews were transcribed. The teachers' answers were categorized and analyzed focusing on teacher knowledge emerged in terms of Mathematical Knowledge for Teaching framework (Ball, Thames, & Phelps, 2008). In order to ensure the validity of the analysis, three researchers crosschecked the categories of teacher interviews. The task was: If a middle school student has asked you to what extent he or she needs to factorize the polynomial $2x^2 + 4x - 6$, how would you answer?

This study shows that teachers who knew about polynomial factorization in college-level explicitly did not just follow the knowledge to teach polynomial factorization in school context. Teachers who had proper common content knowledge (CCK) sometimes modified to suit the level of secondary school mathematics, but recognized how they are treating number factors in explaining polynomial factorization in school context. Based on the results, we identified that horizon content knowledge (HCK) plays significant roles to understand secondary school mathematics from advanced viewpoint. The other teachers who did not show the proper common content knowledge did not consistently and concretely explain the basis for their explanation. So, this study implies that especially HCK and CCK should be emphasized in order to connect between college-level mathematics knowledge and school-level mathematics knowledge in preservice teacher education.

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Upper-division Physics Student Thinking Regarding Non-Cartesian Coordinate Systems

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As part of a larger effort to develop a research-based math methods curriculum for undergraduate physics students, results from a case study probing student thinking on plane and spherical polar coordinates are presented. Using a resources framework, a think-aloud protocol was used to elicit student thinking regarding non-Cartesian coordinates. Findings are consistent with previously published literature regarding student thinking on coordinate systems. Mark, a senior physics major, despite initially clearly identifying and defining the radial and polar unit vectors on a diagnostic 2-dimensional problem, made inconsistent assertions when asked to apply those definitions in three dimensions using spherical coordinates. Additionally, we will address content issues concerning the definition of displacement and position vectors in Cartesian and Non-Cartesian coordinate systems.

Key words: Coordinate systems, Non-Cartesian, Resources, Interviews, Physics

Using non-Cartesian coordinate systems continues to be difficult for undergraduate physics students (Paoletti *et al.* 2013, Montiel *et al.* 2009) even in the upper division where application of the concept is of significant importance (Sayre and Wittmann 2008). As part of a larger effort of PER in the upper division (Caballero *et al.* 2015, Loverude and Ambrose 2015) our collaboration has begun to develop a research-based curriculum for a mathematical methods course for undergraduate physics majors. This paper discusses a portion of that effort by investigating student thinking regarding non-Cartesian coordinates systems, specifically plane and spherical polar coordinate position vectors.

The polar coordinate system, both planar and spherical, is commonly used in a number of upper-division physics courses, including mechanics and electrodynamics (Griffiths 1999, Fowles and Cassiday 1999). Students are initially taught how to convert from the more familiar Cartesian coordinate system into polar coordinates while the expectation for most upper-division courses is that students can think in and use non-Cartesian coordinates fluently; not simply translate. Issues arise as students attempt to map Cartesian thinking into non-Cartesian coordinate systems (Hinrichs 2010). The research presented here analyzes students' in-the-moment thinking using a Resources theoretical framework in an attempt to understand a student's underlying reasoning with respect to non-Cartesian coordinate systems. The nuanced nature of students' reasoning during our interviews informs the use of a Resources Framework, wherein we attempt to identify the kinds of resources students appear to be using as they proceed through the interview.

For this poster, we present a case study of one student, Mark, who initially demonstrates outstanding understanding of unit vectors and their definitions in non-Cartesian coordinates; i.e., that $\hat{r}$ points in the direction of increasing radius. However, he has a self-described "moral dilemma" when deciding whether or not to include $\hat{\phi}$ and/or $\hat{\theta}$ terms in spherical position vectors where only a $\hat{r}$ term was needed. The content details of position vectors versus displacement vectors in three dimensions are not trivial and will be discussed alongside student data and analysis.

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The Mathematics Attitudes and Perceptions Survey: New Data and Alignment with Other Recent Findings

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Student attitudes about and perceptions of mathematics influence their success and learning, and have been of interest for many years in mathematics education. The Mathematics Attitudes and Perceptions Survey is a short, validated Likert-scale instrument that measures confidence, interest, relation of mathematics to the real world, persistence in problem solving, growth mindset, use of sense-making behaviours, and the extent of other novice attitudes towards mathematics. In this poster, we share the complete instrument and its categories, a brief summary of the development process and resulting model statistics, as well as scores across different populations measured so far (3 institutions, variety of courses). The student responses include new data since the publication of the instrument as well as additional analysis of groups, in particular a comparison of attitudes between genders that matches with recent results relating STEM persistence to attitudes and beliefs (Ellis et al., 2016; Wang et al., 2016).

Key words: student attitudes, student perceptions, student beliefs, survey tool

Motivation and Development

Development of the Mathematics Attitudes and Perceptions Survey (MAPS) was prompted by interest in matching up results regarding student attitudes in mathematics to those captured via similar surveys "scored" relative to expert consensus views in physics (Adams et al., 2006), chemistry (Barbera et al., 2008), biology (Semsar et al., 2011), earth sciences (Jolley et al., 2012), and computer science (Dorn & Tew, 2015). These contain many similar statements and share aspects of development (Adams & Wieman, 2011), though for MAPS the quantitative aspects followed more conventional instrument development in terms of factor analysis and model fitting.

The survey consists of 31 Likert-scale statements scored based on alignment with expert views and reported behaviours involving mathematics. The instrument was designed iteratively with interviews of faculty and students, rounds of responses from multiple populations of students leading to factor analysis and model confirmation. Further details and preliminary population scores are available in an initial article (Code et al., 2016).

Results So Far

While our data does not have sufficient detail for comparison in some demographic variables, nor for a relation to teaching methods like Sonnert et al. (2015), we have similar findings to the recent MAA study of calculus programs, including an overall decrease in such attitudes over a first year of a calculus courses, with men overall reporting higher attitudes in most categories including confidence in their mathematical ability (Maciejewski, 2016). The drop in attitudes in introductory post-secondary courses and the gender gap are similar to results in other fields with the instruments which helped inspire MAPS (Hansen & Birol, 2013; Bates et al., 2011). This convergence of results, along with the ease of deployment, suggests the value of MAPS in measuring student attitudes and perceptions in mathematics. Our poster will present the full instrument, categories, and a variety of data from different populations so far.

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Online Course Component and Student Performance

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Abstract: *A study was conducted in a small, private university in Northeastern United States in order to determine if introducing an online component to a first year Calculus course would influence student learning. An online component was presented in two sections of the calculus courses while two sections were taught using the traditional format. Preliminary data suggest a positive correlation between the online component and improved student performance in the course.*

Keywords: online learning, calculus

Online education is rapidly growing in colleges and universities. According to Allen and Seaman (2013), thirty two percent of students in higher education in the United States are taking online courses, which equates to approximately 6.7 million students taking at least one online course each year. The college student population is becoming increasingly diverse and different learning options are necessary to meet the demands of these students.

This poster reports on a study at a small, private university in the Northeastern United States. Prior to the intervention, there were very little opportunities for students to participate in online learning. There has been recent interest from the administration to remain competitive with other institutions by offering online mathematics courses. The request was to transition some mathematics courses to an online format. In order to address this concern, we incorporated an online component to two sections of a first semester Calculus course, a course which was has traditionally been taught without an online component. The remaining two sections of the course were taught in the traditional format without an online component. The research question guiding this study is: *Does an online component increase student performance rates in a first year calculus class?*

The study began in the Fall semester of 2016, and will continue in the Spring of 2016. Students in the online sections participated in weekly discussion questions through a forum using critical thinking skills to review the course material. The other sections were given the same questions, but discussed in a face to face setting during the fourth hour of the course. Themes from the student discussion have been grouped and coded for analysis. Questions from exams and the final were selected to compare results from each section. Student surveys are being distributed at the end of the semester to gather feedback on their experiences. Formal data analysis will take place in the Spring of 2016, but preliminary results indicate a positive correlation between the online component and student performance in the course. The results of the study will be used to implement changes in other courses that will be moved to an online format.

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JITAR Online Modules to Improve Math Preparation of Engineering Students

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Engineering educators are challenged with students at greatly varying mathematical skill levels while needing to quickly bring all students up to the same mathematical mastery level at appropriate points during a semester. To address this problem our team designed a teaching e-tool in WeBWorK called Just-In-Time Assessment and Review (JITAR) to be delivered as an on-line system consisting of a series of individualized mathematics modules inserted within engineering courses at strategic points in the semester, prior to students needing those math skills. JITAR assesses the mathematical competency level of the individual student and provides formative individualized learning opportunities in time for the students to be successful in applying the necessary mathematics to the new engineering material. The new type of WeBWorK assignment was designed to support the desired presentation and flow of the module integrating assessment and e-learning assistance by offering a customized learning path to students. This project is currently funded by National Science Foundation.

Key words: Engineering Education, On-line Assessment, Just-In-Time, JITAR.

There are several factors that influence student retention and success in engineering, the most important being mathematical competency, but there is well documented knowledge gap in the preparation of engineering undergraduates. Engineering programs typically enforce prerequisites to guarantee a certain level of mathematics proficiency before the students enroll in engineering classes. Due to several factors this sequence is not completely effective at preparing engineering students so engineering educators are challenged with students at greatly varying mathematical skill levels, while needing to quickly bring all students up to the same mathematical mastery level at appropriate points during a semester. While the traditional model of integrating engineering applications into the mathematics courses and later reviewing those concepts in engineering courses has benefited the math preparation of the engineering students, it does not completely address the mathematical knowledge gap of engineering students due to its "one size fits all" approach to the problem (Manseur, Leta, & Manseur, 2010).

To address this problem, mathematics and engineering instructors designed a teaching e-tool called Just-In-Time Assessment and Review (JITAR) delivered as an on-line system through WeBWorK consisting of a series of individualized mathematics modules, to be inserted within engineering courses at strategic points in the semester. JITAR assesses the mathematical competency level of the individual student and provides formative individualized learning opportunities in time for the students to be successful in applying the necessary mathematics to the new engineering material. The structure of the modules relies heavily on the fact that the assessment and review content needs to be generated based on individual student's performance.

The first JITAR module has been developed for Complex Numbers and Complex Functions and implemented in Linear Systems for Biomedical Engineers course in the Biomedical Engineering curriculum. The module is an application in WeBWorK that allows any course instructor to import and adopt easily. In this poster, we will present results from the implementation of JITAR modules (Ozturk, Duca, & Raubenheimer, 2015).

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A Framework for Characterizing a Teacher's Decentering Tendencies

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This poster presents a framework for characterizing teachers' decentering during teacher-student interactions when teaching. Analysis of video data of a graduate teaching assistant's (GTA's) precalculus class generated six levels of teacher-student interactions. These levels will be described and illustrated with excerpts from this video analysis.

Key words: Precalculus, Graduate Teaching Assistants, Decentering

Mathematics education policy documents (e.g., NCTM, 2000) have emphasized the importance of teachers' focusing on and using student thinking to inform their interactions with students. In our work to support teachers to consider student thinking **when** teaching, we leveraged Steffe and Thompson's (2000) description of decentering—the manner by which an individual adjusts (or does not adjust) his or her actions **to understand** another individual's thinking. In our group's early work Carlson, Bowling, Moore, and Ortiz (2007) identified five levels of decentering when analyzing teacher-teacher interactions when participating in a professional learning community.

We coded video data collected in a GTA's precalculus classroom using research based instructional materials (Carlson, Oehrtman & Moore, 2016) designed to engage students in constructing deep meanings of the course's key ideas. Our early coding using Carlson et al's (2007) framework resulted in our extending the framework to include a teacher's focus on students' answers and thinking when teaching. These levels are described (Table 1) and will be illustrated in the context of the GTA introducing his precalculus students to the ideas of constant rate of change from a quantitative reasoning perspective.

Decentering Levels	Description
0	Appears interested in student's answers but not in student's thinking.
1	Takes actions to move students to her/his thinking or perspective without trying to understand or build on the expressed thinking and/or perspectives of students.
2	Appears interested in understanding the thinking and/or perspectives of students but actions do not appear to be based on these expressions.
3	Appears interested in understanding the expressed thinking and/or perspectives of students but takes actions to move students to her/his thinking or perspective.
4	Appears to have insight into the expressed thinking and/or perspectives of students and makes general moves based on the students' expressions.
5	Takes action to understand the thinking of others (probes), appears to understand the expressed thinking and/or perspective of students and takes actions that builds on and respects the rationality of these expressions.

Table 1. Characterization of Teacher Decentering Levels (adaption from (Carlson et al., 2007)

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SOME LOGICAL ISSUES IN RUME

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In this communication, we will present various arguments supporting the claim that it is worthwhile taking in account logical issues in research in undergraduate mathematics education. We will provide arguments relying on research on student's difficulties and their links with teachers' practices on the one hand; on relevance for researchers on the other hand.

Key words: logical analysis of language - syntax and semantics – truth *versus* validity

We have shown (Durand-Guerrier 2003) that predicate calculus, with concepts such as variable, open sentence, quantifiers, logical connectors, logical validity, truth in an interpretation, syntax and semantic, is a relevant epistemological reference for mathematics education. We present two sides of these logical issues in RUME: on teaching and learning, and on research.

Students'difficuties and links with teachers'practises

Undergraduates' difficulties concerning logical matters in mathematics are well documented in the research literature (Chellougui 2009; Dubinsky & Yparaki 2000; Durand-Guerrier et al. 2012; Epp 2003; Roh 2010). We hypothesize that they are linked with some teachers' practices. We will provide two examples of such links.

1) While the logical formalism could appear as a mean of conceptual clarification (Quine 1950), it appears for many fresh university students as an unbearable obstacle (Chellougui 2009), and many students are unable to unpack the logic of mathematical statements (Selden and Selden 1995). On another side, some ordinary teacher's practices tend to reinforce these difficulties by neglecting the logical side of students' mathematical difficulties (Durand-Guerrier 2003, 2008).

2) A main concern in undergraduate education is to develop competencies in proof and proving as a clue contribution to mathematical conceptualization. However, for many students, the requirement for proof is seen as a formal demand of teachers. In Durand-Guerrier (2008) we claimed that this could be related to the lack of a clear distinction between truth in an interpretation and logical validity. Durand-Guerrier and Arsac (2005) evidenced that while expert use semantic controls to check the validity of a proof it is not the case for undergraduates

Logical analysis as a tool for research in mathematics education

We claim that logical issues are also valuable for RUME. In our work we use logical tools to enrich and deepen the *a priori* analysis of didactical situation on the one hand, the *a posteriori* analysis of students' productions on the other hand (Durand-Guerrier 2013). We use *logical analysis of mathematical statements* to enlighten ambiguities likely to create difficulties that are reinforced in case of plurilinguism (Durand-Guerrier et al. 2016), and Copi's natural deduction as a tool to identify steps in a proof that could be *a priori* problematic (Durand-Guerrier 2008).

Conclusion

Although logical issues concern all the levels in the curriculum, it is clear that they become more crucial at university where logic and mathematics are closely intertwined in mathematics activities, including proof and proving. We will give examples in this communication showing that taking in consideration logical issues in research design for RUME appears to be fruitful.

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A Comparison of Faculty Expectations and Student Perceptions of Engagement in a Calculus I Class

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This poster describes and includes a discussion of the learning benefits and gains that students, enrolled in a Calculus I course, and their instructor reported as a result of participating in an active learning environment. The alignment between what the instructor valued and what 67 students experienced as they engaged in Calculus I activities was assessed using a survey. The results indicate a positive relationship between perceived importance and reported frequency of engagement, which resulted in benefits and student learning. Opportunities for improvement outcomes were found and can serve to support strategies that improve teaching and learning in Calculus I.

Key words: Calculus I, Engagement, Learner-centered Instruction

Learner-centered teaching is an instructional approach to teaching that is increasingly being encouraged in undergraduate gateway mathematics courses such, as Precalculus and Calculus I. Research (e.g., Fritschner, 2000; Kogan & Laursen, 2013) suggests that students learn best when they are engaged with active learning tasks that promote student thinking and lead to deeper mathematics understanding. Learner-centered teaching reflects several instructional approaches including, but not limited to, actively engaging students in group-worthy tasks, peer-learning, problem-solving, and engagement that includes student effort outside the classroom that will contribute to their academic success. While learner-centered instructional strategies have many reported benefits, research also indicates that it is not easily accomplished and there is often a mismatch between instructor and student expectations (Brophy, 2004). The congruence between what instructors value and what students report doing in their Calculus I actively learning environment is critical to understanding the value instructors place on instructional practices intended to support student learning and understanding of Calculus I concepts.

The question that guided this study was: What benefits and gains do students enrolled in a Calculus I LC course and their instructor report as a result of participating in a LC teaching environment? Student and instructor reported benefits and gains were assessed using a modified Class-Level Survey of Student Engagement (CLASSE) survey. The purpose of the study was to use the results to identify effective Calculus I instructional practices that are beneficial in promoting student success in Calculus I.

A quadrant analysis was used to analyze the data and give meaning to the relationship between perceived importance and benefits of Calculus activities by the instructor and reported frequency of occurrence and learning benefits of those activities by students. The findings indicate a high degree of congruence associated with the completion of in-class higher-order thinking tasks, student effort, and participation in in-class group worthy activities/tasks. Areas for further consideration include student engagement with the instructor (i.e., meeting with the instructor during office hours to review problem solutions) and outside-classroom activities (i.e., participating in a study group or pre-reading/problem solving). The results of the study can be used to shape/transform Calculus I teaching and learning experiences, and inform the design of faculty development activities.

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Calculus Students' Meanings for Average Rate of Change

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This study considers calculus students' conceptualization of average rate of change at a private liberal arts college in the Midwest. Researchers have indicated that undergraduate students do not develop productive meanings for average rate of change. In order to explore undergraduate students' meanings for average rate of change further, we conducted clinical interviews with 10 undergraduate students on a four-item test. Participants were undergraduate students taking Calculus 1 at the time of the study. Interviews were conducted towards the end of the semester to ensure students have learned average rate of change. Qualitative techniques were used to analyze data. We will present and interpret data highlighting the techniques used by the students during the tasks. We will conclude with implications from our findings and questions for future research.

Keywords: Average rate of change, calculus, mathematical meanings, undergraduate students

Mathematics students in the United States are underperforming when compared to other major world powers (Hanushek, 2010). As Byerley, Hatfield and Thompson (2012) discuss in a recent study, the underlying understandings of math concepts that students develop while studying at our schools and universities are an explanation for the gap in performance. By studying the understandings behind a major concept, we can draw conclusions about the quality of education that brings the students to this point. A previous study by Carlson, Jacobs, Coe, Larsen and Hsu (2002) showed that Calculus students have difficulties with problems that require an understanding of average rate of change.

10 undergraduate students from a Calculus 1 course from a private liberal arts college participated in this study. Interviews were conducted towards the end of the semester to ensure students have learned average rate of change. The interviews were task-based clinical interviews (Clement, 2000) and were videotaped. Participants were given a four-item test, two of which were the same that were used by Yoon, Byerley and Thompson (2015), consisting of questions that examined their meanings for average rate of change. Interviews were conducted in a one-on-one environment. Each interview was transcribed and written work was digitized.

We used open and axial coding techniques (Clement, 2000; Strauss & Corbin, 1998) and conceptual analyses (Thompson, 2008) to analyze data. Each researcher analyzed student's responses individually by noting each student's responses for working with average rate of change tasks. Then, we met to discuss our observations, looking for common patterns among the students' techniques for specific tasks. We created codes to identify these patterns we observed. Codes were revised and changed in the process to capture the similarities or differences in techniques we observed both within similar tasks and across varying tasks.

We found some students interpreted average rate of change as the difference of function values only. We also found students conceived average rate of change as the arithmetic mean of rates. We will discuss students' meanings for average rate of change in detail in the poster presentation and will present questions for future research.

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Empowered Women In RUME: What Have We Been Up To?

For the past three years we have run a seminar for 60 – 75 women in RUME the day before the annual conference called MPWR: Mentoring and Partnerships for Women in RUME. Participants included graduate students, post-doctoral fellows, faculty, and researchers outside of academic positions. In this poster, we provide a window into these seminars, specifically addressing the motivation for the seminar, the structure of the seminar, topics discussed in the seminar, research related to the efficacy and transferability of MPWR, and the future of MPWR. Our hope with this poster is to both share what we have been doing and get feedback from the community for what more can be done.

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Colorado State University	Cal Poly Pomona	Texas State University	Portland State University	Virginia Tech

Key Words: Mentoring, Women, Support

The disproportionately low number of women in STEM fields in academia at all stages of a career is well documented (Hill, Corbett, & St. Rose, 2010), as is the role of mentoring, both vertically and laterally, in bringing (and keeping) more women in these positions (Beede et al., 2011). However, mentoring is underutilized (Preston, 2004). As female mathematicians whose expertise is in research in undergraduate mathematics education (RUME), we identified a need in our community for increased support and mentorship. Prior to 2014 (when this seminar began) there was no formal mentoring structure for women specifically, or RUME participants in general. Some women were getting mentoring because of their personal or academic networks, but this was not equitably accessible, especially for women coming from universities with no other RUME researchers, or for women coming into RUME from mathematics or non-undergraduate mathematics education. We need to ensure that support exists for all women at all career stages in their academic development. The MPWR seminar, which stands for Mentoring and Partnerships for Women in RUME, was created to address this need, and, based on participant feedback, is succeeding in doing so.

In this poster we describe the MPWR seminar, including motivations for this ongoing seminar, a description of past seminars, and our vision for future MPWR seminars. We also address the beginning stages of researching what aspects of MPWR are most effective and why, with an eye towards what aspects of MPWR could be adapted by other organizations.

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Designing and Developing Likert Items That Capture Mathematical Problem Solving

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The Mathematical Problem Solving Item Development Project is designing and developing Likert items that capture students' capacity in mathematical problem solving (MPS). The project, now in year two, continues to refine items that capture aspects of MPS. The refinement process included piloting items on over 1000 students in College Algebra and Calculus, hour-long think-aloud interviews with 26 students, and review by experts. The goal of this poster presentation is to provide information about the item development and design and gather feedback and suggestions on further design and development.

Key words: mathematical problem solving, mathematics assessment, gateway courses

Research has identified key prerequisite procedural knowledge and conceptual knowledge linked to foundational preparation for gateway mathematics courses (e.g. Carlson, Oehrtman, & Engelke, 2010); however, there is limited research on connections between foundational MPS capacity and success in gateway mathematics courses (Schoenfeld, 2013). The *Mathematical Problem Solving Item Development Project* aims toward eventually developing an instrument that provides a profile of an undergraduates' MPS capacity.

Research Questions

This project has created and tested over 100 Likert items linked to research-based aspects of MPS such as sense-making, justifying, representing and connecting, and looking-back (Epperson, Rhoads, & Campbell, 2016). Discussion with RUME attendees will assist us in determining future item development as well as in identifying any issues in research design and methods. This will help us address the questions: (1) What design issues need to be resolved to create items intended to minimize the effects of domain knowledge while bringing MPS capacity to the forefront in measurement? (2) To what extent can Likert items linked to research-based MPS behaviors capture undergraduates' MPS capacity?

Discussion

The refinement process has been informed by results from pre- and post-tests consisting of five open-ended problems and 25-30 associated items to over 1000 students over three semesters of College Algebra and Calculus at a large urban university in the Southwest, analysis of 26 hour-long think-aloud student interviews, and expert review. This poster will provide details on item development and refinement as well as the methods used in item analysis.

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Reducing Abstraction in the Group Concept Inventory

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In this poster we report on results from the Group Concept Inventory (GCI), a conceptual assessment for introductory group theory students. Over 400 students from thirty institutions took the inventory. We use the framework of reducing abstraction (Hazzan, 1999) to situate student responses. We found that students frequently reduced abstraction (in a multitude of ways) when dealing with fundamental concepts in group theory.

Keywords: Abstract algebra, Reducing abstraction, Concept inventory

Abstract algebra is often one of the earliest courses where students "cop[e] with the difficult notions of mathematical abstraction and formal proof" (Weber & Larsen, 2008, p. 139). Hazzan (1999) introduced the framework of *reducing abstraction* in an effort to "describe the mental processes of undergraduate students as they solve problems in abstract algebra" (p. 71). She introduced three ways students cope with abstraction; by (1) retreating to familiar mathematical structures, (2) using canonical procedures, or (3) adopting a local perspective.

We used this framework to situate students' responses to an inventory designed to measure conceptual knowledge in an introductory undergraduate group theory course (Melhuish, 2015). We report on the frequency of responses from 432 students spanning thirty United States institutions. We pair these results with discussion of the role of abstraction level.

Sample Results

1. Let $i = \sqrt{-1}$. Consider the homomorphism $\phi(n) = i^n$, that maps $\mathbb{Z}$ under addition to the set $H = \{1, -1, i, -i\}$ (a subgroup of $\mathbb{C}$ under multiplication). What is the kernel of this homomorphism?

Table 1
Distribution of Responses for Question 1

Response	Frequency (n=432)
a. {1}	120
b. {4}	41
c. multiples of 4	223
d. empty set	48

In this prompt, students are asked to reason about the kernel of an unfamiliar homomorphism. Most students chose the correct answer of 'multiples of 4,' exhibiting an adequate understanding of the concept. From follow-up interviews we found that students who selected the response of $\{1\}$ often relied on an incomplete process of first identifying the identity in the co-domain group. For students who chose $\{4\}$, they were most often adopting a local perspective in recognizing an element in the kernel, not all elements of the kernel. Finally, students who opted for the *empty set* typically did so because 0 was not an option, which to them was a familiar identity for $\{\mathbb{Z}, +\}$.

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Engaging in Abstract Algebra through Game Play: Group Theory Card Game *Groups*

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In this presentation, I discuss the viability of a mathematical game as a learning tool for abstract algebra—specifically, the groups of order four. Throughout 2016, I designed and tested variants of my group theory card game, Groups, *among individuals ranging from no post-secondary mathematics experience to current or prior graduate level mathematics study. Here, I review the design choices and challenges central to working in a game space drawing heavily on abstract algebra, and assess alterations to the game's mechanics influenced by my interactions with players.*

Keywords: Abstract Algebra, Group Theory, Educational Game

Salen and Zimmerman (2004) define a game as a "system in which players engage in an artificial conflict, defined by rules, that results in a quantifiable outcome" (p. 80); this definition is often argued to hold natural parallels to that of mathematical exploration, and a wealth of individuals have already begun designing and utilizing games for mathematical learning both inside and outside of the classroom (Ke, 2008; Kebritchi, Hirumi & Bai, 2010; McCue, 2011; Wijers, Jonker & Drijvers, 2010). However, the majority of these games target adolescents at the elementary and secondary level. In the literature on instruction in abstract algebra, Weber and Larsen (2008) advocate building a strong *informal* knowledge base for student reflection, and later introducing students to the corresponding *formal* mathematical concepts (p. 147). In this regard, Weber and Larsen's approach via mathematical reinvention lends itself to exploring groups and group theory through a game-based lens as a means of preempting formal content treatment. Hoping to evoke a similar sense of reinvention, I detail the ongoing development—design, testing, and critical review—of my group theory card game, *Groups*.

Methods

Participants played several video-recorded rounds of *Groups* in pairs, in a novice-novice, novice-expert, or expert-expert match, participated in a recorded interview, and completed a written questionnaire that facilitated a transition to formal mathematical thinking. Preliminary data shows participants had a strong understanding of inverses and identities within groups, but required further clarification on associativity and closure.

Significance

Game-based approaches to mathematics instruction are an innovation worth exploring in undergraduate mathematics education, and can align with reinvention approaches. Refinement of *Groups* may lead to a classroom-viable group theory learning tool that could extend the accessibility and appeal of abstract algebra—and university-level mathematics, in general—to a broader audience, including secondary-level students. From the data, I plan to further explore the following: (i) the utility of *Groups* for new group theory learners; (ii) the utility of *Groups* for current or prior group theory learners; and (iii) how to improve *Groups* for all students.

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Research on Concept-based Instruction of Calculus

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Abstract: This study, involving 254 college-level calculus students and 3 teachers, investigated the misunderstanding of concepts in calculus and designed concept-based instruction to help students understand concepts. Multiple achievement measures were used to determine the degree to which students from different instructional environments had mastered the concepts and the procedures. The midterm examination and the final examination results showed that the students enrolled in the concept-based learning environment scored higher than the students enrolled in the traditional learning environment and the investigation at the end of the semester showed that most of students like the concept-based learning environment.

Key words: Concept-based Instruction, Misunderstandings, Teaching design

In the context of mass higher education, the ability of college freshmen is generally in a lower level than before. Many college students can do simple works on calculus, but they cannot understand the idea behind the concept, and as a result, usually have fuzzy understanding of the relationship between concepts. Therefore, to find the cognitive difficulties of the students on the concepts of calculus and to design the concept instruction are the keys to the reform of the teaching on Calculus.

This research presented a study on calculus course in three freshmen classes by carrying out the teaching design and teaching experiment. Research methods such as design research, questionnaires, interviews and classroom observation were adopted. There were 3 teachers and 254 students participated in the practice. Based on the findings of this study, the following conclusions could be drawn:

Firstly, college students' concept image of the fundamental concepts of calculus was one-sided, and some even wrong. Some students couldn't define the limit by correct words. Most of the students usually thought of the slope of the tangent when seeing the derivative, rather than the rate of change. There was confusion in the understanding of the geometrical meaning of differential and linear approximation. Some students know that the definite integral can express the area, but they can't make sure the area of what region; some students did not know which amount was sliced when they calculated the integral.

Secondly, we constructed principles on concept instruction in calculus as follows: (1) Concepts were introduced and demonstrated in a genetic way. (2) To help students understand the concepts by means of geometric or intuitive examples. (3) Paying attention to the elaboration of the relations of the concept between them. The results of teaching experiment showed that the students enrolled in the concept-based learning environment scored higher (M=34.42) than the students enrolled in the traditional learning environment (M=30.27) on the 40 point Conceptual Understanding Subscale and the students enrolled in the concept-based learning environment scored significantly higher (M=48.68) than the students enrolled in the traditional learning environment (M=42.65) on the 60 point Procedural Skill Subscale in the examination.

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Equity Issues That (May) Arise in Active Learning Classrooms

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There is an overwhelming amount of evidence that the incorporation of active learning in the classroom benefits all students and can be especially beneficial for women and underrepresented populations. However, our work is not finished when it becomes an integral part of teaching and learning across the nation. Classroom settings that foster group interaction and collaboration may result in an environment that is even more undermining to underrepresented populations. In this poster we illustrate these potential issues that arose in an abstract algebra course.

Key Words: Equity, Abstract Algebra, Inquiry Oriented Learning

Research clearly indicates that active learning is beneficial for students in undergraduate mathematics courses, especially for students from traditionally underrepresented populations (Freeman et al, 2014; Laursen, Hassi, Kogan, & Weston, 2014). There are a number of evidence-based reasons that active learning classes may support a more equitable classroom. For instance, active learning classes often center around "low floor, high ceiling" tasks which allow for multiple entry points and for students to share their thinking (rather than only correct answers). This provides opportunities for students to see each other struggle and emphasizes the process of learning as well as allowing for a "broadened notion of competence" (Esmonde, 2009). In this poster we present on an Inquiry Oriented Abstract Algebra (IOAA) class that employed these strategies. In addition, the instructor was conscientious concerning equity issues and took active measures to create a classroom in which all students were valued contributors. Despite this, the teacher, a participant observer (TA), and an additional observer became acutely aware of differences in the class relating to participation and the nature of the contributions. In the poster, we use observer field notes and reflections to identify and describe issues related to equity that arose in the IOAA class, and consult video of the class for triangulation. We then consider reasons why such issues arose *despite* the active learning environment and conscientious teaching. Specifically we consider two questions: (1) Were there aspects of this execution of active learning that can account for the equity issues, such as shorter class periods, the teacher's lack of experience with this material, the specific students, etc., or (2) Are there more general aspects of active learning classes that expose equity issues that may not be exposed by lecture?

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Conceptual Understanding of Differential Calculus: A Comparative Study

The research community shares a concern for students' conceptual understanding of calculus and commonly advocates for student-centered approaches as a way to promote it. In this study, we investigated the effect of different instructional approaches on 151 undergraduate students' conceptual understanding of differential calculus in context-specific, natural settings. We collected data on the pre- and posttest of the Calculus Concept Inventory in three classes. In one class, most of the time was dedicated to conceptually oriented problem solving, another class implemented practice problems for students, and the third class was a traditional lecture class. The results showed that there was no difference in students' conceptual understanding of differential calculus controlling for their initial understanding. Thus, our findings do not support the research that advocates for student-centered instruction suggesting that the approaches' implementation and contextual differences may be sources of variation in their effectiveness.

Key words: Calculus, Conceptual Understanding, Active Learning, Instruction, Concept Inventory

Mastery of calculus, a desired and necessary student learning outcome (Sofronas et al., 2011), needs to include not only mastery of procedures but mastery of concepts, as well (Zerr, 2010). Multiple attempts have been made to identify instructional approaches that lead to greater conceptual understanding of STEM disciplines (Freeman et al., 2014; Prince, 2004) and specifically of calculus (Laursen, Hassi, Kogan, & Weston, 2014; Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006), typically advocating for student-centered instruction. However, those calculus studies either used measures with limited evidence of validity and reliability or aggregated data across classrooms, potentially different in instruction implementation or contextual factors. With our ex post facto study, we aimed to overcome these limitations and investigate students' conceptual understanding of differential calculus (measured by a validated instrument) in three calculus classes with distinct instructional approaches taking contextual factors into account.

Literature Review

The education research community has been working on identifying instructional approaches effective for students' learning and specifically for their conceptual understanding of content for a long time (Prince, 2004). Many researchers have advocated for student-centered instruction as an effective one, typically contrasting it with the teacher-centered instruction. For example, in physics, one of the largest studies was conducted by Hake (1998) where he compared student conceptual understanding in interactive engagement classes and traditional classes. The results of that study suggested that students in the former classes had higher conceptual understanding than students in the latter.

In undergraduate mathematics, several studies exist that explored the influence of student-centered instruction - specifically inquiry-based learning (IBL) - on student conceptual understanding. One of such studies is a study of Laursen et al. (2014) where learning gains of students in IBL mathematics classes were compared to those of students in non-IBL mathematics classes. The results showed that students' cognitive gains in understanding and thinking, among others, were greater in IBL classes than in non-IBL. However, the measurement of learning gains in this study is important to note. The learning gains were self-reported by students, i.e., the gains were students' subjective perceptions of their learning. Perceived learning, though it has its advantages, might not always be an accurate estimation of actual learning.

Another relevant study was conducted by Rasmussen et al. (2006) where students conceptual understanding of differential equations was explored in IBL and traditional classes. The results also supported the effectiveness of IBL. However, several notes need to be made about the measurement of conceptual understanding in this study, as well. First, the validity evidence for the instrument used was limited (Kwon, Allen, & Rasmussen, 2005). Second, the measure was administered only as a posttest assessment (without a pretest) and only to volunteers after the final exam.

Both studies also used data that were aggregated across classrooms. While data aggregation has its pros in terms of increasing sample sizes and, therefore, increasing the power of statistical comparisons, it may have cons, as well. Our main concern is that by considering students from different classes as one sample, important class-level differences may be overlooked. These differences, may contribute to differences in learning outcomes between classes. Examples of such class-level differences may include different quality of teaching of different instructors or different implementation of the same teaching approach.

Due to the limitations of the studies of Laursen et al. (2014) and Rasmussen et al. (2006) discussed above, we decided to explore the effects of student-centered instruction on student conceptual understanding using a validated content instrument, administered during class time at the beginning and end of the semester. We specifically focused on differential calculus as (1) it is one of the fundamental college mathematics courses, and (2) the content measure for this material was already developed and validated. We also decided to consider each class individually to explore the effects of instructional approaches holistically. In this study, we examined two different student-centered instruction types and one traditional instruction type. The decision to study two different student-centered instruction types instead of one is consistent with the suggestions drawn from the meta-analysis of Freeman et al. (2014). This meta-analysis encouraged further research to focus on "second-generation research," which compares courses that differ in active learning implementation, rather than on "first-generation research," which compares active learning courses with traditional ones. The three studied classes are described in the next section.

Methods

Context

Course. The study was conducted in the three course sections of Calculus I, the first course in the calculus sequence. This mainstream course has the traditional material on limits, derivatives, the integral, and culminates in the fundamental theorem of calculus. All three sections met twice a week for a lecture with a professor (1 hour and 50 minutes each) and once a week for a recitation with a graduate teaching assistant (50 minutes). The study was conducted during the same academic year with the data collection in the first two classes done in the fall semester and in the third class in the spring semester.

Lectures. The lecture portion of the first class was taught in an active learning classroom with most of the class time dedicated to conceptually oriented problem solving (COPS) and whole class discussion (the COPS class). The ALT classroom has 8 round tables with 9 seats at each table (a total room capacity is 72). The room also contains flat screen displays (one per table), and whiteboards that cover the walls. In the COPS class, class periods typically started with resolving any questions or problems that students encountered doing homework or that remained from the last class period. The professor would ask students to write their concerns on a whiteboard, and then have a whole class discussion to address the concerns. Then, the professor would lecture for a short period of time (10-20 minutes), followed by student active work that would take the majority of the class time. The active work typically included student group work on worksheets that consisted of conceptually oriented problem sets. The groups were self-selected and included 4-5 students each. The students were also encouraged to work on whiteboards to show their solutions. During this part of the class, the professor and undergraduate learning assistants walked around the classroom and talked to students to monitor their progress and answer or pose questions. If a common question or misconception arose, the professor would often address it via a whole-class discussion. To wrap up the active work, the professor would ask students to do a gallery walk and/or would hold a whole class discussion. At the end of the class, students typically turned in their completed worksheets.

The lecture portion of the second class was taught in a traditional lecture hall and implemented practice problems (PP) during lectures (the PP class). Similar to the COPS class, this class also started with the professor answering student questions. Then, the professor would present new material and work through an example problem. Next, students were asked to solve a similar problem in groups (i.e., their neighbors) or individually, as they preferred. During this part of the class, the professor and learning assistants circulated around the classroom to monitor student progress and answer questions. After most students finished, the professor would write down the solution suggested by the students and then discuss it with the whole class.

The lecture portion of the third class was also taught in a lecture hall but utilized primarily direct instruction (DI), the DI class. This professor prepared handwritten notes of the material (typically, proofs) and projected them on the screen in class. He/she would talk through the projected notes and then show an example problem on the board. This professor also incorporated graded quizzes in class which usually consisted of true-false questions (typically conceptual) to check student understanding of the material. Answers to the quizzes were discussed during the following lecture. In this class, no group work was utilized.

Recitations. Recitations for the COPS and DI classes were taught by the same teaching assistant in a primarily lecture style. This teaching assistant would typically answer student questions, if any, conduct quizzes if required by the professor, and then explain the material and show solutions for example problems. Recitations for the PP class were taught in an active style with most of the class time dedicated to answering students' questions, addressing their concerns, and clarifying misconceptions.

Participants

The three professors who participated in the study were experienced mathematics faculty members with similar goals for Calculus I classes. All of them aimed for students to have a mastery of both concepts and procedures by the end of the course. In addition, all of them wanted students to be actively involved in class and ask questions. The teaching assistants – recitation instructors – were both graduate students studying mathematics. The recitation instructor for the PP class was an experienced teaching assistant; the recitation instructor for the COPS and DI classes was a new teaching assistant at the university where the study was conducted, though he/she had teaching experience at a different institution.

A total of 151 undergraduate students participated in the study (49 in the COPS class, 64 in the PP class, and 38 in the DI class). The students were enrolled in the Calculus I course at a large, suburban public university located on the east coast of the U.S. Student demographic information is presented in Table 1. In the COPS class, most participants were sophomores; in

Table 1
Sample Demographic Information

Characteristic	COPS (N=49)		PP (N=64)		DI (N=38)		Overall (N=151)	
	Frequency	%	Frequency	%	Frequency	%	Frequency	%
Student classification	N=47		N=63		N=37		N=147	
- Freshman	14	29.8%	43	68.3%	16	43.2%	73	49.7%
- Sophomore	21	44.7%	11	17.5%	14	37.8%	46	31.3%
- Junior	5	10.6%	5	7.9%	6	16.2%	16	10.9%
- Senior	7	14.9%	4	6.3%	1	2.7%	12	8.2%
GPA	N=46		N=56		N=35		N=137	
- 3.5 or better	15	32.6%	22	39.3%	6	17.1%	43	31.4%
- 3.0 up to 3.5	17	37.0%	23	41.1%	10	28.6%	50	36.5%
- 2.5 up to 3.0	11	23.9%	9	16.1%	11	31.4%	31	22.6%
- 2.0 up to 2.5	3	6.5%	2	3.6%	8	22.9%	13	9.5%
Gender	N=45		N=63		N=37		N=145	
- Male	23	51.1%	40	63.5%	25	67.6%	88	60.7%
- Female	22	48.9%	23	36.5%	12	32.4%	57	39.3%
Race/Ethnicity	N=44		N=62		N=37		N=143	
- White	23	52.3%	33	53.2%	18	48.6%	74	51.7%
- African-American	2	4.5%	4	6.5%	7	18.9%	13	9.1%
- Hispanic	4	9.1%	4	6.5%	2	5.4%	10	7.0%
- Asian	13	29.5%	16	25.8%	7	18.9%	36	25.2%
- American Indian or Pacific Islander	0	0%	0	0%	1	2.7%	1	0.7%
- Other/Mixed	2	4.5%	5	8.1%	2	5.4%	9	6.3%
Age	M=20.33 (SD=3.46); N=45		M=19.02 (SD=2.06); N=61		M=20.11 (SD=3.04); N=37		M=19.71 (SD=2.87); N=143	

the PP class, the majority were freshman; in the DI class, freshman and sophomore students were enrolled in about the same proportion. Students' GPAs (self-reported) in all classes varied greatly. In terms of gender, in the COPS class, about a half of students were male, while in the PP and DI classes, the majority of students were male. Students also varied in race and ethnicity. In all classes, about a half of students were White and about a quarter were Asian. Notably, the DI class had more African-American students than the other two classes. Lastly, in the PP class, students were, on average, 19 years old; in the COPS and DI classes, they were, on average, 20 years old.

Procedure

The Calculus Concept Inventory (CCI; Epstein, 2007), a measure of conceptual understanding of differential calculus, was administered in all three classes at the beginning and end of the semester during recitations. Additionally, at the end of the semester, students were also asked to complete a demographic form. Students received a small amount of extra credit for participating in the study. They also received their individual scores on the inventory. The instructors received only class average scores. After the semester was over, the instructors also participated in interviews, during which they were asked mainly about their teaching practices in the classes in question and their teaching philosophies.

Results

Data Exploration

We computed descriptive statistics of CCI scores for each class measured at each point of time (the beginning and end of the semester). The averages and standard deviations are presented in Table 2. First, we were interested in whether students in each class showed growth over time. To answer this research question, we conducted three dependent samples t-tests with a Bonferroni correction ($\alpha=0.017$). The results revealed a significant effect of Time for the PP class, $t(63)=3.303$, $p=0.002$, but not for the COPS class, $t(48)=2.165$, $p=0.035$, or for the DI class, $t(37)=2.371$, $p=0.023$.

Next, we wanted to know if students in the three classes differed in their conceptual understanding on the pre- and posttest. To answer this research question, we conducted two ANOVA tests with a Bonferroni correction ($\alpha=0.025$). The results showed a significant effect of Class for both the pretest ($F(2,148)=5.466$, $p=0.005$) and the posttest ($F(2,148)=5.700$, $p=0.004$). Multiple comparisons – Tukey HSD tests – revealed a significant difference between the PP and COPS classes ($p=0.013$ for the pretest; $p=0.012$ for the posttest) and between the PP and DI classes on the pretest only ($p=0.017$).

Table 2
Descriptive Statistics for CCI

	Mean (SD)			
	COPS (N=49)	**PP (N=64)**	**DI (N=38)**	**Total (N=151)**
Pretest	6.43 (3.03)	8.06 (3.18)	6.37 (2.56)	7.11 (3.08)
Posttest	7.33 (3.60)	9.28 (3.77)	7.37 (3.11)	8.17 (3.66)

Differences in Conceptual Understanding between Classes over Time

To determine whether there was a difference in student conceptual understanding of differential calculus between classes over time, we conducted a mixed design ANOVA with Time as a within subjects factor and Class as a between subjects factor (see Figure 1). The results indicated a main effect of Time (F(1,148)=19.160, p=0.000), i.e., students' conceptual understanding, averaged across classes, was higher at the end of the semester (M=7.11; SD=3.08) than at the beginning (M=8.17; SD=3.66). The results also showed a main effect of Class (F(2,148)=6.811, p=0.001). Multiple comparisons – Tukey HSD tests – revealed that students in the PP class had significantly higher conceptual understanding, averaged across time, than students in the COPS (p=0.005) or DI (p=0.009) classes. No interaction effect between Time and Class was found (F(2,148)=0.187, p=0.830). Thus, the changes in students' conceptual understanding in all three classes were not significantly different from each other.

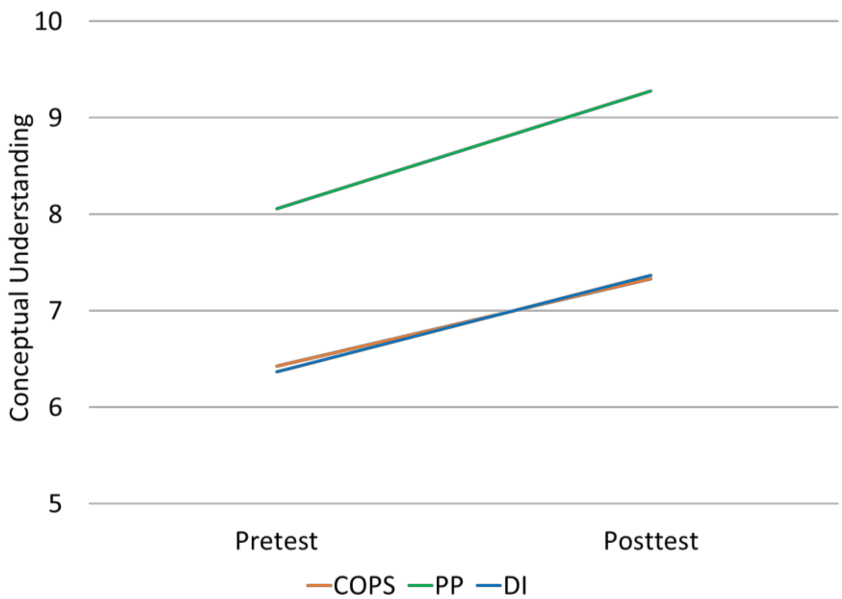

Figure 1. Pretest and posttest means for each class

Differences in Conceptual Understanding at the End of the Semester Controlling for Initial Understanding

To determine whether students differed in their conceptual understanding of differential calculus at the end of the semester controlling for their initial understanding, we conducted an ANCOVA test (see Figure 2). The results showed no difference between the classes in students' conceptual understanding at the end of the semester controlling for their initial understanding, F(2,147)=1.065, p=0.347. The adjusted means were as follows: 7.837 (SE=0.398) for the COPS class, 8.561 (SE=0.353) for the PP class, and 7.924 (SE=0.452) for the DI class.

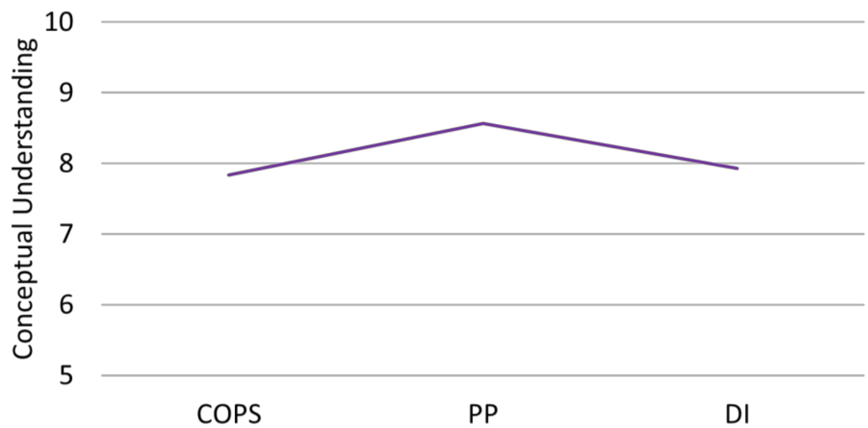

Figure 2. Adjusted posttest means for each class

Discussion

Our findings suggest that students' conceptual understanding of differential calculus is independent from the type of instruction when (1) conceptual understanding is measured by a validated, content instrument, and (2) the study is context-specific, i.e., when each class is considered individually instead of averaging across multiple classes. These nuances may explain why our results appeared to be different from the results of Laursen et al. (2014) and Rasmussen et al. (2006). In other words, implementation characteristics of a particular approach by a particular instructor in a particular course offering may lead to different levels of conceptual understanding and, therefore, need to be taken into account.

Among advantages of the study, we consider its ex post facto design, as no intervention was made. We aimed to explore the effects of teaching approaches in the most natural environment possible, and, therefore, chose to investigate the effects of the approaches typical to the instructors. Thus, this design provides a comprehensive picture of instruction implementation, where all elements of the instructional approaches are considered together. At the same time, a comprehensive picture of instruction has the disadvantage of making those elements with the most influence on the outcome challenging to identify. Therefore, future research should explore more context specific variations in approaches' implementation to determine potential commonalities between the effective ones. Another disadvantage of our ex post facto, context-specific design is a possibility for confounding variables to occur, as no control over the approaches is used. For example, the direct instruction in the recitations of the COPS class may have cancelled out the effect of conceptually oriented problem solving in the lecture periods. Finally, our study design substantially limits the generalizability of the findings.

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Variations of College Algebra Instructors' Presentations of the Mathematics: The Case of Solving Quadratic Inequalities

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The mathematical content presented during instruction has been shown to have an effect on student achievement. To investigate the content presented by instructors during College Algebra instruction, the Mathematical Quality of Instruction (MQI) observation protocol was applied to video recordings featuring instructors' presentations of examples of solving quadratic inequalities. Wide variation was observed in the solution methods chosen by instructors and in the rationale provided for choosing a particular procedure. This poster summarizes the variation in the mathematics that was observed and the ability of the MQI protocol to capture this variation.

Key words: College Algebra, Instructional Activities and Practices, Classroom Observations

College Algebra is cited as one of the most failed courses at universities across the country (MAA, 2007). It is difficult to pinpoint the particular causes for high failure rates in College Algebra, but the literature provides some insight into the relationship between teaching and student understanding: researchers agree that quality of the content offered during instruction is linked to student success (Porter, 2002), and how mathematical content is presented has an effect on student understanding and comprehension (Seidel & Shavelson, 2007; Weinberg, Wiesner, & Fukawa-Connelly, 2014).

Informed by previous observations of wide variation in the mathematics offered during College Algebra instruction (Beisiegel, Gibbons, & Paul, 2016), a study was designed to further investigate the qualities of the mathematics presented by College Algebra instructors. The purpose of the study was to identify what variation in the mathematics occurred in the lessons and how well the Mathematical Quality of Instruction (MQI, Learning Mathematics for Teaching Project, 2011) observation protocol captured this variation. Four College Algebra instructors' lessons at a large university were recorded regularly throughout the duration of the course. To focus on the mathematical content offered during instruction, video clips were chosen from the set of recordings that featured instructors' presentations mathematical examples. In particular, lesson clips containing teachers' presentations of solving quadratic inequalities were selected to allow for comparison between the mathematical content offered during instruction.

The MQI protocol was used as a lens and framework for observing the selected video lessons. This instrument was selected because of its attention to the interactions between the instructor and the content. In particular, one of the MQI dimensions, *Richness of the Mathematics*, is well-suited to capture the mathematics offered during a lesson. This dimension includes codes such as *Explanations, Linking Between Representations,* and *Mathematical Sense-Making*. Two researchers viewed and scored the selected College Algebra video clips using the *Richness of the Mathematics* dimension of the MQI.

Wide variation in the mathematics was observed. First, most instructors presented only one solution method, showing either the graphing method or the table/number line procedure; this was well-captured by the MQI. Second, the instructors differed in their decision of when to convert the inequality to an equality, and this decision affected their choice of solution method; the MQI did not attend to this variation. These findings, along with examples of the procedures observed for solving quadratic inequalities, will be presented in the final poster.

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Investigating Prospective Teachers' Meanings of Covariation Before and After Calculus Coursework

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This study seeks to uncover prospective teachers' construction of mathematical meanings after engaging in a two semester calculus sequence. The research question for this work is: How might the learning of calculus impact prospective teachers' mathematical meanings of functions, and in particular, prospective teachers' meanings of covariation? The purpose of this study is to understand whether, and if so, how, the study of calculus is useful for prospective teachers, and to place a lens on meanings that students develop on covariation, a concept that permeates the secondary school mathematics curriculum.

Because not all secondary school teachers will teach calculus or more advanced mathematics, the importance of success in calculus remains a question for students. Even when it is accepted that calculus may be useful, learning more advanced mathematics continues to be questioned, especially as the undergraduate mathematics courses increase in levels of abstraction. At the teaching university where this study takes place, this is a relevant issue because students who major in mathematics will become middle or high school teachers. This study may contribute to our understanding of the need of calculus as prerequisite knowledge for teaching.

The courses in which this study takes place include Calculus 1 and 2. The site of this study is a small private university with a primary goal of training prospective teachers at the undergraduate and graduate levels. Students who enroll in the calculus sequence are undergraduate double majors in mathematics and in education. Because I teach both courses as well as the mathematics methods courses for middle and high school teaching, I have a unique opportunity to get to know students well as they form mathematical meanings and acquire pedagogical content knowledge in mathematics.

Research Design and Theory

The study is taking place in two calculus courses during the 2016-17 academic year (Calculus 1, Fall 2016; Calculus II, Spring 2017). The first stage of the project involves participants' completion of an assessment that measures meanings students form with respect to specific mathematics concepts related to covariation. Items in this instrument were developed by Dr. Patrick Thompson (2016) and his research team through Project Aspire (*Mathematical Meanings for Teaching secondary mathematics).* This instrument has been designed to assess pre-service and current teachers' mathematical meanings for teaching secondary mathematics.

The second part of the project consists of administering the same assessment tool at the end of Calculus I to understand whether there have been any changes in students' meanings. At this time, I will interview students to check my interpretations of their written work and to allow students to elaborate on their responses. Students will also be asked to reflect on any changes in their responses on the two completed assessments. Students who continue to Calculus II will revisit this assessment at the end of the semester in Spring 2017, as the research cycle is repeated. Theoretical lenses used to guide this work include theory of meanings (Thompson, Carlson et al., 2014), as well as the concept of backward transfer of mathematical meanings that may be productive for teaching (Hohensee, 2014).

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Improving Undergraduate STEM Education Through Adjunct Mathematics Instructor Resources and Support (IUSE-AMIRS)

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The Improving Undergraduate STEM Education Through Adjunct Mathematics Instructor Resources and Support (IUSE-AMIRS) project aims to measure the impact of course coordination and support on adjunct mathematics instructors' knowledge, instructional practices, and job satisfaction. In this project, we use the organization and coordination of Precalculus with the goals of 1) implementing best practices for learning and instruction, 2) improving instructor knowledge, and 3) creating a professional learning community. As a part of this project we measure the impact of Precalculus course coordination and adjunct support on student achievement, leading to student retention in STEM majors. We believe our initiative can be implemented in other departments and institutions that have a similar need for adjunct instructors in math courses with multiple sections.

Key words: [Adjunct, Precalculus, Course Coordination, Professional Learning Community]

Students' persistence in continuing to pursue STEM degrees is heavily influenced by their classroom experiences, especially in the first year mathematics courses (Hutcheson, Pampaka, & Williams 2011; Pampaka, Williams, Hutcheson, Davis, & Wake, 2012). In this regard, the quality of pedagogy can make a big difference in the retention of STEM students beyond beginning mathematics. There is a growing body of research aimed at graduate teaching assistants touting benefits of targeted professional development (PD) (DeLong & Winter, 2001), but much needs to be done with respect to the growing population of adjunct instructors (Austin & Sorcinelli, 2013).

To better address this challenge at our institution, we have developed the project *Improving Undergraduate STEM Education Through Adjunct Mathematics Instructor Resources and Support (IUSE-AMIRS)*. Through this project, inaugurated in summer 2016, we are creating a model for implementing best practices for learning and instruction through adjunct instructor development and support. We have incorporated instructor supports backed by research and provided course coordination of Precalculus to begin a departmental transformation that will ultimately support students in this early mathematics course. Our course coordination includes having a course coordinator, two dedicated Precalculus tutors, and a common pacing guide, syllabus, assessments and rubrics for all instructors. In addition, we provided a summer workshop for Precalculus adjunct faculty and tutors, during which the participants received a comprehensive training on our recently adapted curriculum. This workshop was part of a larger PD effort that has continued throughout the semester through online weekly meetings. These meetings form the foundation of a professional learning community (PLC) aimed to provide content and instructional support for the instructors. Adjunct faculty highly regard such supports as they help improve teaching and integrate adjuncts into their institutions (Lyons & Burnstad, 2007; Bowers, 2013). To better understand the impact of these supports, we have been collecting a wide range of data from both adjuncts and students including instructor and student surveys, instructor interviews and classroom observations. We are measuring the effects of IUSE-AMIRS on Precalculus instruction and instructors since we strongly believe they will positively impact student achievement, thus leading to increased retention in STEM majors.

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Quantitative Learning Centers: What We Know Now and Where We Go from Here

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Given the recent national and international events the need for developing students' quantitative literacy (QL) has taken center stage in the mathematics education community. We are interested in investigating the existing support structures and the impact they have on the development of QL. The purpose of this study is to investigate the literature on quantitative learning centers at institutions of higher education. This poster will discuss the themes that emerged from a qualitative analysis of these works, highlighting what we currently understand and identifying opportunities for growth.

Key words: quantitative learning centers, quantitative literacy, literature review

A desire for quantitative literacy (QL) in college graduates has caused institutions of higher education (IHE) to form quantitative course requirements for their students. These requirements are intended to develop QL, "the ability to adequately use elementary mathematical tools to interpret and manipulate quantitative data and ideas that arise in an individual's private, civic, and work life" (Gillman, 2011). This ability is desired across fields, not just in the sciences, and employers search for candidates who have these reasoning skills. Out of a need for student support, quantitative learning centers (QLCs) have been founded at IHEs across the country. The centers seek to provide support for students enrolled in classes with a quantitative focus through a variety of programs, including drop in tutoring (Black, 2016), scheduled one-on-one tutoring (Mayes, 2016), and review sessions (Grant, 2016). Their ability to effectively support students and their impact on quantitative literacy is something that has not been broadly assessed. The purpose of the study presented in this poster is to investigate the literature on QLCs at IHEs in order to synthesize what we currently understand and identify opportunities for growth.

Methods

In order to address our purpose, a literature review was conducted. The literature included in the review were accessed through ERIC, PsychInfo, and Google Scholar using the keywords "learning center," "mathematics," "undergraduate," "help-seeking," and "peer-tutoring," as well as variations on the term learning center, such as "support center" and "help center". Works that focused on undergraduate student use of a help center for mathematical support were included. A content analysis was performed on the resulting literature to identify emerging themes.

Findings

This poster will elaborate on the results of the analysis. Most of the literature focuses on help seeking behaviors of students, an individual center's effectiveness in meeting its' objectives, or some combination thereof. The following themes emerged: 1) the academic need of students attending the QLCs (Xu, Hartman, Uribe, & Mencke, 2001), 2) the discipline of the students attending the QLCs, and 3) the center as a physical space for students to work together (Solomon, Croft, & Lawson, 2010). Opportunities for growth will also be discussed.

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Characterizing Normative Metacognitive Activity During Problem Solving in Undergraduate Classroom Communities

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Mathematical problem-solving research studies abound, and a significant portion express the role of metacognition as an underlying component of the problem-solving process. Unfortunately, much of the research on metacognition in mathematics does not describe the explicit role metacognition plays during the problem-solving process. Moreover, metacognitive interventions are typically disconnected from natural mathematical activity and discourse within a classroom community. The purpose of this qualitative study is to characterize sociomathematical metacognitive norms within an introductory number theory course intended for pre-service teachers. Utilizing Vygotsky's conception of language-based, mediated action and activity theory as an analytic framework, this study aims to test the use of these methods to investigate "real-time" metacognition with explicit focus on the broader classroom community. Attention is paid to the dynamic relationship between the teacher and students.

Key words: Activity Theory, Metacognition, Problem Solving, Sociomathematical Norms

There has been increased focus to develop problem-solving skills and related 'habits of mind' (e.g., CBMS, 2012; MAA CUPM, 2015). Metacognition is one such habit of mind (Costa & Kallick, 2000), and problem-solving frameworks identify metacognition as a core component of the problem-solving process (e.g., Carlson & Bloom, 2005; Schoenfeld, 1985). Although problem-solving *frameworks* have been heavily studied, research does "not yet offer a *theory* of problem solving" (Schoenfeld, 2007, p. 539, emphasis added). Particularly, metacognition remains undertheorized and under-studied in its application to classroom communities (Carroll, 2008), especially at the undergraduate level (Dumford, Cogswell, & Miller, 2016). Much of the research to date has not described the *explicit* role metacognition plays during the problem-solving *process* (Carlson & Bloom, 2005). A shift of focus to a process view of "real-time" metacognition necessitates an investigation "in the context of natural purposeful activity" (Neisser, 1976, p. 7). Specifically, there is limited research documenting the normative metacognitive behaviors during problem solving of classroom communities. Understanding how metacognition manifests itself in such an environment could help to develop techniques to foster metacognition as normative behavior within the mathematics classroom.

As such, this research aims to address the following research question: *How do sociomathematical metacognitive norms during problem solving develop in an undergraduate mathematics community of practice?* Vygotsky's (1978, 1986) social constructivism is adopted as the theoretical lens and Ernest's (2010) model of sign appropriation/use is incorporated to highlight the reflexive nature between individual and collective. Activity Theory (Engeström, 1987; Leont'ev, 1979) is utilized as a framework to investigate these interactions, as it provides concrete language to describe broader social factors potentially influencing metacognitive norms. To test the feasibility of activity theory, I conducted a qualitative pilot study in a number theory course designed for pre-service teachers. This poster will present analysis, results, and an evaluation of methodological tools. The pilot study also informed data collection and analysis for my dissertation study. Modifications and initial results of this current study will also be shared.

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Research in Courses before Calculus Through the Lens of Social Justice

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The terms equity, diversity, inclusion, and social justice have entered the research lexicon. This theoretically-focused poster presents some recent policy efforts to generate a shared meaning for "social justice" in mathematics education and offers a theoretical framework for making sense of (and making sense with) intercultural interactions as an essential component of rigorous research. The poster includes illustrations for how to use these tools for thinking through and talking about research. To anchor discussion, we focus on research on teaching and learning in the courses before calculus (e.g., algebra, mathematics for pre-service elementary teachers).

Keywords: Social justice, Research in undergraduate mathematics education

This year two organizations, TODOS: Mathematics for All and the National Council of Supervisors of Mathematics (NCSM), issued a position paper, *Mathematics Education Through the Lens of Social Justice: Acknowledgement, Actions, and Accountability*. In it, social justice includes "fair and equitable teaching practices, high expectations for all students, access to rich, rigorous, and relevant mathematics, and strong family/community relationships to promote positive mathematics learning and achievement." Underlying all of these is an understanding of how power, privilege, and oppression contribute to and maintain an inequitable learning system.

As people trained in research in undergraduate mathematics education, we know that our work is about more than identifying a problem and solving it. As citizens of a first-world country in the 21st century, we are keenly aware of societal injustice. And, as a community, we have an opportunity to guide how social justice issues are explored and addressed in collegiate mathematics education research. The opportunity has been there for a while (e.g., Aguirre & Civil, 2016; Adiredja, Alexander, & Andrews-Larson, 2016; D'Ambrosio et al., 2013; Davis, Hauk, & Latiolais, 2010; Gutiérrez, 2013; Nasir, 2016).

According to the TODOS-NCSM position paper, three conditions are necessary to establish just and equitable mathematical education for all learners: (1) acknowledge that an unjust social system exists, (2) take actions to eliminate inequities and to establish effective policies, procedures, and practices that ensure just and equitable learning opportunities for all, and (3) be eager for accountability so changes are made and sustained. How do we increase researcher capacity to do these three things? We must address our own needs – as researchers – for language, concepts, and awareness-building. This will support us in the inevitable struggle to gain and use pertinent understandings related to social justice. The poster offers key ideas and examples from intercultural development (Bennett, 1993; 2004).

Questions driving poster conversation: What is the role of social justice in research in collegiate mathematics education? When we conduct research in the U.S. we make decisions about who participants are – what would be different if decisions in the projects we are in the midst of or just finished had included overt and repeated attention to the three tenets? How might research be framed to provide evidence that supports action to eliminate an inequity? How do we do that? How might the research design or the analysis be different if the results of the work are to be held accountable by research peers *and* judged in a court of stakeholder opinion that values equity as much as excellence in mathematics education? What are some of the concepts and language from intercultural development that can help us address these questions?

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Exploring the Content–Specific Mathematical Proving Behavior of Students: Opportunities for Extracting and Giving Mathematical Meaning

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Studies were conducted to explore the efficacy of a non-traditional transition to upper division proof course using Freudenthal's notion of mathematizing *as a framework. Classroom video data and student work were analyzed using grounded theory methodology. Results indicated that students in the non-traditional course developed better understandings of the role of definition and counter example in proof through engagement in meaning making activities that fostered both the semantic and structural aspects of proof writing.*

Key words: Transition to Proof, Mathematizing, Grounded Theory

These studies were conducted in a large public research university on the West Coast. Participants were 109 undergraduates enrolled in three sections of a transition to upper division mathematics course. Thirty six students were enrolled in a pilot course that had students use first-order languages and their semantics to investigate truth/falsity in a variety of structures as a basis for developing mathematical reasoning. Thirty seven students were enrolled in a traditional section of the course. The remaining students were enrolled in a revised non-traditional course.

Study 1

We asked: What landmark thinking strategies (such as acting upon definitions) occur along the way to successful proof at the collegiate level? All spring non-traditional course sessions and a sample of five traditional course sessions were observed, audio-taped and transcribed. Student work was collected. Data were analyzed using the constant comparative methods of Strauss and Corbin (1998). A learning trajectory of the ways in which students' understanding of the big ideas of mathematical proof developed over time was constructed.

Study 2

We asked: How do final exam responses of students in this non-traditional course compare to those students who took a traditional proof course? Common exam items were coded numerically and qualitatively using the understandings of proof landscape above. Non-traditional students did better on constructions of objects using operable definitions (Bills & Tall, 1998).

Study 3

We asked: In what ways did the revised non-traditional course support the development of understanding of proof? Transcripts of classroom activity and student written proofs were analyzed. Cases of student proof writing in contexts illustrated ways in which the course allowed both formal and natural strategies can be used to engage in meaning making activities (Pinto & Tall, 1999). Future research should address how such development can be fostered in a traditional setting.

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Questioning Assumptions about the Measurability of Subdomains of Mathematical Knowledge for Teaching (MKT)

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The goals of undergraduate mathematics teacher education include developing teachers' content knowledge and pedagogical content knowledge. As a strategy for conceptualizing and assessing these forms of knowledge, researchers have further divided these domains. However, it has proven difficult for research groups to create tasks to reliably capture a specific domain without involving other domains, leading them to question these subdomains. We argue that tasks' inability to measure subdomains separately is not evidence that tasks or theory are flawed. Instead, we propose that assessment tasks are effective in measuring MKT when they represent the work of teaching, rather than when they isolate subdomains. To make this argument, we use an analysis of teachers' thinking in response to nine MKT assessment tasks. Though prior work provides evidence that the tasks measure MKT, the tasks cannot be meaningfully parsed into the subdomains of multiple established MKT frameworks.

Key words: Mathematical knowledge for teaching, pedagogical content knowledge

A common viewpoint in undergraduate mathematics teacher education is that content knowledge is the domain of mathematical knowledge for teaching (MKT) to be taught primarily in content courses and pedagogical content knowledge (PCK) is the domain to be taught primarily in methods courses. Assessment development efforts have focused on measuring content and PCK and theoretically derived sub-components of each separately, with factor analyses confirming (or disconfirming) hypothesized subdomains (e.g. Hill, Schilling, & Ball, 2004; Floden & McCrory, 2007; Herbst & Kosko, 2014; Krauss, Baumert, & Blum, 2008). This approach has yielded at best mixed results (Hill, 2016).

While category theory suggests that factor analysis is a reasonable approach for evaluating theoretical subdomains whose purpose is to inform assessment domain sampling (Kaarstein, 2014), we counter that this need not be the purpose of a theoretical framework. Failure to measure subdomains in isolation does not alone imply that assessment tasks or theory are flawed.

To illustrate this argument, we present an analysis of nine assessment tasks previously shown to assess MKT (Howell, Lai, & Phelps, 2016) across five established frameworks for MKT (Ball, Thames, and Phelps (2008), KAT, COACTIV, TEDS-M, and MUST). Prior work generated knowledge maps for each task specifying the knowledge measured, and verified their accuracy in capturing reasoning in response to the tasks. In the present study, the tasks themselves became the data; we coded them by subdomains of each of the five specified frameworks. Eight of the nine tasks measured multiple subdomains across multiple frameworks, substantiating our hypothesis that tasks that measure MKT well overall may simply not be amenable to the measurement of isolated subdomains of MKT.

Implications of this study substantiate concerns in the field of meaningful inconsistency in the conceptualization and description of MKT, but may temper critiques of theoretical frameworks based on assessments like those studied. A theorized subdomain need not be distinctly measureable in isolation from the larger construct to be useful in informing the field's thinking, pushing policy, or as a heuristic for organizing teacher supports.

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Data Cleaning in Mathematics Education Research: The Overlooked Methodological Step

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The results of educational research studies are only as accurate as the data used to produce them. Drawing on experiences conducting large-scale efficacy studies of classroom-based algebra interventions for community college and middle school students, I am developing practice-based data cleaning procedures to support scholars in conducting rigorous research. The poster identifies common sources of data errors in mathematics education research and offers a framework and related data cleaning process designed to address these errors. I seek feedback on the framework and discussion around data cleaning techniques used by other RUME scholars in their research and in the preparation of future researchers.

Key words: Research methodology, Efficacy studies, Algebra

Screening data for potential errors and ensuring anomalies do not influence analyses is an essential step of the research cycle (Wilkinson, 1999). Despite the importance of data cleaning in rigorous research practice, most methodology courses only give cursory attention to the topic (Osborne, 2012). I am developing practice-based data cleaning processes to support scholars in implementing rigorous research in classroom settings. Specifically, I ask: (1) What are the sources of data errors in educational research studies conducted in authentic mathematics learning environments? and (2) How can a data cleaning process be designed to consistently produce accurate, reliable, confidential, and timely datasets?

The framework presented in this poster was informed by two large-scale efficacy studies. Study A was a three-year, nationwide study involving over 10,000 middle school students and 180 mathematics teachers. Study B is a two-year, statewide study of community college elementary algebra courses. During Study A, a list of data related challenges and their associated resolutions was compiled and used to inform the data cleaning process currently used in Study B. Four common sources of data errors appeared in both studies: variations in assessment administration; participant mobility; multiple participant names; and use of external vendor systems. The following data cleaning process was developed to identify and repair these issues:

1) Create visually distinct instrument forms; indicate administration format in final data sets;
2) De-identify study data as early as possible in the data collection process;
3) Compare record counts against participant lists to identify missing and extra records;
4) Check data files for missing values, missing data columns, and extra data columns;
5) Check identifier columns for duplicate values;
6) Transform categorical values into pre-determined standard values;
7) Flag records with errors;
8) Establish a review process so data cleaning work can be checked by another person.

The data cleaning process and taxonomy of common data error sources offered here can provide a framework for other researchers to evaluate their current data management strategies. Furthermore, I hope this work can spark discussion around more comprehensive methodology training for future researchers. I also seek feedback on ways to communicate the process and information on how others in the RUME community handle data cleaning issues in their work.

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Calculus Instructor Beliefs Regarding Student Engagement

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Student engagement has been identified as a critical element in student learning of mathematics, yet most university math classrooms have very little active content (Olson & Riordan, 2012). Guided by Schoenfeld's (2011) framework for instructional decision-making, this study examines calculus instructor beliefs about, purposes for, and barriers against student engagement. Results indicate instructors utilize active learning primarily for formative assessment and improving student dispositions with development of understanding as a secondary, implicit goal.

Key Words: Active learning, Student Engagement, Teacher Beliefs, Calculus Instruction

The evidence is overwhelming that active approaches to mathematics instruction are more effective than traditional lecture (e.g. Bressoud et al., 2013; Freeman et al., 2014; Kogan & Laursen, 2014). However, effective instructional changes must be in alignment with teacher beliefs (e.g. Henderson, 2011). Schoenfeld's (2011) framework for instructional decision-making served as the theoretical framework to guide this study. According to this framework, a teacher's decisions are based on their goals, orientations (which include beliefs), and resources. In order to enact instructional change, it is necessary to build upon existing teacher beliefs regarding student engagement. Research questions include:

1) How do calculus instructors view student engagement in their classes?
2) What is the purpose of student engagement?
3) What are some of the barriers for student engagement? How are they overcome?

This study is the first stage of a multi-stage project aimed at shifting departmental culture toward more active mathematics instruction. In this study, four calculus instructors at a small (< 8,000) university were interviewed regarding their beliefs about student engagement, its purposes, and potential barriers for implementation. The interviews were transcribed and analyzed according to Braun and Clark's (2006) thematic analysis, which resulted in a set of themes. Results are summarized for brevity.

Results from this study indicate that this group of calculus instructors recognized the importance of student engagement, and offered many forms of student engagement, such as asking and answering questions, and group-work. All instructors recognized the limitations of lecture in terms of student engagement, but all saw it as a necessary part of teaching mathematics. They also recognized most students' discomfort while participating in whole-class relative to participating in small groups and addressed this discomfort. Purposes of student engagement included formative assessment, improved student communication, improved mathematical dispositions, and improved relationships. Notably, learning was not listed as one of the primary purposes; instead it was mentioned implicitly when describing how student engagement can be a means for improving test scores, allowing more processing time, or creating a more personal connection to the material. If reform efforts seek to increase active learning through student engagement, these beliefs will need to be addressed.

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Students' Strategies When Matching a Function's Graph with the Graph of its Derivative

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This study explores a fundamental calculus connection between a function and its derivative by examining and categorizing strategies students use when matching a function's graph with the graph of its derivative. Through interviews with four students using multiple choice (MC) tasks, we wanted to explore whether common mistakes and students' strategies when drawing the graph of the derivative of an original function are consistent with those found when using open-ended tasks. While tendency to find an equation of the graph in order to differentiate was observed, simple replication of the original function was not observed.

Keywords: Calculus, Derivative, Graph, Function

Given the graph of a function, there are several strategies students use when asked to draw the derivative. One common, correct strategy is to find points of local extrema of the function and match those points with the zeros of the derivative. Another correct strategy is to identify intervals on which the function is increasing and decreasing and match those intervals with places where the derivative is above and below the x-axis, respectively. Students often employ inefficient or altogether incorrect strategies. Some of these strategies include simple replication of the original function when asked to draw the derivative (Nemirovsky & Rubin, 1992) and students' tendency to find an equation of the graph of the function before differentiating (Ferrini-Mundy & Graham, 1994; Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997). However, online assessment programs draw upon MC items to assess student understanding of the connections between graphs of functions and their derivatives creating a potentially different set of challenges. Our study explores whether common mistakes and strategies students use on MC tasks are consistent with those when given open-ended tasks.

For this study, clinical interviews (Ginsburg, 1997) using MC tasks were conducted with four current calculus students at a Midwestern university. Tasks provided students with graphs of functions, asked students to choose the correct derivative graphs, and describe their thinking about their choice. Student strategies were varied. One student focused on locations of local extrema as zeros of the derivative as well as intervals of increasing and decreasing to match places where the derivative was above or below the x-axis. A second student incorrectly attempted to use combinations of first and second derivative concepts, while consistently ruling out a simple replication option. While the two remaining students wanted to find the function's equation in order to differentiate, they were equally focused on how the end behavior of the function determines the end behavior of the derivative graph.

While some common mistakes and strategies used on these MC tasks were observed, the mistake of simply replicating the function was not observed. Common mistakes and strategies observed in this study can inform the development and evaluation of MC tasks for assessments. For university math departments that depend on MC tasks to assess student understanding of the graphical connections between a function and its derivative, using assessment items that reflect conceptual connections between functions and their derivatives rather than surface-level characteristics is essential to increasing one's confidence in student understanding.

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Situational Characteristics Supporting Instructional Change

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Instructional practice and decision-making are influenced by a myriad of factors, with both individual instructor characteristics (e.g., beliefs about teaching and learning and personal experience) and departmental/institutional characteristics (e.g., resources and supports) shaping day-to-day teaching practices. However, little is known about which factors are the most influential and how those factors influence pedagogy. In this project, in an effort to identify commonalities in situational contexts and better understand how these commonalities support non-lecture instructional approaches, we look at interviews from fourteen mathematicians who volunteered to implement inquiry-oriented instructional materials.

Key words: instructional practice, contextual factors, departmental influences

There is a growing body of research indicating that traditional instructional methods (e.g., lecture) is both prevalent in undergraduate mathematics education and problematic in terms of STEM retention and student outcomes. Lecture is the most frequently reported instructional approach in undergraduate mathematics – with about 65% of mathematics faculty reporting extensive lecturing in all or most of their courses (Eagan, 2016). Such lecture-based pedagogy has been labeled problematic for undergraduate learning, persistence, and success. For instance, a meta-analysis by Freeman et al. (2014) found that in undergraduate STEM courses "active learning leads to increases in examination performance that would raise average grades by a half a letter" (p. 8410), and that students in lecture classes are 1.5 times more likely to fail than those in classes where active learning methods are used. The growing tension between the predominant instructional practices of mathematics faculty and the growing body of research on the benefits of active learning gives rise to the need to better understand the factors that contribute to pedagogical decision-making and the factors that influence instructional change.

In order to understand instructional practice and decisions, Henderson and Dancy (2007) made the argument for investigating both individual and situational characteristics. While individual characteristics are often the focus of similar research, a recent research report on abstract algebra instructors found individual factors, such as years of teaching experience and previous experience teaching abstract algebra, failed to predict pedagogical format (Fukawa-Connelly, Johnson, & Keller, 2016). Thus, here we decided to focus on situational characteristics by analyzing interviews from fourteen mathematics instructors who volunteered to implement an inquiry-oriented set of instructional materials. By identifying commonalities in departmental contexts, specifically for instructors who are actively using non-lecture pedagogies, our aim is to better understand how situational characteristics support such teaching practices. Situational factors under consideration include: class size, pre-requisite and subsequent courses, content and coverage expectations, and support/constraints expressed by colleagues and department chairs. Our analysis suggests that support for non-lecture pedagogy can range widely from passive (e.g., a department in which there is little oversight on individual courses) to active (e.g., a department in which professional development opportunities are widely circulated, department chairs who help mitigate negative student evaluations when new instructional approaches are tried).

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Graphs Display Lengths, Not Locations

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Students are frequently asked to reason about graphs that they see as geometric shapes, instead of representations that show the relationship between two quantities. This study shows an instructional intervention, using the theory of multiplicative objects (Saldanha and Thompson, 1998) that has great potential for orienting students to the quantities involved and their relationships, by focusing on how graphs display orthogonal lengths whose magnitudes are measures of quantities.

Key words: Graphs, Magnitudes, Representations, Lengths, Covariation

Research on students' understanding of graphs show that students frequently have difficulties with interpreting, manipulating and creating graphs (Baker, Cooley, & Trigueros, 2000; Kieran, Boileau, & Garançon, 1996). Many studies have suggested instructional sequences to help students develop their skills to carry out the aforementioned tasks (Dugdale 1993; Eisenberg & Dreyfus, 1994; McDermott, Rosenquist, & vanZee, 1987) but they frequently rely on asking students to use mnemonics based on graphical shapes, or "shape thinking", that has been shown to be problematic for students (Moore & Thompson, 2015); moreover, these studies are frequently silent on the original causes of the student difficulties that they are seeking to remedy.

While teaching both Precalculus and calculus at a large southwestern university, I noticed that these college-level students had surprising levels of difficulty in reasoning with graphs. We frequently drew vertical and horizontal lengths to represent *changes* in quantities on a graph, but I soon realized these lengths were meaningless if the students did not also have an understanding of the lengths on a graph that represent *total* quantities. A graph on rectangular (Cartesian) axes expresses the relationship between two quantities (Thompson 1993) by displaying points where a point's perpendicular distance from the y- and x-axis represent paired values of the x- and y-quantities, respectively. As such, a point on a graph is what Saldanha and Thompson (1998) called a multiplicative object. I noticed when students' interacted with graphs that they saw points on a graph as locations instead of representing the values of two lengths, the magnitudes (Thompson et al. 2014) of which represented the measurement of two quantities. They must also be able to visualize the covariation of these quantities (Carlson 2002) as the graph develops.

To investigate, I carried out a research project with a single participant, a community college student studying nursing in the American southwest. We used dynamic software to investigate graphs and their properties by highlighting the vertical and horizontal magnitudes (via directed lengths, or vectors) that were paired together in the multiplicative objects that were the points on any given graph. I found that these activities provoked surprising responses from my subject about the nature of graphs and their utility. We then continued to use these dynamic visualizations to investigate transformations of graphs and the student had a significant amount of success in justifying answers that earlier she could not explain.

I feel that this study will be of great interest to math education researchers because the nature of graphs and of how students use, create, manipulate, and reason with graphs is central to our work. Focusing on this aspect of reconceptualizing graphical shapes to points as 'locations' to points as 'paired lengths' may be a strong starting point for instructional innovation and increased student learning.

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Reasoning About Relative Motion: A Frames of Reference Approach

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In introductory physics classes, student frequently experience difficulties with relative motion problems. Previous studies have categorized student difficulties with reference frames, or used computer simulations or experiments to create seeming paradoxes that students would need frames of reference to resolve; however, these studies failed to define what they meant by a "frame of reference" in the mind of a student. In 2016 I carried out a pilot study that used our cognitive definition of a conceptualized and coordinate frame of reference (Joshua, Musgrave, Hatfield, & Thompson, 2015) as well as quantitative reasoning (Thompson, 1993) to guide an instructional intervention and analyze the difficulties the student had with relative motion tasks. Both constructs proved to have great explanatory power, as they revealed aspects of the student's thinking that were not commonly explored in previous studies. Both the results of this study and their implications will be the topic of my poster.

Key words: Relative Motion, Frame of Reference, Velocity, Physics, Quantitative Reasoning.

It is commonly acknowledged that the reason students struggle with relative motion tasks is because they fail to correctly use reference frames. Some studies have focused on categorizing student difficulties with reference frames (Bowden et al., 1992; Panse, Ramadas, & Kumar, 1994). Others have tried interventions based on computer simulation designed to have students experience different points of view (Monaghan & Clement, 1999, 2000) or to build experiences with seemingly paradoxical conclusions that students would need reference frames to resolve (McDermott, 1984; Trowbridge & McDermott, 1980). However, when I looked closely at these studies, I could not help but notice that velocity was frequently seen as an aspect of a single object. Even though students were asked to find an object's speed in "a new reference frame", the task was still framed as *the object's velocity* instead of *the rate of change of distance between an object and a reference point with respect to time*. Moreover, most of these interventions sought to simply build student intuition about velocity as an isolated quantity instead of a intensive quantity composed from distance and time.

To address this gap in the research, I investigated how a single student reasoned about tasks involving relative motion. After a clinical interview where I asked the student to work through several tasks, I used the theories of quantitative reasoning and conceptualized and coordinated frames of reference entail, to analyze the student's reasoning. Among other results, I found that the student was unusually attuned to the necessity of a reference point (though not always sure of how to utilize one) but completely unaware of the idea of a directionality of comparison. My analysis of the student's work guided the content of an instructional intervention, followed by a chance for the student to rework the original relative motion tasks. He showed great improvement in being able to both explain answers he previously thought were correct without justification, and to complete some previously blank tasks. He also used his commitment to reference point to spontaneously coordinate reference frames. I believe that my data, as well as my analysis of the data using my construct of a cognitive frame of reference and quantitative reasoning, will be of interest to the math education community as they showcase an area of applied mathematics where our students commonly struggle, as well as a potential avenue for improved instruction.

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All The Math You Need:
An Investigation into the Curricular Boundaries of Mathematical Literacy

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This project attempts to seek out common threads and analyze discrepancies in the tertiary-level mathematical literacy / quantitative literacy curricula proposed by eight different textbooks and content-providers. Following the framework developed originally in (Harel 1987) we investigate sequencing of content, levels of generality, emphasized applications, introductory material, as well as explicitly stated learning outcomes.

Key words: mathematical literacy, quantitative literacy, textbook analysis.

Mathematical literacy, and related notions and terminologies (such as numeracy, financial / quantitative / statistical literacies), refer to the usefulness of, and ability to use or apply, mathematical and quantitative ideas, broadly viewed, in a range of different life contexts [TSG 23 Call for papers and participants]. In earlier work, researchers have analyzed the overlaps and divergences between these specific terms; for a recent overview of related work, see [Karaali et al. 2016]. In this project, we seek to clarify the boundaries of these terms using curricular materials as a guide, mainly focusing on mathematical literacy as a goal, no matter how a specific term is defined or used. In particular we intentionally avoid fixing a definition for these terms and focus entirely on delineating the boundaries described by curricular content offered.

We analyze eight different textbooks and other curricular content aimed for the tertiary-level. Following the framework developed originally in [Harel 1987] we investigate sequencing of content, levels of generality, emphasized applications, introductory material, as well as explicitly stated learning outcomes.

We summarize our results in the poster using the above framework, using visual and verbal descriptions of each dimension (sequencing of content, levels of generality, emphasized applications, introductory material, explicitly stated learning outcomes). In particular we note that the topics covered do overlap significantly among the texts analyzed, thus offering hope that curricular boundaries might be determined even if there may not be consensus on definitions of terms like mathematical literacy and quantitative literacy. We also note that curricula and textbooks mostly fall into one of two categories that can broadly be described as *pragmatic* and *idealistic*.

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TSG 23 Mathematical Literacy - Call for Papers and Participants, ICME 13.

Student Beliefs About Mathematics in an Inquiry-Based Introduction to Proof Course

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An "Introduction to Proof" or "Transition to Proof" course is widely offered as an essential part of the undergraduate mathematics curriculum at most post-secondary institutions. This poster reports on the iterative development of one such course that used an Inquiry-based approach to the teaching and learning of mathematical proving and proof. Moreover, the changing beliefs of students, about the nature of mathematics and about doing mathematics, in this course, are discussed.

Keywords: Transition to Proof, Inquiry-based learning

The importance of proof has long been emphasized by numerous mathematics and mathematics education organizations, such as the *Mathematical Association of America* and the *National Council of Teachers of Mathematics*. Apart from the general importance placed on proof, the process of proving is also indispensable in the act of doing mathematics. At the higher academic levels (graduate and professional mathematics), proving can be considered as the definite way in which the truth of a claim is established or realized (Hanna, 2000).

In the United States of America, most undergraduate students of mathematics first encounter the process of proving and the related product of proof in an "Introduction to Proof" or a "Transition to Proof" course. Historically, the teaching of proving to students in these courses has followed the custom of presenting the statement of the theorem and following it with a presentation of the finished proof. However, research suggests that this form of mere presentation of proof may not engender understanding of the proving process for the students (Raman, 2002) and may instead promote memorization of the proofs without understanding the proving process. In an effort to correct this trend and to allow for more opportunities for undergraduate students of mathematics to gain understanding into the process of proving, the authors designed and implemented an "Introduction to Mathematical Proof" course that was built upon principles of inquiry-based mathematical teaching and learning. Other undergraduate courses, such as abstract algebra (Larsen & Lockwood, 2013) and linear algebra (Wawro et al., 2012), taught using these principles seem to have helped students gain insight into the relevant mathematical content.

This poster presents the design, development, and implementation of the various iterations of this inquiry-based course in mathematical proving and proof. It will report on the sequencing of tasks that may have helped students gain a deeper understanding of different proving strategies, such as using the Principle of Mathematical Induction. It will also present data from classroom observations and from student interviews about specific features of this course that helped students.

Students self-reported that specific beliefs about doing mathematics and about the nature of mathematics evolved during the span of this course. For instance, students modified their stances on the following beliefs: 1. There is always exactly one solution to mathematical problems and one valid way to solve mathematical problems; and 2. Mathematical proofs or arguments or justifications must adhere to a particular format. Observations of students' work in class and in interview settings provided triangulation of these espoused changes in beliefs by the students.

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Bringing Evidenced-Based Practices to a Large-Scale Precalculus Class: Preliminary Results

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Using evidenced-based practices in a large undergraduate mathematics classroom can be challenging but results of recent research on active-learning demand investigation. Preliminary results show that students already have self-confidence upon entering the classes, but there is a slight gain in perceptions of the value and appreciation of mathematics. Additionally, activities and clickers were considered useful and important by some of the students interviewed.

Keywords: Large scale classrooms, precalculus, evidence-based practices

In general, researchers and curriculum developers have developed numerous student-centered instructional strategies which has been shown to support conceptual learning gains (Freeman et al, 2014), diminish the achievement gap (Kogan & Laursen, 2013; Riordan & Noyce, 2001), and improve STEM retention rates (Rasmussen, Ellis, & Bressoud, 2013). Our objective was to introduce evidence-based active learning practices and conduct a mixed method study to understand how these new practices are perceived and affect student outcomes. Outcomes defined were students' attitudes towards mathematics, interest in mathematics, and self-efficacy. Research questions were: 1) what are students' perceptions of the new instructional strategies? And 2) how are students' self-efficacy and attitudes towards mathematics different after participation in the evidence-based practice course compared to a comparison course?

Study Design

The participants in this study were students in two (one treatment and one comparison) large classes at a southern university. The content in the courses were the same: precalculus, including triangle and unit circle trigonometry, conic sections, and sequence and series. The course included lecture and computer labs where homework, quizzes and exams were taken. In the treatment class, the instructor introduced the use of real-world videos, team activities, and conceptual clicker questions answered in small groups. There were 5 team activities, 3 real-world videos and clicker questions in the summer treatment class. None of these were done in the fall control class.

Data included pre and post surveys of students that asked about their perceptions of themselves as mathematical learners, their perceptions of mathematics, and the course itself. Interviews at the end of the semester with approximately 20 students were conducted and video recorded. Data analysis is ongoing with quantitative methods being used on numerical questions in the surveys and qualitative methods used on the open ended questions and interview videos.

Preliminary Results and Conclusion

Students were already confident in their mathematical skills, but there were advances in their value of mathematics. Students were more engaged, positive about the social value, and showed mathematical gains from participating in the treatment class. Further research is needed to identify connections between the team activities, clickers, and changes in the students. Early results show that there may be ways to implement these strategies in a large-scale mathematics classroom.

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Post-class Reflections and Calibration in Introductory Calculus

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One of the reasons for the exodus in STEM majors is students' experiences in their first undergraduate mathematics course, usually introductory calculus. However, students with high calibration are more likely to be aware of their deficiencies and seek assistance in time for it to be effective. Although there is evidence that students who regularly complete post class reflections are more successful than those that do not, it is not known if such assignment also improves students' calibration. The purpose of this correlational study was to investigate to what extent students enroll in CLEAR calculus become more growth mindset orientated the relationship between post-class reflections, calibration, and achievement in introductory calculus.

Key words: Calculus, calibration, formative assessment, transition to college

Most undergraduates who take calculus have some prior experience and believe they are prepared for the course (Bressoud, Carlson, Mesa, and Rasmussen, 2013). Without acknowledging the framework in which undergraduates learned mathematics in secondary school, undergraduates are more probable than any time in recent memory to struggle with the move to college. These initial struggles leave students more likely than ever to struggle with the transition to college and the advanced mathematical thinking needed to be successful beyond calculus (Kajander & Louric, 2005; Selden & Selden, 2002; Tall, 2008). One possible way to help undergraduates transition to college is the utilization of formative assessments, for example, exit tickets; such assignments can help undergraduates to see their professor as more caring and increases the probability students will seek help when they are struggling (Black and Wiliam 1998, 2009; Dibbs, 2014). However, the number of formative assessments finished is a far larger indicator of understudies' course grade than their weight in the course grade (Dibbs, 2014). One possible latent variable that could account for this impact was that undergraduates who finished more post-labs had better calibration than those that did not. The hypotheses for this study were: (1) There is no relationship between students' calibration on tests and achievement in introductory calculus (2) There is no relationship between the number of post- class reflection assignments students complete and their calibration on tests

Participants were recruited from two sections of calculus (n = 60) at a rural Hispanic-serving research university in the South during the Fall 2016 semester. Students enrolled in calculus most commonly major in engineering, physics, computer science, mathematics, or secondary mathematics education. At the time of consent, students' ACT scores (SAT scores were converted to ACT equivalents), current GPA, native language, gender, and major were collected. Data was collected during the four unit tests of the course and the final exam. At the bottom of each question, students had to circle a face on an affective Likert scale indicated their confidence in their answer. During analysis, the Likert scale was converted to a numerical score, and the difference between student's exam score and Likert score was used to obtain a calibration score for each item. A linear regression model was constructed. Based on pilot data, ACT scores, gender, and native language were all included in the initial model:

$$Score = a_1(calibration) + a_2(ACT) + a_3(gender) + a_4(native\ language)$$

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Error Detection in an Introductory Proofs Course

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We report on an exploratory quantitative study of students' error detection skills. Based on the research on "proof framework" type errors and models for proof comprehension, we classify errors in proofs as internal or external for proof validation. We then test students' ability to detect errors of both types and determine if detection is correlated with success in an introductory proofs course.

Key words: proof validation, proof comprehension

Researchers have shown that students struggle to validate proofs (Selden & Selden, 2003; Inglis & Alcock, 2012; Ko & Knuth, 2013; Weber, 2010). It is known that students are particularly prone to miss "proof framework" errors (Selden & Selden, 2003; Inglis & Alcock, 2012). Researchers have also proposed that one component of comprehension is "local" or line-by-line understanding (Mejia-Ramos et al, 2012). For use in proof validation, we propose a classification of errors as external (e.g., proof framework errors) or internal (e.g., local errors). Here we take a quantitative approach to studying students' error detection skills. That is, instead of focusing on the complex written and mental processes involved in proof validation, we focus on quantifying students' final judgments about purported proofs. We hypothesize that students are more adept at detecting internal errors over external errors and external error detection is more predictive of performance in an introductory proofs course than internal error detection.

Method

In summer 2016, we studied 23 students in a 300-level introductory proofs course emphasizing analysis at a large public university. Students were given 12 "proof feedback activities," each consisting of one claim and one purported proof. Depending on the activity, the claim was false, the proof had an error, both, or neither. The students' task was to list any errors in the claim or proof. We call an error external if it is global or structural. An internal error is not an external error. The activities included a mixture of types of errors.

Results and Conclusions

For each student we recorded identified errors and course grade. The results of a paired samples t test ($t(22) = 2.16$, $p = 0.04$) confirmed that students were more successful in identifying internal errors ($M = 35.40$, $SD = 24.69$) than external errors ($M = 22.71$, $SD = 16.23$). Logistic regression was used to predict the probability that a student would earn a grade of C or better. The predictor variables were participants' rates of success in detecting internal errors and in detecting external errors. A test of the full model versus a model with intercept only was statistically significant, with the Cox & Snell $R^2 = 0.32$. These results confirm that external error detection is more predictive of performance in an introductory proofs course than internal error detection. Ability to detect external errors had significant effect ($p = 0.049$) on success in the course. Future work includes an analysis of students' false negatives (i.e., "errors" listed in parts of proofs that were correct).

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An Active Learning Environment in Introductory Analysis

Brynja Kohler and Patrick Seegmiller

December 8, 2016

Abstract

At Utah State University, the course known as 'Math 4200:Foundations of Analysis' is a requirement for all department majors, and, in addition to introducing real analysis, serves as an introduction to rigorous proof. All too frequently, courses such as these are taught with the typical lecture format: the instructor enters the classroom to deliver polished explanations of definitions, theorems, and their proofs, while leaving students to struggle to follow lectures and then struggle further on their own to make sense of incomprehensible homework problems. This poster includes descriptions of easy-to-implement strategies that change the classroom to an active learning environment, with ample opportunities for formative assessment and feedback without overloading the professor with busywork. The strategies include: name tents, group exercises and quizzes, peer-reviewing of homework, concept quizzes, individual presentations, and growth mindset reflections. We surveyed students to find their reactions to these class activities and found positive and helpful implementation hints.

1 Introduction

Here we will share the course specifics, a description of the text and instructor's experience, the classroom, students and their demographics. Our course and student population are quite typical of Land-grant research 1 universities, and we have class sizes of 40-45 students typically. The authors have researched various textbooks and course materials that encourage an active approach to analysis and these will be summarized and reviewed.

2 Methods

In this section, we will detail the course design and instructional strategies employed.

- **Peer-reviews of Homework** Using an online course organization system (CANVAS) we set up homework assignments (5 proofs each) and a peer-review system, so students could read each other's work, the instructor's solutions, and evaluate and comment on the writing.
- **Group Work** The class has tables so students are students can easily work together in groups. We assigned groups and changed them monthly. In almost every class meeting, groups were prompted to discuss and/or write about specific problems. The instructor gathered written work (only 12 groups reduces the grading load) and provided non-graded formative feedback in the following class meeting.
- Several more simple strategies for an active learning environment.

3 Results

In this final section of the poster, we will summarize comments from students regarding the implementation of these active learning approaches. Generally, students were enthusiastic about mathematics and their learning experiences in the class.

Pre-service Teachers' Use of Informal Language While Solving a Probabilistic Problem

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Muteb Alqahtani
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This exploratory study investigates pre-service teachers' (PSTs) collaborative discus-sions to solve probabilistic problem. The PSTs synchronously collaborated online using Virtual Math Teams with GeoGebra to investigate the fairness of a series of die by using interactive simulation to randomly sample from the die with replacement. While discussing their solution in the chat panel, the PSTs used informal, non-standard language to describe the distribution of the data. In this poster, we present examples of how the PSTs, using informal language, co-constructed their knowledge of different probabilistic concepts while solving the problem. This study contributes one of a series of tasks that were designed to elicit how PSTs build understandings of mathematical concepts without formal introductions to these concepts.

Key words: probability, pre-service teachers, law of large numbers, informal language, statistics

Mathematics is viewed as blending language, symbolism, and representations, and when working on problems, students must articulate their language of the connections between their symbolism and representations (O'Halloran, 2005). Studies (e.g., Francisco & Maher, 2011) have shown that when students engage in open-ended problem solving they tend to use informal, non-standard language as they build ideas. While solving problems in an online environment, students rely on their language to make those connections and make sense of mathematical ideas.

A study on PSTs' informal talk about variation was rich with statistical ideas of spread, modal clumps, and distribution chunks (Makar & Confrey, 2005). Makar & Confrey (2005) noted that when PSTs used the word *range* they referred to measure or location. Also, PSTs who used *spread out* would frequently accompany it with the word *evenly* for the notion of spread.

In our study, four groups of three PSTs from a large university solved an open-ended probability task in a collaborative online environment, VMTwG. The VMTwG integrates a multi-user version of GeoGebra, a dynamic geometry environment, with a shared whiteboard and a chat panel. The teachers met in VMTwG synchronously and collaborated to solve the probability task. In this task, the PSTs were provided with an interactive simulation of six dice that are weighted differ-ently. The simulation allows users to choose the number of trials from 1 to 1,000 rolls and select from three different representations of the data: a fre-quency table, a pie chart, and a bar graph to present the outcomes. The PSTs were asked to roll the dice, explore the different representation of the outcomes, and discuss in the chat panel the fairness of each die.

To analyze teachers' interactions in VMTwG, three researchers openly coded the chat logs for informal language referring to empirical and theoretical probabilistic concepts such as sampling, law of large numbers, distribution, and randomness.

Similar to Makar & Confrey's (2005) findings, PSTs used non-standard language to make sense of distributions. PSTs described various distributions as (*un)even*, *well-distributed*, *bell curve*, and *smallest range*. They justified choosing 1,000 trials to discover *patterns* of the graph such as *more equal*, *steady*, and *more accurate*, which we view as an attempt to establish connections between theoretical and empirical probability. However, our PSTs used *range* differently, to explain how close the frequency bar graphs were to a uniform graph. Our study contributes understanding of the development of PSTs' knowledge of probability and statistics. It also contributes understanding of how non-standard, informal language shapes mathematical understanding in collaborative environments.

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Student Ways of Framing Differential Equations Tasks

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In this research presentation I utilize the theoretical perspective Knowledge In Pieces (diSessa, 1993) to identify the knowledge resources two students utilized while in the process of completing various differential equations tasks. The results provide a fine-grained description of the knowledge students consider to be productive with regard to completing various differential equations tasks. Further the analysis resulted in the identification of five ways students frame differential equations tasks and how these framings are related to the different knowledge resources students utilize while completing the various tasks. These framings did not only provide insight into the students' general approaches to completing the task; differences in the individual students' applications of knowledge across the tasks were accounted for by the different Framings. The results have direct implications with regard to the teaching of differential equations as they inform the ideas students view as productive when completing various tasks involving differential equations.

Keywords: Differential Equations, Knowledge in Pieces, Undergraduate Student Learning

In a recent review of undergraduate mathematics education literature Rasmussen and Wawro (in press) made a call for research on student learning in differential equations. In response to this call, I conducted 5 individual, problem-based interviews with each of 8 students. Two students' complete set of interviews were transcribed and analyzed using methods sharing characteristics with the iterative processes of Knowledge Analysis as discussed by Cambell (2011). Through this process I identified regularities in the language, ideas and actions utilized by the students across various tasks. One result of the analysis was the identification of five ways of framing differential equations tasks, general approaches to completing the differently framed tasks, and sets of resources regularly utilized to complete tasks of a particular framing.

Results

The five ways students framed the various differential equations tasks identified in this study are the differential equation as: a description of the behavior of the quantity of interest, as a relationship between values of quantities and the value of the rate of change of the quantity of interest, an equation that provides a model, as a relationship between values, and, as a relationship between functions.

In the poster I outline in more detail, the similarities and differences between the different framings, the general approaches to the students took with regard to the individual framings, and the regularities with regard to student usage of resources within the framings.

Implications

Based on the analysis of the students learning opportunities may be created by providing students with tasks that prompt them to focus on relationships between the variables (and functions) in the differential equations, the quantities those variables (and functions) represent, and the connections between differential equations and their solutions.

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Using Evidence to Understand and Support an Educational Reform Movement: The Case of Inquiry-Based Learning (IBL) in College Mathematics

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We present a graphic review of a decade of research and evaluation work on inquiry-based learning in college mathematics. Featured studies examine student outcomes of IBL instruction, processes of change as instructors explore and adopt IBL approaches, and the workings of the faculty learning community that has formed to learn and promote IBL ideas. Collectively, these studies highlight the use of evidence to understand instructional practices and change in these practices, and to support evidence-based decision-making by instructors and change leaders.

Key words: inquiry-based learning, faculty change, evidence-based decision-making

This poster offers a graphics-oriented summary of a decade of research and evaluation studies of inquiry-based learning (IBL) in college mathematics. Our group's foray into this topic began with a large, multi-campus research study of student outcomes of inquiry-based learning as compared with primarily lecture-based instruction (e.g., Kogan & Laursen, 2014; Laursen et al., 2014; Hassi & Laursen, 2015). As IBL practitioners began to share their knowledge and experience with other instructors, we evaluated their faculty development activities and used these projects as vehicles to study the processes of faculty change (e.g., Hayward, Kogan & Laursen, 2016) and to develop better ways to measure such change. We continue to work with IBL leaders on IBL professional development and expanding community capacity to deliver skillful professional development. Another current study examines the IBL math ecosystem, seeking to understand the history, opportunities and challenges for the IBL math community as it develops strength as an educational change movement.

The poster will show the intellectual arc of our research questions and findings over time and highlights diverse research methods. The poster will also spark conversation about broader themes, such as selecting research methods, balancing scholarly and applied research, or taking a long view in developing a research program.

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EDUCATIONAL POINT OF NEWTON-LIEBNITZ FORMULA

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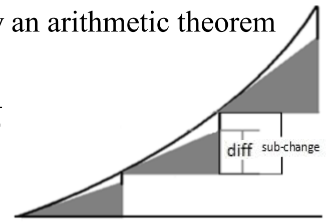
Presenting author: Cao, Rongrong, School of mathematics and statistics, Qingdao University



Newton-Liebniz formula has been stated by a figure and proved by an arithmetic theorem

$$0.9...9 < \frac{diff}{sub-change} < \frac{1}{0.9...9} \quad \Rightarrow \quad 0.9...9 < \frac{sum\ of\ diffs}{total\ change} < \frac{1}{0.9...9}$$

Increasing number of 9s, we get the Newton-Liebniz formula:

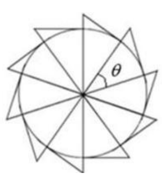
Integral of differentials = total change. See Lin's early book (Jan 1999) or recent paper in US-China Education Review A, March 2016, Vol.6, No. 3.

We discuss in this single page its educational point.

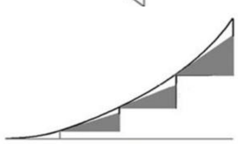
1. Tolstoy's view of history in *War and Peace*: Only by taking infinitesimally small units for observation (the differential of history, that is, the individual tendencies of men) and attaining to the art of integrating them (that is, finding the sum of these infinitesimals) can we hope to arrive at the laws of history.

2. Economy, collection of merchants, or integral of differentials of merchants, the latter are, the individual tendencies of merchants.

3. More exact examples, e.g.



$$0.9...9 < \frac{\text{Arc length}}{\text{Tangent length}}\left(=\frac{\theta}{\tan\theta}\right) < 1 \Rightarrow 0.9...9 < \frac{\text{Circumference}}{\text{Sum of tangent length}} < 1$$



$$0.9...9 < \frac{\text{Tangent length}}{\text{Sub-curve length}} < 1 \Rightarrow 0.9...9 < \frac{\text{Sum of tangent length}}{\text{Curve length}} < 1$$

Increasing number of 9s, we get the circumference and curve length.

There is a similar conclusion for the area under the curve.

The authors thank the reviewer's comments.

Any questions, please contact with presenting author Cao Rongrong, E-Mail: caorrqdu@sina.com.

Student reasoning with differentials and derivatives in upper-division physics

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Abstract: *Students encounter multiple mathematical representations of change in physics courses. In addition to the complexity of the material, students must navigate mathematical notation that can seem arbitrary and can differ from conventions used in mathematics coursework. In this poster we will examine student responses illustrating the challenges of mathematical representations of change, drawn from students in upper-division physics courses in mathematical methods and thermal physics.*

Description

This work is part of a collaboration to investigate student learning and application of mathematics in the context of upper-division physics courses. Our project seeks to study student conceptual understanding in upper-division physics courses, investigate models of transfer, and to develop instructional interventions to assist student learning. Throughout, we seek to go beyond procedural skill and calculation and to probe conceptual understanding and the development of quantitative reasoning skills whose development is often left implicit.

As part of the project, we have examined several key ideas that cut across the upper-division physics curriculum, including vectors, integration, and derivatives. For this poster, we focus on mathematical descriptions of change encountered by students, including derivatives and differentials. Physics and mathematics tend to place different levels of emphasis on differentials, and students often struggle to conceptualize the distinctions between differentials, derivatives, and a finite difference indicated by a delta.

In the poster we will show data from a variety of written questions in which students reason with change. In some problems, students performed calculations of various quantities. In others they were asked for interpretations of expressions. For example, in thermal physics:

> *The definition of enthalpy can be used to derive the differential expression $dH = T\,dS + V\,dP + m\,dN$.*
> *These expressions contain differentials like dH. What does this mean?*
> *What is implied by choosing this notation instead of ΔH?*

A second question, posed in the mathematical methods course, was focused on ordinary differential equations. Students were asked to interpret quantities containing differentials:

The law of conservation of momentum give the differential equation $(M+m)dv+v_P dm=0$. Interpret the quantities dv and dm.

Student responses suggest a lack of distinction between change quantities. We will present examples of students performing inappropriate manipulations that suggest confusion between derivatives and ratios of variables, as well as written explanations. For example:

"dm is a rate that fuel mass is leaving"
"differentials imply integration"

Although this work is preliminary, we are particularly interested in presenting this to a math education research audience due to the disciplinary differences in use of change quantities.

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An alternate characterisation of Developmental Mathematics students

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Developmental – or the antiquated "remedial" – mathematics is a large enterprise in American colleges. For the California State University (CSU) system roughly one-third of all students require developmental mathematics. Placement in these courses in the CSU is determined by a standardized test. Those who fail are required to take some campus-specific variant of developmental mathematics. This poster addresses the question, what more can be said about students enrolled in developmental mathematics programs other than they have failed an exam? An analysis of survey instrument data will be presented that shows San Jose State University developmental mathematics students are fundamentally different, undesirably so, than their non-developmental counterparts on a range of attitudinal, affective, and dispositional measures.

Key words: developmental math, attitudes towards and perceptions of mathematics

Poster Description

Though developmental students failed an exam to be streamed into their developmental courses, this failure may be the symptom and not the cause of their under-performance in mathematics. Institutional records at San José State University (SJSU) indicate that developmental mathematics students never reach the level of academic achievement of their non-developmental colleagues, if they ever complete the developmental sequence – a result contradicting other studies (eg. (Bahr, 2008)) – putting into question the effectiveness of current developmental mathematics education practices at SJSU.

An online survey was sent in Fall, 2016 to all San Jose State University students enrolled in a freshman mathematics course. In addition to general questions about financial concerns of the students, the survey comprised two established survey instruments: the Mathematics Attitudes and Perceptions Survey (MAPS; Code, et al., 2016) and the Abbreviated Math Anxiety Scale (AMAS; Hopko, et al., 2003). The MAPS survey assesses the students on a range of factors known to impact academic achievement in mathematics: growth mindset, seeing connections between mathematics and the real world, confidence, interest, persistence in working with mathematics, sense making, and views of the perspectives on answers. Students' MAPS responses are weighed against consensus responses of mathematicians and relative-to-expert scores are reported for each factor, along with an overall expertise index. The AMAS assesses students on their level of mathematics anxiety. The survey reports an overall anxiety score, the sum of learning and assessment anxiety.

On almost all MAPS and AMAS scores, developmental math students scored less-favourably than their non-developmental counterparts. There are two notable exceptions: i) developmental math students scored higher on the "connections to the real world" subscale, and, ii) there was no statistically-significant difference between the groups on the interest in mathematics subscale. The poster presents these comparisons along with a cluster analysis that further characterises sub-groups of developmental math students.

The results of this study highlights the importance of addressing non-conceptual aspects of learning and performing mathematics in developmental mathematics courses. Doing so, we argue, is the only way to provide effective developmental mathematics education.

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Student Attitudes, Beliefs, and Experiences Related to Counting Problems

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Mathematics textbooks and mathematics education research articles frame counting problems as requiring clever insight and being inherently challenging and especially accessible. In this study, we distributed a survey to mathematics students in order to examine student attitudes about counting problems and the extent to which these attitudes aligned with presentations of counting in the literature. In this poster, we present results from this survey that highlight some surprising ways in which the responses did and did not align with the literature.

Key Words: Counting, Student Attitudes, Combinatorics, Discrete Mathematics

Both mathematics textbooks and mathematics education research articles tend to frame counting in particular ways. For example, Tucker (2002) says that counting requires logical reasoning and clever insights (p. 169), Lockwood (2013) that counting is accessible and has a variety of applications, and Martin (2001) claims that "Counting is hard" (p. 1). While some of these characterizations are based on evidence with students, there have not been studies that measure the extent to which such perspectives about counting align with what students actually believe and experience. We think such work is important because it will allow researchers and teachers to consider students' perspectives about counting. Our research question is, *"What are students' attitudes, beliefs, and experiences with counting and combinatorial reasoning, and how do these affective factors align with how counting is presented in combinatorics education literature and textbooks?"* We developed a survey that targeted students' attitudes, beliefs and experiences about counting. We collected textbooks and mathematics education research papers, and we identified claims about the nature of counting, how counting compares to other disciplines, and applications of counting. We categorized these claims and used them to create a survey that we sent via a listserv to mathematics majors at a large public university.

We currently have approximately 40 student responses (some did not answer all questions), and the survey will continue to run in the coming months. Due to space we briefly describe just a couple of interesting findings that demonstrate the nature of our results. First, we found that students did seem to align with many of the statements in the literature. For instance, 95% of respondents agreed or strongly agreed that "Solving counting problems requires critical thinking skills," aligning with statements by Tucker (2002) and others. Also, less than 25% agree that "Solving counting problems requires prior mathematical knowledge," which aligns with the claims by many researchers that counting is accessible (e.g., Kapur, 1970; Lockwood, 2013).

There were also some surprising results. For instance, only 20% of respondents agreed or strongly agreed with the statement "Solving counting problems is difficult in general." Given the claims by many textbook authors and researchers (e.g., Tucker, 2002; Martin, 2001), we were surprised that more students did not seem to think that solving counting problems is difficult. Another surprising finding was that only 30% agreed with the statement "Solving counting problems requires memorization." Given our experience with students, we expected more students to associate counting with memorization. In our final poster, we will provide a more comprehensive report of our results and will paint an overall picture about students' attitudes, beliefs, and experiences with counting problems.

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Algebra Instruction at Community Colleges:
An Exploration of its Relationship with Student Success

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We present the research design and data collection strategies for a federally funded project (Watkins, Duranczyk, Mesa, Ström, & Kohli, 2016) that investigates the connection between instruction and student learning and performance in algebra courses at community colleges. The poster focuses on measurement issues we face in identifying the characteristics of mathematics instruction and students' learning gain, specifically we address questions encountered from the pilot data collection (six community college faculty and nearly 150 students) that need to be resolved prior to data collection.

Key words: instruction, community colleges, teaching quality research, algebra, observational measures

Although we like to think that teaching influences learning, the truth is that such connection has not been established empirically (Hiebert & Grouws, 2006). As a first attempt to establish this connection, we investigate the extent to which there is an association between what occurs in the classroom and what students learn in a one-semester course of algebra at a community college. Whereas there is some research documenting how individual and institutional characteristics (e.g., prior achievement, family support, financial aid, learning support and tutoring centers, and ratios of full- to part-time instructors) contribute to failure rates and other performance measures (Bradburn, 2002; Feldman, 1993), there is little information about the fundamental work of teachers in the classroom, and the interactions that occur between instructors, students, and the mathematical content. The Mathematics Quality Instruction (MQI) protocol a video analysis tool used in P-12 settings (Learning Mathematics for Teaching Project, 2011) and the Algebra Instruction Protocol (Litke, 2015) have been adapted to measure faculty and student interaction at the community college. The Algebra and Precalculus Concepts Readiness (APCR) test (Madison, Carlson, Oehrthman, & Tallman, 2015) was used to measure student learning. We also collected information on instructors' mathematical knowledge for teaching algebra, and instructor and student beliefs, and attitudes towards the teaching and learning of algebra which can also moderate the relationship between instruction and learning. Research in the K-12 arena documents that the association between quality of instruction and student performance on standardized tests can be moderated by instructors' knowledge and attitudes towards innovative teaching practices, knowledge of algebra for teaching, and their beliefs about mathematics, its curriculum, and students' learning (Hill, Rowan, & Ball, 2005). In this project we seek to test such relationships, in a different setting and grade level. The first phase, presented in the poster, is the testing of the instruments and initial assessments of the measures done with six instructors at three different community colleges in three states, Arizona, Michigan, and Minnesota.

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Bridging the Gaps between Teachers' and Students' Perspectives of a Culturally Inclusive Classroom

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Creating a culturally inclusive classroom has been suggested to help minority students improve their achievement in class. However, evidence shows gaps between teachers and students about what a culturally inclusive classroom should be. We propose a framework for investigating the differences between teachers' and students' beliefs on such a classroom.
Key words: Minority, Culturally Inclusive Classroom, Mathematics Achievement

Evidence shows that minority students' mathematics achievement tends to rapidly fall behind during fifth to eighth grade (Beaton et al., 1996; Schmidt et al., 1999). In the majority of the states in the U.S., there was a 30% to 50% difference in performance between White students and the largest minority groups (Hispanic and Black students) at the basic level of mathematics on the eighth-grade National Assessment of Educational Progress Exam (Blank & Langesen, 1999). One possible cause of this disparity in performance is the various forms of discrimination minority students experienced in the classroom (Dovidio, 2001; Salvatore & Shelton, 2007).

A culturally inclusive classroom is a learning space that encourages students, and especially teachers, to acknowledge, appreciate, and use diversity as a tool to enhance learning experiences (Nieto, 2004; Montgomery 2001). Unfortunately, evidence shows that there are gaps between teachers and students on what a culturally inclusive classroom is and how to construct such a culturally inclusive classroom (Tyler, Boykin, & Walton, 2006).

The purpose of this study is to develop an instrument to investigate the differences and their extent between teachers' and students' beliefs of what a culturally inclusive classroom should be. We also intend to investigate how these differences are related to teachers' and students' ethnicity, age, and gender. We hope to offer more specific recommendations to bridge the gaps once we understand the relationship between teachers' and students' demographic and their culturally inclusivity beliefs.

From the students' perspective, it has been suggested that a culturally inclusive classroom has no superiority of intelligence based on race or culture, no authentic behavior expectations on culturally different minorities, a willingness to accept a different racial reality in communications, and validating feelings of minority students in class (Sue et al., 2009). While from the teachers' perspective, a culturally inclusive classroom provides equal attention to all students (Gay, 2002), an equal expectation of high achievement for all students (Foster, 1997; Kleinfeld, 1975), and an equal opportunity of learning for all students (Oakes, 1990; Banks et al., 2001).

Based on these teachers' and students' perspectives, we constructed a framework, in tabular form, to guide the construction of the instrument. Mathematics contexts are given in each cell to help generate a questionnaire specifically for a mathematics class.

	Attention	**Expectation**	**Opportunity**
Knowledge	Math topics	Math topics	Math topics
Behavior	Problem solving	Problem solving	Problem solving
Communication	Questioning & Ans.	Questioning & Ans.	Questioning & Ans.
Emotion	Positive & Negative	Positive & Negative	Positive & Negative

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Mathematical Knowledge for Teaching & Cognitive Demand: A Comparative Case Study of Precalculus Examples that Involve Procedures

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In 2010, Charalambous published an article that examined the relationship between mathematical knowledge for teaching (MKT) and task unfolding. As a result of this study, Charalambous found evidence to support the claim that there is a positive relationship between a teacher's MKT and the cognitive level of task presentation and enactment. Drawing upon this finding, the purpose of this case study is to utilize unfolding and cognitive demand as a lens through which to examine mathematical knowledge for teaching at the undergraduate level. While MKT has been studied extensively at the K-12 level, there are relatively few studies that focus on MKT at the collegiate level. In order to help fill this gap, this case study first identifies how Precalculus instructors unfold examples that involve procedures and then examines the MKT that is involved in this unfolding.

Keywords: Mathematical Knowledge for Teaching, Cognitive Demand, Case Study, Precalculus, Examples, Procedures

Mathematical knowledge for teaching (MKT) has been defined as "the mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball, Thames, & Phelps, 2008, p. 399). At the K-12 level, MKT has been studied extensively, but few studies exist at the collegiate level. The purpose of this study is to examine MKT at the collegiate level from the perspective of practice while still drawing upon previous research. In particular, Charlambous (2010) found that there was a positive relationship between teachers' MKT and task unfolding. Charalambous used the MKT measurement developed by Ball for elementary teachers, the Mathematical Task Framework (Stein, Grover, and Henningsen, 1996, p. 469) to analyze the unfolding (i.e., selection, preparation, and enactment) of tasks, and the Task Analysis Guide (Stein & Smith, 1998) to analyze the cognitive demand. While it would be desirable to replicate this study to look for similar results at the collegiate level, no comparable measure of MKT exists. However, it is possible to use the Mathematical Task Framework and the Task Analysis Guide as lens to help examine what MKT at the collegiate level might look like.

Instead of examining collegiate MKT at large, this study focuses specifically on Precalculus examples that involve procedures. The larger purpose of this study is to contribute to research on Precalculus courses (Hastings, 2014; Saxe & Braddy, 2016). Examples were chosen as the specific task of teaching of interest because of their centrality to math instruction. Procedural knowledge is often characterized as superficial memorization of algorithms and therefore less important than conceptual knowledge. However, procedures are an integral part of mathematics and there is a need for students to develop deep procedural knowledge (Star, 2005) that is connected and requires high-level cognitive demand (Smith & Stein, 1998). This comparative case study seeks to answer the following research question: What mathematical knowledge do Precalculus instructors draw upon when selecting, presenting, and enacting examples that involve procedures? In order to answer this question, the Mathematical Task Framework will be used to analyze the unfolding of the examples while the Task Analysis Guide will be used to analyze cognitive demand. Finally, similar and different cases (in terms of the cognitive demand) will be compared in order to examine the MKT involved in unfolding examples.

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Coping with the Derivative of an Atypical Representation of a Common Function

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250 calculus students were asked to evaluate $f'(2)$ when $f(x) = x^3$ *if* $x \neq 2$ *and* $f(x) = 8$ *if* $x = 2$. *Responses were coded, and eight students were interviewed about their answers. The data provide insight into students' understandings of function, derivative, and graph.*

Key words: Function, Derivative, Graph

250 students upon completion of a calculus course were given the following task: "let f be the function defined by $f(x) = x^3$ *if* $x \neq 2$ *and* $f(x) = 8$ *if* $x = 2$. Evaluate $f'(2)$ and explain your answer". This task was motivated by my experience as a calculus teaching assistant, in which students insisted that for $g(x) = \frac{\cos x - 1}{x}$, $g(0) = 0$, the value of $g'(0)$ must be 0 due to the constant rule. Harel and Kaput (1991) and Sfard (1992) share a similar anecdote and attribute students' incorrect responses to not viewing a function as an object. By working with a familiar function with a familiar graph, students might recognize that the function provided to them is the same as the cubed function, and then successfully solve the problem. This recognition is, in theory, available to students who view the derivative as no more than the slope of a tangent line.

100 of the 250 students were in a reform calculus course. 16% of the students answered correctly (including students whose explanations were incoherent or inconsistent), 64% wrote that $f'(2) = 0$, 9% wrote "undefined" or "does not exist", and only 52% of the students who acknowledged that $2^3 = 8$ arrived at the correct answer. Some students wrote that $f'(2) = 0$ *if* $x = 2$ and $f'(2) = 12$ *if* $x \neq 2$. A common way of thinking can be summarized as "take the derivative of each equation, and then plug in 2. The derivative of 8 is 0," which was indicated in students' responses to questions about other piecewise functions during the interviews.

All eight of the students interviewed were enrolled in a reform calculus course that stressed the meaning of derivative at a point as rate of change over a small "essentially linear" interval (Thompson, Bryerly, & Hatfield, 2013). Only one student correctly answered the question on the exam, but during the interview, she immediately changed her answer. The remaining 7 initially claimed that $f'(2) = 0$. In fact, many of the students who graphed *f* correctly tried to explain why their initial answer made sense with the graph. Two students claimed the graph of *f* has the same points as the graph of x^3 and is flat around $x = 2$. Some students claimed that $f'(2) = 0$ because the point (2,8) is separate, and there is no rate of change at a single point. Two students stood out in particular. By the end of the interview, they concluded that *f* has the same graph as the cubed function, but also that $f'(2)$ is not 0, since *f* is increasing around $x = 2$. However, they claimed they could not find its exact value. Each student said that she could only "approximate" $f'(2)$ by finding dy/dx for small intervals around $x = 2$. When asked "what do you think you would end up getting?," they could not provide an answer. They both claimed that it was illegitimate to use the power rule on x^3.

It was common for students to claim that they were graphing "two functions" and not know how to incorporate information about the domain. The data collected reveals that many students have weak conceptions of function, derivative, and graph. Since several of the subjects were enrolled in the reform course, these results have implications for instructional design research in the reform setting.

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Speaking with Meaning about Angle Measure and the Sine Function

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Researchers have reported on the difficulties K-12 students, pre-service and in-service teachers experience in reasoning and communicating about angle, angle measure and trigonometric functions. This work extends the existing literature to highlight that even mathematically sophisticated individuals (e.g., PhD mathematics students) often struggle to speak with meaning about these ideas sans targeted interventions to support them in doing so. We share tasks and data from semi-structured clinical interviews conducted with graduate teaching assistants to highlight differences in communication about these ideas pre- versus post-intervention.

Keywords: Graduate teaching assistants, Precalculus, Speaking with meaning

Researchers have reported on the difficulties K-16 students, pre-service teachers and in-service teachers experience in reasoning and communicating about angle, angle measure and trigonometric functions (Moore, 2013; Moore, 2010; Thompson, 2008). This work extends the existing literature to highlight that even mathematically sophisticated individuals (e.g., PhD mathematics students) may struggle to *speak with meaning* about these ideas sans targeted interventions to support them in doing so (Clark, Moore & Carlson, 2008).

We share tasks and associated data from one-hour, semi-structured clinical interviews conducted with graduate teaching assistants (GTAs) (Clement, 2000). Research participants were mathematics PhD students who had taught (either as lead instructor or recitation leader) at least one college-level course at a public, state university in the Southwestern United States. Twenty-two of the 24 research participants had undergone one or more years of an intervention designed to support them in constructing deeper meanings and speaking with meaning about the ideas in an undergraduate precalculus class. Their participation in this intervention coincided with teaching a precalculus course using research-based curriculum materials that were designed to support quantitative and covariational reasoning (Carlson, Oehrtman, & Moore, 2015). The other two research participants had no exposure to the intervention or the research-based curriculum materials, but rather they underwent the standard university GTA training and taught calculus courses (Calculus I or Calculus II, as lead instructor and as recitation leaders).

There was relative uniformity in response characteristics from participants who had undergone the intervention. As such, the data presented will showcase two representatives from that group and the two participants sans intervention experience. Our preliminary analyses of the data suggest the latter two participants struggled to communicate meaningfully about *how* certain procedures (e.g., measuring an angle using a protractor) work, *why* formulas (e.g., relating arc length, angle measure and radius length) are what they are, and *how* to think about the sine function in a way that works in different contexts (as opposed to thinking of values in a table in one context, to having a triangle-based method in another context, and a x,y-coordinate meaning in yet another). In contrast, post-intervention participants spoke with relative fluency about generating formulas from definitions and applied consistent ways of thinking about the sine function to tasks throughout the interview. Excerpts from interview transcripts and written work will be shared to demonstrate the above distinctions between the two groups in conceptual versus procedural orientation of responses, as well as coherency and consistency in reasoning from task to task.

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Examining Prospective Teachers' Justifications of Children's Temperature Stories

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Part of the work of mathematics teacher educators (MTEs) are to provide authentic experiences to prospective teachers. We showed four temperature story problems involving integers to prospective teachers (PTs), and asked them if the stories matched given number sentences. While each of the stories were similar to the number sentence, none of them matched exactly. We examine the reasons PTs gave for saying that the stories matched the number sentences, and discuss implications of their thinking for mathematics content courses for prospective teachers.

Key Words: Prospective Teacher Education, Integers, Mathematical Knowledge for Teaching

In working with prospective teachers (PTs), part of the work of the mathematics teacher educator (MTE) is to help them develop the mathematical knowledge that they will need for teaching (MKT) (Ball, Thames, & Phelps, 2008). This is true, even if the PTs are taking a course primarily focusing on mathematics content, rather than methods for teaching. Research has shown that providing PTs with tasks authentic to the work of teaching will help motivate them to want to learn the mathematics (Newman, King, & Carmichael, 2007). Thus, as MTEs, we try to provide PTs in our mathematics content courses with tasks that will ask them to perform activities that they will need to do while teaching. This poster focuses on a task designed around using temperature as a context to support the learning of integer operations.

Part of the MKT that PTs use in their work as teachers is being able to evaluate contextual problems that relate to the mathematics that they are teaching. The *Common Core State Standards for Mathematics* (Council of Chief State School Officers & National Governors Association, 2010) suggests using contexts for teaching integers which include temperature, elevation, credit/debits, and electron charges. However, studies have shown that students, including PTs, have difficulty using negative integers in these contexts (e.g., Whitacre et al., 2015). This study looks at the ways that PTs' evaluated temperature problems designed by children.

In this study, we provided opportunities for elementary and middle school PTs (N=100) to explore four stories written by fifth graders that were intended to correspond to given integer number sentences. We investigated the PTs' responses to the children's problems, noting what they attended to in order to decide which stories made sense with the given number sentences. Each of the four stories contained aspects that were not a perfect fit with the number sentence (e.g., changing structure of number sentence, unrealistic situations). However, a majority of the PTs in our study indicated that two of the stories matched the number sentences perfectly—neglecting some important nuances. For example, one problem fits the number sentence 17 – 13, although the given problem was 17 + -13. Many of the PTs incorrectly argued that these number sentences were equivalent, because when evaluating a subtraction problem, students are often taught to "add the opposite."

In the poster, we will share the four temperature story problems as well as the data from the PTs who said that the story matched the problem. We will discuss the reasons that PTs gave for saying that the stories matched the problems, and the implications that this has for teacher education, particular in the context of mathematics content courses for prospective teachers.

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Classroom Participation as a Socialization Agent for Identity shaping of Preservice Mathematics Teachers

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Abstract
This study explores classroom participation as an agent of socialization for preservice elementary mathematics teachers. Each character, the teacher, student and curriculum in the classroom plays various roles in the socialization process. However, the teacher plays a major role because s/he is responsible for creating the environment where participation is possible. In this work, the analysis of data shows the teacher's teaching method(s), questioning and listening skills, as well as her background understanding of the specialized mathematics knowledge needed for teaching and the students all help to create such an environment. Also, as students explore mathematics by *doing* through group work, class discussions and individual work, they experience growth in mathematics classroom practices that results in change in the mathematics identity of the students.

Key words: Socialization agent, mathematics knowledge for teaching (MKT), and mathematics identity.

Setting the Stage

Recent educational reforms have increased the amount of attention paid to the preparation of elementary school teachers, particularly in subject areas such as mathematics. Low rankings by U.S. students in international comparisons, slow growth on national assessments such as the National Assessment of Educational Progress, and high levels of remediation among university and college students have helped to signal alarms about the ability of elementary school teachers to teach mathematics effectively. Also, research on teacher knowledge has begun to document specialized forms of mathematics knowledge that are unique to teaching (Ball, D. L., Hill, H. C., & Bass, H., 2005; Hill, Rowan, & Ball, 2005). For students who are products of schools in these contexts, and who desire to become teachers, it is important to understand how universities can support such students. Inadequate preparation in mathematics at the K-12 level should not automatically disqualify these students from the teaching profession. Principled, research-based understandings of how to best support students with less than ideal mathematics backgrounds are critically important. This study shows how university designed mathematics courses and classroom practices affect mathematics identity and socialization of preservice elementary mathematics teachers.

Findings

This poster reports the preliminary findings of classroom participation as a socialization agent for identity shaping of preservice mathematics teachers. Quantitative results include changes in identity using Martin's (2000) mathematics socialization and identity frameworks. The analysis of field notes, audio and video recordings of the classroom activities of the spring semester show consistent changes in participants' self-confidence, their use of appropriate language and a stronger mathematics identity.

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Investigating Student Learning Through Team-Based Learning Calculus Instruction

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We have a collection of ongoing studies designed to investigate the impact of Team-Based Learning (TBL) in calculus instruction on student learning. The first study involves the implementation of TBL in Calculus I and II. Initial findings suggest that TBL students have larger score gains on the Calculus Concept Inventory (CCI) than students receiving traditional instruction. However, there seems to be a gender gap as women tended to have smaller CCI gains than men. The second and third studies investigate the transfer of calculus to major courses, one by asking calculus content questions in subsequent major courses and the other through the educational setting of first-year student Learning Communities.

Key words: [Calculus, Team-Based Learning, Instructional Design, RUME]

We present a collection of studies conducted at Iowa State University on the impact of Team-Based Learning (TBL) in calculus on student learning. The project addresses three research questions: (1) Is TBL more effective than traditional instruction in Calculus I and II?, (2) Does calculus instruction using TBL promote transfer of calculus knowledge to downstream major courses?, and (3) Do calculus enrichment activities in first-year student Learning Communities (LCs) promote transfer of calculus knowledge?

In Fall 2015, three members of our group taught Calculus I in large and small lecture settings using Larry Michaelsen's TBL approach. This teaching strategy based on a constructivist learning theory involves students first engaging with introductory material individually and then at a higher level in teams (Hrynchak & Batty, 2012). The students do preparatory work outside of class using reading guides and instructional videos before completing a five question assessment individually and then again as a team. The majority of class time is spent working on application exercises in teams. We collected data to determine the impact of TBL on Calculus Concept Inventory (CCI) score gains, midterm exams, final exams, and final grades. In particular, we calculated individual normalized gains for the CCI (Epstein, 2013). The TBL group had a larger normalized gain than the traditional group. We also looked at individual gains by gender and section size. Women tended to have smaller CCI gains than men. The students enrolled in the small (under 50 students) TBL sections had the highest CCI gains. The second highest CCI gains were in the large (over 100 students) TBL sections. The midterm and final exams were not commonly graded but data for Fall 2016 and Spring 2017 will be from uniformly graded exams. Currently (Fall 2016), we are in our second implementation of TBL in Calculus I. We will implement TBL in Calculus II during the 2017-2018 academic year.

The second component of our project involves assessing students' retention and facility with calculus in downstream math, science, and engineering courses. Instructors in these courses will include a calculus based application problem on exams. The third component of our project involves the first-year student LCs. During Fall 2016, students completed discipline specific calculus projects related to concepts currently discussed in their Calculus I classes. To assess near transfer of calculus to STEM courses, we will administer a discipline specific calculus problem to Chemical Engr. and Aerospace Engr. students in Spring 2017. By transfer, we mean the ability to apply knowledge or procedures learned in one context to new contexts, with near transfer occurring when the learning situation is similar to the previous learning situation (Mestre, 2002).

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Analyzing Focus Groups of an Experimental Real Analysis Course: ULTRA

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As part of a larger study, we analyzed focus groups of students discussing their perceptions of an experimental real analysis course. The aim of this course was to teach real analysis to prospective and practicing teachers in a way that improved their future teaching. This poster analyzes data from four focus group interviews from 20 students after they completed the experimental course. The majority of comments from the participants' comments about the course, both in general and in regards to informing their future teaching of secondary mathematics, were favorable. We present commonalities in the participants' responses.

Keywords: secondary mathematics, focus groups, real analysis, teacher preparation

Rationale

Many secondary math teachers are required to complete a substantial number of courses in advanced mathematics. The efficacy of advanced mathematics courses for improving future teaching has been questioned (Darling-Hammond, 2000; Monk, 1994). In particular, many prospective and practicing secondary teachers see little to no value in the advanced mathematics courses that they take (e.g., Goulding, Hatch, & Rodd, 2003; Ticknor, 2012; Wasserman et al, 2015; Zazkis & Leikin, 2010).

In our current work, we helped develop a real analysis course, ULTRA (Upgrading Learning for Teachers in Real Analysis), that aimed to be simultaneously faithful to teaching the content of real analysis while also informing teachers' future pedagogical practice. The rationale for this course and sample lessons are described in Wasserman et al. (in press). We implemented this course with 32 prospective and in-service teachers in the Spring of 2016; 20 of these teachers agreed to participate in a focus group interview on the efficacy of this course and how this course can be improved.

Research Questions

The research question addressed in this proposal is: What were students' perceptions of this innovative real analysis course? To what extent did they find this course relevant to their teaching and why?

Method of Analysis

We answer these questions by exploring participants' utterances regarding the efficacy of the course in (i) learning real analysis, (ii) learning secondary mathematics, and (iii) teaching secondary mathematics, using each participant utterance as the unit of analysis. Analysis is ongoing (there were over 1000 participant utterances), but preliminary work demonstrates that (i) participants found the real analysis course beneficial to their teaching and (ii) participants could describe specific ways that particular modules used in the real analysis course would change or improve their practice. We will elaborate on these themes in our poster.

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Extreme Apprenticeship

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Extreme Apprenticeship is a novel, student-centred teaching method that is designed for teaching large courses with hundreds of students. It is based on Cognitive Apprenticeship. In this poster, we present the Extreme Apprenticeship method and data collected from courses taught with it.

Key words: Extreme Apprenticeship, Cognitive Apprenticeship, student-centred

Extreme Apprenticeship (XA) is a student-centred teaching method for organising instruction in an effective and scalable manner. The method was originally created for teaching university-level computer programming (Vihavainen, Paksula, & Luukkainen, 2011), and later employed in university mathematics education (Hautala, Romu, Rämö, & Vikberg, 2012; Rämö, Oinonen, & Vihavainen, 2016). Its theoretical background is in situated view on learning and Cognitive Apprenticeship (Collins et al., 1991). The focus is in supporting students in becoming experts in their field by having them participate in activities that resemble those carried out by professionals (Hautala et al., 2012; Rämö et al., 2016).

In XA, teaching consists of weekly tasks given to the students, course material, guidance and lectures. For each course, there is a teaching team whose members are the responsible teacher and undergraduate/graduate teaching assistants. The key elements are instructional scaffolding and continuous bi-directional feedback. Instructional scaffolding is implemented by providing constant support in the weekly tasks. The teaching team guides the students in drop-in sessions, and the students may spend there as much time as they need to. The members of the teaching team lead the student subtly towards the discovery of a solution through a process of questioning and listening. They model to the students how mathematicians work and thereby help them in gaining the kinds of skills mathematicians need. The drop-in sessions take place in a collaborative learning space that is easy to access for the students.

The tasks have been divided into small and approachable goals, which are then merged together as the students start to master a topic. This enables students to tackle tasks they are not yet able to complete by themselves. The students receive continuous feedback on their work, and at the same time the teachers receive feedback on the progress of the students. Some of the tasks are pre-lecture problems, which force the students to read the course material and prepare for the lectures. This way, the lectures can be made to support discussion and more active participation from the students, and time can be allocated for finding links between the course topics and building a broader picture of the subject.

The regular and close interaction between students and teaching staff guides the students in learning approaches and metacognitive skills used by experts (Burton, 2001). Further, it supports the students to establish relations within the communities of practice which enhances the students' integration into the community (Lave & Wenger, 1991).

Alongside the XA method, we present data collected from courses taught with the XA method. The data indicates that passing rates and student satisfaction have not dropped even though the workload has been significantly increased and the requirement level raised. In addition, students complete more coursework than before introducing the method.

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Hypophora: Why Take the Derivative? (no pause) Because it is the Rate

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***Abstract**: Part of a larger study of the development of teaching among novice college mathematics instructors, this report focuses on one participant, Disha, and her use of a questioning technique called hypophora. At the beginning of the observations, 25% of her questions were hypophora. After video-case based activities during weekly coordination meetings her use of hypophora decreased to about 10% of questions. Although Disha rejected the idea that her teaching had changed in any way, she acknowledged that she began "breaking things into smaller pieces" to help students understand.*

Keywords: questions, calculus, hypophora, professional development

Background. This study examined the teaching of five novice instructors (four graduate student Teaching Assistants (TAs) and one Ph.D. graduate) teaching calculus when their weekly course coordination meetings included video case-based professional development activities. For each instructor, research data gathering included audio and video capture of four class sessions, short surveys about practice, and multiple interviews with the researcher. The poster focuses on one participant, Disha, and how she used hypophora. *Hypophora* are questions that speakers pose and then immediately answer themselves (e.g., "Why would we want to take the derivative? (no pause) So that we can find the critical points for the function."). Disha, who was in her fifth semester teaching, asked an average of 128 questions per class. Most of these (74%) were comprehension checks (e.g., "Is that ok?", "Do you see what I am trying to say?") The next most common question type was hypophora. While it was unusual for students to attempt to answer, in at least one instance a student attempted to answer a hypophoric question. Disha did not acknowledge the student in any visible way and continued talking, answering it herself. The poster will include details about the types and uses of questions noted during instruction.

Influence of Video-case Professional Development. Disha was impressed with the "wait time" of one of the instructors in a video case and seemed to relate this "wait time" to "breaking things into smaller pieces" or scaffolding. Evidence from observation indicated her types and uses of questions changed after this case. Disha rejected the idea that her teaching had changed, though she did state that during her office hours she would "break things into smaller pieces" and wait for an answer when working with students. Kung (2010) observed that TAs learned about student thinking through interacting with students, watching them work problems and listening to them discuss mathematics, as one would during office hours. The influence of the video case may have been indirect: as a moderator of her perception of her own office hour experiences, which were in turn a moderator of her classroom practice. After video case activity, Disha spent more time exploring incorrect answers with students and asked questions of a greater depth. By exploring incorrect answers and asking deeper questions, Disha may have gained further insight into student thinking (Ball, 1997; Fennema et al., 1996; Kung & Speer, 2009).

Discussion Points at the Poster Session. Can hypophoric questions be valuable and in what ways? Is its use related to mathematical discourse? How might language acquisition be connected to and supported by a phase of hypophora use? How might the use of hypophoric questions can be changed to allow students time to think, process, and respond?

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Individual and Group Work with Nonstandard Problems in an Ordinary Differential Equations Course for Engineering Students

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We explore understanding of the Existence and Uniqueness Theorems (EUTs) by a group of engineering students working on nonstandard problems. Students presented three sets of solutions: individual solutions produced in the first tutorial, individual solutions submitted as a homework, and solutions submitted after the discussion with peers in small groups during the second tutorial. The focus of the study is on the role of individual and group work with nonstandard problems. The results show that students gained a deeper understanding of EUTs and appreciated the experience.

Key words: existence and uniqueness theorems, design research, individual work, group work, nonstandard problems.

Description of the Study

The importance of the subject of ordinary differential equations (ODEs) for engineering students is supported by the majority of engineering lecturers (Francis, 1972). It is known that students experience difficulties with ODEs and even with the very concept of a differential equation (Arslan, 2010). It has been emphasized that it is essential to teach engineering students EUTs (Roberts Jr., 1976); an innovative approach in differential equations called the Inquiry Oriented Differential Equations (IO-DE) project has been shown to be effective in developing students' more conceptual understandings of ODEs (Rasmussen and Kwon, 2007). However, for students, understanding and the correct use of the EUTs remain as serious challenges (Raychaudhuri, 2007).

Nowadays EUTs are among very few theoretical results included in standard ODE courses for engineering students. The lecturer in this study devised a set of six nonstandard questions to challenge students' conceptual understanding of the EUTs. The tasks were embedded into the course design and used in two tutorials in the final part of the course when students had acquired sufficient theoretical knowledge and developed good computational skills. Students were requested first to work on the problems individually in the tutorial and at home. One week later the students discussed their solutions in small groups and presented revised solutions to their peers.

Research Questions

1) How can nonstandard questions be used to challenge students, develop analytical skills and further conceptual understanding of important concepts and ideas in an ODE course?
2) To what extent have individual work and group discussions contributed to students' conceptual understanding of the EUTs?

Conclusions

Lecturers should include more nonstandard questions that they know their students will find difficult and may not be able to answer, and do it more often. Our research has shown that students valued the experience, were not distressed by it, and gained a deeper understanding of the EUTs.

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A Topological Approach to Formal Limits Supported by Technology: What Concept Images do Students Form?

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The formal definition of the limit of a function was taught in a first-year calculus course using open intervals and a topological approach. Student understanding was supported with computer-based visualization tools. The concept image framework was used to interpret results of a pilot study in which data was gathered through concept maps and analyzed using categorical content analysis. Results indicate some bridging between students understanding of formal limits presented via open sets and their informal limit conceptions; absolute values inequalities did not appear in the students' concept images.

Key words: formal definition of the limit, calculus

The standard formal definition of the limit of a function includes absolute value inequalities, which are difficult for students to master (*e.g.*, Almog & Ilany, 2012). This hinders student understanding of the concept (Quesada, Einsporn, & Wiggins, 2008). In addition, the standard formal definition is far removed from the initial intuitive approach to limits (Nagle, 2013).

Student understanding of the limit has been studied using the concept image framework (*e.g.*, Tall & Vinner, 1981) to uncover the cognitive structures that students associate with the limit. Results indicate that students have difficulty forming the mental constructs required for full understanding (Cottrill, Dubinsky, & Nichols, 1996; Maharaj, 2010; Tall & Vinner, 1981).

While graphing technologies have been used to support visualization of the formal definition of the limit (Cory & Garofalo, 2009; Verzosa, Guzon, & De Las Peñas, 2014), students nonetheless have difficulty connecting the graphical presentation to the absolute value inequalities in the standard definition (Quesada et al., 2008).

For functions between general topological spaces the formal limit is defined in terms of open sets. When restricted to $\mathbb{R}^1$, open sets correspond to open intervals. A rigorous definition of the limit using open intervals can replace the standard absolute value formulation; absolute value inequalities can be introduced at a later stage as a computational tool.

Open intervals are more easily understood and visualized than their equivalent absolute value counterparts. The potential benefits of an open interval approach, motivated by David Tall (2008), include more effective use of visualization technologies, minimization of the algebraic/visual disconnect, and a bridge between the intuitive notion of "close" and its formal presentation. This prompts the research question: *what concept images of the formal limit are formed by students when it is defined in terms of open intervals and supported by technology-based visualization tools?*

The formal definition of the limit of a function was taught in a first-year calculus course using the outlined approach. Preliminary data was collected from students in the course through structured concept maps (Ritchart & Perkins, 2008) and analyzed using categorical content analysis (Shkedi, 2005; Strauss & Corbin, 1990) to uncover students' concept images. Results indicate some bridging between formal limits presented via open sets and informal limit conceptions; absolute values were not part of students' concept images as revealed by the concept maps. A full study including clinical interviews is in progress. An intriguing question arises: are absolute value inequalities a necessary component of a complete image of the formal definition of the limit?

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Leveraging Research to Support Students' Quantitative and Co-variational Reasoning in an Online Environment

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Quantitative and Co-variational reasoning have been shown to be important facets of a student's mathematical learning. We are proposing an online workbook as a tool for supporting students in reasoning quantitatively and co-variationally. In this poster, we will briefly present a section on understanding graphs as representing the co-variation of two quantities' values.

Key words: Online learning, Quantitative reasoning, Graphing

Introduction and Theoretical Framework

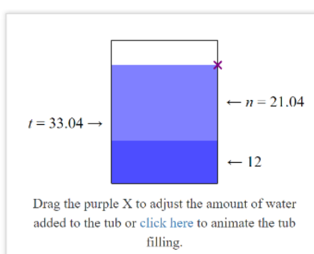
Thompson (1990) has described *quantitative reasoning* to be the analysis of a situation into quantities and relationships between quantities, and Saldanha & Thompson (1998) and Carlson et. al. (2002) have described *co-variational reasoning* to be attending to how two varying quantities change together. Researchers have found that students' ability to reason quantitatively and co-variationally is crucial for them to write formulas and draw graphs to represent how quantities vary together (Moore & Carlson, 2012). However, there has been little research on how modern technology can be leveraged to support students in: reasoning quantitatively and co-variationally, and making connections between different representations of a situation.

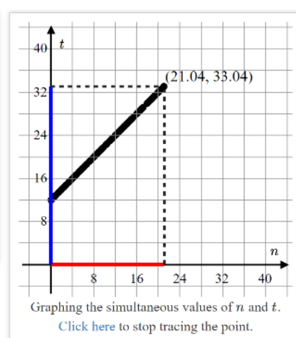
Mathematical Meanings and the Workbook

Thompson & Carlson (2017) have proposed that for students to use *variables* in meaningful ways, they must be able to reason about variables as representing the varying values of quantities. If a student reasons about variables as representing the varying values of quantities, she is in a position to reason about *formulas* as representing the value of one quantity *in terms of* another, i.e. a formula describes how the two quantities are related and vary together. Another form of representing how two quantities are related and vary together is the use of graphs.

As described by Thompson & Carlson (2017), students should reason about a point on a graph as representing the simultaneous values of two quantities at some moment in their co-variation. If one imagines two quantities varying together while tracing the point that represents their simultaneous values, a *graph* emerges that represents their co-variation (Moore & Thompson, 2015, in press).

To support students in reasoning in the aforementioned ways, and making connections between mathematical representations, we have developed online workbook content that contains text, interactive questions, videos, and animated applets. In this poster presentation, we will share select pieces of the online workbook. The image below is a snapshot of one of the animated applets portraying a tub being filled with water, while representing the volume of added water and the total volume of water with variables, and graphically.





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Problem Posing and Developmental Mathematics Students

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Engaging in mathematical problem posing activities can have positive effects on students' mathematical thinking and can advance students' understanding of mathematical concepts. Knowing how underprepared undergraduate students pose problems informs the use of problem-posing activities for helping these students advance their understanding of mathematics as they transition to college-level mathematics courses. Forty-five undergraduate students enrolled in a developmental mathematics course participated in a written problem-posing assessment to describe what underprepared undergraduate students' problem posing looks like. Students' written responses were assessed for whether the response was a mathematical question, whether the responses were solvable, and the connections between each response a student provided. Results of the assessment indicate students at all levels of course performance posed solvable mathematical problems and commonly posed problems by changing the objective for each problem created.

Key words: Problem Posing, Undergraduate Students, Developmental Math, Exploratory Analysis

Posing mathematical problems is an important mathematical activity, playing a foundational role in mathematical problem solving (Polya, 2009), mathematical exploration (Cifarelli & Cai, 2005), and in people's everyday interactions (Kilpatrick, 1987). Engaging in mathematical problem-posing activities can have a positive effect on students' mathematical thinking, such as advancing students' skills at analyzing mathematical problems (English, 1997) and advancing conceptual understanding of operations on fractions (Toluk-Ucar, 2009). As knowledge of mathematics is a mediating factor while posing problems (Silver, Mamonda-Downs, Leung, & Kenney, 1996; Kontorovich, Koichu, Leiken, & Berman, 2012), undergraduate students who are underprepared for college mathematics stand to benefit from participating in problem-posing activities. Engaging these students in problem-posing activities could provide these students an opportunity to advance their mathematical thinking.

Forty-five students enrolled in an undergraduate developmental mathematics course participated in a written problem-posing assessment to investigate what undergraduate developmental mathematics students' problem posing looks like. Participants were given thirty minutes to pose three responses each for four problem-posing tasks, consisting of varying amounts of given information. Participants written responses were analyzed to determine if a response was a mathematical question, if the mathematical question solvable, and the connectedness between responses written for each task. Additionally, participants' final course grades were obtained to examine for differences in problem posing performance based upon level of course performance.

Analysis of students' written responses revealed that undergraduate developmental students at all levels of course performance posed solvable mathematical problems, and students were more likely to pose problems by changing the objective of a previous problem or changing the situational context of another problem than they were to pose problems by varying numerical quantities in each posing situation. These findings suggest that problem-posing activities are an accessible venue for undergraduate developmental mathematics students at all levels of achievement to engage in mathematical thinking.

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Students' Ways of Thinking About Transformational Geometry

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As part of a proof-of-concept project, we created multi-media activities and instructor support materials for secondary mathematics teacher preparation. One focal topic was transformational geometry. Data collection included undergraduate and secondary school students responding to tasks in surveys and in interviews. Despite its prominence in the Common Core State Standards for Mathematics, little is known about how students think about ideas in transformational geometry or about how they engage with items used on assessments for this topic. The poster reports findings on student thinking and invites discussion to inform future work.

Keywords: Transformational geometry, Teacher preparation

Using tasks based on sample CCSSM assessment items, we gathered data from undergraduate and secondary school student via written surveys and interviews to inform answers to the question: What productive and unproductive ways of thinking do students exhibit when working on transformational geometry tasks involving translation, reflection, rotation and dilation? Findings are based on data from 137 written surveys and 12 task-based clinical interviews containing multiple-choice and free response questions modeled after sample CCSSM items. During interviews, to gain insights into student thinking, participants were asked to carry out all transformations described in each of the answer options. In addition to coding responses for correctness, constant comparative analysis was used to identify ways students thought productively and unproductively as they completed the tasks.

Students were successful in selecting the correct sequence of transformations in the multiple-choice question. However, further examination revealed that many who correctly performed the transformation sequence given in the correct answer choice struggled with the transformations described in the other answer choices. This suggests that having students work all answers of a multiple-choice item may be valuable to teachers as a part of formative assessment of students.

We documented unproductive ways of thinking that confirm some noted by others (e.g., Portnoy, Grundmeier, & Graham, 2006; Thaqi & Gimenez, 2012; Yanik, 2011) and augment the set. These include thinking: order of transformations does not matter (and they can be done in the "easiest" order), a reflection "over" the x-axis means the image must remain above the x-axis (a horizontal shift), absence of a stated rotation direction implies a clockwise rotation, and dilation refers only to a shrinking in size. Some students also did not draw intermediate steps in a transformation sequence. On the other hand, many students demonstrated productive ways of thinking, such as performing transformations point-by-point (though this was only productive if students applied the rigid motion to each point correctly) and creating rectangles between points and the origin to help when performing rotations about the origin.

We seek advice from RUME poster session attendees to identify implications for practice and formulate future directions for research. We also hope to solicit focal topic ideas for development of additional activities for pre-service secondary teacher development as well as gather input from college instructors, department chairs, and other stakeholders.

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Linear Algebra Laboratory: Transitioning Between three Worlds of Mathematical Thinking

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The aim of this study is to investigate students' transition between the three worlds of mathematical thinking and the challenges that they face in making these transitions. We anticipate that by creating more opportunities to move between the worlds we will encourage students to think in multiple modes of thinking and hence gain richer conceptual understanding.

Key words: three worlds, linear algebra, transition

Linear algebra is a core subject for mathematics students and is required for many STEM majors. Research on linear algebra has revealed that many students struggle to grasp the more theoretical aspects of linear algebra which are unavoidable features of the course. Linear algebra is made out of many languages and representations. Instructors and text books often move between these languages and modes naturally and rapidly, not allowing students time to discuss and interpret their validities. They assume that students will pick up their understandings along the way, but the linguistic and epistemological studies show how these assumptions are rather deceiving. As Dreyfus (1991, p. 32) declares "One needs the possibility to switch from one representation to another one, whenever the other one is more efficient for the next step one wants to take… Teaching and learning this process of switching is not easy because the structure is a very complex one." We hypothesize that most students do not have the cognitive structure to perform the switch that is available to the expert.

In order to examine the nature of the switch between the representations in linear algebra, we employed Tall's (2013) three-world model of conceptual embodiment, operational symbolism, and axiomatic formalism. In Tall's view, the embodied world involves mental images, perceptions, and thought experiments; the symbolic world involves calculation and algebraic manipulations; the formal world involves mathematical definitions, theories and proofs.

The aim of this study is to investigate students' transition between the three worlds of mathematical thinking and the challenges that they face in making these transitions. As part of the design of this study we have created a set of linear algebra tasks that are specifically crafted to move learners between the worlds. In particular we will examine where students get "stuck" and which direction would be more challenging for students (e.g., embodied to symbolic or symbolic to embodied, etc.). The data will be generated from students' laboratory work. Students will also write a reflection on each lab, specifically describing their thought processes using the 3-world model theory. They need to particularly emphasize if the task helped them to move between the worlds and if yes, how. We will integrate Geogebra, into our teaching materials (slides, applets, etc.) to demonstrate a variety of concepts, and into our lab worksheets to create a geometrically rich environment that will have a significant effect on understanding linear algebra concepts that are difficult to access otherwise. Our working hypothesize is that by creating more opportunities to move between the worlds we will encourage students to think in multiple modes of thinking and hence gain richer conceptual understanding.

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Schema as a Theoretical Framework in Advanced Mathematical Thinking

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In this talk we present a theoretical framework based on Skemp's idea of schema. According to Skemp, concepts are embedded in a hierarchical structure of other concepts, these levels in the structure being classifications of concepts. As the concepts are paired together, relations between them as well as classifications are also possible. The complexity of this hierarchical structure comes from the fact that these classifications of concepts and relations are not unique, giving way to multiple hierarchical structures, which can be interrelated. When components of these conceptual structures come together to make a structure that would not be realized by only looking at the individual components, the resulting structure is called a schema.

Key words: schema, Topology, understanding

Teaching and learning of advanced mathematics topics are often challenging. The aim of this study is to create a theoretical framework based on the notion of *Schema*, in order to investigate undergraduate students' difficulties in Topology.

Skemp (1987) gives a detailed definition of schema in his chapter, "The Idea of a Schema". He describes a system where concepts are embedded in a hierarchical structure of other concepts, these levels in the structure being classifications of concepts. For example, a train can be classified as a mode of transportation and can contribute to one's concept of transportation. There are not only single concepts, but when we pair concepts together, we can have a relation between them, for which a classification is also possible. We can also look at transformations of concepts, which can be combined to make other transformations. What makes this hierarchical structure of concepts, relations, and transformations so deep and complex is the fact that these classifications are not unique, giving way to multiple hierarchical structures, which can be interrelated. When components of these conceptual structures come together to make a structure that would not be realized by only looking at the individual components, we call this resulting structure a schema. Skemp (1987) claims that a schema integrates existing knowledge, serves as a tool for future learning, and makes understanding possible. Without a suitable schema in position, students will have difficulty in understanding or making sense of new concepts. The proposed framework will promote schematic learning and seek to identify whether the presence or absence of a certain schema will have an effect in understanding new knowledge in Topology. Skemp (1987) used Topology for the reason that "the relevant schema can be quickly built up, whereas most mathematical ones take longer." (p. 30)

Piaget and Garcia's (1989) triad framework gives a starting point in developing a suitable schema in Topology:

Intra Stage: Working purely within a definition of a topology; basic examples
Inter Stage: Connecting definition with previously knowledge
Trans Stage: Coherent structure (e.g. Skemp's *understanding*); Viewing a topology as how open sets are defined for a topological space

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Beyond the Exam Score: Gauging Conceptual Understanding from Final Exams in Calculus II

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Instructors often want to evaluate their students' degrees of conceptual understanding in their mathematics courses, but are typically limited to course assignments and exams. In this research we ask: To what extent can mathematics instructors recognize conceptual understanding of their students based on final exam responses? During a summer REU program we examined pre-existing exams along with other course materials to address this question. We developed codes for student responses that were guided by Anderson and Krathwohl (2001), Mejia-Ramos et al. (2011), Thurston (1994), and the APOS framework. Student responses more clearly and often demonstrated lower level understanding than deeper, conceptual knowledge because few problems called for explanations or justifications. The goal of this research was to improve the effectiveness of assessments in evaluating conceptual understanding. In future exams, we suggest that prompting students to display behaviors typical of varying levels of understanding with justification would make evaluations more accurate.

Keywords: Conceptual understanding, Calculus, Exams

Instructors want their students to have a "conceptual understanding" of the topics in their mathematics courses, but do not always know how to find evidence for this. Undergraduate Math Education Researchers have found evidence in in-depth interviews with students, classroom observations, and student responses on tasks created specifically for an educational study. However, an instructor is typically limited to information that can be conveyed through written work such as homework and exams. Thus, we asked: 1. How can we use information about the problem-solving behaviors that students demonstrate when solving final exam problems to make inferences about their degrees of conceptual understanding? 2. How can we create new instructional tools that would provide more evidence for conceptual understanding? We coded responses from 40 final exams randomly selected from four Calculus II sections taught in Fall 2015 at a large, land-grant university. The construction of our codes was guided by the frameworks of Krathwohl (2002), Mejia-Ramos et al. (2012), Thurston (1998), and Dubinsky & McDonald (2001), and an in-depth analysis of first-year calculus. After two rounds of revision, the final codes ordered from lower to higher conceptual understanding were: Execute Computations, Recognize Definitions, Represent Visually, Recognize and Apply Procedure, Recognize Details, Generate Examples, and Analyze Relationships. Success on the exam was largely determined by performance on low-level tasks. Students most often struggled with higher-level behaviors based on error proportions and frequency. Compared to students with lower levels of understanding, the lack of justification in student work made differentiating between medium and higher levels more difficult. Finally, we often coded only one or two behaviors per question, which emphasized the exam's lack of opportunity to demonstrate higher-level understanding. In future exams, designing problems that require varying types of justification, or prompt students to display specific behaviors would make coding more accurate. We would like to expand upon this research to create assignments designed to build conceptual understanding of specific calculus concepts through tasks increasing in conceptual complexity.

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Categorizing Teachers' Beliefs About Statistics Through Cluster Analysis

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The CCSSM emphasize statistical concepts for grades 6-12. The paper will attempt to answer the following question: How do middle school mathematics teachers in a professional development program differ from each other with regard to how they view statistics?

Key words: Statistics Education, Statistics

The emphasis on teaching statistics standards in The Common Core State Standards for Mathematics (National Governors Association, 2010) is much greater than in previous standards (Tran, Teuscher, Dingman, & Reys, 2014). Since beliefs and conceptions influence how teachers teach (Thompson, 1992), it is important to attend to the beliefs that teachers possess about statistics to study how these teachers would teach statistics. In this paper, the researcher will answer the following question: How do middle school mathematics teachers in a professional development program differ from each other with regard to how they view statistics*?* This paper will include analysis on teachers' responses for seven open-ended survey items about their beliefs on statistics.

Methodology

Instruments for assessing statistical beliefs and attitudes tend to be Likert-scaled items with no room for the respondents to elaborate on their answer choices (Gal & Ginsburg, 1994; Gal, 2003). The researcher designed this survey to be open-ended, so that teachers would have the freedom to answer the items as they wished.

The following is the list of the survey items that were given to a population of 50 in-service middle school teachers in the beginning of their second year in a two-year professional development project: a) Briefly give a definition for statistics. What do you take this to mean? What comes to mind when you see or hear the word statistics? b) How would you describe your personal background in statistics? c) To you, what are the differences, if any, between statistics and mathematics? What are the similarities, if any, between statistics and mathematics? d) I am looking forward to statistics content in year two because: e) I am not looking forward to statistics content in year two because. f) I see statistics being useful to my students because. g) I see statistics as not being useful to my students because.

The teacher response data were not ordinal or interval scale in nature, so the researcher used partitioning around the medoid clustering which does not depend on Euclidean distance (Kaufman & Rousseeuw, 1987). The researcher determined the number of clusters by choosing the value for the number of clusters, $k(\leq 6)$, that would maximize the silhouette distance (Charrad, Ghazzali, Boiteau, & Niknafs, 2014).

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Student Understanding of Elements of Multivariable Calculus

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We present preliminary findings from written, post-instruction surveys to gauge student understanding of various elements of multivariable calculus. The content addressed includes contour plots, partial derivatives, representations of gradients and slopes, construction of volume difference integrals.

Key words: Multivariable Calculus, Difficulties

While student understanding of single-variable calculus has received a great deal of attention in the literature, overall few studies have been done on multivariable calculus topics. This area has rich research potential, including characterization of the student understanding landscape as well as how students generalize various concepts such as domain and range (Dorko & Weber, 2014) and integrals (Jones & Dorko, 2015) from single- to multivariable calculus. We present preliminary findings from written, post-instruction surveys to gauge student understanding of various elements of multivariable calculus.

Methods

The written questions discussed here were administered in a survey administered at the end of multivariable calculus classes, typically as part of the final exam. Results come from multiple institutions, with variations in class size and instructional approach. Our findings have not sorted among these variables, and thus are described qualitatively.

The content addressed includes partial derivatives; representations of gradients, slopes, and contours; and volume integrals. The surveys were developed as part of the *Raising Calculus to the Surface* project (Wangberg & Johnson, 2013), which seeks to build a stronger geometric understanding of multivariable calculus using a pedagogically modern approach. Results on student concept images of one-dimensional integration pre-instruction have been reported previously (Fisher *et al.,* 2016).

Results

Here we summarize the most significant findings. Basic computational tasks (partial derivatives, gradients, directional derivatives) yield low success rates, primarily due to what appear to be executive or arbitrary errors (Orton, 1983). A question asking for an estimate of a function value along the direction of the gradient vector demonstrates significant misinterpretation of the graphical representation: students interpret the magnitude of the gradient vector according to the grid spacing. Students are successful interpreting the general meaning of a partial derivative, consistent with prior findings in physics (Thompson *et al.,* 2006). However, results from a subsequent question asking for the meaning of an equation for the numerical value of a partial derivative at a particular point suggest that application of the general meaning is not straightforward. Finally, very few students in this sample are able to set up an integral for the volume enclosed between two surfaces; despite many different approaches to the integral, the most common error is the determination of the correct limits.

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Designing a Richer Flipped Classroom Calculus Experience

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The study presented here is an illustrative example of an action based research project, which was focused on broadening student partition in the flipped classroom experience in order to address issues of equity and social justice in the calculus curriculum. While flipped classrooms have gained recent notoriety within the literature, they rarely incorporate or addresses other critical perspectives. Our study highlights how using design principles such as realistic mathematics education (RME) and culturally responsive pedagogy can effectively target the hidden curriculum and shape the norms and classroom discourse features (Sfard, 2008).

Keywords: Flipped, Realistic Mathematics Education, Culturally Responsive, Calculus

In this action based research, we designed and implemented a two-week classroom teaching experiment for calculus students covering topics in trigonometry and vectors. Based on the current best practices in flipped classroom (Bagley, 2015) we created three different types of instructional videos (*expository, inquisitive and illustrative*) and used the design principles of RME to create the paired classroom activities (Gravemeijer & Doorman, 1999). The *expository* videos introduced the mathematical topic, provided an overview of the content, and made connections to prior mathematical topics. The *inquisitive* videos provided a single mathematical problem that was discussed in length, giving probing questions and possible solution paths, but did not present a final solution to the problem. The problem presented in the *inquisitive* videos was then the focus for the classroom discussion. The *illustrative* videos were designed to show procedural techniques and operations to solve particular problems.

The study took place with 27 students enrolled in a first years course in Mathematics at a Northern Norwegian University. This unique setting provided the opportunity to include design elements to support students with varying degrees of fluency in English, a challenge that is faced by many educators in the United States (Mosqueda & Maldonado, 2013). We draw on several curriculum constructs (formal, observed and hidden) to frame the complete life cycle of this classroom teaching experiment (Stein, M.K., Remillard, J.T., & Smith, 2007). We highlight how our unique role as both the designer of the materials and the teacher of the content allowed for us to use a critical lens to address and shape the underlying norms and beliefs in the classroom through the hidden curriculum (Yackel & Rasmussen, 2003).

For example, we were compelled by culturally responsive pedagogy and RME such that our goal was to have students critically analyze the world they live in, and attend to how mathematics is used within that realm (Aguirre, 2009). We designed the curriculum in this way such that example problems were based in the local context of the town (e.g. using vectors to describe the movement from the local airport to the university) and at a national level (e.g. modeling wolf population zones in Norway). We also wanted students to use mathematics to engage with 21st century issues so we designed a curriculum unit around modeling historical temperature data in Norway to make arguments for climate change. We analyzed features of the mathematical discourse that were made salient within the curriculum (Sfard, 2008) as well as using interview data, to showcase the impact the inquisitive videos may have had on fostering commognitive conflict related to the mathematical topic. Finally, we provide some recommendations, insights and further areas of research based on the outcomes measured in this study.

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Putting on the Uniform: Coordination within the Calculus Curriculum

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The study presented here examines the types and relative frequency of uniform course components (exams, textbooks, etc.) currently in place in the Precalculus through single variable calculus sequence at graduate universities and how those components are effected by the presence of department factors such as regular course meetings, instructor type, and the presence of a course coordinator. Our results indicate that while the total number of uniform course components decline throughout the Precalculus through single variable calculus sequence, its effect is mitigated by the presence of a course coordinator and regular course meetings. In addition, student success is significantly related to the presence of both a course coordinator and regular course meetings.

In this current study, we provide a comprehensive report regarding the coordinated course structures in place to support university students in introductory mathematics sequences across the country – both in terms of uniform course elements and departmental factors. Bressoud, Mesa and Rasmussen (2015), during their initial study examining the characteristics of successful programs in college calculus identified that a "system of coordination" such as having a course coordinator or uniform exams in a calculus programs resulted in a powerful net effect promoting student success. While that study's design did not allow for documenting the frequency of such options across the country, the Progress through Calculus (PtC) project does, including precalculus and the entire single-variable calculus sequence.

During phase 1 of the PtC project, a national census survey was distributed to all 330 American mathematics departments that offer a graduate degree in mathematics and closed with a 68% response rate. The survey asked for details (including enrollment data, delivery format, etc.) of all courses in the mainstream Precalculus to Calculus 2 (P2C2) sequence. Mainstream refers to any course in this sequence that is part of student preparation for higher-level mathematics courses. Using the survey data collected from 265 institutions, we identified factors that quantified uniformity and coordination within and across the P2C2 sequences. In this presentation, we focus on coordination within a courses.

Our descriptive results showcase that the only uniform course elements held by a majority of courses in the P2C2 sequence are textbooks and course topics. While still relatively common we observed less uniformity with regards to graded elements (Final Exam, Midterms, and Grading) and even less with regards to course materials (handouts, videos, and graphing calculators). We note that besides the exception of gateway exams, the prevalence of uniform elements decreases between preparation for calculus and calculus 1, and again between calculus 1 and calculus 2. However, this decline in the total number of uniform elements is not significant when controlling for the frequency of course meetings and the presence of a course coordinator.

In addition to presenting the relative frequency of the uniform course components, we also note that regular meetings and instructor type (tenured faculty) were significant predictors for student success, as measured through DFW rates. These finding suggests that having an

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organized structural system in place increases the overall uniform experience for students regardless of the prescribed uniform components.

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** The Project through Calculus PI team consists of Linda Braddy, David Bressoud, Jessica Ellis, Sean Larsen, Estrella Johnson and Chris Rasmussen. Graduate students include Naneh Apkarian, Dana Kirin, Kristen Vroom, and Jessica Gehrtz.*

Who is teaching the Precalculus Through Single-Variable Calculus Sequence and How are they Teaching it?

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It is well documented that the precalculus through single-variable calculus sequence (P2C2) acts as a barrier for many STEM intending students. Students often cite poor instruction as a primary reason for switching out of STEM programs (PCAST, 2012; Seymour & Hewitt, 1997), which leads to questions about what instructors and instruction look like across the country. This poster presents findings from national census survey data collected as part of a larger study, Progress through Calculus (PtC). In particular, we answer: (1) What types of instructors are currently teaching courses in the P2C2 sequence and how prevalent are they nationally? (2) What relationship exists (if any) between instructor type and primary instructional format?

Key words: Instructors, Instructional Practices, Precalculus, Calculus

Sample Results

To address the first question, we consider the types of instructors participants identified as *frequently* teaching courses within the P2C2 sequence. 205 institutions provided detailed course information on instructors of 894 mainstream courses. At these institutions, full-time faculty frequently taught 67.3% of the courses within the P2C2 sequence; tenured and tenured-track faculty frequently taught 60.4% of the courses; part-time teaching faculty, visiting faculty, or postdoctoral researchers frequently taught 42.5% of the courses; and graduate teaching assistants/associates (GTAs) frequently taught 29.1% of the courses. Note that these percentages do not add up to 100%, because more than one type of instructor might frequently teach any given course.

Our second question considers the relationship between instructor type and primary instructional format. Participants identified the primary instructional format for the regular class meetings of each course. The following results are based on the four categories of instructors identified above and four categories of instructional format: lecture, lecture incorporating some active learning, minimal lecture with mainly active learning techniques, or lecture plus computer based instruction. 200 institutions provided information about the instructional format for 881 courses. Of these, 66% were identified as being taught using mostly lecture, 16% used some active learning in tandem with lecture, 2.5% of courses were taught using mostly active learning, and 3.6% of these courses used computer-based instruction alongside lecture. This trend is consistent across instructor type. While all instructor types utilized lecture most often for their courses, when conditioning on instructor type, proportionally GTAs employ lecture less often in their courses than other instructor types. For example, 57.5% of the courses GTAs frequently taught were lecture based while part-time teaching faculty, visiting faculty, or postdoctoral researchers utilized lecture in 71.1% of their courses.

Along with the results reported here, we will present a more detailed description of instructor type and instructional formats for all courses, including findings using similar analysis separated by course type (e.g., Precalculus, Calculus 1) and highest mathematics degree awarded.

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The Effects of Graphing Calculator on Learning Introductory Statistics

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Abstract: Graphing calculators have been used for teaching introductory statistics for decades. They helped students to obtain accurate statistical analysis results. However, heavily relying on graphing calculators may hinder students' understanding of certain statistical concepts such as the normal distribution and p-value. In this study, we focused on the effects of using a graphing calculator on students' conceptual understanding of normal distribution and p-value, and their performance of calculating normal probabilities and conducting a hypothesis test.

Methods

This study included four sections of an introductory statistics course from two instructors. Each instructor taught one section with the use of graphing calculator and another section without graphing calculator. The goal was to test if the average grade was significantly higher from the graphing calculator section (referred to as TI section) than that from the section without using graphing calculator (referred to as non-TI section).

Two quizzes and three final exam questions were included in this study. Quiz one contained one multiple choice question of the conceptual understanding of standard normal transformation, and one calculation question covering the normal probabilities. Quiz two contained one multiple choice question of the conceptual understanding of p-value, and a short answer question covering a hypothesis test of difference in proportions. The three final exam questions were used to test on retention, including two multiple choice questions of the concepts of standard normal transformation and p-value, and a calculation question of hypothesis testing of a single mean. Each quiz was given to both the TI section and non-TI section at the same time.

Mantel-Haenszel analyses were used to analyze the multiple choice questions and 2 by 2 ANOVAs with two independent factors, instructor (instructor one and instructor two) and the use of calculator (TI and non-TI), were used to analyze short answer questions.

Results

The Mantel-Haenszel analysis for quiz one multiple choice question showed that the proportion of correctness was significantly higher for the TI section than that for the non-TI section when testing on the conceptual understanding of standard normal transformation. The Mantel-Haenszel analysis for quiz two multiple choice question showed that there was no significant difference in proportions of correctness between the TI and non-TI sections when testing on the conceptual understanding of p-value. The 2 by 2 ANOVA of quiz one calculation question showed that the average score was significantly higher for the TI section than that for the non-TI section when students did the calculation of normal probabilities. The 2 by 2 ANOVA of quiz two short answer question showed that the average score was also significantly higher for the TI section than that for the non-TI section on performing hypothesis testing. All of the Mantel-Haenszel analyses and 2 by 2 ANOVAs of the final exam questions did not show significant differences between the TI and non-TI sections for retaining the knowledge of standard normal transformation, normal probability calculations, p-value and hypothesis testing.

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Exploring Conceptions of Mathematics: A Comparison of Drawings and Attitudinal Scales

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This study explores beliefs about doing math held by pre-service teachers. Pre-service teachers in a mathematics content course drew pictures of a person doing math. Additionally, modified Fennema-Sherman Mathematics Attitude Scales (FSMAS) (Ren, Green, & Smith, 2016) were administered to the students. The drawings were analyzed using a framework developed from the Farland-Smith (2012) rubric. In addition to exploring beliefs about doing math held by the students as evidenced by the drawings, the study considers the validity of the drawing methodology through a comparison to the FSMAS results.

Key words: Mathematical beliefs, Pre-service teachers, Drawing methodology

Teachers play a key role in helping students form perceptions about mathematics and what it means to do math. A teacher's beliefs can influence the mathematical experiences they have with their students and so can influence the conceptions about math that the students form (Mewborn & Cross, 2007). As teachers can have such impact on the impressions about mathematics held by future generations, and as these impressions may dissuade students away from mathematical disciplines, understanding teachers' conceptions of mathematics is an important area of study. The focus of this current study is to better understand pre-service elementary and middle school teachers' conceptions of doing mathematics. That is, when someone thinks about "doing mathematics," what comes to mind? It could be that very specific applications are imagined or that feelings or emotions are evoked.

To better understand this question, two types of data were collected from 49 students in a capstone mathematics content course for pre-service teachers at a regional university in the southeastern United States. In one data collection phase, participants responded to the prompt, "In the box below, draw a picture of a person doing math. Place as much detail into the drawing as you can." Participants then used several sentences to explain their drawing. In the other data collection phase, students completed a version of the Fennema-Sherman Mathematics Attitude Scales (FSMAS) as modified and validated, for use with lower primary teachers, by Ren, Green, and Smith (2016). The scales used by Ren, Green, and Smith were *Confidence*, *Effectance Motivation*, and *Anxiety*. The collection phases were spaced one week apart.

A framework developed by Wescoatt (2016) will be used to explore the drawings. The framework is a refinement of a rubric developed by Farland-Smith (2012) and modified by Bachman, Berezay, and Tripp (2016). To explore the "doing" of math, each drawing will be analyzed for elements of Action (what is being done), Mathematics (mathematical elements such as symbols), Appearance (the person's physical appearance), Location (a description of surrounding elements), and Affect. Each image will be assigned a score from 1 to 7 for the Affect category, with 1 representing an extremely negative image and 7 representing an extremely positive image. The written explanations will be used to verify interpretations.

One of the drawbacks of the framework is that it has not been completely validated. As a participant draws his or her image, the possibility exists that he or she is merely drawing an image of stereotypical elements that are not reflective of his or her actual beliefs. In an attempt to begin rectifying this shortcoming, a comparison between the participants' FSMAS results and the Affect scores from the drawings will be explored.

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Developing, Implementing, and Researching the Use of Projects Incorporating Primary Historical Sources in Undergraduate Mathematics

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Mathematics faculty and education researchers increasingly recognize the value of the history of mathematics as a support to student learning. There is an expanding body of literature in this area which includes direct calls for the use of primary historical sources in teaching mathematics. The current lack of classroom-ready materials poses an obstacle to the incorporation of history into the mathematics classroom. Transforming Instruction in Undergraduate Mathematics via Primary History Sources (TRIUMPHS) is a seven-institution collaboration that will design, implement, test, and publish curricular materials based on primary historical sources, train approximately 70 faculty and graduate students on their development or implementation, and conduct and evaluation-with-research study. We present an overview of the project, including activities and research to date.

Key words: Primary Sources, History of Mathematics, Meta-discursive Rules, Case Study

Introduction

In addition to the general benefits of inquiry-based learning, particular advantages incorporating primary sources include providing context and direction to the subject matter, honing students' verbal and deductive skills through reading the work of some of the greatest minds in history, and the invigoration of undergraduate mathematics courses by identifying the problems and pioneering solutions that have since been subsumed into standard curricular topics. By working collaboratively to develop Primary Source Projects (PSPs) while training faculty across the country in their use, TRIUMPHS will ensure these materials are robustly adaptable to a wide variety of institutional settings, while simultaneously developing an ongoing professional community of mathematics faculty. Additionally, our evaluation-with-research study will directly contribute to a greater understanding of (a) how student perceptions of the nature of mathematics evolve, (b) the potential of particular PSPs to promote student learning of meta-discursive rules in mathematics, and (c) how to support faculty in developing and implementing this research-based, active learning approach in undergraduate STEM education.

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Online STEM and Mathematics Course-Taking: Retention and Access

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Using survey data and interviews from a large urban university system, this study explores factors that impact student decisions to take math classes online. The results suggest that access to online math courses likely impacts student course taking patterns, with significantly more students taking a different course if their desired math course is not offered online, compared to non-math courses.

Keywords: Online learning, student characteristics, access.

In 2013, over 40 million college students took online classes worldwide; by 2017, that number is expected to triple (Atkins, 2013). Online courses are often thought to provide access to education, particularly for student parents, and students who work, but little research has been conducted to explore how the availability of online courses impact student course-taking patterns (Jaggars, 2011).

Research question: When online sections of a particular course are not offered, do students take the same course face-to-face, and does this differ for math classes?

Methodology: This research used a sample of students who enrolled in online or comparable face-to-face courses at one of the two- or four-year colleges at the City University of New York (CUNY) from 2004 to 2017. These results focus on survey results for those students enrolled in spring 2016, for a total sample size of 14,689. In addition, 49 interviews were conducted. Student responses to survey and interview questions were coded using grounded theory (Glaser & Strauss, 1967). For student responses to survey questions related to their enrollment, z-scores were calculated.

Results: What factors impact student decisions to enroll in online math classes? In interviews, students expressed different perceptions explaining whether and why they preferred to take math classes online or face-to-face. Many students specifically felt that it was better to take math classes face-to-face: *"Some courses can be done online... The complicated courses, you have to be there—like math—to ask questions directly. Like math—you can't miss a day, and you really need a face-to-face course for something like that… something where you have to read and remember and take a test, you know that's okay for [an] online course."*

But other students had the opposite opinion, feeling that math was particularly well-suited to online learning: *"[I took] math [online] as opposed to English, where I can truly benefit from class discussion. Math is what it is. So I figured that would be a good course to take [online]."*

Even more striking, almost half of all students in an online math course would have taken a different course if that course had not been offered online; this is significantly higher compared to students who took other subjects online (see Table 1).

	all	math	
different course	37%	46%	***
* $p<0.05$, ** $p<0.01$, *** $p<0.001$; Significance calculated using two-tailed z-test for proportions			

Table 1: Student reports of alternative course selection if the course had not been offered online

Students who took math classes online often needed them to complete a specific degree, so if the class had not been offered online, this could have been a barrier to degree completion.

"[If the computer math class had not been offered online that semester], I don't know—[I would have taken] any other math—calculus or algebra—online. [If I had not been able to take an online math class that semester,] I wouldn't be able to finish my college during the fall."

Limitations: CUNY's highly diverse, urban population of low-income, foreign-born, and first-generation students may not be generalizable to other less-diverse college populations.

Conclusion: These results suggest that students take different courses if the math class that they are interested in is not offered online. Delaying developmental math and progressing more slowly through course sequences are both associated with college dropout (Attewell, Heil, & Reisel, 2012; Fike & Fike, 2012). Because of these patterns, colleges should be cautious about restricting access to online math courses or offering insufficient math course sections online to meet student demand.

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Service-Learning and a Shift in Beliefs about Mathematical Problem Solving

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During academic service-learning experiment, students in an experimental Precalculus class regularly tutored basic algebra to middle-schoolers. At the end of the quarter, student-tutors demonstrated academic improvement and a shift in beliefs about importance of conceptual understanding in problem solving. These manifested benefits can motivate mathematics departments to implement service-learning as part of academic curriculum.

Key words: Precalculus, service-learning, beliefs, social justice in mathematics education

Academic service-learning combines coursework and community service: students gain or enhance subject knowledge by participating in structured outreach (Hadlock, 2013). Prior studies on math and service-learning showed gains in tutors' subject knowledge, decrease in mistakes (Yurasovskaya, 2016), and reduction of math anxiety (Henrich & Lee, 2011).

Beliefs about mathematics influence student learning (Leder, Pehkonen & Torner, 2002). Our research explores impact of academic service-learning on student beliefs about mathematical problem solving. This work is motivated by the need to improve Calculus preparation and retention (Bressoud, Mesa & Rasmussen, 2015), and the necessity for more research on the effect of service-learning on mathematics education.

Our study explores two research questions: Will students who engage in tutoring algebra pre-requisites to middle-school students demonstrate a significant shift in beliefs about mathematical problem solving? Will students-tutors demonstrate significant improvement in course performance? Following a pseudo-experimental setup of McKnight, Magid, Murphy, and McKnight (2000), control and experimental sections are randomly chosen for the experiment. For theoretical background, we refer to the work of McLeod (1989) on the impact of beliefs on student performance in mathematics.

Methodology

Our study takes place in a Precalculus class at an urban Catholic university: twenty university students tutor algebra to middle-school kids 2-3 hours per week for the duration of the quarter. Control section receives traditional course instruction. At the start and the end of the quarter, students in both sections take a diagnostic test to establish a baseline, and fill out a survey composed of Indiana Mathematics Belief Scales (Kloosterman & Stage, 1992). Exam performance and the shift in attitudes and beliefs are analyzed at the end of the quarter.

Results

Experimental class showed a shift in the average score for the question group "Understanding concepts is important in mathematics", from 22.23 to 24 with t-value 1.78 and p=0.1, along with a change in standard deviation: from 4.07 to 1.8 in pre vs post scores. Control section did not show any statistically significant shift in any of the belief groups. Experimental section made a significant shift on individual items measuring importance of computational skills and of conceptual understanding why a given numerical answer is correct. Initially, experimental section showed lower averages on the diagnostic and exam one: 59% vs 76% in the control section. By exam two and the final, the difference in averages was no longer statistically significant (74% experimental to 76% control).

Positive results of our research can serve to provide departments of mathematics with incentives to implement outreach and service learning as part of academic curriculum.

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Interpretive Reading of Mathematical Propositions for Proving: A Case Study of a Mathematician Modeling Reading to Students During Joint Proof Production.

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Successful proof production in advanced mathematics relies on meaningful apprehension of the to-be-proved proposition, yet we know that this is a challenge for many students. This study examines the ways in which a mathematician-instructor, addressing this issue, models the practice of interpretive reading of mathematical propositions to a pair of students in the context of joint proof-production in a Real Analysis course. Taking a social practice perspective on reading and adopting Sfard's commognitive framework as a theoretical lens, I identify three aspects of the discursive work the expert engages with to demonstrate processes of active meaning-making to students: (1) re-reading of text with grammatical shifts, (2) posing comprehension monitoring questions, and (3) narrative enactment of text.

Keywords: Reading; Proof Production; Expert-Novice Interaction; Real Analysis.

Many students have difficulties understanding to-be-proved propositions (Selden & Selden, 1995). A key challenge is that relevant information is not readily available in the text, and thus interpretive work is required to make propositions sensible for proving (Weber, Brophy, & Lin, 2008). Not enough is known about the processes by which successful proof writers read propositions with understanding, or how such practices can be taught.

Theoretical Framework. This study relies on two central assumptions. First, I posit that reading of advanced mathematical text is a social practice (Rasmussen, Zandieh, King, & Teppo, 2005). Second, following Sfard (2008), I consider mathematical meaning to be constituted by the discursive work within which text is embedded.

A Study of a Mathematician Modeling Reading; Methods, Results and Implications.

Methods. Data for the present study are taken from a larger video corpus collected to investigate the teaching practices of a highly experienced mathematician-instructor of a Real Analysis course. This paper focuses on a single classroom event in which the instructor performs interpretive-reading while assisting a pair of undergraduate students in constructing a proof to a statement about uniform convergence during a review session. The focal episode was analyzed using micro-ethnographic techniques (Erickson, 1992). A stimulated-recall interview with the instructor served as an additional data source to inform analysis.

Results. Three aspects of the instructor's demonstrative reading emerged as important meaning-making devices: (1) repetitive re-reading of text with shifts in the grammatical structure of the natural language within which signifiers are embedded (e.g. "*it*'s a sequence that is bounded" "*they*'re all bounded"); (2) posing of rhetorical, comprehension-monitoring questions (e.g. "And what do we know about it?"); and (3) narrative enactment of text using multi-modal symbolic devices (e.g. "Here are the f_ns ((draws overlapping curves on the board)) ... and they converge to something down here"). In a stimulated recall interview the mathematician-instructor described his performance as "forcing them to read slowly in order to learn to breakdown the sentences. So they begin to understand what's in the question".

Implications. This case study sheds light on the complexity involved in enacting mathematical meaning from text, and can thus contribute to our understanding of the difficulties students experience. The analysis can also help position the practice of interpretive reading, tacitly assumed in current models of proof production (Weber & Alcock, 2004), as an explicit goal of instruction in advanced mathematics courses.

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Mathitude: Precalculus Concept Knowledge and Mathematical Attitudes in Precalculus and Calculus I

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The concept knowledge and attitudes and perceptions about mathematics of students in precalculus and calculus were measured using the Precalculus Concept Assessment (PCA) and Mathematical Attitudes and Perception Survey (MAPS). We found significant differences in these measures among several subgroups, and found correlations between these measures and student success.

Key words: Concept Knowledge, Perceptions, Attitudes, Calculus

There is a pattern of students citing calculus as a major factor in their decision to leave STEM majors (Rasmussen and Ellis, 2013). Industry and government leaders are calling for colleges and universities to understand and address this problem as it has implications in the hindrance of our nation's economy and intellectual power. In this study we have explored students' conceptual knowledge and mathematical attitudes in precalculus and calculus, using two instruments. First, the Precalculus Concept Assessment (PCA) instrument can assess student learning, effectiveness of curricular treatment, and determine student readiness (Carlson, Oehrtman & Engelke., 2010). Second, Mathematics Attitudes and Perceptions Survey (MAPS) instrument can provide information about how well student beliefs of mathematics align with expert beliefs (Code, Merchant, Maciejewski, Thomas & Lo, 2016). Concept knowledge and expert-like attitudes and perceptions about mathematics are both critical components of mathematical expertise. By studying these components, we can help students to reach higher levels of expertise.

Students enrolled in Precalculus and Calculus I during Spring 2015, Fall 2015, and Spring 2016 were invited to participate in the two surveys described above at the beginning of the semester prior to any instruction or opportunity to attend SI. Both surveys were administered online. After the 15 week semester was nearly over, students were asked to complete the same two surveys again. The matched data set used in this study included N = 181 participants which are all students who provided complete data for both pre and post surveys.

We found several interesting results from our study. First, when we examined concept knowledge as measured by the PCA, we found that although Calculus students began the semester with a higher score on the PCA than precalculus students (as expected), at the end of the semester, the precalculus students had caught up to the calculus students, whereas the calculus students PCA scores improved only slightly. Second, we examined the MAPS scores of different subgroups of students. We found that male students were significantly more likely than female students to agree with expert beliefs on mathematics' relation to the real world; and that students who attended a peer-led supplemental instruction (SI) program were more likely to improve their confidence throughout the semester. We also found a weak but statistically significant relationship between PCA scores and agreement with expert opinions on the MAPS instrument, especially with the MAPS subscores of Confidence and Exploration in Problem Solving. Finally, both the post-semester PCA and MAPS scores are positively correlated with higher grades.

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