# Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education

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# Preface

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematics Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its twenty-first annual Conference on Research in Undergraduate Mathematics Education in San Diego, California from February 22 - 24, 2018.

The program included plenary addresses by Dr. Joanne Lobato, Dr. Marcy Towns, and Dr. Juan Pablo Mejia Ramos and the presentation of 158 contributed, preliminary, and theoretical research reports and 95 posters.

The conference is organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education.

The proceedings include several types of papers that represent current work in the field of undergraduate mathematics education, each of which underwent a rigorous review by two or more reviewers:

- Conference Papers are elaborations of selected RUME Conference Reports
- Contributed Research Reports describe completed research studies
- Preliminary Research Reports describe ongoing research projects in early stages of analysis
- Theoretical Research Reports describe new theoretical perspectives for research
- Posters are 1-page summaries of work that was presented in poster format

The proceedings begin with the winner of the best paper award, the paper receiving honorable mention, and the paper receiving meritorious citation; these awards are bestowed upon papers that make a substantial contribution to the field in terms of raising new questions or providing significant or unique insights into existing research programs. These papers are followed by the pre-journal award winner, which was selected based on its potential to make a substantial contribution to the field; this award is limited to authorship teams that only include graduate students, recent PhDs (within 2 years of graduation), and/or mathematicians who are transitioning to mathematics education research.

The conference was hosted by San Diego State Univerisity and the University of California San Diego. Their faculty and students provided many hours of volunteer work that made the conference possible and pleasurable, and we greatly thank them for their support.

Many members of the RUME community volunteered to review submissions before the conference and during the review of the conference papers. We sincerely appreciate all of their hard work.

We wish to acknowledge the conference program committee for their substantial contributions to RUME and our institutions. Without their support, the conference would not exist.

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# E-IBL: An Exploration of Theoretical Relationships Between Equity-Oriented Instruction and Inquiry-based Learning[1,2]

Stacy Brown
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*The purpose of this theoretical report is to further current discussions of the relationships between Equity-Oriented Instruction (EOI) and Inquiry Based Learning (IBL) pedagogies. Specifically, it proposes a framing of Equity-Oriented Inquiry Based Learning (E-IBL) that foregrounds equitable practice, as opposed to viewing equitable practice as a gratuitous outcome of IBL pedagogies. Drawing on data from teaching experiments conducted in IBL-Introduction to Proof courses, the inter-relationships between knowledge, identity and practice (Boaler, 2002), Pickering's 'dance of agency,' Gutiérrez's dimensions of equity, and Bourdieu's notion of habitus, this paper explores why intentional attention towards the critical axis of equity – that which links identity and power – is necessary, if IBL pedagogies are to promote equity.*

*Key words*: Inquiry based learning, equity oriented instruction, identity, agency

## A Question and Some Answers

At the end of the academic term, students in an Inquiry Based Learning (IBL) Introduction to Proof course were asked: *Imagine you are talking to another student and that they've asked you, "What's it like being in an IBL course?" What would you say?* Below are five responses.

> *Student 1*: It may seem that group work may be tedious and unwanting but for [Intro to Proof] it works very well. You get to meet other individuals that interpret the course material differently, which leads to a better understanding of the class. Plus, you may end up making genuine friends. Again, it works well and I don't think I'd be performing as well if I didn't have group members.

> *Student 2*: IBL classes are really, really fun! It makes it so you're not bored with listening to a professor talking for two hours. Although it disturbs nap time in class, you do learn quite a lot more. If you enjoy working in groups and not listening to someone talk for two hours, these types of classes are perfect.

> Student 3: Me gusta. It's fun, engaging and effective. Keeps me involved and learning from my and other people's mistakes. A lot of people can help.

> Student 4: As an introvert, I hate groupwork. I would rather learn based off the book than being forced to talk.

> *Student 5*: I don't like it […]. I feel like I'm thrown to the wolves and told to just figure it out with no guidance to start me off. I think if I had been aware I would be taking an IBL class, I would've tried to do one that wasn't.

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[1] This paper came about, in part, from conversations with Aditya Adiredja, Luis Leyva, and William Zahner. I would like to thank them, Gail Tang, Robin Wilson, Tim Fukawa-Connelly, Darryl Yong, and the Equity Working Group for their feedback during the early stages of this work.
[2] I dedicate this paper to Uri Treisman, for seeing potential in a nearly illiterate, blue collar first-gen kid from LA and for teaching her how to engage in critical inquiry around questions of equity in and out of the mathematics classroom.

These responses are of interest for two reasons. First, the students were enrolled in the same IBL class. Second, the worrisome responses, those of students 4 and 5, were from female minorities. Thus, the comments came from students who are, statistically speaking, the least likely to complete a mathematics degree (NSF, Science & Engineering Indicators, 2016). And, I have shared them here not to argue that the remarks represent the views of all female minorities in IBL Introduction to Proof Courses but rather as a rationale for exploring the theoretical relationships between *Inquiry Based Learning* (IBL) and *Equity-Oriented Instruction* (EOI).

## The Purpose of the Paper

The purpose of this theoretical report is to further current discussions about the relationships between *Equity-Oriented Instruction* (EOI) and *Inquiry Based Learning* (IBL) pedagogies. Specifically, this report proposes a framing of *Equity-Oriented Inquiry Based Learning* (E-IBL) pedagogies that foregrounds issues of equity, as opposed to viewing equity as a gratuitous outcome of IBL. To understand this framing, current characterizations of EOI and IBL are considered and used to explore rationales for viewing IBL as a pedagogy that promotes equity. Then, drawing on data and field notes from teaching experiments in IBL courses, I examine why efforts to co-enact IBL and EOI pedagogies might create tensions and, therefore, dilemmas for instructors. The paper concludes with a framing of E-IBL.

## A Framing of Equity-Oriented Instruction

Over the past two decades, researchers interested in student learning in school contexts have begun to reconceptualize equity in mathematics education. These researchers (Gutiérrez, 2008; Martin, 2009) challenge our practice of "gap gazing" and argue for the *de-essentialization* of disparities in students' academic achievement; i.e., against "the framing of mathematics achievement …(as) a kind of individualistic accomplishment" (Gutiérrez, 2008, p. 361). Indeed, drawing on Bourdieu's notion of habitus[3] (Bourdieu, 1984), researchers are illustrating the ways in which practices of structural exclusion enacted in students' mathematics education function to marginalize working-class and culturally diverse students (Jorgensen, Gates, & Roper, 2014). This marginalization occurs through schooling practices that align with the habitus of some students but not others by requiring the linguistic capital and practices of particular classes.

Working in ways that align with arguments both for de-essentialization and attention to habitus, Boaler (2002a) has sought to describe the situated nature of learning in schools, arguing not only that students' knowledge, practices and identity are inter-related (Figure 1) but that these inter-relationships "constitute the learning experience." This model of the inter-relationships between identity, practice, and knowledge emerged through Boaler's studies of learning in diverse school settings. It posits that one's knowledge is interactively constituted with one's practices. In particular, Boaler found "practices such as working through textbook exercises, in one school, or discussing and using mathematical ideas, in the other, were not merely vehicles for the development of more or less knowledge, they shaped the forms of knowledge produced" (p. 43). Speaking to the different instructional practices employed in schools and students' identities, Boaler notes that direct instruction places the student in a hierarchical relationship with the teacher, where the teacher is the authority and the students are "received knowers" (Boaler, 2002). In contrast, in discussion oriented classrooms students are

[3] The term *habitus* refers to the informal knowledge and skills that are developed through one's socialization "within the family, home, and immediate environment" so that one learns how to "act in and interpret their worlds" (Jorgensen, Gates, & Roper, 2014, p. 223). *Habitus* is sometimes thought of as, 'the way society becomes deposited in persons in the form of lasting dispositions, or trained capacities and structured propensities to think, feel and act in determinant ways, which then guide them" (Wacquant 2004, p. 316, cited in Navarro 2006, p. 16).

called on to engage in acts of interpretation, expression, and agency. Hence, the practices do not promote students' passive acceptance of mathematical ideas but rather called on them to "contribute to the judgment of validity, and to generate questions and ideas." Thus, enacted practices fostered distinct relationships between students' identities and the "knowledge to be taught." Drawing on Wenger (1998), Boaler argued these findings exemplify his claim that "learning transforms who we are and what we can do, it is an experience of identity" (p. 215).

Beyond those practices enacted in the classroom and the ways they involve an inherent positioning of students, one must consider the possibility that the discipline itself can foster particular disciplinary relationships and, therefore, identities. This point was made both by Boaler (2002) and Pickering (1995), who argue learning involves a *dance of agency*: an interplay of human and disciplinary agency. Specifically, while disciplinary agency refers to the ways that established practices and artifacts (e.g., proving practices, linguistic conventions, syntax, etc.) interact with and affect one's work with mathematics, human agency refers to the ideas, symbols, terms and practices humans develop that impact the discipline. Engaging in mathematics therefore involves a dance of agency in which one both asserts agency on the discipline and surrenders to the "agency of the discipline" (Boaler, 2002, p. 49). And, it is by consideration of disciplinary relationships that we see yet again the ways in which the inter-relationships between identity, knowledge, and practice constitute the learning experience.

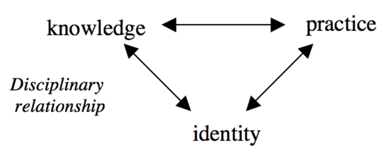


Figure 1. Adapted from Boaler (2002a).

Taken together the works of Gutiérrez (2008), Jorgensen, Gates, & Roper (2014), Bourdieu (1984), Boaler (2002), and Wenger (1998) collectively point to several key tenets of *Equity-Oriented Instruction* (EOI). EOI necessarily disrupts the reproduction of the structural inequities that are shored up and replicated through students' mathematics education. Intentionally, it attends to and broadens the forms of habitus that afford participation in schooling by valuing, among other things, the practices and "linguistic repertoires," that is the social *capital* (Bourdieu, 1984), of those who are further marginalized by schooling (Jorgensen, Gates, & Roper, 2014). It affords the development of identities that enable rather than inhibit participation in the dance of agency and, therefore, students' engagement in authentic mathematical practices. As practices are enacted in discourses (Gee, 2001), EOI requires students be afforded opportunities to engage in work that forestalls the impact of one's social capital while also affording access to rich mathematics. It requires instructors avoid essentializing students while working to provide students with "opportunities to draw upon their cultural and linguistic resources (e.g., other languages and dialects, algorithms from other countries, different frames of reference) when doing mathematics, paying attention to the contexts of schooling and to whose perspectives and practices are 'socially valorized' (Abreu & Cline, 2007; Civil, 2006)" (Gutiérrez, 2009, p. 5).

## A Framing of Inquiry Based Learning Pedagogies

Inquiry Based Learning (IBL) pedagogies have been defined in a variety of ways. Often IBL pedagogies are defined as any form of instruction in which students actively pursue knowledge through activities and discussions (Rasmussen & Kwon, 2007). According to the *Academy for inquiry based learning,* IBL is a "big tent" term for, "Teaching methods in mathematics courses

… where students are (a) deeply engaged in rich mathematical tasks, and (b) have ample opportunities to collaborate with peers (where collaboration is defined broadly)."[4]

IBL pedagogies differ in (at least) two key ways from traditional, lecture-based mathematics instruction.[5] First, curricular activities are often inverted. By this I mean that rather than introducing institutionalized knowledge and having students practice using that knowledge, IBL curricular tasks elicit students' ways of understanding[6] and then through task sequences provide opportunities for students to accommodate their understandings and develop disciplinary practices. The introduction of institutionalized knowledge is the final rather than first step in learning. Second, students are expected and encouraged to act with intellectual autonomy within collaborative settings. In other words, they are called on to engage in specific forms of human agency: (a) generating and proposing problem solving strategies; (b) comparing and contrasting approaches; and (c) justifying and validating solutions. Lastly, it is important to note in relation to these forms of agency, the instructor is expected to elicit, build on, and manage individual student contributions, student-to-student interactions, and whole class discussions.

## Why researchers have argued IBL promotes EOI

The association between active learning and equity has a long and well warranted history. The results of the Treisman (1992) studies demonstrated to many in the mathematics community that opportunities to collaborate around rich mathematical tasks could change the outcomes of students who are disadvantaged by structural inequities. More recently, Freeman et al. (2014) conducted a meta-analysis of 225 studies that compared active learning pedagogies to lecture-based instruction. They found that active learning pedagogies significantly decreased failure rates and that "active learning confers disproportionate benefits for STEM students from disadvantaged backgrounds and for female students in male-dominated fields." In a study that specifically focused on IBL pedagogies, Laursen et al. (2014) found not only did enrollment in IBL classes positively impact student success in subsequent courses but also that the IBL courses reduced the gender gap, with female students not only showing equal or greater learning gains but also higher levels of intention to persist than those in non-IBL courses.

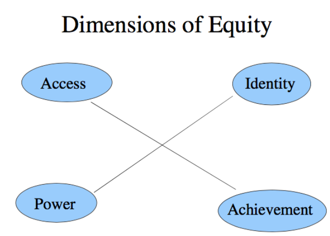


Figure 2. Gutiérrez's (2009) Dimensions of Equity.

Beyond these empirical studies, supports for IBL's potential to promote equitable outcomes can be found in recent theoretical analyses. Tang, Savic, El Turkey, Karakok, Cilli-Turner, and Plaxco (2017) provide a detailed analysis of IBL and its relationship to the dimensions of equity proposed by Gutiérrez (2009) (See Figure 2). Specifically, Tang et al. argue that in collaborative learning environments, *all* students are invited to engage in the "doing, discussing, and

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[4] Retrieved from http://www.inquirybasedlearning.org on July 29, 2017.

[5] Kuster and Johnson (2016) proposed a four-component model of IBL that aligns with that proposed here. Cook, Murphy and Fukawa-Connelly (2016) have proposed a six-component model. Due to space limitations, these models are not discussed in this theoretical report.

[6] I am not using the phrase *ways of understanding* as it is used by Harel (1998, 2005) but rather to collectively refer to students' knowings, their ways of inscribing and their ways of discussing the mathematics at hand.

presenting" of mathematics. The implication here is that IBL pedagogies increase access to rich mathematics, while also promoting achievement (Freeman et al., 2014; Laursen et al., 2014). Thus, IBL pedagogies act along the dominant *access-achievement* axis. Building on the findings of Hassi's (2015) qualitative study, Tang et al. also discuss how collaborative learning environments in which students assert agency, foster growth in self-esteem and self-confidence and, therefore, students' sense of power. Thus, according to Tang et al., IBL pedagogies act not only along the dominant axis but also the critical *identity-power* axis.

## Why IBL might not gratuitously promote EOI

> *… identity has as much to do with others as it does with self … A large part of who we are is learned from how others interact and engage with us.*
> *(Pierson Bishop, 2012, p. 38)*

It is not the purpose of this section to argue that IBL pedagogies do not promote more equitable learning *outcomes* than traditional lecture-oriented pedagogies. Certainly, it would be a fool's errand to do so given recent research (e.g. Laursen et al., 2014). Instead, the purpose is to argue that IBL pedagogies are not necessarily EOI pedagogies and, consequently, do not produce equitable learning *environments* "for free." Instead, intentional attention to equity is required.

To explore the ways in which IBL might fail to function as a form of EOI, I will discuss three data excerpts drawn from field notes and student work samples collected during a series of five IBL teaching experiments. These experiments occurred in IBL-*Introduction to Proof* courses taught at a designated Hispanic-serving university, where the majority are first generation students eligible for need-based financial assistance. The classes were majority-minority classrooms: on average 67% were ethnic minorities and one-third were students who identify as female. Students classified as Hispanic by institutional categories were the dominant minority group, with many preferring the terms Latino/Latina or Chicano/Chicana rather than Hispanic.[7]

### The First Example

The first example is drawn from field notes. It concerns an event of *othering*: viewing or treating an individual as distinct from or alien to oneself or one's group (possibly without intent).

**Field Notes Excerpt.** The class begins with a whole class discussion about the theorems the class will focus on proving that day and a target time for discussing their proofs. Students are asked to move into their small groups, which have been assigned by students counting off the numbers 1 through 7. Mariella[8], a Latina, begins to move her desk towards her group. She stops a few feet short of her group because the other members (three male students) have already moved their desks together and left no space for her desk (as shown in Figure 2). She works quietly on her own, occasionally looking at the male students who do not appear to notice her exclusion.

The instructor observes Mariella's situation for approximately 20 minutes in an effort to provide adequate time for someone, either Mariella or the male group members, to rectify the exclusionary situation. The instructor speaks with Mariella to confirm that the group of three male students is, in fact, her assigned group. Mariella requests of the instructor that she be allowed to work alone. The instructor respects her request, observing that she is uncomfortable. The classroom learning assistant (an advanced undergraduate) is asked by the instructor to check

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[7] Following Gutiérrez (2013), I use the terms Chicano/Chicana to refer to people with indigenous ancestry in the western United States. I recognize its use by students (and researchers) as intentional and political. Hence forth, I will use the gender-neutral terms Chicanx and Latinx.
[8] All names are pseudonyms.

in with Mariella periodically. Several extended mathematical conversations are observed between them. The instructor also checks in with Mariella (it is a 2-hour class so there is time for both to visit without "hovering" around her). After the class, the instructor speaks with two other female students individually and asks each how she would prefer instructors respond in such situations. Unprompted, both of the female students share similar experiences where they were either physically excluded or "invisible" during group work. Both suggest moving Mariella to a group with another female.

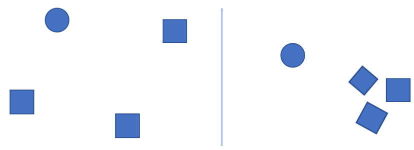



Figure 2. Pre- and post-grouping desk arrangements (Mariella's desk is shown as a circle)

**Post Observation Notes:** The next day Mariella is asked to change her group and, shortly thereafter, observed assisting the other female student. Instances of Mariella actively engaging with her new group while engaging in proving efforts are observed in several subsequent classes.

**Discussion of the Excerpt.** Why is this an instance of IBL not gratuitously promoting EOI? To be certain, some might argue that the students described in the vignette were not engaging in IBL because a central tenet of IBL is collaboration and the students weren't collaborating. There are two issues with this response. First, the male students were collaborating. Second, Mariella had tried to join the group to collaborate but had been excluded. Another critique might center on the fact that the instructor could have remedied the situation by reminding the male students of the participation norms that were discussed extensively at the beginning of the course or that Mariella should have acted to end her exclusion, since participation is an expectation of *all* IBL students. Such responses, however, assume that the tenets of IBL (e.g., collaboration) should be privileged to such an extent that they are enacted in lieu of the tenets of EOI (e.g., practices focused on supporting and valuing students' identities). Instead, they ignore the costs marginalized students pay to participate when students are called on to enforce IBL practices (or be the object of an enforcement) and, potentially, act against their own identities, dispositions, or cultural practices. Moreover, privileging collaboration while ignoring these potential costs does little to mitigate marginalized students' sense of exclusion or the potential for acts of enforcement to create the *illusion of participation*. And it is here that the problem lies. Even if *all* IBL students are expected to advocate for their own participation it is not the case that *all* students are called on to do so (often again and again). More importantly, it is not the case that *all* will have the cultural habitus, disposition, or identity that will support them doing so. Indeed, a post-class discussion with Mariella confirmed after she was publicly othered that she felt extremely uncomfortable "forcing" herself into the group and preferred to work alone.

I will conclude this example with a note of caution: *The purpose of the first example was not to suggest that instructors move othered students to a new group. The purpose was not to suggest a quick fix for the manifestation of exclusionary practices in our institutional spaces.* The purpose was to highlight a classroom situation in which those seeking to co-enact IBL and EOI pedagogies may feel an irreconcilable tension between the two and, necessarily, feel they have to decide whose tenets are privileged.

## The Second Example

LANGUAGE*: a system of verbal symbols through which humans communicate ideas, feelings, experiences. Through language these can be accumulated and transmitted across generations. Language is not only a tool, or a means of expression, but it also structures and shapes our experiences of the world and what we see around us. (Sibley, 2003, p. 1)*

*Mom, how do you say quesadilla in Spanish?*
*-Sebastian, Age 7*

As noted above, EOI requires students be afforded opportunities to engage in collaborative work that supports students' identities through the broadening of the forms of social capital that are recognized and valued. The position taken in this paper is that one's linguistic practices are not secondary to one's identity but rather are an integral component (Bishop, 2012; Gee, 2001, 2005). The extent to which one's language, culture, and practices are valued in an environment determines the extent to which one's identity is valued. Since 1998, Latinx and Chicanx students have had to deal with the educational fallout of California Proposition 227.[9] This proposition codified a stance towards bilingualism that views students' use of non-English languages as a deficit rather than an asset to the students and their communities. It is one of the reasons Californian dialects that heavily integrate Spanish words are often practiced without speakers recognizing their use of another language – a point illustrated by Sebastian's remarks.

Gee (2001, 2005) and Sfard and Prusak (2005) argue that identities are constructed through discourse. Others, such as Bishop (2012), argue discourses "play a critical role in enacting identities" (p. 44). Most who have taught university mathematics courses in environments where the majority of students are first generation urban students can readily attest to the varied and at times colorful dialects used. These languages stand in stark contrast to that employed with relative continuity for thousands of years among the practitioners of the discipline of mathematics, especially when writing proofs. To illustrate this continuity, I ask the reader to consider the resemblance between linguistic practices evident in the two proofs in Figure 3. The first is from *Euclid's Elements* (c. 350 BCE, T.L. Heath's 1909 translation) and the second is from *Mathematical Proofs: A Transition to Advanced Mathematics* by Chartrand, Polimni, and Zhang (2008) (see pp. 145-6). This continuity is highlighted here to provide some evidence for the claim that a disciplinary discourse exists, which members adopt as they are enculturated into the discipline of mathematics.

| | |
|---|---|
| ***I.6 If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.***<br><br>***Proof:*** Let ABC be a triangle having the angle ABC equal to the angle ACB; I say that the side AB is also equal to the side AC. For if AB is unequal to AC, one of them is greater. Let AB be greater; and from AB the greater let DB be cut off equal to AC the less; let DC be joined. Then since DB is equal to AC, and BC is common, the two sides DB, BC are equal to the two sides AC, CB respectively; and the angle DBC is equal to the angle ACB; therefore, the base DC is equal to the base AB, and the triangle DBC is will be equal to the triangle ACB, the less to the greater: which is absurd. Therefore, AB is not unequal to AC; it is therefore equal to it. ■ | ***6.17 For every nonnegative integer n, $3 \mid (2^{2n}-1)$.***<br><br>***Proof***: Assume, to the contrary, that there are nonnegative integers $n$ for which $3 \nmid (2^{2n}-1)$. By Theorem 6.7, there is a smallest nonnegative integer $n$ such that $3 \nmid (2^{2n}-1)$. Denote this integer by $m$. Thus $3 \nmid (2^{2m}-1)$ and $3 \mid (2^{2n}-1)$ for all integers $n$ for which $0 \le n < m$. Since $3 \mid (2^{2n}-1)$ when $n = 0$ it follows that $m \ge 1$. Hence, $m = k+1$, where $0 \le k < m$. Thus $3 \mid (2^{2k}-1)$ which implies that $2^{2k} - 1 = 3x$ for some integer $x$. Consequently, $2^{2k} = 3x + 1$. Observe that $2^{2m}-1 = 2^{2(k+1)}-1 = 2^{2k+2}-1 = 2^2 \cdot 2^{2k}-1 = 4(3x+1) - 1 = 12x + 3 = 3(4x + 1)$ Since $4x+1$ is an integer $3 \mid (2^{2m}-1)$, which produces a contradiction.<br>■ |

Figure 3. Proofs from Euclid (c. 350 BCE; 1909 translation) and Chartrand et al (2008).

[9] After its long and damaging reign, CA Proposition 227 was repealed in the late fall of 2017 after this paper was submitted.

How is this point related to IBL pedagogies and their goals in *Introduction to Proof* classrooms? If our goal in the IBL *Introduction to Proof* classroom is to inquire into the practice of proving and in so doing enculturate students into the discipline of mathematics then our goal is to foster students' growth by developing their awareness of and capacity to engage in disciplinary practices. One of the ways we do this is through our efforts to move students' discursive practices towards disciplinary discursive practices. We argue that students' informal language develops as students work collaboratively, gradually refining their discourse into that of the discipline. We aim to create institutional spaces where the unrefined can interactively be transformed into the refined as students acquire the necessary "cultural dispositions through enmeshment in a cultural community" (Kirshner, 2004; see also Kirshner, 2002). And it is here that a tension between EOI and IBL arises. IBL pedagogies are necessarily unidirectional *enculturation pedagogies* (Kirshner, 2004), which implicitly enact discourse hierarchies by positioning the students' discourses as that which is to be "transformed" and "developed." This point is illustrated in Kuster and Johnson's (2016) discussion of IBL pedagogies:

> 2) *Teachers support formalizing of student ideas/contributions*. In inquiry-oriented instruction, as the students reinvent the mathematics, their reinventions build to be commensurate with formal mathematical ideas. The instructor must be able to promote the students' ability to connect their mathematical ideas to more formal mathematics.

Yet, what of the constitutive role of discourses in relation to students' identities? If discourses are constitutive of identities are we not, in fact, placing students' identities in a hierarchy?

To see how this issue manifests itself in the IBL *Introduction to Proof* classroom, I ask that the reader consider two classroom artifacts. These artifacts were selected because they are examples of the students' own discourses – those which were not taught in but rather brought into the mathematics classroom –and considered in relation to the students' identities. When considering these artifacts, I ask that the reader pay careful attention to his or her initial reactions, thoughts and physical responses.

**Classroom Artifact 1.** The first artifact is from a brief survey administered prior to students' IBL activities focused on standard mathematical logic. The survey was meant to elicit students' ways of thinking and definitions prior to working towards shared definitions that would not only aid communication in the classroom but also develop into the definitions used in the discipline.

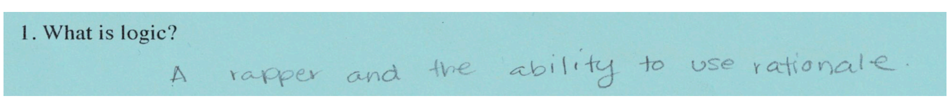
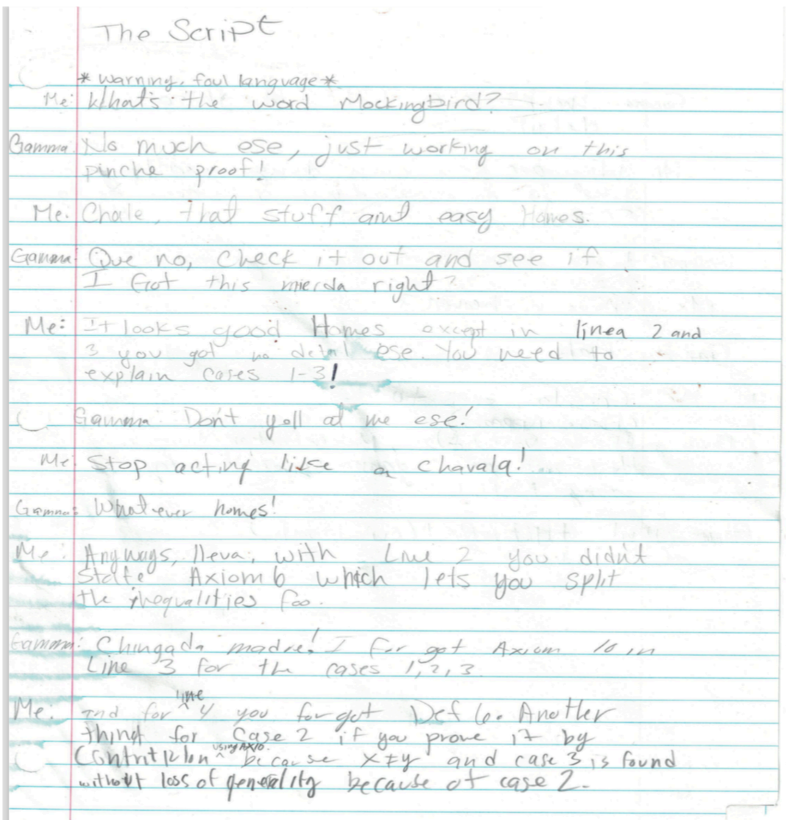



Figure 4. Student Survey Response

In my experience, many laugh when they read this response. Why is it humorous? In the worst-case scenario, it is because the response "a rapper" is viewed as absurd. In the best, it is because we are surprised and entertained by the unexpected. Thus, the laughter is not necessarily a positioning of the student's response in the realm of the ridiculous. It depends. Furthermore, it is possible our laughter is an intended outcome of the student – a purposeful breaching of norms (Herbst & Chazen, 2011). Then again, it may be that the humor was unintentional and that we are witnessing a moment of deliberate candor. It may be that, regardless of the costs that can arise from revealing one's identity through discourse, the student is in fact declaring: "these are my meanings, in my world, in my everyday life." In this case, when the response is shared and laughter is heard, what do students learn about their meanings? And, what do they learn as their meanings are placed in a discursive hierarchy that relegates certain forms of social capital to the realm of that which must be changed?

**Classroom Artifact 2.** The second classroom artifact is a student proof script drawn from a sample of 43 *proof scripts*: written dialogs in which a student and a fictional peer discuss a proof so as to promote the peer's understanding of any gaps or key points in the proof (See Appendix A). This particular proof script (Figure 5) was chosen because it exemplifies one of the many instances of students describing deep mathematics using their own discursive practices and is reflective of the spoken discourses observed in the IBL *Introduction to Proof* course. Indeed, field notes indicate that throughout the IBL *Introduction to Proof* course students grew increasingly accustomed to intensely discussing proofs in their everyday vernaculars.


The Script

*warning, foul language*

Me: What's the word Mockingbird?

Gamma: No much ese, just working on this pinche proof!

Me. Chale, that stuff and easy Homes.

Gamma: Que no, Check it out and see if I Got this mierda right?

Me: It looks good Homes except in linea 2 and 3 you got no detail ese. You need to explain cases 1-3!

Gamma: Don't yell at me ese!

Me: Stop acting like a chavala!

Gamma: Whatever homes!

Me: Anyways, lleva, with Line 2 you didn't state Axiom 6 which lets you split the inequalities too.

Gamma: Chingada madre! I forgot Axiom 10 in Line 3 for the cases 1,2,3.

Me: And for line 4 you forgot Def 6. Another thing for Case 2 if you prove it by Contradiction using Ax10. because $x \neq y$ and case 3 is found without loss of generality because of case 2.

Figure 5. Joseph's Proof Script Excerpt[10]

The point here is that if one looks at the script and expects to see a student's engagement in the valorized discourse – the disciplinary discourse – then one will find a bar that has not been reached. Yet, if we step away from the hierarchy and de-hegemonize the discourses, we see something else. We see an instance of a student *enculturating mathematics*. And in so doing, we can recognize a dialog focused on accounting for mathematical acts that are not only non-trivial for novices but are also valued by the discipline. Take for example the reference to Axiom 6 in the eighth stanza,[11] where we find the student employing the disjunctive definition of ≤ as a warrant for the use of cases – a mathematical act that is non-trivial at this stage of development. Consider also the last stanza. Here we are provided a rationale for why one of the cases is not

[10] At the time of data collection, I was unaware of the potential exclusionary effect of choosing a Greek letter, *Gamma*, as the student name.
[11] The student meant Definition 6, as indicated by subsequent remarks. This typo is not a point of concern in this paper.

needed; namely, that due to similarities in the logic structure we can argue "without loss of generality." And, it is by seeking an enculturated mathematics, rather than a student's enculturation, that we are afforded an opportunity to recognize the pervasive code-switching as exemplifying a student's masterful *blending* of two dialects – the personal and the mathematical – rather than as indicating a lack of participation in implicitly demanded, disciplinary discourse practices. This is *his dance of agency*.

Having witnessed his dance of agency we can ask, like we did with the first artifact, was the unexpected breaching of norms meant to be humorous? Certainly, this is a possibility. It is also possible, however, that humor was not his primary aim but rather was something secondary to asserting agency over that which seeks to place his discourse and, therefore, his identity in a hierarchy. It is because of the latter possibility that I ask again: What do students learn as their discourses (and therefore, their identities) are placed in a hierarchy that relegates their social capital to the realm of that which must be changed during their enculturation into mathematics?

**Discussion of the Artifacts.** The two student artifacts were shared due to their similarity to students' in-situ spoken discourses and in an effort to demonstrate a tension between IBL and EOI. A key tenet of IBL is that students' move towards institutionalized knowledge (and therefore, normative disciplinary discursive practices) through their collaborative activities (Kuster & Johnson, 2016; Cook, Murphy, & Fukawa-Connelly, 2016). It privileges mathematical discourses by calling on instructors to enact discourse hierarchies in lieu of attending to the integral role discourses play with respect to students' identities. Consequently, enacting IBL pedagogies means working to modify students' discourses, as they progress towards a valorized discourse. In contrast, practitioners who privilege EOI practices over those central to IBL must attempt to navigate a precarious balance between respecting students' means of expressing identity and supporting their development of disciplinary discursive practices. Hence, privileging EOI means rejecting discourse hierarchies while simultaneously providing students opportunities to become knowledgeable of disciplinary discourses through their own *dance of agency*. Thus, drawing on Gutiérrez (2009), this paper argues that privileging EOI when enacting IBL means valuing instances when students "change the game" (e.g., by valuing the student's *bridging* of his own and disciplinary vernaculars and his enculturation of mathematics) while also valuing the student's success "playing the game" (e.g., by valuing the mathematical sophistication that underlies the detailed and precise mathematical refinements embedded in the student's dialog). It means creating institutional spaces where *reciprocal enculturation* is possible.

As with the first example, I will conclude this example with a note of caution: *The purpose of the second example and its discussion of the student artifacts was not to suggest that we create classrooms were anything goes. It was not a call to discard disciplinary discourse practices.* Instead, the purpose was to argue for a rejection of discourse hierarchies and to call for forms of praxis that de-hegemonize our institutional spaces. Implementing such praxis is likely to be difficult and requires further exploration. Moreover, such praxis is likely to defy efforts towards lists of "good teaching practices" since, in essence, it is calling for instructors to enact something akin to a *pedagogy of solidarity* (Gaztambide-Fernández, 2012). Consequently, it requires the enactment of practices and perspectives that attend to past histories of colonization and oppression and are, therefore, not only highly situated and temporal, but also critically intertwined with who we (ourselves) are in relation to others.

### The Third Example

*Competence:* the ability to do something well. Synonyms: skill, talent and ability
– Cambridge Dictionary

*I am invisible, understand, simply because people refuse to see me.*
– Ralph Ellison, The Invisible Man

The third example is drawn from field notes. It is like the first example in that it can be interpreted as an instance of othering. It is not, however, an instance of othering through physical exclusion. Instead, it is an instance that brings to our attention the possibility of othering obscured by the illusion of participation. With this said, the purpose is not to illustrate this difference. The purpose is to consider (hopefully with care) the ways in which IBL classroom environments may create spaces where students, who have been marginalized and oppressed through schooling and other facets of our society, can face *demands for competency* – demands that are not experienced by all who enter these spaces.

**Field Notes Excerpt:** The students have counted off, moved their desks, and are working on Propositions 20-25. In the center aisle is a group that has arranged their desks in a triangular formation. All of the students in the group are male, one is Black. His name is Maxwell.[12] He is the only Black male in the class. The instructor checks on the group while circulating around the room and notices Maxwell is not working on the same tasks as the other two in his group. Though from afar the group was observed talking periodically, up close it is observed they are not collaborating around a task but instead occasionally checking in by asking "where are you at?" The instructor encourages the group to work on the same task and reminds them why this is important. The instructor observes that the situation does not change. The students are sitting together but not working together.

**Post Observation Notes:** Throughout the term, reincarnations of the situation are observed, not with every group but in more than one. In some groups, Maxwell and his groupmates collaborate, in others either he is excluded or isolates himself. It is not clear which is the case. Maxwell periodically attends office hours and asks good questions. From the instructor's viewpoint, Maxwell is a good student, he works hard and cares about his academic performance.

**Discussion of Excerpt.** Why is this an instance of IBL not gratuitously promoting equity? The answer to this question is long and I ask that the reader bear with me. As noted in relation to the first example, one might argue that the students described in the vignette were not engaging in IBL because a central tenet of IBL is collaboration and the students weren't collaborating in the ways expected. But rather than rehash issues related to the costs some pay to participate and rather than hypothesize about the potential reasons for the recurring instances of Maxwell's isolation (be they voluntary or involuntary), I ask that the reader consider another issue – one which may be observed in IBL classrooms but may remain unobserved in lectures. In most areas of America, societal structures are in place that lead to people's homes and K-12 schools being separated along racial and socio-economic lines. By experience and habitus people become accustom to separation. One consequence of this separation is that we can fail to grow in ways that foster the dispositions and perspectives necessary to bridge and move past our recurring experiences of separation and societal messages of difference. Placing students in multi-racial (or multi-religion or multi-gendered, etc.,) groups in a university class and expecting students to

[12] All names are pseudonyms.

freely collaborate, therefore, is in essence calling on students, if they have not yet already done so, to spontaneously generate the dispositions and perspectives necessary to bridge their recurring experiences of separation. While we can argue that it is by calling on students to do so that we begin to foster that which is necessary for students to collaborate across societal divides, (and I agree with this claim) we must also recognize that there are grounds for asking, "Is calling on students to do so enough to promote more equitable institutional spaces?"

Those whose habitus and practices have been valorized need not worry about the isolation and/or exclusion of *othered* individuals, for they can rest assured that they will not be excluded in the grand scheme of things. Neither isolation nor exclusion are part of their prevailing storyline. Consequently, changing the prevailing storyline from one of exclusion and isolation to one of inclusion and collaboration often falls on the *othered* individuals. This can take the form of having to prove one's competence in settings where one must overcome harmful stereotypes. While inquiry, discussion, and the sharing of ideas (activities promoted in IBL environments) provide more opportunities for one to demonstrate one's competence than do lectures, they can also heighten the need for some students to exhibit and establish competence if they are to participate. Thus, it can be the case that the demands placed on *othered* students are not the same as those who are not *othered* in our institutional spaces. When recognized, instructors can support *othered* students by "assigning competence" to them (Cohen & Lotan, 1995) but such actions only further the *othered* students' efforts to meet *the demand to establish competency*. These actions do not mitigate the fact that *othered* students face this demand – that they are called on, often again and again, to establish their competence.

Thus, it is by applying an EOI lens that we recognize collaborative learning environments may place different demands on marginalized students than those who are not marginalized. We recognize that the situation described in this excerpt may or may not have been due to racism, while at the same time recognizing that some of our students (our students of color, our religious minorities, our LGBQT+ students, …) are more likely to experience exclusionary situations repeatedly. The playing field is not level. There are forces inside and outside of the classroom that work to maintain this disequilibrium. We can work to mitigate the impact of these forces while at the same time recognizing they are unlikely to go away easily or be susceptible to quick fixes. Moreover, such work may entail allowing a student to opt-out and focus on a different task rather than engage, yet again, in the fight against the forces that continually seek to make some students invisible. Or, it may entail something else. The point isn't to find answers (or quick fixes) but rather to argue that privileging EOI when engaging in IBL rather than assuming IBL is gratuitously equitable may mean enacting a perspective and praxis that does not privilege collaboration at the expense of ignoring the demands some face when asked to "participate."

## A framing of E-IBL

In this paper, I call into question the assumption that IBL pedagogies gratuitously promote EOI and argue E-IBL requires intentional attention to equity. I posit that intentional attention to equity calls on practitioners to employ EOI as a lens when viewing IBL learning environments. Applying such a lens necessarily entails foregrounding issues of structural exclusion and acting to disrupt the social mechanisms that result in their reproduction in institutional spaces (Jorgensen, Gates, & Roper, 2014; Battey & Leyva, 2016). It means privileging students' identities and habitus when IBL practices call on students to act against either; e.g., by valuing varied forms of social capital (e.g., linguistic resources (Zahner & Moschkovich, 2011)) or addressing instances of *othering* by first attending to students' identities and habitus and to the costs and demands some students face. At its core, this framing posits E-IBL instructors must be willing to recognize that, as argued by Wenger (1998), learning is "an experience of identity" and that identity and power are negotiated in institutional contexts (Adiredja & Andrews-Larsen, 2017). Consequently, privileging the demands of EOI over the tenets of IBL, may require instructors to navigate the tensions present in institutional spaces that support students not only "playing the game" but also "changing the game" (Gutiérrez, 2009) by pushing back against the costs and demands they face or asserting agency through acts of *reciprocal enculturation*.

Lastly, the purpose of this theoretical report was to further current discussions of the relationships between EOI and IBL pedagogies by examining the ways in which IBL pedagogies might not gratuitously promote equitable learning environments. Here is it important to note that I have NOT argued that IBL pedagogies are inequitable. To say that the paper argues so is to deeply misinterpret what was written. Instead, the point is to highlight how persistent broader inequities which are an artifact of our societal structures and practices can manifest themselves in the IBL classroom. Indeed, as argued by Tang (2017),[13] "IBL may not be causing these situations but rather may be making them visible." And, it is through our intentional attention that we can see this visibility and begin to question and explore what it means to enact an equity-oriented IBL pedagogy, which attends to the potential tensions between *Equity-Oriented Instruction* and *Inquiry Based Learning*.

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## Appendix A.

**The Script Assignment**

***Instructions*:** You are being asked to create a 1- 2 page dialog between two students, you and a mathematics student named Gamma, who is not enrolled in MATH 310. In the dialog you and Gamma should discuss the proof provided below. Your goal is to help Gamma understand the proposed proof and why it is or is not a valid proof.

***Assignment:***
***Part 1***. Start by reading the proof and identifying what you believe are the "*problematic points*" for a learner when attempting to understand the theorem and its proof. A *problematic point* is anything you think is incorrect, is confusing, or is correct but warrants further discussion. List these "*problematic points*" in a bulleted list.

***Part 2***. Write a dialogue between you and Gamma in which you and Gamma discuss the theorem and its proof. The dialog should address the *problematic points* you identified and listed in your bulleted list through questions posed either by you or Gamma.

***Theorem*:** For any real numbers $x$ and $y$, if $x \leq y$ and $y \leq x$ then $x = y$.

***Proof*:**
1) Assume $x$ and $y$ are real numbers such that $x \leq y$ and $y \leq x$.
2) Then ($x < y$ or $x = y$) and ($y < x$ or $y = x$).
3) We will consider four cases
   Case 1. $x < y$ and $y < x$.
   Case 2. $x < y$ and $y = x$.
   Case 3. $x = y$ and $y < x$.
   Case 4. $x = y$ and $y = x$.
4) In Cases 1 through 3 our assumptions contradict the Law of Trichotomy.
5) We are left with Case 4.
6) Case 4. $x = y$ and $y = x$.
7) Therefore, $x = y$.
8) The result follows. ♦

Self-Regulation in Calculus I

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*Improving STEM retention is a major focus of universities and studies have shown calculus to be a barrier for STEM intending students. Prior to this study, local data indicated students did not pursue STEM fields because they were not passing calculus. In this work, I report on factors that impacted student success in Calculus I. In particular, I examined the relationship between final grades and self-reported self-regulatory aptitudes after accounting for incoming math aptitude. Results indicate self-regulatory aptitudes predict final grades above and beyond math aptitude. In addition, measures of self-regulation differed amongst high and under achievers as well as low and over achievers. This indicates self-regulation plays a role in student success. Furthermore, gender differences were present in measures of self-regulation which may be of importance for improving retention of women in STEM.*

*Keywords:* Calculus, Motivation, Self-Regulation

Calculus I is known to be a barrier to success for students desiring a career in science, technology, engineering, and mathematics (STEM) fields (National Academies of Sciences, Engineering, and Medicine, 2016). Recent national data shows that little more than half of students in calculus I receive a grade of an A or B and DFW rates are around 22-38% depending on the type of institution in which the course is taken (Bressoud, 2015). Of particular concern is the number of women who do not persist into calculus II with 20.1% of females switching their calculus II intention at the end of calculus I while only 11.1% of males switch their calculus II intention (Ellis, Kelton, & Rasmussen, 2014).

Research has correlated student self-regulation with final grades (Pintrich P. R., Smith, Garcia, & McKeachie, 1991). Self-regulation can broadly be defined as the planning, monitoring, controlling, and reflection on one's progress toward a goal in the areas of cognition, motivation/affect, environment, or behavior (Pintrich, 2000). With particular reference to calculus, recent studies have shown self-regulation measures can predict exam scores in Calculus I (Worthley, 2013) and a calculus based engineering analysis course (Hieb, Lyle, Ralston, & Chariker, 2015). This suggests that addressing self-regulation factors may be important aspects of the curriculum that could potentially improve success for some. However, there is a gap in the literature regarding achievement group differences in self-regulatory aptitudes. Prior regression models indicate self-regulation predicts grades above and beyond incoming math aptitude when considering the sample as one group (Hieb, et al., 2015; Worthley, 2013). However, when classified into four achievement groups based on performance relative to the median incoming math aptitude and median final grade (see Figure 1), it is not known if achievement groups report the same type of self-regulation. In a similar study on college chemistry students, Yu (1996) found many differences in self-regulation amongst achievement groups. The same may be true for calculus.

Furthermore, it remains unclear what role gender may play in the relationship between self-regulatory aptitudes and final grades in Calculus I. Prior studies have shown gender differences among self-regulatory aptitude measures (Pintrich & DeGroot, 1990; Zimmerman & Martinez Pons, 1990; Yu, 1996). In addition, although prior studies have shown aspects of self-regulation impacts success after taking into account incoming math ability, there is a gap in the literature regarding if a model of success for males would differ from a model for females. Better understanding of differences in performance according to gender and achievement groups can aid in designing interventions that cater to specific student populations. To address these gaps in the literature, three main research questions guided data analysis for the current study:

1) Are gender differences present in self-reports of self-regulation among students enrolled in Calculus 1?
2) Is there a relationship between final grades and self-regulation according to gender?
3) How do achievement groups differ in their self-reports of self-regulation?

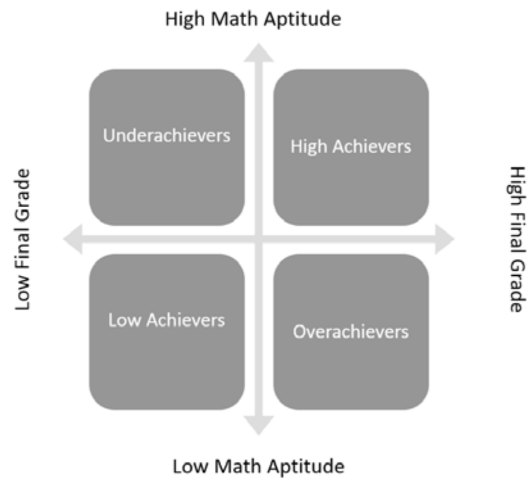


*Figure 1. Achievement groups in calculus. High achievers scored above the median on both math aptitude and final grade. Overachievers scored below the median on math aptitude but above the median on final grade. Underachievers scored above median on math aptitude but below median on final grade. Low achievers scored below median on both math aptitude and final grade.*

## Theoretical Framework and Literature Review

Broadly, self-regulation involves setting a standard or goal, monitoring progress toward the goal, controlling oneself to make adjustments if needed, and reflecting on one's performance (Pintrich, 2004). Self-regulation is rooted in social cognitive theory, examining reciprocal interactions between the individual, their behavior, and their environment (Zimmerman, 1989). Specific to the academic context, Pintrich and Zusho (2007) argue classroom contexts such as academic tasks and instructor behavior impact students' self-regulatory processes which in turn impacts student outcomes. For example, the individual may realize they are not making adequate progress toward their goal leading them to put forth more effort (behavior) or change their study location (environment). Reciprocally, the tasks one is provided to work on (environment) changes the way one cognitively engages with a course and the study strategies and effort (behavior) one engages in, impacts learning.

This study draws upon Pintrich and Zusho's (2007) and Pintrich's (2004) frameworks for self-regulation. Pintrich and Zusho's model places self-regulation within the context of the classroom. They argue students' personal characteristics and the classroom context impact students' motivational and self-regulatory processes. While some self-regulation models consider motivation to fall under self-regulation, Pintrich and Zusho distinguish motivational processes from self-regulatory processes. They argue motivation only becomes self-regulatory when there are active attempts to monitor and control motivation. In Pintrich and Zusho's model, motivational and self-regulatory processes then affect student outcomes. The outcomes feed back into the model to impact future classroom context, motivation, and self-regulatory processes. According to Pintrich and Zusho's model, interventions to alter the classroom context could lead to changes in motivational and self-regulatory processes. However, it must first be understood which motivational and self-regulatory processes are impacting outcomes.

Pintrich's (2004) framework provides a means of examining motivational and self-regulatory processes within categories. Pintrich classifies self-regulation as occurring in four areas: cognition, motivation, behavior, and environment. In addition, he considers self-regulation to occur over four phases: forethought and planning, monitoring, control, and reflection. While Pintrich acknowledges that self-regulation does not necessarily occur linearly through the phases and some aspects of self-regulation don't neatly fit into one area, thinking of self-regulation in terms of phases and areas does allow for distinction among self-regulation processes.

Pintrich's (2004) framework stems from his work developing the Motivated Strategies for Learning Questionnaire (MSLQ). The MSLQ is a questionnaire designed to measure students' course specific self-regulatory aptitudes (Duncan & McKeachie, 2005). The MSLQ has 15 subscales which Pintrich (2004) later mapped onto his classification framework.

In recent years researchers have used the MSLQ to consider the role of self-regulation in success among calculus students. In particular, some studies have attempted to utilize models that predict student success in calculus considering variables such as self-regulatory factors. For instance, Worthley (2013) and Hieb, et al. (2015) used subscales of the MSLQ in their models. Worthley found MSLQ subscales of test anxiety and self-efficacy for learning and performance were good predictors of first midterm grades when combined with math placement test results. Hieb, et al. found that of the select MSLQ subscales administered to their subjects, time and study environment management, internal goal orientation, and test anxiety were good predictors of exam scores. These studies indicate self-regulatory factors play a role in student success and should be examined in more detail.

Furthermore, studies have shown males and females differ in their mathematics interest and self-efficacy beliefs as early as middle school (Pajares, 2005) and the trend continues into college (Pajares & Miller, 1994). In addition, females maintain higher test anxiety than males (Hong, O'Neil, & Feldon, 2005; Pajares & Miller, 1994). Considering these results, it seems plausible that different gender groups may need attention on different areas of self-regulation. Thus it is necessary to examine whether the impact of self-regulation aptitudes on grades vary by gender.

## Method

### Participants

All autumn 2016 Calculus I students at a large Midwestern university were invited to participate in the study. All students, regardless of consent to participate in the study, were given the opportunity to complete all measures. The Calculus Concept Readiness test was a graded quiz assessment for the course. The Motivated Strategies Learning Questionnaire was one of the three surveys given during the course. Students were required to complete all three surveys to earn bonus points. No credit was given to students for consenting to allow their data to be used for the study.

Of the 2539 students enrolled in the course on the 15$^{th}$ day of class, 603 consented to have their data be used in research. Among these 603 students, 29 withdrew from the course. Of the 573 remaining students, 36% ($n$ = 149) of students had missing data leaving a complete data set for 424 students. Of the 424 remaining students, 50.5% ($N$ = 214) were female and 49.5% ($N$ = 210) were male.

### Measures and Procedure

This study consists of five measures: the Calculus Concept Readiness (CCR) assessment (Carlson, Madison, & West, 2015), ACT/SAT Math scores, scores on the Motivated Strategies for Learning Questionnaire (Pintrich P. R., Smith, Garcia, & McKeachie, 1991), and students' final course grades. All sections of the Calculus 1 course were coordinated meaning all students completed the same quizzes, midterm exams, and final exams which minimizes grade differences between sections. The CCR assessment and ACT/SAT math scores were used as a measurement of students' incoming math ability. The MSLQ was used to measure students' motivation and use of cognitive, behavior, and environment self-regulation. Final grades were used as an outcome measure of success in calculus.

**Missing data and insufficient effort response.** Of the 573 students who consented to participate in the study and finished the course, math aptitude scores are missing for 15.9% ($n$ = 91) students due to either a missing ACT/SAT Math score or a missing CCR score. For the MSLQ subscales, scores on each subscale were averaged and thus a student who missed, for example, one question on a subscale, still had a computed average score. However, 9.6% ($N$ = 55) students did not take the MSLQ. An additional 4% ($N$ = 23) students had their MSLQ scores removed due to an insufficient effort response indication of spending less than 3 seconds per question (Kong, Wise, & Bhola, 2007). Overall, examining list-wise missingness, 36% ($n$ = 149) of students have missing data leaving a complete data set for 424 students.

**Calculus Concept Readiness assessment.** The Calculus Concept Readiness (CCR) assessment is a research-based instrument developed with an aim to assess the reasoning and understandings required for Calculus I (Carlson, Madison, & West, 2015). The CCR stands in contrast to many placement exams as the CCR focuses on conceptual understanding rather than procedural knowledge. The assessment consists of 25 multiple choice items which focus on covariational reasoning, the function concept, proportional relationships, angle measure, and trigonometric functions (Carlson, Madison, & West, 2015). The instrument has established reliability as well as internal and criterion validity (Carlson et al., 2015).

The Calculus Concept Readiness assessment was administered to students in the first week of class. Students took the assessment in a timed online environment. While the assessment consists of 25 items, due to error, the first item was omitted leaving an assessment of 24 items.

A Q-Q plot confirmed normality of CCR scores. The average CCR score was $M = 13.37$ ($SD = 3.87$). On average, males ($M = 14.35$, $SD = 3.80$) performed better than females ($M = 12.41$, $SD = 3.71$) on the CCR. This difference, 1.94, BCa 95% CI [1.30, 2.69], was significant $t(422) = 5.32$, $p < .001$ and represents a medium effect size $d = .52$.

**ACT/SAT Math scores.** Student ACT and SAT Math scores were collected from the university's database system. For students with no ACT Math score but with an SAT Math score, SAT Math scores were converted to ACT Math equivalent scores (Dorans, 1999). A Q-Q plot indicated scores are normally distributed. The average ACT/SAT math scores was $M = 29.63$ ($SD = 2.77$). On average, males ($M = 30.08$, $SD = 2.84$) performed better than females ($M = 29.20$, $SD = 2.64$) on the ACT/SAT math test. This difference, .88, BCa 95% CI [.31, 1.42], was significant $t(422) = 3.29$, $p = .001$ and represents a small effect size $d = .32$.

**Math aptitude.** In order to create a single composite incoming math aptitude score, ACT/SAT math and CCR scores were combined. ACT/SAT math and CCR scores were chosen as math aptitude measures due to their correlation with final grade. There was a significant relationship between CCR scores and finals grades, $r = .525$, $p < .001$ as well as between ACT/SAT math scores and final grades, $r = .487$, $p < .001$.

University Math Placement test scores were also considered as an additional component to the incoming math ability score. Math placement test scores were significantly related to final grades, $r = .186$, $p < .001$. However, a hierarchical regression showed math placement test scores did not significantly contribute to the model after accounting for ACT/SAT math scores (Table 1). In the model, ACT/SAT math scores were input as the first step since they are known to be correlated to math grades. Math placement and CCR scores were entered in the second block using forced entry. Due to the non-significant contribution of math placement scores, math placement scores were not included in composite math aptitude score. The CCR scores contributed significantly to the model even after accounting for ACT/SAT Math scores.

*Table 1. Linear model of predictors of final grades, with 95% bias corrected and accelerated confidence intervals.*

| Model | | B | Bootstrap[a] | | | | |
|---|---|---|---|---|---|---|---|
| | | | Bias | Std. Error | Sig. (2-tailed) | 95% Confidence Interval | |
| | | | | | | Lower | Upper |
| 1 | (Constant) | .048 | -.003 | .073 | .509 | -.103 | .190 |
| | ACT/SAT Math | .025 | <.001 | .002 | .001 | .020 | .030 |
| 2 | (Constant) | .106 | -.005 | .066 | .105 | -.034 | .228 |
| | ACT/SAT Math | .016 | <-.001 | .002 | .001 | .011 | .021 |
| | CCR Score | .013 | <.001 | .002 | .001 | .010 | .016 |
| | Math Placement D Score | .001 | .000 | .001 | .226 | .000 | .003 |

a. Unless otherwise noted, bootstrap results are based on 1000 bootstrap samples

The composite math aptitude score was computed by transforming scores on the CCR and ACT/SAT math test into z-scores and then summing the scores. These composite math aptitude scores ranged from -5.24 to 4.00 with an average of $M = .068$ ($SD = 1.65$).

On average, male math aptitude scores ($M = .467$, $SD = 1.65$) were higher than female scores ($M = -.324$, $SD = 1.55$). This difference, .791, BCa 95% CI [.480, 1.095], was significant $t(422) = 5.073$, $p <.001$ and represents a medium effect size $d = .49$.

**Motivated Strategies for Learning Questionnaire.** Students completed 12 of the 15 Motivated Strategies for Learning Questionnaire (MSLQ) subscales during the fifth week of the semester. The MSLQ is a self-report questionnaire designed to measure students' motivation and learning strategy use (Pintrich, Smith, Garcia, & McKeachie, 1991). Three of the original subscales were excluded due to a combination of both low inter-item reliability and final grade correlation as originally reported by Pintrich et al.. All other subscales have reasonable reliability and predictive validity (Pintrich, Smith, Garcia, & McKeachie, 1993). Of the 81 original items on the questionnaire, this left 69 items.

The first section of the MSLQ focuses on student motivations as measured by the subscales of intrinsic motivation, task value, control of learning beliefs, self-efficacy, and test anxiety. Intrinsic motivation is measured by four items such as "In a class like this, I prefer course material that really challenges me so I can learn new things". Students who are intrinsically motivated seek learning opportunities for the sake of learning rather than solely for a grade. Task value is measured by six items and seeks to determine if a student finds the course material interesting and useful through items such as, "I think I will be able to use what I learn in this course in other courses" and, "I like the subject matter of this course". Control of learning beliefs is comprised of four items such as "It is my own fault if I don't learn the material in this class". Students with high control of learning beliefs believe they can succeed through effort. They attribute their success or failures to effort rather than uncontrollable attributes such as innate ability or the instructor. Self-efficacy is measured by eight items asking students about their confidence and belief that they learn the material and do well in the class. Two examples of a self-efficacy items are, "I'm confident I can understand the basic concepts taught in this course" and, "I expect to do well in this class". The final motivation scale is test anxiety. The five items ask students about their anxiety during testing with such questions as, "When I take tests I think of the consequences of failing" or "I have an uneasy, upset feeling when I take an exam". These five motivational subscales assess the inner-person motivational or affective aspects of self-regulation.

The second section of the MSLQ focuses on learning strategy use. These four scales are designed to measure use of cognitive and metacognitive strategies; elaboration, organization, critical thinking, and metacognitive self-regulation. The elaboration subscale is comprised of six questions asking students if they try to make connections between what they are learning and prior knowledge. For example, "I try to relate ideas in this subject to those in other courses whenever possible" or "When I study for this class, I pull together information from different sources, such as lectures, readings, and discussions". The organization subscale's four questions ask students if they attempt to organize their thinking through use of outlines, diagrams, and other organizational techniques. The five items on the critical thinking subscale ask students if they try to think beyond and question what they have learned. Two examples of a critical thinking item are, "Whenever I read or hear an assertion or conclusion in this class I think about possible alternatives" and, "I treat the course material as a starting point and try to develop my own ideas about it". Finally, the metacognitive self-regulation subscale contains 13 items asking if students monitor their understanding. This can take the form of losing focus, "During class time I often miss important points because I'm thinking of other things", or attempts to change strategy when losing focus or not comprehending the material, "If I get confused taking notes in class, I make sure I sort it out afterwards". These subscales focus on the inner cognitive aspects of self-regulation such as monitoring understanding and strategy use.

The second set of learning strategies subscales relates to resource management. The three scales are time and study environment, effort regulation, and peer learning. The time and study management subscale's eight questions asks students if they attend class, keep up with assignments, and study where they can concentrate. The four effort regulation subscales ask students how much effort they put into the class when they find it boring or hard. Finally, the peer learning subscale is comprised on three items asking if they work with their peers when studying. For example, "When studying for this course, I often try to explain the material to a classmate or friend". These subscales address students' attempts to regulate their environment and behavior as they learn.

**Final grades.** Final grades as a decimal percentage were collected from the university's learning management system gradebook. The average final grade was $M = .791$, $SD = .125$. A Q-Q plot gives some indication of non-normality, however, given the large dataset ($N = 424$), the Central Limit Theorem applies. On average, male final grades ($M = .805$, $SD = .126$) were higher than female final grades ($M = .777$, $SD = .124$). This difference, .028, was significant $t(422) = 2.28$, $p = .023$ and represents a small effect size $d = .22$.

## Results

This study examines two phenomena. First, gender differences in motivation and self-regulation self-reports as well as gender differences in how these self-reports impact success in calculus. Second, how self-reports of self-regulation differ amongst achievement groups.

### Gender Differences

A multivariate analysis of variance (MANOVA) was performed to determine gender differences in MSLQ subscale scores. Using a Wilks's Lambda, there was a significant effect of gender on MSLQ subscales, $\Lambda = .800$, $F(12,411) = 8.548$, $p < .001$. The MANOVA was followed up with one-way ANOVAs. Adjusting for Bonferroni's correction, significant differences in gender were found on intrinsic motivation, self-efficacy, test anxiety, critical thinking, organization, intrinsic motivation, and time and study environment (*ps* < .004) (Table 2). Females reported significantly lower intrinsic motivation, self-efficacy, and critical thinking than males. Females reported significantly higher test anxiety, organization, and time and study environment structuring than males.

*Table 2. Univariate effects for gender*

| Dependent Variable | df | df error | F | Sig. | Gender | Mean | 95% Confidence Interval Lower Bound | 95% Confidence Interval Upper Bound |
|---|---|---|---|---|---|---|---|---|
| Intrinsic Motivation | 422 | 1 | 12.543 | <.001 | Female | 4.437 | 4.278 | 4.596 |
| | | | | | Male | 4.844 | 4.683 | 5.004 |
| Task Value | 422 | 1 | 4.829 | .029 | Female | 4.981 | 4.809 | 5.154 |
| | | | | | Male | 5.256 | 5.082 | 5.430 |
| Control | 422 | 1 | .146 | .703 | Female | 5.406 | 5.273 | 5.539 |
| | | | | | Male | 5.443 | 5.309 | 5.577 |
| Self-Efficacy | 422 | 1 | 21.045 | <.001 | Female | 4.841 | 4.671 | 5.010 |
| | | | | | Male | 5.403 | 5.232 | 5.574 |
| Test Anxiety | 422 | 1 | 13.368 | <.001 | Female | 4.545 | 4.349 | 4.741 |
| | | | | | Male | 4.027 | 3.830 | 4.225 |
| Elaboration | 422 | 1 | .221 | .638 | Female | 4.575 | 4.429 | 4.721 |
| | | | | | Male | 4.525 | 4.378 | 4.673 |
| Organization | 422 | 1 | 16.318 | <.001 | Female | 4.479 | 4.312 | 4.645 |
| | | | | | Male | 3.992 | 3.824 | 4.161 |
| Critical Thinking | 422 | 1 | 26.816 | <.001 | Female | 3.221 | 3.053 | 3.390 |
| | | | | | Male | 3.852 | 3.682 | 4.022 |
| Metacognitive Self-Regulation | 422 | 1 | .079 | .779 | Female | 4.564 | 4.453 | 4.675 |
| | | | | | Male | 4.541 | 4.429 | 4.654 |
| Time and Study Environment | 422 | 1 | 10.014 | .002 | Female | 5.488 | 5.376 | 5.600 |
| | | | | | Male | 5.232 | 5.119 | 5.345 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Effort Regulation | 422 | 1 | 1.142 | .286 | Female | 5.726 | 5.595 | 5.856 |
| | | | | | Male | 5.625 | 5.493 | 5.757 |
| Peer Learning | 422 | 1 | .000 | .987 | Female | 3.548 | 3.352 | 3.745 |
| | | | | | Male | 3.546 | 3.347 | 3.745 |

A hierarchical regression was performed in order to determine predictability of final course grade. Math aptitude was entered in the first step. Then all ten MSLQ subscale scores were entered in the second step via forced entry. Finally, gender was entered as the third step. In the first step, math aptitude was a significant predictor of final grades, $R^2 = .352$, $F(1,422) = 228.82$, $p < .001$. In the second step, the MSLQ subscales were added to the model and contributed a significant change in $\Delta R^2 = .136$, $F(12,410) = 9.076$, $p < .001$, for a total model $R^2 = .488$. The third step, entering gender, did not result in a significant change in $R^2$ ($\Delta R^2 = .002$, $F(1,409) = 1.534$, $p = .216$). This final step indicates that after accounting for math ability and MSLQ scores, gender does not significantly predict final grade.

In addition, a secondary analysis was applied to determine if the same hierarchical regression model, with predictors of math aptitude and MSLQ subscales and outcome final grades, works equally well for men and women. To compare the fit of the model, $R^2$ values were compared. The hierarchical regression was run for only males and only females. Then, Fisher's Z-test ($z = 1.45$, $p = .147$) compared the $R^2$ values of the models. The results were not statistically significant indicating that the predictors work equally well for men and women. Next, the structure of the models was compared using Steiger's Z. Using the hierarchical regression model for males, expected female outcomes, final grades, were calculated. Correlations between observed final grades and expected final grades for men and women, using the model based on male data, were then compared using Steiger's Z, ($Z_H = -2.11$, $p = .034$) with correlation between male and female expected grade used as the third correlation in the test. The results of Steiger's Z indicate the structure of the hierarchical regression model is not significantly different for both men and women.

**Achievement Level Differences**

In order to determine how self-regulation may differ amongst achievement groups, students were categorized into four clusters. Students were ranked according to both their math aptitude and final grade scores. Students below the median in math aptitude and final grade were categorized as low achievers. Overachievers were those students below the median in math aptitude but above the median in final grade. Students above the median in math aptitude but below the median in final grade were categorized as underachievers. Finally, students above the median in math aptitude and above the median in final grade were categorized as high achievers (Figure 1).

There were 167 low achievers, 73 overachievers, 67 underachievers, and 176 high achievers in the sample. The average math ability score for low achievers was $M = -1.50$ ($SD = 1.00$) with an average final grade of $M = .6555$ ($SD = .123$). For overachievers, the average math ability score was $M = -.973$ ($SD = .768$) with an average final grade of $M = .8552$ ($SD = .038$). The average math ability score for underachievers was $M = .94$ ($SD = .83$) with an average final grade of $M = .7178$ ($SD = .0762$). For high achievers, the average math ability score was $M = 1.52$ ($SD = 1.12$) with an average final grade of $M = .8893$ ($SD = .0521$).

A multivariate analysis of variance (MANOVA) was performed to determine achievement group differences on MSLQ subscale scores. Using a Wilks's Lambda, there was a significant effect of achievement group on MSLQ subscales ($\Lambda = .699$, $F(36,1209) = 4.336$, $p < .001$). The MANOVA was followed up by post hoc Hochberg's GT2 tests and confirmed with Games-Howell tests. Group differences at a $p < .05$ level are indicated in Figure 2. When comparing to low achievers the post hoc tests indicate both high achievers and overachievers have greater intrinsic motivation, task value, and self-efficacy but lower test anxiety. Only high achievers have greater metacognitive self-regulation and control of learning beliefs than low achievers. When comparing underachievers, both high achievers and over achievers have greater task value, self-efficacy, time and study management, and effort regulation. Only high achievers have greater intrinsic motivation and metacognitive regulation but lower test anxiety than underachievers. At a $p < .05$ level, no statistically significant differences were found between high achievers and over achievers or under achievers and low (Figure 2).

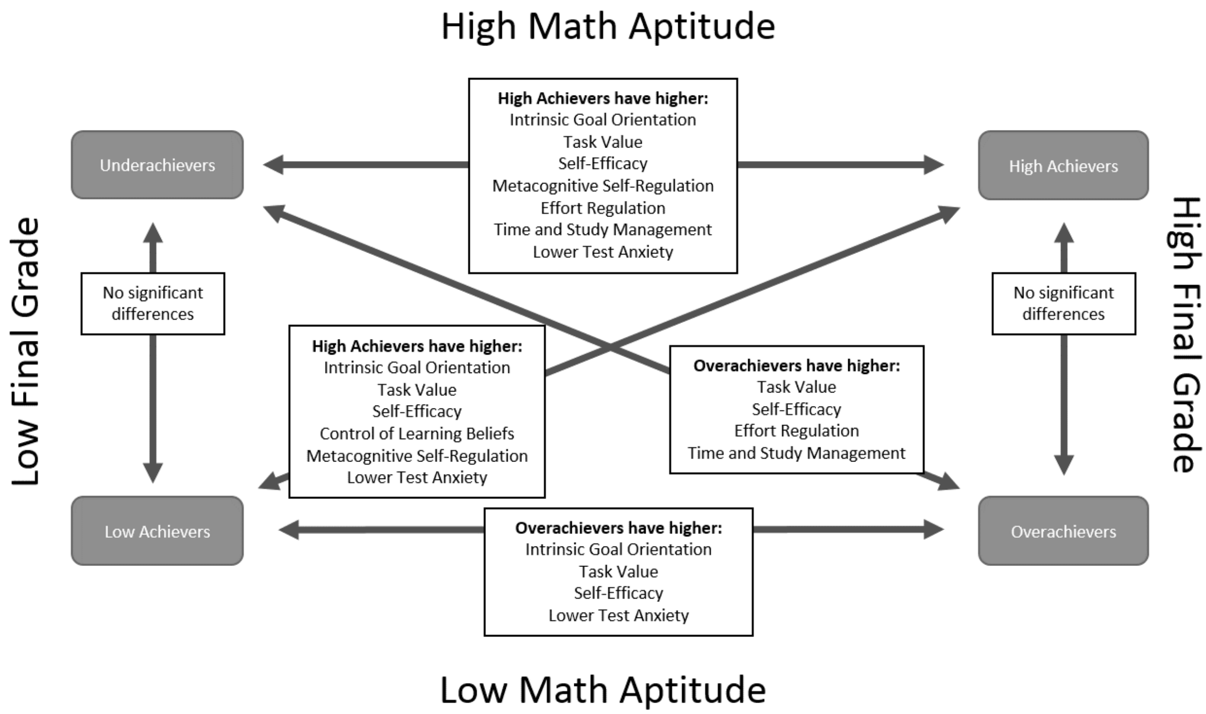


*Figure 2. Differences in MSLQ subscales by achievement group, Group differences significant at p<.05 level.*

## Discussion

Results indicate that the CCR adds significant predictive power when used in combination with ACT/SAT math scores. Combined, these scores can account for 32% of variance in final grades. In addition, adding MSLQ measures of self-regulation, the model accounts for 48% of variance in final grades. This indicates that self-regulation attributes are important for success in calculus I. Incoming math aptitude or pre-requisite knowledge are not enough to ensure success. This assertion is supported by the differences in MSLQ scores amongst achievement groups. While both high achievers and underachievers came in with above median math aptitude, underachievers ended the course with a grade below the median. Differences in self-regulation may account for the underperformance of underachievers as these students differed significantly on several MSLQ subscales compared to high achievers. Furthermore, low incoming math aptitude does not necessarily doom a student to failure. Self-regulation may again play a role as overachievers and low achievers scored significantly differently on several MSLQ subscales.

In particular, the motivational scales seem to play a role in student achievement. The only differences between low achievers and overachievers were in motivational areas. Overachievers reported higher intrinsic motivation, task value, self-efficacy, and lower test anxiety than low achievers. The same subscales, in the same direction, distinguish high achievers from both underachievers and low achievers. However, overachievers only reported higher task value and self-efficacy than underachievers. This indicates that students who are motivated to learn the material for the sake of learning rather than, or at least in addition to, for a grade, do better in the course. In addition, students who see the value in learning the material and how it will relate to other course work or their career perform better. Further, students who believe they can learn the material and believe they can perform well in the class achieve higher grades. Finally, test anxiety plagued both groups who finished the course below the median. However, there was no significant difference between overachievers and underachievers on test anxiety and intrinsic motivation. This indicates that while test anxiety and intrinsic motivation may be important factors for success, they are not the only reason for underperformance.

In fact, effort and time and study regulation may play a larger role in underperformance than anxiety. Both high achievers and overachievers reported higher effort and time and study regulation than underachievers. Students who performed well in the course were able to continue working even when they found assignments boring, plan their time wisely, and find environment conducive to studying. The overachievers scored highest on both the effort regulation and time and study environment subscales followed by high achievers, low achievers, and underachievers. This seems to indicate overachievers may have been able to surpass underachievers via their persistence in the face of boredom, elimination of distractions, and planning of their time. Given that the other significant subscale differences between overachievers and underachievers were task value and self-efficacy, it is possible overachievers persisted because they felt their effort would be fruitful both in terms of their efforts leading to immediate success in the class and the long term benefits of the course.

Finally, metacognitive self-regulation, monitoring one's own understanding, was the only cognitive learning strategy subscale which differed amongst achievement groups. High achievers reported higher metacognitive self-regulation than either underachievers or low achievers. This indicates the ability to accurately monitor one's understanding of the material and then take steps to correct misunderstandings contributes to calculus I success.

One of the most surprising results of this study is the lack of differentiation in learning strategies (elaboration, organization, and critical thinking) amongst achievement groups. These learning strategies are meant to promote deep, conceptual learning in which students take the knowledge they are learning and integrate it into their previous knowledge by making connections, organizing the information, and thinking about whether or not it makes sense given what they already know. The low mean scores for these scales indicates students may not be not be engaging in these learning strategies. This seems to lead to two possible conclusions. Either the coursework does not require students to engage in deeper learning, or the items on the questionnaire do not appropriately measure the constructs of elaboration, organization, and critical thinking as they apply to mathematics and calculus I in particular.

Additionally, organization showed a non-significant yet negative correlation to final grade. Many of the questions related to organization deal with underlining and highlighting of information. It may be the case that students who engage in these types of activities do so because they understand themselves to be weaker students and these are known strategies for learning. On the other hand, students may be engaging in ineffective underlining and highlighting strategies but yet believe themselves to be using productive strategies for learning, which leads to low scores despite what they perceive as effort. This may be particularly troublesome given females reported higher use of organization strategies.

Results show that after taking into account incoming math ability and MSLQ scores, there are no additional significant gender effects on final grade. In addition, math aptitude and MSLQ scores predicts final grades for men and women equally well and the structure of a hierarchical regression model for women is not be significantly different than a model for men. This indicates that gender is not an interaction variable that affects the strength or direction of the relationship between the predictors (math aptitude and MSLQ scores) and final grade.

However, woman entered the course with lower math ability scores than men. This means females are more likely to be below the median in math ability starting them in the low ability or overachiever group. However, females also reported lower intrinsic motivation, self-efficacy, and time and study environment management, as well as high test anxiety than men. Given the results on the relationship between self-regulation and achievement group, this means females are likely to remain in the low achievement group.

## Conclusions

Overall, results show incoming math ability is not enough to guarantee either success or failure. Students' motivations, cognitive learning strategies, and resource management also play a role in their course outcome. As such, any efforts to improve student success in calculus I should address students' self-regulation as well as pre-requisite mathematical understandings. In fact, it is not only students who enter with low math ability who are at risk of failure and need self-regulation addressed. Results indicate the difference in a high math ability students' outcome may also be attributable to self-regulation. Furthermore, addressing self-regulation may be particularly helpful for females whose self-reports are lower than their male counterparts. Efforts to increase women in STEM must take into account self-regulation.

Given that the majority of differences between achievement groups were not in cognitive strategies (i.e. elaboration, organization, critical thinking), addressing cognitive skills or learning strategies, which have often been the focus of self-regulation interventions (Hattie, Biggs, and Purdie, 1996), does not seem warranted. In addition, general study skills courses (e.g. Hofer, Yu, & Pintich, 1998) might not provide students with the domain-specific skills they need. Current research emphasizes the need for curricular integration of self-regulation and classrooms (Boekaerts & Corno, 2005; De Corte, 2000; Schunk, 2005). Therefore, efforts to address self-regulation in calculus should be integrated with the course.

Supplemental instruction models in which self-regulatory needs are addressed directly in relation to the mathematics content (Peterfeund, et al., 2007) may be beneficial. However, these programs may not adequately support underachievers who, according to their incoming math ability, may not believe they need the extra assistance upon entering the course. Furthermore, due to their low effort regulation, underachievers may not put forth the effort to partake in extra programs.

Addressing self-regulation within the classroom may have greater potential to reach underachieving students. For example, Cook, Kennedy, and McGuire (2013) found spending a lecture directly discussing self-regulation in a chemistry course significantly improved final grades for students who attended the lecture. Alternatively, De Corte, et al. (2011) advocate for an even deeper integration of self-regulation into the mathematics classroom creating what they call "powerful learning environments for self-regulation and constructive beliefs" (p. 163).

These powerful learning environments include variation in teaching methods and learning activities as well as a classroom culture that encourages active problem solving and self-regulated learning. Working with elementary school children on mathematics, Schunk's (1998) intervention incorporating these features led to increased achievement beliefs, including self-efficacy. Given self-efficacy was a distinguishing characteristic amongst achievement groups, this type of learning environment shows promise.

**Limitations**

The sample for the study leads to issues of external validity. The sample is from one university, during one semester, during which all students completed the same assignments. In particular, the exams which were 80% of the final grade were identical, only with slight form variations. Additional sampling from other universities would strengthen the conclusions of this study.

In addition, only 603 of the 2539 students in the course consented to participate in research. It is unknown if there was bias in students' self-selection into the study. Additionally, 29.68% ($N = 179$) students in the study were missing data. Withdrawn students were removed so the power of the math aptitude and MSLQ results to predict students to withdraw is unknown. In addition, much of the missing data is due to student lack of completing surveys or insufficient effort in responding to the MSLQ. These behaviors may indicate that a certain subpopulation of students who are unwilling to put forth effort were missing from the study. Given that assessing student effort is part of this study, missing this subgroup could potentially decrease validity.

**Future Research**

One area for future research is understanding why certain subscales do not show strong correlations with final grade. According to self-regulation theory, all of these subscales should be tied to final grades and learning gains. Why is it that the cognitive learning strategy scales have such low correlations with final grade? Future research could couple use of MSLQ subscales in a course with exams known to be conceptual in nature. In addition, future research should examine the MSLQ subscales themselves to determine if they are measuring elaboration, organization, and critical thinking as they apply to mathematics. The MSLQ was designed to measure self-regulation in any subject but it could be the case that mathematics requires skills that are not captured in the general language. The original MSLQ study showed higher final grade correlations on all three subscales when used with students from 14 different subjects. However, none of these subjects were mathematics. It appears as though work may need to be done on domain specific learning strategies.

Finally, how can self-regulated learning be supported in the calculus classroom? There is existing research on interventions for non-domain specific self-regulation instruction as well as interventions for elementary school mathematics (Hofer, Yu, & Pintrich, 1998; Schunk, 1998). Which of these interventions are transferable to the calculus classroom? What additional interventions may be successful? It is not enough to know that motivations, learning strategies, and resource management leads to better grades. The community must be able to intervene with students in order to help them develop these dispositions.

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How Positioning as Teacher or Student May Change Validation of the Same Proofs

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*Recent results such as by Wasserman, Weber, and McGuffey (2017) show the promise of using a particular type of task in undergraduate mathematics courses for secondary teachers: tasks that situate mathematics in pedagogical context. Yet their design deviated from previous recommendations that tasks necessitate pedagogical knowledge; some of their tasks can be solved with purely mathematical knowledge, even if pedagogical knowledge may be beneficial. We examine the phenomenon they observed, that the presence of pedagogical context appears to change the work of a mathematical task. We presented 17 practicing secondary teachers with the same set of proofs to validate, once in the context of teaching secondary mathematics, and then in taking a university mathematics course. We argue that the construct of social positioning – as a student or teacher – explains differences in secondary teachers' proof validations as well as the problem of disconnect between undergraduate mathematics and secondary teaching.*

*Keywords:* Mathematical Knowledge for Teaching, Proof Validation, Secondary Teachers

One enduring question of undergraduate programs for preparing secondary teachers is: how do undergraduate mathematics experiences inform future secondary teaching? As Felix Klein lamented in 1908, for the typical schoolteacher, "university studies remained only a more or less pleasant memory which had no influence upon his teaching" (Klein, 1932, p. 1). Many teachers of this century, particularly secondary teachers, may still concur that undergraduate mathematics do not much impact their current teaching (Goulding, Hatch, & Rodd, 2003; Ticknor, 2012; Zazkis & Leikin, 2010), whether or not they agree that undergraduate studies were "pleasant".

To mend the discontinuity between undergraduate mathematics preparation and teaching practice, mathematics educators advocate for using tasks that situate mathematics in pedagogical contexts. The recommendation spans elementary (e.g., Stylianides & Stylianides, 2010) and secondary levels (e.g., Lai & Howell, 2016; Wasserman, Fukawa-Connelly, Villanueva, Mejía-Ramos, & Weber, 2016). Empirically, pilot programs based on this recommendation show promise, with teachers perceiving undergraduate mathematical content and practices as relevant (Wasserman, Weber, & McGuffey, 2017), and teachers doing mathematics that may not have happened without a pedagogical context (Stylianides & Stylianides, 2010). From these authors' work, along with others (e.g., Biza, Nardi, & Zachariades, 2007), we can conclude that pedagogical context does draw out mathematical reasoning and utility in a way that tasks without pedagogical context do not.

What mechanism lies behind these optimistic results, especially in face of prior findings of teachers' perceptions that their undergraduate preparation is inapplicable to their teaching? One explanation is the process of mathematization, meaning that teachers derive from the pedagogical context a "set of conditions with which a possible solution to a task needs to comply" and reason through these constraints mathematically (Stylianides & Stylianides, 2010, pp. 164–165). Figure 1(a) shows an example of a task featured in their study, on judging the appropriateness of a definition of even number. Doing this task well involves coordinating mathematical reasoning, such as about characteristics of a good mathematical definition, with pedagogical reasoning, such as about what is developmentally or curricularly appropriate for students. It is impossible to do the tasks without bringing in some pedagogical knowledge.

Stylianides and Stylianides argued that the necessity of both mathematical and pedagogical reasoning motivates teachers to see mathematical knowledge as useful for and usable in teaching.

Yet mathematization due to the necessity of pedagogical reasoning cannot explain the results of Wasserman et al. (2017), whose tasks do not require pedagogical reasoning, even if it may be beneficial. A representative task from their study is shown in Figure 1(b), on constructing examples with particular mathematical properties. It is possible for a teacher to respond to this task based on knowledge of the curriculum; however, it is also possible that the teacher can rely strictly on mathematical knowledge accrued through standard secondary and undergraduate coursework. To explain why working on such tasks helped to convince teachers of the utility of undergraduate mathematics, Wasserman and colleagues cited Lobato's (2012) conceptualization of transfer. Wasserman and colleagues argued that tasks that situate mathematical practices in teaching scenarios help teachers see the practices as part of teaching. Consequently, in accordance with Lobato's (2012) theory, Wasserman and colleagues' findings that teachers did knowingly transfer mathematical ideas from undergraduate mathematics to teaching practice was possible because they perceived these mathematical ideas as applicable to their teaching.

(a)
Use the two considerations [of continuity of mathematics in the curriculum, and of intellectual honesty, meaning honest to mathematics as a discipline and honoring of students as learners] to discuss the appropriateness of the following textbook definitions for elementary school students:

(1) An even number is a number of the form $2k$, where $k$ is an integer.
(2) An even number is a whole number that it is a whole number times 2.
(3) An even number is a natural number that is divisible by 2.
(4) An even number is a number that has 0, 2, 4, 6, or 8 in the ones place.
(5) An even number ends in 0, 2, 4, 6, or 8.
(6) An even number is a number that is not odd.
(7) A whole number is even if it is another whole number times 2.

Stylianides and Stylianides (2010, pp. 167–168)

(b)
[Consider each statement, made by a teacher:]

- "The perimeter is just the sum of all side lengths"
- "Remember, to multiply a number by ten, just add a 0 to the end"

Determine for what set(s) of objects the statement is true. If there are any, provide an example of a set(s) of objects for which the statement is not true.

Discuss in what mathematical contexts might the statement by the teacher be appropriate, if ever? When might it be inappropriate, if ever?

Wasserman, Weber, and McGuffey (2017, p. 9)

*Figure 1. Examples of tasks with pedagogical context, from (a) Stylianides and Stylianides (2010) and (b) Wasserman, Weber, and McGuffey (2017).*

The common feature of both the above studies is tasks that place mathematics in the context of teaching. Stylianides and Stylianides contended that for tasks to promote mathematical knowledge as applicable to teaching, solutions "cannot be sought in a purely mathematical space, but rather in a space that intertwines content and pedagogy" (p. 164). However, Wasserman and colleagues' tasks do not satisfy this criterion, while still resulting in teachers seeing mathematical knowledge as applicable to teaching. *The differences in their tasks raises the question of whether the most salient characteristic of tasks with pedagogical context is that they have pedagogical contex*t. In other words, it may be possible that simply placing a task in pedagogical context is enough to change how a teacher perceives the work of the task.

## The Present Study

Our purpose is to investigate the phenomenon that *the presence of pedagogical context changes the work of a mathematics task*. We approach the problem of disconnection between undergraduate mathematics preparation and secondary teaching practice by seeking to understand why being situated in teaching appears to make a difference, even when the task does not necessitate pedagogical reasoning. Throughout this paper, we use *pedagogical context* refer to contextual elements of teaching practice, such as as student talk or curriculum materials. In contrast we use *university context* to refer to tasks that are set in the context of an undergraduate mathematics course, and do not have contextual elements related to teaching. Distinguishing these two contexts explicitly highlights the potential differences in teachers' undergraduate mathematical preparation and the mathematical work of their teaching.

We conducted an interview-based study in which we presented 17 practicing secondary teachers with two parallel tasks, one in the context of teaching high school mathematics, and one in the context of taking a university mathematics course. The prompt for the tasks are shown in Figure 2. Although neither version requires pedagogical reasoning, we hypothesized that the presence of pedagogical context would prompt teachers to activate different resources due to norms and expectations of either teaching or mathematics. Our research questions were: (1) *In what ways does the presence of pedagogical context change teachers' reasoning on a mathematics task?* (2) *What norms and expectations are used in reasoning about these tasks, and in what ways do these norms and expectations relate to the context of learning undergraduate mathematics or teaching secondary mathematics?*

| (a) Pedagogical Context | (b) University Context |
|---|---|
| In a unit on mathematical justification, *you ask your high school students to prove* the following statement: | In a unit on mathematical justification, *your mathematics professor asks you to consider proofs of* the following statement: |
| "When you multiply 3 consecutive numbers, the product is a multiple of 6." | "When you multiply 3 consecutive numbers, the product is a multiple of 6." |
| Below are three responses. Determine whether each *student's proof* is valid. | Below are three responses. Determine whether each *proof* is valid. |

*Figure 2. Tasks used in the present study. Three candidate proofs of the statement were provided to participants, labeled as "Kate's answer", "Leon's answer", and "Maria's answer" in Task (a), and labeled as "1", "2", and "3" in Task (b). The same candidate proofs were presented both times. Differences in the tasks are indicated using italics here; they were presented without emphasis in the study.*

Based on the results of these interviews, we argue that *social positioning* – as a student or as a teacher – can influence teachers' reasoning on a mathematical task. We elaborate two key implications of this phenomenon for research and practice.. The implication for research, particularly on teachers' knowledge of mathematics, is that researchers should be alert to how priming as a teacher education student or as a practicing teacher can shape the way that mathematical knowledge is activated. Researchers must consider the potential effects of positioning in designing protocols and in interpreting the resulting data; the validity of conclusions drawn may depend on how teachers understand the purpose of a task. The implication for the practice of teacher education is that teachers' thinking in mathematics coursework may not predict how teachers' mathematical knowledge for teaching is activated in the classroom, even when the content is directly related to what they teach. Instructors of mathematics courses for teachers should attend to how teachers' responses may prioritize perceptions of what is "expected" in a mathematics course and how these expectations may contrast with expectations for K-12 teaching and the instructor's intended expectations.

Below, we begin with an overview of the construct of position and then discuss why proof validation tasks were a strategic choice for this investigation. We then review literature on teachers' conceptions of proof, focusing particularly on studies where secondary teachers were asked to validate proofs. After discussing our data, methods, and results, we turn to how our finding of the role of position can be used to explain results of prior literature, and we conclude with implications for research and practice.

## Theoretical Perspective

### Position

To interpret responses to the tasks in this study, we turn to the construct of *position*, as it is used in the theory of practical rationality. Herbst and Chazan (2003) introduced the term *practical rationality* to describe the grounds on which teachers' actions can be justified or critiqued. As they observed, the rationality behind teachers' actions "cannot be reduced to individual wisdom, gift, sensibility or skill, since these are common to people who perform the same job; yet they are not all part of the explicit regulations that describe this job" (p. 2); actions in teaching are shaped by combination of sources including teachers' own convictions and knowledge – and also, importantly, the demands that their *instructional system* places on the persons who take on the *position* of teacher and student.

The notions of instructional system and position are both relevant to our work. As Herbst and Chazan conceptualize it, an *instructional system* involves teachers taking student work on a task and assigning value to it in terms of the mathematical knowledge it represents. For example, a student's work on a proof task might represent knowledge of a specific proof technique, ability to cite a particular theorem correctly, or perhaps understanding of what constitutes proof. It is up to the teacher to determine what knowledge the student's work can represent and to what degree.

**Position of teacher.** No matter the instructional system, a person's position is defined by the social rules they must adhere to. Based on empirical work, Herbst and Chazan (2011) argued that the social rules for teachers come from four main obligations: the *discipline* of the subject taught, the *individual* students, the *interpersonal* culture of the class, and the *institution* where the teacher teaches. In brief, the disciplinary obligation demands that a teacher's actions are consistent with the practices of mathematics as a discipline; the individual obligation says that every student has the right to be treated according to their individual being and feeling; the interpersonal obligation asks that the teacher share the class space in culturally appropriate ways; and the institutional obligation demands that the teacher comply with the norms and expectations of their department, school, district, or any larger system such as professional unions.

**Position as student.** While Herbst and colleagues have focused primarily on identifying and describing the practical rationality of teachers, Aaron has built on this work to develop a theory of practical rationality of students. Aaron (2011) argued that being a student is itself a cultural practice and proposed four key obligations of the position of student. Three coincide with Herbst and Chazan's (2011) obligations: *individual*, *interpersonal,* and *institutional*. However, because students are learning the discipline, they are unlikely to be committed to the discipline; however, they are committed to *truth* based on personal notions of true and false.

**Activities concerning proof and their subjectivity**

Mathematical proof is a predominant focus of many undergraduate mathematics courses, including those required for secondary teachers (Conference Board of the Mathematical Sciences, 2012; Tatto et al., 2012). Proof-based courses may also be where secondary teachers feel most disconnected to future teaching (e.g., Ticknor, 2012; Wasserman et al., 2016), despite the centrality of proof and formal reasoning to secondary mathematics (National Council of Teachers of Mathematics, 2000). For our study, tasks with proof activities are a strategic site for studying the influence of pedagogical context because of the potential for connection to teaching, the empirical documentation of disconnection, and also – as we discuss next – the inherent subjectivity of proof activities. Subjectivity is key to this study as it means that judgement can be shaped by a person's norms and expectations, which are shaped by that person's position..

In a review of literature on mathematical proof, Inglis and Mejía-Ramos (2009) organized mathematical activities concerning proof into three categories. These activities are constructing proof; reading proof, consisting of comprehending and validating a purported proof; and presenting proof. As Selden and Selden (2003) argued, validation – determining whether a purported proof of a statement actually establishes that statement mathematically – is "inextricably linked" to constructing proof as well as presenting a proof; both the latter activities involve the ability to determine whether a statement is adequately justified (p. 9).

Understanding the process of proof validation is helpful for understanding mathematics learning and teaching from a socio-cultural perspective. How students and teachers determine what constitutes an acceptable mathematical explanation and justification is at least in part socially negotiated (Yackel & Cobb, 1996). Recent studies have demonstrated the subjectivity of validation across different populations, including mathematicians (Mejía-Ramos & Inglis, 2009; Miller, Infante, & Weber, 2018). These studies call attention to how validation is shaped by how much expertise or competence is attributed to the author who produced the proof. As Weber and Alcock (2005) argued, "Determining whether a warrant would be considered acceptable by the mathematical community may inherently involve a degree of subjectivity" (p. 38).

It has been claimed that mathematical truth is objective, because deductive reasoning is purely analytic (e.g., Hempel, 1945). From this perspective, then, why has proof validation been shown empirically to be subjective? One explanation is how proof validation is operationalized, as observed by Bass (2015). Even if there is such a thing as an ideal proof,

> For mathematical claims of any reasonable complexity, mathematicians virtually never produce complete formal proofs. Indeed, requiring that they do so would cause the whole enterprise to grind to a halt ... Proving a claim is, for a mathematician, an act of producing, for an audience of peer experts, an argument to convince them that a proof of the claim exists. (Bass, 2015, p. 5)

In sum, we hypothesize that validation of proofs with potential gaps, such as those produced by students, would be particularly susceptible to subjectivity. Moreover, reading and validating student work is an inherent part of teaching. For these reasons, we designed the present study around validation of student proofs.

## Literature Review

Prior research has demonstrated that teachers have wide variation in their conceptions of proof. For example, teachers may emphasize different criteria when determining whether or not an argument counts as a proof (Knuth, 2002a), and these conceptions may change when intentionally set in a school context (Knuth, 2002b). Buchbinder (2018) suggested that teachers are likely to bring both mathematical and pedagogical considerations to bear when evaluating proofs. Three factors appear to be particularly salient in influencing teachers' conceptions of and validations of proof: (1) the presence or absence of algebraic notation; (2) the use of verbal representations; and (3) the use of examples in proof.

**Presence or Absence of Algebraic Notation.** Both students and teachers have expressed sensitivity to algebraic notation in proofs in particular contexts. Healy and Hoyles (2000) found that students believed that proofs with algebraic notation would receive the best scores from their teachers, even though it was not the type of proof they would have written for themselves. When asked to predict what their students would value in proof, teachers "appeared to overestimate the extent to which their students would make judgments that were based on mathematical content rather than simply on form" (p. 407).

Algebraic notation can give proofs a recognizable form or structure, and it can also contribute to proofs being perceived as general. Tabach et al. (2011) found that some teachers ascribed more generality to a proof when it was represented using algebraic notation than verbal representations. This emphasis on algebraic notation, expressed by teachers, highlights the implicit values of structure and generality. In the case of the students in Healy and Hoyles' (2000) study, it also highlights their perception that more explanatory proofs were less valued by their teachers, as these students did not perceive the algebraic notation as being helpful in understanding a proof.

Investigating the relative value placed on algebraic notation in different contexts using parallel tasks would shed light on the ways in which these views about algebraic notation may relate to proof validation.

**Use of Verbal Representations.** Despite believing that algebraic notation would earn the highest scores, students may acknowledge their own preference for verbal proofs, which they saw as being more explanatory (Healy & Hoyles, 2000). Yet Tabach et al. (2010) found that teachers tended to treat verbal proofs as "mere examples" and did not necessarily recognize explanatory value that students saw in them. One hypothesis is that teachers found the verbal proofs less transparent than algebraic proofs (Tabach et al., 2011). However, most teachers in Healy and Hoyles' (2000) study selected a verbal representation as the way they would approach a particular proof. In interviews, they expressed that "it was more important that the argument was clear and uncomplicated than that it included any algebra" (p. 413).

These somewhat contradictory results related to the role of verbal representation in proof leads to a question about how context may influence teachers' values related to the purpose of proof, and, in turn, their approach to proof validation.

**Use of Examples.** Teachers and students also attend to the role examples play in proof. The students in Healy and Hoyles' (2000) study found examples to be helpful in developing understanding, even while they recognized them as not sufficient for proof. Buchbinder (2018) investigated both students' and pre-service teachers' perceptions of the role of examples in proofs in the context of proving or refuting universal statements. She found that the pre-service teachers valued student work that tested multiple examples, even while acknowledging the limitations of that approach.

We explore these factors influencing proof validation by engaging in a direct comparison of tasks situated in pedagogical and university contexts. To our knowledge, no existing study of proof validation compares reasoning about the same proofs across different contexts. Such a comparison may shed light on how previous results on proof validation by teachers fit together.

## Data & Method

### Rationale

To determine the ways in which the presence of pedagogical context changes teachers' reasoning on mathematics tasks, we asked teachers to each work on the same set of parallel tasks. One task featured the pedagogical context of teaching secondary mathematics; the other situated the participant as a student in a university mathematics course. We chose to contrast the pedagogical secondary context with a university context because the most recent and intensive context in which secondary teachers experience proofs is as university students. These two contexts thus serve as productive contrasts to inform future work in teacher education.

Figure 3 shows the set of tasks used to address the research questions. The university context could be considered a pedagogical tertiary context; however, we note that the task situates the participant as a student, not a professor. Moreover, responses from our participants indicate that they were reasoning from the stance of student, not university instructor.

| Set of Tasks with Pedagogical Context | Set of Tasks with University Context |
|---|---|
| In a unit on mathematical justification, you ask your high school students to prove the following statement:<br>*When you multiply 3 consecutive natural numbers, the product is a multiple of 6.*<br>Below are three responses. Determine whether each student's proof is valid. | In a unit on mathematical justification, your mathematics professor asks you to consider proofs of the following statement:<br>*When you multiply 3 consecutive natural numbers, the product is a multiple of 6.*<br>Below are three responses. Determine whether each proof is valid. |
| **Kate's Answer**<br>A multiple of 6 must have factors of 3 and 2. If you have three consecutive numbers, one will be a multiple of 3. Also, at least one number will be even and all even numbers are multiples of 2. If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2. | **1.**<br>A multiple of 6 must have factors of 3 and 2. If you have three consecutive numbers, one will be a multiple of 3. Also, at least one number will be even and all even numbers are multiples of 2. If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2. |
| **Leon's Answer**<br>$1 \times 2 \times 3 = 6$.<br>$2 \times 3 \times 4 = 24 = 6 \times 4$.<br>$4 \times 5 \times 6 = 120 = 6 \times 20$.<br>$6 \times 7 \times 8 = 336 = 6 \times 56$. | **2.**<br>$1 \times 2 \times 3 = 6$.<br>$2 \times 3 \times 4 = 24 = 6 \times 4$.<br>$4 \times 5 \times 6 = 120 = 6 \times 20$.<br>$6 \times 7 \times 8 = 336 = 6 \times 56$. |
| **Maria's Answer**<br>$n$ is any whole number.<br>$n \times (n + 1) \times (n + 2) = (n^2 + n) \times (n + 2)$<br>$= n^3 + n^2 + 2n^2 + 2n$<br>Cancelling the n's gives $1 + 1 + 2 + 2 = 6$. | **3.**<br>$n$ is any whole number.<br>$n \times (n + 1) \times (n + 2) = (n^2 + n) \times (n + 2)$<br>$= n^3 + n^2 + 2n^2 + 2n$<br>Cancelling the n's gives $1 + 1 + 2 + 2 = 6$. |

*Figure 3. Parallel tasks for validating mathematical proofs used in this study, based on the TEDS-M released item #MFC709 (TEDS-M International Study Center, 2010).*

### Data Source

**Participants.** We interviewed 17 practicing secondary mathematics teachers who had 1 to 14 years of experience teaching, and who had worked with a variety of grade levels and courses. These teachers represent a convenience sample across two sites. We, the two authors, conducted interviews at our own sites, and we each had pre-existing relationships with or connections to the interview participants. As such, we acknowledge the role these relationships may have played in our data collection. We both identify as mathematics educators, but we take on different roles in that space. The first author interviewed ten participants and was a mathematics methods instructor for nine of these participants during their teacher preparation program. The tenth participant was a cooperating teacher at a partner school site. The second author interviewed seven participants and was a mathematics content course instructor for five of these participants. The other participants taught in the same department as one of the second author's former students. Throughout the paper, we refer to participants from the first site using the letter A (e.g., A07) and from the second site using the letter B (e.g., B02).

**Tasks.** To ensure that the pedagogical context was realistic, we used existing tasks that had been extensively reviewed as representing mathematical knowledge for teaching. For the research question reported, we used tasks, shown in Figure 3, based on the TEDS-M released item #MFC709 (TEDS-M International Study Center, 2010), which represents pedagogical content knowledge (Tatto et al., 2008). Healy and Hoyles (2000) used a similar task in their interviews about proof validation with secondary students and their teachers. Healy and Hoyles' (2000) use of the task shows the potential inherent in the task for illuminating different values and norms that might influence proof validation.

**Protocol.** All participants completed the pedagogical context first and the university context second, with personal questions before each context to prime their identities in that context. Prior to the tasks with pedagogical context, we asked the participants how many years they had taught and what courses they were currently teaching. Then, after the tasks with pedagogical context and before the university contexts, we asked participants to name the courses they took in university and identify their favorite course. In each context, after the initial validation of the proofs, we asked parallel follow-up questions to probe participants' thinking further. Two questions we asked included: (1) Would you judge the proof as "partially" valid or invalid? [If the participant concurred] What makes this judgement better than "valid" or "not valid"? (2) Would you agree or disagree with the statement, "Kate's Answer/Proof 1 is less valid because it does not use algebraic notation." This question targeted potential belief in the importance of algebraic notation in proof (e.g., Knuth, 2002a, 2002b; Tabach et al., 2011).

### Data Analysis

Our analysis occurred in four phases. We first coded all of the interview responses related to the pedagogical context. We identified the reason each proof was judged valid or invalid, and reasons for agreement or disagreement about the role of algebraic notation in proof. We developed codes based on existing literature and discussion of interviews from each site. During the coding process, any cases that were unclear were discussed and consensus was reached on all codes. The second phase of analysis involved applying the same coding process to all interview responses related to the university context. Coding each context separately enabled us to interpret the reasoning expressed in each context independently, rather than inferring a participant's reasoning in the university context based on the reasoning they expressed in the pedagogical context. In the third phase of analysis, we looked across contexts for differences in the determinations about the proofs, participants' reasoning, and agreement or disagreement about the role of algebraic notation. We identified changes at the level of considering all participants and at the level of individual participants. Finally, we used the patterns evident in the third phase of analysis to generate themes connected to the research questions.

## Findings

Clear differences emerged in teachers' validations of proof based on context. First, we report teachers' validations of the empirical proof. Next, we explore the role explanation might play in proof validation. Third, we unpack teachers' validations of verbal proofs. Finally, we describe the importance of algebraic notation in proof by context.

**Examples are Not Proof**

As shown in Table 1, teachers consistently judged Leon's Answer/Proof 2 to be not valid in both contexts. They reasoned that even though the examples he selected were correct, examples were not sufficient for provide the statement in all cases.

*Table 1. Validation of Leon's Answer/Proof 2, by context*

| | Pedagogical Context | University Context | Total |
|---|---|---|---|
| Valid | 3 | 0 | 3 |
| Not Valid | 14 | 17 | 31 |
| Other | 0 | 0 | 0 |
| Total | 17 | 17 | 34 |

The teachers who judged Leon's Answer valid in the pedagogical context did so because they wanted to give Leon credit for the elements of the proof that were correct, while acknowledging that it was not a general proof. For example, one teacher said, "It's valid for what he did, but not valid for every number" (A07). Another said,

> It's not a proof *per se* but giving multiple examples and showing that they are all correct, and unfolding them, showing that 24 is 6 times 4, I just find that his reasoning is very strong. It's like induction. I would absolutely consider both of these [Kate and Leon] to be equally strong, but that they are different thinkers. (B02)

The third teacher in this category said, "Leon had concrete examples but didn't generalize it. He didn't take it a step further and show why it always works." This participant continued:

> I'm used to grading on, "You've got a good start," but I guess if I had to grade on a passing/not passing, then I'd have to have a better idea of the expectations for this class? You know, there's times that at the beginning, I'd say, "Yes! Good job! You're showing instances, you're understanding the concept." (B07)

In terms of the professional obligations identified by Herbst and Chazan, these responses negotiate between the obligation to the *individual* student and the obligation to the *discipline*. The three teachers wanted to give credit to Leon for the aspects of his work that were correct, while also acknowledging that in mathematics, empirical proof schemes do not establish a mathematical statement. In the university context, there is no obligation to fellow students, as the proofs were presented without authorship; however, the universal judgment of Proof 2 as not valid can be interpreted as obligation to the *truth*.

This was emphasized further by teachers who expressed a reluctance to commit to calling a proof either valid or not valid. When given the choice, many teachers revised their judgements from valid or not valid to partially valid or invalid. For example, one teacher who judged Leon's Answer to be invalid in the context of teaching when forced to choose between just valid and invalid described more of a middle ground when given the opportunity. Almost none of the teachers revised their response to partial validity in the university context.

**Role of Explanation Varies by Context**

Teachers' validations of Maria's Answer/Proof 3 revealed interesting differences around the reasons for judging the proof to be valid or invalid. For the most part, as shown in Table 2, participants in both contexts found this proof to be invalid. Those who judged this proof to be valid still identified problematic elements of the proof.

*Table 2. Validations of Maria's Answer/Proof 3, by context*

| | Pedagogical context | University Context | Total |
|---|---|---|---|
| Valid | 2 | 3 | 5 |
| Not Valid | 14 | 14 | 28 |
| Other | 1 | 0 | 1 |
| Total | 17 | 17 | 34 |

Looking more closely into the reasons participants judged these proofs to be invalid, we found variation by context. As shown in Table 3, in the pedagogical context, the teachers focused primarily on the incorrect algebraic step of "canceling the *n*'s". In the university context, the lack of explanation of steps in the proof became more salient. This difference could be explained by an obligation to the *institution*, which determines curricular expectations. Algebraic properties are developed in middle school and secondary mathematics, whereas proof-based courses in university may emphasize justifying steps of deductive arguments more than particular algebraic properties. These changes in rationale demonstrate the potential for pedagogical context to influence the process of validation, as well as suggest how position may influence reasoning.

*Table 3. Reasons given for judging Maria's Answer/Proof 3 to be invalid, by context*

| Not valid because… | Pedagogical context | University Context | Total |
|---|---|---|---|
| Incorrect algebraic step ("canceling") | 11 | 6 | 17 |
| Does not explain all the steps | 1 | 4 | 5 |
| Incorrect algebra AND does not explain all the steps | 0 | 1 | 1 |
| Participant assumed algebra would eventually work out, but not enough steps are shown | 1 | 2 | 3 |
| Other reason | 1 | 1 | 2 |
| Total | 14 | 14 | 28 |

**Role of Algebraic Notation Varies by Context**

The majority of participants changed their validations for Kate's Answer/Proof 1 based on context. Overall, participants found the proof valid in the pedagogical context and not valid in the university context. Table 4 shows these shifts. Ten teachers changed from valid in the pedagogical context to invalid in the university context. Among remaining participants, five judged the proof valid in both contexts, and one judged the proof invalid in both contexts.

*Table 4. Validations of Kate's Answer/Proof 1, by context*

| | Pedagogical context | University Context | Total |
|---|---|---|---|
| Valid | 16 | 5 | 21 |
| Not Valid | 1 | 11 | 12 |
| Other | 0 | 1 | 1 |
| Total | 17 | 17 | 34 |

The most common rationale for determining the proof was valid in the pedagogical context was that teachers judged each part of Kate's reasoning to be true (11 participants). As one teacher said, "The argument's pretty solid, talking about the different factors and what it means to be consecutive numbers and why there has to be a multiple of 2 and a multiple of 3" (A03).

In the university context, of the 11 participants determining the proof was invalid, 8 participants reasoned that the proof was invalid because it did not contain algebraic notation. As one participant said,

> Even though I know what's going on, in general terms, at a college level you should use general terms, your *n*, *n* + 1, *n* + 2 ... So, I know the understanding's there, but *at the college level, I feel like your ability to prove things formally is more at stake*. (A09, emphasis ours)

Participants A09 shows sensitivity to the obligation of *institution*, that in the university context, what is expected of students is proving things formally.

We now report the results of asking participants whether they agreed or disagreed with the statement that "Kate's Answer/Proof 1 is less valid because it does not use algebraic notation". Participants' responses to this question also varied by context. Table 5 shows these results.

*Table 5. Teachers' views on whether "Kate's Answer/Proof 1 is less valid because it does not use algebraic notation", by context*

| | **Pedagogical context** | **University Context** | **Total** |
|---|---|---|---|
| Agree: The proof is less valid | 1 | 10 | 11 |
| Disagree: The proof is NOT less valid | 15 | 5 | 20 |
| Other | 1 | 2 | 3 |
| Total | 17 | 17 | 34 |

These responses show the strong role that algebraic notation played in judgements about Kate's Answer/Proof 1. In the pedagogical context, one participant said "The whole point of doing proofs *in high school* is that they know the math, *the goal isn't to show the algebra*. Normally for me, it's *more about the explanation* being the important part of what they are doing" (A02, emphasis ours). This teacher and others explained that their goal for proofs is not having students state formal theorems or use particular mathematical notation. Instead, they are interested in the logical and explanation the student provides. The theme of the logic of the argument as distinguished from the notation used, in the pedagogical context, is echoed in Healy and Hoyles' (2000) findings, where teachers prioritized the meaning of the argument above the notation, due to the obligation to *individual* students. In their study, teachers justified their preference for Kate's proof by saying the explanation would likely make most sense to students.

In the university context, participants in our study reasoned differently about algebraic notation. As A05 said, "I feel like that *by the time you get to a university math course* your algebra skills are already there, you're strong with algebra, so *you should be able to manipulate the numbers and variables* so you could do that" (emphasis ours). A05 is saying that algebraic notation should be understood by the time a person was in college math. Other participants emphasized the perceived judgements of their own university professors: "In the college level, I would agree with that. *I don't think any of my college professors would have given me credit for something without algebraic notation*" (B07, emphasis ours). These responses parallel that of Healy and Hoyles' (2000) secondary students participants, who estimated that their teachers would award more credit to proofs with algebraic notation.

The university context also revealed a sense of negotiating between the obligation to *truth* and the obligation to *institution* expectations. One teacher said of Proof 1, "I think it's valid. But a college professor would call it invalid" (A06). This teacher judged Kate's Answer valid and Proof 1 as not valid. As another participant with the same validation responses put it, "It's very much playing the game of how your professor likes proofs" (B02).

The teachers' recognized that they had different responses for different contexts. For example, one said,

> *Knowing what was expected of me at that level*, *I would agree with that, because it doesn't have formal use of variables and notation*. It's hard to say because *I disagree so much at the high school level*. I agree with it because when you get into that level of mathematics, the thinking should be there for sure, but knowing how to formally construct a proof I think is one of the expectations you need to follow. *That doesn't take away from how important I think the understanding is.* (A09, emphasis ours)

These responses reflect different obligations as a university student as opposed to a secondary mathematics teacher.

Some teachers were mostly comfortable with how their opinions had changed based on the context, because they felt that it was appropriate to have different expectations. For instance, one participant said, "I want to have high expectations for high school students, *but they haven't seen this kind of reasoning before*, whereas university students should be more familiar for this, at least in theory. So there needs to be a higher expectation there" (B03). This response can be seen as based on teachers' obligation to the *individual* students in the pedagogical context.

## Discussion

Our findings corroborate previous findings that pedagogical context does change the work of a mathematics task, and in particular, that they influence the process of proof validation. We make three points in this discussion. First, the construct of position, from the theory of practical rationality, can be used to explain differences in proof validations, both in our own findings and in the literature. Second, to researchers, we suggest how and why the role of position must be taken into account in both research design and data interpretation. Finally, to instructors of mathematics courses for teachers, we suggest why position may play an important role in secondary teachers' perception of disconnect between their university mathematics experiences and their teaching practice.

In the theory of practical rationality, a person is subject to the obligations of their context and must sometimes negotiate between them. In this study, when participants were positioned as teachers, they negotiated between obligations to *individual* students and to the *discipline*, and at times brought in obligations to the *institution*. When participants were positioned as university students, they negotiated between obligations to *truth* and obligations to *institution*. The different sets of obligations led the participants to come to different conclusions about the validity of proofs in different contexts.

The results related to Leon's Answer/Proof 2 highlight the tendency of teachers, especially in a pedagogical context, to draw on pedagogical considerations for an individual student when they are thinking about validity. Many teachers talked about valuing what was correct in Leon's work, even while acknowledging it was not a proof. This difficulty with the binary scale of valid or invalid for judging proofs connects to findings from Buchbinder (2018). Including the option of partial validity highlights some of the pedagogical considerations teachers recognize when doing proof validations. This result is even more striking because almost none of the teachers opted to judge proofs as partially valid in the university context. Participants' discussions of

Maria's Answer/Proof 2 reveal what teachers perceive is valued by the *institution* of university, and how this differs from teaching. While in the pedagogical context teachers wanted to acknowledge individual students' progress, in the university context they were much stricter in their validations. Many participants talked about how Proof 1 needed "more math". This is another way in which position influences validation. The cultural expectation of proofs requiring particular organizational structures or particular notations, especially in the *institution* of university, plays a role in validation separate from judging the mathematical content.

The role of cultural expectations brings us to the importance of position in research design and data interpretation. Cultural expectations by position explain differences between our findings and those of Tabach and colleagues who found that teachers rejected proofs written with verbal representation (Tabach et al., 2010), and over-valued proofs with algebraic notation while under-valuing verbal proofs (Tabach et al., 2011). Our results from the university context corroborate theirs, and contradict theirs in the pedagogical context. An analysis of the protocols in both these studies suggests that the teachers may have been put in the position of student, or at the very least, participants were *not* positioned as teachers. In these studies, teachers began their participation by solving pure mathematics problems, and then asked to validate proofs.

It is well-known that stereotype threat can impact results in mathematics education studies, due to invoking beliefs about identity; similarly, we argue that positioning can impact results, as shown in our study where we asked participants to validate the same set of proofs and only varied the task by position. The different positions activated different obligations, which led to judging mathematical arguments differently. In research in undergraduate education, findings about future teachers as undergraduate students are often used to draw conclusions about what teachers are capable of doing in the field. If it is unclear how participants are positioned by the research design, the conclusions drawn about the teachers may well be incorrect.

Finally, the construct of position may be used to make sense of the problem of disconnection between the undergraduate mathematics experiences and teaching practice of secondary teachers. The obligations that an undergraduate mathematics student is subject to may be very different from the obligations of a secondary teacher. Our findings suggest that when engaging in validation, a proof activity that is vital to success in any proof-based course, teachers may interpret the activity differently in different contexts. Thus the reasoning that future secondary teachers use in undergraduate coursework, and thus the reasoning that instructors are evaluating, may not actually predict how the teachers would reason when they are teaching a high school class. In personal conversations with mathematicians, the authors of this paper have heard laments that undergraduate students, including future secondary teachers, exaggerate the importance of algebraic notation and do not focus enough on the logic of an argument. What if this behavior is more due to expectation than to inclination or ability?

Our findings underscore and qualify the promise of using tasks with pedagogical context to bridge the gap between undergraduate mathematics experience and secondary teaching practice. Positioning future teachers as teachers engages them in reasoning that may be more predictive of how they reason when they teach. This means that the feedback that they receive on these tasks is more likely to be beneficial and influential to their future teaching. At the same time, there is also more work to be done to understand how future teachers, who do not yet have experience teaching, interpret the obligations of pedagogical context, as well as how pedagogical context layered into a university context may differ from the pedagogical context of actual teaching.

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Red X's and Green Checks: A Preliminary Study of Student Learning from Online Homework

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*Homework is thought to play an important role in learning of mathematics. Undergraduate mathematics students spend more time doing homework than they do in class. Many students' homework is at least partially online. Because homework accounts for the majority of students' interaction with content, it has the potential to be a rich learning environment. However, we know little about the nature of students' activity as they complete online homework. This paper proposes an empirically-based model of students' activity in an online homework context. The model was developed from analyses of (1) video recordings of 9 calculus II students completing an online homework assignment about sequences and (2) follow-up interviews with those students regarding their activity. The model characterizes nuances in students' activity regarding reasoning that leads to their answers, the points at which they submit parts or the whole problem and why, and how they leverage immediate feedback.*

*Keywords:* online homework, instructional triangle, didactic contract

## Introduction and Background Literature

Homework is thought to play an important role in students' learning of mathematics. University calculus I students spend more time doing homework than they do in class (Ellis et al., 2015; Krause & Putnam, 2016). Many students' math homework is at least partially online. Because homework accounts for the majority of students' interaction with content, it has the potential to be a rich learning environment.

Most research about online homework has had an achievement focus. LaRose (2010) found that online homework increases students' procedural competence on integration problems. Researchers have sought to determine if differences in homework formats (online, written, or a combination) affects course and/or exam grades. A preponderance of evidence indicates an online homework format has either a slight positive effect or no effect on student exam grades and course grades (Dedic, Rosenfield, & Ivanov, 2008; Halcrow & Dunnigan, 2012; Hauk & Segalla, 2005; Hirsch & Wiebel, 2003; LaRose, 2010). However, researchers have found that students are more likely to complete online homework than they are pencil-and-paper homework (Halcrow & Dunnigan, 2012; Hauk & Segalla, 2005; Hirsch & Weibel, 2003; LaRose, 2010; Roth, Ivanchenko, & Record, 2008).

Perhaps in part as a result of findings that online and written formats are comparable in terms of student achievement, and that students generally seem to like online homework systems and find them helpful for their learning (Halcrow & Dunnigan, 2012; Hauk & Segalla, 2002; Krause & Putnam, 2016; Roth, Ivanchenko, & Record, 2008), online homework systems have proliferated. Their use will likely continue to expand. However, beyond achievement studies and some information about students' perceptions of online homework, we know little about how students engage with these systems and what benefits they might derive from them. Given that homework represents a majority of the time that students interact with content, research about student thinking and learning in an online context can go a long way toward bolstering student learning outcomes. As such, this paper seeks to answer the following research question: what is the nature of calculus II students' activity as they complete an online homework assignment about sequences?

## Theoretical Framework

Homework is one component of an instructional system (Ellis et al., 2015; Figure 1). Ellis et al. (2015) build on Herbst and Chazan's (2012) elaboration of the instructional triangle to describe the relationships between teachers, students and knowledge at stake (content). Online homework is a milieu, or an environment through which students can learn knowledge at stake and through which students receive feedback on their actions (Artigue, Haspekian, & Corblin-Lenfant, 2014; Herbst & Chazan, 2012).

The didactic contract (Brousseau, 1997) governs the interactions between components of the instructional system. The didactic contract is "a set of reciprocal obligations and mutual expectations [that is] the result of an often implicit negotiation" (Artigue, Haspekian, & Corblin-Lenfant, 2014, p. 53). For example, students are expected to do homework, and the instructor is expected to provide opportunities for them to learn the knowledge at stake (via homework or other milieu).

This framework is useful for the study at hand because the nature of students' activity while doing homework is likely influenced by other components of an instructional system, such as what they might learn in other milieu (e.g., class) and what expectations students might have for an online homework assignment. Additionally, the idea of a milieu as a feedback-providing environment aligns with a distinguishing feature of an online homework system: the immediate feedback it provides for each problem. The research question about the nature of students' activity in the context of an online homework system, then, focuses primarily on elaborating what occurs in the arrow between 'student' and 'milieu' in Figure 1, while acknowledging the other facets of an instructional system that influence that activity.

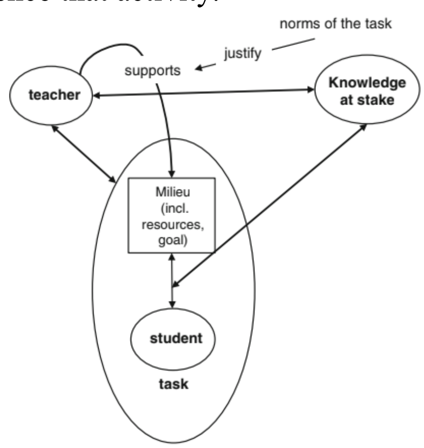


*Figure 1. Instructional triangle (Ellis et al., 2015, p. 270)*

## Data Collection and Analysis Methods

The data presented here come from video recordings and follow-up interviews with 9 calculus II students who completed an online homework assignment about sequences. The data were collected in the fall and spring semesters at a large public university in the US. Calculus II at this university is a coordinated course and so while the students were from five different sections, each with a different instructor, they all completed the same online homework assignment. The homework assignment used the platform WebAssign, and corresponded to section 10.1, sequences, from Rogawski and Adams' (2015) *Calculus: Early Transcendentals*. The problems students answered are shown in Table 1; numerical values in all but Q1 were randomized such that students had the same questions with slightly different numbers. The students had three attempts to answer each question that required a numerical value (e.g., Q4) and one attempt for each multiple choice question that asked if a sequence converged or diverged (e.g., Q7). The course coordinator had disabled the 'try a similar problem' feature. Once a student submitted an answer, the online platform provided a green check mark to indicate a correct answer and a red X to indicate an incorrect answer.

Table 1. Selected WebAssign questions from section 10.1, sequences

| Name | Question |
|---|---|
| Q1 | Match each sequence with its general term. (Assume $n \geq 1$)<br>(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ (b) -1, 1, -1, 1, … (c) 1, -1, 1, -1, … (d) $\frac{1}{2}, \frac{2}{4}, \frac{6}{8}, \frac{24}{16}, \ldots$<br><br>*(Each part had the following multiple-choice answers)*<br>$\cos(n\pi)$ $\frac{n!}{2^n}$ $\frac{n}{n+1}$ $\sin(n\pi)$ $(-1)^{n+1}$<br><br>*(Note: students could submit each a, b, c, and d individually)* |
| Q2 | Let $a_n = \frac{1}{2n-1}$ for n = 1, 2, 3, … . Write out the first three terms of the following sequences.<br><br>(a) $b_n = a_{n+1}$ $b_1 =$ _____ $b_2 =$ _____ $b_3 =$ ______<br><br>(b) $c_n = a_{n+3}$ $c_1 =$ _____ $c_2 =$ _____ $c_3 =$ ______<br><br>*(Note: students could submit each $b_1$, $b_2$, $b_3$, $c_1$, $c_2$, and $c_3$ individually)* |
| Q3 | Calculate the first four terms of the sequence, starting with n = 1. $c_n = \frac{5^n}{n!}$<br><br>$c_1 =$ _____ $c_2 =$ _____ $c_3 =$ ______ $c_4 =$ _____<br><br>*(Note: students could submit each $c_1$, $c_2$, $c_3$, and $c_4$ individually)* |
| Q4 | Calculate the first four terms of the given sequence, starting with n = 1.<br>$$c_n = \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \cdots + \frac{1}{3n+2}$$<br>$c_1 =$ _____ $c_2 =$ _____ $c_3 =$ ______ $c_4 =$ _____<br><br>*(Note: students could submit each $c_1$, $c_2$, $c_3$, and $c_4$ individually)* |
| Q5 | Find a formula for the $n^{th}$ term of the following sequences (with a starting index of n = 1)<br>(a) $6, -\frac{7}{8}, \frac{8}{27}, \ldots$ $a_n =$ ______________<br><br>(b) $\frac{6}{5}, \frac{10}{9}, \frac{14}{13}, \ldots$ $a_n =$ ______________ |
| Q6 | Determine the limit of the sequence and state if the sequence converges or diverges.<br>$$a_n = \frac{7 + n - 3n^2}{12n^2 + 1}$$<br>$$\lim_{n\to\infty} a_n = \_\_\_\_\_\_\_$$<br>___ The sequence converges.<br>___ The sequence diverges. |

| | |
|---|---|
| | *(Note: students could submit the limit and the 'converges / diverges' part individually. The converges/diverges part was a multiple-choice question in which students selected one of the two answers.)* |
| Q7 | Determine the limit of the sequence and state if the sequence converges or diverges. $$a_n = \left(\frac{2}{3}\right)^n$$ $$\lim_{n\to\infty} a_n = \_\_\_\_\_\_\_\_$$ ___ The sequence converges.<br>___ The sequence diverges.<br><br>*(Note: see note for Q6.)* |
| Q8 | Determine the limit of the sequence and state if the sequence converges or diverges. $$a_n = -\left(\frac{5}{8}\right)^{-n}$$ $$\lim_{n\to\infty} a_n = \_\_\_\_\_\_\_\_$$ ___ The sequence converges.<br>___ The sequence diverges.<br><br>*(Note: see note for Q6.)* |
| Q9 | Determine the limit of the sequence and state if the sequence converges or diverges. $$a_n = \frac{n}{\sqrt{n^3+2}}$$ $$\lim_{n\to\infty} a_n = \_\_\_\_\_\_\_\_$$ ___ The sequence converges.<br>___ The sequence diverges.<br><br>*(Note: see note for Q6)* |
| Q10 | Determine the limit of the sequence and state if the sequence converges or diverges. $$a_n = \ln\left(\frac{11n+2}{-9+6n}\right)$$ $$\lim_{n\to\infty} a_n = \_\_\_\_\_\_\_\_$$ ___ The sequence converges.<br>___ The sequence diverges.<br><br>*(Note: see note for Q6.)* |
| Q11 | Use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges. (If the quantity diverges, enter DIVERGES). $$c_n = 1.01^n$$ $$\lim_{n\to\infty} c_n = \_\_\_\_\_\_\_\_$$ |
| Q12 | Use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges. (If the quantity diverges, enter DIVERGES). $$b_n = n^{2/n}$$ |

| | |
|---|---|
| | $\lim_{n\to\infty} b_n =$ ________ |
| Q13 | Use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges. (If the quantity diverges, enter DIVERGES). $$c_n = \frac{10^n}{n!}$$ $\lim_{n\to\infty} c_n =$ ________ |
| Q14 | Use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges. (If the quantity diverges, enter DIVERGES). $$b_n = \ln(n^2+5) - \ln(n^2-1)$$ $\lim_{n\to\infty} d_n =$ ________ |

The data were collected in two phases. First, I video-recorded each student as (s)he completed the homework. I made copies of students' scratchwork, class notes, and any other materials they referred to while completing the assignment. I then watched each video and took notes about students' observable actions (e.g., what they wrote for scratchwork, what they typed into calculators, what they submitted for answers, and when these actions occurred in relation to one another). This formed the first phase of data analysis. Most broadly, all students' interactions with any given problem fell into three components: pre-answer submission, answer submission, and post- answer submission. For the pre-answer submission component of each problem, I made notes about what the students did before submitting an answer (e.g., computed terms of a sequence, looked at class notes). This portion of students' activity represents their first interactions with the knowledge at stake.

For the answer submission component, I made notes about what the student submitted. Because most questions had multiple parts, the "what students submitted" focused on which part(s) of the question and the student's actual numerical answer. This was important because it forms what students received feedback on, and feedback is a component of the homework as a milieu. My notes about post- answer submission focused on whether the answer was correct or incorrect and what the student did following that feedback. This was important because it represents another opportunity in which students might (re)-engage with the knowledge at stake.

In the second phase of data collection, the student and I watched the video and I asked questions pertaining to my notes. I asked questions such as "can you explain how you arrived at that answer?" to gain insight into the pre-submission component of students' work. To gain insight into how students made use of the immediate feedback, I asked questions such as "can you explain why you submitted just this part of the question?" or "can you explain why you submitted all of these parts at once?" and "what did you think when you saw that your answer was correct/incorrect?" This interview was also video-recorded and was transcribed for analysis.

The data were analyzed via a constant comparative analysis (Strauss & Corbin, 1994). I sought to elaborate the students' activity in the previously-identified pre-answer submission, answer submission, and post- answer submission components of the students' work. Using my initial notes and the transcripts, I first sought to categorize what students did before submitting an answer. The themes that emerged were that students did some mathematical thinking (e.g., computing terms), formed guesses, or engaged in a combination of the two. In the answer submission component of students' work, I observed that sometimes students submitted parts of a question and sometimes they submitted all parts at once. Because questions had multiple parts, the feedback students received could be [fully] correct, partially correct, or incorrect. Further, because students had multiple tries on most questions, the feedback could be intermediate (if students had more tries on any part of the question) or final (if students had all parts of the question correct, or if they had run out of tries). Per the post- answer submission component, I observed the students continuing to work on the question, guessing an answer without doing more work, and moving to the next question. Based on students' answers to interview questions, I was able to categorize these as students cycling forward, cycling back, or not cycling (defined below). I made each of these themes into a code, then went through all of the data applying these codes. This allowed me to refine them and 'collapse' them into the model presented in the results section. Finally, I re-coded all of the data using this model to ensure that the model accurately accounted for the phenomena observed in the data. The result, presented in the next section, is an empirically-based model that describes the nature of these students' activity in an online homework context.

## Results: Preliminary Model of the Nature of Students' Activity While Completing Online Homework

In this section, I present the result of this study: an empirically-based, preliminary model of the nature of students' activity while completing online homework. I then provide examples of different paths through the model.

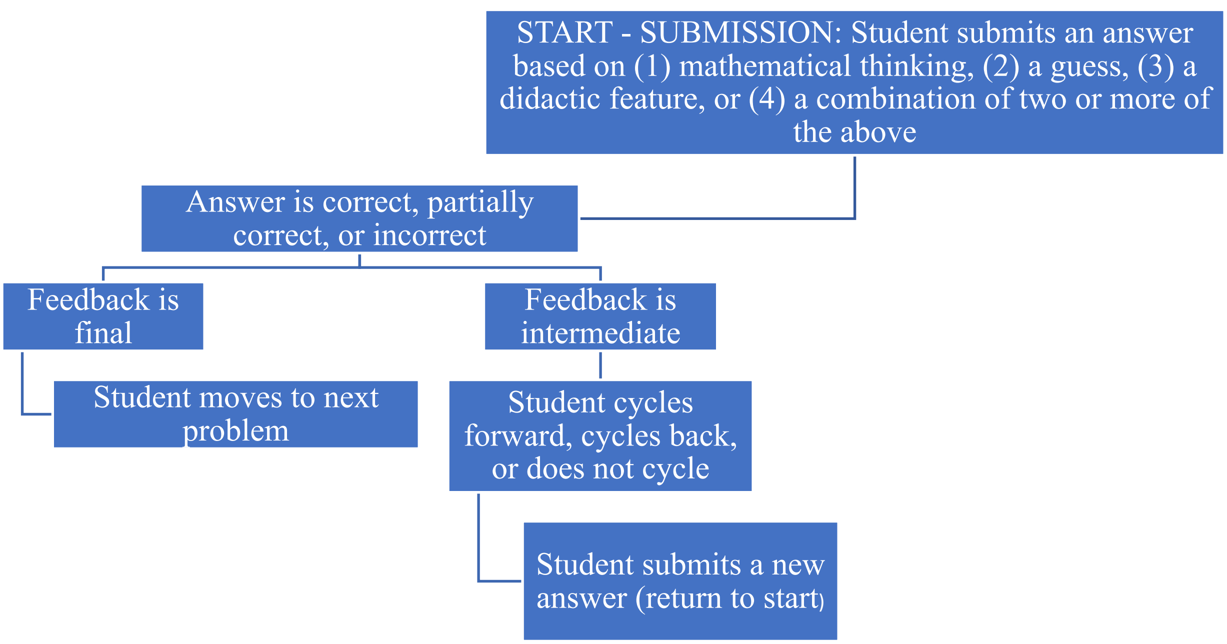


*Figure 2. Preliminary model of the nature of students' activity while completing online homework*

## Unit of analysis

The first step in coding was to determine the unit of analysis. I made this decision by question type. Question 1 was analyzed as one unit, since the multiple choice answers were the same for each sequence. In questions 2 and 5, parts A and B were coded separately because they were fairly distinct problems. Questions 3 and 4 were each treated as a single unit. For example, if a student submitted a value for $c_1$ but not $c_2$, $c_3$, or $c_3$, that was coded as 'intermediate' feedback because there were more parts of the question. This decision was informed in part by the fact that students frequently described submitting an answer to one part of a question to make sure they were on the right track; that is, the students saw $c_1$ as a fraction of a question, not an entire problem in itself. Similarly, for a multi-part problem like questions 6, 7, 8, 9, and 10, the two parts were considered together. For example, if a student submitted the value of the limit (but not the converge/diverge part) and it was correct, the feedback was coded as intermediate. Questions 11, 12, 13, and 14 were each a single unit in analysis.

## Submission

The data indicated that students submitted answers based on either mathematical thinking, guessing, a combination of mathematical thinking and guessing, or didactic features. *Mathematical thinking* is characterized as applying a general principle that is within the learning objectives of the course to a specific problem. Examples include applying a theorem, creating a graph, computing, doing algebraic manipulation, differentiating, and symbolizing. I defined mathematical thinking in terms of applying principles within the learning objectives of the course to exclude reasoning such as 'I submitted 0 because sequences often converge to 0'. While we might argue that a student has engaged in generalizing based on their experience, and generalization is a critical component of mathematical thought, I wanted the definition of mathematical thinking to reflect the particular mathematics we might want students to engage in for these problems; noticing that many of the sequences they saw converged to 0 was not a goal of the course.

Submitting an answer based on *didactic features* means that rather than focusing on the mathematics, the student reasons about what she is 'supposed to learn'. This follows from the idea of the didactic contract as a set of mutual obligations and expectations; according to the theory framing the study, students have particular expectations about the opportunities to learn provided to them in a homework assignment. I follow Merriam-Webster's definition of *guessing* as 'estimating or suppose (something) without sufficient information to be sure of being correct.'

## Feedback

Feedback is *final* if the student does not submit another answer for a problem, and *intermediate* otherwise. A student might obtain a correct answer and move to the next problem, in which case the green check is final feedback. If a student runs out of attempts on a problem, the green check or red X corresponding to the last attempt is final feedback.

It might seem that a correct answer would be final feedback because a student is unlikely to submit another answer. However, I repeatedly observed students submitting each part of a multi-part question individually and explaining that they did so because they could use the feedback to guide their work on other parts of the question. Correct answers in these cases affirmed for students that they had the right method for solving the problem. Hence a key finding is that one way students interact with an online homework system is that they use the feedback as formative assessment. As such, it is important to distinguish between types of feedback within the model.

## Cycling forward / cycling back / not cycling

One emergent theme in the data was that students often submitted one part of a multi-part problem to obtain feedback regarding whether they were on the right track. In some cases, if the answer were correct (such as having computed a term correctly), they took the green check mark as evidence that they had the correct method, and they applied that method to the remainder of the problem. In other cases, such as submitting the correct numerical value for a limit but not the multiple choice converge/diverge, students would interpret a green check mark as evidence that they understood the problem at hand, then submit the multiple choice part. Contrastingly, students often interpreted a red X as meaning they needed to find a new process. Finally, one students obtained a red X and appeared to guess a new answer without considering any mathematics (see Example 2).

I borrow from Carlson and Bloom (2005) problem-solving taxonomy in characterizing these actions as *cycling forward* and *cycling back*. Carlson and Bloom (2005) observed that mathematicians' problem-solving involved a 'checking' phase in which they "verified the correctness of their computations and results" (p. 63). The mathematicians would then "accept the result and move to the next phase of the solution" (*cycle forward*) or "reject the [incorrect] result and cycle back" (Carlson & Bloom, 2005, p. 63). The online homework system does the verification for students, who then made decisions to cycle forward, cycle forward, or not cycle.

The next section provides examples of student activity as framed in this model.

## Examples of Student Activity in Online Homework Problems

In each example below, the header represents the code for the reasoning behind a student's initial submission; the parentheses indicate the students' name[1] and the question number. The first paragraph lists the codes for the student's work on the question. Subsequent paragraphs provide details and evidence that support the codes. These examples contain quotes from only three of the nine students, but they are representative of the ways in which all the students acted. By focusing on three students, I hoped to allow the reader to compare how one student had different 'paths' through the model for different problems (e.g., in Example 1 we see how Susie relied on mathematical thinking on Q10, and in Example 2 and 3 we see multiple reasons she combined mathematical thinking and guessing). The final example is meant to demonstrate the ways in which students leveraged immediate, intermediate feedback, and how important it was for them in their thinking.

## Example 1: Mathematical thinking (Susie Q10)

Susie's work on Q10 (Figure 2) is coded as: student submits answer based on mathematical reasoning, answer is partially correct; feedback is intermediate; the student cycled forward; the student submitted a new answer based on mathematical thinking; the answer was incorrect (because of the format); feedback is intermediate; the student cycled forward; the student submitted a new answer based on mathematical thinking; the answer was incorrect (again because of the format); feedback is intermediate; the student cycled forward; the student submitted a new answer based on mathematical thinking; the answer was incorrect (again because of the format); feedback is final; the student moved to the next question.

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[1] Names are gender-preserving pseudonyms based on students' self-identified gender on a demographic survey completed at the end of the study.

Susie began this problem by computing $a_2 = 2.08$, $a_4 = 1.12$, $a_6 = 0.92$, and $a_8 = 0.83$. She submitted $\lim_{n\to\infty} a_n = 0$ and 'converges' at the same time; WebAssign marked the former as incorrect and the latter as correct, meaning in the model we code this as *partially correct*. Susie described that she reasoned the limit was 0 "because like on the previous ones like if they continually decrease, they all went to 0. But this one that wasn't the case." Susie's computations and comment about paying attention to the trend are evidence that she submitted her answer based on *mathematical thinking*. The feedback was *intermediate* because Susie tried the problem again. She *cycled back*. As evidence of this, she described in the second interview

*Interviewer*: And can you tell me about, so here we have part of the question right, so you know it converges, but you didn't know what the limit was. Can you tell me what you were thinking?

*Susie*: That I needed to find, I needed to go further down in like the n values.

Susie computed $a_{12} = 0.75$, $a_{100} = 0.62$, and $a_{1000} = 0.607$. This, taken with her comment that she needed to 'go further down in like the n values', is evidence of mathematical thinking. Susie submitted 0.6 as an answer. Because WebAssign wanted the exact answer ln(11/6), the system marked this *incorrect*. The feedback was *intermediate* because Susie tried the problem again, submitting 0.607 (*incorrect*). While finding these decimals, Susie *cycled forward.* She explained in the second interview, after viewing her third incorrect submission *(final feedback)*,

*Interviewer*: Do you feel like you understood what was going on?

*Participant*: I thought so, but then I got it wrong, so I wasn't sure what it was converging to…

*Interviewer*: So you kept trying bigger and bigger values…

*Participant*: I felt like it was a good method but, so then I thought it was just like the decimals, so it might be a fraction. But yeah natural logs are not my forte.

These quotes are evidence that Susie cycled forward. She felt her method was appropriate, and she continued to try larger values of n. However, she submitted another decimal when the system wanted an exact answer, and obtained a red X. This feedback was final because Susie was out of attempts and moved to the next problem.

Like example 5 (below), this example demonstrates how students leverage immediate, intermediate feedback. Susie had a green check mark for the 'converge' part. She said this told her "I needed to go further down in the n values". That is, she leveraged this feedback by coupling it with her prior work and using it as a hint toward what future action she should take. Students did this frequently, both with the questions that had a converge/diverge multiple choice part and with questions that required them to compute terms (See Example 5).

**Example 2: Mathematical thinking and guessing (Susie, Q1)**

Susie's work on Q1 is coded as: student submits answer based on a combination of mathematical thinking and guessing; answer is partially correct; feedback is intermediate; the student did not cycle; the student submitted a new answer based on a guess; the answer is correct; the feedback is final.

Susie's first attempt at question 1 involved a *combination of mathematical thinking and guessing*. She correctly wrote the first term or two for the given general terms $\frac{n}{n+1}$, $\frac{n!}{2^n}$, and $(-1)^{n+1}$ and these computations are evidence of mathematical thinking. However, she made some errors in writing the values of trig functions, which resulted in her guessing for one part of the question. Susie wrote, $\sin(\pi) = 0$ , $\sin(2\pi) = 1$, $\cos(\pi) = -1$ and $\cos(2\pi) = 0$. Susie then matched the terms she wrote to the given sequences. Her work corresponded exactly for 1a, 1c, and 1d.

However, on 1b, because the incorrect terms for the trig functions did not match any of the sequences, Susie guessed between the two remaining answer choices. Watching the video of herself doing the problem, Susie explained "I knew it was between one of those two [sine and cosine] but apparently I couldn't figure out which one it was… but I mean I like multiple choice because you can free guess."

Susie submitted all four parts of the problem at once. Her submission and the feedback she received are shown in Table 2. Because the feedback was a mix of correct and incorrect responses, this is coded as *partially correct*. Susie submitted a different answer to 1b after obtaining this feedback, so the feedback shown in Table 1 was *intermediate*.

Table 2. Susie's first submission for Q1

| Problem | 1a | 1b | 1c | 1d |
|---|---|---|---|---|
| Sequence | $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ | -1, 1, -1, 1, … | 1, -1, 1, -1, … | $\frac{1}{2}, \frac{2}{4}, \frac{6}{8}, \frac{24}{16}, \dots$ |
| Susie's submission | $\frac{n}{n+1}$ | $\sin(n\pi)$ | $(-1)^{n+1}$ | $\frac{n!}{2^n}$ |
| Feedback[2] | Correct | Incorrect | Correct | Correct |

When Susie saw that her answer to 1b was incorrect, she immediately clicked $\cos(n\pi)$, submitted the answer, and moved to the next problem. Hence she *did not cycle*. The evidence for this is twofold. First, there was a four-second gap between when the red X appeared on her screen for 1b and her clicking and submitting $\cos(n\pi)$. Second, she did not return to her scratchwork to re-compute any values of cosine. This submission, then, was based on a *guess* because Susie did not have sufficient information to be sure she was correct. Her answer was correct, and the feedback was *final* because Susie did not submit any more answers. Susie's guessing was an affordance of the multiple-choice nature of the question. The next example describes guessing as an affordance of the number of chances students are allowed.

**Example 3: Mathematical thinking and guessing (Susie, Q12 and Calvin, Q4)**

Susie's work on Q12, $\lim_{n\to\infty} b_n$ where $b_n = n^{3/n}$, was coded as: student submits answer based on a combination of mathematical thinking and a guess; answer is correct; feedback is final.

Susie computed $b_2$, $b_{10}$, $b_{20}$, and $b_{50}$. She submitted the answer $\lim_{n\to\infty} b_n = 1$, which was correct. In the second interview, Susie explained her thinking as she was computing the terms.

*Susie*: I think I was starting [inaudible] if it was diverging to 1 or 0.

*Interviewer*: Okay, so when you typed 1 here [in the answer box] -

*Susie*: I was, I was thinking it was 1, but it could have also been 0 if you went down further…

*Interviewer*: So was this, do you think this was one of those things where you like have three chances, so –

*Susie*: Yeah.

*Interviewer*: - try one and then –

*Susie*: If it's wrong try another one… cuz like just a lot of graphs converges [sic] to 0. But it was, like it was going to 1, like it had 1 point something, so I figured it was 1, but there was also the possibility that if you went further if it was going to converge to 0 or not.

[2] The students see a green check mark for a correct answer and a red X for an incorrect answer, not the word 'correct' or 'incorrect'.

*Interviewer*: So it sounds like you were paying attention to both number patterns like the 1 and the experience that things go to 0 often.

*Susie*: Yeah.

Susie's computation of $b_2$, $b_{10}$, $b_{20}$, and $b_{50}$ and attention to the fact that the values were decreasing to 1 is evidence of mathematical thinking. Her comments about submitting 1 and then submitting 0 if the 1 were wrong are evidence of guessing. This example sheds some light on student guessing behavior. First, Susie guessed in part because the particular constraints of the system afforded a guess or two. That is, Susie had three chances to obtain full credit for the problem, and this made her feel comfortable guessing. Second, the example highlights that students' guesses may be based on some sort of reasoning (as opposed to guessing random numbers). Susie had in mind 0 and 1 because she had noticed that in her experience, sequences frequently converge to those numbers.

Another question in which guessing was afforded by multiple chances per problem, and based on some sort of sensemaking, was in question 4. Several students submitted $c_1 = 1/3$, $c_2 = 1/5$, $c_3 = 1/7$, $c_4 = 1/9$. Calvin, one of the students who did this, described

*Calvin*: I looked at it and it looked like they were just asking me to write down what they had already typed out. So I was a little confused on why it was so easy. So I just, since I knew I had multiple tries, I just decided let's just go with it, assuming that I understood it right.

Calvin's submission of $c_1 = 1/3$, $c_2 = 1/5$, $c_3 = 1/7$, $c_4 = 1/9$ was based in part on mathematical thinking, as evidence by his comments that "it looked like…" and "assuming that I understood it right." These comments show that he had engaged in some sensemaking about what the problem was asking. However, he guessed because he did not have sufficient information to be sure he was correct. We know he was unsure because he said he was "confused on why it was so easy" and he was "assuming he understood it." Calvin's comments about multiple tries indicates that he guessed in part because the system afforded it. Like Susie, he felt the multiple tries allowed for guessing. The feedback Calvin obtained for these answers was partially correct ($c_1$ was correct and the others were incorrect). He cycled back, as he describes below:

*Interviewer:* So one was right, and three of them were wrong… do you remember what you were thinking?

*Calvin:* This isn't as easy as it looks.

*Interviewer:* So I'm pretty sure here you were just looking at the computer screen… do you remember what you were looking at or trying to figure out?

*Calvin:* I was just trying to soak up as much information from the problem to figure out what I was supposed to do. I probably re-read the question and then made sure I understood what the question was asking and then I went back to the equation or numbers they gave and tried to figure out what part I wasn't understanding… I think I figured out that the terms were, like they were trying to get me to add the previous term to it.

Here, the immediate feedback was a cue for Calvin that he did not understand what was going on (as he had assumed), and he needed to think through the question in a different way. He realized that he might need to add the terms, and computed $c_2 = 1/5 + 1/8$. He submitted $c_2 = 13/40$. He submitted just that one answer because

*Calvin:* I knew I only had two tries for all three of them… I figured if I did one of them I'd limit the amount of tries I lost.

This is evidence that the number of tries influenced the way in which Calvin worked, but it is also evidence that the multiple tries allowed Calvin to try an idea. While in some cases the multiple chances afforded guessing, in many more cases, students used the multiple chances to submit answers as multiple chances with which to engage in the mathematics.

**Example 4: Mathematical thinking, guessing, and didactic features (Rosalyn, Q11) and Guessing and didactic features (Rosalyn, Q12)**

Rosalyn's work on questions 11 and 12 show evidence of attending to the didactic contract when submitting answers in online homework.

In Q11, Susie's work is coded as: submitted an answer based on mathematical thinking, guessing, and the didactic contract; the answer was correct; the feedback was final; the student moved on to the next problem. In the next question (Q12), Rosalyn's second submission involved guessing based on attention to didactic features. Her work in this question is coded as: submitting an answer based on mathematical thinking; answer is incorrect; feedback is intermediate; the student did not cycle; the student submitted an answer based on a guess and didactic features; the answer was incorrect; feedback is intermediate; and the student stopped working on the question (she wanted to ask for help, and saved her last attempt for later).

In question 11, Rosalyn appeared to have (mis)applied the power rule to take a derivative. She wrote $\lim_{n\to\infty} 1.02^n = \lim_{x\to\infty} 1.02^x = \lim_{n\to\infty} x(1.02)^{x-1} = \infty^{\infty-1}$. She then said,

*Rosalyn*: Infinity to the infinity minus 1 would be, that doesn't make any sense. I'm going to enter diverges and hopefully that's it. [Submits 'DIVERGES', which is correct] Okay, it diverged. I don't know how to solve it, but I guessed. So – and probably one more of these diverges. Don't know which one though but most likely.

*Interviewer*: Why do you think probably one more diverges?

*Rosalyn*: Usually like in WebAssign like the fact that there's four of them [problems that read 'If the quantity diverges, enter DIVERGES] I want to say like probably at least another one diverges… because the whole point is you know you learn like, you know a little bit of each. So I feel like if they have one that diverges they most likely will have another one diverges.

Although Rosalyn said she "guessed," I coded the submission as involving mathematical thinking because Rosalyn did some computation and I believe the $\infty^{\infty-1}$ may have influenced her decision to guess 'diverge'. This guess also appears to have been influenced by Rosalyn's expectation that some of the sequences diverge, as evidenced by her comment that "the whole point is… you know a little bit of each." Rosalyn appeared to believe that if diverging was something she was supposed to learn, it would be an answer to at least one problem. She applied this reasoning in the next question.

Rosalyn's first submission on question 12 was $\lim_{n\to\infty} b_n = 2$. This was based on mathematical reasoning because she did some (incorrect) algebra, either trying to simplify $n^{2/n}$ or take a derivative, and writing $\lim_{n\to\infty} \frac{2}{n} * n = \lim_{n\to\infty} 2 = 2$. She submitted 2, which was incorrect. Rosalyn explained,

*Rosalyn*: I don't know this one, and I'm going back to the whole I'm-hoping-two-of-them-diverge, so I'm going to check if it diverges.

She then typed in 'diverge' without doing any additional scratchwork. This was a guess because she did not have sufficient evidence to be sure that it diverged. It was a guess based on a didactic feature because she relied on her reasoning in question 11: divergence was something she was supposed to learn, so she expected 'diverge' would be a correct answer for at least one question.

**Example 5: Mathematical thinking (Calvin, Q2A and Q8)**

Calvin's work on questions 2 and 8 provide evidence of solving problems by engaging in mathematical thinking and leveraging immediate, intermediate feedback to support his solution attempts. Calvin's work on Q2 and Q8 followed the same coding: submitting an answer based on mathematical thinking; the answer was correct; the feedback was intermediate; he cycled forward; he submitted more answers based on mathematical thinking; those answers were correct; and the feedback was final.

On Q2, Calvin wrote $b_1 = a_2$. He described,

*Calvin*: I was just writing out what I was thinking… $b_n$ is $a_{n+1}$ so if you have $b_1$ you have to add one when you're doing $a_n$ so it's going to be $a_2$. [Submitted $b_1 = 1/3$]

*Interviewer*: I noticed here that this question has 3 boxes to fill in [points to part A] and you filled in one and then submitted it. Do you remember why you did that?

*Calvin*: Just to make sure I was doing it right.

Submitting one term to check that they were on the right track was very common; nearly all students did this. In particular, here Calvin did some mathematical thinking (as evidenced by his sensemaking of how $a_n$ and $b_n$ were related), submitted one answer, and obtained the intermediate feedback that it was correct. He reflected on the fact that his method was correct and went on to submit the correct answers for $b_2$ and $b_3$ (cycling forward)

In Q8, Calvin did some algebra, writing $-(8/5)^n$. He described, "to remove that negative from the exponent, you just flip whatever the coefficient is." Calvin engaged in mathematical thinking because he was doing algebraic manipulation and paid attention to the magnitude of the fraction (see below). He then clicked the 'diverge' multiple choice box, but not the numerical limit. He explained,

Calvin: I saw that the fraction was greater than 1 so I was like okay it has to diverge. But I wanted to make sure. But I don't know why I did it with that [the multiple choice part] because I only have one shot.

Calvin then submitted $-\infty$ in the box for the numerical limit, which was correct. Like question 2, he described that he employed the immediate feedback as a 'check.' The comment about being unsure why he submitted the multiple choice part (for which he had one chance), instead of the numerical limit (for which he had three chances) is evidence of one of the ways in which the structure of the homework system influences students' work. Like Susie and Rosalyn, Calvin felt the multiple chances afforded trying answers he was unsure of. Moreover, in many cases, these students and the others in the study felt the multiple chances afforded feedback on their work that they could, and often did, employ in directing their future work.

## Discussion

I have proposed a preliminary, empirically-based model of the nature of students' activity while completing online homework. The components of this model are (1) the basis of the student's answer submission, (2) the correctness of the submission, (3) whether the feedback is immediate or final, (4) whether the student cycles forward, back, or does not cycle and (5) whether or not the student submits a new answer. It is my hope that this model serves as an entry point for others studying student thinking and learning in the context of homework. Because homework forms the majority of students' interaction with content (Ellis et al., 2015; Krause & Putnam, 2016), it is a rich area for studying student thinking and learning. It is also an area in which we may be able to employ particular structures that engender desired action. For example, a key finding in this study is that students often employed their multiple chances per problem as ways to obtain feedback on their thinking. That feedback served as a formative assessment. This finding supports allowing students more than one attempt on 'open-ended' online homework problems.

One key finding is that in coding the data, I found that most students' initial submissions, and re-submissions, were based at least partially on mathematical thinking. In future work, I intend to attempt to quantify this more specifically. Another key finding in this study relates to why students guess. The students in this study only guessed randomly in multiple-choice settings (e.g., Susie's guess described in Example 2). Their guesses had some sort of reasoning behind them, such as a generalization based on experience that 'sequences often converge to 0', or an implicit understanding of the didactic contract that they would be provided with problems that represented the scope of knowledge at stake. These findings support looking at student activity in homework as part of an instructional triangle, as Ellis et al. (2015) propose.

These findings raise a number of avenues for future research. First, this study was done with a very specific population of students. An important question is whether or not the model accurately represents the nature of *other* students' online homework activity. Do students in different courses interact with an online homework assignment in similar ways? Another avenue for research is further investigation in the number of attempts students have per problem. My findings support allowing students multiple tries. However, is a particular number of tries better than some other number? How, if at all, would the nature of students' activity change if they had unlimited tries? Finally, the students in this study had particular constraints built into their WebAssign system, one of which was that the instructor had disabled the 'see similar example' feature. Is the nature of students' activity different when they have access to a 'try a similar problem' or 'see similar example' feature? How do students make use of such features? Answering these questions can tell us what conditions and resources within an online homework system promote desirable student activity such as engaging in and reflecting on mathematical thinking.

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# An APOS Study on Undergraduates' Understanding of Direct Variation: Mental Constructions and the Influence of Computer Programming

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*This study explores undergraduates' understanding of direct variation before and after instruction using computer programming to teach generalization over the concept. Based on an initial genetic decomposition for direct variation, the four math/CS researchers developed a research design that included lessons featuring computer programming and mathematical proof writing activities. This study shares results from an application of the instructional research design to N=33 undergraduates interested in teaching. Lessons were from a secondary education math methods course. Follow up interviews were conducted with seven participants. The analysis, using APOS as a framework, categorized mathematical behaviors at the Action, Process or Object level. The study identified obstacles that may have prevented progression through deeper levels of understanding such as deficient prerequisite skills and an affinity for routine algebraic manipulation rather than considering underlying relationships. The student data demonstrated how computer programming activities influenced undergraduates' mental images.*

*Key words:* Direct Variation, Generalization, Computer Programming, Pre-Service Teachers

## Introduction

The ability to generalize is considered an essential skill for reasoning about and deeply understanding mathematical concepts by mathematics education researchers. Many researchers have investigated how to explicitly induce students to develop generalizations in the context of mathematical explorations (Tall et al., 1991). In this paper, we extend our previously published preliminary study into student's understanding of direct variation using computer programming exercises (Stenger et al., 2017). Many mathematics education researchers believe that using computer programming activities designed to parallel the construction of an underlying mathematical process may stimulate or accelerate the development of the associated mathematical construction (Dubinsky and Tall, 2002; Jenkins et al., 2012). In prior work, we developed an explicit method for motivating students to generalize into mathematical constructions using computer programming exercises and proof writing based on the theoretical perspective of APOS theory (Stenger et al, 2017). The research questions we investigate are: (1) Does our genetic decomposition of direct variation adequately describe the observed students' constructions; and (2) Do our instructional treatment's computer programming activities influence students' mental constructions as described in the genetic decomposition?

## Proportional Reasoning

The ability to reason using proportions is an important and pervasive requisite for many academic and daily pursuits. In developmental psychology, proportional reasoning is considered "a major landmark in the passage from concrete operational to formal operational thinking" (Light, et. al., 2016; Inhelder and Piaget, 1958). Consequently, investigating how learners acquire and practice proportional reasoning has been the subject of numerous studies. Lessons on proportions are typically considered the responsibility of middle school mathematics educators, yet proportional reasoning problems occur and cause difficulties for students throughout their academic and non-academic life (Ben-Chaim, 2012; Hilton, 2016). Scenarios that are addressed by proportional reasoning are not only math classroom problems, they also appear in other STEM classes in a wide range of scientific contexts, including, for example, trajectories, fuel consumption and the expansion of gases (Light, 2016), as well as economics and geography (Hilton, 2016). Additionally, professions such as architecture, nursing, and pharmacy require the ability to reason proportionally (Hilton, 2016). The necessity to employ proportional reasoning skills in increasingly complex and diverse problem solving situations continues to present a teaching and learning challenge for mathematics educators and students.

Mathematics education researchers have dedicated considerable energy to proportional reasoning with elementary and middle school students, high school students, undergraduates and graduates. Collectively these researchers have shown that students and adults have difficulty with problems involving proportional reasoning (Noelting 1980; Vergnaud, 1983; Hart, 1988; Lesh et al., 1988; Caput and West, 1994). When proportions are "treated as procedural computational problems where the goal is to find missing values using a certain technique, such as 'cross-multiplication'" (Warshauer, 2014; Heinz and Sterba-Boatwright, 2008), students may be missing the appropriate struggle required for deeper learning (Warshauer, 2014; Liu, 2013). Frith, et. al. (2016) performed a study among undergraduate law students that suggests that even semester-long interventions may make only modest improvements in understanding. Ben-Chaim, et. al. (2012) noted that often both pre-service and in-service teachers have insufficient understanding for subjects taught in elementary and middle schools. Courtney-Clarke and Wessels (2014) found that in a study of pre-service teachers in Namibia, only 25% could determine the "relative size of two common fractions (a comparison problem)."

There is widespread recognition that proportional reasoning is important and students and teachers may struggle. We found that explanations for the difficulties undergraduates experience with the concept of direct variation are sparse in existing literature. Hence this study will contribute to the literature on how undergraduates' understand direct variation by examining student's mental constructions and exploring how computer programming activities support the development of mathematical constructs.

## Theoretical Framework

The theory of reflective abstraction was described by Piaget (1985) as a two-step process, beginning with reflection on one's existing knowledge, followed by a projection of one's existing knowledge onto a higher plane of thought. Further, Piaget (1985) and Dubinsky et al. (2005a, 2005b) wrote that during the process of cognitive development, reflective abstraction could lead to the construction of logico-mathematical structures by the learner. The conviction that reflective abstraction could serve as a powerful tool to describe the mental structures of a mathematical concept led Dubinsky to develop APOS theory.

In APOS theory the mental structures are Action, Process, Object, and Schema. A mathematical concept develops as one acts to transform existing physical or mental objects. Actions are interiorized as Processes and Processes are encapsulated to mental Objects. It is tempting to view the progression as linear, but APOS practitioners hold that learners move back and forth between levels and hold positions between and partially on levels. In other words, the progression is not linear. This nonlinear behavior and the resulting mental structures may explain the different ways learners respond to a mathematical situation (Arnon et al., 2014).

## Genetic Decomposition for Direct Variation

The genetic decomposition was developed as a conjecture of the mental constructions, Actions, Processes, and Objects, that may describe the construction of mental schema for the concept of direct variation as it develops in the mind of the learner. The genetic decomposition served as a model for the design of this research study as well as the analysis of the results. It was also the basis for the computer activities in the lessons that were developed for the students. The pervasive impact of the genetic decomposition is consistent with an APOS theoretical framework (Asiala, et al., 1996).

The prerequisite concepts to start the construction of direct variation are an Object conception of multiples of a number, a Process conception of a variable and an Object conception of constant. The notion of equality (=) needs to be used as a relation between corresponding elements of two sets. The learner must have a Process level conception of one-to-one correspondence between two sets X and Y, and be able to recognize and compare corresponding members.

### Action

The Actions needed are simple algebraic manipulations involving division and/or multiplication of numbers. The learner will apply the Actions to substitute in known values and solve for an unknown value in the equation. For example, she or he might find a constant ($k$) by dividing the first value ($x$) by the second value ($y$), and then multiply a subsequent number by $k$ to find the answer. Each activity is viewed by the learner as a single instance, isolated from subsequent similar instances. At this level, $k$ is viewed as a specific value, not as an arbitrary constant. The learner may or may not see the relationship between $x$ and $y$, they may work several examples without seeing a general pattern.

The same Actions described above can take place in different settings with different representations of the relation, such as a table, mapping, graph, and an analytical example.

### Process

These Actions are interiorized into Processes as the learner repeats the Action with different values of $k$ or different values of $x$ or $y$. They might iterate through values of $x$, but instead of checking specific numbers, the student can determine in general and in his or her imagination, for example, that as values of $x$ increase, corresponding values of $y$ will increase. The learner recognizes a general behavior that $x$ and $y$ vary.

As the learner iterates over $x$, this Process with $x$, $y$, and $k$ is coordinated into a new Process where the learner can view a sequence of numbers X and can determine if elements $x$ in a set X vary with corresponding values $y$ in a set Y without multiplying each value of $x$ by $k$ but by imagining each value of $x$ as a multiple of its corresponding value of $y$. While they imagine multiplying by $k$ or dividing by $x$ and $y$ to get $k$, they may not see that $y$ is locked into a value by $x$ and $k$, into a pattern that is carried out no matter what value is given. They may or may not see the rate of variation as a constant rate.

### Object

The Process of checking if elements of a sequence of numbers X are equal to a constant multiple $k$ of corresponding values of Y, (or quotient of $x$ and $y$ is constant) encapsulates into an Object when the individual is able to apply Actions or Processes to it. The Actions that can be carried out on the Process conception of direct variation include comparing and contrasting it with other generalized properties of multiples such as doubling or halving, and to interpret the role of varies directly in the possibility that the two sets X and Y have a constant $k$ when any corresponding elements are divided. For example, they may understand that the ratio between corresponding elements of X and Y is a constant $k$. They may double the values in X and observe that values in Y are doubled. Then they may halve the values in X and observe that values in Y are halved, and so on. The learner may generalize the process that the subsequent values are determined by $k$, the constant of proportionality, locked in a pattern that is carried out no matter what value of $x$ you select. Another Action on the process may be reversing the process to determine X when $k$ and Y are known.

## Instructional Treatment Overview

We have developed an explicit approach to teaching abstraction and generalization in the mathematics classroom using computer programming exercises (Jenkins et al., 2012). The instructional treatment is grounded in APOS theory and considers the mental processes by which abstract concepts are acquired and utilized in mathematics (Dubinsky, 1984). Dubinsky is an advocate of students writing computer programs where the constructs in the program parallel the constructs of a mathematical topic under inquiry. By using computer programming exercises where the programming activity specifies the procedure for the computer to carry out, the student is motivated to reflect upon the enactment of the concept. Dubinsky states that the act of programming is a generic process which carries out what may be viewed as a more general construct in specific cases and induces the student to move towards a generic abstraction of the concept (Tall et al., 2002). Our instructional treatment is built upon this notion that programming is a vehicle for building abstractions in the mind of the learner. Numerous researchers in APOS theory have shown that computer programming is a viable tactic for teaching a variety of topics in undergraduate mathematics (Asiala et al., 1998, Weller et al., 2008). Consequently, it is a commonly held belief that computer constructions are an intermediary between concrete objects and abstract entities (Dubinsky, 2000; Asiala et al., 1996; Dubinsky, 1995).

Our instructional treatment is composed of four primary phases: essential characteristics (ESS), programming activities (PROG), general expressions (GEN), and conjectures and convincing arguments (CA). This structure is depicted in the flowchart shown in Figure 1. Designing a lesson using the instructional treatment begins with an initial, naive genetic decomposition of the concept. The essential characteristics the learner needs to understand are identified and decomposed. Using the prerequisites identified in the ESS stage, programming activities and motivating questions are documented. These are used to create a guided lesson with response sheets for the PROG phase. During the PROG stage, participants are taught how to recognize general expressions in their programs and employ them as they explore the lesson topic. The PROG and GEN steps are iterated an appropriate number of times based on the requirements identified in the ESS phase. Each iteration results in the learner identifying additional general expressions. The participants are taught to write the generalizations as mathematical statements leading to more general expressions and generalizations. Finally, learners are motivated to make conjectures about relationships between concepts using the discovered general expressions and make convincing arguments using what they've discovered. Throughout the PROG, GEN, and CA phases data is collected in the form of participant response sheets. Participants are pre-tested and post-tested over the lesson concept. All of the collected data is analyzed using the APOS framework. Our instructional treatment has been applied to a variety of mathematics concepts such as parity of integers, functions, and proportional reasoning (Stenger et al., 2016).

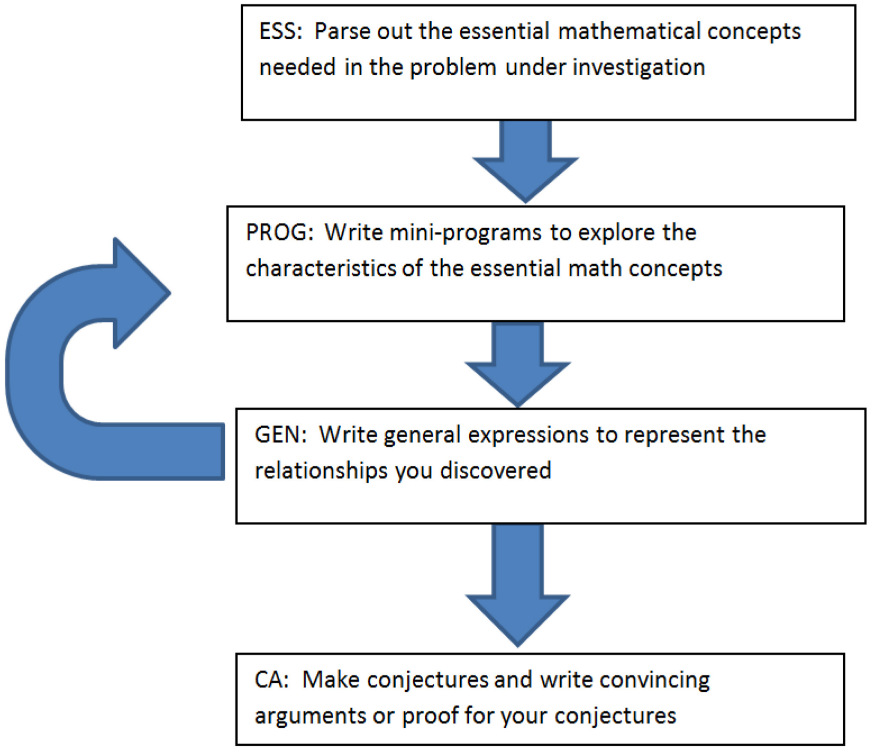



*Figure 1. The four phases of the Instructional Treatment*

## Methodology

We applied our instructional treatment to the concept of direct variation for this study. Our investigation was carried out with 33 upper level undergraduates who were interested in teaching mathematics. Each subject participated in a complete lesson including the pre-test, response sheets, and post-test. The format of the lesson was as follows. A brief introduction to the programming environment was given along with the code template shown in Figure 2. A cursory review of the relationship distance is rate times time ($d=rt$) was also presented. Using the code template with an increasing rate and fixed time, participants were asked to complete the program to output the associated distance. Learners were encouraged to experiment with their computer programs and make observations about any relationships. Once this initial table was constructed, the participants were ushered through a series of program modifications and written responses. For example, they were asked to add columns to their programs to depict the doubling or halving of the rate with time fixed and the resulting distance. Programs were modified to show the results of doubling, tripling, and halving the rate with time fixed. Written responses to questions and reflections on their observations were recorded by the participants on their response sheets including generalizations of behavior. Observations on variation and direct variation were solicited as general expressions and participants were taught how to denote the general expressions in mathematical language. For example, participants might observe that if rate doubles and time is fixed, then distance doubles. The instructional treatment was designed so that repetition with various program modifications would stimulate the desire to generalize the observed behavior and make conjectures about the mathematical construct. The final stage of the lesson involved making conjectures and convincing arguments. Participants were shown how to use general expressions to support, or refute, a conjecture using mathematical language. They were then asked to attempt their own convincing arguments with the general expressions they recorded during their inquiry. All of the participant's responses were collected on written response sheets during the lesson. Additional data was collected in the form of interviews. We recorded interviews with seven of the participants which were then transcribed and analyzed. All of the collected data was reviewed and scored using APOS theory. We devised a ranked set of scores to denote pre-action, action, process, and object levels for the direct variation concept based on our genetic decomposition and recorded scores for each subject's pre-test, response sheets, post-test, and where applicable interview data. In the event that authors disagreed, a discussion and further analysis of the data was used to reach consensus.

```
print("r", "t", "d", sep="\t")
t = 5
r = 1
while r < 11:
  print(r, t, "?", sep="\t")
  r = r + 1
```

*Figure 2. Computer programming template for lesson*

## Results

For ease of reporting and discussing results we have adapted the following convention. When referring to elements from a first set , say X, that vary with elements of a second set, say Y, we call values in the first set x and the corresponding values in the second set y.

In the discussion that follows, R denotes the researchers and U0001 to U0033 identify undergraduates. Results are presented that show how student mathematical behavior correlated to the genetic decomposition. Results also illustrate the influence of computer programming on students' ability to generalize over the concept of direct variation.

### Overall Results

Table 1 shows the number of undergraduates who were rated at the Pre-Action, Process, and Object level prior to instruction and after instruction. Forty-two percent of the students were at the process level prior to instruction. (14/33). Seventy-six percent (25/33) were either Action or Process. Only two students demonstrated an Object conception before instruction. Table 1 shows the number of undergraduates who were rated at the Pre-Action, Process, and Object level prior to instruction and after instruction. Forty-two percent of the students were at the process level prior to instruction. (14/33). Seventy-six percent (25/33) were either Action or Process. Only two students demonstrated an Object conception before instruction.

| | Before Instruction | After Instruction |
|---|---|---|
| Pre-Action | 6 | 0 |
| Action | 11 | 5 |
| Process | 14 | 9 |
| Object | 2 | 19 |

*Table. APOS level before and after instruction*

**Action**

Student responses to questions were scored at Action level based on the description in the genetic decomposition. Action level responses were analyzed by the authors for common mathematical behaviors. Student responses during the lesson and in interviews following the lesson fell into three categories of mathematical behavior:

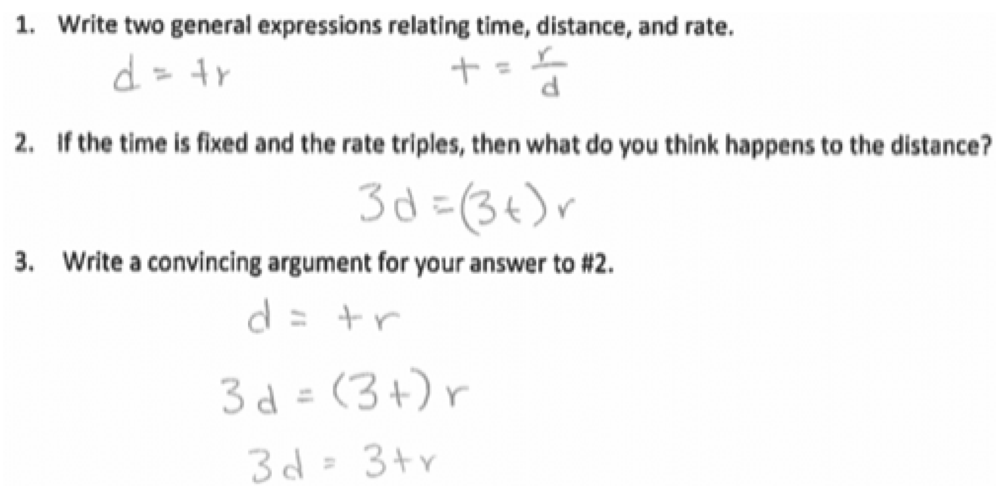
- Category 1. Using specific values or thinking about specific instance
- Category 2. Balancing the equation
- Category 3. Substituting a value in the equation

**Using Specific Values or thinking about specific instance.** In the follow-up interview, the researcher asked the student to explain their thinking on a response.

R: What were your thoughts on this? (pointing to post-test response)

U0001: I like having values just cause[sic] it helps distinguish what we're already going over like variables are fine but when I actually have a number to place with the variable it makes it easier to keep up with where I'm going and what I'm doing. So I would place a random value somewhere just so I know how to get from point A to point B.

**Algebraic manipulations of a general expression.** It is not unexpected that students at the Action level for a concept would use specific values to direct their problem solving. Surprisingly, this study found that ten of the eleven Action level students did not rely on specific values but performed algebraic manipulations on a general formula. What looked like a general argument, which might imply an Object conception, was instead an explicit, step-by-step procedure to balance the equation. This is similar to Frith, et al. (2016) who found students could work proportion problems applying "mechanical knowledge or algorithmic procedures" without actually reasoning about the relationship. Mechanics of algebra included either trying to balance the equation (9 instances) or to substitute general expressions into the equation (8 instances). Students at this level did not meet the prerequisite skills, as defined in the genetic decomposition, two students were at the pre-action level for the concept of multiples, eight did not meet the prerequisite process level for the concept of variable, two did not meet the prerequisite for constant, and one did not meet the prerequisite for the concept of one-to-one correspondence.



1. Write two general expressions relating time, distance, and rate.

$d = tr$ $\quad$ $t = \frac{r}{d}$

2. If the time is fixed and the rate triples, then what do you think happens to the distance?

$3d = (3t)r$

3. Write a convincing argument for your answer to #2.

$d = tr$

$3d = (3t)r$

$3d = 3tr$

*Figure 3. Action Category 2 – Balancing the equation*

**Balancing the equation.** The snip of student U0005 in Figure 3 shows a typical response in Action Category (2). The student carried out the step by step procedure, multiplying both sides of the equation by a constant, e.g., if $d=tr$ then $3d=(3t)r$. This student wrote in their response of a "need to balance", as they multiplied both sides by 3.

**Substituting a value in the equation.** The snip of student U0029 in Figure 4 shows a typical response in Action Category (3). The student carried out the step-by-step procedure, substituting $3r$ for $r$ in the equation $d=rt$. This work demonstrates a lack of the prerequisite requirement for a process understanding of variable, as $d$ takes on the role of the first distance and the second distance.

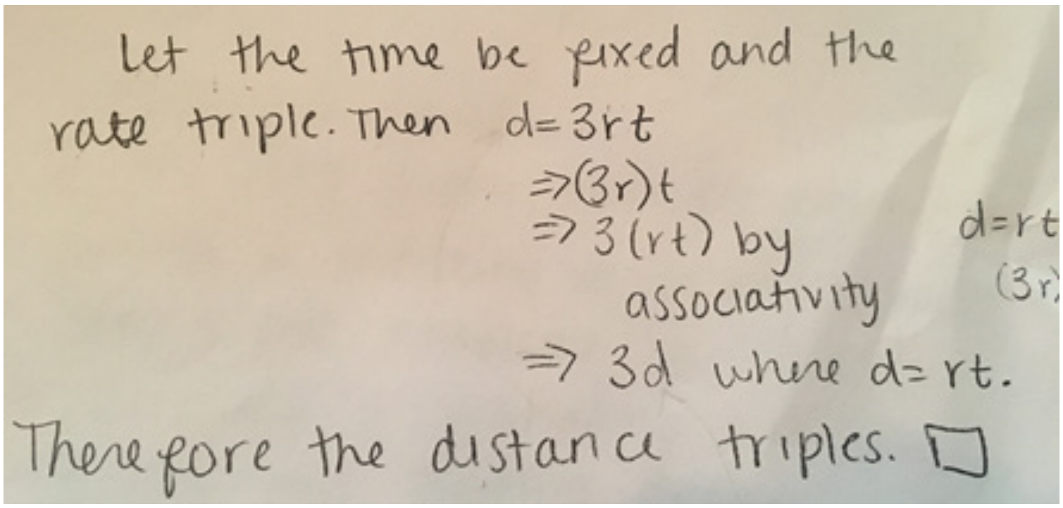



*Figure 4. Action Category 3 – Substitution*

**Process**

Of the 14 students at the process level, 10 demonstrated the notion of varies without demonstrating a notion of varies directly. Students' concept of varies fell in two categories:

- Category 1. Varies - $x$ increased (or decreased) then $y$ increased (or decreased)
- Category 2. Varies by some multiple - $x$ increased (or decreased) by some multiple, then $y$ increased (or decreased)

In either case, whether or not they repeated the given information about $x$, for their part in the solution they did not mention the multiple. They did not indicate an awareness of the "locked relationship" between $x$ and $y$ that is determined by the constant of proportionality $k$.

**Varies.** The snip of work from U0003 in Figure 5 below shows a typical response for varies in Process Category (1). The student described a dependence between rate and distance where the rate increased then the corresponding distance "will increase as well". The parenthetical statement by the student "The same time frame in a quick pace" indicated they were imagining a process in their mind, where rate and distance varied in a coordinated way.

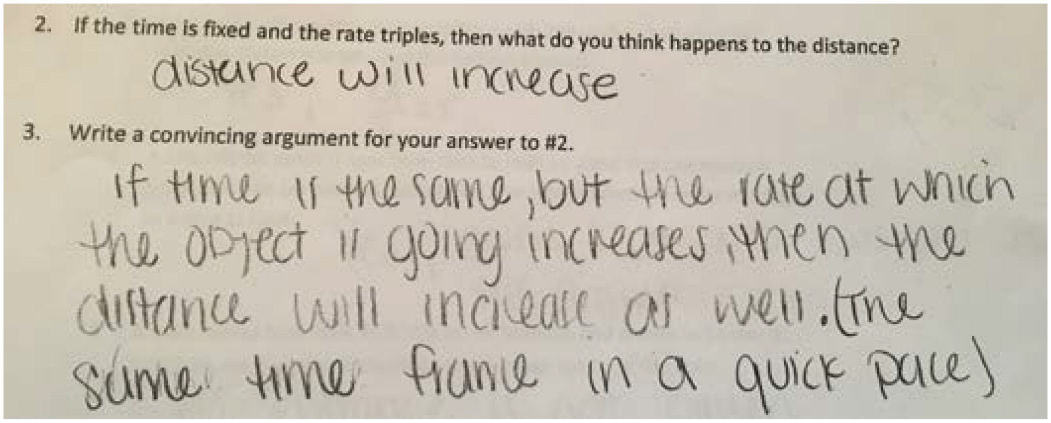



*Figure 5. Varies – Process Category (1)*

**Varies by some multiple.** The snip of work from U0007 in Figure 6 shows a typical response for varies in Process Category (2). The student described a dependence between rate and distance where the rate tripled then the corresponding distance traveled increased. They are imagining a process where an object is moving at a faster speed so "a greater distance would be covered in a fixed amount of time". There was a Process in their mind where rate and distance varied in a coordinated way.

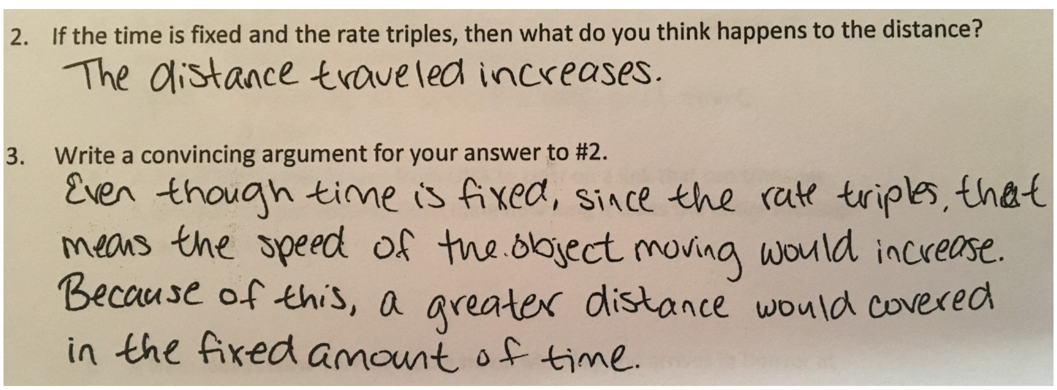



*Figure 6. Process Category (2) - Varies*

In neither case did the students in Process Category (2) demonstrate a knowledge of the "locked in" relationship between $x$ and $y$ that is a part of direct variation and is fully determined by the constant of proportionality.

The other mathematical behavior that was common among students at the process level was incorrect use of substitution. Students were stuck performing learned, routine algebraic manipulations and did not look for a general relationship. When they engaged in computer activities designed to act on the varies relationship such as doubling or halving elements of X, they did not see the effect on the relationship between corresponding elements in Y as locked in place by $k$. Eight students used flawed substitution to attempt to solve the general expression algebraically, in the same manner observed in Action level students (see Figure 4 above).

**Object**

The responses that indicated Object level understanding of direct variation, according to our genetic decomposition, fell in two categories:

- Category 1. Relationship between X and Y locked in place by $k$

- Category 2. Elements of X were dependent on values in Y and the dependency determined by *k*

Verschaffel et al. (2000) found that students' natural, naïve understanding of proportions were a hindrance to deeper understanding of the concept. Although 32 of our 33 students correctly identified two general expressions relating *d*, *r*, and *t*, only two students demonstrated an Object level knowledge of the fixed relationship between X and Y, determined by *k*, before the instructional treatment.

**Influence of Computer Programming on Generalization**

The influence of writing computer programs to explore the concept of direct variation was demonstrated by 16 of the 33 students. These students referenced their programming activities in their responses, in multiple instances, even though neither the question (nor the instructor) suggested responding with program code. Students naturally and intuitively adopted language from their programs. Twelve of the sixteen, who referenced programming in their responses concerning general expressions, improved at least one level during the instructional treatment, while two stayed the same and two went down a level. The students who referenced their programs when asked to give a general expression fell into two categories: (1) Computer Input: Print Statements and (2) Computer Output. In both cases illustrated below by typical responses, the students imagined generating code in their mind, and copied their imagined code onto their response sheet.

The response from Student U0007 in Figure 7 shows a typical response for Computer Category (1). The student imagined writing a computer program with the displayed print statement as an input statement. The response below was after the first computer programming activity. The print statement was stuck in between the answers for Response #3 and Response #4. It appears as a transition between the English statement "the distance is also doubled" and the general expression *(2r)t*. The transitory work is seen as the student wrote "*d = r2*t*" above the print statement "*(r*2)*5"*.

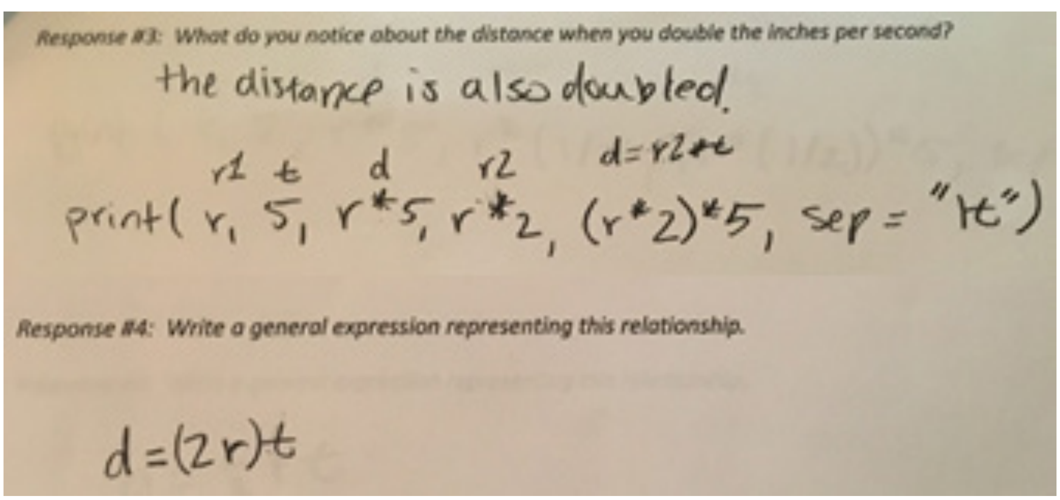


*Figure 7. Category (1) – Computer program*

Two students demonstrated evidence that running a computer program in their mind and reflecting on the output in table form was a transition from English language to mathematical language. The response from Student U0002 demonstrated the typical response for Computer

Category (2) by constructing a table for Response #5 in the left margin after concluding, "halving the rate also halves the distance". The same student then responded in Response #6 with "d2 = $\frac{rt}{2}$.".

The response from Student U0002 in Figure 8 shows the typical response for computer category (2). The student wrote the table over to the side of Response #5 and appeared to use it as a transition between the responses. This student worked #3 and #4 in a similar manner, using a table from a computer program they ran in their imagination.

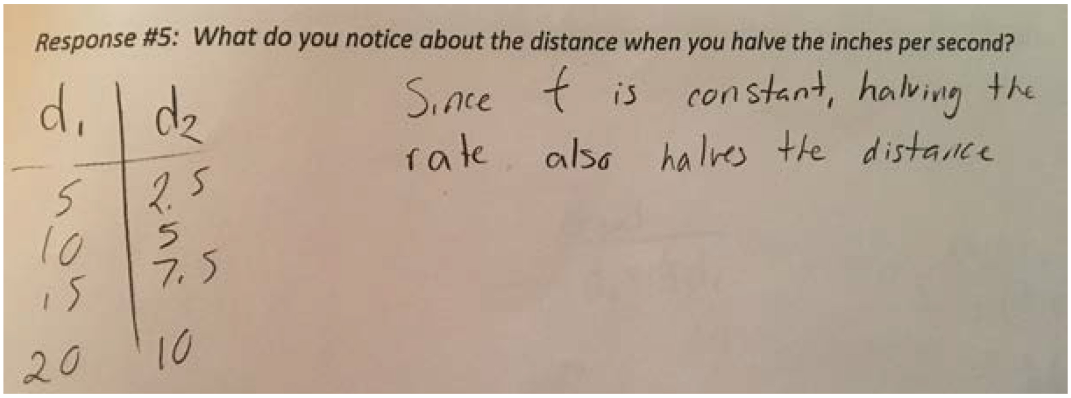



*Figure 8. Computer Program – Category (1)*

Students U0006 and U0007 were probed about the relevance of the programming in a follow-up interview:

R: Did you think the programing and the coding helped with the proofs and seeing the general expressions? Do you feel like it contributed in any way?
U0007: I was it does make it easier to look at because you can see everything broke[sic] down and spread out.
U0006: I agree, totally.

In the following interview snip, U0004 described how they developed a "mindset of generalizing" during the instruction. They did not demonstrate a general notion using variables in their expression until after the first programming activity.

R: Just describe when you were writing the last couple of proofs or either one of the proofs
U004: I was thinking more of just the letters and generalizing it after we had done those together and the ones on the other response sheets because I think I was in a mindset of generalizing it…
R: Right
S: So the way I wrote it out, I put more notation the second time on the post-test.

## Conclusion

In this study, students explored direct variation through an explicit method for teaching generalization that uses computer programming and convincing arguments. The researchers found that scoring and assessing undergraduates' conception of direct variation was complex due to the task-dependent and context-dependent nature of conception. The genetic decomposition adequately described the students' constructions observed in the data. We noticed students at the

Action level tended to manipulate algebraic expressions without understanding the underlying structure. We found many students in our study have a notion of vary but not directly varies. We observed some students who needed to construct the property vary, at the Process level, before constructing the property varies directly, at the Object level and conjectured that a notion of vary is a prerequisite to directly vary. Therefore, we have modified our genetic decomposition to account for this in future studies. We found that prerequisite deficiencies corresponded with the inability to progress through levels of understanding as measured by APOS. We found that students naturally turned to their computer programs to help find general expressions for the concept. Some students considered the inputs to their programs and others reflected on the outputs of their programs when asked to write general expressions for observed relationships. The programming activities influenced students and served as a catalyst to move from purely English descriptions of their conceptions to using mathematical symbols and a "mindset of generalizing". The results of this study can facilitate further analysis of using computer programming and proof writing to overcome cognitive obstacles in undergraduates' understanding of proportional reasoning.

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**First-generation Low-income College Student Perceptions about First Year Calculus**

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## Abstract

*The purpose of this study was to explore first-generation low-income students' experiences with first-year calculus, including their self-belief in being successful in math. As part of the Progress though Calculus project, one STEM-focused institution was studied with survey results from students enrolled in first year calculus, and interviews and a focus group of three first-generation low-income students who completed first year calculus. Quantitative results illustrated similar rates of faculty and student interaction and increased self belief in being successful in math while taking first year calculus for first-generation students, in comparison to their continued generation peers. Qualitative findings emphasized the value of creating interactions with faculty and other students, and faculty's impact on students' sense of belief in being successful in calculus. Promoting non-cognitive factors such as student support and self-belief in math success may influence math completion of first-generation low-income students.*

**Key Words**: First-generation low-income students, self-belief, first year calculus

Along with innovative pedagogies and curriculum enhancements to improve math education, it is also important to consider the increasingly diverse student population, gaps of math completion among marginalized students, and the impact of non-cognitive factors such as self belief and support networks as part of the formula for student success in mathematics. In this study I explore the experiences of first-generation low-income students in first year mathematics from an asset approach, meaning I focus on what these students bring from their identities to support their success, as well as attending to barriers these students face. For decades college administrators and researchers have viewed the first-generation and low-income identities as "at risk", which is reinforced by the well documented national graduation gaps of these students. For instance, among 4.5 million college students from 1995-2002, six-year graduation rates for first-generation low-income students were 44% lower than continuing-generation higher-income students (Engle & Tinto, 2008).

To address these graduation gaps and move beyond the focus on the disparities of underserved students as a disadvantage to being successful in college, a paradigm shift is needed to to support students that attend college rather than require students to adapt to college. One way to provide support is focusing on first-year mathematics completion since it is highly correlated to graduation rates (Colorado State University, 2015). By understanding what assets students bring to first-year mathematics success, we can better understand how to support a higher graduation rate among this population.

## Shifting to an Asset Approach to Understanding First-generation Low-income Students

Nationally, 28% of all undergraduate college students are first-generation, and 27% are low-income (Cook & King, 2007). Most prominent research on first-generation low-income students has focused on deficits of this student population, as a disadvantage to being successful in college. Academic deficiencies of these students include: higher need for remedial courses (Chen & Carroll, 2005), undeveloped student success skill sets (Collier & Morgan, 2008), less academic and co-curricular engagement (Pascarella, Pierson, Wolniak, & Terenzini, 2004; Warpole, 2003), and lower educational aspirations (Pike & Kuh, 2005). For example, in considering the success in math of first-generation low-income students, lower levels of math completion have been documented. An analysis of first-generation student college transcripts from 1992 to 2000 shared that 55% of first-generation students took at least one math course in college compared to 81% of students whose parents had a bachelor's degree (Chen & Carroll, 2005). Additionally, at Colorado State University (2016) after controlling for prior academic preparation, first-generation, students of color, and Pell eligible students were significantly less likely to place into college algebra and to complete three credits of math in the first year compared to their peers. Non-cognitive disparities include a lack of parental support (Ward, 2012), not as much social capital (Lin, 2011), lower levels of a sense of belonging (Aires & Seider, 2005; Ward, 2012), and a cultural mismatch with the university (Roberts & Rosenwald, 2001; Stephens, Fryberg, Markus, Johnson, & Covarrubias, 2012).

A different approach to defining deficits and expecting students to compensate for deficits is research done within the perspective of promoting the strengths and assets of students as an advantage for collegiate success. Although not as prevalent, research within an asset framework focused on higher self-authorship with first-generation students, which is a transition from relying on others to defining oneself to more internal thinking in determining one's life path (Pizzolato, 2003), high levels of motivation to attend college with an emphasis on hard work (Martin, 2012) , and a desire to contribute to society (Olive, 2009).

Research has also shown the importance of meaningful individual connections to support first-generation low-income students. For instance, a strong network of faculty who care and have high expectations plus peers who offer encouragement has been found to help first-generation college students transition to college (Coffman, 2011), and obtain a college degree (Lourdes, 2015).

Broader interventions that centered on creating an institutional culture to support these students' success have also been impactful. This supportive culture was promoted by emphasizing interdependence of being part of a community, which positively impacted first-generation students' academic performance prior to the fall semester (Stephens et al., 2012). This research represents a paradigm shift of supporting students that attend college rather than requiring students to adapt to college, has been defined as becoming a student ready college (Brown McNair, Albertine, Cooper, McDonald, & Major, 2016) .

## Self-belief Factors Affecting Math Completion

To begin to reflect on ways to enhance math completion with first-generation low-income students, one non-cognitive factor to consider is self-belief based on the power of positive psychology, which is the study of conditions that influence the optimal functioning of people (Gable & Haidt, 2005). Theories to inform this perspective of developing student assets are stereotype threat (Steele, 1997), which challenges college success, and self-belief (Bandura, 1977; Dweck, 2006), which can potentially mediate challenges and promote academic success. Stereotype threat theory asserts that negative stereotypes of one's performance based on his or her social group can put individuals at risk of lower performance (Steele, 1997).

In response to the negative influences of stereotype threat, positive psychology theories of self-belief are used with Bandara's theory of self-efficacy and Dweck's theory of a growth mindset. Bandara's theory of self-efficacy is a social cognitive theory based on the belief that one can achieve his or her goals (Bandura, 1977). Expanding upon self-efficacy is growth mindset, which is the belief that one may improve through engagement with the learning process (Dweck, 2006).

### Self-belief and Math Achievement

Research on the relationship of self-efficacy and math achievement is evident both with students who have not performed well in math along with engineering students with high levels of math performance. Investigating students who were repeating a developmental math course, they identified high self-efficacy as the essence of their persistence despite a low self-concept in mathematics (Canfield, 2013). For engineering students who usually excel in math, self-efficacy was correlated with mathematics achievement scores and cumulative grade point averages (Loo & Choy, 2013).

Research has demonstrated greater course completion rates in challenging math courses (Yeager & Dweck, 2012), which supports an approach emphasizing a growth mindset. Many studies have also focused on the growth mindset as a mediating factor to stereotype type threat of marginalized populations in math performance. Dar-Nimrod and Heine (2006) studied math achievement and gender, and illustrated that females with a growth mindset performed better than females with a fixed mindset on math assessments similar to the Graduate Record Examination.

### Purpose of this study

Lower math completion rates of first-generation low-income students along with the positive impact of self-belief and math achievement, warrant further investigation into ways that self-belief can enhance success in mathematics. The purpose of this study is to explore first-generation low-income students' experiences with first-year calculus, with particular focus on their self-belief in being successful in math. First-generation students are defined, using the TRIO definition, as students whose parents have not obtained a college degree (Nunez, Cuccaro-Alamin, & Carroll, 1998), and low-income is defined as Pell recipients, which are government grants for college students with exceptional financial need (Dynarski & Scott-Clayton, 2013)

Specifically, the following research questions guide this work: (1) How do first-generation low-income college students experience first year calculus at a STEM focused institution? (2) How does first year calculus influence the self-belief of first-generation low-income college students to be successful in math?

### Methodology

To provide context of this research, a broad overview of the Project through Calculus (PtC) research that is studying ways to enhance student calculus completion rates is summarized. A part of the PtC research was a pilot study at one STEM institution, which is the focus of this paper.

#### Progress through Calculus Research

The Progress through Calculus study is sponsored by the Mathematical Association of America and funded by the National Science Foundation (NSF) to research student success in calculus. Twelve higher education institutions were identified by the research project team as institutions using structural, procedural, curricular, and pedagogical approaches to the pre-calculus and calculus program that has been successful in higher math completion rates, especially with underrepresented students. Specific approaches include math placement, course structure, active learning, student support, instructor coordination, graduate teaching assistant training, integration between math and other STEM disciplines, and local data analysis. Prior to researching the twelve institutions, three pilot studies were held at institutions based on geography, convenience, and access; to refine data collection content and procedures.

One of the pilot studies as the focus of this study was done at an institution that implemented initiatives including developing a teaching faculty track, coordinating efforts with calculus faculty, and implementing active learning practices in calculus courses. Additionally, this institution was of interest since the first year calculus D FW rates have decreased from 22% in fall 2006 to 10% in spring 2015.

**Research Approach**

This case study mixed methods design was used as a means to understand a complex social unit problem holistically (Merriam, 1988) which for this research was student experiences in math, and to study "how" and "why" questions (Yin, 2003) that informed these experiences. This study utilized multiple sources (Merriam, 1988), focused on the contextual environment (Fryvbjerg, 2011), and gained multiple perspectives with a progressive focus reconsidering issues throughout the research process (Stake, 1995). Multiple data sources, varying perspectives, and in-depth information for this study included student interviews, a student focus group, and a student survey. Additional contextual information about the environment was gathered through faculty interviews, classroom observations, and information from the local coordinator. The mixed methods design was a convergent parallel design with both interviews/focus group and survey results gathered and analyzed independently, and then the results interpreted together (Creswell & Plano Clark, 2011).

**Study Participants and Data Collection**

In spring 2017, this pilot study took place at a four year public STEM focused institution (4,600 undergraduate students, 11% first-generation, 11% Pell recipients, and 26% students of color) with a two day site visit. Students that participated in the interviews and focus group were recruited with assistance from the multicultural support program director. Three first-generation low-income (defined as financial struggles being central to their college experience) students participated including a first year white female majoring in engineering, a third year African American female majoring in computer science, and a third year Asian male majoring in computer science. The student survey was developed by the Progress through Calculus Research team, and questions on self-belief and resources for academic success from students enrolled in first-year calculus were utilized for this study.

The data collection is outlined in Table 1 focusing on one hour individual interviews at mid-semester and one focus group with the same three students at the end of the semester. The interview responses and institutional context informed the focus group questions by developing both themes and areas of differing perspectives. Two researchers were present at the focus group, with one facilitator and one who took reflective notes. The math instructors facilitated the completion of the student surveys during one class period.

**Table 1. Data Collection Process**

| Data Method | Data Means | Data Collection Timeline |
|---|---|---|
| Student Interviews | Audio-recorded and notes | Mar-17 |
| Institution context | Faculty & Staff interviews<br>Classroom Observations | Mar-17 |
| Student Focus Group | Audio recorded and notes | Apr-17 |
| Student Survey | Completed in class | Apr-17 |

### Data Analysis

Qualtrics survey results on student self-belief and interactions with faculty and peers as support resources were analyzed with SPSS using chi-square analysis. For the interviews and focus group, holistic data analysis was accomplished with an inductive process to identify relevant emerging themes (Yin, 2003), making sense out of the data collection (Miles, Huberman, & Saldana, 1994).

The interviews and focus groups were transcribed and then coded with MaxQDA, a qualitative coding software program. The researcher began with a first-cycle coding process and then reviewed each code and coded segment to illuminate connections between the categories in the second-cycle coding process, and used a second coder to review the coding process (Miles et al., 1994). Throughout the data analysis process, high quality data verification was implemented including attending to all the data, and using the researcher's expert knowledge (Yin, 2003). For instance, the researcher's expertise of creating generalizations was balanced with including description of vicarious experiences (Stake, 1995). Collectively the data analysis of breaking apart and piecing together data themes, illuminated further understanding of the math experiences of first-generation low-income students (Merriam, 1988).

### Trustworthiness

The core principles of trustworthiness for this study were triangulation of data sources, incorporation of various evaluators with different theoretical perspectives, and continual maintenance of a chain of evidence (Yin, 2003). Triangulation of multiple sources including interviews, focus groups, and survey data enhanced the richness of the study. Additionally, working together with four researchers with mathematics backgrounds provided an additional perspective to my higher education background, which broadened the collective lens of this research. Field notes were taken by all the researchers and compared to each other's documents for a collective chain of evidence.

## Quantitative Survey Results

On the survey, there were 335 respondents, with a 69% response rate. Demographics of the students were first year 317(95%), first-generation 36 (11%), and Pell eligible 44 (13%). The questions analyzed in this study asked about faculty and student interactions, and self-belief in mathematics.

## Faculty and Student Interaction

Focusing on faculty interaction outside of class, survey frequencies found similar percentages of first-generation students ≈20% (7) compared to continuing generation students (57). Concerning interactions with peers for all students, there were higher percentages of working with peers outside of class than instructors, and similar percentages with first-generation ≈58% (21) compared to their continuing generation (173). In general low income students had similar percentages than first-generation students.

## Student Self-Belief

Aspects of self-belief studied were confidence, ability to do math, and growth mindset. The survey results indicated that most first- generation ≈65% (24), and continuing generation ≈62% (184) significantly or moderately increased their confidence in math by taking calculus, which are similar percentages amongst first-generation and continuing generation students.

Along with confidence, findings about a student's ability to learn mathematics revealed that most first-generation students ≈72% (26) said that their math ability "moderately or significantly increased" with taking calculus, which was slightly higher than continuing generation students ≈67% (198). Additionally, ≈ 75% (27) first- generation students shared that their growth mindset "significantly or moderately increased" while taking first year calculus, which again was slightly higher than continuing education students ≈70% (208). In general, low income students had similar percentages than first-generation students. Overall there were fairly similar percentages of student self-belief in math success with first-generation and continuing education students.

# Qualitative Interview and Focus Group Results

The interview and focus group results emphasized the importance of interactions with faculty and students, highlighted faculty's impact on students' sense of belief and students growth mindset in being successful in math, and explored students' identities around generational and income status.

## Faculty Interaction

In exploring faculty interactions, key factors that emerged were the importance of how faculty responded to questions, and the value of small individual interactions. An example of how faculty reacted positively to questions is illustrated by the following statement by one student, "When you ask a question and a faculty member is really supportive and they don't look down on what you ask, they just answer this is what it is." A less supportive response is illustrated by the statement by another student, "if we ask a question that is dumb he looks down on us, so it's really intimidating."

In addition to responding to questions, short interactions with faculty had a big impact on the student's experiences in math courses. One student illustrated the impact of a faculty interaction as being the most positive experience in calculus.

> "My math teacher was sitting outside on one of the picnic tables and I didn't want to sit with him and talk about math….so I sat on a bench …. He was going back into his office and he stopped by and was talking to me….How are your classes going? Then he said he didn't care about the other classes just mine, it kinda made me laugh… It was kinda of like your cool and we joke around now. I feel like I know him a lot better. Listening to him lecture I have that connection: you know what you're talking about I will believe what you are saying. I

mean I guess it showed because I did a lot better on my last test. That was the best positive experience that I have had in calculus. It was getting that connection."

**Student Interaction**

Along with faculty interactions, the importance of peer support was highlighted, describing how students worked with other students in math courses. Two students shared that they looked for students that were doing better than them, and then would ask them to be in a study group. Another example was a network of students beginning with two students working together, each branching out to other friends, and then coming back together to complete the homework. Illustrating the interdependence of working with another student and becoming more of an independent learner is illustrated below.

> "Last semester I had that one student who was really good at math I feel that she pushed me along and unfortunately I'm not in the same section with her this semester, kinda of realizing that I'm alone here. I need to figure it out but we still meet up to do the homework since we have the same class. I've been able to go, I solved 1 and 2 what did you get? Seeing if we have the same answers really helped me to learn more, being able to say I learned this so I can tell her I learned it."

A common theme of their math experiences was the major significance of working with faculty and other students outside of class.

**Faculty Impact on Student Self-Belief**

The other prominent research finding was the tremendous impact that faculty had on student's self-belief in being successful illustrated by the quote below.

> "I went in [to her office] and said I can't do Calc II, I'm a fraud, and she said yes you can. She said we are going to sit down and go through this exam and she went question by question and she said what did you do wrong? It's not like you don't understand what's going on sometimes you are reading the question incorrectly. You know the material you just need to interpret the question and answer it correctly. Okay that clicks. She didn't give up she didn't brush me aside as one of twenty students. She remembered my name which was important."

**Growth Mindset**

Beyond faculty and peer support, another theme of all of the student's experiences was some sort of struggle in a course. Strategies of how they responded to academic struggles were studying earlier before a test, and going to office hours. More importantly was the constructive reflection from those experiences. Two students shared,

> "Failing, it gives me more motivation. I can't study one week in advance, let's go 2 weeks. Question yourself and say I did that wrong, you kinda go through with that self evaluation I know what I did wrong and I know how to improve after a bad test".
>
> "I know how to pick myself up. It's not something that is going to crush me and I am going to succeed. Where other people have dropped out because they earned their first B on an exam, I was just bewildered, so I definitely know how to push through the hurt".

These reflections illustrated a growth mindset valuing the importance of continued engagement in the learning process to improve performance.

**First-generation Low-income Identities**

One of the interesting aspects of the first-generation identity is that it may be an emerging identity that becomes apparent as a young adult differing from a lifelong identity. For instance one student in this study shared that she learned about first-generation from a TRIO staff member who said that being first-generation would help her get to college, and another student discovered that he was first-generation from his high school principal. In reflecting on this newly discovered first-generation identity, a student described first-generation with two words, story and determination. "Where you come from is your story and where you go is your determination."

In addition to the first-generation identity, all of the students spoke about financial struggles, which was a more long-standing identity when growing up. A high saliency was shared that being first-generation and low-income impacted how they experienced college in terms of navigating this new adventure, working experiences, and living arrangements. Additionally other intersecting identities such as being a student of color, and having immigrant parents were also interwoven into their perceptions and experiences.

Overall, the students' comments about being first-generation and low-income were more focused on the assets of these identities rather than the deficits. Advantages of being first-generation included having more "ump" to get through school, being acknowledged by career representatives as being hard workers, and having more job experiences. One student shared below that as a low-income student she had better work experience than her higher income peers.

> "We say we wish we had rich folks and then we say no we don't because we know where we would be at. We would take things for granted… I finally got an internship but I had experiences that other students didn't. I have had a job for 4 years and worked with teams, and other students that come from a linage family you have never had to work. You don't know what it means to really work."

The challenges of being first-generation low-income students focused on the uncertainty of college, and dealing with the imposter syndrome of questioning the capability of being successful. Beyond the internal reflections about being first-generation low-income students, there was also a theme that continuing higher income students were not only unaware of these identities, but lacked an appreciation that some students may be different from the majority identities as illustrated below.

> "Everyone thinks that their parents are the same as theirs. That they came from a good family, parents that went to college have good jobs. I think that is why a lot of students who do have that- think that everyone else has it. Predominately like the continuing generation students here look down upon other students."

These results illustrate increased self belief in mathematics at this institution, the power of faculty and student interactions as integral to first-generation low-income student experiences in math, and especially the impact of faculty believing in student success.

## Discussion

A major highlight of this research was the positive impact that calculus had on students' increased confidence in math, their ability to learn mathematics, and their sense of hope in being successful in calculus, which was mostly similar for first-generation low-income in comparison to their continued higher income peers. Additionally students illustrated a growth mindset in sharing the ability to bounce back from academic struggles. All of these factors may suggest a positive impact on the self-belief in being successful in math for these first-generation low-income students at this institution. This is an important finding considering stereotype threat which is well documented with students having marginalized identities in negatively impacting students beginning with studying women's performance in math (Spencer, Steele, & Quinn, 1999), African Americans' performance on intelligence exams (Steele & Aronson, 1995), and intellectual achievement of low-income students (Croizet & Claire, 1998). Learning even more about how faculty and institutions can provide an environment for enhancing student self-belief is recommended.

Another highlight of this study was the importance of faculty and student interactions. Although strong faculty and student interactions reinforce well established high quality teaching practices, it is an important reminder to keep these qualities at the forefront especially in college courses. It is also essential to note that faculty interaction was similar for first-generation students compared to continuing generation students. This finding conflicts with predominate research that suggests less curricular engagement among marginalized identities, such as first-generation and low-income. For instance, Soriaa and Stebleton (2012) found less academic engagement with first-generation students measured by faculty interactions and contributions to class discussions, and Goodman et al. (2006) found that low income students did not experience faculty contact and active learning at the same levels as higher-income students. Ensuring faculty interaction may be even more imperative for first-generation low-income students, which is reinforced by Lohfink and Paulsen (2005) who found that first-generation student participants who had higher levels of academic engagement focusing on faculty-student interaction persisted in college at higher rates than students with lower levels of academic engagement.

Additionally, in light of the research findings that first-generation low-income students are working with peers outside of class at higher rates than instructors, perhaps more intentional integration of student study groups would be impactful. The impact of faculty and student interactions found in this study may be an illustration of a supportive culture that fosters academic success.

In regard to first-generation and low-income identities, students focused on the advantages of being first-generation low-income students including increased motivation, more work experiences, and being acknowledged by others as hard workers. There was also a theme that continuing higher income students lacked an understanding of first-generation low-income identities. Consequently, it may be important for institutions to strategize ways that encourage all students to gain a stronger appreciation of varying student identities and to emphasize how diverse intersecting identities contribute to the campus community.

Repeating this same study at other institutions as part of the Progress through Calculus research project with larger sample sizes, will provide cross institutional results and additional insights. Although this study illustrates some promising results in enhanced student self-belief in math at one institution, additional more comprehensive findings will continue to explore ways to create an environment that promotes self belief in math in developing the talent of first-generation low-income students.

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Conventions or Constraints? Pre-service and In-service Teachers' Understandings

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*We draw on data from pre-service (PST) and in-service (IST) teachers to characterize relationships between what we perceive to be conventions common to U.S. school mathematics and individuals' meanings for graphs and related topics. We use PST responses during clinical interviews to illustrate three themes: (a) some PST responses implied that things we perceive to be conventions were instead inherent aspects of PSTs' meanings (habitual use of "conventions"); (b) some PST responses implied they understood certain practices as customary choices not necessary to represent particular mathematical ideas (conventions qua conventions); and (c) some PST responses exhibited what we perceive to be contradictory actions and claims. We then focus on data collected with ISTs against the backdrop of these themes to highlight similarities across these populations and to provide implications of our findings.*

Keywords: Conventions, Graphs, Preservice Teacher Education, In-service Teacher Education

Hewitt (1999, 2001) distinguished between arbitrary and necessary information in mathematics curriculum and learning. He described arbitrary information as that which students need to be informed about by an external source (e.g., the name of an object or representational conventions), whereas necessary information students can deduce for themselves. In addressing graphs and coordinate systems, Hewitt (1999) described aspects of coordinate systems that are necessary (e.g., the need for a starting point or origin) and noted:

> These are some aspects of where mathematics lies within the topic of co-ordinates, rather than with the practising of conventions. I am not saying that the acceptance and adoption of conventions is not important within mathematics classrooms, but that it needs to be realised that this is not where mathematics lies. So I am left wondering about the amount of classroom time given over to the arbitrary compared with where the mathematics actually lies. (p. 5)

Whereas we imagine mathematicians and mathematics educators likely agree with Hewitt's distinction, the extent to which students and teachers hold meanings consistent with his description is an open question. Hence, in this report, we address the question, "In what ways do pre-service and in-service teachers understand graphing conventions?" Specifically, we highlight the extent to which what we perceive to be conventions are pervasive in PSTs' and ISTs' meanings for graphs and related ideas (i.e., function).

## Theoretical Perspective: Conventions, Constraints, and Habits

Conventions—or those practices an individual perceives as customary choices within a community of individuals—play an important role in mathematics, with notable examples including notational systems (e.g., function notation), order of operations, and representational systems (e.g., the Cartesian coordinate system). With respect to representational or notational systems, a primary reason individuals in a community establish or adopt conventions is that conventions afford consistent, simplified, or efficient ways to capture, convey, and constrain aspects of ideas and reasoning. Hewitt (2001), however, noted, "The learning of names and conventions plays a vital role in engaging with mathematics and communicating with others about mathematics, but is not mathematics itself" (p. 44).

The conventions established by a community typically do not originate at the community level, and for this reason the emergence and use of conventions cannot be reduced to strictly issues of notation and communication. Conventions predominantly emerge through a process of negotiation, wherein a community collectively adopts, rejects, or modifies the ways in which *individuals* originally attempt to capture and convey aspects of their thinking (Ball, 1893; Cajori, 1993; Eves, 1990; Menninger, 1969). Importantly, negotiation at the community level is not only about a choice of physical notation or representation, but it is also a negotiation of interpretation and meaning (Thompson, 1992, 1995). The emergence of conventions involves individuals of a community simultaneously negotiating, clarifying, and choosing customary forms of expression *and* intended constraints on the interpretations of those expressions. An implication of this is that when an individual enters a community, to understand something as a convention of that community requires that the individual address issues of notation and representation while simultaneously becoming aware of the implied constraints on interpretations and meanings. Although it is easy to speak of conventions as if they exist in a community independent of a knower, we contend conventions are personal constructs. A *person* understands something as a convention of a community when *that person* understands something as a customary or arbitrary choice made by some perceived community and with respect to some idea or concept.

When discussing students' use of notation and representational systems, Thompson (1992) described two ways in which an individual can use a convention: (a) using a convention unthinkingly and possibly unknowingly and (b) using a convention with an awareness that she is conforming to a convention (i.e., convention *qua* convention). Thompson (1992) elaborated, "To understand a convention *qua* convention, one must understand that approaches other than the one adopted could be taken with equal validity. It is this understanding that separates convention from ritual" (pp. 124-125). We leverage Thompson's distinction in the context of teachers' graphing activity, arguing a teacher's use of graphs entails a *convention qua convention* if the teacher's meaning includes an *awareness* of maintaining a convention (i.e., understands the convention as one way to represent some idea among other *equally valid* choices). We claim a teacher's use of graphs entails the *habitual use of "convention"* if the "convention" is a necessary or inherent aspect of a teacher's meanings. In this case, what we as researchers perceive to be a convention is not a convention qua convention with respect to that teacher's meanings; hence, we intentionally use quotations to indicate this difference in perception. As we illustrate in the results section, what an observer understands to be a convention can instead be habitual to a PST's or IST's use of graphs to the extent that the teacher unknowingly assimilates situations in ways that entail the "convention". Alternatively, the teacher might consider using graphs in some different way, but the teacher does not conceive such a way equally valid due to her or his system of meanings necessitating that the "convention" be maintained.

## Relevant Literature

Speaking on various conventions practiced in U.S. and international school mathematics, Mamolo and Zazkis (Mamolo & Zazkis, 2012; Zazkis, 2008) argued that students (and teachers) are not supported in understanding certain conventions as customary choices if educators unquestionably maintain particular conventions. Mamolo and Zazkis hypothesized that a potential outcome of educators unquestionably maintaining conventions is that students do not develop meanings that enable them to understand novel and unconventional situations (e.g., alternative coordinate systems). Mamolo and Zazkis's stance echoes Thompson's (1992) claim, "to ignore convention in our teaching can lead students to think of mathematics ritualistically" (1992, p. 125).

International and U.S. education researchers who have investigated students' meanings for function and other related areas have reported findings that are compatible with Mamolo, Zazkis, and Thompson's sentiments. Researchers (Akkoc & Tall, 2005; Even, 1993; Montiel, Vidakovic, & Kabael, 2008; Oehrtman, Carlson, & Thompson, 2008) have documented that students' meanings for function foregrounds the ritual application of the *vertical line test*. Montiel et al. (2008) identified students who applied the vertical line test when investigating relationships in the polar coordinate system. Doing so resulted in those students claiming that relationships such as $r = 2$ do not define a function. In this, and other (i.e., Breidenbach et al., 1992) examples, the researchers posed graphs that they understood to be representative of functions, yet the students' meanings for functions and their graphs did not afford their assimilating the graphs as such.

Our purpose here is not to rehash the documented claim that students often understand function in ways constrained to the application of the vertical line test (see Leinhardt, Zaslavsky, and Stein (1990) and Oehrtman et al. (2008) for more extensive reviews). Rather, our purpose is to draw attention to a particular feature of students' meanings that, as we illustrate in subsequent sections, is more deep-rooted and wide-spread than researchers have previously reported. Namely, we infer that one explanation for the students' actions in our colleagues' studies is that the students drew on meanings in which a particular coordinate system and what we perceive to be conventions of that coordinate system had become intrinsic to and inseparable from those meanings. For instance, what we perceive to be the convention of representing a function's input along the Cartesian horizontal axis did not appear to be a convention to those students reported on by Breidenbach and colleagues (i.e. habitual use of "convention").

We interpret several other researchers' findings to imply students' habitual use of "convention". For instance, Sajka (2003) detailed a student's use of function notation. Sajka argued that, to the student, function notation was more about what "we usually write" (2003, p. 247) than about using the notation to represent her ideas and reasoning. Using Thompson's (1992) language, the student was more focused on a ritualistic use of notation than on using notation as an act of personally expressing meaning under certain constraints. A consequence of this was that the student deemed examples that did not conform to her image of what "we usually write" as incorrect. Or, the student assimilated examples in ways that were consistent with her image of what "we usually write" but inconsistent with or inattentive to the researcher's intentions. Sajka noted that by conflating what "we usually write" and essential aspects of a mathematical idea, the student produced numerous inconsistencies in her use of function notation, some of which the student was aware of and others that were only inconsistencies from the researcher's perspective. As another example that we interpret to imply some students' habitual use of "convention", Moore, Paoletti, and Musgrave (2014) discussed how students' meanings for the polar coordinate system can be influenced by "conventions" of the Cartesian coordinate system. Notably, the students involved in their study were perturbed by contradictions in their reasoning that stemmed from coordinate pairs in the Cartesian coordinate system being conventionally of the form $(x, y)$ :(input, output), whereas coordinate pairs are conventionally of the form $(r, \theta)$ :(output, input) in the polar coordinate system.

## Methods

We interpret the collection of findings discussed above to indicate the need for a deeper examination of the extent to which individuals', and especially teachers', meanings entail the habitual use of "convention" versus a convention *qua* convention. In cases that teachers' meanings entail the habitual use of "convention", we would expect that their meanings become problematic in situations that an observer considers to break particular conventions or consist of alternative representational systems (Moore, Paoletti, et al. (2014) and Breidenbach, Dubinsky, Hawks, and Nichols (1992)). We would also expect them to (consciously or subconsciously) impose particular "conventions" in order to make sense of their experiences (Montiel et al. (2008) and Sajka (2003)). In order to explore and better understand this phenomenon in the context of teachers' meanings for graphs, we designed and conducted task-based semi-structured clinical interviews with PSTs (Ginsburg, 1997). We used similar tasks in an on-line open-ended survey to explore the extent to which ISTs held meanings compatible with those of the PSTs.

### Participants and Setting

Our work with PSTs involved 31 participants enrolled at a large state university in the U.S. The PSTs were entering their first semester in a four-semester preparation program for secondary mathematics teachers. Each PST began the program during her or his junior year (in credits), and each PST had completed at least two mathematics courses past Calculus II before beginning the program. We chose the PSTs by asking for volunteers from their initial meetings of a secondary mathematics content course. We drew participants from four different sections of the course.

In order to better understand teachers' meanings, we gathered similar data from ISTs. We adapted our interview tasks for an on-line survey completed by 45 ISTs enrolled in a fully online graduate mathematics course at a private U.S. research university as part of a master's degree program in mathematics education. The ISTs were geographically distributed across the U.S. They all had more than three years of experience teaching middle or secondary mathematics and had completed at least one mathematics course beyond Calculus III. All students in the program were invited to complete the survey during their third quarter in the program.

We worked with both undergraduate PSTs and ISTs due to our interest in understanding relationships between individuals' meanings and what we perceive to be conventions common to U.S. instruction. Because the chosen teacher populations had completed at least 14 years of mathematics schooling and, in the case of the ISTs, at least three years of teaching, we conjectured that we would gain insights into the extent to which teachers' meanings entail the habitual use of "convention" or convention *qua* convention. We note that we extended our work to include ISTs in order to explore if the themes identified with PSTs: (i) were similar to those of the ISTs, (ii) could viably explain ISTs' responses to an on-line survey, and (iii) would be ameliorated or otherwise affected by middle or high school teaching experience.

### Data Collection and Analysis

In the initial study with PSTs, we conducted task-based semi-structured clinical interviews (Ginsburg, 1997) during which the PSTs worked on tasks we had designed as discussed in the next section. Each PST participated in one interview lasting approximately 90-120 minutes, with the interview occurring during the first two weeks of the content course. During the interviews, a member of the author team asked that the PSTs verbalize their thinking as much as possible. Although we designed each interview task with particular purposes, the clinical interviews were semi-structured in that we asked questions formulated in the moment and on the basis of our interpretations of a PST's response (Merriam & Tisdell, 2005). We posed follow-up questions for the purpose of gaining deeper insights into the PST's thinking while also attempting to minimize shifts in the PST's thinking due to the researchers' questioning.

We videotaped each interview and digitized all written work. We analyzed the data following a selective open and axial analysis approach (Strauss & Corbin, 1998) for the purpose of modeling the PSTs' thinking on the basis of their utterances and observable actions, which Thompson (2008) described as a conceptual analysis. This process first involved identifying instances of PST activity that offered insights into his or her meanings. We used these instances to develop hypothesized models of the PST's meanings. With these initial models developed, we compared a PST's activity across instances and tasks in order to test and improve our interpretations of her or his activity, including identifying themes across instances and tasks. Lastly, we compared across PSTs to identify compatible and contrasting aspects of their meanings. The research team met throughout the data analysis phase in order to discuss analysis efforts, including differences, uncertainties, and refinements in interpretations of PSTs' activity.

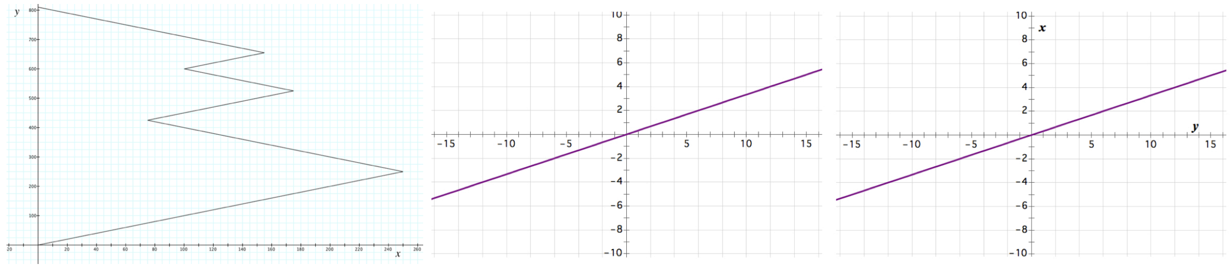
IST responses to the online survey items were analyzed through an iterative process that began with a first review of the entire data corpus and the subsequent development of a coding scheme heavily informed by our work with the PSTs. Members of the research team analyzed a subset of the ISTs' responses and we met to discuss our observations, identify commonalities across ISTs' responses, and adapt or create new codes to capture more ISTs' responses. We iterated this process four times as we refined our codes to capture all ISTs' responses. After obtaining final codes, a second researcher recoded approximately 65% of the data to check for inter-rater reliability. We obtained Cohen Kappa values of 0.78 and 0.85, indicating a high level of agreement, for the two tasks we present in this paper, described in more detail below.

**Task Design**

We designed each task to include what we perceive to be an unconventional feature with respect to the use of graphs in U.S. school mathematics. Although we designed these tasks to be unconventional, we also intended each task to include a graph that is mathematically viable as presented with respect to a particular claim (i.e. a graph in its *given orientation*). Because we did not expect the PSTs or ISTs to spontaneously interpret the displayed graphs as entailing unconventional aspects, we designed tasks to include specific claims (e.g., hypothetical student responses) with respect to features that we intended to be unconventional. By including hypothetical responses focused on such features, we were able to infer the extent that something was an inherent or habitual aspect of the PSTs' and ISTs' uses of graphs.

To illustrate, we provided the graph in *Figure 1a* and posed a variant of, "What about a student who claims that this graph represents $x$ is a function of $y$?" With respect to *Figure 1b*, we presented the graph as the work of a hypothetical student who graphed the relationship $y = 3x$. We asked the participants to describe how the hypothetical student might have been thinking when creating the graph. The follow-up prompt included a graph with the axes labeled (*Figure 1c*), and we explained that a hypothetical student clarified his graph of $y = 3x$ by labeling the axes as given in the second graph (i.e. $x$ on the vertical axis and $y$ on the horizontal axis). Both tasks illustrate our intent on designing graphs that can be conceived as mathematically viable (albeit unconventional) as presented with respect to the given prompts and claims.

The IST online survey was modeled after the PST interview protocol using virtually identical prompts. Due to the temporal nature of the images and prompts, multiple part items were displayed on multiple pages. Following the format of the PST interview, online survey participants were initially shown the graph in *Figure 1a* and asked "Is the following graph a function? Why or why not?". They were subsequently presented with the student response, "Sure, it can be a function… $x$ is a function of $y$", and asked to provide a response to the student.




(a) (b) (c)

*Figure 1.* (a) Is x a function of y? (b)-(c) A hypothetical student's work to graphing y = 3x.

## Results

We structure the results section around the teachers' responses to the two aforementioned tasks due to the teachers' responses being salient representations of their responses to other tasks. For each task, we first provide excerpts to illustrate themes in the PSTs' responses to the tasks. We also present summative PST data for each task to offer the reader a sense of the variety of PST responses. We then highlight instances in which teachers' actions indicated particular contradictions, as we describe below. We then provide results from our analysis of the ISTs' responses and compare the PSTs' and ISTs' propensity to respond in certain ways.

### *x* is a Function of *y*

We asked 25 of the 31 PSTs the entire sequence of prompts associated with *Figure 1a*. On the initial pass, 24 of 25 PSTs claimed that the graph did not represent a function either because of the graph not passing the vertical line test, the graph failing to have a unique $y$ value for each $x$ value, or a combination of the two. The remaining PST claimed that the graph did not represent a function because, "I don't like it [*referring to the cusps*]." With respect to the subsequent prompt (i.e. "Is there some way that we or a student could consider the graph as representative of a function?"), we provide a summary of the PST responses in Table 1. We draw attention to the fact that no PST provided a viable way to think about the graph as representative of a function in the *given* orientation. In Table 2, we present a summary of the PST responses to the student's claim that the graph is a function as, "$x$ is a function of $y$." In the sections that follow we discuss themes in the PST responses to this claim.

Table 1
*PST responses to the question, "Is there some way that we or a student could consider the graph as representative of a function?"*

| PST Response Category | # out of 25 |
|---|---|
| Yes, if rotated counter-clockwise 90-degrees | 6 |
| Yes, if rotated counter-clockwise 90-degrees and axes relabeled so that $y$ and $x$ were represented along the vertical and horizontal axes, respectively, in the new orientation | 5 |
| Did not determine how a hypothetical student might claim that the graph represents a function; maintained that the graph does not represent a function | 14 |

Table 2
*PST responses to the statement, "x is a function of y."*

| Category code | PST Response Category | # out of 25 |
|---|---|---|
| Habitual use of "convention" | Unsure | 1 |
| | Not true | 9 |
| | True, if graph is rotated counterclockwise 90-degrees | 7 |
| | True, if rotated counterclockwise 90-degrees and axes relabeled so that *y* and *x* were represented along the vertical and horizontal axes, respectively, in the new orientation | 1 |
| Convention *qua* convention | True | 7 |

**Convention *qua* convention.** We interpreted 7 of the 25 PSTs' responses to suggest that they did not require *x* or the horizontal axis to represent a function's input. Although 5 of these PSTs described they had a tendency to imagine the graph oriented so that the values defined as the function's input were represented along the horizontal axis, ultimately, each of the seven PSTs understood the graph *as given* to be consistent with the claim "*x* is a function of *y*" (*Excerpts **1***).

*Excerpts 1. x is a function of y; convention qua convention.* [1]

S1: I want to look at this and say this is a function *y* of *x* because that's how I would traditionally view a graph but I think it's valid to view it as *x* of *y*. And then you're still [*pause*] obeying what a function is. But you just have to be cognizant that your axes have changed so I guess it's like, valid.

S13: Rather than *y* being a function of *x*…Yeah I guess if you do it this way [*writes* '*x*(*y*)' *on paper*] …for every *y* there is exactly one *x*. And for every *y* [*puts marker on vertical axis on graph and moves it horizontally to a point where it hits the curve*] yeah, there's exactly one *x*…I've never thought about it that way but yeah, he's right.

**Habitual use of "convention".** Two notable characteristics emerged from our analyses of the PST responses to the sequence of prompts associated with *Figure 1a*: 16 of the PSTs either maintained *x* as representing input values *or* maintained the horizontal axis as representative of input values. These characteristics of the PSTs' meanings were most apparent when we posed the claim, "*x* is a function of *y*."

Collectively, 11 of the 25 ('Not true', 'Unsure', and 'True, if rotated counterclockwise 90-degrees and axes relabeled') PST responses suggested they assimilated the phrase "*x* is a function of *y*" no differently than "*y* is a function of *x*" (*Excerpts **2***). More specifically, these PSTs maintained that the graph does not represent a function because the graph does not pass the vertical line test, because there exists *x*-values for which there is not a uniquely associated *y*-value, or a combination of the two. To each of these 11 PSTs, "function" drew to mind an action that entailed treating (implicitly or explicitly) *x* and the quantity represented along the horizontal axis as the input quantity (i.e. habitual use of "convention").

*Excerpts 2. x is a function of y; habitual use of "convention".*

S7: Okay. Um [*pause*] *x* is a function of *y*. [*long pause*] …Well you know something's not a function if [*placing her marker in a vertical line over the given graph*], two different outputs can give you the same, I mean if two different inputs can give you the same output… Which you have here obviously that, you know, these one two three four five six *x*-values give you different *y*-values [*using her marker to mark points on the graph in a vertical line*]. I mean these, the same *x*-value can give you six different *y*-values.

[1] For space purposes, we use "…" to indicate the removal of spoken words and actions that we did not interpret to alter our interpretation of the PSTs' activity.

S24: [*laughing*] Oh gosh, um, well [*pause*] if $x$ is a function of $y$, well you can't [*pause*] for it to be a function you can't have more than one $y$-value for the $x$ [*motioning the marker over the graph indicating points vertically from one another*]. So, like if I wanted to know what, umm [*pause*] $f$ of one hundred was, or something, like I would get a bunch of different [*begins to mark points on the graph for a specific x-value*], I mean, yeah I would get a bunch of different $y$-values for it, you know [*has marked multiple points on the graph with the same x-value*] …you can't get more than one $y$-value per $x$-value. It's not a function.

Turning our attention to the 7 PSTs who maintained that the statement "$x$ is a function of $y$" is true only on the condition that the graph is rotated 90-degrees counterclockwise and axes *not* relabeled (Table 2), these PSTs understood the statement "$x$ is a function of $y$" in two parts. The phrase both defined the axes orientation and presented a statement to be considered with respect to a relationship between paired values (*Excerpts **3***). Because they understood the given phrase to necessitate a particular axes orientation—an orientation in which input values are represented horizontally—they required that the graph be rotated to orient $y$-values horizontally before considering the validity of the claim with respect to properties of the $x$-$y$ pairing (i.e. habitual use of "convention").

*Excerpts 3. x is a function of y; habitual use of "convention".*

S4: I guess she doesn't understand what graphs represent…so she said $x$ is a function of $y$. That'd be, that'd be looking at it this way [*turning the paper 90-degrees counterclockwise*] and saying look there's no [*motioning hand over the graph as if doing the vertical line test*], there's no crossing…you'd have to flip the whole graph… [*redraws graph in rotated orientation, labeling the horizontal axis as y and the vertical axis as x*] That'd be $y$ and that'd be $x$. So $x$ is a function of $y$. And that's a function… [*Interviewer returns S4's attention to the graph in its given orientation*] No, because $x$ isn't a function of $y$. This is the graph of $y$ as a function of $x$ [*motioning to her sketch*].

S14: Okay so $x$ is a function of $y$. That's trueeee [*turning the paper 90-degrees counterclockwise*]. [*Turning the paper back to the given orientation*] But $y$ is not, $y$ is not a function of $x$…That's what we're looking at here…So you want $y$ is a function of $x$. Is that what you said to me, no you said $x$ is a function of $y$…That's backwards [*laughing*]…because like $x$ is a function of $y$, so that, I think of that as, like the graph like kinda this way [*turning the paper 90-degrees counterclockwise*] … [*motioning over the horizontal–now y–axis*] like if this is our horizontal that's true. Because for every $y$ [*pointing to y-label on horizontal axis of turned graph*] there is one unique $x$ [*pointing to x-label on vertical axis of turned graph*] but [*turns the paper back to given orientation*] for every $x$ [*pointing to x-label on horizontal axis*] there is not [*pointing to y-label on vertical axis*] one unique $y$… she's incorrect because it's like backwards…that's not what we're looking at [*referring to graph in given orientation*].

**A Graph of $y = 3x$**

We asked all 31 PSTs the sequence of prompts associated with *Figure 1b* and *c*. With respect to the first prompt (*Figure 1b*), we provide a summary of their responses in Table 3. No PST encountered difficulty attributing a viable approach to producing the hypothetical solution. Several PSTs provided multiple explanations as to how a student might create the graph.

Table 3

*PST responses to the prompt and graph associated with Figure 1b.*

| PST Response Category | # out of 31 |
|---|---|
| Hypothetical student held some misunderstanding of slope (e.g., 'rising 1 and running 3') | 16 |

| | |
|---|---|
| Hypothetical student graphed $x = 3y$, $y = (1/3)\,x$, or interpreted the equation to mean $x$ is three times as large as $y$ (e.g., variables as literal objects) | 13 |
| Hypothetical student graphed $y$ on the horizontal axis and $x$ on the vertical axis | 13 |

When we presented *Figure 1c* and asked the PSTs to interpret the hypothetical solution, we asked them to comment on the correctness of the solution (i.e. "Does the graph represent $y = 3x$?") in order to gain insights into the extent to which they considered the graph a viable representation of $y = 3x$. We summarize their responses in Table 4. In the sections that follow we discuss themes in the PST responses to the graph and prompts.

Table 4

*PST responses to the prompt and graph associated with Figure 1c.*

| Category code | PST Response Category | # out of 31 |
|---|---|---|
| Habitual use of "convention" | Hypothetical student did not construct a correct graph | 5 |
| | Hypothetical student constructed a graph that is both correct and incorrect | 11 |
| | Uncertain if the hypothetical student constructed a correct graph | 4 |
| Convention *qua* convention | Hypothetical student unquestionably constructed a correct graph | 11 |

**Convention *qua* convention.** 11 PSTs maintained that the graph as oriented in *Figure 1c* unquestionably represents $y = 3x$ (*Excerpts **4***). These PSTs identified the graph's departure from convention, and specifically its departure from a customary axes orientation. They also claimed that the departure does not influence the correctness of the represented relationship between $x$ and $y$ (i.e. convention *qua* convention). Several of the PSTs mentioned the creativity of the hypothetical student's solution (see S30).

*Excerpts 4. A graph of $y = 3x$; convention qua convention.*

S21: Ohhhh…this graph is saying…$y$ is three times bigger than $x$…so where $x$ is one, $y$ is three times bigger [*checking graph*]. Yes. Where $x$ is two, $y$ is three times bigger [*checking graph*]. So this graph is correct… $y$ is three times bigger than $x$.

S30: He graphed it completely right. That's $y$ equals three $x$…he's not wrong. He just has a different perspective than the traditional $x$-$y$…that's just counter to tradition and normal classroom settings. But I think it's smart of him to understand that it's not glued.

**Habitual use of "convention".** We interpreted 20 PSTs (Table 5, first three response categories) who deemed *Figure 1c* incorrect or who expressed uncertainty about the hypothetical solution to hold meanings that entailed the habitual use of "convention". These "conventions" included assigning $x$-values to the horizontal axis, maintaining particular axes directions for positive and negative values (which arose after rotating the graph to obtain $x$-values oriented horizontally), using the horizontal axis to represent a function's "input" (and inferring the given graph contradicted an equation defining $x$ values as "input"), or a combination of these (*Excerpts 5 Excerpts 5*). In some cases, PSTs discarded the hypothetical student's solution or deemed the solution incorrect because of its departure from these "conventions", thus treating "conventions" as unquestionable rules of a coordinate system and graphing (see S20). In other cases, PST responses to the hypothetical solution suggested they drew on meanings for slope or rate in which the habitual use of a particular Cartesian orientation was embedded (see S23). For instance, after rotating the graph 90-degrees counterclockwise so that the $x$-axis was oriented horizontally, some PSTs understood the slope as negative because the line is directed downward left-to-right. This statement is true under conventional Cartesian orientations.

*Excerpts 5. A graph of $y = 3x$; habitual use of "convention".*

S19: I feel like you should know your $x$ and $y$, and like, know which one is which. And, yeah, you're going to get it all wrong I think.

S20: The horizontal axis should always be $x$ and the vertical axis should always be $y$.

S23: Because if you turn it this way [*referring to graph rotated 90-degrees counterclockwise*] then this [*traces left to right along the x-axis which is now oriented horizontally*] and this [*traces top to bottom along the y-axis*] and it would be still not right though…this [*laying the marker on the line which is sloping downward left-to-right*] is negative slope. So I would…show them like the difference between positive and negative slopes also. Because that's something that, like, when I was in middle school we, like, learned kind of like a trick to remember positive, negative, no slope, and zero [*making hand motions to indicate a direction of line for each*].

**PSTs' Contradicting Actions and Claims Across Both Tasks**

Returning to the task associated with *Figure 1a*, we draw attention to S4 and S14's responses (*Excerpts **3***). S4 and S14 claimed the graph as given was not such that $x$ is a function of $y$, and they claimed the rotated graph was such that $x$ is a function of $y$. From our perspective, there is a contradiction that exists in their responses: the graph is such that each $y$-value has a uniquely associated $x$-value no matter the rotated orientation of the paper. But, we stress that the PSTs' responses were not a contradiction from their perspective. As we described above, we infer that these PSTs' meanings for functions and their graphs were such that the ways they conceived $x$-$y$ pairings were dependent on the axes orientations (i.e. habitual use of "convention"). The variable values represented along the horizontal axis were necessarily representative of a function's input, even when presented with interviewer utterances asking them to consider otherwise.

More generally, several cases entailing their habitual use of "convention" included the PSTs exhibiting contradicting actions and claims. We focus on the task associated with *Figure 1a* to provide additional illustrations this phenomenon. First, we note the contradiction an observer can perceive when PSTs claim that rotating the given graph changes the represented relationship or slope (see Excerpt 5, S23). Regardless of orientation, an observer can understand the graph so that each $y$-value is three times as large as the associated $x$-value and that any change in $y$ is three times as large as the change in $x$. To S23, however, slope was as much, or more, an indicator of direction constrained to particular Cartesian "conventions". Thus, they did not perceive a contradiction in claiming that the "slope" changes as the given graph and paper is rotated.

Second, numerous PSTs claimed that the graph in *Figure 1c* was *both* correct and incorrect in its representation of $y = 3x$. In some cases, a PST claiming that the graph is both correct and incorrect was only a contradiction from our perspective. Those PSTs held meanings that enabled them to claim the graph is both correct and incorrect without experiencing a sense of perturbation (*Excerpts **6***, S2 and S9). Namely, the PSTs understood that the graph as given entailed coordinate points satisfying $y = 3x$. At the same time, they held meanings for coordinate systems that entailed the habitual use of "convention" in the form of axes orientations, thus requiring those orientations in order to claim a graph is how it "should be written."

In other cases, PSTs experienced a perturbation that stemmed from their awareness of claiming that the graph satisfies the equation $y = 3x$ but that it is incorrect due to its orientation (*Excerpts **6***, S17). These PSTs did not resolve their perturbation during the interviews, which led to each PST expressing uncertainty about whether particular axes orientations must be maintained in order to have a solution that is mathematically correct.

*Excerpts 6. A Graph of y = 3x; contradicting actions and claims.*

S2: [*S2 is addressing how he would respond to the student who produced Figure 1c*]… I mean I would tell him that this is the correct graph because it technically is. But I would just explain to him, and I don't know how I would explain but how, like when graphing functions $y$ is always going to be the vertical axes and $x$ is always going to be the horizontal axis… explain to him that next time he needs to change his axis. And why [the graph] is right but wrong at the same time.

S9: It's wrong with like how we normally write graphs…So he should lose points because he wrote the graph in like really incorrectly to what, how the graph should be written. Like the horizontal axis should always be $x$ and the vertical axis should always be $y$. But if you're looking at it based on did he understand that, when $y$ equals three, $x$ equals one, like he understood that, um, relationship between $x$ and $y$.

S17: He got them mixed up… So he did the problem correctly…he didn't understand how the graph worked…that the $x$ is always on this [*referring to the horizontal axis*], and $y$ is always on the vertical axis…[his graph] is correct [*making finger quotations surrounding correct*] but it's not mathematically correct. [*S17 then draws canonical graph and illustrates how the graph is correct using points*]…it's not wrong, it's just not what graphs are supposed to be…I don't know. I've always just done what I was told. I don't really know why it has to be that way…I never really questioned why $x$ is there and $y$ is there.

**IST Responses**

We do not present the IST responses to each task as they are compatible with the PST responses. Table 5 provides codes we created to characterize the IST responses to the last stage of hypothetical student work for both tasks, example responses to the hypothetical student prompt associated with *Figure 1a*, and counts of the number of IST responses coded within each category for each task. 12 of the 45 IST responses for the task associated with *Figure 1a* indicate the ISTs understood a convention *qua* convention. 25 IST responses for the task associated with *Figure 1c* indicate the ISTs understood a convention *qua* convention. The remaining 33 and 20 ISTs, respectively, maintained meanings which entailed a habitual use of "convention".

Table 5
*Codes description, counts, and sample responses of IST online survey (n = 45).*

| Convention category | Code | Example Responses to the task in *Figure 1a* | *Figure 1a* | *Figure 1c* |
|---|---|---|---|---|
| Convention *qua* convention | The student's mathematical statement is correct despite breaking from conventions. | That's great! I am so glad you were able to apply the "vertical line test" in a horizontal orientation and realize that you would have a function. You are correct in saying that $x$ is a function of $y$. | 12 | 25 |
| Habitual use of "convention" | The student's mathematical statement is true but the student is incorrect because he/she broke from conventions. (contradicting actions and claims) | I think the student is understanding that $x$ can be a function of $y$ but they are not displaying it correctly through the graph. | 9 | 7 |
| | The student's mathematical statement is incorrect or the IST did not address the student's mathematical statement. | It was not a good explanation and $x$ is not a function of $y$, $y$ is a function of $x$. The value of $y$ depends on $x$. They also did not describe what would make it a function. | 24 | 13 |

**Comparing PST and IST Responses**

To identify similarities between PST and IST responses, and to consider the possible persistence of PST meanings into teachers' professional careers, we assigned numerical values to the two convention categories within each coding scheme (Convention *qua* convention a value of 1, and habitual use of "convention" a value of 2). Table 6 provides the average scores for PSTs and ISTs across both tasks. We used a two-tailed Mann-Whitney U-test to examine the coded responses from the PST and IST populations. Analysis indicates no statistically significant difference between the populations for either task ($p > 0.05$ in both cases), meaning that there is no evidence that PSTs and ISTs provide different responses in relation to the tasks used in our study. Hence, it appears that these meanings for graphing conventions (or "conventions") likely persist into teachers' professional careers, highlighting that experience teaching does not necessarily support shifts in teachers' conceptions of graphing conventions (or "conventions").

Table 6
*Average scores of PST and IST responses and p-values obtained from a Mann-Whitney U-test.*

| | *Figure 1a* | *Figure 1c* |
|---|---|---|
| PSTs | 1.72 | 1.65 |
| ISTs | 1.73 | 1.44 |
| *p*-value | 0.9283 | 0.1416 |

## Discussion and Concluding Remarks

Our analysis provides inferences relative to the meanings teachers maintain in relation to particular graphing conventions (or "conventions"). Our analysis also highlights the pervasiveness of such meanings in both PSTs and ISTs. We interpret our findings to underscore the importance of future work that supports students and teachers in developing meanings that differentiate between what is critical to a mathematical idea and what is customary or arbitrary.

**Significance of Findings**

Our results support the claims of Breidenbach et al. (1992), Montiel et al. (2008), and Sajka (2003), who provided data that we interpret to imply their participants' meanings of function and their graphs entailed the habitual use of particular coordinate systems, orientations, or variable symbols. Our work extends their work in an important way. We are not aware of other researchers who have used task-based clinical interviews or on-line surveys (as opposed to an instructional setting) to present participants explicit claims—through written or spoken words—designed to be unconventional with respect to the notation and coordinate orientations used here. Our findings in this regard shed insights into the extent that some teachers have (or have not) differentiated what we perceive to be conventions from essential aspects of particular mathematical ideas and representational systems. Most notably, we show that despite providing explicit claims and ideas represented in ways compatible with these claims, many of the teachers assimilated the situations in ways that implied their habitual use of "convention". The persistence of some teachers' habitual use of "convention" after a repeated sequence of interview questions and explicit prompts addressing the same "convention" is particularly noteworthy.

At the most fundamental level, it is significant that both PSTs and ISTs—who have completed advanced mathematics courses and many of whom have several years of teaching experience—have developed mathematical meanings that, at best, limit their ability to attribute mathematical viability to school mathematics ideas presented in unconventional formats. Also significant is that some teachers (or soon be teachers) held meanings that led to claims and actions that, although potentially internally viable to the teachers, were contradictory from our perspective and suggested their habitual use of “convention”.

In our study, it was clear that when conventions (or “conventions”) were not violated, the PSTs and ISTs were able to respond in ways that were sensitive to the mathematical viability of students’ solutions. However, this study reveals, perhaps unsurprisingly, that being able to respond appropriately in one context does not necessarily indicate coherence in teachers’ mathematical meanings. Specifically, we document that in many cases, particular aspects of teachers’ meanings were not apparent until their engagement in tasks that we designed to be unconventional. That is, it was not until we violated particular “conventions” that we were able to gain insights into the extent that particular representational features were inherent to teachers’ meanings (i.e. the habitual use of “convention”). We argue that a significant contribution of this paper is that it provides a mechanism to identify previously invisible aspects of learners’ meanings of core mathematical ideas. This is important as it is only when these aspects of meanings are identified that there will there be a chance of supporting learners’—including teachers—development of internally consistent, coherent and generative mathematical meanings.

**Future Work**

Our current work indicates that a non-insignificant number of PST meanings entail the habitual use of “convention”, with some of these cases involving their exhibiting contradicting claims or actions. This trend is consistent with the participating ISTs, suggesting that classroom experience with students does not ameliorate the issue. If we accept that teachers (and students) understanding mathematical ideas in ways that entail conventions *qua* conventions is desirable, then an important question for teacher education is *how might the desired meanings develop?* We believe that our work, in combination with that by previous researchers (Mamolo & Zazkis, 2012; Thompson, 1995; Zazkis, 2008), provides initial guidance in this area. Specifically, for teachers holding meanings that entail the habitual use of “convention”, we hypothesize that one way to support the transition to understanding convention *qua* convention is to develop instruction that supports teachers in raising and reconciling contradictions between claims and actions. Examples provided in this paper provide some viable contexts for these conversations. Moore, Silverman, Paoletti, and LaForest (2014) and Johnson (2015) share additional strategies that speak to Thompson’s (1995) suggestion of placing an emphasis on synthesizing issues of convention, quantitative reasoning, and notation. Mamolo and Zazkis (Mamolo & Zazkis, 2012; Zazkis, 2008) provide other examples that include using unfamiliar coordinate systems. Each of these strategies can be used as design and implementation principles for teacher educators and researchers interested in supporting and understanding PSTs’ and ISTs’ development of meanings that are consistent with convention *qua* convention.

Before closing, we acknowledge the limitations of using on-line survey data to draw inferences about IST meanings. We also acknowledge that our work with PSTs was limited to one university, thus limiting the diversity of the participant pool. There is thus a need for additional studies into both PST and IST populations, and we suggest that investigations of IST populations include other qualitative methods (e.g., various forms of clinical interviews) in order to provide nuanced insights into their meanings, particularly when confronted with contradicting actions and claims. The PSTs in our study did not reconcile these contradictions when they became aware of them. It remains to be seen if other PST and IST populations do so or if interventions are necessary to support their reconciliation of these contradictions.

## Closing Remarks

We close by underscoring that we do not intend to discredit conventions, nor to convey that conventions are unimportant. We hope we have been clear that a convention is important to the extent an individual understands it as a convention *qua* convention. We also do not contend that educators, whether teachers or curricula designers, can realistically address every convention they perceive to constitute some mathematical community. We argue, however, that our results respond to and strengthen calls for a more detailed consideration of how educators and researchers treat and understand conventions (or "conventions"). We agree with Thompson's claim, "…[we] can attempt to make explicit those conventions that are assumed and treated as given, those conventions that are assumed and presented as conventions, and those conventions that are meant for students to recreate or to create in some idiosyncratic form" (1992, p. 125). Educators and researchers must be sensitive to the negotiation of conventions among students, teachers, and any member of a perceived community. Given the complexity of learning and teaching, understanding what this sensitivity might look like will take concerted efforts at many levels with particular emphasis given to students' meaningful creation and use of notation and representations (e.g., diSessa, Hammer, Sherin, & Kolpakowski, 1991; Meira, 1995; Thompson, 1995; Tillema & Hackenberg, 2011). In short, if students and teachers are to understand a convention *qua* convention, then they need opportunities to come to understand mathematical ideas in ways that include the negotiation of customary choices within the context of those ideas that remain invariant among those choices.

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# Stepping Through the Proof Door: Undergraduates' Experience One Year After an Introduction to Proof Course

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*Navigating the transition from computing to proof remains a key challenge for mathematics departments and undergraduate students. Many departments have developed courses to introduce students to proof and proving (David & Zazkis, 2017), but research on the impact of these courses has just begun. This paper reports the experience of four mathematics majors in a semester of real analysis. Each participated in our prior study of students' experience in the introduction to proof course. Where the students' work in that course supported their success in real analysis, they experienced real analysis in quite different ways. Two recognized and exploited the structures common to real analysis proofs; the other two relied on extensive practice with example problems. The results have re-oriented our view of the computation-to-proof transition and where and how students experience proof as problem solving.*

*Keywords*: Transition to Proof, Real Analysis, Students' Experience, Qualitative Analysis

This paper extends our prior research that examined undergraduate students' experience in one introduction to proof course taught at a research-intensive university (Smith, Levin, Bae, Satyam, & Voogt, 2017). Most of the N = 14 participants in that study clearly indicated that they found the work to write proofs different from their prior mathematical work to compute numerical or symbolic answers. Where the majority found proof writing challenging, most were relatively successful in the course, as judged by final grades and self-reports. But the success of courses designed to introduce students to proof and proof-writing cannot be judged locally. As the goal for such courses is to increase learning and achievement in upper-division mathematics, success depends on how well students learn and perform in subsequent proof-focused courses that explore major areas of mathematics (e.g., analysis and algebra).

Here we report on the experience of four "graduates" of an introduction to proof course in their first semester of real analysis. All were successful in that course, as judged by both grades and self-reports. But their descriptions of their work in real analysis, offered in comparison to the introduction to proof course and prior mathematical work through calculus, revealed a more complex pattern of similarities and differences in how students see and carry out mathematics work. For some students, the differences between following procedures to compute answers and writing effective proofs in real analysis were less stark than we initially conjectured—and than they experienced in their introduction to proof course. If true for a small sample, analyses characterizing students' transition to proof and subsequent mathematical development may need to attend to important *continuities* as well as *discontinuities* with prior mathematical work through calculus.

## The Transition to Proof and Proving

Understanding the challenges that undergraduate students face in learning to prove (or disprove) mathematical statements and designing courses and experiences that support their efforts to address those challenges have become major foci of research in undergraduate mathematics education. Recent work has focused on the nature and diversity of courses intended to introduce students to proof and proving (David & Zazkis, 2017; Selden, 2012), specific cognitive challenges in understanding and writing proofs (Antonini & Mariotti, 2008; Selden & Selden, 2003; 2013; Sellers, Roh, David, & D'Amours, 2017; Weber & Alcock, 2004; Zandieh, Roh, & Knapp, 2014), and following students' proving work and reasoning after an initial introduction to proof (Benkhalti, Selden, & Selden, 2017; Weber, Brophy, & Lin, 2008). In this regard, real analysis, which is one of the first proof-based content courses taken by students after introduction to proof, has been investigated by researchers interested in development of students' understanding of particular concepts in real analysis such as the epsilon-delta definition of limits (e.g., Alcock & Simpson, 2005; Oehrtman, Swinyard, & Martin, 2013, Swinyard & Larsen, 2014), and understanding and improving teaching practices of real analysis courses (e.g., Weber, 2004; Pinto, 2013; Wasserman, Weber, & McGuffey, 2017). In particular, some studies have reported students' experiences in real analysis in terms of how they adjusted in new learning environments and changed their learning activities. For example, Weber (2008) interviewed a student in a real analysis course bi-weekly throughout the semester and reported changes in her emotional states in relation to changes in learning strategies. Specifically, the student entered with a rote learning strategy and negative experience with proving from her introduction to proof course. This study described how she changed her learning strategy to an understanding-oriented approach and began to enjoy proving work in the real analysis course through social interactions with peers and the instructor.

As a contribution to this growing literature, we previously interviewed N = 14 undergraduate students after they completed a one-semester introduction to proof course. Our analysis focused on four issues: (1) how the students saw the introduction to proof course as different from prior courses, (2) the activities they reported undertaking to learn the course content, (3) how they characterized their thinking during work on proof (proof reasoning), and (4) their sense of success in the course (Smith et al., 2017). None reported any prior work on proof in high school or their calculus sequence. Most were clear that the course made different demands than their prior work in mathematics, and in response, many initiated new patterns of work. Despite the challenges they reported, most completed the course relatively successfully, leading us to conclude that this particular introduction to proof had successfully brought these students up to and through the doorway to proof. In particular, the course placed students into the work of solving mathematical problems—tasks for which a solution path is not immediately clear (Schoenfeld, 1992)—and supported their adjustment to that work.

But the merit and impact of introduction to proof courses lie as much—if not more—in how students perform *after* they complete such courses. Introduction to proof courses typically emphasize foundational issues of logic and syntax (the grammar of mathematics) and introduce different methods of proof but either cover no additional topics from specific mathematics content or simply provide an introduction to the advanced topics near the end of the semester (David & Zazkis, 2017). The main task of working in proof and proving to understand more about the fundamental structure of a specific domain of mathematics lies ahead of them. If the gap between carrying out known procedures to compute numerical and symbolic answers and proving statements is wide and deep (Selden & Selden, 2013), the transition to proof and proving will not be accomplished in a single semester. Consequently, it makes sense to follow successful introduction to proof "graduates" into their content-specific proof-based courses: Where does reasonable initial success with proof lead? How do students experience their first proof-based course situated in a particular content area? How do they compare their experience in that course to their "preparation" in the introduction to proof course and to their prior mathematical experience? Is it possible to chart students' experience, work, and views of mathematics from computing to proving at a reasonable level of precision, to understand more specifically the challenges that undergraduate students face?

## Conceptual Framework

As in our prior study, the theoretical stance that framed this study was constructivist. From this perspective, students bring forward mathematical "resources" (knowledge, skills, learning practices) developed and proven useful in prior courses and attempt to use the resources to address the tasks of their present courses. Some resources are individual in nature (e.g., students' understanding and mastery of mathematical induction); other resources are inherently social and interpersonal (e.g., how students organize their work with peers outside of class). New challenges, at any scale, mean that some resources will work well as is, some must be adapted, and some must be developed, more or less *de novo*, in the new setting. The view of students as agents in their own learning is also central to our perspective, especially with respect to learning activities outside of class.

The present analysis was more specifically structured by the main concepts that oriented our prior work (Smith et al., 2017). Informed by prior work to understand pre-college and college students' experience of work in "reform" and "traditional" courses (e.g., Smith & Star, 2007), we expected the shift from computing single answers to proving statements to set the stage for major transitions in students' experience of mathematics, where their understanding of the nature of their work, how they feel about their experience and their abilities, and what they do to carry out that work would change in quite substantial ways. Where mathematical transitions are not determined by the external environment, some features of the environment make them more likely, such as the presentation of fundamentally new types of mathematical tasks and solutions. Our prior study conceptualized students' experiences in terms of (a) the differences they saw between the work in their introduction to proof course and their prior mathematics, (b) their sense of the task of writing proofs, (c) their learning activity, in and outside of class, and (d) their subjective "sense of success" in the course.

In the present study, we focused on students' experience in real analysis in relation to their work in the introduction to proof class a year earlier. The above four foci again informed the development of our interview questions and the direction of our analysis. For the first focus (differences with prior courses), we were particularly interested in how students compared the introduction to proof course to real analysis.

Both our prior and present studies foreground what may seem like a vague psychological notion—*students' experience*. Where it is arguably less analytic than more familiar constructs of knowledge and skills that can structure studies of proof learning and understanding (e.g., students' understanding of and ability to use a particular proof method), the broader notion of experience merits researchers' attention. Following Dewey (1916, 1938), we see experience as reflective, interpretive, and consequential for students' engagement in mathematics. In contrast to experiences (plural) that are more particular and happen daily, experience (singular) is inherently summative, formed by students reflecting back and interpreting a corpus of many experiences over time. Experience is interpretive in the sense that its truth depends on how the student herself has read and made sense of her experiences. It is consequential because the narrative of experience that students construct for themselves from their aggregate experiences orients their subsequent activity. For example, mathematics students' experience in proof-based mathematics is consequential for their choices and commitments to complete their major and their thinking and decision-making about post-college work.

## The Program, Courses, and Participants

In the university where our research was conducted, undergraduates—both mathematics majors and minors—complete a calculus sequence, the introduction to proof course, a linear algebra course, and two or more proof-focused content courses. After a linear algebra that does not emphasize proof, the first semester of real analysis and the first semester of abstract algebra are two common sites where students experience proof-intensive work in a specific content area. These two courses are required for both majors and minors, and both require successful completion of the introduction to proof course. Majors are required to complete additional proof-based courses, including the second semester of both real analysis and abstract algebra, as well as other courses—some of which are proof-based.

In the semester of our study, two sections of real analysis were taught by different instructors; both sections used the same text (Kenneth Ross, *Elementary analysis: The theory of calculus*, 2nd edition, 2013). The two sections differed somewhat in their in-class activities, homework, and assessments. In this department, real analysis is widely seen by students, instructors, and advisors as among the most, if not the most challenging undergraduate course. One of the two instructors this semester stated this explicitly to her students in class early in the semester.

In Spring 2017, six of the 14 student participants in our previous study responded positively to our invitation to participate in a follow-up study. All six were mathematics majors or minors and had taken the first semester of real analysis and/or abstract algebra in 2016-17. The other eight initial participants either did not respond or indicated they had taken neither course, had changed majors, or left the university. Two respondents took both real analysis and abstract algebra; the other four took only one. For those who had taken both courses, our interview focused on the more recent course (either real analysis or abstract algebra) to reduce concerns about (re)constructed memory. With four participants, the interview focused on real analysis; with the other two, on abstract algebra. In this paper, we will focus on the former group, who are described in Table 1 below.

*Table 1. Overview of participants*

| Student | Gender | Standing | Home | Major | Career Obj. | Other proof-based courses |
|---|---|---|---|---|---|---|
| S1 | Female | 3 | US | Mathematics | Teaching | Higher geometry (F16) |
| S2 | Female | 3 | US | Mathematics | Actuary | None |
| S3 | Female | 3 | US | Mathematics | Uncertain | None |
| S4 | Male | 4 | Int. | Mathematics | Grad school | Abstract algebra I (F16)<br>Abstract algebra II (Sp17) |

All four participants took the first semester of real analysis in the same semester (Spring 2017), and we interviewed them just after they completed it. S1, S2, and S3 had the same instructor; S4 was taught by the other instructor. Though we did not directly observe either instructor's teaching as we had in the previous study, we asked all students a series of questions about the nature of their in-class activities—both what the instructor did and what students did.

S1, S2, and S3 provided very consistent descriptions of their course, their instructor's teaching, the assigned homework, the use of the text, and the course assessments.

The interviews were semi-structured around focal questions, about an hour in duration, and conducted either face-to-face or via video conference. In two cases (S1 and S3), a follow-up interview was used to clarify students' responses from the first (for S1) or to pose all the questions we needed to ask (for S3). The overall goal of the interviews was to understand students' experience in real analysis relative to their experience in the introduction to proof course—with particular attention to the task of writing effective proofs.

After checking basic information (e.g., major/minor, standing, career plan, other math courses), we asked about the students' sense of how well the introduction to proof course prepared them for the real analysis course (and any other proof-based courses they had taken). Making no assumptions about how participants saw other mathematics courses they took that year (e.g., linear algebra), we asked how they viewed each such course relative to its focus on proof (very little, somewhat, strongly). All four participants indicated that real analysis was strongly proof-based. For the course(s) characterized as somewhat or strongly proof-based, we asked students to compare the difficulty of that course(s) to the introduction to proof course.

Then we explored their experience in each course, but with greater attention to real analysis. For that class, we asked specifically about assignments and instruction, learning activities in class (e.g., group work) and outside of class (e.g., work with peers in any context), and their view of proof tasks and work to produce acceptable proofs. These interviews also provided the opportunity to return to the students' presentation of their experience in the introduction to proof course reported in our prior study, affording a check on consistency in their characterizations.

Toward the end of the interview, we asked them to draw a Confidence Graph to represent the contours of their confidence with that semester of real analysis. As in the prior study (Smith et al., 2017), these graphs helped us understand the challenges participants faced at different points in the course and how they addressed the challenges. The graphing activity was useful because, in addition to the graphical representation it provided of one affective component of students' experience, it also created more space for students to relay their experience verbally, as they explained the features (e.g., location and rate of change) of their graphs.

Figure 1 below represents the comparisons of the different mathematical experience that were supported in the two studies, the previous (Phase 1) and the present (Phase 2). The interviews in the present study supported comparisons between real analysis and the introduction to proof course, but also with participants' experience prior to the introduction to proof course.

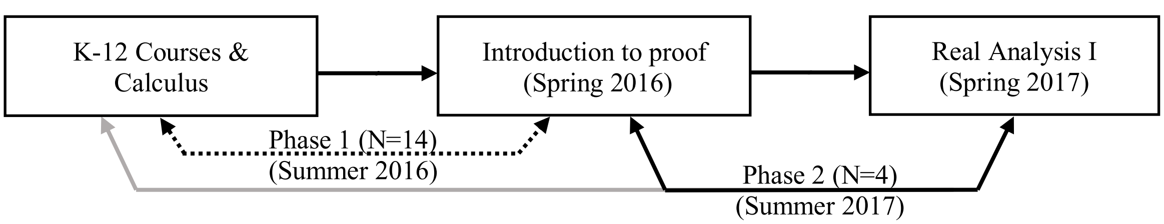


*Figure 1. The previous (Phase 1) and present (Phase 2) studies across the sequence of the courses*

## Results

All four students reported success in real analysis, represented both in final grades (all received 4.0) and their sense of having mastered the content. Also, as indicated in Table 2 below, all four acknowledged and valued the preparation for real analysis they received in the introduction to proof course. However, S1 and S3 made a stronger case for their preparation in the introduction to proof course, where S2 and S4 indicated they did not learn some things that they wished they had. S2 stated that she was not required and taught how to build up the structure of a proof; S4 stated that some methods (e.g., epsilon-delta proofs) were not taught in sufficient detail in the introduction to proof course. All four participants noted that the introduction to proof course moved frequently between content areas (making the course more difficult in the process), where real analysis focused on one set of related ideas. S1, S2, and S3 each indicated that they appreciated learning in real analysis why theorems and rules they learned in calculus were true.

*Table 2. Summary of students' sense of preparation of the introduction to proof course for Real Analysis I*

| Student | Preparation | Relative Difficulty |
|---|---|---|
| S1 | Very well | Introduction to proof > Real analysis |
| S2 | Well | Real analysis > Introduction to proof |
| S3 | Very well | Introduction to proof > Real analysis |
| S4 | Well | Real analysis > Introduction to proof |

Because the interviews at the end of real analysis provided an appropriate context, we also asked the students to compare the difficulty of real analysis to that of the introduction to proof course. S1 and S3 judged the latter as more difficult than real analysis, despite the fact that prior reports led both to expect that real analysis would be very challenging. In contrast, S2 and S4 indicated that real analysis was more difficult than the introduction to proof but cited different reasons for their judgments. S4 indicated that insufficient example problems in real analysis contributed significantly to its difficulty, where S2 found the concepts as well as proof construction in real analysis was more challenging than it was in the introduction to proof. Beyond these top-level judgments about "preparation," we found the two pairs of the participants (S1 & S3 and S2 & S4) provided two quite different narratives about the challenges of the course and how they had worked to address the challenges.

**The Introduction to Proof Course as Preparation for Real Analysis**

Before characterizing the differential experience of the two pairs of students in real analysis, we first buttress our claim that the introduction to proof course provided these students with a reasonable preparation for the challenge of real analysis. Here we summarize evidence that have we reported in greater detail elsewhere (Smith, et al., 2017). Our core argument is that the introduction to proof course represented a relatively rigorous entry into a very different form of mathematical work than the students had experienced in prior coursework. First, none of the four students reported any experience with proof prior to that course (i.e., either in their high school mathematics courses or their university mathematics—that was for most, their calculus sequence). Consequently, all four described how the work in the course seemed quite different from that in their prior courses. They all cited important differences between computing particular answers and constructing proofs of general statements. With different degrees of explicitness, each cited the difficulty involving in learning this new form of mathematical thinking. S3 was most explicit about this challenge, telling us in both interviews that she felt she had "no idea what she was doing" in the beginning of introduction to proof course. They all reported that because of that challenge, they undertook new learning activities outside of class (e.g., attending the Math Learning Center for the first time). Three of the four (S1, S2, and S3) received lower grades (3.0 out of 4.0 for all three) than they had in their previous mathematics courses. Substantively, they all reported learning specific proof methods (e.g., mathematical induction) and working for a few weeks on issues in real analysis (e.g., the convergence [or not] of sequences) in the introduction to proof course. We concluded that the course prepared these students reasonably well for real analysis because (1) the challenge of learning to prove statements in any mathematical domain was important preparation for work in real analysis, (2) the students addressed this challenge in their work and earned reasonable grades, and (3) the tools and content of the introduction to proof course were specific preparation for work in real analysis.

**The Introduction to Proof Course Was Helpful in Different Ways**

As we described in the previous section, all four participants reported that they agreed with the crucial role of the introduction to proof course in preparing for their success in real analysis. Interestingly, however, they seemed to have benefitted from that preparation in different ways, as they sustained or adjusted their learning activities formed in the introduction to proof course and prior mathematics courses.

**S1 and S3: Work together and exploit similarity across tasks.** In explaining their success in real analysis, S1 and S3 both emphasized the quality of their instructor's teaching, citing four main similarities to instruction in their introduction to proof course: (a) group work in class, (b) regularly assigned and graded homework, (c) weekly quizzes, and (d) the instructor's encouragement. Both S1 and S3 also reported they could reasonably predict the general nature of exam questions that were aligned with the regular homework and weekly quizzes. They completed their homework each week, whose content predicted the weekly quizzes, which in turn predicted the content of exams. Their instructor also gave a practice final exam, described to resemble the actual final. But this shared experience with instruction was coupled with changes in their learning activity. Whereas both S1 and S3 attended the Math Learning Center (MLC) at the university for the first time during their introduction to proof course and benefited from the activities and relationships supported there, neither attended in the MLC during real analysis. Instead, they worked remotely outside of class with the other members of their classroom small group that they maintained for the entire semester. When they got stuck on homework problems, they messaged with each other, sent pictures of the status of their solution attempts, and asked each other for suggestions.

Their voluntary group work outside of class in real analysis was influenced by their positive experience of social interactions in the introduction to proof course. Describing the different nature of the introduction to proof course in our prior study, both S1 and S3 reported that social and interpersonal resources were one of the key factors of their success in that course. S1 stressed the interaction with her peers and the instructor in and outside of class to be successful in the course.

> *Interviewer*: Okay. So, what do you think it takes to be successful in [the introduction to proof course]?
>
> *S1*: […] And I think it definitely helps to make friends and connect to other people who are in the classroom, or other people who have that class that you know and work together. Because you might not notice little errors in your proof, but they would. Like you can help compare. (Introduction to proof course, Summer 2016)

In addition to getting help from others on homework, the regular small group work in the introduction to proof course provided her with emotional support. She felt "more like high school classes with the interaction between the students and the professor" in comparison to her calculus classes in large lecture halls. She enjoyed the introduction to proof course because "there is a lot of interaction between everyone in the class." Similarly, S3 stated that the MLC provided opportunities to form a learning community supporting her success in the introduction to proof course. In our prior study, she stated that she "became friends" with others who regularly visited the MLC from her section and other sections of the course and worked together with them on weekly homework problems. Though she did not find tutors or peers at the MLC for real analysis, her prior experience in the introduction to proof course influenced her strong belief in collaboration with peers on mathematical work and confidence in real analysis.

> S3: It's okay if, you know, if they don't know something and if you don't know something because then you talk about it and then figure it out. So yeah, I mean if you had a group of people that were always on their phones, like you probably would be a little bit screwed over but if you talk things out and take all of your notes and just be like, okay let's figure this out. Three, four brains working on it, like everyone's in this class for a reason, we got here somehow.

In sum, the positive experience from the social and interpersonal resources of the introduction to proof course transferred to real analysis. Although they reported fewer resources in real analysis (e.g., less intensive group work in class, no TAs at MLC for real analysis) than there had been in the introduction to proof course, they adjusted to the new environment by finding a group of students who wanted to work together and support each other. They strongly endorsed the importance of collaboration with their peers and interactions with the instructor including her feedback and encouragement.

**S2 and S4: Work independently on lots of examples.** In contrast, S2 and S4 emphasized the importance of repeated practice on numerous example problems for each course topic, as practice increased the likelihood of mastery and success on course assessments. In addition, both carried out this practice-focused work on their own. S4 expressed frustration that his real analysis instructor (different from S1-S3's) did not provide a sufficient number of examples comparable to his experience in the introduction to proof course. Instead, he actively searched the internet for examples, explaining that he looked for problems that were related to those worked in class and had complete solutions (proofs). He would then work the problem and compare his proof to the one provided. If he was unsure how to start, he reviewed the provided proof and then attempted to complete it on his own—comparing his proof to the one provided when he finished. He never went to the MLC during real analysis (in part because he did not think that Center personnel were prepared to help with that content), though he had done so regularly during his introduction to proof course. Also, he stated that he did not need to get help from MLC or office hours to complete the homework problems in real analysis, which are similar to what his instructor showed in class. By contrast, he reported in the prior study that he was not able even to start some of the homework problems in the introduction to proof so he had to go to the MLC.

S2 did not complain about the supply of example problems; she found the combination of problems worked in class, homework problems, and problems in the text not assigned for homework sufficient. Though she was part of the in-class group that S1 and S3 cited as important, S2 seldom contacted her group outside of class and solved most course problems on her own. She described her method of study for exams to involve "just doing lots of problems." Like her peers, S2 did not attend the MLC during real analysis, though she had done so repeatedly and productively during her introduction to proof course. She was also able to complete almost all the homework problems using what her instructor showed her in class, whereas she reported that in the introduction to proof course there were significant gaps between problems worked in class and homework and between homework and exam problems.

**Common Structure Among Real Analysis Proofs**

One common thread in these results is the importance of noticing and abstracting a structure common to many real analysis proofs (what Selden & Selden [2013] have called a "proof framework"). Although S1 and S2 took different approaches to their work in real analysis, principally in how they engaged their peers, both spoke to the common structure they saw among the real analysis proofs their instructor and they produced. S4 spoke to this issue in different terms, and S3 did so only obliquely and without emphasis.

S1 saw the common structure among epsilon methods with some variation (e.g., epsilon-delta, epsilon-N) depending on the concepts involved in the statements (e.g., functions, sequences, and series). She learned to always start with specific sentences in the structured way of proving the statements using the epsilon methods.

> *S1*: In 320 [real analysis] so it's always epsilon greater than zero be given and then, you know, use the definition. So we did plenty of examples to know the structure of that, so in that case, she [her instructor] would, she would always tell us this is how you should write it, I guess.

She liked her instructor's practice of assigning similar problems using same approach/structure in the homework and stated that her instructor's proof writing in class emphasized this pattern. Her perception of common structure contributed substantially to her confidence going into major course assessments. It is notable that she found the introduction to proof course difficult partly because she saw no explicit common structure among the proofs she wrote. She addressed this issue at the interview in our prior study where she just finished the introduction to proof course.

> *Interviewer*: But was there any variation in your sense of the difficulty on the final exam by content? Or was it mostly just problem-by-problem; it wasn't so much whether it was number theory or sequences and series or whatever the content was?
>
> *S1*: It was more problem-by-problem. I was just afraid there was gonna be one I would look at [on the final exam] and just not know how to start. Because that's how a few of the homework problems had gone before. So that's what I was worried about. (Introduction to proof course, Summer 2016)

S2 also stated that the real analysis proofs were a lot more structured than those in her introduction to proof course. She asserted this pattern ("the proof was basically the same for every type of like, every type of problem") and indicated that real analysis proofs had an "introduction" that stated an arbitrary epsilon, the body of the proof, and a "conclusion" that related the particular case to the definition. In particular, her description of this structure corresponds to how Selden and Selden (2013) described the proof framework— "the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results" (p. 308).

S4 did not specifically describe common structural aspects among real analysis proofs like S1 and S2 did but instead emphasized the epsilon-delta method in real analysis. He responded that he thinks it is really important because "320 [real analysis] is all about that [epsilon-delta method]." Contrasting to other participants, S4 also described the process of completing a real analysis proof after setting up its structure as "computation." He used that term to indicate the repeated process of determining appropriate values for delta or N in epsilon arguments. Though he used different terms than S1 and S2, we interpret his assertion as similar to theirs: All three are citing structural regularities across many different real analysis proofs.

**Decrease in Problem-Solving Activity**

This abstraction of common structure across many different proofs is significant for many reasons, not the least of which is that it narrows considerably the "problem solving space" students found themselves in during course work. In describing their work in proof production in real analysis, none of the participants except S4 spoke to specific challenges in "filling in the blanks" of the common structure proof—Selden and Selden's (2013) "problem-centered part" of a proof.

S1 identified her work in proof production as executing known procedures on familiar tasks and connected to mathematical work prior to the introduction to proof course. Though she mentioned the variations in the common structure of epsilon methods according to different definitions being used (e.g., limit of a sequence, limit of a function at a point, etc.), she did not report specific challenges after setting up such structures in proof production. Rather she emphasized the importance of access to this common structure, compared to her proof work in the higher geometry course she took concurrently with real analysis.

> *S1*: And looking at the problems [in real analysis], you kind of had an idea already about what kind of structure you could use or you had several options whatever one you wanted to but in higher geometry, sometimes it was a lot harder to just look at a problem you didn't really know what kind of structure and do the proof. It wasn't a whole lot. It wasn't like use contradiction to prove this, it was just prove the statement. You know, there wasn't a lot of certain structure to it.

Although we did not ask her in detail about her work in higher geometry, the instructional style, or course content, it is notable that she characterized her reasoning involved in producing analysis proofs as relying on the explicit connection between the statements to be proven and the common argument structures introduced in the course. Comparing her work in real analysis to her prior work before the introduction to proof course, she did not dispute the parallel between "common structure" in real analysis proof and procedural work through in her prior calculus courses.

S4 described his struggles with proof production, even after setting up common structures. Responding to our question about the emphasis on proof in real analysis, he described the mathematical work in that course as "half proof, half calculation." Later in the interview describing his struggles in proof production, his response indicated what he meant by the calculation in real analysis as follows.

*S4*: The epsilon-delta method, the proof was kind of confusing and also the calculation part was kind of tricky sometimes. Because when you see it, like it's not obvious because you have to think about you're gonna use the ratio test or … like comparison test, that wasn't obvious, so you need to think a bit.

S4 specifically indicated that choosing an appropriate test among the series convergence tests taught in the course (e.g., ratio test, comparison test, root test, integral test) was a "tricky" part for him because it was not procedural and forced him to "think a bit." The reasoning involved in this calculation part corresponds to the problem-centered part of a proof in the sense that it may call on students' mathematical intuition with respect to the particular examples of the concepts (e.g., series or functions) given in the problem (Selden & Selden, 2009). Though his description of the calculation part indicates that he recognized a problem-solving aspect of proof production in real analysis, he seemed to distinguish this part from what he originally described as "problem-solving" nature of his prior work in the introduction to proof course. In our prior study when he was interviewed after the course, he emphasized a role of conceptual understanding in proof writing that differed from procedural work in prior calculus course.

*S4*: It's important to understand for the calculus is really important but even without understanding, you can just get used to do it, how to solve the problems, for calculus courses so, but for 299 [the introduction to proof course], you have to understand it. That's the difference. (Introduction to proof course, Summer 2016)

He recognized a problem-solving aspect of proof production in the introduction to proof course that necessitated understanding key concepts in the statements to be proven. He described this crucial role of understanding in characterizing the different nature of mathematical work in moving from calculus to the proof and proving work. On the other hand, he did not emphasize conceptual understanding for proof production in real analysis, rather stressed the importance of repeated practices that cover various examples.

## Discussion

This study produced three main results; all concern transition to proof "outcomes" from one introduction to proof course. First, the introduction to proof course prepared all four participants relatively well for proof-based work in real analysis, one major content area of advanced mathematics. If the goal of introduction to proof courses is to increase students' achievement in upper-level coursework, this course succeeded, at least for some students. Note that the introduction to proof course covered the basics of proof and proving and situated students' work in three different content areas. As such it fell into David and Zazkis's (2017) "Standard + Sampler" category of such courses. Only five of the 176 courses they reviewed across the R1/R2 institutions in the U.S. were of this type.

Second, even in our small sample, we have examples of students pursuing and achieving success in real analysis in different ways, even after "the same" introduction to proof. In particular, our four students took up group-work from their introduction to proof course in quite different ways—from substantially to not at all. The changes in how they used the resources of other people in real analysis appeared to depend jointly on (a) the absence of expertise in real analysis in the MLC, (b) the perceived ease or difficulty of real analysis problems, and (c) the individual orientations of the four students.

Third, returning to our opening metaphor of proof as a doorway, mathematical work on the other side of that door can be similar in important ways to the computational focus of prior mathematics work. Three of the four participants reported regularities across real analysis proofs that resemble in some ways the mathematical work that preceded the focus on proof—to recognize problems and apply the appropriate procedure to produce answers without significant effortful problem-solving. Though their introduction to proof course regularly asked these students to solve real problems (that is, tasks that were problematic for them), the tasks in real analysis significantly reduced the problematic nature of their mathematical work, as noted by S1, S2, and S4.

One major limitation of this study is our small and "correlated" sample. Three of the four participants experienced real analysis with the same instructor and engaged each other in the same small group—though they indicated no knowledge of their joint participation in the study. Our two different approaches to mastery (engage one's peers vs. repeated individual practice) are likely not the only narratives of mastering real analysis. Variation among students (e.g., in prior mastery experiences) and among instructors both likely contribute to the diversity of students' experience in real analysis. A second limitation leads to our next steps in this research: Most of the "graduates" of the introduction to proof course in this study have thus far had only modest experience in proof-based courses. Their journey will continue into new content areas and under the direction of different instructors. In the next phases of the research, we intend to track their experience in these new contexts (e.g., abstract algebra, real analysis II) and extend the reflective comparison of present and past experiences that we initiated in this study. We also hope to increase our sample size as more participants in our previous study enroll in proof-based content courses.

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# Mathematical Reasoning and Proving for Prospective Secondary Teachers

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*The design-based research approach was used to develop and study a novel capstone course: Mathematical Reasoning and Proving for Secondary Teachers. The course aimed to enhance prospective secondary teachers' (PSTs) content and pedagogical knowledge by emphasizing reasoning and proving as an overarching approach for teaching mathematics at all levels. The course focused on four proof-themes: quantified statements, conditional statements, direct proof and indirect reasoning. The PSTs strengthened their own knowledge of these themes, and then developed and taught in local schools a lesson incorporating the proof-theme within an ongoing mathematical topic. Analysis of the first-year data shows enhancements of PSTs' content and pedagogical knowledge specific to proving.*

*Keywords:* Reasoning and Proving, Preservice Secondary Teachers, Design-Based Research

Mathematics educators and education researchers agree on the importance of teaching mathematics in ways that emphasize sense making, reasoning and proving. In fact, teaching practices that emphasize mathematical reasoning have been shown to be more equitable, and associated with higher knowledge retention (Boaler & Staples, 2008; Harel, 2013). These practices encompass a wide range of processes such as exploring, conjecturing, generalizing, justifying and evaluating mathematical arguments (Hanna & deVillers, 2012; Ellis, Bieda & Knuth, 2012). Policy documents also encourage integrating reasoning and proving throughout grade levels and mathematical topics (CCSSI, 2010; NCTM, 2009). However, the reality of many mathematics classrooms has been far from the vision put forth by researchers and policy makers. Reasoning and proof is often confined to high school geometry, tied to a particular format and used to show already known results. Consequently, students and teachers alike often view proof as redundant, rather than a means of deepening mathematical understanding (Knuth, 2002). Although teachers often recognize the importance of reasoning and proof beyond high school geometry, they tend to choose skills-oriented activities over proof-oriented ones for their own classrooms (Kotelawala, 2016). One reason for this might be, as Bieda (2010) found, that even experienced teachers struggle to implement reasoning and proving in their classrooms.

These studies suggest that enacting teaching practices that emphasize reasoning and proof is a complex process that requires awareness and intentionality on behalf of the teacher. Preparing teachers who are capable of implementing such teaching practices and cultivating positive attitudes towards mathematical reasoning and proving is a critical objective of teacher preparation programs (AMTE, 2017). Yet, the theoretical and practical knowledge in this area of teacher preparation has been scarce (e.g., Ko, 2010; Stylianides & Stylianides, 2015).

To address this knowledge gap, we utilized a design-based-research approach (Edelson, 2000) to develop and study a novel capstone course *Mathematical Reasoning and Proving for Secondary Teachers*. The goals of the course were to improve the PSTs mathematical knowledge for teaching proving, including subject matter knowledge and pedagogical content knowledge. The goals of the research study were to (a) explore how PSTs' knowledge and dispositions towards the teaching and learning of reasoning and proof developed as a result of participating in the capstone course, and (b) to identify course design principles that afforded PSTs' learning. In this paper, we illustrate the overall structure of the course; we describe in greater detail one of the four modules, specifically, the module on Conditional Statements (CS); and we provide details on how PSTs interacted with the components of the CS module. We also show evidence of growth of PSTs' mathematical knowledge for teaching proof, following the completion of the course.

### Theoretical Framework and the Course Design

Researchers have conjectured that engaging students in reasoning and proving, that is, exploring, generalizing, conjecturing and justifying, might require a special type of teacher knowledge: Mathematical Knowledge for Teaching of Proof (MKT-P) (e.g., Lesseig, 2016; Stylianides, 2011). Building on their work, we theorize that MKT-P consists of four interrelated types of knowledge, two specific to content, and two related to pedagogy (Fig. 1).

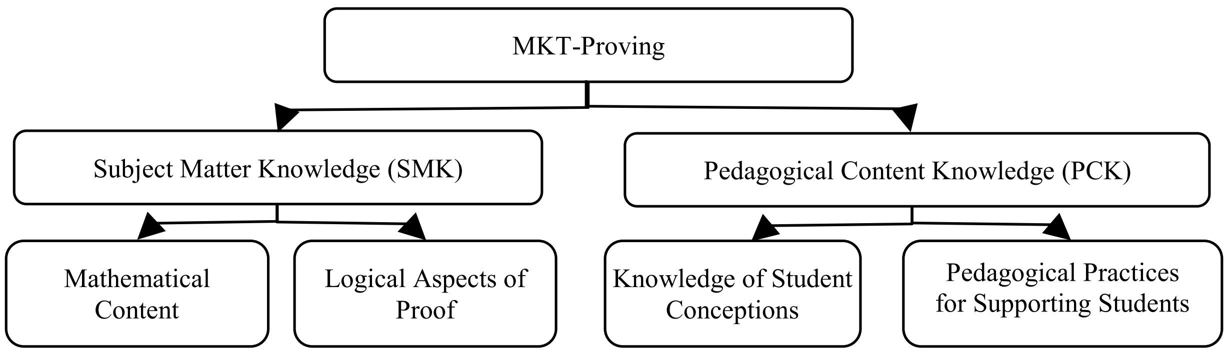



**Figure 1.** Mathematical Knowledge for Teaching of Proving (Buchbinder & Cook, 2018)

As suggested in Figure 1, to enact reasoning and proving in their classrooms, we hypothesize that teachers must have robust Subject Matter Knowledge (SMK) of mathematical concepts and principles, and also knowledge of the logical aspects of proof, which includes knowledge of different types of arguments, proof techniques, knowledge of logical connections, valid and invalid modes of reasoning, the functions of proof, and the role of examples and counterexamples in proving (Hanna & deVillers, 2012). Teachers also need strong Pedagogical Content Knowledge (PCK) specific to proving, such as knowledge of students' proof-related conceptions and common mistakes, and knowledge of pedagogical strategies for supporting students' proof activities. Thus, we built into the course structure opportunities for PSTs to develop and practice these four types of knowledge (Fig. 1).

The course consists of four modules, each three-weeks long, corresponding to four proof-themes: quantified statements, conditional statements, direct proof and indirect reasoning. These themes were identified in the literature as challenging for students and PSTs (Antonini & Mariotti, 2008; Weber, 2010). Thus, each module was designed with activities to enhance PSTs' knowledge of the logical aspects of proof related to the proof themes, followed by developing and teaching lessons at a local school integrating that proof-theme with current mathematical topics. This combination of the activities was intended to address both subject matter knowledge and pedagogical knowledge specific to proving, as mentioned above. Figure 2 shows the structure of the course (top) and the general structure of a single course module (bottom).

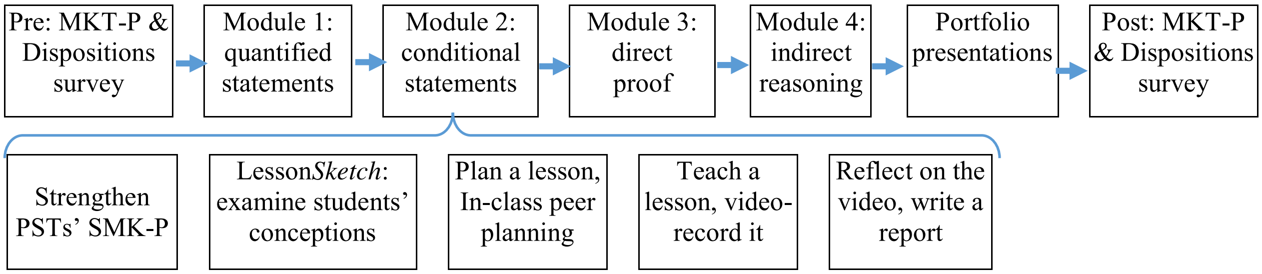



**Figure 2.** Design of the course *Mathematical Reasoning and Proving for Secondary Teachers*.

The theoretical background underlying the course design also draws on a situated perspective of learning (Peressini, Borko, Romagnano, Knuth, & Willis, 2004) which views learning as patterned participation in social contexts. In particular, Borko et al. (2000) assert that learning to teach occurs across multiple settings through active participation in the social contexts embedded in them, and should not be confined to a university classroom. Building on the literature on practice-based teacher education (e.g., Ball & Forzani, 2011; Grossman, Hammerness & McDonald, 2009) we designed opportunities for PSTs to enhance their proof-related pedagogical knowledge in an environment of reduced complexity and risk, through the virtual learning platform Lesson*Sketch* (Herbst, Chazan, Chieu, Milewski, Kosko, & Aaron, 2016). The learning experiences created and administered through this platform allowed the PSTs to engage in practices such as interpreting sample student work, identifying students' conceptions of proof, evaluating students' mathematical arguments, and envisioning responding to them in ways that challenge and advance students' thinking. We also engaged PSTs in planning and implementing, in local schools, lessons that combine the proof themes with the regular mathematical content. We see this as a critical component and the unique feature of our course design, that aims to bring together the aspects of proof and secondary curriculum.

As PSTs enacted their lesson, they recorded it using 360° video cameras, which captured simultaneously the PSTs' teaching performance and the school students' engagement with proof-oriented lessons. PSTs then watched and analyzed their lesson, wrote a reflection report and received feedback from the course instructor to inform future lesson planning.

The design-based research format of this project requires that we take a careful look at the various course components and the participants' interaction with them. Thus, in the methods and results sections below, we describe one module, Conditional Statements (CS), in order to demonstrate how we address the first goal of our project: to develop and study the capstone course. We focus on the following research questions:

1. *How did the PSTs interact with the CS module?*
2. *What mathematical and pedagogical ideas addressed in the CS module were implemented in the PSTs' lessons?*

At the same time, a second main goal of our project was to improve the preservice teachers' knowledge of and disposition toward teaching proof and reasoning through this course. Thus, we explored the research question:

3. *How did the PSTs' MKT-P and dispositions toward proof develop throughout the course?*

## Methods

Participants in the first iteration of the course were 15 PSTs in their senior year (4 middle-school, and 11 high-school track; 6 males and 9 females). The PSTs had completed the majority of their extensive mathematical coursework, and two educational methods courses, one focused on general mathematics education topics that are common across grade levels, and one specific to teaching secondary mathematics.

Multiple measures were used to collect data on PSTs interaction with the CS module. We collected and analyzed PSTs' responses to home- and in-class assignments, and video-recordings of all in-class sessions to answer the first research question. To answer the second question, we analyzed the PSTs' cumulative teaching portfolios containing four lesson plans, video-records of the lessons taught, reflection reports, and sample school students' work. We used a modification of Schoenfeld's (2013) TRU Math rubric to analyze 360° video-records of the PSTs' lessons.

In addition, we investigated how PSTs' knowledge of content and pedagogy, and their dispositions towards proof evolved throughout the course. For that, we used two instruments: pre- and post- measures of mathematical knowledge for teaching proof - the MKT-P questionnaire, and dispositions towards proving survey. Prior to the study, we identified four existing instruments from the literature (Corleis et al., 2008; Kotelawala, 2016; Lesseig, 2016; Nyaumwe & Buzuzi, 2007), that partially matched our research focus. We combined elements of those instruments and supplemented with our own items to create an MKT-P instrument containing four sets of questions corresponding to the four types of MKT-P (Fig. 1). The dispositions towards proof survey included six sets of questions, some open and some closed, addressing PSTs' notions of proof, the purpose and usefulness of proof, suitability of proof for secondary students, PSTs' own confidence and comfort with proving, as well as confidence in teaching proof to students. The survey questions were a combination of items from related instruments (e.g., McCrone & Martin, 2004; Nyaumwe & Buzuzi, 2007). Both the MKT-P instrument and the dispositions survey were reviewed and analyzed by experts in educational assessment. The data collected with the MKT-P questionnaire and dispositions towards proof survey were analyzed quantitatively to answer the third research question.

## Results

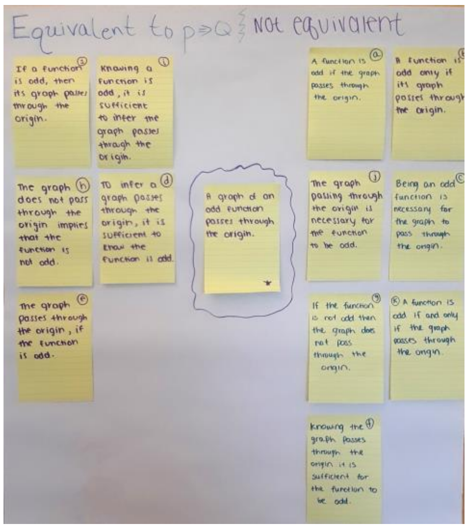
### PSTs' Interaction with the Conditional Statements Module

In this section we take a closer look at the Conditional Statements (CS) module and the PSTs interaction with the module components. The Conditional Statements module comprised the following activities: (1) sorting conditional statements, (2) Lesson*Sketch* experience *Who is right?*, (3) analysis of conditional statements in the secondary curriculum, (4) planning and implementing a lesson that incorporates some ideas about conditional statements, and (5) implementation reflection.

#### Sorting Conditional Statements.

For this in-class activity, PSTs broke up into three groups and each group received one conditional statement. Their task was to identify the statements' hypothesis $P$ and conclusion $Q$, and use the $P$ and $Q$ to write statements in 11 logical forms such as: $P\ if\ Q$, $P\ only\ if\ Q$, $If\ {\sim}P\ then\ {\sim}Q$, and others. The PSTs were to write each statement on an index card using different colors for true and false statements, and then sort the cards into statements equivalent to $P \Rightarrow Q$, and non-equivalent to it. The original statements given to the groups were: "A graph of an odd function, defined at zero, passes through the origin" (group 1), "A number that is divisible by six is divisible by three" (group 2), and "Diagonals of a rectangle are congruent to each other" (group 3).

Each group created a poster showing how they sorted the cards, and presented their work to others. Figure 3 shows a poster by group 1. The statements on the cards (Fig. 3-a) are written in terms of the original statement, which is located in the middle of the poster. Figure 3-b shows which logical forms PSTs identified as equivalent or non-equivalent to $P \Rightarrow Q$, the incorrect answers are marked with *. Note that the PSTs correctly identified all equivalent forms and all non-equivalent forms. However, they also wrongly identified two equivalent forms (j) and (b) as non-equivalent.




(a)

Types of statements as sorted on the poster.
P: $f$ *is an odd function, defined at zero*
Q: *the graph of* $f$ *passes through the origin*

Equivalent to $P \Rightarrow Q$:
If P then Q.
d) To infer Q, it is sufficient to know P
e) Q if P
h) Not Q implies not P
i) P is sufficient to infer Q

Nonequivalent:
a) P if Q
c) P is necessary for Q
f) Q is sufficient for P
g) If not P then not Q
k) P if and only if Q
j) Q is necessary for P*
b) P only if Q*

(b)

**Figure 3.** (a) Group 1 poster, sorting equivalent and non-equivalent statements; the original statement is in the middle; (b) logical forms represented on the poster.

The form that the PSTs in all three groups found most challenging to interpret was $P$ *only if* $Q$. Some PSTs realized that it is equivalent to a contrapositive, $\sim Q \Longrightarrow \sim P$, and therefore equivalent to the original statement. But other PSTs rejected this idea by arguing that $P$ can be true "not only when $Q$ is true", concluding that $P$ *only if* $Q$ is not equivalent to $P \Longrightarrow Q$. Group 2 argued that a statement "A number is divisible by 6 only if it is divisible by 3" is both untrue and non-equivalent to the original statement, because a number is divisible by 6 *not only* when it is divisible by 3, but also when it is divisible by 2. Since the PSTs found the arguments for equivalence and non-equivalence of the two statements to be equally appealing, it required facilitator intervention to clarify the equivalence using the contrapositive argument. In this explanation, the abstract logical notation appeared to be more useful than the contextualized one.

After all groups presented their posters and the discrepancies were discussed and resolved, PSTs received additional prompts to grapple with, such as: What is the relationship between truth-value of a statement and equivalence of statements? Identify inverse and converse among the given forms, and rewrite the language of necessary and sufficient conditions symbolically. PSTs discussed these prompts in their groups first, and then as a whole class. The follow-up homework assignment (discussed below) introduced the PSTs to students' conceptions of conditional statements.

**LessonSketch experience *Who is Right*?**

Lesson*Sketch*.org is an interactive-media web-based platform for teacher education, which allows a teacher educator to represent classroom interactions as cartoon sketches, which PSTs can analyze. Such representations preserve much of the authenticity of the real classroom, but allow PSTs more time to interpret student thinking and plan a response (Herbst et al., 2016). The interactive tools of Lesson*Sketch* allow teacher educators to create rich learning experiences for PSTs, that provide PSTs with opportunities to experience situations that resemble classroom interactions, and envision themselves participating in them as teachers. These experiences can involve analyzing samples of student work; watching classroom scenarios in the form of video, animation or story board; responding to prompts about these scenarios; creating their own depictions of classroom interactions; and participating in discussion forums.

The experience *Who is right?* is based on real student data, and was field-tested in prior studies (Buchbinder, 2018). In this experience the PSTs were presented with a false mathematical statement: "If $n$ is a natural number, then $n^2 + n + 17$ is prime," and asked whether it is true, false, sometimes true, or cannot be determined. It is important to note that although a mathematical statement can only be true or false, we discovered in pilot implementations of the experience that PSTs struggled to commit to a dichotomous answer. Instead, they were trying to hedge their responses in the comment box. This tendency to avoid a dichotomous response is reminiscent of "fuzzy logic" (Zazkis, 1995), the form of reasoning that allows one to assign to a statement a value that qualifies one's confidence in its correctness. By providing PSTs four, rather than two, options to choose from when evaluating the truth-value of the statement, we allowed greater flexibility for PSTs to respond to the prompt. As researchers and teacher educators, we were able to assess who among our PSTs had not yet developed bivalent logical reasoning, which aligns with conventional mathematical logic.

After PSTs evaluated the statement on their own, they viewed a set of slides depicting arguments of five pairs of students evaluating the truth-value of that given statement. The arguments were developed to reflect common student misconceptions about conditional statements. For example, "proving" the statement by testing a set of strategically chosen examples, requesting more than one counterexample to disprove a statement, asserting that the truth-value of a statement cannot be determined when both supportive examples and counterexamples exist (Buchbinder & Zaslavsky, 2013). The PSTs' task was to evaluate the correctness of these five students' arguments, identify instances of expertise and gaps in student reasoning and pose questions to advance or challenge that reasoning. In a similar vein, we provided four, rather than two, response options for evaluating students' arguments: correct, more correct than incorrect, more incorrect than correct and incorrect. The design decision to allow PSTs to assign partial correctness to student arguments stemmed from pilot implementations of this Lesson*Sketch* experience (Buchbinder, 2018).

Figure 4 shows the distribution of PSTs' ratings of each of the five arguments. Each row corresponds to one of the five arguments, and the numbers in the rows correspond to the number of PSTs choosing a particular rating (incorrect, more incorrect than correct, more correct than incorrect, or correct). To validate the numerical data, we examined PSTs' justifications of their ratings, which revealed that that PSTs scores are not solely dependent on mathematical correctness of the evaluated argument, but are strongly influenced by pedagogical considerations.

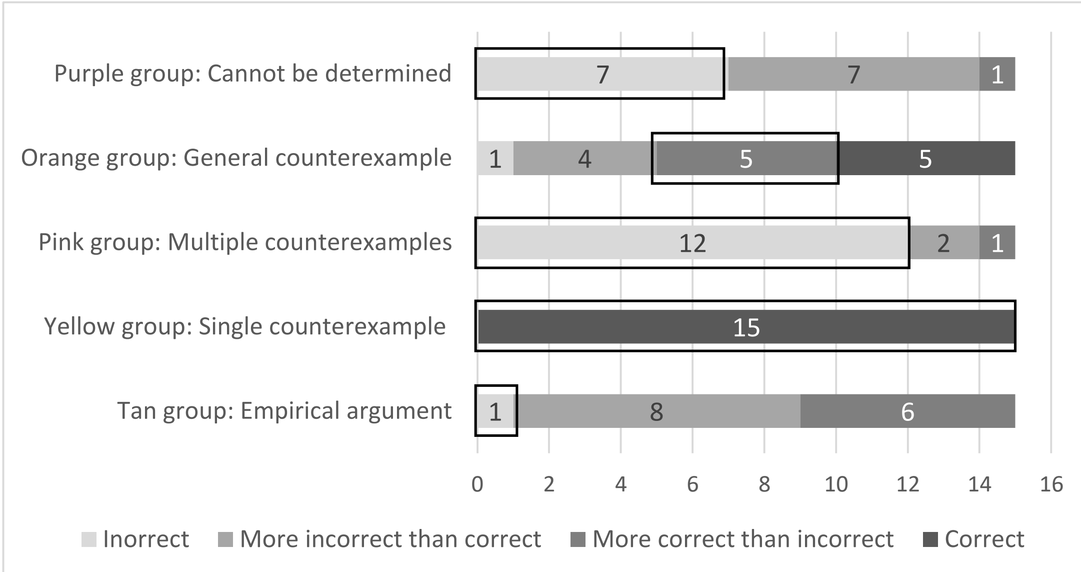



**Figure 4:** Distribution of PSTs' ratings of five student arguments in the LessonSketch experience *Who is right?* The dark frame indicates expected correct answer.

Figure 4 shows that all PSTs accepted a disproof by a single counterexample, and the vast majority of PSTs rated as incorrect the request for multiple counterexamples. Also, 14 out of 15 PSTs rated negatively the assertion that the truth-value cannot be determined when both supportive and counterexamples exist. The one PST who rated this student response as more correct than incorrect justified it with a pedagogical consideration. She wrote:

> This statement should be written with a universal quantifier to make it easier for students to understand it, but since it is not I can see where the students are confused and are not sure if they can prove the statement true or false. However, finding one counterexample is enough to prove this statement false, which they don't understand yet.

This kind of response from the PSTs was not an isolated instance. Overall, across all PSTs' responses to the variety of student arguments we observed that mathematical correctness of the argument was not the ultimate evaluation criterion; PSTs' evaluations were strongly affected by pedagogical considerations, reflecting the PSTs' desire to award partial credit for correct students' work or computation. In particular, the eight PSTs who rated Tan group's empirical argument as more incorrect than correct, noted that it is not an appropriate way to prove a conditional statement, but acknowledged the correctness of student calculations and validated their efforts.

We note that that while valuing student contributions is an important pedagogical practice, it is crucial for PSTs to recognize which arguments are mathematically valid, and help students to progress towards more mathematically accepted logical arguments. This is, however, heavily dependent on the PSTs' own mathematical reasoning. In our sample, there were five PSTs who believed the given statement to be true because they, themselves, could not find a counterexample. These PSTs also rated the students' empirical argument as more correct than incorrect, mimicking their own invalid reasoning.

PSTs completed the *Who is right?* activity at home, and then shared and discussed their responses as a whole class. The main focus of the discussion was on the students' conceptions of proof and the type of evidence needed to determine the truth-value of conditional statements.

PSTs also discussed the ways to acknowledge student effort, while pointing students towards more valid modes of argumentation.

**Conditional statements in the secondary curriculum.**

The final in-class activity in the Conditional Statements module focused on analyzing where conditional statements appear in the secondary curriculum. Working in groups, PSTs analyzed excerpts from glossary sections in a few textbooks. Their task was to write some of the rules or theorems stated there in the form of conditional statements or in some of the equivalent forms. The goals of this activity were (1) to demonstrate the prevalence of conditional statements in high school mathematics curricula aside from geometry, and (2) to anticipate student difficulties in reasoning about such statements. The discussion questions for this activity addressed the importance of understanding conditional statements, potential student difficulties related to conditional statements and brainstorming ways to support student thinking.

**Planning and implementing a lesson on conditional statements.**

The lesson planning process included several steps. About a week prior to the lesson, PSTs contacted their cooperating teacher to find out the mathematical topic for their lesson. Based on this information, the PSTs developed a lesson incorporating that topic with some ideas about conditional statements. During an in-class session, the PSTs worked in small groups sharing the lesson plans, testing out ideas, giving and receiving feedback with their peers and the course instructor. After improving the lesson plans through this process, the PSTs implemented their lessons in middle school and high school classrooms participating in the study.

PSTs in the middle school worked with one teacher and class, throughout the semester. These PSTs could develop closer relationships with the students, but all their lessons were tied to one mathematical unit, namely exponents. The high school track PSTs rotated among different classrooms and teachers. This allowed exposure to a variety of mathematical topics, but limited the PSTs ability to establish long-term connections with students. This also complicated their lesson planning, since the PSTs struggled to envision the students' mathematical background. The high school track PSTs taught lessons in Pre-algebra, Algebra 1, and Geometry on a range of topics such as order of operations, variable expressions, linear equations, classifying triangles, and parallel lines.

The analysis of PSTs' lesson plans showed that PSTs came up with a variety of creative ways to integrate conditional statements in their lessons while appropriately adjusting them to the students' level. Almost all PSTs used real-world examples, such as, "If I do my homework, I will get good grades" to introduce students to the general structure of a conditional statement, and to identify the hypothesis ($P$) and the conclusion ($Q$). One of the PSTs, Sam, came up with a particularly creative way to introduce conditional statements: She showed students a few product advertisements, asked them to turn the slogans into conditional statements and analyze their structure in terms of $P$ and $Q$.

There was a great variation among PSTs' use of mathematical vocabulary in the lessons. For example, although both Cindy and Audrey developed a lesson on exponents asking 8$^{th}$ grade students to evaluate the truth-value of several conditional statements, Cindy did not introduce any vocabulary in her lesson, while Audrey mentioned explicitly conditional statements, and used $P$ and $Q$ notation. Nate, who taught 10$^{th}$ grade geometry, included in his lesson on parallel lines the definitions of converse, inverse and contrapositive. The majority of PSTs used a softer approach, explaining that statements of the form "if__then__" are called conditional statements, and used language of given and claim, instead of hypothesis and conclusion for P and Q. These kinds of adjustments were discussed by PSTs throughout the in-class sessions of the module as possible ways to support students' engagement with proof, particularly at the lower grade levels.

The most utilized types of tasks implemented by the PSTs were True or False, and Always-Sometimes-Never, in which the PSTs had students identify the hypothesis and the conclusion in each statement, determine whether the statement is true or false, and provide justifications or counterexamples. But there were other types of tasks utilized by the PSTs. For example, Bill created two sets of notecards: one set contained hypotheses (e.g., a triangle is not equilateral) and another set contained conclusions (e.g., a triangle is isosceles). First, Bill asked his students to match hypotheses to conclusions to produce conditional statements about triangles. After determining as a group which statements are true and which are false, Bill asked the students to change the order of cards by physically switching between the hypothesis and the conclusion. Then he had students examine the relationship between the converse and the original statement.

Another PST, Logan, modified a common Algebra 1 task into a proof-related activity on conditional statements. First, he asked his Algebra 1 students to produce an algebraic expression describing a certain sequence of operations: pick a number, quadruple it, subtract 6 and divide the result by 2. Then, he asked students to determine the truth value of several statements about the resulting expression, for example: "If your output is 34, then your input had to have been 8," or "If your input is even, then your output will be odd." Overall, except for four PSTs who only minimally addressed conditional statements, the majority of PSTs successfully and creatively integrated conditional statements in their lessons.

**Reflection on lesson implementation.**

As mentioned above, the PSTs recorded their lessons using 360° cameras to capture both the teacher and the student interactions. Each PST watched and reflected on their video by (a) annotating it, and (b) writing a report on how the lesson went, according to a given set of prompts. To support their claims, PSTs were required to provide time-stamps in the video. Some reflection prompts were common across all lessons, such as the following questions: In what ways did you engage students in making sense of mathematics? What aspects of students' thinking did you find particularly interesting or surprising? On the scale 1 (low) – 5 (high) evaluate your own performance in the lesson and explain the rating. In addition, each module had questions specific to the proof-theme. In the CS module, the PSTs were asked to reflect on aspects of their lessons that were specific to conditional statements. The following excerpt is taken from Grace's reflection report in which she responded to a prompt: What ideas about conditional statements do you think students understood by the end of your lesson? How do you know? Grace wrote:

> The pair work allowed [the students] to bounce ideas off each other and I could tell that they understood how to write the converse and contrapositive based on their discussions. For example, I overheard things like, "Converse, ok we need Q then P" or "This is the contrapositive, right? … If not Q, then not P?" I was happy to hear these conversations because not only were the students engaged in the activity, but they were working and communicating well together.

Reflecting on the video-recording of the lesson was quite a time consuming process, however when asked to reflect on this aspect of the course at the end of the semester, the majority of PSTs indicated that this contributed to their learning. In the summative course evaluation Ellen wrote:

> I felt that the video recordings were extremely beneficial for my learning. Even when there were parts of the lesson that I thought went very smooth at the time, I later found when watching the videos that things did not always go as smooth as I had thought.

This idea of occasional mismatch between one's feeling of their teaching performance and the lesson as recorded on camera was a recurring theme in our data. For example, Audrey, who was initially happy with her lesson, wrote after watching the video: "I realized that I am so quick to answer students' questions, that I am not stepping back and asking for other students' ideas." And concluded: "I want to make sure I am reaching all my students." Overall, the 360° video capturing allowed the PSTs to see how students reacted to their teaching in general, and to the specific proof-theme.

**Evidence of PSTs' Learning**

In this section we provide data from the MKT-P and disposition questionnaires to examine research question three: *How did the PSTs' MKT-P and dispositions toward proof develop throughout the course?* To trace changes in PSTs' knowledge and dispositions as a result of the course, we administered pre- and post- measures of MKT-P and pre- and post-surveys on dispositions towards proof (Fig. 2). The MKT-P contained 12 questions, 3 in each of the four areas of MKT-P (Fig. 1). In line with Hill and Ball (2004) all MKT-P items were embedded in pedagogical contexts, that is, as representing student mathematical work. The items in our measures, therefore, called for analyzing, interpreting and responding to students' conceptions of proof, similar to activities described above in the CS module.

Out of the three items measuring PSTs' Knowledge of the Logical Aspects of Proof (or Logical Knowledge, LK, for a shorthand), two items addressed knowledge related to conditional statements. In one item, a geometrical statement about quadrilaterals, and its converse were given, accompanied by a set of four claims about these statements, for example, "to prove statement (1) is false it is sufficient to prove statement (2) is false", or "to prove statement (2) is true, it is sufficient to prove statement (2) is false". The task for PSTs was to identify the correct claim about the given pair of statements. Almost all PSTs performed well on this item in both pre- and post- questionnaires, indicating that the item was too easy for PSTs in our sample, and did not sufficiently discriminate among them. We plan to modify this item in the future.

The second item dealing with conditional statements had several parts. It first introduced four statements about real numbers: (a) If $x < 1$, then $x^2 < x$, (b) If $x^2 < x$ , then $x < 1$, (c) If $x^2 > x$ , then $x > 1$, and (d) If $x > 1$ then $x^2 > x$ . PSTs were asked for each statement to determine whether it is true or false, and if false, provide a counterexample. Next, we referred to statement (a) as "If P then Q", and asked the PSTs to identify the logical form of the rest of the statements, accordingly. Lastly, we asked the PSTs to identify an equivalent statement to (a) from a given list of distractors. Across all parts of the item, we documented a 14% score increase from pre- to post-test, with most gains occurring in identifying equivalent statements.

Overall, the data analysis revealed that PSTs had relatively high initial scores on three out of four types of MKT-P, that is Knowledge of Logical Aspects of Proof, Knowledge of Students' Conception of proof, and Knowledge of Pedagogical Practices for supporting students (Fig. 5). Because of the high initial scores we calculated the calculated the percentage of possible growth in lieu of examining the point increase. That is, we calculated what percent of the possible growth (difference of maximum score and pre-test score average) constitutes the observed growth (difference in post-test and pre-test averages). For example, although the increase in the Knowledge of Logical Aspects of Proof was only 1.15 points, it constitutes 48% of possible 2.4

points needed to obtain the maximum score. The highest gain of 66% occurred in the Knowledge of Pedagogical Practices for supporting students' learning of proof. This was reflected in the items, among others, that called for interpreting students' conceptions of proof related to activities from the CS module.

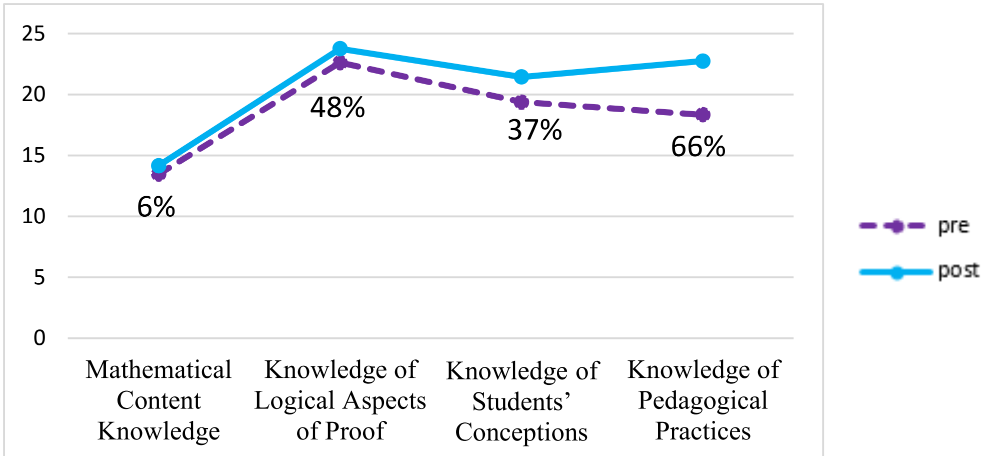



**Figure 5:** Sub-score averages across all participants (out of 25 points) for the four components of the MKT-P survey, with percent increase.

The dispositions towards proof survey included five categories of questions: (1) the PSTs' notions of a proof, (2) the purpose and usefulness of proof, (3) confidence and comfort with proving, (4) the suitability of proof in the school curriculum, and (5) confidence in and knowledge about teaching proof to students. In general, the pre- and post-test results did not show much change in the PSTs' thinking around categories 1, 2 and 4, but a few noteworthy results had to do with categories 3 and 5. Prior to the course, about 65% of the PSTs agreed or strongly agreed that they felt confident in their ability to prove mathematical results from the school curriculum, whereas 92% responded positively after completing the course. A slightly greater change was noted in confidence of teaching proof to students, from 50% agreement in the pretest responses to 84% agreement in post-test responses. The fact that PSTs' overall growth of dispositions was relatively modest may be attributed to the not necessarily justifiable feeling of confidence at the start of the semester due to the PSTs' prior mathematical coursework.

## Discussion

This paper described the first iteration of a 3-year design-based-research project aimed to enhance PSTs' knowledge and dispositions for teaching proof at the secondary level. We have described the overall structure of the course *Mathematical Reasoning and Proving for Secondary Teachers*, the theoretical underpinnings of its design, and provided details on the Conditional Statements module, one of four course modules. Our first two goals in this paper were to describe how PSTs interacted with the Conditional Statements module, and to examine what mathematical and pedagogical ideas addressed in the Conditional Statements module were implemented in PSTs' lessons. Towards this end we provided descriptions of the various components of the module, such as an activity on sorting conditional statements and the Lesson*Sketch* experience *Who is right?* which required the PSTs to evaluate the logical reasoning of students in relation to determining the validity of a conditional statement. We also presented data on how the PSTs interacted with these components of the module, and identified particular strengths and weaknesses in PSTs' knowledge. Our data show that PSTs interacted with different components of the CS module in productive and meaningful ways. Although the scope of the paper does not allow presenting the full detail of the interactions, the analysis of classroom videos showed that PSTs engaged in rich discussions around the logical aspects of conditional statements actively seeking to clarify their meaning, especially when presented in the alternative forms to "If P then Q". The PSTs also were deeply concerned with the pedagogical side of teaching middle- or high-school students about conditional statements, as their lesson plans and reflections attest.

The data on PSTs' interactions with the CS module serve as a backdrop for understanding the gains in PSTs' content and pedagogical knowledge of proof, following their participation in the capstone course. The comparison of PSTs' performance on pre- and post- measures of MKT-P shows that the areas in which the PSTs' growth of MKT-P was most evident are those that were emphasized in the course, namely, the logical aspects of proof and pedagogical knowledge specific to proving. Due to the small sample size we were unable to test whether the gains were statistically significant, however, we are encouraged by these outcomes, especially since they are based on evidence beyond PSTs' self-report.

One of the critical elements of design-based-research, according to Edelson (2002), is to treat a particular study as an instance of a more general phenomena to develop educational design theories that go beyond the specific research context. This can be achieved by examining the relationship between the design features of the intervention – the capstone course – and the PSTs' learning. In the descriptions above we highlighted the range of practice-based elements in the course design: Analyzing students' conceptions of proof; devoting course time and resources to lesson planning and sharing; teaching in middle school and high school classrooms; and reflecting on one's teaching, supported by video technology. We assert that all these elements contributed to enhancement of PSTs' MKT-P and dispositions towards proof. One particular aspect where this can be seen is the PSTs' pedagogical knowledge for proving. Throughout the course, and in the final course reflection, many PSTs acknowledged that they often found it challenging to integrate the proof-themes with pedagogical practices. However, as we showed in the case of CS module above, the PSTs' lesson plans and classroom implementations clearly reflect the proof-themes addressed in the course (although there was obvious variation among the PSTs). This is also visible in the increased PSTs' scores on the MKT-P portion related to the Pedagogical Aspects of Proof. This suggests to us, that the repeated cycles of lesson development, implementation and video-supported reflection contributed to PSTs' pedagogical knowledge for proving.

Our data analysis is still ongoing, as we examine video of on-campus sessions and of the PSTs' teaching to create a more fine-grained description of how PSTs' content and pedagogical knowledge evolved throughout the course, and to match this growth to the design principles of the course. The results of this analysis will inform future iterations of our project. In particular, we plan to further conceptualize and enhance instructional scaffolding of the course in the subsequent iterations of the study, to better support PSTs' learning.

Through our data analysis we seek to generate an evidence-based instructional model, and four proof-modules that can be adopted by other courses or institutions to improve preparation of secondary mathematics teachers, and, potentially advance the field's understanding of how to support PSTs' development of mathematical knowledge for teaching proof.

## Acknowledgments

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# Exploring Pre-service Elementary Teacher's Relationships with Mathematics via Creative Writing and Survey

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*Thirty-two pre-service elementary teachers completed a survey regarding their beliefs and attitudes towards learning and teaching mathematics and two creative writing tasks. In the first writing task participants described their personal relationship with personified mathematics and in the second, they introduced personified mathematics to their future students. By interpreting the survey and writings, different aspect of attitudes towards mathematics were discovered. A main finding was that the two writing tasks were able to provide a more nuanced view of how pre-service teachers' beliefs about mathematics changed over time. We found that their past experiences with mathematics often affected the way that they talked about mathematics to their students, though they did attempt to suppress their negative experiences when introducing mathematics to their students.*

*Keywords*: affect, pre-service teachers, creative writing

## Introduction

Negative experiences in learning mathematics can cause a person to develop mathematics anxiety. Unfortunately, research suggests that teachers can contribute to these negative experiences, which can have an adverse effect on a student's learning (Bekdemir, 2010). Therefore, the mathematical experiences of pre-service teachers are critical not only for themselves but also for their future students.

In addition to mathematical content knowledge, teachers' views, beliefs, and preferences about mathematics influence their instructional practice (Thompson 1984). While surveys can measure pre-service teachers' views and attitudes toward mathematics, they can be limited because participants are responding to specific items rather than expressing their own thoughts. Creative writing tasks in which participants write about their experience with mathematics can provide researchers with information about different aspects of their attitudes and beliefs about mathematics. Using Zazkis' (2015) method of eliciting personification, I gave thirty-two pre-service teachers opportunities to describe their relationship with mathematics as though mathematics were a person in two creative writing tasks. The first writing task was the same as Zazkis' task in which they tell about their personal relationship with mathematics, and in the second one, they introduced mathematics to their future students. Participants also completed a 14 question survey assessing their beliefs and attitudes regarding mathematics. The survey questions were modified from the Mathematics Anxiety Rating Scale (MARS)-abbreviated version (Alexander & Martray, 1989) and the Mathematics Teaching Efficacy Belief Instrument (Enochs & Riggs, 2002).

I analyzed these two writing tasks using conceptual blending (Fauconnier & Turner 2003) to explore pre-service teachers' beliefs about mathematics and how it could affect their teaching. Moreover, using a survey about their beliefs about mathematics, I saw how two different types of tasks could bring some common ideas and different perspectives regarding beliefs and attitudes toward mathematics.

## Literature Review

Several studies used conventional surveys to measure pre-service teachers' perception of mathematics. Survey methods were used to investigate both the relationship between participants' attitudes and anxiety level of mathematics and to see how those results are related to their teaching of mathematics.

A teacher's personal mathematics anxiety can affect their anxiety about teaching mathematics. Hadley & Dorward (2011), surveyed 692 elementary school teachers in a three-part survey. The first part was MARS-R (MARS-Revised) and the second part consisted of 12 Likert-scale items which mirrored the MARS items but assessed anxiety about *teaching* mathematics. The third part of the survey asked about the elementary teachers' mathematics instructional practices. It included questions regarding how the teachers use writing mathematics in class, manipulatives, group work and so forth. Also to investigate relationship between teachers' mathematics anxiety level and student achievement, the class average scores on the state mathematics test for each teacher was collected. The result showed that teachers who were anxious about mathematics tended to also be anxious about teaching mathematics. However, upon further analysis the data indicated that for teachers with higher levels of anxiety about mathematics, there was no relationship with anxiety about teaching mathematics. Some of these teachers had very high anxiety about teaching mathematics, while others had moderate or even low anxiety about teaching mathematics. Some low anxiety teachers were experienced teachers. The study also found that regardless of teachers' anxiety level, when teachers were comfortable teaching mathematics, students achieved somewhat higher test scores. No relationship was found between anxiety about teaching mathematics and mathematics instructional practices.

Mathematics anxiety also influences confidence in teaching mathematics. Bursal & Paznokas (2006) measured sixty five pre-service elementary teachers' math anxiety levels and confidence levels for teaching elementary mathematics and science. They used Revised-Mathematics Anxiety Survey (R-MANX), and the Math Teaching Efficacy Belief Instrument (MTEBI) for mathematics. The result showed that participants who scored low and moderate mathematics anxiety showed confidence in teaching mathematics and participants who scored high mathematics disagree with MTEBI statements.

Teachers' prior experiences with mathematics can influence their beliefs more than their teacher education programs. Raymond (1997) observed six first and second year elementary school teachers for 10 months to see the relationship between mathematics beliefs and teaching practice. The study involved observing and interviewing six teachers to categorize their beliefs about mathematics, beliefs about learning mathematics, beliefs about teaching mathematics, and teaching mathematics. He evaluated each teacher as "traditional", "nontraditional" or "even mix" for each category. All six teachers named their prior school experiences of mathematics being the main influence on their beliefs about mathematics and teaching experiences. Teacher education programs were viewed as having a slight influence on their teaching and beliefs about mathematics. This showed how beliefs about mathematics are affected by prior school experiences, and for teacher how it could affect their teaching practice. In addition, the study showed that the effectiveness of a teacher education program had on teachers' beliefs about mathematics. The authors mentioned that it is possible that a teacher education program cannot directly affect teaching practices beyond certain level. Therefore, the result of the study suggests that focusing on pre-service teachers' beliefs about mathematics could indirectly affect pre-service teachers practice.

These studies highlight the current issue of pre-service teachers' relationship with mathematics and the effects that could have on their teaching mathematics. Having mathematics anxiety can affect how a teacher teaches mathematics and the teacher's level of comfort with teaching math can affect student learning. It is also evident that a teacher's personal beliefs about mathematics have an effect on his or her teaching. Therefore, having less mathematics anxiety, being comfortable teaching mathematics, and general positive relationships with mathematics are important for pre-service teachers to gain in order to become better mathematics educators.

Non-standard methods were used in some studies to investigate people's perception of mathematics. Zazkis (2015) assessed pre-service teachers' attitudes towards mathematics by using a creative writing task in which they described their relationship with mathematics as though mathematics were a person. Then, he used conceptual blending to interpret participants' human description of mathematics by mapping their descriptions from the human relationship space to the corresponding characteristic in the mathematics space. This method of data collection and analysis provided a "vivid window" into the beliefs of the pre-service teachers.

Stereotypes about what kind of people do mathematics can play into students' beliefs about mathematics. Picker and Berry (2000) asked 12-13 year olds in five countries to draw a mathematician. By analyzing the drawings, they found that children of this age relied on stereotypical images from the media to portray a mathematician. Students who possess these beliefs may feel mathematics anxiety if they do not conform to the stereotype.

Several studies regarding students' relationship or view about mathematics have been measured by some kind of survey. Survey methods are limited because a participant has prescribed choices for the items. It is possible that the way she or he expresses anxiety about mathematics could be different from the wording of the survey items. Hence, chosen vocabularies in a survey could limit participants' ways of expressing views on mathematics. Likert-like surveys have similar limitations. It could be ambiguous for a person to describe their anxiety on a numerical scale. Therefore, using two writing tasks, I explored pre-service teachers' relationships with mathematics through concepts and feelings they portrayed to gain a different perspective of preservice teachers' relationship with mathematics.

## Methodology

### Data Collection: Personification of Mathematics and Survey

Before we make an effort to change a pre-service teacher's beliefs about mathematics, we need to be able to assess their relationship with mathematics. In a study of pre-service teachers' beliefs, Zazkis (2015) utilized the method of eliciting personification when he had 36 pre-service teachers write a story about their relationship with mathematics. The data were analyzed with conceptual blending. Conceptual blending involves taking the elements of two mental spaces and blending them together to form a new space. Through conceptual blending, Zazkis made connections between the personal relationship space and the mathematics space to form relationship with mathematics space. For example, a student described math as a "terrible beast" which can be connected to one's level on enjoyment of mathematics. Then using conceptual blending, Zazkis identified that a "terrible beast" in the relationship space maps to their level of enjoyment in the mathematics space, which implies a fear of mathematics of that student possess in the combined space. Zazkis concluded that personification is not a replacement for the methodology of past studies but it offers a vivid window into study participants' relationships with mathematics.

In my study, using Zazkis' method of eliciting personification, I explored thirty-two pre-service elementary school teachers' relationships with mathematics and how these relationships could affect the way that they teach mathematics to their future students. Thirty-two pre-service elementary teachers completed a survey assessing their beliefs and attitudes regarding the learning and teaching of mathematics along with two creative writing tasks (see Table 1). The survey questions were modified from Mathematics Anxiety Rating Scale-abbreviated version (Alexander & Martray, 1989) and Mathematics Teaching Efficacy Belief Instrument (Enochs & Riggs, 2002).

*Table 1: Writing Tasks*

| Writing Task 1 Prompt (W1) | Writing Task 2 Prompt (W2) |
|---|---|
| Your assignment is to personify Math. Write a paragraph about who Math is. This paragraph should address things such as: How long have you known each other? What does he/she/it look like? What does he/she/it act like? How has your relationship with Math changed over time? These questions are intended to help you get started. They should not constrain that you choose to write about. | You are introducing Math for the first time to children. Describe who Math is as a character to children using following questions as guidelines. What kind of Personality does it/she/he has? What does Math look like? What does Math act like? How one can get along with Math? How one can get to know Math? How should one treat Math? These questions are intended to help you get started. These should not constrain that you choose to write about. |

Using conceptual blending, I formed connections between the pre-service teachers' relationships with mathematics to their portrayal of mathematics to their students. The results of this study display how teaching mathematics and relationships with mathematics are related without using a quantitative survey. I also compared my analysis of written assignments with the survey to find relationships between the two different data collection methods. I found a "vivid window" to investigate pre-service teachers' self-reflections on mathematics and their portrayal of mathematics for their future students so that they can be aware of the potential conflict.

**Data Analysis: Conceptual Blending**

Conceptual blending is a basic mental operation that leads to new meaning, global insight, and conceptual compressions useful for memory and manipulation of otherwise diffuse ranges of meaning. (Turner & Fauconnier, 2003, p.57). We can bring two things together mentally in various ways. For example, consider "land yacht" as a reference to a large, luxurious automobile. The word "land" and "yacht" come from different domains. But combination of two words from different domain provides a different but relatable meaning. Despite the fact that "yacht" is associated with water, it gives large luxurious meaning to the "land yacht". And the "land" implies that the vehicle moves on the ground which gives the meaning of automobile. Notice that blending two spaces enable us to relate land and yacht to what we already had an idea of, which is a large, luxurious automobile. In this paper, using conceptual blending, I attempted to connect the participants' written product and mathematical level to view their relationships with mathematics. Moreover, I assessed participants' relationships with mathematics and their portrayal of mathematics for children through conceptual blending.

## Results

The results are separated into three different categories describing how the writing tasks and survey relate to each other, and one case study of a participant in which we highlight the different contributions of the two data collection methods.

### Category 1: W1 and W2 Offered a Different View

A significant portion of students described mathematics differently between personification of mathematics and portrayal of mathematics. Most of the data in this category showed that pre-service teachers had a negative personal relationship with mathematics but portrayed mathematics positively to children.

Mary wrote on W1 that her relationship with mathematics is changing as she encounters math but when reflecting upon taking a statistics course in college she concludes “math betrayed me once again and made me feel inadequate.” Her expression of “betrayal” in the human relationship space is connected to level of enjoyment in mathematics space. By connecting human relationship space and mathematics space, Mary’s relationship with mathematics appeared to be negative. But despite her disappointing encounter with math, on her W2 she wrote “Things may not go your way when associating with Math, but it always has your best interest in mind”. Her portrayal of mathematics having “your best interest in mind” corresponds to level of importance of mathematics in mathematics space which leads to value of mathematics in Portrayal of mathematics space. This shows a hope that she wants children to have a positive experience with math, though she did not appear to have a positive experience with math.

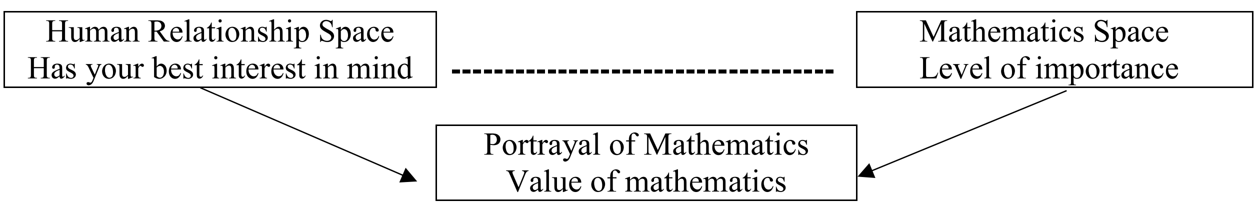


*Figure 1. “Best interest in mind” conceptual blending diagram.*

Jamie described math as a “scary monster that thrives on the tears of children” and extended her bad relationship with math by adding “I have never understood the purpose of forcing kids to take math classes throughout their entire school careers”. But on W2, she wrote “Math is a super chill dude. Do not be afraid to laugh with math and ask questions because math is always there. Math actually gets excited when you ask questions because that means you are learning.” Her W2 describes positive characters of math despite her bad relationship with math from W1.

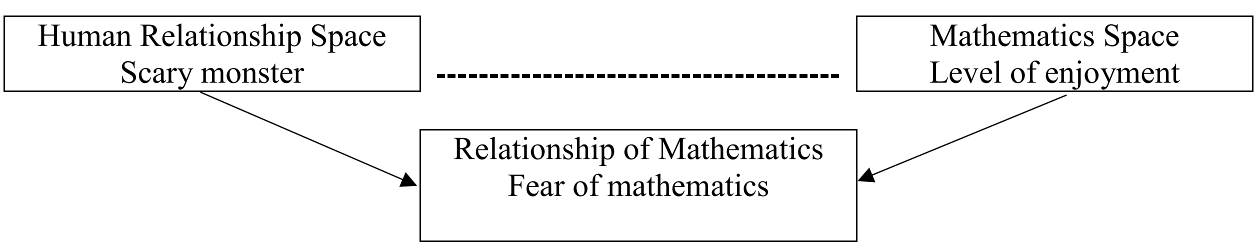


*Figure 2. “Scary monster” conceptual blending diagram.*

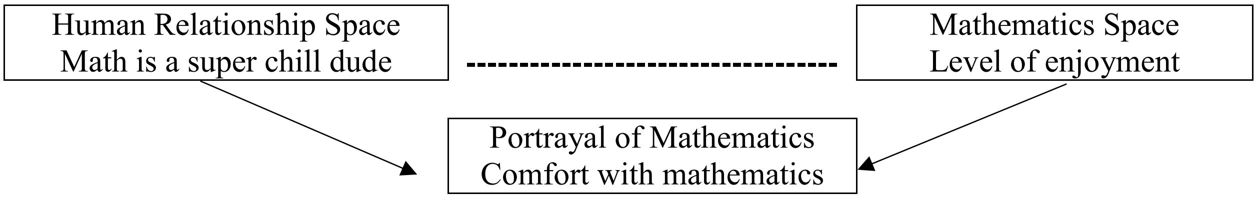


*Figure 3. "Super chill dude" conceptual blending diagram.*

A surprising finding here is that Jamie's personal level of enjoyment of mathematics appears to be completely opposite of her portrayal of mathematics to her students. Even more interestingly, she understood no benefits of kids learning math in W1, but on W2 she encourages kids to ask questions to math emphasizing learning.

**Category 2: W1, W2 and Survey Offered Similar Views**

It was common for participants to have similar views on mathematics on W1 and W2. It appeared that a student's experience with mathematics affects when she or he introduces mathematics to children in below examples.

Fiona described her challenging relationships with mathematics on W1 and added "I still try to make things better between us, because clearly this isn't at all my fault" then concluded by saying "I still have hope that I'll at least understand him one day, especially in the days that I have to introduce to him others" Then on W2, she portrayed mathematics with fearful caution for children. She wrote "listen closely, and if you still have questions, ask him! However don't speak negatively of him as that will affect your relationship with him." Fiona's experience with mathematics directly impacts her portrayal of mathematics. Then she concluded W2 by saying "My poor friend-such a bad reputation for such a useful skill" There writings align with her responses in the survey as she indicated that solving a problem that involves mathematical reasoning is a frustrating experience.

Jamie indicated on a survey response that "Learning mathematics requires a special talent." On W1 she wrote that having a good relationship with math is like having a "magic power." The description of "magic power" corresponds to complexity in the mathematics space which shows her relationship with mathematics was difficult due to her perception of mathematics being complex. She also agreed that "Solving a mathematics problem more than once is a waste of time" and "reasoning skills that are taught in mathematics course can be helpful to me if I were to major in math or a related field" on the survey.

**Category 3: W1 and W2 Offered Something Different From Survey**

There were a few examples of when two writing tasks differed from the survey responses. Bailey indicated that "a teacher's own feelings about mathematics are independent of a teacher's teaching practice" on the survey, but her writing tasks showed that her personal beliefs did affect her portrayal of mathematics to students. On her W1, she wrote "math is picky and I wish that he and I could get along better, math is too confusing for me to be around for long periods of time, so I can only handle him in small doses". On her W2 she wrote "as you both grow up, it will be harder to build a strong relationship with him (math)." She also portrayed math as more of a self-centered character by saying "He loves to talk about himself, and likes to joke with you and often tries to trick you" Which shows Bailey's challenging relationship with mathematics impacting on her portrait of math to children.

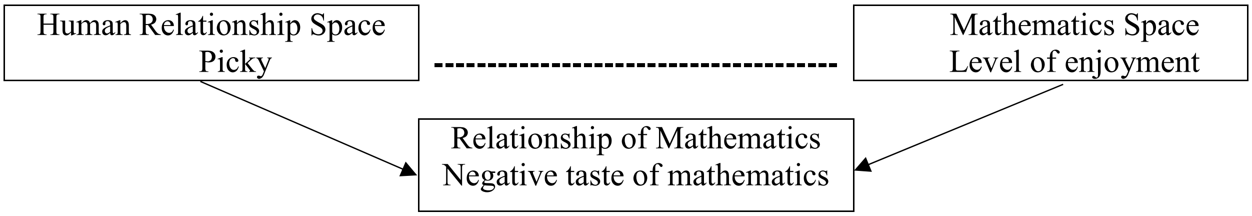


*Figure 4. "Picky" conceptual blending diagram.*

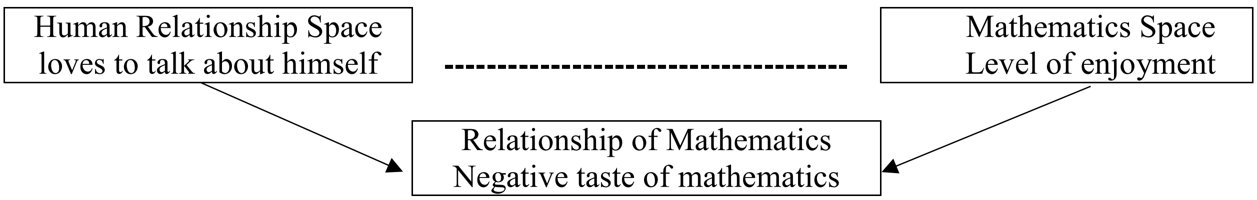


*Figure 5. "Self-Centered" conceptual blending diagram.*

Gabby also indicated that "a teacher's own feelings about mathematics are independent of a teacher's teaching practice" on her survey. But on her W1 she expressed that she and math did not get along. On W2, her introduction of math to children was a warning in which she said "you may get tired of seeing math everyday but make the most of it" and "Math does get harder as time goes on and the friendship will get more complicated."

Jamie indicated that that "a teacher's own feelings about mathematics are related to how well a teacher can teach mathematics to students." But her negative experiences with mathematics on W1 did not seem to affect her portrayal of mathematics on W2 which was more optimistic and positive. It appeared that Jamie's personification of mathematics on W1 seemed completely different from her portrayal of mathematics on W2 as we saw in Figures 2 and 3.

**The Case of Emily**

Zazkis (2015) claimed that the method of eliciting personification of mathematics through writing cannot replace survey methods, but can offer a different viewpoint. Here we will discuss the case of Emily, in which we can clearly see that the survey and writing tasks assessed different aspects of her beliefs.

Emily indicated on the survey that "solving a problem that involves mathematical reasoning is an enjoyable experience". Several of her survey responses indicated that she has positive views about mathematics. Then she personified mathematics using four different descriptions:

> Math has the "bad boy" look – tattoos that you don't understand, chains with missing links, and he always has an answer for everything. He's kind of a dork with his dark rimmed glasses and a shaggy hair cut. Math is as wide as he is tall, and you would think he was a linebacker for the school football team. He looks mean, but she's really a big teddy bear.

These personifications shows the changes in her relationship with mathematics over time. She indicated that at first mathematics appears mysterious and intimidating, but after learning more about mathematics that view changed. The diagram for these character traits are shown in Figures 6 and 7. Elsewhere in her writing she clearly stated that "Math used to scare me and I dreaded the time of day when I had to spend an hour with him, After years of dreading him, I finally knew it was time to get to know the real him, once I did, I really found my time with him to be fun and interesting."

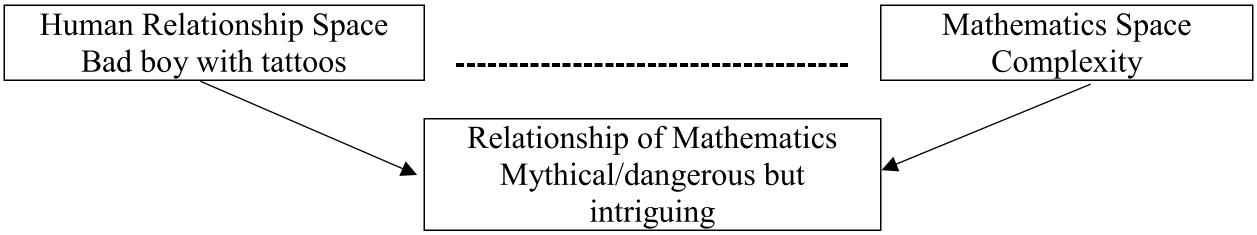


*Figure 6. "Bad boy" conceptual blending diagram.*

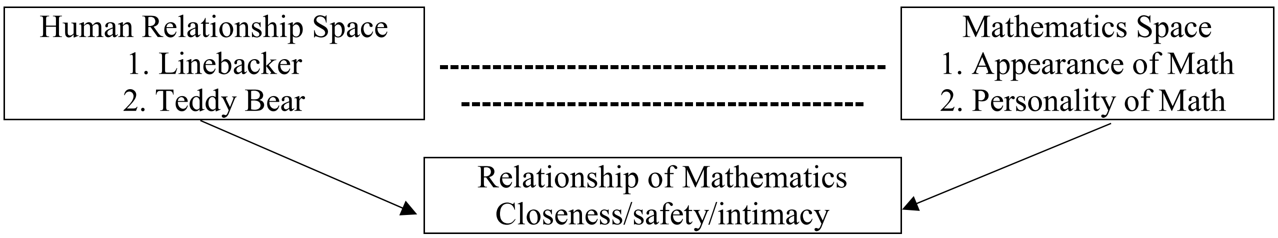


*Figure 7. "Teddy bear" conceptual blending diagram.*

Understanding Emily's struggle in the past with mathematics helps us analyze what she wrote on W2; "He thinks it's funny but does not realize he's scary. You might see one of Math's problems and it might look very scary because you don't understand it! But don't worry!" She expressed caution to children based on her experience. It also appears that her overcoming struggles with mathematics on W1 contributed to her positive attitude towards mathematics in W2; "He wants you to understand all of his different identities and for you to enjoy spending time with him!" This could explain why her portrayal of mathematics for her future children was different in W2 because of her current positive views of mathematics. She portrayed mathematics by saying "Math is like a superhero too! He can do almost ANYTHING!"

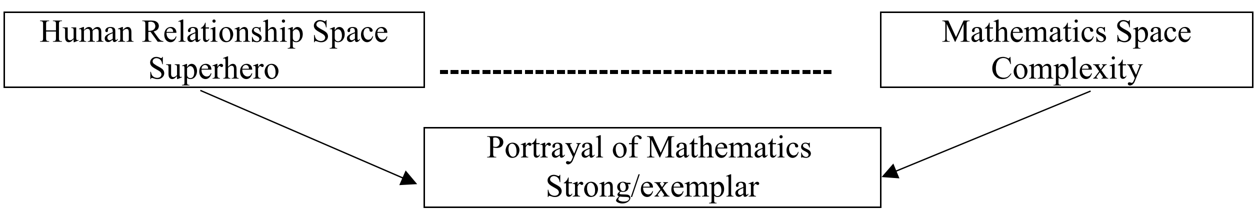


*Figure 8: "Superhero" conceptual blending diagram*

As shown above, W1 and W2 offered more than the survey item since it showed Emily's past relationship with mathematics and how that changed over time. Survey questions are often not designed to capture this type of change in a person's relationship with mathematics, but only how a person currently feels about mathematics.

There were also aspects of her relationship with mathematics that appeared on the survey but were not captured in her written assignment. On the survey, she indicated that "learning mathematics requires special talent" and "For me, doing well in mathematics course depends on how well the teacher explain things in class." These two ideas were not presented in her writings. This result provides more evidence that the two writing tasks do not provide the same information as the survey when measuring beliefs and attitudes towards mathematics.

**Notable Trends**

Thirty-two percent of participants agreed that “teacher’s own feeling about mathematics is independent of a teacher’s practice.” However, among that 32%, some participants’ two writing tasks were not completely independent. It appeared that their own experience with mathematics affect how they portray mathematics for children.

Thirty-four percent of participants agreed that “Reasoning skills that are taught in mathematics courses can be helpful to me if I were to major in mathematics or a related field” contrary to “in my everyday life” Despite this response rate, almost all of the participants wrote something on W2 emphasizing how children should get along with mathematics because it is good for them. But it is notable to point out that significant number students did not write anything on W1 regarding usefulness of mathematics. W1 mostly described their relationship with mathematics respect to their performance or understanding of mathematics.

If W1 showed positive relationship with mathematics, W2 also showed positive attitudes towards mathematics when introducing to children. For example, Emily wrote “I really found my time with math to be fun and interesting!” and then introduced math by saying “I would like to introduce to you my favorite, dear, friend, Math!” then she explained how math can do anything and how a great guy he is.

Most of the participants expressed importance of mathematics in W1 and W2 regardless of their past experience with mathematics. Many of the participants wrote something close to “you should try to get along with him because it is good for you,” which shows that they value mathematics. Although many of participants mentioned why mathematics is important, some of them did not specify the reason why children should get along with mathematics.

## Discussion and Further Research

Zazkis (2015) concluded in his paper that the tool of personifying mathematics in writing offers another perspective of pre-service teachers’ relationship with mathematics than previous data collection methods. This study has given evidence that the two writing tasks did offer a different perspective than the survey. Although some of the participants showed similar views on all three tasks, other participants showed a different and more detailed attitude towards mathematics and portraying mathematics in the two writing tasks. In particular, the case of Emily showed that the writing tasks can capture the changes in a pre-service teacher’s relationship with mathematics over time, whereas many of the existing mathematics attitude surveys designed to measure a person’s current beliefs about mathematics. On the other hand, because of the open-ended nature of the writing tasks, there may be survey items that would address particular aspects of a person’s beliefs about mathematics that they might not think to include in their writing.

Another contribution of this study is the comparison between the writing task in which they described their own relationship with mathematics (W1) and the writing task in which they introduced mathematics to their students (W2). The most common difference that we found was that the teachers would have a more positive attitude about mathematics when presenting it to children, although we did see that their prior experiences affected the way that they would talk about mathematics to their future students.

Further investigation on factors that cause pre-services teacher’s beliefs and attitudes toward mathematics is necessary to improve their views about mathematics that could impact their teaching practice. Although many of the participants express the value of mathematics on W2, it was unclear in many cases if the valuing mathematics was because they wanted to prevent frustration for students struggling with mathematics or because they believed in the benefits of mathematics as a discipline.

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# Supporting Prospective Teachers' Understanding of Triangle Congruence Criteria

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*We report on the results of pre-post written assessments of our college geometry students' justifying SAS to a hypothetical 10th grade student. We hypothesized that investigating transformations in the taxicab metric would perturb students' understandings of the relationships between triangle congruence criteria and isometries, so they would more explicitly identify the properties of transformations as a necessary part of their justifications of triangle congruence criteria. We describe how the ways our geometry students' responses on the pre/post assessments did not reveal the understandings we anticipated. We discuss conjectures of ways of introducing transformations and sojourning to taxicab geometry that might be more productive.*

*Keywords:* Geometry, Congruence, Mathematical Knowledge for Teaching

## Introduction

Research has demonstrated that supporting college geometry students' understandings of transformational geometry remains a challenge (e.g., Hegg & Fukawa-Connelly, 2017). Because many of our college geometry students are prospective secondary teachers, supporting their mathematical knowledge for teaching secondary geometry is an important course goal. The United States' Common Core State Standards for Mathematics [CCSS-M] state that high school geometry students should be able to justify triangle congruence criteria (angle-side-angle, side-angle-side, side-side-side) as a consequence of properties of rigid motions (NGACBP/CCSSO, 2010). Hegg and Fukawa-Connelly (2017) found that college geometry students struggle with explicitly using relevant properties of transformations in such justifications, and they suggest that "asking for the kinds of explanations of ideas that [college geometry students] would give [secondary geometry] students has value in both giving researchers insight into their understanding of the content and giving policy-makers a better understanding of what additional supports will be needed going forward" (p. 8). This paper presents a study that investigated an instructional intervention aimed at perturbing preservice secondary teachers' understandings of congruence in order to better support their understanding of transformational proofs for the triangle congruence criteria. The assessment we used followed the suggestion by Hegg and Fukawa-Connelly by situating a question about justifying the side-angle-side triangle congruence criteria in a classroom scenario.

## Theoretical Background

### Transformations and Intellectual Need

The CCSS-M advocate for a definition of congruence that is based on the rigid motions of the plane (i.e. isometries in Euclidean geometry) (NGACBP/CCSSO, 2010). The standards then call for the development of congruence criteria for triangles to follow from this definition of congruence. According to the CCSS-M, a *transformations-first* approach would develop ideas of geometric congruence in the following ways:

1. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

2. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
3. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. (NGACBP/CCSSO, 2010)

While there is some historical precedent to this pedagogical approach (Sinha, 1986), this approach has been little researched in the mathematics education literature (Jones & Tzekaki, 2016). It should be noted that this is a slight departure from the development of geometry as presented in Euclid's *Elements*, since Euclid himself did not have the tools to define rigid motions, but defining congruence as a result of rigid motions is roughly mathematically equivalent to Euclid's approach of defining congruence through superposition in Euclidean geometry (Sinha, 1986). Moreover, most students experience an approach to Euclidean geometry in high school that relies on the (mathematically flawed) axiomatic system developed by Euclid himself in the Elements, with the possible addition of the SAS criterion for triangle congruence to his original five axioms (Portnoy, Grundmeier, & Graham, 2006). However, as Portnoy and colleagues found, students struggle to construct proofs using geometric transformations. Specifically, students often focus less on the properties of rigid motions and instead view them as actions that they can perform with geometric objects. Students also often view the result of a transformation as the resulting image of the geometric figure, instead of the result of transforming the entire plane (Hollebrands, 2003). Nevertheless, a transformations-first approach to geometry can provide accessibility for students to reason about their own actions (Simon, 1996). Moreover, connections between synthetic (Euclidean) and analytic (transformational) geometry can help students eventually reason about calculus (Hollebrands, 2003).

We identified the CCSS-M approach to introducing triangle congruence criteria due to properties of rigid motions as potentially problematic for our students. We had encountered SAS as a postulate in our own high school teaching and learning experiences. Anticipating that our students' encounters with SAS would likely also be as a postulate, we were concerned about motivating a justification for congruence criteria based on properties of transformations. After all, if two triangles in the plane are congruent, there is a rigid motion that maps one to the other that preserves *all* corresponding side lengths and angle measures. How might we motivate them to consider a justification for congruence *criteria* based on *properties* of a *composition* of isometries?

Harel (2014) emphasizes this problem for secondary students learning geometry in his discussion of the CCSS-M's transformations-first approach:

> In the [CCSS-M's] approach the transition from middle school geometry to high school geometry is to be carried out through the rigid motions of translation, reflection, and rotation and the motion of dilation. In middle school geometry, these motions are delivered informally, and in high school they are defined as functions on the plane. In both levels, the motions are merely described, not intellectually necessitated through problems the students understand and appreciate… (p. 26).

Harel discusses the lack of *intellectual need* for using transformations as a method of justification for K-12 students. Harel informally defines intellectual need as follows:

> When people encounter a situation that is incompatible with or presents a problem that is unsolvable by their existing knowledge, they are likely to search for a resolution or a solution and construct, as a result, new knowledge. Such knowledge is meaningful to

them because it is a product of their personal need and connects to their prior experience. (p. 23.)

Keeping in mind that the prospective teachers in our courses were likely not exposed to this approach in school, we investigated tasks and approaches that might provide this intellectual need.

**Taxicab Geometry**

Taxicab geometry is an excellent non-Euclidean geometry to introduce as students' first experience outside of Euclidean geometry. It has an axiomatic structure that is close to Euclidean geometry (Krause, 1986), it has real-world applications, and it is close enough to Euclidean geometry to allow for students' to retain and apply some intuitive and embodied understandings from Euclidean geometry. Taxicab geometry develops from changing the metric for measuring distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ on the Cartesian coordinate plane from the traditional Euclidean metric, $d_E = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, to the taxicab metric, $d_T = |x_1 - x_2| + |y_1 - y_2|$. An important consequence of changing the metric in this way is that not every "rigid motion" in taxicab geometry is also a taxicab *isometry* (that is, not every rigid transformation in taxicab geometry preserves distance). Moreover, two segments of the same length under the taxicab metric may not be congruent, if there does not exist an isometry that maps one to the other (see Figure 1).

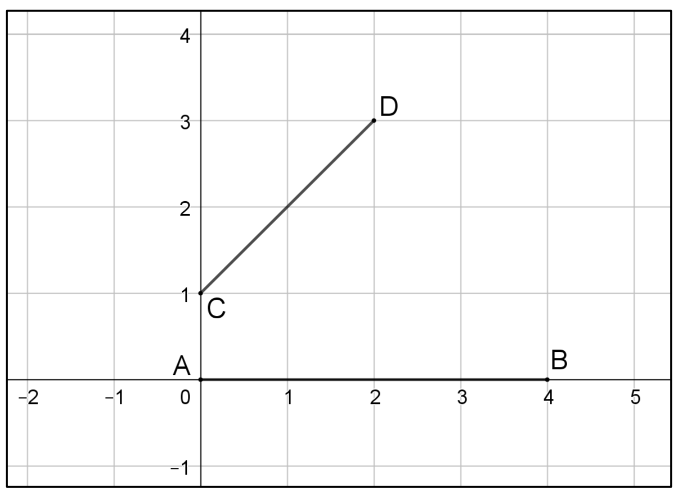



*Figure 1. In the taxicab metric, AB and CD have the same length, but there is no isometry mapping AB to CD.*

One important aspect of taxicab geometry to note is that in an internally consistent treatment of taxicab geometry, taxicab angles do not necessarily have the same measure as Euclidean angles. Thompson and Dray (2000) describe how taxicab angles may be defined in relation to Euclidean angles to support taxicab trigonometry. However, a right angle in taxicab geometry is the same as a right angle in Euclidean geometry. By focusing on transformations of right triangles, discussion of taxicab geometry does not need to include this added complication.

Thus, working in taxicab geometry provides a rich context for problematizing students' association of congruence with equal measure in a way that working in Euclidean geometry cannot. Since it is a theorem of geometry under the Euclidean metric for distance that two segments have the same measure if and only if they are congruent, students can assume that having the same measure implies congruence without referring to a definition of congruence which relies on isometries. However, this conflation of equal measures and congruence may lead a student to assume that Figure 2 shows a counterexample to the SAS triangle congruence criterion in taxicab geometry (lengths are labeled based on the taxicab metric). In fact, there is no taxicab isometry that will take one of the legs of one of the right triangles to either of the legs of the other right triangle, so there are no corresponding sides of these triangles that are congruent. Hence, the SAS triangle congruence criterion does not apply, although students who continue to associate congruence with equal measure will think it does.

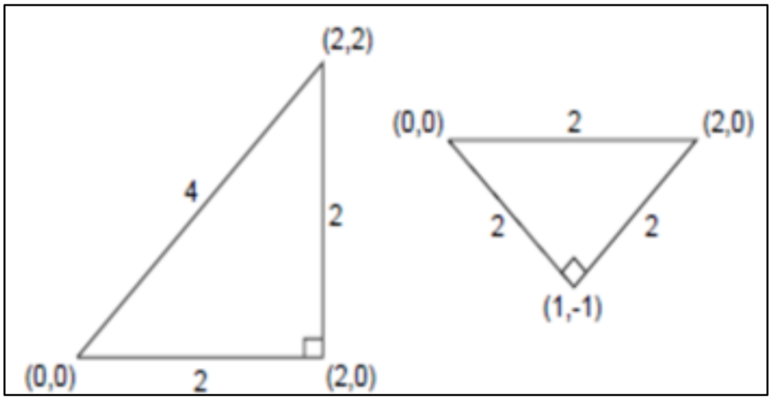



*Figure 2. In the taxicab metric, these pair of triangles appear to violate the SAS criterion*

In this way, taxicab geometry provides a context in which students can establish an intellectual need (Harel, 2013) to separate their notions of congruence and measurement, and therefore appeal to other notions (such as transformations) to find justifications for the triangle congruence criteria.

## Research Question

We theorized that we could provide an intellectual need for our students to reconsider their understanding of congruence by prompting them to explore the possible ways in which a rigid motion may not preserve distance. After such an experience, in our thinking, students would understand congruence (and therefore triangle congruence criteria) in Euclidean geometry as depending on the *properties* of the transformations and thus be better-prepared to support future secondary students' in constructing that way of thinking about triangle congruence. Thus, our goal was not to teach them a technique or procedure; rather our goal was for them to realize that their preconceptions of triangle congruence criteria would not always hold if one figure was the image of another under a transformation that was not an isometry. We viewed this as foundational for motivating a disposition to teach their students to justify triangle congruence criteria as a result of properties of isometries. However, since all rigid motions are isometries in Euclidean geometry, we created an instructional intervention using taxicab geometry to serve as this perturbation. We anticipated that while students would begin to understand congruence as properties of compositions of translations, rotations, and reflections, they would not attend to the importance of these rigid motions preserving distance (that is, that they are isometries). By moving to the context of taxicab geometry wherein not all rigid motions preserve distance, they must attend more closely to the analytic concepts of distance preservation, they cannot just rely on their intuitive notions of superposition.

## Methods

### Study Setting

We were motivated to collaborate in designing instruction as we were each preparing to teach a Spring 2017 course that would satisfy the geometry content knowledge requirement for teaching licensure programs at our institutions. Prasad taught Fundamentals of Geometry, a junior-level course required for prospective middle and high school teachers with 41 students. Boyce taught College Geometry, a junior-level course required for prospective high school mathematics teachers with 29 students. Both courses also served as math electives for other math majors. Boyce's course was four credits, taught in the ten week quarter system, whereas Prasad's course was three semester credits. Thus, Boyce's course began when Prasad's course was approximately halfway complete. Neither of the courses included a "transition to proof" course as a prerequisite. The topics of both courses included Euclidean and transformational geometry, as well as non-Euclidean (hyperbolic and taxicab geometry).

### Instructional Sequence

Our approach was to first introduce SAS as a postulate in Euclidean geometry. Because this was likely to be PTs own experience with SAS prior to the course, and the perspective of a majority of colleagues, parents, and even curricula likely to be encountered in their teaching, we felt it was important for them to first understand the meaning and application of the SAS congruence criterion. At the beginning of our courses, students completed Euclidean proofs about properties of isosceles triangles and quadrilaterals by relying on triangle congruence criteria. We later introduced plane transformational geometry, emphasizing the definition of an isometry as a distance-preserving automorphism (of the Euclidean plane) rather than a mapping from one domain to a distinct domain. In both courses students explored fixed point and orientation properties, the results of composing isometries, and analytic formulas for particular plane isometries using Cartesian coordinates. See sample tasks in Figure 3.

- Let line $m$ be the positive x-axis and let the line $k$ be the line $y = x$. If $R_{C,\alpha} = r_k \circ r_m$ is the rotation resulting from reflection across the line $m$ followed by reflection across the line $k$, then find the center $C$ and angle α of this rotation.
- How could you describe a translation as a composition of two reflections? Provide an example to illustrate your reasoning.
- How could you describe a translation as a composition of two rotations? Provide an example to illustrate your reasoning.
- Recall that we left congruence as an undefined term. In Common Core Geometry, two geometric objects are defined to be congruent if there is an isometry that maps one figure to the other. How might you prove two triangles are congruent using isometries? Create an example.
- Let *S* be the square with vertices (-1,1), (1, 1), (1, -1), (-1, -1). Identify the 8 isometries that leave S invariant (the *symmetries* of the square).

*Figure 3.Sample tasks in transformational geometry*

At this point in our courses we administered a written task prompting our college geometry students to justify SAS to a hypothetical 10th grade student (see Figure 4). Our aim was to assess how and whether students would evoke reasoning about isometries in their explanations. Students in Boyce's course were assigned the task as a homework assignment, for which they were accustomed to working in groups but submitting an individual write-up, whereas Prasad's students were instructed to complete the prompt individually.

> You are teaching your tenth-grade class about the SAS Congruence and after the lesson, a student comes up to you to tell you that he is still struggling with understanding why it works. *Even though you may be treating it in class as a postulate (thereby just assuming that it is true), how could you help your student better justify SAS Congruence?*

*Figure 4.Task prompt for eliciting reasoning about transformations*

We then embarked upon a week-long introduction to taxicab geometry. After introducing the taxicab metric, students explored relationships between Euclidean and taxicab geometry. As part of their explorations, students were tasked with determining whether triangles congruent in Euclidean geometry would be congruent in taxicab, and vice versa. As part of this process, students explored which Euclidean transformations would preserve taxicab distance and thus be taxicab isometries.

- Identify all taxicab isometries
- How can we define congruence in taxicab geometry?
- Come up with examples of each, or explain why such an example is not possible:
    - 2 triangles that are congruent in both Euclidean geometry and Taxicab geometry
    - 2 triangles that are congruent in Euclidean geometry but not in Taxicab geometry
    - 2 triangles that are congruent in Taxicab geometry but not in Euclidean geometry

*Figure 5.Sample tasks in taxicab geometry*

**Data Sources and Coding**

Following the week-long unit on taxicab geometry, we administered a follow-up prompt, again tasking students with explaining how they might help a 10th grade student justify the SAS criteria. In both classes, the length of time between the pre-assessment and post-assessment was between two and three weeks. However, because of the staggered implementation of assessments and the course sequences, Boyce administered the post-assessment to his class weeks after Prasad had finished this project with her class. Since both authors had analyzed the post-assessment responses from Prasad's class by then, Boyce decided to slightly modify the post-assessment prompt (this is discussed further in the Results section). On the post-assessment, both groups of students completed the assignment individually. The sequence of data collection and analysis is displayed in Figure 6.

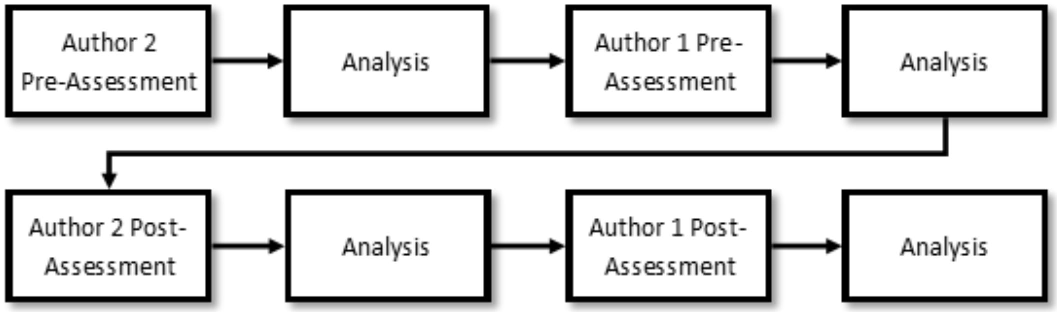



*Figure 6. Sequence of data analysis*

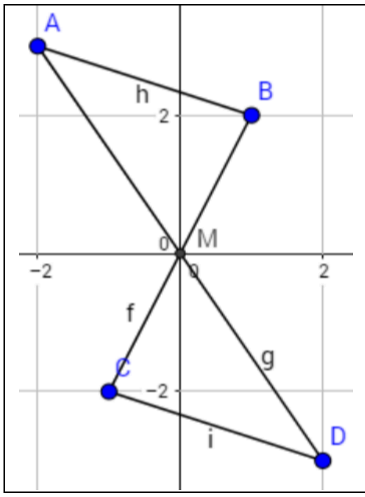
Prasad's course started approximately 8 weeks before Boyce's, thus her students completed the written assessments first. We coded the responses to this first set of assessments independently, using the constant comparison method (Glaser, 1965), before Boyce's class completed the same prompt for the pre-assessment. We selected 10 of the 34 responses to code independently, and we then met to discuss commonalities in our codes and developed a shared set of codes and meanings. After some additional codes were added after coding the remaining assessments (as well as Boyce's pre-assessments) there were eighteen different codes.

For example, one student responded on the pretest,

*I would grab two sticks and hold them together at a single angle and ask students if there was any way to make two triangles out of it. If need be I would grab two more of the same length and show that there is still only one triangle that can be formed. If I can't find sticks I can draw it out.*

This student's response was coded as having an *instructor orientation*. This code meant that the prospective teacher (PT) focused upon hypothetical interactions with a student in their response. The PT *understood the premise and conclusion of SAS*, but their *justification was missing* – the PT seemed to infer that the hypothetical student in the prompt would be convinced by lack of counterexamples. Their response was also coded as referring to *construction*, since the PT modeled one way of constructing a figure to address the prompt.

Other students' responses indicated understanding of the meaning of the SAS congruence criterion, and attempted to justify SAS by referring to other Euclidean theorems or axioms, such as the SSS criterion. The response depicted in Figure 7 was coded this way as well, as the student referred to vertical angles and the alternate interior angles theorem. This student *created a specific example* to discuss and *provided a figure* to reference. This student did not have an instructor orientation and did not describe a construction.




Let $M$ be the midpoint of $\overline{AD}$ and $\overline{BC}$. Then $\overline{AM} \cong \overline{DM}$ and $\overline{BM} \cong \overline{CM}$. Then, by the vertical angle theorem, $\angle AMB \cong \angle DMC$. By SAS, $\Delta ABM \cong \Delta DCM$. Since corresponding parts of congruent triangles are congruent, $\overline{AB} \parallel \overline{CD}$ and $\overline{AB} \cong \overline{CD}$, and by the alternate interior angles theorem, $\angle BAM \cong \angle CDM$ and $\angle ABM \cong \angle DCM$.

*Figure 7. Student response on pre-assessment showing justification of SAS using other Euclidean theorems*

After we had completed coding Boyce's pre-assessments, we collapsed some of the initial eighteen codes, using axial coding (Charmaz, 2006). For instance, we originally had separate codes for whether students responses demonstrated understanding the premise of SAS, whether students demonstrated understanding the conclusion of SAS, whether students' attempted to justify SAS, and whether students attempted to justify something. After axial coding, we created a new code, whether students' understood the premises and conclusions of SAS.

**Coding schemes**

We focused on the following aspects: (a) whether students focused their responses on hypothetical interactions with a 10th grade student, (b) students' understanding of the premises and conclusions of SAS, (c) students' describing a construction to justify SAS, (d) students' use of other Euclidean axioms or theorems to justify SAS, such as SSS, and (e) students' use of transformations to justify SAS.

Prasad's class took the post-assessment next. As discussed in the Results section, we modified the task prompt before Boyce's class completed the post-assessment (see Figure 10). We coded the results on each of our post-assessments using the codes (a) - (e) listed above, marking in a spreadsheet whether or not they were exhibited with 1s or 0s. We then reconciled our codes so that each student had single indicator of '1' or '0' for each of the five codes.

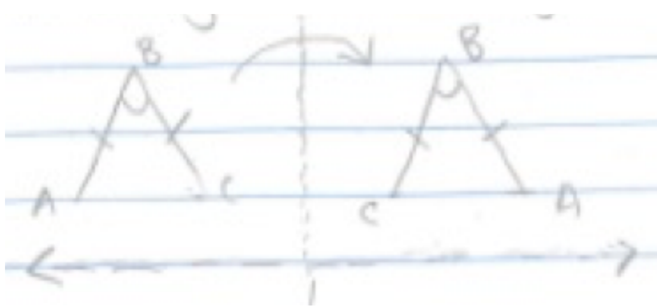
Following the administration of the post-assessments, we returned to the data to focus more closely on how students were using transformations in their justifications. For this purpose, we developed the following rubric:

(1) substantive, valid, and precise justification
(2) substantive, mostly valid, missing some precision
(3) correct, but imprecise and lack of attention to SAS
(4) describes transformations, but informal and not specific
(5) scant or incorrect use of isometries
(6) no mention of isometries

We independently rated each of the responses across all four assessments using this rubric to understand whether students' experiences with taxicab geometry influenced not only the frequency, but the qualities, of their justifications for SAS involving transformations. We coded "use of transformations" quite broadly, to include situations in which a transformation was described informally (e.g., a description of "moving" triangles akin to Euclid's superposition principle) as well as situations in which the name for a transformation (e.g., translation) was used, however inappropriately or without details.

**Coding examples**

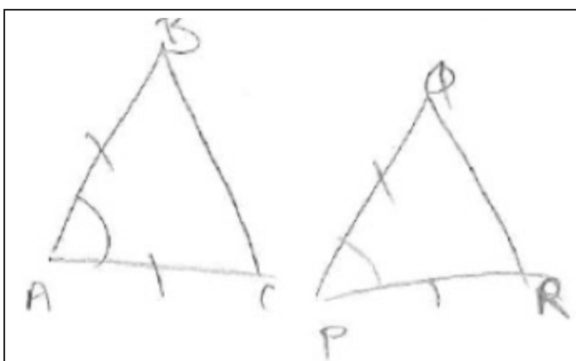
Figures 8-10 exemplify our coding of students' use of transformations in their justifications. In Figure 8, the student describes a reflection as preserving distances and angles, but the justification of SAS is about the (in)ability to manipulate sides or angles rather than relying on properties of reflections. The student also labels corresponding sides and points imprecisely. This response was coded as justifying SAS using transformations, but informally and not specifically.




*"If I reflect the vertices A, B, and C over the y-axis, I preserve the distance between the points and the shape of the figure.* By preserving the distance between the points we get that all the sides and angles are congruent. If I preserve two sides and an angle between them, there is no way for me to manipulate the last side or the other two angles without changing the entire triangle."

*Figure 8. Pre-assessment response coded as (4), describes transformations, but informal and not specific*

The response in Figure 9 exemplifies an instructor orientation, as the focus of the response includes choice of manipulatives and explicit attention to students' thinking. In contrast to the response in Figure x, the student describes a particular example for which a translation maps, not one triangle to another, but the pairs of sides and their included angle to corresponding pairs of sides and an included angle. Although the response was imprecise, it shows attention to the premise of SAS and was thus coded as (2).

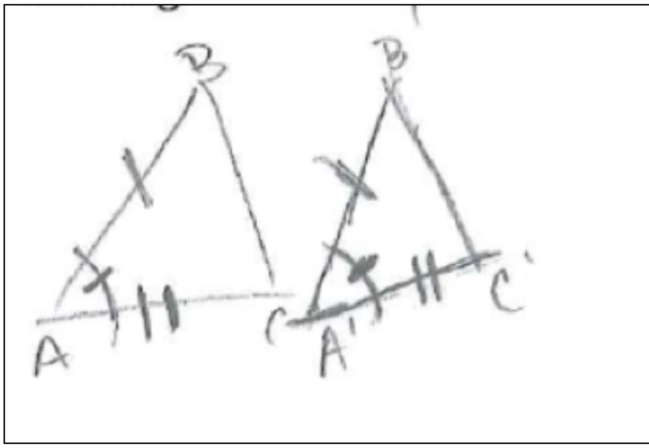



I could do a similar example like we just did in class. We could draw a triangle on a transparency and *notice if we do a translation then the two sides and angle we are claiming to be congruent will line up.* This could help them see that the two triangles are congruent given that two sides and the angle they share are congruent.

*Figure 9. Pre-assessment response coded as (2), substantive, mostly valid, missing some precision*

The response depicted in Figure 10 is in some ways more precise than that of Figure 9 – the student chose a more generic, scalene triangle as an example and described the premises of the

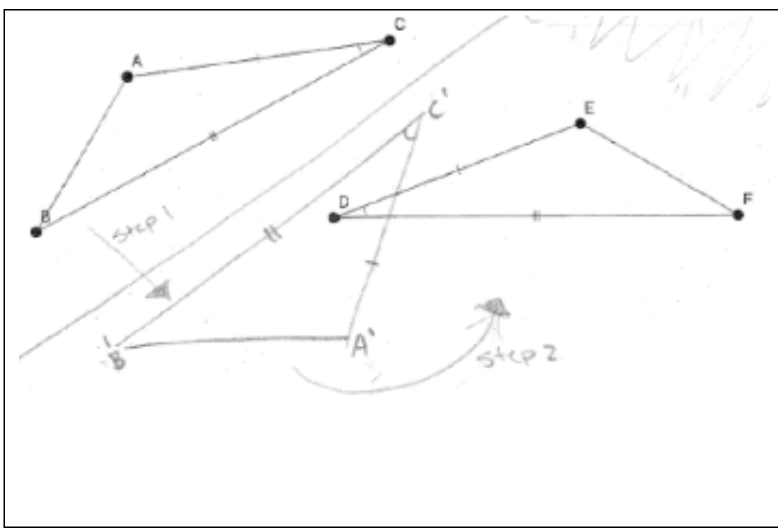
SAS postulate. However, as was also exhibited in the response in Figure 8, the student's description of the transformation involves mapping one triangle to another triangle. There is no justification for why the isometry that maps AB to A'B', AC to A'C', and ∠BAC to B′A′C would necessarily map the remaining sides and angles of $\Delta ABC$ to the corresponding sides and angles of $\Delta A'B'C'$.




Well if we assume/use SAS Congruence as a postulate, we would know that AB≅A′ B′, AC≅A′C′, and ∠BAC≅∠B′A′C′. We could show that *if we translate triangle* $\Delta ABC$ *onto* $\Delta A'B'C'$ *that the sides and angles in fact are all congruent to their counterpart.* Thus proving that SAS holds true.

*Figure 10. Post-assessment response coded as (3), correct, but imprecise and lack of attention to SAS*

The response depicted in Figure 11 does not indicate an instructor orientation. The student first states that isometries preserve angles and side lengths, and then proceeds to describe a sequence of three isometries that map one triangle to another. There are several aspects of precision missing. For one, the conclusion should be that $\Delta ABC$ is mapped to $\Delta EFD$, rather than $\Delta EDF$. More substantively, the student does not provide justification for the existence and uniqueness of parallel lines that underlie their approach, and their description of isometries refers to a sequence of transformations of triangles, rather than transformations of the entire plane.




I would explain SAS by using isometries since isometries preserve distance in Euclidean geometry. The angles and side lengths will stay the same. *First I would reflect* $\Delta ABC$ *over a line parallel to BC.* Now I have ΔA′B′C′. *If I take* $\Delta A'B'C'$ *and rotate it till C′B′ is parallel to DF and then translate it so all the points overlap with* $\Delta EDF$, then I can show the triangles are congruent.

*Figure 11. Post-assessment response coded as (2), substantive, mostly valid, missing some precision*

### Analysis of Coding Across Pre/Post Assessments

After completing the coding of the results, we examined the corpus of data to look for common patterns and themes across pre/post assessments within each class or within pre/post assessments across classes. In particular, we tested whether there was an increase in students' substantive use of transformations in justifying SAS across pre/post assessments. For example, consider that the response depicted in Figure 7 includes an example for which the properties of rotation centered at *M* could justify the congruence criteria. Might we find more instances of substantive justifications using transformation criteria in the post-assessment, following the introduction of taxicab geometry?

## Results

### Pre-assessment Results

On the pre-assessment, the majority of our students focused on how they would interact with hypothetical secondary students in their responses. They described details such as the particular triangular examples or manipulatives they might introduce (e.g., Figure 7). When students included justifications, they were as apt to focus on constructions, or on justifications of SAS using SSS, as they were to describe transformations (see Table 1). This was the case across both of our courses. Students in Prasad's class were more likely to respond with a focus on interacting with a hypothetical student to the extent that they did not provide a justification for SAS (23/34 students versus 6/18 students). However, some of the students in Boyce's class provided multiple ways they might respond to the prompt, reflecting that they had discussed the task with classmates prior to writing up their responses.

*Table 1. General coding results*

| General Results Across Pre/Post Assessments | | | | |
|---|---|---|---|---|
| Codes | Prasad (pre) N=34 | Prasad (post) N=34 | Boyce (pre) N=18 | Boyce (post) N=25 |
| Instructor orientation | 17 | 20 | 12 | 3 |
| Did not justify SAS | 23 | 22 | 6 | 2 |
| Described a construction | 4 | 4 | 5 | 7 |
| Used other Euclidean axioms/theorems | 5 | 6 | 2 | 4 |
| Use of transformations | 3 | 7 | 6 | 16 |

*Note:* There were three students in Prasad's class that completed only the pre-assessment, three students in Prasad's class that completed only the post-assessment, one student in Boyce's class that completed only the pre-assessment, and eight students in Boyce's class that completed only the post-assessment.

### Post-assessment Results

After Prasad administered the post-assessment in her class and examined the results, she noticed that although there was an increase in the number of students mentioning transformations in their responses, it seemed the majority of students responded in a similar manner on the pre-assessment as the post-assessment. There were still many students who did not provide a justification of SAS as part of their response to the prompt, and the majority of students maintained an instructor orientation.

One of the issues with eliciting a response including transformational reasoning was that the examples that students chose to include often involved cases for which a single translation, rotation, or reflection would map one triangle to another, or an example involving isosceles triangles (e.g., Figure 10). We modified the task prompt for Boyce's post-assessment to include more explicit directions that the prospective teacher was to provide a justification by referring to particular diagram that involved mapping an (ostensibly) scalene triangle to an image that would require consideration of a *composition* of translations, reflections, or rotations (see Figure 12).

Boyce's students were not able to collaborate on the post-assessment, as they had on the pre-assessment, as the task appeared on their final exam. Only 3/25 students adopted an instructor orientation, suggesting that the revised prompt suggested that they focus more on describing a justification for SAS rather than describing how they would help a hypothetical student. That 23/25 of the students provided a justification for SAS also suggests that revisiting triangle congruence criteria in the context of taxicab geometry may have supported their understanding of the premises and conclusions of SAS. The majority of students in Prasad's class (16/25) used transformations to provide a justification on the post-assessment, suggesting that students may have been more apt to consider transformational reasoning as well. However, as we discuss in the next section, many of the justifications using transformations were lacking substance and precision.

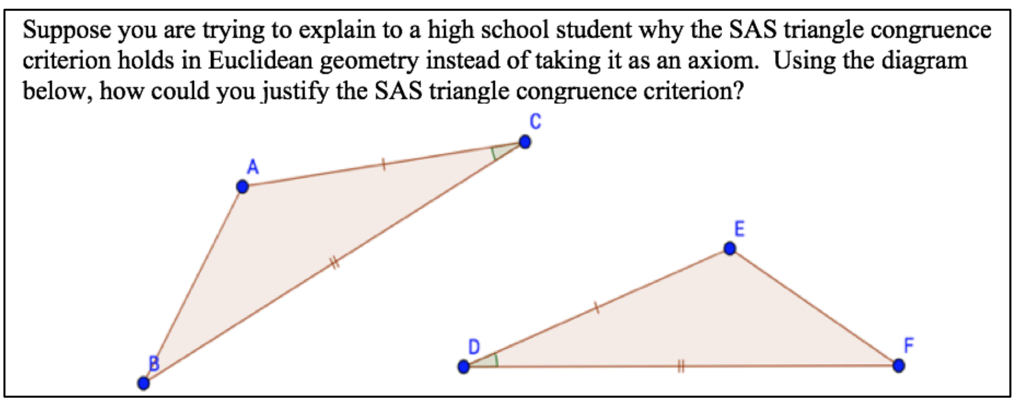



*Figure 12. Revised task prompt for eliciting reasoning about transformations*

**Use of Transformations**

When we revisited the data to rate the use of transformations in our students' justifications, we conceived of several aspects we valued: substance, precision, and validity. When we compared our independent ratings of the 32 responses that had been coded by either of us as having something to do with transformations, we considered the robustness and granularity of our rating scheme. After calculating inter-rater agreement ($\kappa = 0.6118$, using linear weighting), we found our rating scheme to be on the borderline of moderately/substantially useful (Landis & Koch, 1977). By collapsing codes to (1-2), (3-4-5), and (6), we found agreement in all but 4 of 32 cases. After reconciling these 4 cases, we considered the results across the pre/post assessments.

The results depicted in Table 2 show greater substantive use of transformations in PTs' justifications in response to the revised task prompt. On the pre-assessment, only 1 of the 6 responses mentioning transformations in Boyce's class had been rated as '1' or '2', whereas on the post-assessment 8/16 of responses had such ratings. Still, only about a third of the responses to the revised task prompt included substantive justifications of the SAS using properties of transformations (8/25). Although there were also more PTs in Prasad's class that mentioned transformations on the post-assessment than the pre-assessment, their justifications in response to the original prompt tended to be less precise and more informal—none of the responses were rated as '1' or '2'.

*Table 2. Transformation coding results*

| Transformational Reasoning Across Pre/Post | | | | |
|---|---|---|---|---|
| Codes | Prasad (pre) N=3 | Prasad (post) N=7 | Boyce (pre) N=6 | Boyce (post) N=16 |
| 1 or 2 (substantive) | 1 | 0 | 1 | 8 |
| 3, 4, or 5 | 1 | 5 | 5 | 5 |
| 6 (lack of isometries) | 1 | 2 | 0 | 3 |

## Conclusions

The premise of our study was that after our students' explorations in taxicab, they would be more likely to describe properties of transformations in their justifications of SAS. We were surprised that on the pre-assessment, before exploration of taxicab geometry, there were so few students providing a valid justification for SAS or even attempting to justify SAS (as opposed to using it to justify something else). With consideration that our students were assigned the prompt directly after explorations with isometries, we had anticipated that more students would provide justifications that referred to translations, rotations, or reflections, at least informally. Our students interpreted the assessment prompt in interrelated ways that potentially obscured their mathematical thinking: using transformations to prove a congruence theorem, how to justify a theorem in a mathematically valid way, and what forms of justification are appropriate for secondary students.

The revised post-assessment seemed to shift their focus away from the third interpretation, but it was also less likely to elicit how they would operationalize their knowledge of transformations in a classroom situation. Still, there were enough promising answers to the pre-assessment to suggest that perhaps shifting to taxicab geometry might not be the most effective way to perturb their understanding of congruence – instead of providing intellectual need for properties of isometries, it may have just confused them. Although we observed many interesting features of students' thinking about transformations, justifications, and triangle congruence criteria, the results of this study have made it clear that investigating students' justifications of triangle congruence criteria was too broad an assessment for the learning goal of perturbing their understandings of congruence.

## Discussion

The results of this study and our more broad discussion about taxicab geometry suggest a clear axiomatic system that includes the isometries as axioms may be fruitful for engendering college students' justifications for triangle congruence criteria. While transformational reasoning has long been considered a part of analytic geometry (which is perhaps emphasized in K-12 more than it is in college-level geometry), a reliance on an axiomatic system usually paves the way to synthetic geometry in curricula. However, since the CCSS-M relies on the isometries to be treated as axioms, we believe that the resultant axiomatic system needs to be explicitly defined for teachers who are expected to follow the CCSS-M. If mathematics teacher educators are to expect preservice teachers to spontaneously call upon properties of transformations to justify congruence criteria, it also seems we may need to spend more time focused on applying transformational reasoning in our courses, further necessitating a well-defined axiomatic system within analytic geometry.

Moreover, such an axiomatic system has the advantage of connecting clearly to the axiomatic system that underlies taxicab geometry: instead of taking it as axiomatic that every rigid motion is an isometry, taxicab geometry restricts the possible rotations and reflections that can axiomatically be considered isometries. Thus, an excursion into taxicab geometry may focus students' attention more clearly on the consequences of defining congruence as the result of compositions of isometries. For the next step of our investigation, we would like to run another iteration of tasks (revised again) the next time we have the opportunity to teach these courses, perhaps exploring a more elementary idea of using taxicab isometries to perturb the notion of what it means for two segments to be congruent.

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A Preservice Mathematics Teacher's Covariational Reasoning as Mediator for Understanding Global Warming

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*In this paper, I discuss the role of covariational reasoning in mediating one preservice mathematics teacher's (PST's) understanding of the link between carbon dioxide ($CO_2$) pollution and global warming. I used Thompson and Carlson's (2017) levels of covariational reasoning to inform the discussion of the results. Jodi, the PST, completed a mathematical task I created for the study during an individual, task-based interview. The analysis of Jodi's responses revealed that Jodi's covariational reasoning supported increasingly sophisticated images of the link between $CO_2$ pollution and global warming as it developed from No Coordination Level to Chunky Continuous Level. My findings also suggest that covariational reasoning can involve two quantities changing simultaneously but in different intervals of conceptual time, which I have not seen previously reported in the literature. Finally, the study suggests that the implemented task is a suitable point of entry to study the mathematics of global warming.*

*Keywords:* Covariational Reasoning, Global Warming, Preservice Teachers, Modeling

## Introduction

Global warming refers to an increase in the mean global surface temperature caused by human emissions of greenhouse gases. This phenomenon is a contemporary and pressing issue with important social, economic, and ecological consequences around the world (Intergovernmental Panel on Climate Change [IPCC], 2013). Global warming also offers a potentially motivating and engaging context to learn integrated mathematics and science. The planetary scale of global warming, however, makes it difficult for a single person to experience such phenomenon in its entirety. Mathematical modeling can provide students and teachers with more easily accessible representations and objects such as quantities, graphs, or equations, that they can manipulate and reason with in order to further their understanding of the phenomenon (Barwell & Suurtamm, 2011; Barwell, 2013a, 2013b; Mackenzie, 2007). Mathematics teachers, however, are unlikely to incorporate global warming into their instruction. They may believe it is too politically charged, be unfamiliar with concepts related to climate science and global warming, or feel unprepared to teach the mathematics of global warming (Dahlberg, 2001; Lambert & Bleicher, 2013; Leiserowitz, Smith, & Marlon, 2010; Pruneau, Khattabi, & Demers, 2010). Therefore, there are both societal and cognitive needs for studies regarding global warming and mathematical reasoning.

In my research, I investigate how preservice teachers (PSTs) make sense of introductory mathematical models for global warming in which the mathematics involved can be accessible to high school students. The models require PSTs to think about a dynamic situation in terms co-variation between quantities. Existing research in mathematics education has demonstrated that students and future mathematics teachers can have persistent difficulties comprehending and mathematically expressing co-variation between quantities (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2012; Oehrtman, Carlson, Thompson, 2008; Thompson, 2011). In this paper, I discuss the role of one PST's covariational reasoning in mediating her understanding of the Earth's energy budget and the link between carbon dioxide pollution and global warming.

## Background Information

The Earth's climate system is powered by the sun, and there is a continuous exchange of heat between the sun, the planet's surface, and the atmosphere. The *Earth's energy budget* accounts for the direction and magnitude of this continuous heat exchange (Figure 1). The sun warms the planet's surface at a relatively constant rate S. As the surface warms up, it radiates (infrared) energy to the atmosphere at a rate R. A small portion L = (1 – g) × R escapes to space, while the majority B = g × R is absorbed by *greenhouse gases* (GHG) in the atmosphere such as water vapor ($H_2O$), carbon dioxide ($CO_2$), and methane ($CH_4$). The atmosphere re-radiates the absorbed heat in both directions toward space and toward the surface, at a rate A. The continuous heat exchange between the surface and the atmosphere is known as the *greenhouse effect* and influences the planet's mean surface temperature. The rates of radiation or heat exchange S, R, B, L, and A (Figure 1) are usually measured in Joules per second per square meter ($J/s/m^2$), while the concentration of GHG in the atmosphere is usually measured in the same units of volume (e.g., $m^3/m^3$) or in parts per million by volume (ppmv). The parameter $0 \leq g \leq 1$ is related to the greenhouse effect and represents the proportion of surface radiation absorbed by GHG in the atmosphere. Quantifying changes in heat exchange due to changes in the concentration of GHG is central to accurately model global warming. My study focuses on how an instantaneous increase in the atmospheric $CO_2$ concentration produces variation in heat exchange over time, and how that variation affects the planet's mean surface temperature.

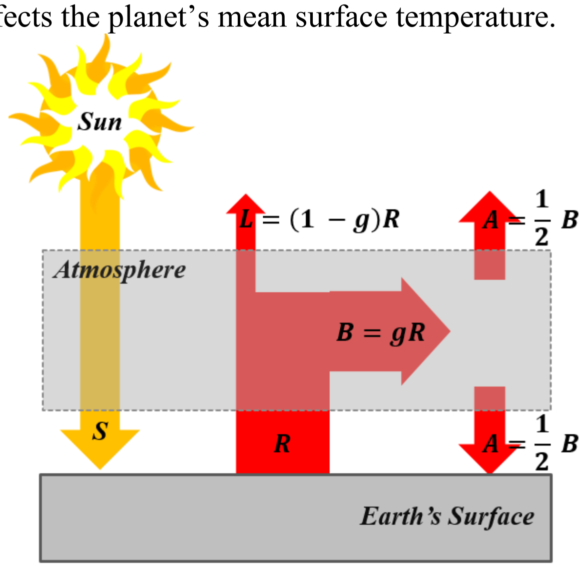



*Figure 1: The Earth's energy budget, assuming a single-layered atmosphere*

The *planetary energy imbalance function* N(t) is a measure of the energy imbalance in the Earth's energy budget over time. In particular, N(t) can be defined as a difference between the rate of downward radiation and the rate of upward radiation at the planet's surface, or mathematically N(t) = (S + A(t)) – R(t). The Earth's energy budget is said to be in *radiative equilibrium* when N(t) = 0 (rate of downward radiation equals rate of upward radiation), which implies that the *planet's mean surface temperature function* T(t) remains constant. However, there are factors or *forcing agents* that can push the energy budget out of equilibrium, producing N(t) ≠ 0. The present study focuses on how N(t) and T(t) vary over time after an instantaneous increase in the atmospheric $CO_2$ concentration at t = 0. Such increase would result in an atmosphere with more capacity to absorb radiation or heat from the surface. This translates into a value for B(0), and consequently A(0), such that N(0) = (S + A(0)) – R(0) > 0; this is known as a *positive forcing by $CO_2$*. Since the rate of downward radiation exceeds the rate of upward radiation, the planet's surface starts warming up as time increases (T(t) increases); a hotter surface radiates heat at a higher rate (R(t) increases). Since the surface is radiating heat at a higher rate, the atmosphere absorbs heat at a higher rate (B(t) increases). As a result, the atmosphere starts radiating heat back to the surface at a higher rate (A(t) increases), further warming the surface. The planet's mean surface temperature T(t) and the rate of radiation R(t) will continue to increase until a new radiative equilibrium is reached. Notice that the latter implies that N(t) will decrease towards zero as time increases. By using the equality $A(t) = \frac{g}{2}R(t)$ (Figure 1), the expression N(t) = (S + A(t)) – R(t) can be reduced to N(t) = S – $\beta$R(t), where S is the solar constant and $\beta = 1 - g/2$. That expression shows that N(t) decreases as R(t) increases. This variation indicates that the planet's mean surface temperature T(t) increases at a decreasing rate and asymptotically approaches to a new equilibrium value. As long as N(t) > 0, T(t) would increase because the rate of downward radiation S + A(t) exceeds the rate of upward radiation R(t). Since N(t) is decreasing, T(t) would slow down as it increases.

## Literature Review

Thompson (1994a) used *image* to refer to a set of fragments of experiences involving sensory-motor inputs (e.g. vision, smell, movements, touch, taste, etc.), affective states (e.g. fear, joy, struggle, etc.), and cognitive process (e.g. imagining, inferring, deciding, etc.), that one collects and coordinates when reasoning in particular ways about particular situations. In the context of co-variation, talking about an image allows us to elaborate upon that "something" in the student's mind when the student talks about something changing or something accumulating.

*Co-variation* refers to two quantities changing simultaneously and interdependently. Thompson (2011) provided a definition of co-variation in terms of a person's images of variation. For Thomson, to say that a quantity's value varies "is to say that one anticipates its measure having different values at different moments in time. So a varying quantity's value might be represented as $x = x(t)$, where t represents (conceptual) time" (p. 46). Variation therefore always occurs over an interval. The student anticipates D = domain(t), the values of *conceptual time* over which t ranges, as covered by intervals of length ε. Then, the variation in quantity $x$ can be represented with $x_\varepsilon = x(t_\varepsilon)$, where $t_\varepsilon$ represents the interval [t , t + ε) and t can be any value of conceptual time. The student envisions $x$ varying in intervals of conceptual time with the understanding that the quantity also varies within any interval of completed variation. Co-variation can then be defined by extending that definition of variation. Co-variation can be represented as $(x_\varepsilon , y_\varepsilon) = \big(x(t_\varepsilon) , y(t_\varepsilon)\big)$, where the pair $(x_\varepsilon , y_\varepsilon)$ represents a student's image of uniting in mind two quantities, and then varying them simultaneously over intervals of conceptual time.

Co-variation represents a more intuitive approach to teaching and learning functional relationships (Thompson, 1994b). Researchers in mathematics education use the term covariational reasoning to refer to someone's ability to envision co-variation. Saldanha and Thompson (1998) have defined covariational reasoning as:

> Someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either

quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (pp. 299).

Saldanha and Thompson also conjectured that someone's images of co-variation or covariational reasoning undergoes several developmental stages. Carlson and colleagues (Carlson et al., 2002) build upon Saldanha and Thompson's conjecture. They developed the Covariation Framework as a theoretical instrument to examine and assess a student's covariational reasoning abilities. The framework describes five levels of development, each more sophisticated than and built upon the previous one: *Dependency of Change* or Level 1 ($y$ changes when $x$ changes), *Direction of Change* or Level 2 ($y$ increases as $x$ increases), *Amounts of Change* or Level 3 (a change $\Delta y$ in $y$ correspond to a change of $\Delta x$ in $x$), *Average Rate of Change* or Level 4 ($y$ increases more rapidly for successive changes $\Delta x$ in $x$), and *Instantaneous Rate of Change* or Level 5 ($y$ increases more rapidly as $x$ continuously increases). Carlson and colleagues emphasized that if a student's covariational reasoning was classified at a particular level, then it is implied that the student's covariational reasoning can also operate at any of the previous levels.

The way that a student might perceive a quantity's value varying can shape their covariational reasoning. Castillo-Garsow and colleagues (Castillo-Garsow, Johnson, & Moore, 2013) theorized that students can develop two different types of images of variation: *chunky images* and *smooth images*. The chunky thinker's way of envisioning variation has two main features: (a) a unit or chunk building up the variation through iteration and (b) no variation occurs within this unit or chunk. The chunky thinker reasons in discrete or atomic units (chunks) and only attends to what is *happening* at the chunk's ends (what an observer would describe as the lower and upper limit of the interval) so that no variation occurs within the chunk. Castillo-Garsow and colleagues argued that this way of envisioning variation "makes it seemingly impossible for a student using chunky thinking to imagine a situation dynamically while simultaneously imagining the mathematics of it" (p. 34). In contrast, the smooth thinker can envision ongoing change as occurring progressively and continuously.

> Ongoing change is generated by conceptualizing a variable as always taking on values in the continuous, experiential flow of time. A smooth variable is always in flux. The change has a beginning point, but no end point. As soon as an endpoint is reached, the change is no longer in progress. (Castillo-Garsow et al., 2013, p. 34)

The smooth thinker can also envision ongoing change in chunks but, unlike the chunky thinker, the smooth thinker can envision change occurring within the chunk. Castillo-Garsow and colleagues emphasized that smooth images are not a refinement of chunky images; the chunky thinker will not develop smooth images by solely considering smaller and smaller chunks in the independent variable.

The literature discussed in this section have introduced important aspects regarding a person's covariational reasoning. First, covariational reasoning is recursive since change is envisioned in intervals and within intervals (Thompson, 2011). Covariational reasoning also involves the development of a multiplicative object linking change between two quantities (Saldanha & Thompson, 1998). In addition, covariational reasoning develops through levels, each more sophisticated than and built upon the previous levels (Carlson et al., 2002). Finally, envisioning variation can involve two different types of images: chunky images or smooth images. These images may support or constrain the way that co-variation is envisioned (Castillo-Garsow et al., 2013).

## Conceptual Framework

I make use of Thompson and Carlson's (2017) levels of covariational reasoning to inform the discussion of the study's results. They suggested that "a researcher could use [the levels] to describe a class of behaviors, or she could use it as a characteristic of a person's capacity to reason … covariationally." For the current study, I used the framework as a way to characterize Jodi's covariational reasoning relative to a particular mathematical task involving two quantities—planetary energy imbalance and mean surface temperature—varying over time. Thompson and Carlson describe six distinctive levels of covariational reasoning (Table 1), each more sophisticated than and encompassing the previous levels.

*Table 1. Major levels of covariational reasoning*

| Level | Description |
|---|---|
| Smooth continuous covariation | The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously. |
| Chunky continuous covariation | The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables changing by intervals of a fixed size (not necessarily of the same size). The person imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 (and so on), like laying a ruler. Values between 0 and 1, between 1 and 2, between 2 and 3, and so on, "come along" by virtue of each being part of a chunk—like numbers on a ruler—but the person does not envision that the quantity has these values in the same way it has 0, 1, 2, and so on, as values. |
| Coordination of values | The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y). |
| Gross coordination of values | The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values. |
| Preccordination of values | The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects. |
| No coordination | The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values. |

## Methods

### Context and Purpose

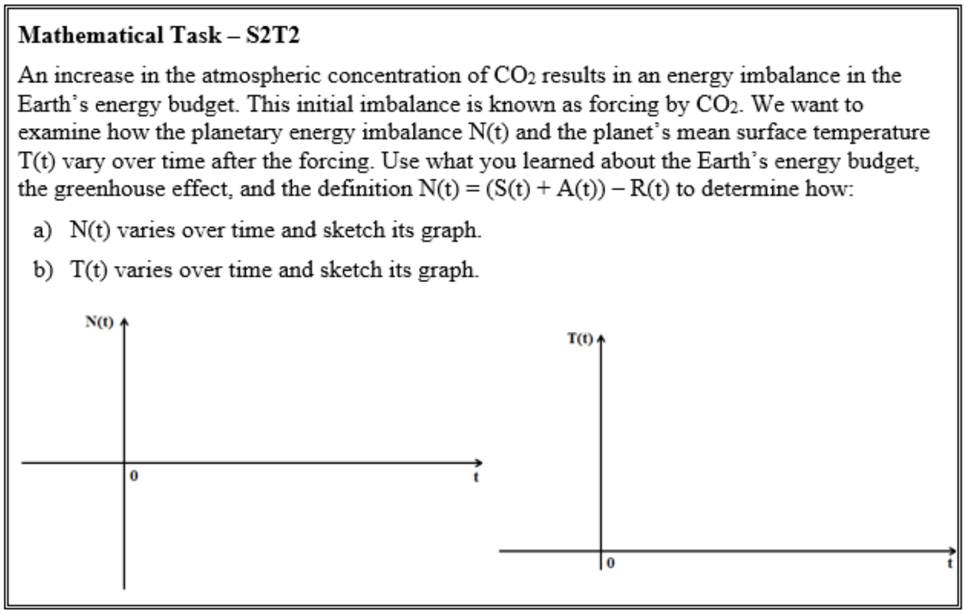
The current study is part of a larger research project that investigated how PSTs made sense of introductory mathematical models for global warming. The research project was conducted in two stages. The first stage was an exploration of the PSTs' conceptions of three quantities commonly used to model global warming: concentration, energy density, and heat flux density. The second stage was an examination of the role of covariational reasoning in making sense of three functions commonly used to model global warming: the forcing by $CO_2$ function F(C), the planetary energy imbalance function N(t), and the mean surface temperature function T(t). To accomplish the research goals, I created an original sequence of six mathematical tasks, four regarding the quantities and two regarding the functions previously mentioned. I then conducted four individual, task-based interviews with each PST as they worked through each task in the sequence. The current study focuses on the case of one PST, Jodi, and her responses to one mathematical task regarding the functions N(t) and T(t).

### Participants

Three PSTs enrolled in a secondary mathematics education program at a large Southeastern university participated in the larger research project. The PSTs have completed two calculus courses covering topics such as limits, derivatives, linear approximation, curve sketching, optimization, indeterminate forms, integrals, Fundamental Theorem of Calculus, areas, volumes, arclength, techniques of integration, Taylor series, and separable differential equations. In addition to calculus, the PSTs completed an Introduction to Higher Mathematics course covering topics such as logic, sets and relations, functions, and proof writing. The PSTs were also completing the course Connections in Secondary Mathematics, which covered topics in secondary mathematics such as algebra, functions, conics, linear systems, and sequences and series. The sequence of tasks designed for the research project required PSTs to reason about ratios, rate of change, functions, curve sketching, and modeling. Therefore, participants with experience in these topics were preferred. All three PSTs had no previous experience learning about concepts related to global warming. They, nonetheless, believed that global warming was real and that human activity was at least partially responsible for it. All three PSTs completed all six mathematical tasks in the sequence. The current study focuses on Jodi's responses to the sixth mathematical task. I chose Jodi's case because: (a) her responses were markedly different from her peers, which represents a unique case for discussion, and (b) her case provided explicit examples of how covariational reasoning can mediate the understanding of global warming.

### Data Collection

The second stage of the larger research project included two tasks (the fifth and sixth tasks) involving function commonly used to model global warming. The fifth task required Jodi to make sense of the forcing by $CO_2$ function F(C), while the sixth task described a situation in which the planetary energy imbalance function N(t) and the mean surface temperature function T(t) were involved. Before Jodi began working on those two tasks, I showed her a 7-minute long video introducing the concepts of *Earth's energy budget*, *radiative equilibrium*, and *greenhouse effect*. The video was retrieve from the NASA YouTube channel *NASAEarthObservatory*. Once the video ended, I answered any questions that Jodi had concerning the concepts discussed in the video. I then presented Jodi with the fifth task. The goal of that task was for Jodi to make sense of the forcing by $CO_2$ function, F(C). F represents the energy imbalance caused by an instantaneous increase in the atmospheric $CO_2$ concentration C. Thus, the value of F depends on the value of C. Jodi was tasked with sketching the graph of that relationship. A week later, Jodi participated in another individual interview in which she completed the sixth task in the sequence (Figure 2). The current study focuses on Jodi's responses to that sixth task.



**Mathematical Task – S2T2**

An increase in the atmospheric concentration of $CO_2$ results in an energy imbalance in the Earth's energy budget. This initial imbalance is known as forcing by $CO_2$. We want to examine how the planetary energy imbalance N(t) and the planet's mean surface temperature T(t) vary over time after the forcing. Use what you learned about the Earth's energy budget, the greenhouse effect, and the definition $N(t) = (S(t) + A(t)) - R(t)$ to determine how:

a) N(t) varies over time and sketch its graph.

b) T(t) varies over time and sketch its graph.



*Figure 2. The sixth mathematical task in the sequence.*

Jodi completed the task during a 60-minute, semi-structured, task-based interview (Goldin, 2000). I created interview protocols containing pre-defined questions to guide the interview process and to ensure all participants received similar probes during the interviews (structured part). Additionally, I spontaneously reacted to participants' responses and reasoning by asking additional follow-up questions not included on the protocols (unstructured part). Throughout the interview, I adopted a facilitator role, avoiding intervening in their reasoning or directing their thinking in any particular way. I carefully listened to their responses and asked for clarifications, further explanation, or arguments for their claims. The interview was video recorded and transcribed for analysis. All of Jodi's work on paper was collected as well.

**Data Analysis**

Videos and transcripts were analyzed through Framework Analysis (FA) method; this method consists of five inter-related stages: familiarization with data, developing an analytic framework, indexing and pilot charting, summarizing data in the analytic framework, and synthesizing data by mapping and interpreting (Ward, Furber, Tierney, & Swallow, 2013). Through these stages, the researcher creates and refines framework analysis' distinctive feature: the *matrix output*, a table arrangement into which the researcher systematically reduces, summarizes, and analyzes the data. I began the analysis by watching the interview videos. While watching the videos, I took notes regarding the video's general content at different moments in the interview and moments when the conversation changed topics (familiarization with data). I used the notes to focus my analysis on the episodes of the videos showing information relevant to the larger research project's goals. I then watched those particular episodes and coded them using the major levels of covariational reasoning described by Thompson and Carlson (2017) as themes (developing analytic framework). I then transcribed these episodes and organized text excerpts according to levels of covariational reasoning. I next read all text excerpts categorized under a particular level, selecting the excerpts that were more representative of each particular level (indexing and pilot charting). I repeated this process until I selected representative excerpts for each level the participants demonstrated. Then, all selected excerpts were organized into a matrix output containing six columns –one for each level– and two rows: one for N(t) and another for T(t) (summarizing data in analytic framework). Charting data in the matrix output allowed me to observe and analyze Jodi's responses in light of: (a) her level of covariational reasoning, (b) moments when her covariational reasoning advanced or progressed, and (c) her images of the Earth's energy budget and the link between $CO_2$ pollution and global warming (synthesizing data by mapping and interpreting).

## Results

Jodi's responses to the task suggest her level of covariational reasoning served as mediator of her understanding of the link between carbon dioxide ($CO_2$) pollution and global warming. In this section, I present and discuss the evidence supporting that finding. I divided the result section into two subsections: Part 1 refers to Jodi's construction of the graph of N(t), while Part 2 refers to Jodi's construction of the graph of T(t).

### Part 1: Sketching the Graph of The Planetary Energy Imbalance N(t)

During the first part, Jodi's covariational reasoning advanced from No Coordination Level to Coordination Level. At the beginning of the interview, I was interested in assessing whether Jodi could reason about the situation in a non-numeric and dynamic way. I thus asked Jodi to think about and describe how N(t) would be varying after a positive forcing without using particular values of N. Jodi's response suggested a covariational reasoning at the No Coordination Level; she had not yet developed images of values of N and values of time varying together. She stated the following regarding the variation of N(t):

> I think that the value of N will stay the same after we have [*pauses*]. If we increase [the concentration of $CO_2$] by just some number, then we would increase N by some number. And, it wouldn't increase or decrease if $CO_2$ is kept stable.

In a previous interview, Jodi worked on the task involving the forcing by $CO_2$ function F(C), where C represented the atmospheric $CO_2$ concentration. F(C) measures a change in the planetary energy imbalance caused by changes in C. The task assumed a single increased in atmospheric $CO_2$ concentration at t = 0, hence F(C) = N(0). Jodi's response suggests she has not yet made that distinction between F(C) and N(t). This might explain why Jodi did not think of N as a function of time and imagined N remaining constant if the atmospheric $CO_2$ concentration remained constant. Jodi's image of the energy budget involved realizing that a change in atmospheric $CO_2$ concentration would produce an energy imbalance. Because she did not coordinate N with t.

I decided to let Jodi use particular values to assess whether she could develop images of N and t varying together. I showed her a diagram of the Earth's energy budget with the initial values S = 240 J/m$^2$/s, R(0) = 390 J/m$^2$/s, B(0) = 300 J/m$^2$/s, and A(0) = 150 J/m$^2$/s to illustrate the idea of radiative equilibrium –notice that N(0) = (S + A(0)) – R(0) = (240 + 150) – 390 = 0. Next, I explained that an instantaneous increase in atmospheric $CO_2$ concentration at t = 0 would result in different initial values for B(0) and A(0). I showed Jodi a different diagram with the new values B(0) = 310 J/m$^2$/s and A(0) = 155 J/m$^2$/s to illustrate the idea of departing from radiative equilibrium – notice the positive forcing F = N(0) = (S + A(0)) – R(0) = (240 + 155) – 390 = 5. To further assists Jodi, I told her to use the following recursive rules: $B_i = 0.794 \times R_i$, $A_i = 0.5 \times B_i$, and $R_{i+1} = S + A_i$ for i = 0, 1, 2, … . The recursive rules shaped the way Jodi visualized variation in the rates of radiation R(t), B(t), and A(t). In particular, Jodi envisioned these rates varying one after the other –B(t) would vary first, then A(t), and finally R(t)– all occurring during a single time interval of fixed length h. She conceived of these intervals as chunks of time and referred to them as *cycles* or *revolutions* –probably because the order in which the rules are used resembled a cyclic process: determine R(t), then A(t), then R(t), then back to B(t) and so on. Jodi registered the values of R(t), B(t), and A(t) on the diagram of the energy budget (Figure 3).

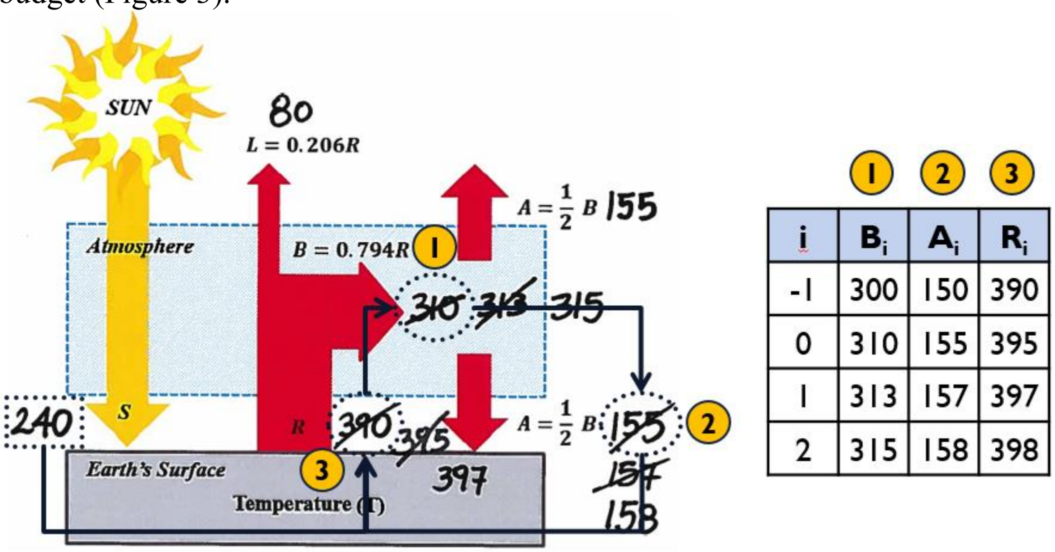



*Figure 3. The way Jodi determined the values of the rates of radiation in the diagram of the energy budget.*

Since Jodi determined particular values for the rates of radiation for three different cycles, I decided to assess whether she could describe how N(t) was varying as time increases. When asked, Jodi replied "N [was] increasing as time goes on," without attempting to determine values of N(t) and to coordinate them with values of time. Although Jodi incorrectly anticipated that N(t) would increase over time, her response unveiled that she developed a gross image involving values of N and t varying together. This Gross Coordination Level supported the notion of N(t) varying as time increases, even if the atmospheric $CO_2$ concentration was to remained constant after its initial instantaneous increase. Jodi's image of the energy budget expanded to include the idea of variation after a change in atmospheric $CO_2$ concentration (forcing). In other words, Jodi can envision an energy budget *doing something* or changing (as opposed to statically remaining out of equilibrium) after a forcing.

Jodi did not demonstrate envisioning individual values of N and t going together. Therefore, I asked Jodi to use the rule $N_i = (S + A_i) - R_i$ to determine individual values of N(t) (S is assumed to be constant). Jodi determined the values $N_0 = 5$ J/m$^2$/s, $N_1 = 2$ J/m$^2$/s, and $N_2 = 1$ J/m$^2$/s. She used these values to create a discrete collection of pairs (i , $N_i$); she represented each pair as a point in the coordinate plane and join all points by a concave-up, decreasing curve in order to draw the graph of N(t) (Figure 4). The way Jodi constructed the graph of N(t) suggests she coordinated individual values of N(t) with individual values of time. I should clarify that, for Jodi, an individual value of time can be described (from the point of view of an observer) as a time interval of fixed length. This Coordination Level allowed Jodi to realize that N(t) was decreasing as time increased, a fact that she was not expecting as suggested by her statement "I thought N would be larger." When I asked her to interpret her graph, Jodi stated that "[The graph means] that we are going back to an equilibrium, or we are not as far from equilibrium as we were." Jodi's interpretation of her graph appeared to be a product of *discovering* that N(t) was actually decreasing as time increased. She anticipated that N(t) would be increasing. When she saw a decreasing graph of N(t), Jodi had to make sense of the situation by relating the graph to the idea of radiative equilibrium: N(t) is decreasing to show that the energy budget is *returning* to an equilibrium. I used the word *returning* to indicate that Jodi envisioned the energy budget as *going back* to the original radiative equilibrium (i.e., before the forcing occurs at t = 0), rather than reaching a new one. This conjecture is further supported when I asked Jodi to interpret her graph in terms of how the heat content in the surface was varying over time. Jodi stated that the heat content was decreasing since "we would need to be losing energy so that we can go back to equilibrium." Jodi's image of the energy budget expanded to include the idea of moving towards radiative equilibrium. Jodi envisioned an energy budget *returning* to the original radiative equilibrium as time increased (after the forcing occurred at t = 0).

| i | $N_i = (S + A_i) - R_i$ |
|---|---|
| 0 | (240 + 155) – 390 = 5 |
| 1 | (240 + 157) – 395 = 2 |
| 2 | (240 + 158) – 397 = 1 |

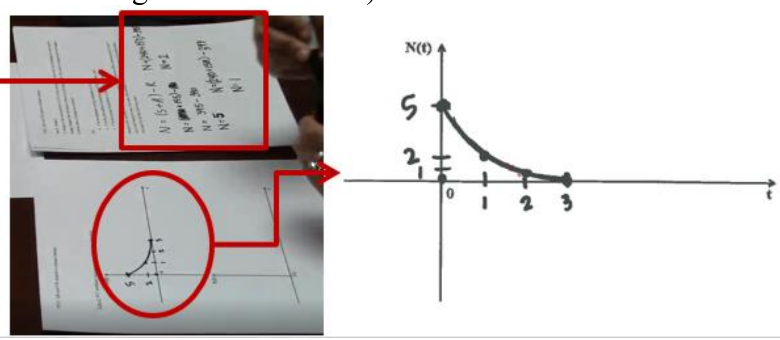



*Figure 4. Jodi's construction of the graph of the planetary energy imbalance N(t)*

Jodi's responses suggest that her covariational reasoning developed from the No Coordination Level toward the Coordination Level as she worked through Part 1 of the task. Initially, Jodi did not demonstrate an image of N and t varying together. This No Coordination Level supported an image of the energy budget involving the realization that a change in $CO_2$ concentration produce an energy imbalance (forcing). The image, however, did not include envisioning an energy budget changing after the forcing occurred. The use of recursive rules to determine individual values for the rates of radiation R(t), B(t), and A(t) appeared to help Jodi develop a gross image of N and t varying together. This Gross Coordination Level supported an image of the energy budget involving change after a forcing: the budget did not statically remain out of equilibrium after a forcing but changed as time increased. Once Jodi determined individual values for N(t), she was able to coordinate them with values of time, creating a discrete collection of pair. The Coordination Level was followed by the realization that N(t) was decreasing as time increased. It also supported an image of the energy budget returning toward radiative equilibrium after a forcing.

**Part 2: Sketching the Graph of The Mean Surface Temperature T(t)**

During the second part, Jodi demonstrated covariational reasoning at the Chunky Continuous Level. Jodi envisioned the rates R(t), B(t), and A(t) varying one at a time within a single time interval of fixed length h (cycle): B(t) would vary first, then A(t), and finally R(t), all occurring during a single cycle. She would repeat this reasoning for three cycles in order to visualize how the energy budget was changing over time. Jodi followed the same strategy to reason about how the planet's mean surface temperature T(t) was varying as time increased. This time Jodi attended to the amounts by which R(t) and A(t) were increasing from cycle to cycle.

*Jodi*: Well, I don't expect the temperature of the Earth to be zero. So, I wouldn't think it would start here [*points at the origin*] … Let's say that the temperature of the Earth is like here [*points at the middle of the vertical axis*]. So, the temperature of the Earth at equilibrium is there [*makes a mark on the vertical axis*].

*Interviewer*: So, is that the initial surface temperature?

*Jodi*: Uh-huh. I am trying to look at the differences. So here, the change was five (A increased from 150 to 155). Then, the change was two (A increased from 155 to 157). [*Long pause*] is it not changing?

*Interviewer*: What isn't changing?

*Jodi*: [The heat content] increased by five (A increased from 150 to 155), then it decreased by five (R increased from 390 to 395). Then, it increased by two, and then it decreased by two [*pauses*]. So, it is almost as if there was no change in temperature because I associate energy as kind of having a relationship with temperature. So, if the energy increases, then the temperature increases. But, in this scenario, an equal change in energy [input] was an equal change in [energy] output [*simultaneously points at A and R*].

*Interviewer*: Would you be able to draw the graph of the surface temperature now?

*Jodi*: I want to say that [T(t)] stays the same, but maybe it like [*pauses*]. OK, [a] cycle started here [*points at B*], and here the Earth's temperature would've been something … So, when the cycle started, there was an input, and then it got released [*circles her hand over the diagram in the B-A-R direction*]. Then, another cycle starts: input of energy, release of energy. So, it would almost be like [*draws a periodic curve (Figure 5)*]

The excerpt above unveils two interesting aspects of Jodi's reasoning regarding the situation. One aspect is the way she envisioned variation in the rates of radiation as time increased. Specifically, the order in which Jodi used the rules $B_i = 0.794 \times R_i$, $A_i = 0.5 \times B_i$, and $R_{i+1} = S + A_i$ shaped the way she envisions such variation. When using the rules, Jodi started by determining a value of B(t) (first rule), then a value of A(t) (second rule), and finally a value of R(t) (third rule). Jodi envisioned the rates varying one after the other in the B-A-R direction. A second aspect involved Jodi's conception of R(t) and A(t). In particular, Jodi appeared to conceive of A(t) and R(t) as heat entering and leaving the surface, respectively. This may explain why Jodi attended to the amounts by which A(t) and R(t) were increasing rather than the values they took. Let $\Delta_i A$ and $\Delta_i R$ be the amount by which A(t) and R(t) increased within Jodi's cycle i, respectively. From Jodi's perspective, the heat content is increasing during the *first half* of the cycle (when she perceived A(t) increasing), and then decreasing during the *second half* of it (when she perceived R(t) increasing). Since she noticed that $\Delta_i A = \Delta_i R$, Jodi envisioned the heat content having the same value at the beginning and at the end of the cycle. Jodi therefore envisioned T(t) varying in the same way: T(t) would increase during the first half of the cycle and decrease during the second half of it such that T(t) would take the equilibrium value T(0) at the beginning and end of the cycle. Jodi represented this variation in T(t) with a periodic curve (Figure 5).

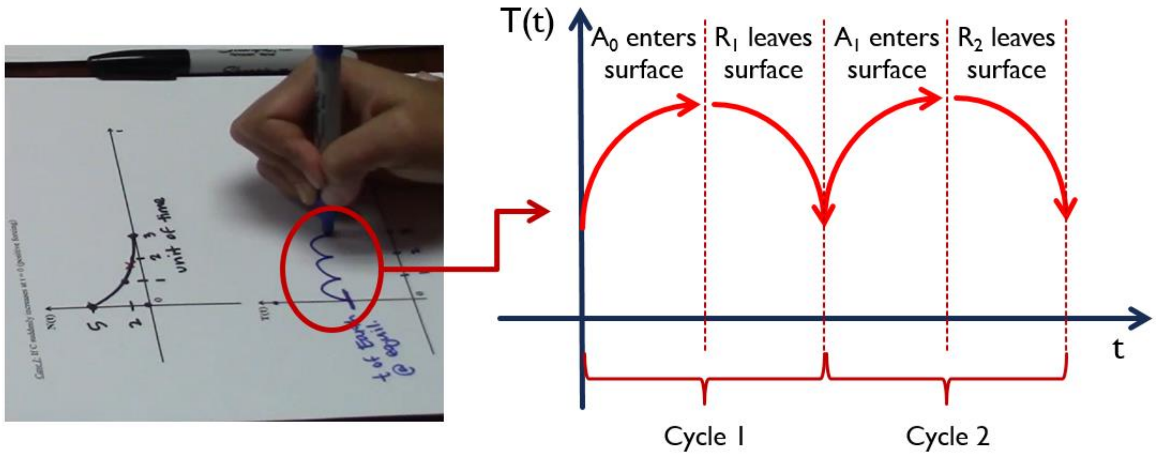



*Figure 5. Jodi's first periodic curve to represent T(t) (left) and the way she constructed that curve (right).*

Jodi drew three different curves representing T(t): *Periodic Curve 1*, *Periodic Curve 2*, and a *Quasi-periodic Curve* (Figure 6). Jodi drew Periodic Curve 1 in the way described in the previous paragraph. Periodic Curve 2 was the result of Jodi changing the way she defined a cycle. Jodi stated that if a cycle is defined in the R-B-A direction, then T(t) would increase throughout a cycle and would return to equilibrium value T(0) at the beginning of the next cycle.

> If we started the cycle here [*points at R*], then it would be like zero cycle. [*Circles her hand in the R-B-A direction*]. At the end of cycle 1, we would have an increased temperature. But then, we would go back and it would almost be something that looks like this [*draws Periodic Curve 2*], where at the beginning of a cycle, we will be back to a normal temperature, equilibrium temperature.

Unfortunately, I did not ask Jodi to explain why she drew linear segments to represent the variation in T(t). I, however, do not think that Jodi drew a linear segment to represent T(t) increasing at a constant rate during a cycle. She did not seem to attribute such meaning to her curve choice (linear versus non-linear) since she unproblematically moved from non-linear (Periodic Curve 1), to linear (Periodic Curve 2), and back to non-linear (Quasi-periodic Curve) as she drew the graph of T(t) (Figure 6).

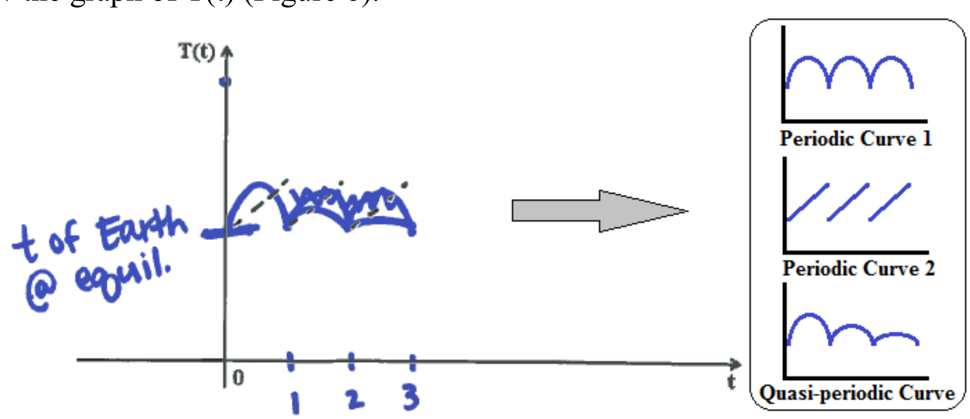



*Figure 6. Jodi drew three different curves for T(t): two periodic curves and one quasi-periodic curve*

The Quasi-periodic Curve was the result of Jodi attending to the variation in the amounts $\Delta_i B$ by which B(t) was increasing from one cycle to the next. Jodi conceived of $\Delta_i B$ as measures of (what an observer would describe as) the largest difference $\Delta_i T = \max\ \{T(t) - T(0)\}$ for Jodi's cycle i or, put in another way, the amplitude of the *arc* for that cycle i. The amounts $\Delta_i B$ were decreasing from one cycle to the next. Jodi interpreted that variation as indicating that T(t) was quasi-periodically approaching to the original equilibrium value T(0).

> The amount of [heat] was decreasing, like each time. Because in here, [B(t)] increased 10, then 3, and then 2 (B(t) took the values 300, 310, 313, and 315). So, maybe [the arcs] should be like smaller [*points at Periodic Curve 1*]. You know, like they wouldn't be the same size. Because, the Earth's temperature wouldn't increase that much [*draws the Quasi-periodic Curve*]. Because the increases in temperature are smaller during the cycles.

Regardless of the type of curve, Jodi demonstrated Chunky Continuous Level of covariational reasoning while constructing the graph of T(t). Jodi envisioned changes in T(t) are occurring simultaneously with changes in time (i.e., T(t) taking particular values for particular values of time). Jodi also envisioned changes in T(t) and changes in time as happening in intervals of fixed size. For instance, for Periodic Curve 1, T(t) increased by an amount $\Delta T$ during the *first half* of a cycle and then decrease by $\Delta T$ during the *second half* of the cycle. Thus, T(t) varies in intervals of fixed length $\Delta T$. The time t also increased in intervals of fixed length $h > 0$, where h is the duration of a Jodi's cycle. This Chunky Continuous Level supported an image of the planet's surface returning to its original equilibrium temperature as time increased (i.e., after the forcing occurred at $t = 0$). Jodi may have seen the variation in $\Delta_i B$ as consistent with her image of the energy budget returning to its original radiative equilibrium. Notice that Jodi's conclusion contradicts the notion that an increased in atmospheric $CO_2$ concentration would result in a warming effect over the planet's surface. This may become an obstacle to understand the link between $CO_2$ pollution and global warming.

## Conclusions

The study's findings suggest that Jodi's covariational reasoning mediates her understanding of the Earth's energy budget and the link between carbon dioxide pollution and global warming. Jodi's responses suggest that her covariational reasoning developed from the No Coordination Level toward the Coordination Level as she constructed the graph of N(t) (Part 1 of the task). Initially, Jodi did not demonstrate an image of N and t varying together. This No Coordination Level supported an image of the energy budget involving the realization that a change in $CO_2$ concentration produce an energy imbalance (forcing). The image, however, did not include envisioning an energy budget changing after the forcing occurred. The use of recursive rules to determine individual values for the rates of radiation R(t), B(t), and A(t) appeared to help Jodi develop a gross image of N and t varying together. This Gross Coordination Level supported an image of the energy budget involving change after a forcing: the budget did not statically remain out of equilibrium after a forcing but changed as time increased. Once Jodi determined individual values for N(t), she was able to coordinate them with values of time, creating a discrete collection of pair. The Coordination Level was followed by the realization that N(t) was decreasing as time increased. It also supported an image of the energy budget returning to the original radiative equilibrium after a forcing.

Jodi demonstrated a Chunky Continuous Level of covariational reasoning as she constructed the graph of T(t) (Part 2 of the task). Her Chunky Continuous Level was characterized by: (a) the order in which Jodi imagined the rates of radiation taking values, (b) Jodi's conception of A(t) and R(t), and (c) the way she envisioned T(t) and t varying in different intervals of conceptual time. Jodi imagined the rates B(t), A(t), and R(t) as taking values one after the other –same order as the recursive rules $B_i = 0.794 \times R_i$, $A_i = 0.5 \times B_i$, and $R_{i+1} = S + A_i$. For instance, R(t) could take a value *only after* A(t) had already taken a value. Determine the values of these rates in the B-A-R direction represented a Jodi's cycle. Jodi conceived of A(t) and R(t) as heat *entering* and *leaving* the planet's surface, respectively. Jodi thus imagined the surface heat content *first* increasing by $\Delta_i A$ *and then* decreasing by $\Delta_i R = \Delta_i A$. For Jodi, this meant that T(t) was *first* increasing *and then* decreasing within a single Jodi's cycle. In particular, T(t) would increase during the *first half* of a cycle and then decrease during the *second half* of it. Using Thompson's (2011) definition of co-variation, I describe Jodi's image of co-variation between T(t) and t in the following way. If τ represents conceptual time, then a Jodi's cycle can be denoted as $t_\varepsilon = t(\tau_\varepsilon)$ where $\tau_\varepsilon$ represents the interval $[\tau, \tau + \varepsilon)$ for $\varepsilon > 0$. Similarly, the mean surface temperature can be denoted as $T_{\varepsilon/2} = T(\tau_{\varepsilon/2})$. Thus, $(t_\varepsilon, T_{\varepsilon/2})$ represents Jodi's image of uniting in mind t and T(t), and then varying them simultaneously over different intervals of conceptual time. This finding, to the extent of my knowledge, has not been previously reported in the literature regarding covariational reasoning. Jodi's Chunky Continuous Level supported an image of the planet's surface as cooling down after the atmospheric $CO_2$ concentration increased (positive forcing). In particular, she imagined the surface returning to its original equilibrium temperature as time increased. Jodi may have found this consistent with her image of the energy budget returning toward radiative equilibrium after a positive forcing. This images contradict the long-term impact of $CO_2$ emissions on the planet's mean surface temperature, which increases as a response to an increase in atmospheric $CO_2$ concentration.

The study's results also suggest that modeling the dynamics of the Earth's energy budget can be a suitable point of entry to teach mathematics of global warming. Jodi identified relevant quantities and established quantitative relationship between them as she modeled the dynamics of the budget. The results, however, suggest two modifications might be needed for future implementations of the task. First, avoid using recursive rules to explore variation in the energy budget. Such rules appeared to have been an obstacle for Jodi envisioning smooth co-variation. Second, explicitly relate change in N(t) with change in T(t). This may have helped Jodi see that the rate of change of T(t) is proportional to N(t) (Widiasih, 2013). Nonetheless, the study offers a way in which mathematics and science can be studied together in the context of global warming.

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# The Sierpinski smoothie: Blending area and perimeter

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## Abstract

In this report, we present an analysis of 10 individual interviews with graduate mathematics education students about the area and perimeter of the Sierpinski triangle (ST) and the resulting paradoxical situation. We use conceptual blending as a theoretical and methodological tool for analyzing students' reasoning to investigate how students encounter and cope with the ST having zero area and infinite perimeter. Our analysis documents the diverse ways in which the students reasoned about the situation. Results suggest that an infinite perimeter is more accessible to these students than zero area, that encountering the paradox is dependent on how blends are composed, and that resolution of the paradox comes through completion and elaboration. The analysis contributes to what we know about how students think about infinite limit processes and furthers the theoretical/methodological framing of conceptual blending as a useful tool for revealing the structure and process of student reasoning.

**Keywords:** Conceptual blending, Infinite processes, Fractal, Paradox, Student thinking

*It's still hard for me to wrap my mind around the Sierpinski triangle, and that there's infinite perimeter and no area. It makes sense to me individually, but both together at once, I'm still, it's still mind-boggling. – Carmen, graduate mathematics education student*

## Introduction

Straightforward notions of the area and perimeter of geometric shapes are first learned in elementary school, and are revisited and leveraged throughout secondary and postsecondary mathematics. Despite such familiarity, counterintuitive situations involving these ideas can arise when working with fractals. One such situation, a region with zero area and an infinitely long perimeter, was encountered by a class of mathematics education master's students in a chaos and fractals course when investigating the Sierpinski Triangle (ST). As seen in Carmen's introductory quote, this was a non-trivial exercise and caused some students serious consternation.

The ST is a fractal, and is the result of an infinite iterative process that begins with an equilateral triangle. Connecting the midpoints of its sides results in another equilateral triangle with sides half the length of the original's and area that is one-fourth of the original's, which is then removed. Repeating this process on the remaining three triangles and so on, ad infinitum, results in the ST (Figure 1). At each step of the process, the area of the object shrinks by a factor of 3/4 and the perimeter grows by a factor of 3/2. Thus, the ST has a perimeter of infinite length and an area of zero. This counterintuitive situation is a consequence of the fact that it is a fractal with Hausdorff dimension $\log_2(3)$, putting it between one- and two-dimensions.

We chose to focus on student reasoning about the ST for three reasons. First, the familiar concepts of equilateral triangle, area, and perimeter are central to the ST which makes it a mathematical object accessible to a wide range of students, while at the same time offering opportunities for quite sophisticated analyses including self-similarity, fractal dimension, infinite processes, and a paradoxical situation. Second, there is little research on ho about the ST, despite its mathematical significance and prominence. Third, of student work on the ST illuminated the surprising complexity of student and what happens "at the end".

To investigate student reasoning about the ST we conducted individual interviews about three weeks after the in-class investigation of the ST in a chaos and fractals course. Based on the interview data, and using the ideas of conceptual blending, we address the following two related research questions: (1) How do students make sense of (a) area and (b) perimeter of the ST? (2) How do students coordinate the area and perimeter of the ST and cope with the resulting paradoxical situation?

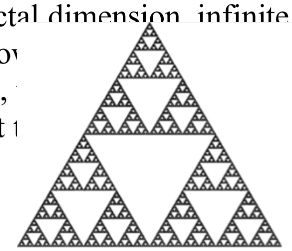

*Figure* 1. *The ST*

## Theoretical Background and Literature Review

### Infinity and Paradoxes

One of the first places students are asked to work with the infinite is when they are introduced to limits. Convergence and limits, especially related to sequences and series, are a notoriously challenging topic for students, and many believe that impoverished understandings of infinity contribute to that challenge. Many researchers have made use of paradoxical tasks to investigate and promote student understandings of infinity (e.g., Dubinsky et al., 2005ab; Ely, 2011; Radu & Weber, 2011; Wijeratne & Zazkis, 2015).

The relevant paradoxes that have been used in mathematics education research are infinite iterative tasks. The combination of physical steps (e.g., moving halfway to the door, drawing a triangle) with something physically impossible to complete is a situation which is not easily resolved, even by students with extensive mathematics training (Ely, 2011). The above research in this area has revealed potential infinity conceptions, the projection of finite patterns onto the completed state, conceptions of limits as unreachable, and an urge to preserve consistency with the physical world.

### Conceptual Blending

We use conceptual blending theory (Fauconnier & Turner, 2002) as a theoretical and methodological tool for analyzing students' coordination of two infinite processes, one increasing (perimeter) and one decreasing (area). Blending is based on the notion of mental spaces, which are "small conceptual packets constructed as we think and talk, for the purposes of local understanding and action" (p. 40). According to the theory, these mental spaces "organize the processes that take place behind the scenes as we think and talk" (p. 51). Conceptual blending is defined as the conceptual integration of two or more mental spaces to produce a new, *blended,* mental space. An important feature of this new blended space is that it develops an emergent structure that is not explicit in either of the input mental spaces. This emergent structure is generated by three processes: composition, completion, and elaboration.

*Composition* is the selective projection of elements from input spaces into a common space. During composition, distinct elements may be projected on top of each other or *fused,* and common elements may be projected separately. The composition process develops a new space, with the potential for structure not available in either input space. *Completion* is the process of recruiting familiar frames to the blended space, along with their entailments. That is, an individual recognizes certain aspects of a blended space as parts of a familiar frame and brings in additional knowledge, scripts, assumptions, etc., to complete the frame and prescribe structure for the blended space. These frames can serve as tools for *elaboration*, which is sometimes called *running the blend*. Elaboration is the process that leads to the emergence of something new within the blended space, using the tools of the completion process and the elements that compose the blend. These processes, composition, completion, and elaboration, do not necessarily take place sequentially.

This theory has been applied to the learning of mathematics by a number of researchers. For example, Lakoff and Núñez (2000) propose that most of the important ideas in mathematics are metaphorical conceptual blends. Alexander (2011) goes further and discusses how one can see conceptual blending within the formal structure of mathematics, and that the actualization of blends is a cognitively challenging but critical part of the evolution of the discipline. However, while a number of researchers have used conceptual blending to explain mathematics and mathematical thinking in general, it has been only minimally used in empirical studies of student thinking.

The few examples of empirical studies include the work of Edwards (2009) and (Yoon, Thomas, & Dreyfus, 2011) who analyzed how people invest their real gesture space with mathematical meaning; the use of grounded blends and physical space separates that work from what we present here. Gerson and Walter (2008) used the theory to look at the emergence of calculus concepts for individuals during small group work, but did not leverage the elements of the blending process as we do. Megowan and Zandieh (2005) and Zandieh, Roh, and Knapp (2014) do leverage the processes of composition, completion, and elaboration to investigate students' reasoning and proving activities in a geometry course, but they used small groups as their unit of analysis while we saw distinctions between groupmates. As these are the most relevant studies we could find, we situate our work as part of a new movement to leverage conceptual blending as an analytic tool in empirical mathematics education research.

## Methods

### Setting and Participants

The study took place in a graduate level mathematics course of 11 students (10 of whom participated in individual interviews). All students in the course were instructors and/or tutors of secondary or tertiary mathematics. Their master's degree program required a substantial mathematics component, and the chaos and fractals course studied here fulfilled part of that requirement. The course was taught by one of the research team members. Students sat in four groups throughout the course: 1) Carmen, Joy, and Jackie (Jackie did not participate in interviews); 2) Shani, Soo, and Kay; 3) Mia, Kevin, and Elise; 4) Sam and Curtis. Students regularly worked on mathematical tasks during class time in their groups and then discussed their thinking with the whole class. Data was collected as part of a larger study and included video-recordings of each class session, individual task-based interviews conducted at the middle and end of the semester, and copies of all student work.

### Methods for Data Collection

The focus of the analysis in this paper are students' responses to the following question from the mid-semester interview: *In class, we discussed the Sierpinski Triangle. How do you think about* *what happens to the perimeter and the area of the ST as the number of iterations tends to infinity?* This question was accompanied by a printout of the ST (as seen in Figure 1), with a follow-up prompt to tell us what they thought about the following claim of a fictitious student, "Fred": The computation shows that the perimeter goes to infinity because the perimeter is given by $3 \times (\frac{3}{2})^n$ which increases to infinity as n tends to infinity. But, the perimeter can't really be infinitely long, because there is nothing left to draw a perimeter around, since the area goes to zero.

This interview task was designed based on the classroom discussion of the ST. At that time, students seemed to agree that the area went to zero but were unsure of what happened to the perimeter. They publicly considered the possibilities that it went to infinity, converged to some value, or did not exist because there was nothing left for a perimeter to go around. We included a sequential expression for the perimeter in the hopes of foregrounding the paradoxical situation by helping students see that the perimeter diverges. The interview was structured so that we would first gain insight into the students' reasoning about the area and perimeter of the ST, followed by an opportunity for them to respond to Fred's claim. All interviews were conducted by the same member of the research team, with one of the other researchers present to video-record and ask occasional follow-up questions. Each interview lasted roughly an hour, 5-20 minutes of which were spent on the ST segment.

As noted, Fred's claim is based on an argument heard in class, presented first by Carmen, amended to include a (correct) algebraic expression. This ensured that the argument did not feel contrived to the students, and in fact several of them recognized this and noted that some students in class struggled with this same scenario. So, students had previously seen the ST and considered, to some extent, the same paradoxical situation we brought up in Fred's argument. This means that when we consider students' thinking in the interviews, we are gaining access to a semi-retrospective account of their original thinking. As conceptual blending is not a linear process, and in fact mental spaces coexist for extended periods of time, this gave us a better chance of seeing fully blended spaces, but reduced our ability to access the completion process or identify failed blends along the way.

**Methods for Analysis**

The transcripts and student work produced during the interviews were coded and organized in two rounds. The first round consisted of identifying the elements of each student's input spaces and blended space, extending and expanding our previous work (Rasmussen, Apkarian, Dreyfus, & Voigt, 2016). The second round was a fine-grained analysis of the blended spaces to identify the blending processes.

To identify a student's input space for area (similarly for perimeter), we first marked which of their utterances were about the area. Next, we categorized these utterances into sets of ideas about the area of the ST - including the process by which it is created and the resulting product. In the spirit of grounded theory (Strauss & Corbin, 1998), these ideas were coded and compared iteratively until a coherent set of idea codes emerged. The interviews were divided into two groups and analyzed by different members of the research team. These analyses were then swapped, compared, and vetted. The multiple, iterative bouts of discussion among the research team members provided many occasions to share and defend interpretations, thereby minimizing individual bias and keeping interpretations grounded in the data.

We investigated students' blending by identifying each of the three processes: composition, elaboration, and completion. To see how a student's blend was composed, we identified which elements of the student's input spaces were brought up as they considered the coordination of area and perimeter (prompted by Fred's paradox). We identified the ways students elaborated their blended spaces by identifying ideas which were not in the input spaces, but emerged as they worked to make sense of the task. For completion, we asked ourselves what frames (and entailed tools) students used (other than input space elements) in order to make their elaborations. The analysis of the composition of blended spaces was carried out in the same fashion as identifying the input space elements. Interpretation of completion and elaboration was done first as a group, with all four authors debating each point, then a more detailed pass was made by two members of the team in close comparison with the transcripts, and these analyses were then discussed again among the four authors until agreement was reached.

## Results

In this section, we address our two research questions. We begin with the first: how students made sense of (a) area and (b) perimeter of the ST. In particular, we specify the various elements of students' respective input spaces for area and perimeter, including both their conclusions about the area and perimeter as well as the ideas they use to justify their conclusions. These results are organized by idea, with mention of how widespread they are, rather than by student. Following this analysis, we present results for the second research question: how students cope with the paradoxical situation that arises from coordinating area and perimeter for the ST. We address this question by considering students' blending processes. These processes were individualized, and hence we present these analyses organized by student rather than process. In doing so, we present blending diagrams and note the composition of blends from students' input spaces, discuss the completions we were able to see, and identify the resulting elaborations.

### Results 1: Area and Perimeter

During the in-class discussions about the ST there was widespread agreement that the area would go to zero but less agreement that the perimeter would diverge to infinity. We expected similar claims in the interview, and were surprised to find that only six of the ten students concluded that the area of the ST goes to zero. Soo indicated that area shrinks unendingly but was adamant it would never actually reach zero; Shani and Kay said that the area converges to something nonzero; and Kevin said only that it converged, but he had not worked out what it converged to. On the other hand, all ten students concluded that the perimeter tends to infinity.

Among students' justifications for their conclusions, we identified seven qualitatively different mental space elements for area and seven qualitatively different mental space elements for perimeter. We display these different mental space elements side-by-side, with descriptions of the elements and, in most cases, illustrative quotes. This is intended to highlight the parallel nature of these ideas and to relate to the diagrams (Figures 2-7) that accompany the blending results.

| **Area** | **Perimeter** |
|---|---|
| <u>Infinite decreasing process</u><br>Common among all 10 students was the element that area is the result of an infinite, decreasing process. For example: | <u>Infinite increasing process</u><br>All 10 students conceived of the perimeter of the ST as the result of an infinite |

| | |
|---|---|
| Carmen: *So, ok eventually the area gets to zero, but that's if you could do it infinitely many times. And if you actually conceptualize doing infinitely many times you're never gonna stop.* | increasing process. Elise's reasoning is typical of this thinking:<br><br>Elise: *You're just like forever adding length to your perimeter, so I feel like your perimeter is forever increasing.* |
| <u>Area removed at each step</u><br>All students except Curtis there was explicit use of the justification that area is removed at each step. Two students computed the first few steps during their interviews.<br><br>Kay: *We're always taking out the middle triangle of each equilateral triangles and we're doing that infinitely so it's like we're taking away area with each iteration.* | <u>Perimeter is added at each step</u><br>All students except Curtis also pointed to the fact that perimeter is added at each step. Four students accompanied this with computation for the first few iterations.<br><br>Joy: *I think it goes towards infinity because each iteration you're creating more triangles and so you're creating, you're adding to the perimeter.* |
| <u>Change in the rate of change</u><br>Shani and Kay, who were in the same group, were the only two students who concluded that the area tended to something non-zero. They were also the only two who shared what we refer to as the change in the rate of change for area element, as exemplified in this excerpt.<br><br>Shani: *As we keep taking off little pieces and more become white, it's getting smaller and smaller. Or the amount that it's increasing is getting smaller and smaller and smaller.* | <u>Change in the rate of change</u><br>Two other students, Elise and Carmen, gave some consideration to the rate at which the perimeter increases and to changes in this rate. For example, Elise argued that<br><br>Elise: *Every time after the first iteration I'm adding more perimeter than I added before. So if I keep adding more then I think it's going to keep going to infinity because I'm just going to keep adding bigger and bigger.* |
| <u>Each step decreases by a factor of ¾</u><br>Only one student, Curtis, reasoned about the area in a multiplicative manner. All other students focused on the removal of area at each step, which suggests an additive conception.<br><br>Curtis: *Each time, [the area is] 3/4 the previous, so, I mean, I know that when you keep multiplying 3/4 by itself you get, you get zero out.* | <u>Each step increases by a factor of $^3/_2$</u><br>Only two of the 10 students, Curtis and Carmen, pointed to the fact that at each step the perimeter increases by a factor of 3/2. Curtis went further and expressed this formally with a limit as follows:<br><br>Curtis: *The whole thing increases by three-halves at each stage. So, you can just say that's three-halves to the n give you the perimeter at n.* |

Three other pairs of mental space elements were identified, but as they were used infrequently we mention them only briefly. Curtis was the only student to express a conception of area and perimeter as each composed of congruent components, not just as numeric sequences. This way of reasoning coincides with him being the only student to describe the area as decreasing by a factor of 3/4 at each step and one of only two who described perimeter increasing by a factor of 3/2 at each step. Kevin and Mia (groupmates) both indicated that "r>1" indicated divergence and "r<1" indicated convergence, with Kevin explicitly motivated by conceptualizing perimeter and area as geometric series. Finally, Mia conceived of the area of the ST as computed from "leftover" triangles and the perimeter as computed from "removed" triangles during the recursive creation process explored in class. In her interview, she drew these sets of triangles as distinct figures from which to compute area and perimeter.

**Summary for RQ1: Making sense of area and perimeter**

Students made sense of the area and perimeter of the ST, first and foremost, as infinite iterative processes. This in and of itself is no surprise, given the construction process students were introduced to in class. What *did* surprise is the fact that, except for Curtis, students used informal additive reasoning to reach their conclusions. The few students who did some computations did so only for the first few iterations and did not generalize the adding of perimeter or removal of area into algebraic expressions from which to take limits. While some students used limit language or referred to convergence criteria, it was not done concretely, despite their mathematics experience.

Given students' informal ways of reasoning, the parallelism between area and perimeter ideas is noteworthy. As seen in the previous section, each element of reasoning about area had a corresponding element of reasoning about perimeter. While some of these ideas were common (infinite processes, adding area, removing perimeter), others were not. In several cases students' idiosyncratic ways of thinking were consistent within students across area and perimeter. Curtis is the most obvious case, standing out by using multiplicative and more formal reasoning than the other students, but also Kevin's mentioning of geometric series, and Mia's reference to a convergence test. However, there were some discrepancies. For example, the two students who referred to change in the rate of change of area did not refer to it in the case of perimeter, whereas two students who referred to it in the case of perimeter did not refer to it in the case of area.

Despite the idiosyncrasies, there was quite a lot of consistency in ways of reasoning across students, both with respect to area and with respect to perimeter. Based on the fact that the students had previously discussed the area and perimeter of the ST in class, this might have been expected; however, we note that the interviews presented substantial differences to what we expected on the basis of the classroom discussions. Further investigation is needed to conclusively explain these shifts, but we speculate that conceptualizing the infinitely large (e.g., ST perimeter) may be easier than the infinitely small (e.g., ST area).

The two students who claimed that the area tended to something non-zero had been in the same group; they were also the only two students taking change in the rate of change into account when discussing area, and they might have discussed these issues at some stage. While a causal claim between this justification and conclusion that the area tends to something non-zero is not warranted, we suspect that focusing on the relative size of pieces removed suggested for these two students coupled with a strong grounding in the figure itself (as opposed to the result of an infinite process) contributed to their conclusion of non-zero area. On the other hand, the two students who referred to change in the rate of change regarding perimeter were in different groups. We also note that the two who referred to a convergence test (r >1, r<1) were in the same group, though Kevin was more specific than Mia. Themes within group members' reasoning are also seen in the following blending section, but again they are not totally consistent.

## Results 2: Blending Area and Perimeter

As the previous section detailed the input space elements, we do not revisit the nature of those elements in any detail but instead focus on the blended spaces. One element appears in each student's blended space which did not appear in the area/perimeter section: *infinite creation process*. This element is a result of fusion, wherein two input space elements (here, infinite increasing and infinite decreasing) are projected onto one element. As students were introduced to the ST as something created through an iterative, recursive process affecting both area and perimeter, in a sense the students are re-fusing elements which they originally separated. To organize these ideas, a three-part diagram is used: rectangles represent mental spaces, with the upper rectangles representing the input mental spaces and the lower rectangle representing the blended mental space, and the lines show mappings between the spaces (Figures 2-7).

**Joy.** We gained access to Joy's blending process primarily through her response to Fred's argument. Her blended space (Figure 2) is composed of the infinite process of creating the Sierpinski Triangle, the area tending to zero, and perimeter tending to infinity. Completion brought into the blended space a metaphor of *perimeter as fence*, along with several entailments. One such entailment is that fences should remain, even if the space they enclose is no longer there. Part of Joy's elaboration based on this frame, as she worked to resolve Fred's paradox, was to say that "we don't count their space, but there is still a perimeter associated with it." Another entailment of the fence framing is that not only do fences have length, but they also take up space. This contributed to another element of Joy's elaboration, that the perimeter will fill in the Sierpinski Triangle, "so eventually in a sense it's all fence." Some parts of Joy's elaboration were grounded in a physical metaphor, and she recognized this when responding to Fred. She added to her elaboration that the Sierpinski Triangle is "not a real object," and identified the juxtaposition of an infinite mathematical process with the physical world as "where the disconnect is."

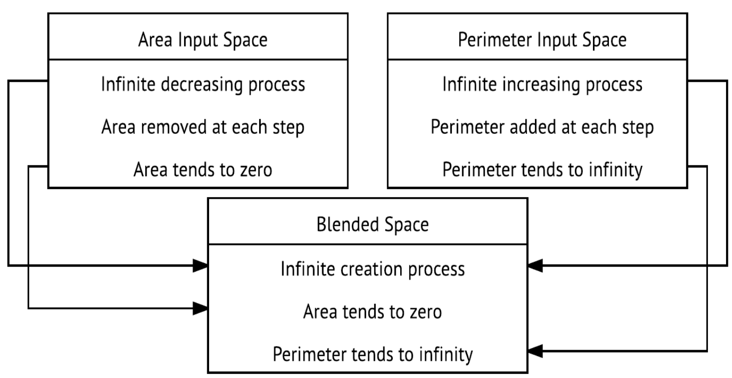


*Figure 2. Blending diagram for Joy's reasoning.*

**Elise.** Like Joy, Elise's blended space is composed of the infinite process of creation for the

ST, perimeter tending to infinity, and area tending to zero (Figure 3). However, the framing metaphor that completes Elise's space is one of a skeleton, not a fence. She elaborated her blend, saying, "I'm thinking of our perimeter as like, like I guess I think at the end of this I have this skeleton, so I have no area, nothing is left inside" This skeleton metaphor brings with it entailments of bones remaining when flesh has gone, clearly mapping perimeter to bones and area to flesh. In addition, we note that Elise mentioned "at the end" in her elaboration, perhaps hinting that she sees the ST as an abstract object at the end of a generating process.

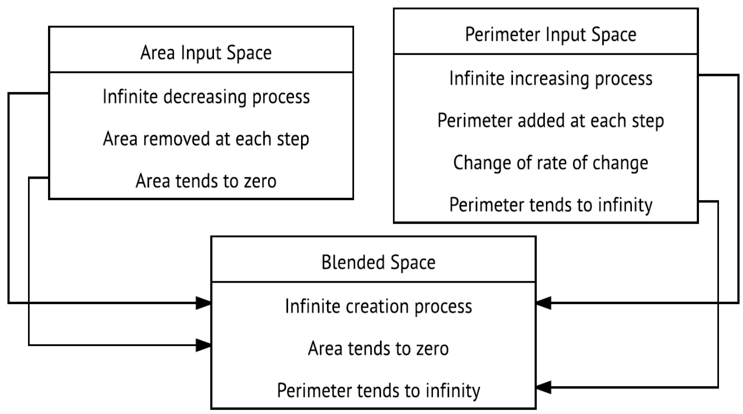


*Figure 3. Blending diagram for Elise's reasoning.*

**Curtis.** As with Elise and Joy, Curtis's blended space is composed of an infinite creation process, perimeter tending to infinity, and area tending to zero (Figure 4). Unique to his blended space, however, is his formulation of these tendencies. He wrote area as $lim_{n\to\infty}\ (\frac{3}{4})^n$ and perimeter as $lim_{n\to\infty}\ (\frac{3}{2})^n$ and computed the limits of each sequence, obtaining 0 and $+\infty$ respectively. The completion process brought in a zooming frame, saying, "we could say you could zoom in for infinitely, as much as you want, and you could get like these as tiny and tiny as you want, there's still more perimeter to draw" when prompted with Fred's paradox. The second frame we saw Curtis leverage is one related to mathematics classes (e.g., Calculus, Analysis) where symbolic manipulations are sufficient. Evidence of this comes from the fact that Curtis did not encounter a paradox when considering an object with zero area and an infinite perimeter on his own, something he elaborated by saying "this isn't like, not physically drawing something like a perimeter, it's kind of just a concept."

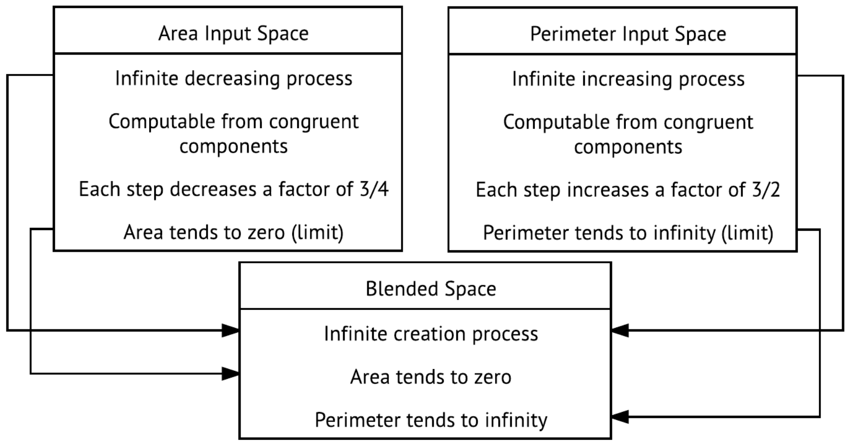


*Figure 4. Blending diagram for Curtis's reasoning.*

**Carmen.** Carmen's blended space is, like several others', composed of an infinite process of creation, area tending to zero, and perimeter tending to infinity (Figure 5). The completion of her blend, however, was particularly distinct. She brought in a calculus frame and identified "analogies to calculus or real analysis," including Riemann sums, that she saw as similar to Fred's paradox. The "calculus arguments" that she referenced seem to imply, to Carmen, that Fred's paradox is like other paradoxical situations that she has seen in previous mathematics courses. Upon reading Fred's arguments during the interview, Carmen stops to query whether "the perimeter can't really be infinitely long" implies zero perimeter or some non-zero finite length (for Fred). She proceeds to resolve the dilemma by eliminating each, leaving only the possibility that the perimeter is indeed infinite and Fred is wrong. During this episode, two more frames appeared. Like Joy, she brought in a fence metaphor for the perimeter and the entailment that fencing should remain, but did not use the idea that fences take up space. Her elaboration using the fence frame, "you have sort of your old triangle fences that you had before [...] we still have this fence around, that big triangle and the center, and we still have those other ones we made before," is how she argued that the perimeter of the ST cannot be zero. Finally, she brought the frame of self-similarity, with the entailment that "we can keep zooming in." The elaboration using this frame was that the perimeter cannot be a finite value, which she explained using a contradiction. Carmen said, "I think if we could [stop] then you could say ok it's this number," but the zooming goes on forever, "so that's kind of why it can't be a number."

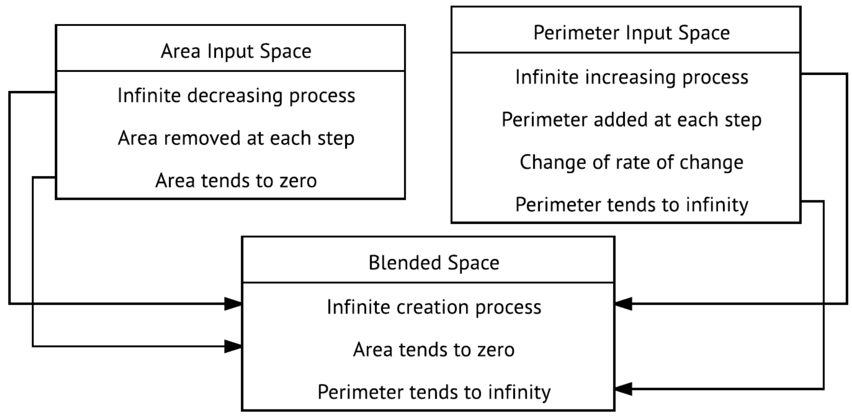


*Figure 5. Blending diagram for Carmen's reasoning.*

**Sam.** Sam's blended space is composed of the infinite process of creation, area tending to zero, and perimeter tending to infinity (Figure 6). The composition also includes a unique fusion of removing area at each step and adding perimeter at each step, so that one element of his blended space is the *simultaneous* addition of perimeter and removal of area. He clearly said that "as you keep adding triangles you're taking chunks from the area [...] and because you keep taking chunks out of it you're adding triangles you're adding perimeter." Sam completed his blend by bringing in a frame about the nature of infinity and infinite iterative processes, saying about area that "*at infinity* it's going to 0. It's not before infinity." These tools allow him to elaborate his blend, establishing that Fred's paradox only exists if the process stops at a finite stage, or "if it's before infinity [Fred's] statement will be right."

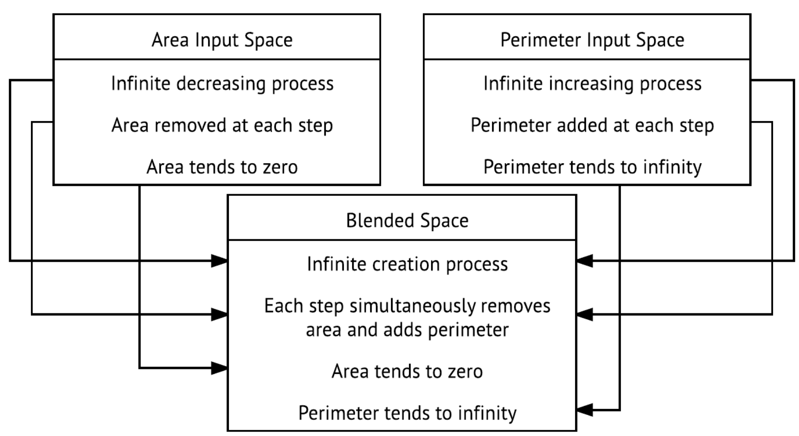


*Figure 6. Blending diagram for Sam's reasoning.*

**Kevin.** Kevin's blended space is composed of the infinite process of creation, the perimeter tending to infinity, and an area that converges without committing to a value (Figure 7). When asked about the area and perimeter of the ST in the interview, Kevin immediately responded with: "perimeter can be defined as a geometric series that diverges, and then the area converge and to me that indicates it's not dimension 1 or 2 but somewhere in between." Kevin's thinking was difficult to unpack, particularly differentiating between the completion and elaboration of his blend. We have evidence that he saw a relationship between Fred's paradox and the non-integer dimension of the ST, which is an irregular object - but is difficult to say what is the frame and what is the elaboration. Kevin is one of the students whose blending process was difficult to access, maybe because it had solidified in the time between the class activity and the interview. Nevertheless, we can see the results of the blending process. The blend we can identify supports his interpretation of the root of the paradox, explaining that "it seems like Fred is assuming that [the ST] has to be a natural number dimension. So, if it has no area, then it can't have a perimeter because it wouldn't make sense for it to have one, but not the other."

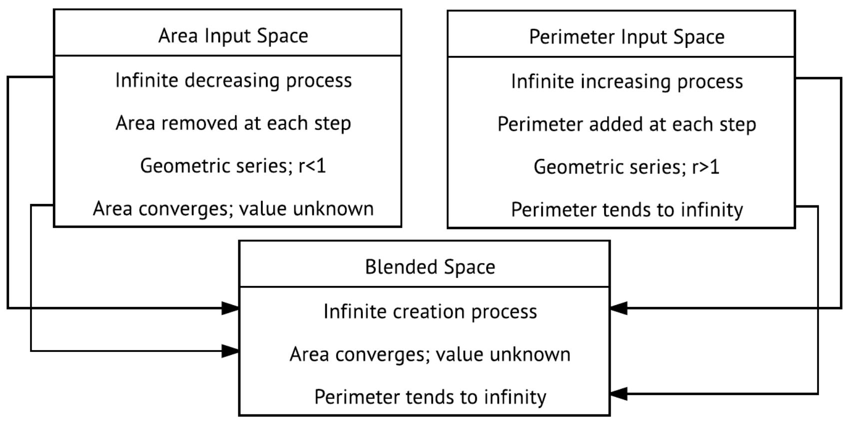


*Figure 7. Blending diagram for Kevin's reasoning.*

**Shani, Kay, and Soo.** The group of Shani, Kay, and Soo had similar blending processes and so we address them together. In particular, this group saw the perimeter as infinitely decreasing, but not converging to zero. Shani and Kay described perimeter as converging to some "infinitely small" yet nonzero value, while Soo described an unending decreasing process which never reaches zero. While researchers can see the nuanced differences between these two conceptualizations, both resulted in students composing similar blended spaces that did not include "zero area," and this contributed to the fact that none of the three encountered Fred's paradox. Shani and Soo's completion processes were framed by their understanding that the process does not end, and we have no evidence of elaboration on their part. Kay's completion, however, brought in a frame of perimeter with the entailment that "a perimeter encloses something."

These different frames were particularly evident in their responses to the interviewer's prompt to consider, as a thought experiment, whether or not Fred's argument would make sense *if* he was correct that the perimeter "went to zero." Shani did not engage with the thought experiment, instead returning to her statement that the area of the ST did not go to zero. Kay engaged with this, and her frame of perimeter as something that encloses led her to an elaboration that "[the perimeter] could be just kind of wrapping either around itself or so kind of close together, it basically is like almost a single line or something." Soo, however, more deeply considered the possibility of the perimeter going to zero and in fact displayed a second round of conceptual blending. This second round, which included the idea that the area of the ST goes to zero, triggered a different completion process of regular and irregular triangles, in which she declared that "[Fred's argument] doesn't make sense in general if you are looking at just regular triangle and then see perimeter and then area relationship. But we are looking at Sierpinski triangle." While she engaged with Fred, and was the only one of her group who recognized and grappled with the paradox, we saw no further elaboration.

**Mia.** Mia was a special case. As noted in the discussion of the first research question, she conceptualized the ST as two figures: the triangles that remain and the triangles which were removed during the recursive creation process. In talking about the coordination of area and perimeter, she clearly stated that "when we're talking about the perimeter, we're looking at the triangles that we're taking out, and when we're looking at the area, we're looking at the area of the triangles that are left over," and accompanied this with sketches of both 'sets' of triangles. This composition, with two distinct figures, does not seem to make sense to Mia - she was unable to move past this contradiction, she did not engage with Fred's paradox, and we did not identify any completion or elaboration.

**Summary for RQ 2: Coordinating area and perimeter and coping with the paradox**

Our analysis of students' blending processes, especially as provoked by encountering Fred's argument, revealed how students deal with the paradox of coordinating infinite perimeter and zero area associated with the ST, and how they cope with, or resolve, the cognitive dissonance it provokes. It was sometimes challenging to unpack and distinguish the completion and elaboration processes. We attribute this difficulty in part to the fact that this was the second opportunity in which the students were prompted by Fred's paradox. Having detailed each individual student's blend previously, we now comment on what we learned.

All students *composed* a blended space from their area and perimeter input spaces following Fred's prompt, and most of them also *completed* their blended space with additional frames, which then supported *elaboration* of the blend - leading to new implications. In two students' interviews we saw evidence of completion but not elaboration (Shani and Soo); only for one student (Mia) we do not have evidence of completion.

We saw one commonality across all students' composition processes: the fusion of *infinite (increasing) process* and *infinite (decreasing) process* into a unified *infinite creation process* for the stepwise creation of the ST. This is not to say that there was a shared conception of exactly what happens at each step, only that the process is infinite. This can explicitly be seen in the case of Mia, who envisioned an infinite process that created two separate figures. We purposefully refer to these elements as *infinite*, with all the ambiguity about potential/actual infinity it entails, because our data does not support conclusions about the nature of students' conception of the infinite.

We found that four students did not encounter the paradox on their own, and this seems related to the composition of their blended spaces. That Mia's blend was composed of processes based on two distinct figures (removed vs. leftover triangles) prevented her from encountering a paradox, since she did not see one figure with infinite perimeter and zero area. Shani and Kay's input spaces for area included a non-zero limit for area, and their completions allowed them to coordinate this without experiencing a paradox. Soo is the fourth who did not encounter the paradox on her own, because she did not see an end where area would equal zero, but she was able to entertain the idea of an object where the paradox might exist.

While composition is an important part of the coordination of area and perimeter, and explained who encountered the paradox or not, it was not enough to explain students' different ways of reasoning. For example, the blending spaces of Joy, Elise, Curtis, and Carmen included the same three elements, but they completed these spaces with different frames and/or different entailments of similar frames. For each student (except Mia), we have evidence of 1-3 distinct frames being used to complete their blended spaces. In all the cases, one of the frames has to do with the nature of mathematics – e.g., the nature of infinite processes. This might be expected, as the paradox itself is rooted in a mathematical context. However, four students also used physical frames (fence, skeleton, zooming-in) and their entailments to coordinate area and perimeter and to make sense of that coordination.

Elaboration of the blend was the most varied of the processes we analyzed, due in part to its dependence on both the composition and completion of the blended spaces. We observe that students could arrive at similar conclusions based on very different lines of reasoning (e.g., Joy and Curtis concluding that the ST is a mathematical concept, not a real object) and that students with superficially similar starting points could bring in different frames and reach different conceptualizations. Our approach, using conceptual blending, allowed us to see these nuances as students' lines of reasoning separated and coalesced in a non-deterministic way.

## Discussion

The in-class Sierpinski Triangle activity was intended to be a brief interlude, a relatively simple yet interesting task that would serve as motivation for a discussion about self-similarity and later fractal dimension. However, it proved to be challenging for the students. The classroom discussion, and the interview question it prompted us to ask, proved very rich for exploring how students reason about features of the ST.

The design of our interview question sequence is worth revisiting. We included an actual classroom episode as a hypothetical question about Fred's claim for students to engage with. This allowed us to investigate the extent to which students maintained the ideas expressed in a classroom discussion, pick out nuances of individual thinking that were not accessible in the larger group, and see how students would defend and elaborate their ideas in the face of an alternative view (Rasmussen, Apkarian, Dreyfus, & Voigt, 2016). Students' responses to Fred's argument were, in general, deeper than their original responses to the interview task. This may be a function of our participants' career paths as instructors, their enculturation into a classroom where mathematical arguments were normative, or something else entirely – nevertheless it was a rich source of data about students' reasoning and understanding of the topic.

As the 10 students we interviewed were in the same graduate program, part of the same class, had worked together and discussed the Sierpinski Triangle (including Fred's argument), we expected to see some consistency in their responses. As seen at every stage of our analysis, this was not the case. To be sure, some ideas about the nature of the infinite iterative process were present in all interviews. But while in class students seemed comfortable with the idea that the area of the ST goes to zero, and concerned about what happens to the perimeter, *all* students' input spaces for perimeter included that it was infinite, and only six of the ten spaces included area going to zero. There were other idiosyncratic elements present in students' input spaces such as Curtis's multiplicative reasoning about components and Mia's conceptualization of the ST as two separate figures.

There were also idiosyncrasies in terms of the composition of blended spaces. Though most students had the ideas of perimeter being added and area being removed at each step, only Sam fused these ideas and mapped them to his blended space. While several students had blended spaces composed of the same elements, the development of those spaces showed more variation. Some students completed their blends with ideas from calculus or analysis (e.g., Carmen, Curtis), fractal dimension (e.g., Kevin), and metaphors (e.g., Joy, Elise). These frames resulted in varied elaborations. Some related to the nature of the ST, such as "it's not a real object" (Joy), its non-integer dimension (Kevin), or that is only the remaining outline (Elise's skeleton, Carmen's fence); others framed the nature of the paradox itself, such Sam's statement that the paradox only exists "before" infinity.

Some of the less common elements of students' mental spaces may be attributable to groupwork. Kay and Shani were the only two to insist that area converged to a nonzero value, and the third member of their group, Soo, was not entirely certain; Kevin and Mia are the two who mentioned "r>1 and r<1;" Carmen and Joy are the two who used a fence metaphor. However, groupmates' blending processes were not always parallel in structure. While Carmen and Joy both use a fence metaphor to complete their blends, the entailments of that metaphor and how they elaborated using that frame were markedly different. Altogether, we conclude that while classroom conversations and small group work affected students' thinking about the ST, there are individual idiosyncrasies that cannot be attributed to these collaborative efforts.

More generally, our analysis methods allow us to point to some of the precise points of departure, from initial ideas to completing frames and final elaborations, one of the methodological implications of our work for future researchers. Along with Zandieh et al. (2014), our articulation of the component process of conceptual blending in a mathematical context allow for nuanced analysis of students' reasoning – though they looked at group blends and types of blends, while we look at more individualistic reasoning. This is particularly relevant for situations where students must bring together multiple ideas. Identifying all three processes - composition, completion, and elaboration - allows us to examine not only the main ideas students mention, but how they are used and enacted, or what leverage they give students in thinking about mathematical objects. This is in contrast to other lenses which make claims about the level of students' understanding, the extent to which their ideas are normative, or the conceptual structures that they might "possess." We are particularly impressed with the analytic power of the completion process, allowing us to articulate the tools by which students elaborate their blends. Thus, our analyses lie fully within the domain of enacting ideas.

Our use of these processes as analytic tools allowed us insight into student thinking that, on the surface, appeared as a jumble of ideas and conclusions with little connection. We further see that the completion process, that of recognizing elements and bringing in a frame and its entailments, was critical in students' ability to think deeply about the ST. Regardless of how a blended space is composed, it was completion which allowed for elaboration and the formation of new understandings.

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Graduate Student Instructors' Growth as Teachers: A Review of the Literature

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*In recent years, providing teaching professional development for graduate student instructors has become more common in mathematics departments in the US. Following this trend, mathematics education researchers have begun to conduct studies on professional development programs and on graduate students as future mathematics faculty. The purpose of this literature review is to examine the current status of research in this field and make recommendations for future research on graduate student instructors and professional development. In examining the literature, we found that there are few studies that have researched graduate student instructors' growth as teachers. However, those studies that do address growth attend to it in three ways: by examining growth over time, by taking a stance on what constitutes desirable growth, and by using models or theories of growth.*

*Keywords:* Graduate Student Instructors, Professional Development, Teacher Growth, Literature Review

## Introduction

In recent years, undergraduate mathematics educators have focused on helping graduate students develop as teachers. Most large universities rely on graduate students as an integral part of their teaching staff and therefore, many undergraduate students are impacted by the quality of their teaching (Belnap & Allred, 2009). Additionally, many graduate students go on to teach after graduation; hence we can view graduate school as a training ground for their work as teachers as well as researchers. To help prepare graduate students as teachers, an increasing number of mathematics departments across the U.S. have begun designing professional development programs specifically focused on teaching.

Indeed, at the 2017 Conference for Research on Undergraduate Mathematics Education there were at least eight different presentations on graduate student instructor (GSI) professional development. A striking aspect of these presentations was the variety of approaches, methodologies, and theories used. One theme we noted in many of the presentations was a focus on how graduate students grow as teachers over time[1].

With this in mind, we reviewed the research literature on how mathematics GSIs learn to teach. Our goal was to identify and characterize the models and theories that have informed studies of GSIs' growth as teachers. We sought to take stock of what is known about improving GSIs' teaching, what gaps there may be, and how to move forward. To do this, we asked: *How have researchers studied mathematics GSIs' growth as teachers?*

In this literature review we report on the results of a Grounded Theory analysis of three major research databases from 2005 to 2016: Education Resources Information Center (ERIC), PsycINFO, and Web of Science, as well as the RUME proceedings from 2010 to 2016 and the AMS Notices from 2005 to 2016. We describe the current state of research in the field of

[1] Here we define growth as the process of changing along an identifiable trajectory. For something to be considered growth, it must be true *that* something has changed and that exactly what has changed can be *identified.*

mathematics GSI professional development, describe emergent themes, and discuss the implications of these themes for future research. Finally, we make the central claim: *graduate student instructors' growth as teachers is a largely unexamined practice*.

## Using a Grounded Theory Approach

We felt that grounded theory was the most appropriate research design to answer our research question. According to Creswell (2013), "the intent of a grounded theory study is to move beyond description and to generate or discover a theory, a 'unified theoretical explanation' (Corbin & Strauss, 2007, p. 107) for a process or action" (p. 83). We chose this design because we wanted our review to move beyond just describing the literature on GSIs' growth as teachers and generate a theory for how researchers have studied GSIs' growth. In other words, the process or action that we wish to study is the ways in which researchers study growth.

### Need for Theory Development

A key aspect of grounded theory research is that "this theory development does not come 'off the shelf,' but rather is generated or grounded in data from participants who have experienced the process" (Creswell, 2013, p. 83). There has been an increase in the number of researchers studying GSIs' growth as teachers, but researchers have examined this phenomenon in very different ways. While studying a phenomenon from different perspectives can bring light to different aspects, it is also important for research to be connected and not isolated. Therefore, we thought it was important to conduct this literature review in order to examine and build these connections.

### Researcher Positioning

Researcher positioning, or reflexivity, is another important characteristic of grounded theory research. In order to make it clear how the researchers are positioned in the study, they should "convey...their background (e.g., work experiences, cultural experiences history), how it informs their interpretations of the information in the study, and what they have to gain from the study" (Creswell, 2013, 47). Our background, as a research team, is in studying how professional development impacts graduate students' growth as teachers. Therefore, we are particularly interested in and aware of research in this area. This personal interest influences how we interpret the literature and our sensitivity to ways in which our research is similar to and different from other, related, research. While one of the articles included in our literature review was co-authored by a member of the authorship team, this individual was not responsible for coding this article. Also, the work that we have published together was not included in the review, since it was published in 2017.

### Sampling

Another key aspect of conducting grounded theory research is utilizing theoretical sampling. Since the purpose of grounded theory is to develop a theory of a process or action, theoretical sampling is used to choose participants that would best help the researcher form the theory (Creswell, 2013). In our study, we started by sampling from databases that index the major peer-reviewed publications in mathematics education. When this sampling did not return as many results as we expected, we decided to sample from some other peer-reviewed publications that we knew published research on GSIs' growth as teachers but were not indexed by the original databases that we searched.

## Coding

Open coding is an essential part of grounded theory analyses. Open coding refers to "coding the data for its major categories of information" and then using constant comparison to identify and refine emergent themes (Creswell, 2013, p. 86). Once we identified articles that studied GSIs' growth as teachers, we open coded them using descriptive codes (Miles, Huberman, & Saldaña, 2014) in order to summarize how researchers were studying GSIs' growth as teachers. After this first cycle of coding, we went through a second cycle of pattern coding in order to group the summaries produced by the first cycle of open coding "into a smaller number of categories, themes, or constructs" (Miles, Huberman, & Saldaña, 2014, p. 86). Finally, we shared the memo summaries that we wrote for each article with the original authors in order to conduct member checking and validate that our interpretation of the article matched their original intent.

## Methods

All articles considered for inclusion in the review were peer-reviewed and contained at least one search term from each of the following four categories: teaching, domain, level, and participants (see Table 1 for exact search terms). This yielded 1,889 articles. We read each abstract to determine if an article could reasonably address our research questions. We double-coded until we reached consensus on the criteria for inclusion, with an inter-rater reliability of 97%. After discussion, we agreed to include only articles that focused on novice instructors teaching collegiate mathematics. This excluded, for example, articles that discussed GSIs teaching STEM classes in general. As a result, we identified seven articles that were relevant. To capture other relevant research on this topic, we then turned to the proceedings of the RUME conferences, as we were aware that relevant articles were published there. We read the abstracts of the RUME proceedings for the years 2010 through 2016 (we restricted our time frame due to infrequency of relevant articles), again coding for inclusion, and found 17 relevant articles. Finally, we searched the AMS Notices using an advanced Google Scholar search for the years 2005 through 2016 using the same search terms in Table 1 but excluding the "domain" category. This yielded an additional two articles, which gave us a total of 26 articles that focused on novice instructors teaching collegiate mathematics.

Each article was then open coded for teaching practices and whether or not the authors used an explicit or implicit model or theory of how GSIs grow as teachers. Six articles were double-coded, at which point the team discussed preliminary findings and how to adjust the coding procedure. After consensus was reached, the rest of the articles were coded.

*Table 1. Search Terms*

| Category | Terms |
|---|---|
| Teaching | teach*, instruct*, "professional development", PD, training, TD |
| Domain | STEM, math* |
| Level | undergrad*, collegiate, tertiary, college |
| Participants | "graduate student", novice*, "future faculty", beginning, GST, GSI, GI, GTA, TA |

## Findings

Of the 26 articles that addressed novice mathematics GSIs' teaching, 16 addressed some aspect of growth. Among these 16 articles, three themes regarding growth emerged: studying growth over time, taking a stance on growth, and using a model or theory of growth. The "studying growth over time" theme took the form of studies that addressed some change in teaching over time among GSIs. In accordance with our definition of growth, these articles both *identified* something that changed and described *how* it changed. The second theme, which we call "taking a stance," can be described as any article that explicitly stated what sort of change would be desirable among GSIs. It is important to note that some articles that addressed change over time did not take a stance on growth. Lastly, the theme called "using a model or theory of growth" describes the papers that call upon theoretical or empirical models of growth from the literature to support their arguments about growth in teaching. We will elaborate on these three themes below by providing thick descriptions of articles that illustrate each theme.

**Studying Growth Over Time**

In total, ten of the 26 articles (38.5%) addressed growth over time. These studies addressed a range of topics, including change in beliefs, knowledge, and teaching practices. Here, we describe three of them. First, in 2005, Speer, Gutmann, and Murphy called upon the community to produce more longitudinal research to examine how GSI's instructional practices, beliefs and expectations evolved over time. In an effort to foster collaborations between K-12 and undergraduate mathematics education, the authors reviewed work related to GSI professional development and drew parallels to work in K-12 teacher education. The authors emphasized the importance of GSIs in undergraduate mathematics education, reviewed current trends and developments in GSI professional development programs, drew connections to K-12 research, highlighted research in progress on GSI professional development, and provided suggestions for future directions of research. While these authors did not use the language of growth, their discussion of identifying and advocating for change in teaching is compatible with the definition of growth used in this literature review.

Another example of an article addressing growth over time is Raychaudhuri and Hsu's (2012) longitudinal study. The authors explored how math GSIs' ontological and pedagogical beliefs regarding mathematics changed over the course of teaching for a year. Their initial findings indicated that these two types of beliefs could conflict with one another. In their paper, they argue that GSIs' knowledge of students can be classified as behaviorist-teacher-centered or constructivist-student-centered. Again, we see the authors identifying and studying an aspect of teaching that changed over time; in this case, the focus was on GSIs' beliefs about teaching and learning.

More recently, in 2016, Musgrave and Carlson showed that GSIs can develop deeper conceptual mathematical knowledge, which they argue helps them to become better teachers. In particular, the authors looked at how GSIs conceptualize average rate of change (AROC) before and after a professional development intervention. The authors found that before participating in the intervention, GSIs focused on computational or geometric interpretations of AROC. After the intervention, some GSIs were able to express a more conceptual understanding of AROC. We view this as an example of studying how knowledge can grow over time. Notably, these authors also take a stance on knowledge, claiming that deeper conceptual knowledge of AROC is desirable; we explore this theme more thoroughly next.

**Taking a Stance on Desirable Growth**

Eleven of the 26 articles (42.3%) took a stance regarding desirable growth. Many authors drew upon established frameworks or standards for teaching in order to form their stance. For example, in their 2005 *AMS Notices* article, Deshler, Hauk, and Speer advocated for increased pedagogical content knowledge. In this theoretical article, the authors summarized different types of GSI professional development and provided suggestions for what it should include. In their model for GSI professional development, they recommend an initial "intensive" experience followed by sustained follow up sessions spaced out across time. The authors argue that developing pedagogical content knowledge is a key part of effective instruction as it helps GSIs to anticipate and use student thinking, attend to both computational and conceptual understanding, and orchestrate productive class discussions. Finally, the authors emphasize that in order to evaluate GSI professional development, it is important to pay attention to the changes in GSIs' knowledge and beliefs about teaching, learning, and doing mathematics.

Yee, Rogers, and Sharghi (2016) studied the development of GSIs' teaching practices by comparing them to the NCTM's *Principles to Actions* (2014). They used these teaching practices because they "provide a framework for strengthening the teaching and learning of mathematics" (NCTM, 2014, p. 9). In this qualitative multiple case study, the authors examine the teaching practices of ten GSIs as they implemented an intensive iterative lesson study process across two universities. Yee et al. examined whether or not GSIs' mathematical teaching practices evolved over the span of a two-week involvement in an iterative lesson study process and used their stance regarding the development of teaching practices to argue that Lesson Study was a useful form of GSI professional development.

Finally, Friedberg (2005) argued in his theoretical article that while the primary focus of math graduate school is for students to become mathematicians, it is also important that students also become effective communicators and teachers of math. Working under the presumption that "the analysis of experience can contribute to good judgment" in teaching, Friedberg argues that case studies provide an opportunity for math GSIs to improve their judgment in teaching-related issues. He and his colleagues developed 14 different case studies as part of the Boston College Mathematics Case Studies Project. In the article, Friedberg discusses some of the GSIs' experiences learning from the case studies and reiterates how important it is for GSIs to improve their teaching, since developing the communication, listening, and group work skills required for effective teaching is important for their future careers. Ultimately, Friedberg takes the stance that mathematics graduate students should grow in their ability to effectively communicate mathematics.

**Using a Model or Theory of Growth**

Three of the 26 articles (11.5%) called upon a model or theory of growth; we will describe each of them here to illustrate this theme. First, a 2011 study by Beisiegel investigated obstacles to teacher education among GSIs. In this empirical qualitative study, the author sought to uncover issues and difficulties that impact mathematics graduate students' views of their role as an undergraduate instructor. The author found that graduate students feel the need to replicate teaching practices and emulate the identities of other members of the mathematics teaching community, which results in feelings of resignation. Using Lave and Wenger's (1991) theory of legitimate peripheral participation in relation to communities of practice, the author argues that the attention to legitimate peripheral participation in a mathematics department prevented graduate students from adopting alternate modes of teaching. Here, we see the author explicitly drawing upon a theory of growth to inform the arguments made in the study. A stance regarding growth is also taken; the author argues that novices should gain knowledge and understanding about the practices of a community (desirable growth) but that legitimate peripheral participation prevents graduate students from adopting alternate modes of teaching (undesirable growth).

Both of the remaining articles that use a model or theory of growth were written by Nepal (2014; 2015). In the earlier article, Nepal studied graduate students who were not yet teaching a course and investigated how their teaching philosophies changed over a semester of professional development. The author found that all of the beliefs that were present at the beginning of the semester were still present at the end, but that new beliefs were also formed throughout the semester. The second article extended this work by following the GSIs through their first three semesters of teaching and investigated how their beliefs changed during this period of time. The author argued that while beliefs changed little during the pre-service phase of professional development, more significant changes occurred during the in-service phase; specifically, several of the GSIs expressed a belief that they should become more authoritative in the classroom. In both of these studies, Nepal framed his studies around context-based adult learning theory, which is an extension of Vygotsky's (1978) sociocultural theory. He claims that GSIs' beliefs are "developed, changed, or reinforced as they learn more about teaching and learning, and these changes are reflected in their teaching philosophies" (Nepal, 2014). As in Beisiegel's article, an explicit reference to models or theories of growth is made in order to support the overall arguments that are made.

Above we gave in-depth descriptions of the three themes; our overall findings are summarized in Table 2. Note that ten of the articles in the original pool did not address growth at all and are therefore not included in this table.

*Table 2. Articles Addressing Growth*

| Authors and Date | Type | Growth Over Time | Stance | Model or Theory |
|---|---|---|---|---|
| Speer, Gutmann & Murphy (2005) | Theoretical | 1 | 0 | 0 |
| Friedberg (2005) | Theoretical | 0 | 1 | 0 |
| Soto-Johnson, King & Haley (2010) | Empirical (Quant) | 0 | 1 | 0 |
| Beisiegel, (2011) | Empirical (Qual) | 0 | 1 | 1 |
| Raychaudhuri & Hsu (2012) | Empirical (Qual) | 1 | 0 | 0 |
| Speer & Firouzian (2014) | Empirical (Qual) | 0 | 1 | 0 |
| Miller & Wakefield (2014) | Empirical (Qual) | 1 | 0 | 0 |
| Nepal (2014) | Empirical (Qual) | 1 | 0 | 1 |
| Firouzian (2014) | Empirical (Qual) | 0 | 1 | 0 |
| Deshler, Hauk & Speer (2015) | Theoretical | 1 | 1 | 0 |
| Nepal (2015) | Empirical (Qual) | 1 | 0 | 1 |
| Firouzian & Speer (2015) | Empirical (Qual) | 0 | 1 | 0 |
| Reinholz, Cox, & Croke (2015) | Empirical (Qual) | 1 | 1 | 0 |
| Yee, Rogers & Sharghi (2016) | Empirical (Qual) | 1 | 1 | 0 |
| Musgrave & Carlson (2016) | Empirical (Qual) | 1 | 1 | 0 |
| Duncan (2016) | Empirical (Qual) | 1 | 1 | 0 |
| Totals | | 10 (38.5%) | 11 (42.3%) | 3 (11.5%) |

## Discussion

In 2005 Speer, Gutmann, and Murphy called on the community to conduct more longitudinal research on GSIs. One means of conducting further longitudinal research is by looking at how GSIs grow as teachers. In our literature review, we found only 26 articles that address novice mathematics GSIs' teaching. Of that small pool, ten of these articles (38.5%) did not focus on growth at all. Of the 16 articles that addressed growth in some way, none of them reflected all three emergent themes:  growth over time, taking a stance on growth, and the use of a model or theory of growth. Moreover, only three of the articles called on explicit models or theories of growth to support their arguments. It is important to note that ten of the studies addressing growth over time also demonstrated that growth is possible. Considering the importance of such results and the relatively small number of studies examining GSI growth, it is evident that more research is required to understand this phenomenon.

**Limitations**

While we made an effort to find as much of the literature on GSI professional development as possible, there are some limitations of our search that warrant discussion. We searched major research databases (ERIC, PsycINFO, Web Of Science) for peer reviewed articles as well as the RUME proceedings and *AMS Notices*; however, there are journals and proceedings from other conferences that are not included in these indices. We are also aware of other publications that are relevant, but they were included in books, which we did not include in our search. Another limitation is that we realized recently that we did not include the terms "post-secondary" and "postsecondary" in our search, so we may have missed some relevant articles.

**Recommendations for the Field**

Consistent with Speer, Gutmann, and Murphy (2005), we suggest that the field would be greatly enhanced by additional longitudinal studies exploring how GSIs grow as teachers. In addition, it would be beneficial to begin developing an accepted definition of GSI growth. We argue that part of this process is clearly articulating our stances as a research field on teaching quality and how they relate to models or theories of growth in teaching. It is striking that only one article took an explicit stance on both teaching quality and called on an existing model or theory of growth. We call for future research to take an explicit stance on teaching quality and how teaching quality is related to models and theories of GSI growth to better inform professional development for GSIs.

Finally, our findings suggest the need to develop explicit models or theories of growth in teaching that are linked to stances on teaching quality. There has been some progress on this (e.g., development of MKT by Thompson, Carlson, & Silverman, 2007), but more development is needed. We call for the research community to begin developing models of growth that will allow research on GSI professional development to grow into a richer body of literature.

## Annotated Bibliography[2]

Beisiegel, M. (2011). Obstacles to teacher education for future teachers of post-secondary mathematics. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education, 3*, (pp. 19-23), Portland, OR. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html

*In this qualitative study, the author sought to uncover issues and difficulties that impact mathematics graduate students' views of their role as an undergraduate instructor. After interviewing graduate students over a nine-month period and conducting hermeneutic study with thematic analysis, the author found that graduate students feel they need to replicate teaching practice and identity and result in feelings of resignation. Using Lave and Wenger's*

[2] Below we include summaries of the sixteen articles that were in our literature review. These summaries were shared with the lead authors, who were given the opportunity to suggest edits and validate our interpretation of their work.

*(1991) theory of legitimate peripheral participation in relation to communities of practice, the author argues that the attention to legitimate peripheral participation in a mathematics department prevented graduate students from adopting alternate modes of teaching.*

Belnap, J. K., & Allred, K. N. (2009). Mathematics teaching assistants: Their instructional and preparation opportunities. In L. B. Border (Ed.), *Studies in graduate and professional student development: Research on graduate students as teachers of undergraduate mathematics, 12*, 11-38.

Creswell, J. W. (2013). *Qualitative inquiry and research design: Choosing among five approaches*. Thousand Oaks, CA: SAGE Publications.

Deshler, J. M., Hauk, S., & Speer, N. (2015). Professional development in teaching for mathematics graduate students. *Notices of the AMS, 62*(6), 638-643.

*This article provides an overview of professional development efforts for mathematics GSIs in the US. The authors summarize different types of professional development models being used and provide suggestions for what it should include. In their model for professional development, they recommend that an initial "intensive" experience followed by sustained follow up sessions spaced out across time. The authors also highlight the importance of extending and adapting results from the K-12 literature on good instructional practices and teacher knowledge development to help inform research at the undergraduate level. In addition, they argue that developing pedagogical content knowledge is a key part of effective instruction as it helps GSIs to anticipate and use student thinking, attend to both computational and conceptual understanding, and orchestrate productive class discussions. Finally, the authors emphasize that in order to evaluate GSI professional development, it is important to pay attention to the changes in GSIs' knowledge and beliefs about teaching, learning, and doing mathematics.*

Duncan, A. (2016). Investigating a mathematics graduate students' construction of a hypothetical learning trajectory. In T. Fukawa-Connelly, N. E. Infante, M. Wawro, & S. Brown (Eds.), *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education,* (pp. 696-703), Pittsburg, PA. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html

*In this report Duncan investigates how a GSI's instructional planning decisions and mathematical conceptions about angles and angle measurement changed as she worked through a provided hypothetical learning trajectory (HLT), and then created one for her students. The researcher began by using the literature to generate a HLT involving angle, angle measurement, and radius. She then used an initial interview to probe into the GSI's mathematical conceptions of these topics and modify the HLT accordingly. In two additional interviews she had the GSI work through the HLT, and then in a final interview she had the GSI create her own HLT that could be used in a lesson on angle and angle measurement. Duncan found that the GSI's conceptions of angle and angle measurement evolved as she worked through the HLT, and that these informed her construction of an HLT for her own students. From these results Duncan highlights that in order to improve instruction teacher educators must take into account teachers' prior mathematical meanings. Moreover, she argues that this process of having a GSI work through an HLT and then create one themselves can be useful in helping GSIs make accommodations to their conceptions of math and use this to better inform their instructional goals.*

Firouzian, S. S. (2014). Graduate students' integrated mathematics and science knowledge for teaching. In T. Fukawa-Connelly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education*, (pp. 617-623), Denver, CO. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html

*In this report Firouzian examines how GSIs use their mathematical and scientific knowledge when teaching the concept of derivative, as well as applied derivative problems. Using the results from two task-based interviews, Firouzian describes how the calculus GSIs utilized "integrated knowledge" of math and science when completing the teaching tasks. The purpose of this study was to determine whether this knowledge fits within existing frameworks for teachers' mathematical knowledge; in particular Firouzian draws upon Ball, Thames, and Phelps' (2008) egg model when making this determination. He found that this framework "did not capture all the elements of the GSIs' mathematical knowledge for teaching," but rather that the graduate students also utilized their knowledge for teaching science in responding to the interview tasks. Firouzian concludes by proposing that professional education for math GSIs should include opportunities for students to improve their scientific and mathematical knowledge for teaching.*

Firouzian, S., & Speer, N. (2015). Integrated mathematics and science knowledge for teaching framework. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*, (pp. 524-536), Pittsburgh, PA. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html

*In this study Firouzian and Speer examined math GSIs' knowledge of student thinking when solving applied derivative problems. They interviewed eight math GSIs using a task-based protocol. In these interviews GSIs discussed difficulties students might encounter in a given applied derivative problem, as well as analyzed sample student work illustrating typical difficulties. Throughout the interview the GSIs were asked various questions meant to elucidate what type of knowledge they used in their responses. When analyzing these responses, Firouzian and Speer found that Ball, et al.'s (2008) MKT framework did not capture all of the knowledge for teaching that the math GSIs used. They instead offer preliminary theoretical constructs that might be useful to classify such knowledge, and call for more research into the nature of this knowledge. They conclude with a recommendation that GSI professional development programs include examples of student work on applied problems, and incorporate information on non-mathematical content knowledge needed to teach applications to science.*

Friedberg, S. (2005). Teaching mathematics graduate students how to teach. *Notices of the AMS, 52*(8), 842-847.

*In this article Friedberg introduces case studies as a tool for helping mathematics graduate students develop their teaching and communication skills. Working under the presumption that "the analysis of experience can contribute to good judgment" in teaching (p. 843), Friedberg argues that case studies provide an opportunity for math GSIs to improve their judgment in teaching-related issues. He and his colleagues developed 14 different case studies as part of the Boston College Mathematics Case Studies Project (published in the CBMS). These materials were then piloted at around 20 different post-secondary institutions. The goal of these materials was to provide GSIs with the opportunity to discuss teaching*

*issues with others, especially more experienced graduate students who could enrich the discussions by sharing their own experiences. The project received positive feedback from graduate students and faculty, and the materials have since been published. Friedberg concludes the article by reiterating how important it is for GSIs to improve their teaching, as the communication, listening, and group work skills required to teach effectively will be important in their future careers.*

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York, NY: Cambridge University Press.

Miles, M. A., Huberman, A. M, & Saldaña, J. (2014). *Qualitative data analysis: A methods sourcebook*. Thousand Oaks, CA: SAGE Publications.

Miller, N., & Wakefield, N. (2014). A mentoring program for inquiry-based teaching in a college geometry class. *International Journal of Education in Mathematics, Science and Technology, 2*, 266-272.

*This paper describes a mentoring process that took place over two consecutive semesters between an experienced teacher (an associate professor) and a novice GSI. During the first semester, the GSI observed an inquiry based learning (IBL) class that was taught by the professor, helped GSI for the class by holding office hours and grading papers, and taught a few lessons. During the second semester, the GSI was responsible for teaching the course while the professor attended the class and observed his teaching interactions. The professor and GSI would then meet weekly to discuss how the class was going. The experience changed how the GSI viewed teaching and learning and provided him with an opportunity to use IBL while being mentored by an experienced teacher. In addition to moving from lecture-based teaching to a more inquiry based teaching style, the GSI also changed the types of questions he used while helping students work. Overall, the mentoring experience helped the GSI develop as a teacher and influenced both the GSIs' teaching practices and beliefs about teaching and learning mathematics.*

Musgrave, S., & Carlson, M. P. (2016). Understanding and advancing graduate teaching assistants' mathematical knowledge for teaching. *The Journal for Mathematical Behavior, 45,* 137-149.

*This study investigated how mathematics GSIs conceptualize average rate of change before and after a professional development intervention. The authors found that before participating in the intervention, GSIs' focused on computational or geometric interpretations of average rate of change. After the intervention, GSIs were able to express a more conceptual understanding of average rate of change, suggesting that the intervention was somewhat effective in shifting GSIs' meanings for average rate of change. However, GSIs still varied in their ability to fluently express the meaning behind average rate of change, which led the authors to conclude that the impact of the intervention was not "uniform" across the GSIs.*

National Council of Teachers of Mathematics (NCTM). (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: NCTM.

Nepal, K. (2014). An exploration of mathematics graduate teaching assistants' teaching philosophies. In T. Fukawa-Connelly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics*

*Education,* (pp. 942-946), Denver, CO. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html

*This study examined teaching statements from four different GSIs during a semester-long professional development program. Three teaching statements from each GSI were analyzed (written at the beginning, middle, and end of the program). None of the GSIs were teaching during the semester in which they participated in this professional development program. A fundamental assumption of this study was that GSIs' beliefs are "developed, changed or reinforced as the learn more about teaching and learning, and these changes are reflected in their teaching philosophies" (p. 942). Preliminary findings showed that the four GSIs expressed varying beliefs about teaching and learning with some common themes appearing, such as having high expectations for students and encouraging them to work hard, having a positive attitude toward teaching, and relating mathematics to real life problems. Common themes expressed in the teaching philosophies of the two international GSIs were that teachers should treat their students equally and that teachers' content knowledge is key to their success. Common themes present in the teaching philosophies of the two domestic GSIs were that instructors should keep students engaged and motivate their students to think, learn, and succeed rather than just transferring their own knowledge to students. None of the GSIs changed their earlier opinions from their first teaching statements, but they did express additional opinions in their second and third teaching statements. In addition, "all the [GSIs] were influenced more than anything by the teaching they had experienced during their undergraduate or high school times, especially by the role model teachers they had" (p. 945).*

Nepal, K. (2015). An investigation of beginning mathematics graduate teaching assistants' teaching philosophies. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education,* (pp. 830-837), Pittsburgh, PA. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html

*This study examined both pre-service and in-service teaching philosophies from four different GSIs during a semester-long pre-service professional development class and their first three semesters of teaching. GSIs beliefs evolved from the pre-service phase to the in-service phase. The main factor that influenced their pre-service teaching philosophies were their past experiences as students, while the main factor influencing their in-service teaching philosophies was their experiences as teachers. During the pre-service phase, GSIs held simplistic views of teaching and wrote very little that was specific to mathematics teaching. They also mainly described the teaching behaviors of their past teachers rather than writing about their own beliefs and opinions. Overall, the teaching philosophies of the GSIs changed very little during the pre-service professional development program. During the in-service phase, a major change that was detected was that GSIs said they would like to become more authoritative to gain the respect of their students. In addition, several of the expressed beliefs from the pre-service phase carried over into the in-service phase.*

Raychaudhuri, D., & Hsu, E. (2012). A longitudinal study of mathematics graduate teaching assistants' beliefs about the nature of mathematics and their pedagogical approaches toward teaching mathematics. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics*

*Education,* (pp. 522-525), Portland, OR. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html

*This preliminary study explored how math GSIs' ontological and pedagogical beliefs regarding mathematics changed over the course of a year. Initial findings indicate that these two types of beliefs can conflict with one another. The authors plan on conducting further analysis to examine how this affects a GSI's instructional approach. Moreover, they found that GSIs' knowledge of students can be classified as behaviorist-teacher-centered, or constructivist-student centered. The authors hope to examine how a constructivist approach affects the GSI's teaching practice with regards to planning, performing, and assessing.*

Reinholz, D. L., Cox, M., & Croke, R. (2015). Supporting graduate student instructors in calculus. *International Journal for the Scholarship of Teaching and Learning, 9*, 1-8.

*This report presents how the instruction of two GSIs teaching changed over the course of a semester while using the Peer-Assisted-Reflection (PAR) method of teaching. The PAR method has students solve difficult problems, reflect on their work, provide feedback on their peers' work, and revise their own work using their peers' feedback. The GSIs facilitated this work in their recitations (there were 14 PAR assignments in one semester), as well as met with a working group six times in the semester to discuss instructional goals related to PAR: developing students' communication, collaboration, and persistence. This group focused on using student-centered strategies for instruction. In their analysis the authors focused on two GSIs: Beth (who changed her instruction from lecture-based to more student-centered) and Wong (who attempted to do so, but still retained a teacher-focused practice). Using interviews with the GSIs, the authors argue that this discrepancy may be due to differing GSIs' beliefs about the nature of learning (traditional view vs. reformed view) as well as prior teaching experiences (Wong had more teaching experience than Beth, and so it's possible that his teaching practice was more established than hers). The report concludes by stating that the PAR method can be useful in developing GSI's student centered teaching, and that even inexperienced GSIs can successfully implement PAR assignments in the classroom.*

Soto-Johnson, H., King, K., & Haley, C. (2010). Does mentoring a graduate student effect student achievement? *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education*, 1-16. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html

*The authors conducted a quantitative study in order to explore the relationship between the mentoring status of instructors and their students' performance. The study included one graduate student who was being mentored while teaching a course for preservice elementary teachers. In designing their study, the authors took the stance that having a teaching mentor should result in improved student achievement. To verify this hypothesis, they conducted a Chi-Squared test on the students' course grades as well as an ANOVA test on the students' final exam grades. While the Chi-Squared test did not reveal any differences, the ANOVA did indicate that there was a statistical significance difference in the final exam grades for students who had an instructor that was being mentored.*

Speer, N., & Firouzian, S. S. (2014). Current and future faculty members' mathematical knowledge for teaching calculus. In T. Fukawa-Connelly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate*

*Mathematics Education,* (pp. 1052-1057), Denver, CO. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html

*In this grounded theory study, the authors examine mathematicians' and graduate students' knowledge of student thinking about core calculus concepts. While they do not study the process of growth, they do take the stance that graduate students should be developing mathematical knowledge for teaching, and knowledge of students in particular. As a result of their analysis, they found that mathematicians were more able to identify known student difficulties and describe common student strategies.*

Speer, N., Gutmann, T., & Murphy, T. J. (2005). Mathematics teaching assistant preparation and development. *College Teaching*, *53*, 75-80.

*In an effort to foster collaborations between K-12 and undergraduate mathematics education, the authors review work related to GSI professional development and draw parallels to work in K-12 teacher education. The authors emphasize the importance of GSIs in undergraduate mathematics education, review current trends and developments in GSI professional development programs, draw connections to K-12 research, highlight research in progress on GSIs professional development, and provide suggestions for future directions of research. In particular, they draw attention to the need for longitudinal research on GSIs in order to "inform the design of exemplary programs that have lasting influence on instructional practices" (p. 79).*

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*, 415-432. doi: 10.1007/s10857-007-9054-8

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Yee, S., Rogers, K., & Sharghi, S. (2016). Graduate students' pedagogical changes using iterative lesson study. In T. Fukawa-Connelly, N. E. Infante, M. Wawro, & S. Brown (Eds.), *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education,* (pp. 1458-1466), Pittsburgh, PA. Retrieved from http://sigmaa.maa.org/rume/Site/Proceedings.html

*In this qualitative multiple case study, the authors examine the teaching practices of ten GSIs as they implemented an intensive iterative lesson study process across two universities. In particular, the authors identified mathematical teaching practices from the NCTM* Principles to Actions *(2014) used in the revision process of lesson study. The authors chose these practices, in particular, because they "provide a framework for strengthening the teaching and learning of mathematics" (p. 1461). As a result of their study, they found that there were mathematical teaching practices that GSIs both used consistently and revised throughout the study and that Lesson Study was an engaging form of professional development for GSIs.*

## Here's What You Do: Personalization and Ritual in College Students' Algebraic Discourse

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*We present results of a discourse analysis focused on college algebra students' uses of personal and impersonal language, references to endorsed mathematical routines, and inferences about mathematical objects in responses to a small-group problem-posing activity. We analyze students' responses with respect to selected dimensions of the arithmetical discourse profile of Ben-Yehuda et al., and provide evidence of a positive association between impersonal language and the presence of statements about mathematical objects and their relationships. We also study the relationship between the mathematical discourse of students and the discourse espoused by curricular resources used in the course.*

*Key words:* College Algebra, Discourse Analysis, Mathematical Routines

At many colleges and universities in the United States, college algebra courses act as gateways into science, technology, engineering, and mathematics (STEM) disciplines. Although students in non-STEM disciplines continue to take college algebra in large numbers, the broad consensus of the mathematical profession is that the primary function of college algebra is to prepare students for success in calculus courses (Herriott & Dunbar, 2009). Research in mathematics education has identified ways of thinking that are propitious for learning concepts and solving problems in calculus, such as attending to algebraic structure (Linchevski & Livneh, 1999; Hoch & Dreyfus, 2004) and reasoning covariationally about functions (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson & Carlson, 2017); college algebra presents some opportunities to develop these ways of thinking if they are addressed intentionally and skillfully. More broadly, some educators aligned with the reform movement in algebra education view algebra courses as responsible for developing "strategic problem solving and reasoning skills, symbol sense and flexibility, rather than procedural fluency" (Drijvers, Goddijn, & Kindt, 2011, p. 4). The report *Adding it up* (National Research Council, 2001) takes a more moderate stance, positing conceptual understanding, strategic competence, and procedural fluency as intertwined strands of mathematical proficiency that develop simultaneously and interdependently. However, there is broad agreement that students' work in college algebra should go beyond the development of rules for manipulating symbols in expressions, equations, and inequalities. Yet Kaput (1995) argues that the content of algebra courses "has evolved historically into the manipulation of strings of alphanumeric characters guided by various syntactical principles and conventions" (p. 71). Though Kaput's assessment is over two decades old and pertains to algebra at both the secondary and postsecondary levels, we hypothesize that it still bears some fidelity to the experience of students who take college algebra.

We are interested in investigating opportunities for students in college algebra courses to explore mathematical objects and their properties and relationships rather than simply performing scripted routines endorsed by a course text or instructor. In this study, we use elements of the *arithmetical discourse profile* of Ben-Yehuda, Lavy, Linchevski, and Sfard (2005) perform a discourse analysis of students' solutions to algebra problems, focusing on how and whether students use routines and how they describe properties of, and actions on, mathematical objects and mediators. We argue that students' uses of language – including both mathematical language and references to human actors and actions – give us insight into their understandings of algebraic objects and relationships. We then compare and contrast students' discourse with the discourse modeled by the course text and comment on how college algebra courses might create additional space for development of conceptual understanding and flexible problem-solving skills.

## Theoretical Perspective and Related Research

In this study, we take a commognitive perspective (Sfard, 2007) in which we regard thinking as a form of communication, and learning as the modification and expansion of one's discourse. Wittgenstein (1953) describes communication as a game in which participants exchange ideas according to certain rules and norms; a *community of discourse* forms when a group of people exchanges ideas under a common set of rules and norms, even if not all members of the group communicate directly with one another. In addition to using words, participants in a community of discourse use *visual mediators*, such as symbols, formulas, and graphs, to identify the objects of discourse and facilitate communication.

Prior research has offered several complementary arguments for the importance of attention to language in classroom instruction in mathematics and science. Chapin, O'Connor, O'Connor, and Anderson (2009) showed that classroom discussions of mathematical concepts and procedures can enhance student learning and provide windows into students' thinking for teachers. Studies in cognitive science have shown that encouraging students to engage in self-explanation while reading examples in mathematics and science texts can enhance the development of problem-solving skills and conceptual understanding (Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Chi, De Leeuw, Chiu, & LaVancher, 1994). Thier (2002) argues that learning of science and development of literacy are interconnected; Fellows (1994) suggests that the process of writing "may force integration of new ideas and relationships with prior knowledge."

Ben-Yehuda, Lavy, Linchevski, and Sfard (2005) developed the *arithmetical discourse profile* as an analytical tool for identifying features of students' mathematical discourse that may facilitate or inhibit their access to rich mathematical thinking. Ben-Yehuda and colleagues divide discourse into a *subject dimension* in which communication focuses primarily on the author of the discourse and an *object dimension* in which communication refers to an external object, as constructed by the speaker. Within the object dimension, the authors analyze learners' word use, use of mediators, use of routines, and inclusion and production of *endorsed narratives*, mathematical facts (as understood or constructed by the speaker) that may be used in exploration. In analyzing word use, the authors consider the extent to which learners engage in *objectifying talk*, mentally constructing intangible, external objects for which mediators (such as numerals, expressions, and formulas) serve as visible "avatars." The authors also analyze the degree of *personalization* in learners' mathematical talk; that is, the extent to which learners' mathematical statements invoke human actors and actions on signifiers. As support for the relevance of personalization as a feature of mathematical talk, the authors cite a large-scale study by Bills (2002) suggesting a negative correlation between mathematics achievement and the use of personal identifiers and past-tense verbs to describe operations.

Subsequent research on students' mathematical discourse has highlighted other salient aspects of the language with which students express mathematical ideas. Sfard (2016) marks a distinction between *ritualized* and *explorative* mathematical discourse. Ritualized discourse has as its primary goal the satisfaction of an external need; this type of discourse typically engages in rote application of routines endorsed by an external source, and utilizes signifiers and mediators without attention to the mathematical objects to which they refer. Explorative discourse, on the other hand, serves the primary goal of knowing more about mathematical objects; participants in this discourse use routines and narratives flexibly and inquire about the source of their endorsement, and engage in *objectifying* talk, referring directly to mathematical objects and their properties. Sfard points out that learners assimilate themselves into these discourses largely through reflective imitation, observing the talk and actions of experts and then studying what aspects of these actions are adaptable to other situations and what aspects must be changed. In an analysis of student discourse about rational functions and their asymptotes, Mpofu and Pournara (2017) suggest that a lack of reflective imitation on the part of learners may lead to incomplete or flawed understandings of mathematical concepts. They observe that while the high school students in their study are able to produce graphs for rational functions correctly, their narratives about these mediators are mostly visual and dependent upon memorized routines, suggesting a ritualized understanding of asymptotes.

Like the authors of prior studies on students' mathematical discourse, we hypothesize that ritualized mathematical discourse – in student talk, in teacher talk, and in curricular materials – may bar students' access to mathematical exploration and understanding. In this study, we analyze specific features of the discourse of college algebra students in response to a problem-posing activity, the discourse of course instructors and curriculum authors in materials used in the class, and similarities and differences between the two. In particular, we investigate the following questions:

1. How do college algebra students use routines endorsed by their textbook or course notes in explaining solutions to problems that they have generated?
2. To what degree do college algebra students' explanations of solutions engage in personalized discourse, referring to human actions on mediators for mathematical objects, and to what degree do they make impersonal statements about mathematical objects and their properties and relationships?
3. What influences might the course text and notes have on students' mathematical discourse?

## Method of Study

### Study Setting and Population

This study was conducted in three large-enrollment sections (with 102, 115, and 114 students, respectively) of a college algebra course for STEM majors at a public university in the south central United States. At this university, most freshmen in science and engineering majors must take the college algebra course as a prerequisite for precalculus and most entry-level chemistry and physics courses. At the time of this study, the college algebra course was the target of a reform effort to shift instruction toward an emporium model (though with some elements of a large-lecture course still in use) and improve student retention and success rates. Students in the course attended class for three hours per week; prior to each class session they were expected to complete a set of "guided notes" in which they copied definitions and procedures and completed example problems from the course text (Abramson et al., 2012). Both the course text and guided notes frequently provided endorsed routines consisting of numbered lists of steps that prescribed how students should approach certain classes of problems.

Even though class sessions took place in a large lecture hall, sessions focused heavily on small-group activities in which students applied procedures to solve problems. For our study, we used a small-group activity because students were accustomed to small-group work and because we anticipated that collaborative work on the activity might lead to greater elaboration in students' solutions.

**Data Collection**

For this study, students in the three sections of college algebra participated in a small-group activity in which each group wrote a free-response algebra problem, including a solution. Students organized themselves into groups, usually based on their location within the lecture hall; groups typically consisted of two to five students. After groups formed, the authors distributed one copy of the Question Posing Activity (Figure 1) to each group. The Question Posing Activity asked each group to identify a mathematics concept covered in class since the first midterm exam and write a question on that concept that could be used as a test question or to review for the upcoming second midterm.[1] Each group was also asked to write a complete solution to that problem that could be understood by other students. Most groups appointed one student to record the group's ideas on the activity sheet. The researchers and college algebra instructors did not intervene in students' work on the activity; if a group asked for assistance or guidance, the researchers and instructors simply reiterated the instructions for the activity.

**Question Posing Activity**

**Names of Students in your Group:**

**Mathematics Concept:**

**Please write a question on this concept that could be used as a question on the upcoming test, or a question that could be used to study for the test:**

**Write a solution to your question using sentences, formulas, and key words that can easily be understood by fellow students. Please write your finished solution here; we've left some room on the back of this page for you to work out your problem and plan what you want to say.**

*Figure 1.* The Question Posing Activity given to study participants.

[1] Students were assured prior to the activity that the questions they wrote would not in fact be used on exams; our purpose in framing the activity in this way was to encourage students to write questions that would have well-defined mathematical solutions.

In posing the activity, we allowed students latitude to select from a variety of algebraic topics and problem types. We designed the activity to be open-ended so that students could select topics and problems that were comfortably within the scope of their understanding and on which they could produce elaborate and clearly written solutions. We anticipated that while responses would cover a variety of topics, the richness of the solutions would afford us greater opportunity to study how students write about mathematical problems that they understand relatively well. Students were allowed to use their notes during the Question Posing Activity, including the guided notes they had completed; however, they were permitted to write questions in their own words and approach problems in ways not specifically endorsed by the notes if they wished.

Over three sections, a total of 72 groups submitted responses to the Question Posing Activity, which we scanned and analyzed. We categorized a response as posing a well-defined algebraic problem if question prompted the reader to answer a mathematical question having a specific solution that could be found using ideas taught in the college algebra course prior to the study. The question itself did not have to follow specific guidelines in sentence structure or phrasing to be considered a question. Of the 72 responses, 70 posed well-defined algebraic problems; of the other two, one asked a question about a general strategy for a broad class of problems ("How do you find rational zeros of a polynomial?") and one wrote a formula for a cubic function but did not ask a question about the function. We limited our analysis to the other 70 responses.

**Data Analysis**

In analyzing the 70 responses that posed well-defined problems, we wished to characterize the mathematical topics addressed by the problems, the ways in which students employed routines from the course text and guided notes in their solutions, and the extent to which students used personal or impersonal discourse in their explanation of the solutions. We developed a three-part coding scheme to address these dimensions of each response.

In order to categorize the questions posed by mathematical topic, we used an open coding scheme to develop an ordered list of topics (Table 1) addressed in the questions that students posed:

Table 1
*Hierarchical list of question topics*

| Abbreviation | Topic |
|---|---|
| **LF** | Linear functions and equations |
| **ARoC** | (Average) Rate of change |
| **QF** | Quadratic functions and equations |
| **FT** | Factor theorem and zeroes of polynomials |
| **FP** | Factoring polynomials |
| **PD** | Polynomial division |
| **PM** | Polynomial multiplication |
| **GP** | Graphs and intercepts of polynomials |
| **O** | Other |

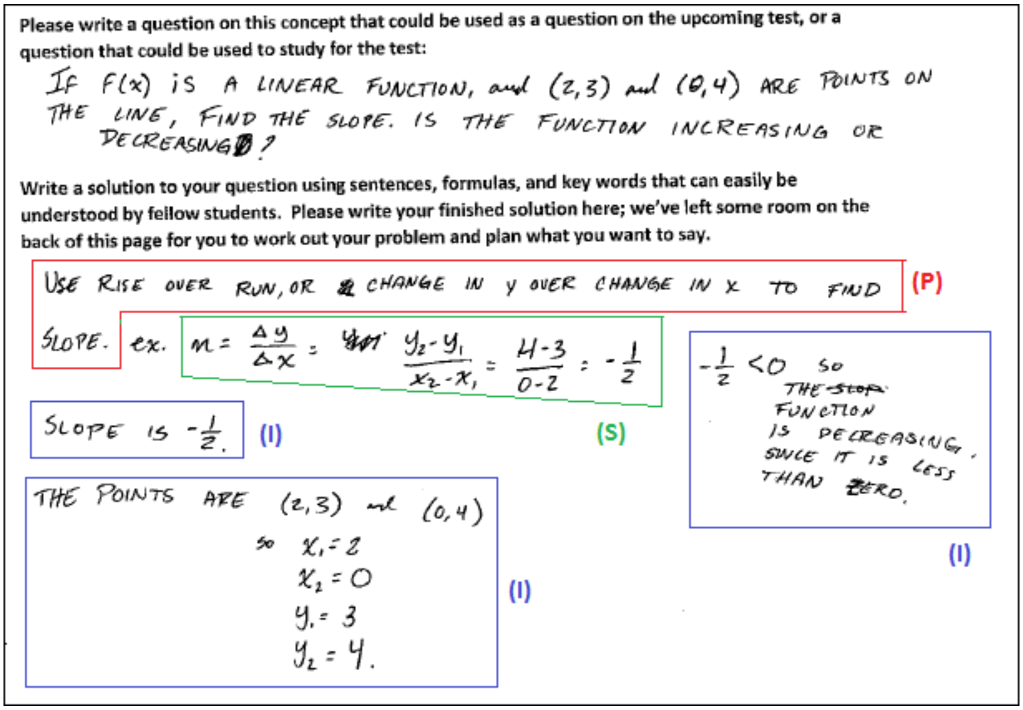
If a question addressed more than one topic in the table, we assigned it the first topic in the table that was relevant to the problem posed. Our rationale for the hierarchical listing of topics was that some topics encompassed others; for example, questions that addressed the factor theorem (FT) tended to address factoring polynomials (FP) as well. However, some questions addressed broader factoring techniques without using the factor theorem; we generally categorized these as FP. We created a separate category for quadratic functions and equations (QF) separate from the category on factoring polynomials; we assigned this topic higher priority because we observed, based on the course text, that factoring of quadratic functions was taken up separately from factoring of general polynomials and at an earlier point in the course.

We also categorized each solution according to its use of a routine from the guided notes or course text and how the routine was presented; we thus coded each response along two dimensions. Each solution was first assigned a code for *routine format*, based on whether the routine used to solve the problem was not explicitly described in the solution (N), was presented as a numbered list or bullet list of discrete steps (L), or was described in paragraph form (P). If a solution did not appear to make clear use of any endorsed routine from the course text or notes, we coded the routine format as (X). For coding reliability, we used symbolic markers to distinguish lists of discrete steps from paragraph descriptions of routines; if a solution presented a routine as a sequence of plain-language statements describing steps without numbering or bulleting the steps, we coded the routine format as (P) regardless of the flow of the steps.

Each solution that used a routine was also assigned a code for *routine fidelity*, based on whether the solution used or described an endorsed routine exactly as presented in the textbook and notes (1), used the same steps as the endorsed routine but reworded some steps (2), consolidated, split, or reordered steps in an endorsed routine but did not introduce any steps not in the endorsed routine (3), or introduced at least one step not in the course text's endorsed routine (4). We thus assigned a composite code, consisting of a letter and a digit, to each response that used a routine from the textbook, based on both routine format and routine fidelity.

Finally, we assigned a *personalization* code to every statement in each written solution. For this analysis, we coded a sentence or phrase in a solution as a *statement* if it expressed a complete thought and served the purpose of explaining or describing a step of the solution. We coded a statement as personal (P) if it invoked a human actor, whether explicit or implied through a description of an action on a signifier, and impersonal (I) if it did not invoke a human actor. We separately coded symbolic (S) statements (equations and inequalities) that occurred "in the flow" of an explanation of a solution that were not part of the scratchwork explained by the written statements. For example, Response 013G (Figure 2) contains one statement that instructs the reader to use a specific formula (P), an in-the-flow symbolic calculation determining the slope of a line (S), a statement of the slope of the line (I), a statement connecting the line's slope to the function's decreasing behavior (I), and a statement explaining how two given points are used in the slope formula (I). We coded this response as (PSIII); because some responses did not readily lend themselves to a linear ordering of statements, we did not perform any analysis based on the ordering of personal, impersonal, and symbolic statements within a solution. Perhaps more notably, we did not conduct any analysis of whether the language in students' statements was mathematically correct or precise; for example, our analysis did not take into account that Response 013G claimed that "the function is decreasing, since it is less than zero," and did not clearly indicate to what object the pronoun "it" referred. Our only aim in this part of the coding process was to capture the extent to which participants' writing referred to human actors or actions on mediators.

When assigning topic, routine, and personalization codes, each author coded each response independently; disagreements were resolved through discussion, which led to iterative refinements of each coding scheme. After these refinements, inter-rater agreement on topic codes, routine codes, and personalization codes reached 100%, 76%, and 88%, respectively. The primary obstacle to inter-rater reliability on assignment of routine codes was the process of deciding whether a step in a student explanation of a routine that differed from a step in the course text's endorsed routine constituted a new step not in the endorsed routine, a consolidation of two steps in the endorsed routine, or a rewording of a step in the endorsed routine.


Please write a question on this concept that could be used as a question on the upcoming test, or a question that could be used to study for the test:

IF F(x) IS A LINEAR FUNCTION, and (2,3) and (0,4) ARE POINTS ON THE LINE, FIND THE SLOPE. IS THE FUNCTION INCREASING OR DECREASING?

Write a solution to your question using sentences, formulas, and key words that can easily be understood by fellow students. Please write your finished solution here; we've left some room on the back of this page for you to work out your problem and plan what you want to say.

USE RISE OVER RUN, OR CHANGE IN y OVER CHANGE IN x TO FIND SLOPE. (P)

ex. $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-3}{0-2} = -\frac{1}{2}$ (S)

SLOPE IS $-\frac{1}{2}$. (I)

THE POINTS ARE (2,3) and (0,4) so $x_1 = 2$, $x_2 = 0$, $y_1 = 3$, $y_2 = 4$. (I)

$-\frac{1}{2} < 0$ so ~~THE SLOPE~~ FUNCTION IS DECREASING, SINCE IT IS LESS THAN ZERO. (I)

*Figure 2.* Personal, impersonal, and symbolic statements in Response 013G.

In categorizing responses by topic, we anticipated that some categories of responses would be more likely to contain impersonal discourse than others. For example, we hypothesized that questions dealing primarily with synthetic division would typically have a higher concentration of personalized discourse, since synthetic division is a procedure that distills the work done in certain cases of polynomial long division into actions on mediators that can be performed rapidly; that is, the goal of synthetic division is to permit the user to compute a quotient and remainder without actively attending to algebraic relationships among the polynomials being divided. On the other hand, we anticipated that responses having to do with the factor theorem might contain more impersonal statements, since in using the factor theorem one makes a connection between a polynomial's zeroes and its factors, and this connection can readily be described in terms of algebraic objects.

## Results

### Students' uses of routines

Responses to the Question Posing Activity addressed a variety of algebraic topics; the topics that occurred the most frequently were polynomial division (PD, 40 responses) and the factor theorem (FT, 10 responses). In discussion with the instructors of the college algebra classes we studied, we determined that students had most recently learned how to use synthetic division to find factors of polynomials; this may account for the high frequency of these topics.

Composite codes for descriptions of routines, along with the frequency of each code, are given in Table 2. Of responses that provided written narratives of routines in their solutions, remarkably few (4 out of 49) described routines exactly as they appeared in the course text, with or without minor rewording. An excerpt of such a response on synthetic division is shown in Figure 3; the textbook's endorsed routine for synthetic division is shown in Figure 4 for comparison. Most responses to the activity either consolidated multiple steps into a single step, split one step into multiple steps, omitted a step, or introduced a step that was not in the endorsed routine provided by the textbook.

Table 2
*Codes for Descriptions of Routines in Solutions*

| Routine Format | Routine Fidelity | | | |
|---|---|---|---|---|
| No clear usage of routine: X (4) | Same as endorsed routine in course text | Rewording of endorsed routine | Consolidates, splits, or omits steps | Includes a step not in endorsed routine |
| Routine not explicitly stated | N1 (13) | N2 (0) | N3 (1) | N4 (3) |
| Number or bullet list | L1 (2) | L2 (1) | L3 (10) | L4 (14) |
| Paragraph form | P1 (0) | P2 (1) | P3 (5) | P4 (16) |

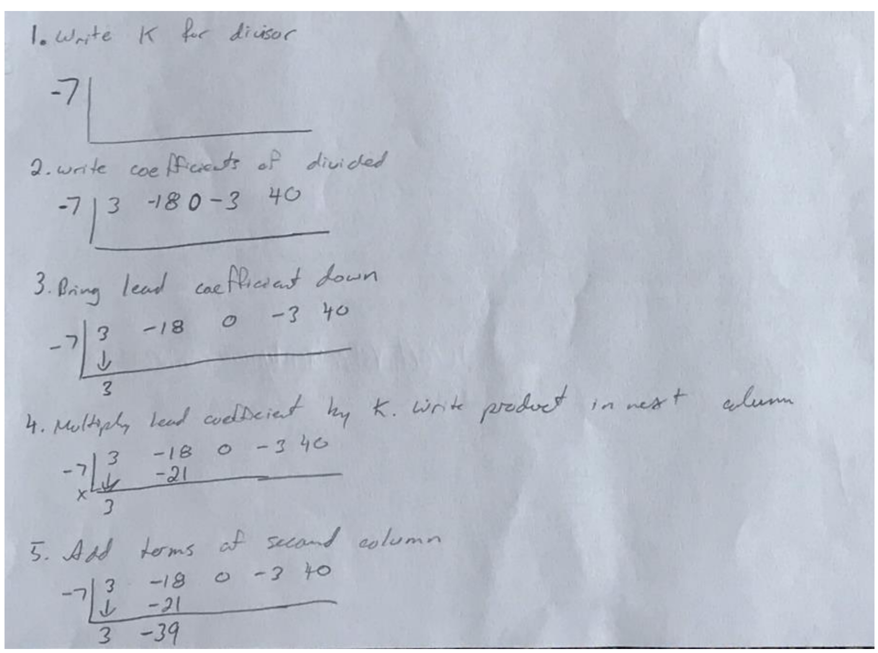



*Figure 3.* Response 009Y, a solution to a synthetic division problem.

*How To...*

Given two polynomials, use synthetic division to divide.

1. Write $k$ for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by $k$. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by $k$. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

From Abramson et al., 2017, p. 397. Used under Creative Commons License. Download for free at https://openstax.org/details/books/college-algebra.

*Figure 4.* The textbook-endorsed routine for synthetic division.

Consistent with Sfard's view of learning as the expansion of learners' discourse to include elements of expert discourse (2016), we frame instances in which a routine is not reproduced with perfect fidelity as possible instances of reflective imitation; that is, instances in which a group of students might decide that a routine as presented in a course text is insufficient or inefficient, and make adaptations to a routine appropriate for a given problem-solving scenario. To investigate this possibility, we performed further analysis on the 16 responses on polynomial division with routine codes of L4 or P4 to determine what additional steps or insights the students added beyond those included in the endorsed routines. We compared each response to the "How To" instructions on synthetic division found in the course text (Figure 4). We found that these 16 responses typically went beyond the textbook-endorsed routine in two ways:

1. Some responses added description of when or how to start the procedure of synthetic division; some described applicability conditions (Ben-Yehuda et al., 2005, p. 203) for synthetic division, and many focused on how to find the constant $k$ that corresponds to the linear divisor.
2. Some responses elaborated on the mathematical significance of the remainder obtained in the synthetic division procedure, stating conditions under which the divisor is a factor of the dividend, or conditions under which the value $k$ in the divisor $(x - k)$ is a zero of the dividend.

As illustrated in Table 3, which shows all polynomial division solutions from one section of the course that were coded L4 or P4, the syntax and wording of these additional contributions were varied; for this reason, we suspect that while students likely had these additional insights as a result of their interaction with the course text or lectures, these contributions were not scripted. As the students' responses suggest, while most students incorporated endorsed routines from the course text into their explanations of solutions, many also took up opportunities to elaborate on the routines by noting key features of problems they created or implications of the outputs of routines they used.

Table 3
*Students' Enhancements of the Endorsed Routine for Synthetic Division*

| Response | Beginning of Procedure | Interpretation of Results of Procedure |
|---|---|---|
| 013D | "Set ($x + 2$) equal to zero to see what you are dividing by. ($x + 2$) set equal to zero is -2." | "set remainder over dividend ($x + 2$)" |
| 013I | "($x + 3$) $\rightarrow x = -3$"<br>"Because our divisor is in ($x - k$) form, where $k$ is a real number, we are able to use synthetic division as a shortcut to divide polynomials." | |
| 013K | | "Put remainder over $x - k$ and simplify." |
| 013L | "First solve for $x$ for the factor ($3x - 1$), which = 1/3" | "If not place remainder over factor ($3x - 1$)" |
| 013M | | "If the last sum is zero, the divisor is a factor. If it is non-zero, it is not." |
| 013O | "It is -7 because $x + 7 = 0 \rightarrow x = -7$" | "If the remainder is zero, then the equation is a factor of the divisor, in this case our remainder was 1090, so it is not a factor of ($x + 7$)" |
| 013Q | "First off, find a number that makes $x + 2 = 0$ (which is -2)." | "Put the bottom numbers into an equation and the last number (if it $\neq 0$) over the root ($x + 2$)" |
| 013V | | "Yes, $x - 2$ is a factor of $2x^4 - 3x^3 - 15x^2 + 32x - 12$ because the remainder is zero."<br>"If there is no remainder, $k$ is a factor. If there is a remainder, $k$ is not a factor." |

**Personalization in students' word use**

In reviewing personalization codes for the 60 solutions that contained at least one verbal statement (coded (P) or (I)), we found that the majority of students' statements were personal (250 personal statements; 38 impersonal statements). For each of these 60 responses, we tallied the number of impersonal statements in the response as well as the percentage of statements that were impersonal. Results of this analysis are shown in Table 4.

Table 4
*Number and Relative Frequency of Impersonal Statements in Verbal Responses*

| Number of impersonal statements | Number of responses | Relative frequency of impersonal statements | Number of responses |
|---|---|---|---|
| 0 | 36 | 0% | 36 |
| 1 | 16 | (0%, 25%] | 9 |
| 2 | 6 | (25%, 75%) | 7 |
| 3 | 4 | [75%, 100%) | 2 |
| More than 3 | 0 | 100% | 6 |

Of the 60 solutions that contained verbal statements, 45 contained at most 25% impersonal statements. In our initial analysis of the data, we anticipated that responses containing a high volume of impersonal statements would be likely to contain objectifying talk; that is, statements about algebraic objects and their properties (as opposed to talk about actions on mediators, such as "bring lead coefficient down" or "Create an 'L' shape and add and multiply the answer given by -2"). To test this hypothesis, we performed an additional analysis on the 38 impersonal statements in the solutions submitted to determine whether each statement, in our judgment, was primarily intended to describe properties of or relationships among mathematical objects (objectifying talk), or to describe the role of an object or mediator in a routine performed by a human (operational talk; Ben-Yehuda et al., 2005, p. 198). Of the 38 statements analyzed, we categorized 25 as objectifying talk and 13 as operational talk; some examples from responses dealing with polynomial division are presented in Table 5.

Table 5
*Examples of Objectifying Talk and Operational Talk in Impersonal Statements*

| Objectifying Talk | Operational Talk |
|---|---|
| "The solution will be $x - 2 - 4/(6x + 2)$, which also proves that $6x + 2$ is not a factor of $6x^2 - 10x - 8$." (009N)<br>"These 3 numbers represent zeros of the polynomial in the problem." (012J)<br>"If the remainder is zero then the $x$ value is also a zero." (012K) | "$k$ is the divisor $x - 6$" (009E)<br>"last number is the remainder over factor" (009R)<br>"$x + 4 = x - k$ which means $k = -4$." (012G)<br>"It is -7 because $x + 7 = 0 \rightarrow x = -7$" (013O) |

As the examples in Table 5 illustrate, students who posed and solved problems on polynomial division frequently made objectifying comments connecting the remainder obtained from the division algorithm to the question of whether the divisor is a factor of the dividend, and connecting factors of polynomials to zeroes of polynomials. On the other hand, some of the impersonal statements they wrote appeared to have the purpose of clarifying the role of the zero of the linear divisor ($x - k$) in the synthetic division routine; we classified these as operational talk since their primary purpose seemed to be to justify the use of a mediator in the routine.

To test our hypothesis that problems on the factor theorem would call for more deductive reasoning and therefore engender greater degrees of objectifying talk, we analyzed the 10 responses that addressed the topic of the factor theorem for their uses of personal and impersonal language in explanations of solutions. Of these 10 responses, only two contained at least one impersonal statement. We observed that in most of the responses on the factor theorem, participants solved a problem that might call for attending to mathematical relationships (such as that between factors and zeroes) and deductive reasoning (for example, ruling out possible rational roots of a polynomial) in a way that suggested the use of a routine, and their description of this routine consisted entirely of personal statements. For example, Response 012D (Figure 5) uses an endorsed routine from the course text to find the zeroes of the polynomial $f(x) = x^4 + x^3 - 13x^2 - x + 12$. This response makes a connection not visible in the endorsed routine (Figure 6); namely, that the purpose of using synthetic division is to find factors that can be used to decompose the original polynomial. On the other hand, it misses some subtleties in this routine; for example, that a fourth-degree polynomial may not split (over the rationals) into linear and quadratic factors. We hypothesize that because the course text's instructions merely suggest that students continue until the quotient is a quadratic "if possible," the object-based reasons for this potential complication are not explored. Thus while the students' response contains some insight not found in the endorsed routine, it largely reflects the operational and personal discourse presented in the course text, and misses a potentially important mathematical understanding as a result.

zeros: -1, 1, -4, 3

Steps:

① Use rational zero theorm to determine possible zeros.

② Pick from list of $\frac{p}{q}$ values and synthetically divide to find factors.

③ Repeat step 2 until you are left with a quadratic function

④ Factor quadratic or use quadratic function

⑤ Set all factors equal to zero and solve.

*Figure 5.* Response 012D describes how to factor a fourth-degree polynomial.

*How To...*

Given a polynomial function *f*, use synthetic division to find its zeros.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.
4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.



*Figure 6.* The textbook-endorsed routine used in Response 012D.

## Discussion

In our analysis of student responses to the Question Posing Activity, we found that students frequently used endorsed routines from the course text and notes as templates for their solutions, even though they often made adaptations to the routines based on the specifics of the problems they were solving. We also observed that personalized mathematical discourse was prevalent in most of the students' solutions; in most of the responses, the majority of written statements contained references to human actors or human actions on mediators. A closer analysis of impersonal statements in students' solutions revealed that the majority of these were mathematical statements about algebraic objects (such as functions, equations, and solutions) and their properties. We found that solutions that avoided impersonal talk often referred to mathematical objects in ways that would seem vague to a third-party reader (for example, referring to mediators without clarifying the objects they represent), and sometimes obscured algebraic ideas that might have led to deeper understanding of the problems they addressed.

While recognizing the limitations of our study, which was situated in a specific instructional context in which student discourse was influenced by a common set of curricular resources and likely by specific teaching practices that were not fully visible to us, we suggest that a prevalence of personalized discourse at the expense of impersonal and objectifying talk may inhibit students' access to certain worthwhile mathematical ideas. We do not mean to suggest that personal statements in a written solution to a problem are indicative of an impoverished understanding of the underlying mathematics; indeed, we expect that virtually any explanation of an algorithm such as synthetic division would contain some description of actions on mediators, since the effective use of mediators is key to the efficiency of a written algorithm. Rather, we wish to suggest that mathematical talk can be enriched with substantive discussion of the mathematical objects to which mediators refer, and that impersonal statements can focus attention on these objects.

In observing students' responses to the activity, we also found that the mathematical discourse of students often resembled the discourse exemplified by the course text and guided notes, though with some modifications. On the one hand, we found that students often made adjustments to the textbook's endorsed routines that added meaning and context; for example, many student solutions to polynomial division problems went further in analyzing the output of the division algorithm and its relation to factors and roots of polynomials than the textbook's description of the routine did. On the other, students frequently incorporated all of the steps in the textbook's endorsed routine into a solution (while possibly consolidating or splitting some steps or providing additional detail), and in doing so they rarely described mathematical objects with any more clarity than the book's explanation contained. This suggests that in a course in which students are required to interact regularly with an assigned text, the text serves as an important influence in the development of students' mathematical discourse. In our commognitive perspective, in which the expansion of mathematical understanding is the expansion of discourse, the discursive norms and practices of a textbook set standards for students' understanding and for the epistemic approach students should take in assimilating new knowledge. While instructors can supplement textbooks with additional explanation, more explorative activities, or richer assessments, a textbook presents a permanent physical artifact of ways in which students are expected to understand the mathematics they know. A textbook that promotes explorative discourse can serve as a resource that helps acquaint students with the objects of the mathematical activity in a course and invites students to participate actively in the development of mathematical ideas. If students have opportunities to engage in mathematical exploration in which they have access to the objects of study and motivation to articulate relationships among these objects, they can develop ways of thinking and explaining that can be distilled into personalized and ritualized mathematical talk – but of whose rituals they are the developers and owners.

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# Assessing the Development of Students' Mathematical Competencies: An Information Entropy Approach

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*We report on the use of a modification of the Danish KOM framework of mathematical competencies for monitoring the development of mathematical competencies of freshmen biology students at a Norwegian university. Preliminary analysis of the data indicated the need for the adjustment of the original framework and development of the appropriate scaling for competencies' intensity which may differ significantly even within one session. We retain only five out of eight groups of competencies in the KOM framework and suggest the scaling scheme for the strength of each of the sixteen competencies in the five groups. In addition, three different dimensions for each competency are introduced and scaled. After all the data have been processed, the need for the development of techniques for processing large data sets arose. We started by converting a large corpus of qualitative data into quantitative data and used recent developments in the information theory to initiate the discussion regarding monitoring of the development of students' mathematical competencies.*

*Keywords:* mathematical modeling, competencies, quantitative data, scaling and evaluation tools, information entropy.

## Introduction

What exactly do we mean by saying "he/she knows" mathematics? One may attempt to answer this question by viewing mathematical knowledge as a complex combination of a number of complementary mathematical competencies. An individual possessing all (or some) competencies at certain levels could be considered as the one possessing some mathematical knowledge. For this purpose, we need: (i) to define what is meant by a mathematical competency; and (ii) to explain how these competencies can be meaningfully assessed.

A mathematical competency should be clearly recognizable and distinct as one of the constituents of the overall mathematical competence. Different approaches to this operationalization have been proposed in the literature by Boesen et al, 2014, Maaß 2006, Niss, 1999, Weinert, 2001 and others. We follow Niss (1999) who suggested the framing of a set of mathematical competencies within the KOM-project (KOM: Competencies and the Learning of Mathematics), initiated by the Ministry of Education and other official bodies for reforming Danish mathematics education. The main objective in the KOM project was to replace traditionally expressed lists of topics, concepts and results in mathematics curricula with a list of mathematical competencies students should possess. The eight groups of competencies suggested in Niss (1999) are: (1) thinking mathematically; (2) posing and solving mathematical problems; (3) modelling mathematically; (4) reasoning mathematically; (5) representing mathematical entities; (6) handling mathematical symbols and formalisms; (7) communicating in, with, and about mathematics; (8) making use of aids and tools (IT included). The twenty four competencies in these eight groups are further classified with regard to abilities "to ask and answer questions in and with mathematics" (1-4) and "to deal with and manage mathematical language and tools" (5-8).

The purpose of this paper is to address the following questions. (1) How efficiently can the KOM framework of mathematical competencies be used for monitoring and assessing the development of mathematical competencies of freshmen biology students? (2) How can information entropy be used to facilitate the process of big data analysis?

## Research setting

This research took place in a department of biology at a Norwegian university. Two Norwegian centres for excellence in higher education, the Centre for Research, Innovation and Coordination of Mathematics Teaching (MatRIC) and the Centre for Excellence in Biology Education (bioCEED), developed a joint project aimed at improving biology students' motivation for, interest in, and perceived relevance of mathematics in biological studies using mathematical modeling activities. The project addresses serious concerns regarding mathematical education of future biologists raised recently in the literature. As mentioned by Bialek & Botstein (2004), "The fragmented teaching of science in our universities still leaves biology outside the quantitative and mathematical culture that has come to define the physical sciences and engineering. Even though most biology students take several years of prerequisite courses in mathematics and physical sciences, these students have too little education and experience in quantitative thinking and computation to prepare them to participate in the new world of quantitative biology." A similar concern has been also expressed by Gross, Brent, & Hoy (2004) who pointed out that "the need for basic mathematical and computer science (CS) literacy among biologists has never been greater."

The teaching was planned and performed by MatRIC researchers, whereas bioCEED took care of all organizational aspects and helped with the recruitment of biology students. MatRIC researchers suggested to use biologically meaningful modelling tasks for demonstrating possible uses of mathematics in life sciences and encouraging better engagement of students in learning mathematics. To this end, the first author prepared teaching materials for eight complementary sessions and conducted teaching once a week over a period of two months in addition to the regular lectures and seminars in their standard mathematics freshmen course for natural sciences students.

*Table 1. Task example 1*

| Adam & Eve |
|---|
| How long it takes for a pair of individuals (one may think, for instance, of Adam and Eve) to produce the world population of today (about 6 billion people) at the present rate of growth $r = 2\%$ per year? |

The topics discussed in the eight sessions were: periodic functions (2 sessions), exponential growth and regression (2 sessions), population dynamics (2 sessions), integrals and modeling (2 sessions). Usually students were given two tasks, the first to "warm up" and the second, main task, to see how well they understood main ideas and techniques. Examples of two tasks from one of the sessions (the first session on population dynamics) are provided in Tables 1 and 2.

*Table 2. Task example 2*

| The Andromeda Strain |
| --- |
| Uncontrolled geometric growth of the bacteria *Escherichia coli (E. Coli)* is the theme of the best-selling Michael Crichton's science fiction thriller *The Andromeda Strain*. In a single day, one cell of *E. Coli* could produce a super-colony equal in size and weight to the entire planet Earth. If a single cell of the bacterium *E. Coli* divides every 20 minutes, how many *E. Coli* would be there in 24 hours? The mass of an *E. Coli* bacterium is $1.7 \times 10^{-12} g$, while the mass of the Earth is $6.0 \times 10^{27} g$. Is Crichton's claim accurate? If not, how much time should be allowed for this statement to be correct? |

In addition to our attempt to engage biology students into more active learning of mathematics through the work on modeling tasks with biological content, the goals of this intervention were to create mathematical competencies profiles for individual learners and to follow their development from session to session and in the long run.

## Data collection

Data collection methods included video recordings of participants, researcher's observation/field notes and students' written material obtained using *Livescribe 3* smart pens and notebooks. One group in each classroom was the "focus group" with a camera recording students' work using a vertical view; this enabled the capture of all written actions. A *GoPro* camera with a panoramic view facing the whiteboard and the groups was used to record all interactions in the classroom, including teacher's involvement and groups' activities. All audio recordings were obtained from the two cameras and then transcribed. Some studies (e.g. Spradley, 1980) criticized this method for researcher's involvement and intrusiveness due to the camera's and his/her presence during the data collection. However, since the researcher was also the teacher at the same time, this made him an integral part of the study in terms of involvement. Furthermore, "the effect of video becomes negligible in most situations after a certain phase of habituation" (Knoblauch, Schnettler & Raab, 2006) and the camera's intrusiveness gradually fades away. An important part of data is the written work produced by students during the sessions using smart pens and notebooks.

## Mathematical competencies framework

In 2016, the first author conducted a "pilot analysis" to test the functionality of the KOM mathematical competencies framework with a group of biology students. The very first analysis of the data brought to light several episodes where the competencies significantly overlapped. As mentioned by Niss, "The competencies are closely related - they form a continuum of overlapping clusters - yet they are distinct in the sense that their centres of gravity are clearly delineated and disjoint" (Niss, 2003, p. 9). This indicates a potential problem with the inter-coder reliability which may arise with the adoption of the KOM framework for tracking the development of student's competencies. In order to maximally avoid possible complications with coding, we retain only five basic groups of mathematical competencies (16 competencies in total) out of the eight suggested in KOM: thinking/acting mathematically, modeling mathematically, representing and manipulating symbolic forms, communicating/reasoning mathematically, and making use of aids and tools. All sixteen competencies in these five groups are described below.

*Thinking/acting mathematically*

i. pose questions that are characteristic of mathematics, know possible answers that mathematics may offer;
ii. understand and handle the scope and limitations of a given concept;
iii. attack (take actions towards a solution) mathematical problems.

*Mathematical modeling*

i. assess the range and validity of existing models;
ii. interpret and translate elements of a model during the mapping process;
iii. interpret mathematical results in an extra-mathematical context and generalize solutions developed for a special task or situation;
iv. criticize the model by reviewing, reflecting and questioning results;
v. search for available information differentiating between relevant and irrelevant information;
vi. choose appropriate mathematical notation, represent situations graphically.

*Representing and manipulating symbolic forms*

i. choose a representation;
ii. switch between representations;
iii. manipulate within a representation.

*Reasoning and communicating*

i. understand others' written, visual or oral information having mathematical content; follow and assess chains of arguments put forward by others;
ii. express oneself in oral, visual or written form in mathematical context; provide explanations or justifications to support own results and ideas.

Aids and tools

i. know different tools and aids for mathematical activity and their properties;
ii. use appropriate aids and tools to develop insight or intuition.

Each of sixteen competencies was coded separately and tested for reliability by requiring three researchers to code the same transcript and verifying consistency of their coding by direct comparison. When all data were collected, an initial data analysis took place focusing on the first three sessions and the work of two students, J and E. This preliminary analysis brought to light an important finding that, in addition to a number of instances when a certain competency was activated, its strength at each instance should also be recorded. Even though the same codes appear within the same session at chronologically different solution stages, they do not exhibit the same intensity in terms of competency activation. A positive impact of this adjustment is that by assessing the strength of each competency at each occasion we improve validity and reliability of our research. On the other hand, this approach demands a much more careful treatment of the data which, in our case, includes the need for the analysis of large sets of qualitative data.

Figure 1 provides an extract from the transcript of the third session which we use to illustrate the coding procedure. The group comprised of two students, J and E, and the first author. Since the video recordings were split in three parts, a, b, and c, we use, for instance, the notation 3b for the part b of the recordings of the work of the third group of students. The figure shows the time, names of the actors, extracts from the transcript of the episode (with clarifications, whenever necessary) and the code for each activated competency (in this example, just P.Q., which stands for "posing questions which are characteristic for mathematics"). Since we focus our attention on students' discourse, the codes are assigned only to students. Assignment of codes for the tutor's (researcher's) discourse is a part of the task analysis and is beyond the scope of this paper.

| Time | Who | What was said | Code |
|---|---|---|---|
| 01:30 (3a) | Y | So, what do we count as part of Norway's population in 2017? (pointing to: $N_{t+1} = N_t + B + D + I - E$, where $N_{t+1}$ : the population at $t+1$ year, $N_t$ : the population at $t$ year, $B$: the number of births, $D$: the number of deaths, $I$ : number of people immigrated and $E$ : the number of people emigrated) | |
| | J | Why do we have "+" at deaths? | P.Q. |
| 12:27 (3a) | J | Don't we use ln for all sides so we get $t$ down? (He refers to this equation $N_t = N_0 * e^{rt}$) | P.Q. |
| 16:16 (3a) | J | Since we have ln $e$ can we just delete it? | P.Q. |
| 17:40 (3a) | J | Can I take this down now? (J is writing down the following: $\ln(N_t) = \ln(N_o) + \ln e^{rt}$, and he is asking about the "$rt$" part) | P.Q. |
| 02:12 (3b) | J | Shouldn't we use the same method we used last Monday? (This question was posed when Y introduced the main task for the first time) | P.Q. |
| 07:30 (3b) | Y | How many doublings do we have? | |
| | J | 24 hours divided by 20 minutes? | P.Q. |

*Figure 1. Part of a modeling session transcription. The columns represent the timing, the name of the participant, the transcription and the competency code.*

**Dimensions and scaling**

The example shows that in the transcript of the session we came across the code P.Q. six times, but the frequency gives us little information, if any. The P.Q. code is one of the aspects of mathematical thinking and acting competency and even though all questions asked by the students in the episode clearly exhibit mathematical characteristics, not all of them have the same depth and targeting. Therefore, the need for certain "scaling" became obvious. The authors started with a “rough” scaling (beginning – developing –accomplished – exemplary) for each of the three dimensions described below (task solving vision, use of mathematical language/vocabulary) and proceeded with its further refinement.

| | | GRADING/SCALING | | | |
|---|---|---|---|---|---|
| | | **Exemplary (A+)** | **Accomplished (A)** | **Developing (B)** | **Beginning (C)** |
| DIMENSIONS | **Task Solving Vision** | Activation of the competency with a meta-reflective perspective towards the final solution | Full vision of task solving process with possibly minor inaccuracies | Occasional, regular or consistent mid-range perspective towards the final solution | No clear perspective or short-range perspective towards the final solution |
| | **Use of Mathematical Language/Vocabulary (Flexibility)** | Worthy of imitation, serving as model | Proficiency of mathematical language with possibly some minor omissions | Basic, occasional confident or regular use | Familiar, everyday expressions or gestures |
| | **Prompting** | The student often “replaces” the tutor during the solving process | The student works independently or needs minor hints to bring the competency into action | Prompted adequately (1 to 4 questions) or indirectly to bring the competency into action | Continuous prompting to bring the competency into action |

*Figure 2. The scaling system: the columns represent the levels of competency intensity and the rows the dimensions of the competency*

**Task Solving Vision**

This dimension focuses on the depth of student’s understanding of the task’s solving steps and his/her perspective towards the solution of the task; it also helps to relate the activation of a given competency with actual steps towards solution. The findings from the preliminary data analysis led to the conclusion that there are episodes where a student may activate a particular competency which, however, has no relation with the solution of a given task. On the other hand, at other instances students exhibited specific competencies in combination with clear ideas at all times of how to solve the problem. Developments in competency strength are not linear and differ depending on the moment of activation, task's type, its difficulty, and other factors. Using the data from the preliminary analysis and synthesizing the second author's teaching experience, we decided to further differentiate the beginning, developing and advanced levels coming up with the refined scaling system presented in Figure 3.

| Scaling | Rating | Description |
|---|---|---|
| Beginning | C1 | Students activate the competency having no clear perspective towards the solution. |
| | C2 | Initial signs of short-range perspective towards the solution. |
| Developing | B1 | Occasional signs of mid-range perspective towards the solution. |
| | B2 | Regular signs which indicate that the student activates the competency having in mind future steps of the solution. |
| | B3 | Consistent signs of mid-range perspective towards many steps of the solution. |
| Accomplished | A1 | Evident signs of long-range perspective towards the final solution. Full vision of the process with minor inaccuracies. |
| | A2 | The student's actions during all steps of the solving process are directed by his/her perspective towards the final solution. Full vision of the process. |
| Exemplary | A+ | In addition to A2, the student exhibits non-standard approach(es) to solution synthesizing ideas and techniques beyond the topic and subject discussed. |

*Figure 3. The sub-levels of the Task Solving Vision*

**Mathematical Language and Vocabulary**

Concepts, algorithms, procedures, computation, problem solving, and language are some of critical component skills that should be combined to achieve certain proficiency in mathematics (Riccomini, Sanders, & Jones, 2008). The language of mathematics can be confusing, especially for students who do not have mathematics as their main subject of study (Rubenstein & Thompson, 2002). However, even these students should communicate their mathematical ideas organizing them in some form of reasoning. There is a significant difference between getting the right answer and explaining how one got it. Students can use a vocabulary enriched with mathematical terms or just utilize familiar, everyday expressions and gestures. An undeveloped mathematical language can be the reason for the overall deceleration of their mathematics learning (van der Walt, Maree, & Ellis, 2008). Difficulties with the use and learning of mathematical vocabulary are well documented (Riccomini et al., 2015); language development can be a challenging issue.

| Scaling | Rating | Description |
|---|---|---|
| Beginning | C1 | Use of familiar, everyday expressions with no visible signs of mathematical language knowledge. |
| | C2 | Use of familiar, everyday expressions with gestures that suggest an initial familiarity with mathematical vocabulary. |
| Developing | B1 | Use of basic mathematical expressions at an intermediate level. The student can use simple connected statements while activating the competency. |
| | B2 | Occasional and confident use of mathematical language with a certain degree of fluency, independency and spontaneity. |
| | B3 | Regular and nearly independent use of mathematical vocabulary in a complex context. The student often uses definitions of mathematical notions to express himself/herself. |
| Accomplished | A1 | The student, independently, expresses himself/herself fluently with a rich mathematical vocabulary. |
| | A2 | Proficient use of mathematical language. |
| Exemplary | A+ | In addition to A2, the student expresses himself/herself at a more advanced level beyond the limits of the topic/subject. |

*Figure 4. The sub-levels of Mathematical Language and Vocabulary.*

Communicating mathematically is a demanding task for all students, even for those who display certain familiarity with mathematical concepts and notions. Most of the competencies in our framework are activated through verbal or written communication. Students use mathematical language, for instance, to pose questions (P.Q.) that are characteristic of mathematics or to interpret and translate the elements (Int.El.) of a model during the mapping process. This direct connection between the competencies framework and mathematical language proved to be a decisive reason for introducing mathematical language and vocabulary as the second dimension in our scaling tool.

**Prompting**

During the students' work on the tasks, one has to take into account instructor's involvement in the process assessing the breadth of prompting and scaffolding.

<table>
<tr><th colspan="2">Scaling</th><th>Rating</th><th>Description</th></tr>
<tr><td colspan="2" rowspan="2">Beginning</td><td>C1</td><td>Continuous prompting or use of directive probing question to activate the competency</td></tr>
<tr><td>C2</td><td>Prompting is still continuous but with moments of independence. The student attempts to interrupt guidance.</td></tr>
<tr><td rowspan="3">Developing</td><td rowspan="2">explicit</td><td>B1</td><td>The student needs 3 or 4 basic prompting actions to activate the competency.</td></tr>
<tr><td>B2</td><td>The student needs 1 or 2 basic prompting actions to activate the competency.</td></tr>
<tr><td>non-explicit</td><td>B3</td><td>Indirect prompting, usually through a statement indicating the student what is expected, but not exactly. For example, "What is your next step?" "How can we explain this?"</td></tr>
<tr><td colspan="2" rowspan="2">Accomplished</td><td>A1</td><td>Independent work, the student can perform the task by activating the competency on his/her own, with minor hints.</td></tr>
<tr><td>A2</td><td>Independent work, the student can perform the task by activating the competency on his/her own, with no prompts or assistance.</td></tr>
<tr><td colspan="2">Exemplary</td><td>A+</td><td>In addition to A2, the student anticipates the solution steps constructively "assisting" the lecturer.</td></tr>
</table>

*Figure 5. The sub-levels of Prompting.*

Wood et al. (1976) characterized scaffolding as an interactive system of exchange in which the tutor operates with an implicit theory of the learner's acts to recruit his attention, reduces degrees of freedom in the task to manageable limits, maintains 'direction' in the problem solving, marks critical features, controls frustration and demonstrates solutions when the learner can recognize them (p. 99). They continued by identifying scaffolding as the process that enables a child or novice to solve a problem, carry out a task, or achieve a goal which would be beyond his/her unassisted efforts. The latter definition and especially the expression "beyond his/her unassisted efforts" could raise some justified arguments since one can never be certain about students' responses and reactions to the assigned tasks. Nevertheless, we adopt this definition of scaffolding for the following two reasons: (i) complementary modeling sessions were designed to assist generation of new knowledge; (ii) freshmen biology students had very limited experience with modeling tasks, if at all; they could view many modelling tasks as quite demanding.

The purpose of introducing this dimension is to monitor the extent to which the instructor was using probing questions and assisted students. Bernstein (1967) referred to the importance of reducing the alternative actions during skill acquisition; he considered it as an essential process to adjust and regulate feedback so that it could be used for correction. The latter is strongly related to this study's focus since our goal is to analyze the process of competency acquisition through the solution of modeling problems. We can see in Figure 5 that activation of a competency with low-level prompting results in a high-level competency intensity. Less scaffolding means more stimulation for independent work and thus a higher competency strength.

This dimension has a unique characteristic that distinguishes it in terms of functionality from the other two. In the first two dimensions, we focus attention at a specific discourse episode "cutting it out" from the session. However, scaffolding is a delicate continuous process where one has to choose very carefully both the timing for the intervention and the format of the help offered to students. Even though scaffolding is a rather demanding task, we view it as the only way to achieve prominent levels of trust and credibility for our scaling tool in general and for this dimension in particular.

## Coping with large volume of qualitative data

When the data collected in the sessions are coded, the record of each competency frequency and strength (beginning, intermediate, developed, exemplary) is also kept. This creates a large corpus of qualitative data which we would like to analyze in order to monitor and assess students' competency development. Looking for an appropriate tool to analyze the data, we decided to convert all qualitative data into numerical values (the value 1 is assigned to C1, 2 to C2, 3 to B1, and so on) and employ information entropy for spotting the trends in competencies development.

### What is information entropy?

Entropy is one of the important concepts in thermodynamics characterizing the "amount of disorder" in a system at a given temperature. "Shannon entropy" or "information entropy" is used to evaluate the amount of information encoded in a transmitted message. The Shannon entropy is defined by

$$H(p) = -k \sum_{x_i} p(x_i) \log_b p(x_i) \geq 0,$$

where $p(x_i)$ is a normalized discrete probability distribution function and $k$ is a coefficient (Shannon, 1948).

Some important properties of the information entropy:

- Higher entropy levels correspond to larger disorder in physics and higher information content in information theory.
- Entropy is an extensive property like mass.
- Entropy grows logarithmically with the number of degrees of freedom.

**How do we use entropy to assess competencies' development?**

Step 1. In our example, we follow the development of the three competencies which were evaluated by the lecturer at 10 different instances during one session. We first collect all data in a 10×3 matrix $A$:

$$A^{\mathrm{T}} = \begin{pmatrix} 2 & 2 & 2 & 4 & 3 & 3 & 4 & 4 & 4 & 4 \\ 5 & 3 & 3 & 4 & 6 & 3 & 3 & 3 & 2 & 4 \\ 5 & 5 & 4 & 4 & 3 & 2 & 6 & 6 & 6 & 5 \end{pmatrix}.$$

Step 2. We collect the frequencies of appearance of each of strengths in a table:

| #0 | #1 | #2 | #3 | #4 | #5 | #6 | #7 | #8 |
|---|---|---|---|---|---|---|---|---|
| - | - | 3 | 2 | 5 | - | - | - | - |
| - | - | 1 | 5 | 2 | 1 | 1 | - | - |
| - | - | 1 | 1 | 2 | 3 | 3 | - | - |

Step 3. We fill the table with probability distribution function for each of the three monitored competencies:

| #0 | #1 | #2 | #3 | #4 | #5 | #6 | #7 | #8 |
|---|---|---|---|---|---|---|---|---|
| - | - | 3/10 | 1/5 | 1/2 | - | - | - | - |
| - | - | 1/10 | 1/2 | 1/5 | 1/10 | 1/10 | - | - |
| - | - | 1/10 | 1/10 | 1/5 | 3/10 | 3/10 | - | - |

Step 4. We compute information entropies for each of the three competencies:

$$e_1 = -\frac{1}{\ln 10}\left(3 \cdot \frac{1}{10} \cdot \ln \frac{1}{10} + \frac{1}{5} \cdot \ln \frac{1}{5} + \frac{1}{2} \cdot \ln \frac{1}{2}\right) = 0.44717,$$
$$e_2 = -\frac{1}{\ln 10}\left(3 \cdot \frac{1}{10} \cdot \ln \frac{1}{10} + \frac{1}{2} \cdot \ln \frac{1}{2} + \frac{1}{5} \cdot \ln \frac{1}{5}\right) = 0.59031,$$
$$e_3 = -\frac{1}{\ln 10}\left(2 \cdot \frac{1}{10} \cdot \ln \frac{1}{10} + \frac{1}{5} \cdot \ln \frac{1}{5} + 2 \cdot \frac{3}{10} \cdot \ln \frac{3}{10}\right) = 0.65352.$$

Note that a higher (closer to 1) entropy value indicates larger "impurity" of data and points at a larger volatility in competency strengths.

Step 5. We compute a so called "distinguishing ability" (impact) $\omega_j$ of each competency

$$\omega_1 = \frac{1 - 0.44717}{1 - 0.44717 + 1 - 0.59031 + 1 - 0.65352} = 0.42233,$$
$$\omega_2 = \frac{1 - 0.59031}{1 - 0.44717 + 1 - 0.59031 + 1 - 0.65352} = 0.31298,$$
$$\omega_3 = \frac{1 - 0.65352}{1 - 0.44717 + 1 - 0.59031 + 1 - 0.65352} = 0.26469.$$

Smaller values of the entropy $e_j$ indicate larger impact of the $j$-th competency $\omega_j$ in the overall assessment. In our example, the first competency has the largest impact and the third one the smallest.

Step 6. We compute the consolidated competencies assessment value A for all three competencies as a weighted some of means for the competencies:

$$A = 0.42233 \cdot 3.2 + 0.31298 \cdot 3.6 + 0.26469 \cdot 4.6 = 3.6958.$$

Note that this consolidated weighted assessment value differs from the regular mean value $A = 3.8$. By monitoring the changes in $A$ from session to session and in the long run, we can follow the dynamics of the range of competencies.

## Conclusions and further plans

In this paper, we addressed the question regarding the use of KOM framework for monitoring the development of mathematical competencies of biology students engaged in complementary to the main mathematics course modeling activities. It turned out that the use of a reduced framework with five out of eight groups of competencies is preferable for practical reasons related to overlapping of competencies and coding reliability.

However, the choice of a comprehensive and reliable mathematical competences framework is not sufficient for successful monitoring of the development of students' competencies. It has become clear from the preliminary data analysis that, in addition to the record of the activation of each of the competencies during the teaching sessions, it is very important to develop a solid scaling system for tracing the "strength" (intensity) of each of the competencies which varies significantly at different instants. We believe that the eight-step scaling scheme serves well our purposes for each of the three dimensions; quite significant variations of the scores for different dimensions associated with the same competency confirm our opinion regarding the need of each of them in the analysis and importance of each of these dimensions in its own turn.

The next research question regards processing of large amounts of qualitative data collected in tables with the strength of each competency measured from C1 (the lowest) to A+ (the highest). We decided to convert all data to numerical values and performed preliminary analysis of the data based on the use of information entropy. One of advantages of this approach is related to the possibility to spot competencies with the most "volatile" behavior as those needing more lecturer's attention. We believe that in combination with other indicators (even as simple as median/mean value for the strength of a given competency), a new entropy-based scaling tool opens interesting opportunities for researchers who need a consolidated evaluation for big amounts of data. Information entropy could be efficiently used both to monitor the development of individual students' competencies and to compare performance of different students. One can assign different weights to competencies for monitoring the progress in the development of particular skills. However, using this method of data analysis inevitably leads to the loss of some essential information, the risk we are currently willing to undertake in order to study the potential of this new progress-tracking tool. In our forthcoming research we will test other ways of using information entropy for monitoring the development of students' mathematical competencies, both as a part of an independent scaling tool and in combination with other techniques for the analysis of large data sets (like monotonicity properties of data strings).

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Goals, Resources, and Orientations for Equity in Collegiate Mathematics Education Research

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*Though the terms equity, diversity, inclusion, and social justice have entered the research lexicon, researchers in collegiate mathematics education face significant challenges in gaining a nuanced understanding of the various ideas associated with these words and how those ideas are consequential for research. This report presents a framework for making sense of (and making sense with) equity as an essential component of research content and conduct. It offers tools for thinking and talking about equity and research design and implementation.*

*Keywords:* Equity, Justice, RUME

People trained in research in undergraduate mathematics education (RUME) know that work in RUME starts with diagnosing challenges in teaching and learning. As citizens of a first-world country in the 21st century, researchers in collegiate mathematics education in the United States are keenly aware of social, political, and economic inequity as challenges in education. And, as a community, researchers have an opportunity to define, explore, and address *equity*, *diversity*, *inclusion*, and *justice* in collegiate mathematics education. Rather than a report of empirical research, this piece is a synthesis of information about professional goals, resources, and orientations for working towards equity as a core aspect of both the content and conduct of research in undergraduate mathematics education. We start with the voices of researchers – the sample quotes in Figure 1 are synthesized from people in the RUME community (mostly attendees at the RUME Conferences in 2017 and 2018) when asked "What does it mean to consider equity in research in collegiate mathematics education?"

**Researchers speak their truths about equity and RUME:**

- I don't have a definition for "equity" and I'm not really sure what you mean by it, so I can't consider it in research.
- I include race and ethnicity in my data, that's how I'm addressing equity.
- My team researches equity in math ed, so of course we're addressing equity in RUME.
- I've faced this my entire professional career; I have done my best to 'fit-in' to have a career.
- We need to help others learn what the RUME ways are to do things, then we would be on a level playing field for research.
- The problem is too big, what can I do about it?
- Others don't want to participate in RUME, why should that change? Yes, it is inequitable, but is that necessarily a bad thing?
- Mathematics operates as whiteness. There is no possibility of equity in RUME until the meaning of the M in RUME is interrogated.
- What you do in math ed is so different from what I do, I couldn't possibly understand, review, and evaluate it.
- Dealing with equity means just doing 'good work,' because good work is bias-free.
- I'm a researcher in math ed that pays attention to equity but I accept that not all of my colleagues are going to be interested in that.
- Equity in education research has undergone a slow evolution. The time for evolution is over, the time for revolution is now.
- For RUME to be rigorous, it has to disrupt the inequitable research status quo.

*Figure 1. Sample of researcher thoughts about equity in RUME.*

**What do Equity, Diversity, and Inclusion mean?**

The difficulty in defining equity is that it involves an attempt to define something we have never seen. In what ways is equity a destination? … a journey? …something else? One can start with the Merriam-Webster dictionary and say, "equity is fairness." A starting point, but it begs the questions: What is fairness? Who decides? In the U.S. we have gotten to the point where we can distinguish between equity and equality: equality is everyone having the same thing while equity is based on a person's current situation and goals and means people have what they need to grow from the one to the other. Also, equity can be partially defined by its complement: inequity. Equity is evidenced by the absence of disparities (e.g., membership in a group that has been historically disadvantaged is in no way correlated to access to opportunities, attainment of educational outcomes, or achievement of life goals). So, tracking change in disparity is a way to measure progress towards equity (by measuring reduction in *in*equity). The distinctions between equity, diversity, and inclusion are a little clearer. Diversity is quantitative - a measure of the variation of particular characteristics of interest across people or groups. Like diversity, inclusion (in a group or structure) may provide a metric related to equity in that inclusion "involves an authentic and empowered participation" (Annie E. Casey Foundation, 2014). At best, however, inclusion may be necessary but is not sufficient for a situation or process to be equitable. As reflected in Figure 1, at this moment in the educational research and practice communities (and more broadly) there is not a well-defined, crisp, and shared definition of equity.

**Social Justice and Education**

Last year two organizations, TODOS: Mathematics for All and the National Council of Supervisors of Mathematics (NCSM), issued a position paper entitled *Mathematics Education Through the Lens of Social Justice: Acknowledgement, Actions, and Accountability*. In it, justice in mathematics education included "fair and equitable teaching practices, high expectations for all students, access to rich, rigorous, and relevant mathematics, and strong family/community relationships to promote positive mathematics learning and achievement" (p. 1). Underlying all of these was a call for attention to the ways power, privilege, and oppressions contribute to and maintain an inequitable educational system.

According to the TODOS-NCSM position paper, three conditions are necessary to establish socially just mathematics education and, we argue, to establish just RUME:

(1) acknowledging that an unjust social system exists,
(2) taking action to eliminate inequities and establish effective policies, procedures, and practices that ensure learning for all, and
(3) managing accountability to monitor progress in and from action, so changes are made and sustained.

How do we increase mathematics education *researcher* capacity to do these three things? In writing this report, we (the authors) claim that one step is meeting researchers where they are, addressing the need for language, definitions, and awareness-building about educational equity. A deep investigation of perceptions in Figure 1 is beyond the scope of this report. As a first step, we have organized this report around: (1) potential *goals* for the RUME community, (2) some *resources* for starting the journey towards equitable research – research that attends to equity in its conduct and its content, and (3) *orientations* for doing this work together with others.

Certainly, significant inequity in the United States is rooted in racism, sexism, and other societally institutionalized -isms that structure opportunity and assign value in ways that disadvantage some people and groups and advantage others for reasons that are (now) anathema to the majority of the population. Acknowledging, unpacking, and addressing the inequitable realities of systemic, professional, and personal biases requires courage.

Cognitive dissonance and disequilibration have come to be valued as opening mental room for the generation of disciplinary learning (Piaget, 1963). Similarly, dissonance and disequilibration related to belief, motivation, culture, and communication can pose opportunities for the creation of new knowledge and ways of thinking. Discomfort is frequently an indicator that we are about to learn something (National Research Council, 2000).

**Speaking Differently in Different Ways**

Mathematics education by its very nature is interdisciplinary, pulling from educational psychology, sociology, and mathematics fields, among others. This means that researchers navigate among different professional cultures. Here, we use the word *culture* to indicate a set of values, beliefs, behaviors, and norms in use by a group that are shared with and taken up by others who become members in the group. Therefore, it may be important for RUME researchers to take note of lessons from researchers in other interdisciplinary groups. It is well documented in the literature that equitable research in other fields is carried out by disciplinary experts in multidisciplinary teams (e.g. see Bililign, 2013). In particular, successful cross-cultural collaboration requires *intercultural sensitivity* (Bennett, 2004) and *responsive communication skills*. The development of these relies on (a) the orientations we have to differences we note in both explicit and implicit values and (b) knowledge and use of variation in norms for communication and resolution of conflict (Bennett, 1993; Hammer, 2009).

International and national variation means factors of ethnic, racial, and other types of group and institutional enculturation and socialization are involved in professional intergroup communication. As an example, consider the work related to gender and communication, both within and across groups. One comparison of African American and European American women found a direct communication style to be more common among African American women than the indirect framing most used by their European American peers. Both groups of women had a goal of reducing potential conflict (or, largely in the case of the European American participants, conflict avoidance), but methods for how to articulate and achieve it were different (Shuter & Turner, 1997). As we have noted before from a gender-as-culture perspective (Hauk & Toney, 2016), communication habits emerge from a childhood and adolescence filled with same-sex conversational partners and a lifetime of social expectation (Maltz & Borker, 1982). Review of the literature on studies of language and gender has found that women may have access to power (and more acceptance) in a majority culture context when using indirect language, uncertainty, and hedges in relatively long sentences: “Well, I was wondering if…,” “Perhaps we might…,” while men fulfill expectations by referencing quantity or judgments with direct assertions: “An evaluation of 3.8…,” “It’s good because…” (Mulac, Bradac, & Gibbons. 2001, p. 125).

The fact that interaction about RUME in most universities occurs in the context of historically male discourse makes every interaction between the sexes a doing of gender in some way (Uchida, 1992). Consequently, gendered communication structures can be (dis)empowering depending on context. Additionally, those whose work focuses on teaching tend to value a pragmatic approach and may seek rewards based on personal motivation rather than external distinction (Wang, Hall, & Rahimi, 2015). Some have written about the importance of women seeking to participate in the career reward structures and other status quo value systems in the academy (Nicholson & de Waal-Andrews 2005; Olsen, Maple, & Stage, 1995). However, embracing the status quo without also attempting to change it has the danger of derailing progress in the intellectual and professional work of RUME.

## Goals

Rigor in research is in its authenticity and trustworthiness (Lewis-Beck, Bryman, & Liao, 2003). Implicit in this assertion are (a) who is making the decision about what is worthy of trust and (b) whose trust needs to be sought. Current research practice common in the United States relies on a closed (and exclusive) system: the people making the decision and whose trust is sought are the extensively educated people doing the research. Yet, research involves humans who are not researchers. The *stakeholders* include researchers *and* those who will be researched as "participants" in the work, colleagues in education who are not researchers (e.g., department chairs, other faculty in the department and university, at other colleges), future audiences for the results of the research, and policy makers who rely on research to inform decisions.

In K-12 education, attention to equity in the focus of research and in the methods and reporting of research, particularly an intention to include views of stakeholders who are not also researchers, has existed for a while. Statements from K-12 about equitable and inclusive mathematics and its teaching might be adapted to be about RUME. In Figure 2, we have "marked up" two statements. One is an existing equitable vision for teaching for robust understanding (TRU, Schoenfeld et al., 2014; 2017) and the other a critical vision for just and inclusive mathematics (Gutierrez, 2012). Each indicates how an adaptation for RUME goals can be stated.

**TRU Framework (Schoenfeld, 2014):** Equitable access to ~~mathematics~~ RUME is the extent to which ~~classroom~~ research activity structures invite and support the active engagement of all of the ~~students~~ stakeholders in the ~~classroom~~ research.

**Gutiérrez (2012):** We seek ~~mathematics~~ RUME that squarely acknowledges the position of ~~students~~ stakeholders as members of a society rife with issues of power and domination. Critical ~~mathematics~~ RUME takes ~~students~~' stakeholders' cultural identities and builds ~~mathematics~~ RUME around them in ways that address social and political issues in society, especially highlighting the perspectives of marginalized groups.

*Figure 2. Examples of two existing goal statements from K-12 education and their adaptation into goal statements for RUME.*

In recent RUME, an inclusive and stakeholder-responsive approach has become more frequent (e.g., Aguirre & Civil, 2016; Adiredja, Alexander, & Andrews-Larson, 2015; D'Ambrosio et al., 2013; Davis, Hauk, & Latiolais, 2010; Nasir, 2016). Change takes time and intentionality (Marshak, 2005). Changing our individual and collective mindsets, behaviors, and approaches related to equity and the work of RUME will not happen as a result of one paper, statement, project or movement. However, "immediate and transformative change is necessary" (TODOS, 2016, p. 1). Aiming for the goals in Figure 2 requires change. How can we achieve this change? On the small scale of a local research project, what is needed to pursue goals like those in Figure 2? Certainly, self-awareness and awareness of others as we plan, act, reflect, and communicate. This occurs in the context of researcher orientation (individual and collective) towards the thing about which the planning, doing, and reflecting is happening.

Regardless of whether research is *about* equity in mathematics teaching and learning, it is time for an "equity lens" to inform research questions, design, methods, and reporting in the focusing and doing of collegiate mathematics education research. In a piece that appeared in the

*Journal for Research in Mathematics Education* (JRME), Aguirre and colleagues (2017) asserted that researchers in K-12 education have a responsibility to challenge the ways in which "power, privilege, and oppression tacitly and explicitly play a role in research programs" (p. 126). The authors also noted that researchers have a choice in moving from the *status quo* to intentional collective responsibility for doing "the right thing for current and future generations" (p. 125). They organize their suggestions as four political acts:

1. Conduct mathematics education research with an equity lens
2. Acquire the knowledge necessary to do genuine equity research work
3. Challenge the false dichotomy between mathematics and equity as research topics
4. Expand the view of what counts as "mathematics."

In this essay, we touch on each of these in the context of RUME with a focus on Political Act 2.

In the next section we offer resources for researcher planning, doing, and reflecting on equity – this includes such things as deciding what to study and how to conduct the research. By no means exhaustive, these resources include existing research reports, frameworks for organizing activity and analysis, and a tool to support purposeful reflection. In the subsequent section, we examine the contextual factor of interpersonal orientation. This report ends with a synthesis across the offered goals, resources, and orientations for working towards equity as we do RUME.

## Resources

Given the nature and foci of research in collegiate mathematics education to date, new ways of thinking and new ways of engaging with topics, participants, structures, mechanisms, and results of research are needed for work with an equity lens. The challenge is how to visualize that world we have not yet seen and traverse with purpose and patience the meandering path to educating ourselves for working towards equity. Crucial to realizing the new view is a re-usable process that supports researchers to detect, discuss, and disrupt inequities within research.

How do we engage in conversations about research in mathematics education that may be challenging and uncomfortable? We offer a few resources here. These have emerged from the authors' own work in critical educational research as well as purposeful conversations with RUME colleagues (e.g., poster-side at conferences, during conference (pre)sessions focused on the topic). We rely on effort to be aware of, and transparent about, our own views as well as active effort to become knowledgeable about views across the gamut of stakeholders in RUME.

In what follows, we have chosen to adapt the engineering design cycle (see e.g., NGSS, 2013) as a way to structure the presentation. We made this choice for two reasons. First, the cycle is cross-cultural. Versions of the design cycle appear across time and societies. Second, it is a framework that has proven useful in many human endeavors with varying numbers of stakeholder groups, from developing a cure for smallpox, to successful moon landing, to organizing effective civil disobedience.

### Research Design Cycle

We propose a process for attending to equity and creating transformative change. The process is based on the design cycle common in engineering and design-based research (e.g., Figure 3, next page). There are many other versions of the design cycle. They share the basic features of a process for defining/understanding the problem, designing/testing solutions, and then iterating the process.

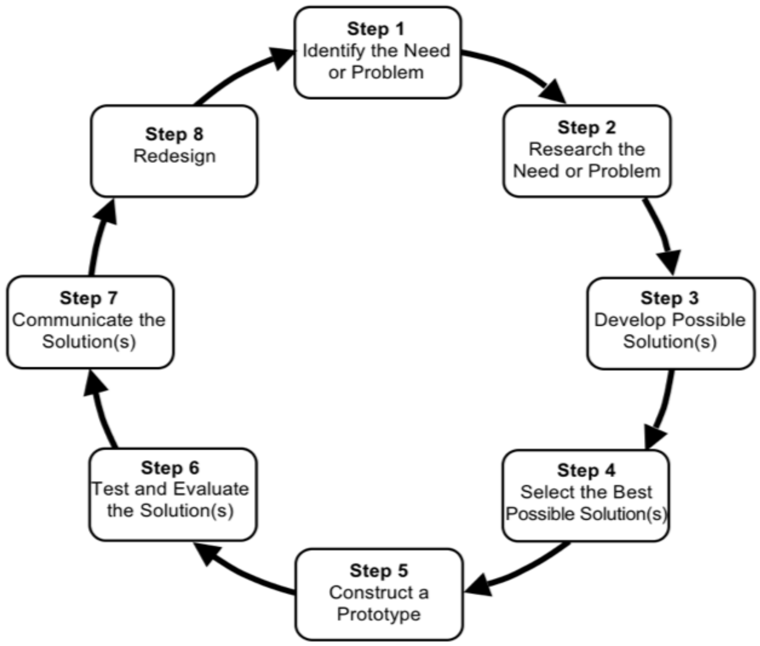


*Figure 3. Example of a design cycle (Massachusetts Department of Education, 2006).*

Just as engineers repeatedly iterate on a design to make a better product or solution, so must we iterate on our strategies, research designs, and methods for creating, using, and reporting on research. Engineers acknowledge there are many different possible solutions and that a solution that meets current needs or goals can always be improved. We also recognize that while there may be many approaches for those attempting research that includes equity as a research focus and/or attends to equity in its conduct, none will be perfect. Individually, and collectively, researchers in collegiate mathematics education must continue to iterate through the cycle. We can pause, but must not become complacent, in the work. Yes, the problem is overwhelming. Yes, it may seem insoluble in our lifetimes. At one point, so was how to create a proof about parallel lines that would stay proved. The important thing is to start, somewhere, and iterate. Intentionally going through the cycle helps to make incremental success visible as we move towards near, subsequent, and far goals.

A design cycle is useful for an individual researcher, in small professional groups, and as a community. However, when they cycle is about work of *research* it is, explicitly, situated in research. No one can do it alone. The researcher version of the design model in Figure 4 (next page) requires at least one thought partner. Keep in mind that in the RUME community, in addition to general opportunities to build professional relationships at the annual conference, there are research working groups. Each of these working groups is a resource. Also, we as the RUME community are stakeholders, and can engage in a cycle to acknowledge, act, and hold ourselves accountable for community progress in making and attaining equity goals.

**Design Cycle Framework as a Researcher Tool**

***Consult with Stakeholders*****:** The cycle begins with developing a focal research goal. A necessary first step is to identify people/organizations that *might be* stakeholders in defining and achieving that goal. Consult with possible stakeholders to determine the degree to which people see themselves as stakeholders and to ask who other stakeholders might be. This consultation is used to shape goals and their framing.

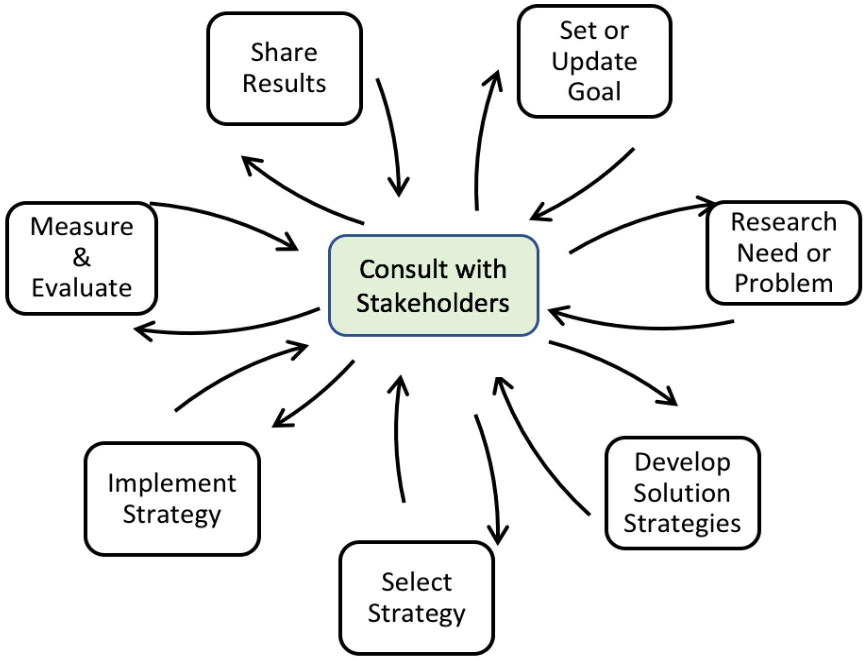

*Figure 4. Research design cycle with centrality of stakeholders.*

For example, is a goal stated as a problem that will be solved, as a situation that will be explored, as a driving question that will be addressed? Also, part of this stakeholder consultation is identification of which mathematics is the mathematics under consideration. Researcher and stakeholder views of what counts as mathematics and what is compelling for research may need considerable negotiation (e.g., determine whose mathematics is the research about and what accountability to stakeholders is explicit in the choice; see, e.g., Balacheff, 2010; Bensimon & Harris, 2012; Sternberg & Ben-Zeev, 1996). Questions about *how to consult with stakeholders* are challenging to pose and to answer. Though in later sections we offer some tools, as noted earlier, this report is an initial tilling of ground for the RUME community. What it means to identify and productively consult with stakeholders will continue to develop as the community builds experience with the idea.

***Set or Update Goal.*** In the context of the acknowledge-act-account framework from TODOS/NCSM, this step acknowledges that there is an issue or area that needs addressing to move towards equity. The goal step also sets up accountability by explicitly stating a goal against which to measure effectiveness. While an overarching goal may be to attend to equity in research in undergraduate mathematics education, it is necessary to create one or more specific goals when engaging in the cycle. In fact, the goals themselves may focus on a target of acknowledgment, action, and accountability. For example, a research team with members across several institutions may set goals to include more about student learning outcomes in the context of student needs, cultures, strengths, and struggles in calculus (generally or, perhaps with a focus on a particularly problematics topic). The research team may have several objectives related to this goal, including (a) focal *research content* that examines what supports equity of outcomes among students (e.g., in addition to equity of opportunities) and (b) situating the research design and implementation in the entirety of student experience as a decision driven by equity in the *conduct of research* (e.g., taking into account the various types of calculus tutoring available to students at each campus, in and outside of the mathematics department; course loads for instructors and students; work hours). Another example, one that is structural, would be setting a goal for the context in which research is conducted and valued. The research team may have the goal of having their respective departments acknowledge, act, and set an accountability structure within the tenure and performance review process, one that incentivizes equitable research practice. Each of these example goals - for professional research design and for professional context design - might be served, together, in the same research project effort. ***Consult with Stakeholders*** to clarify and confirm stakeholder partners/informants and prioritize areas of focus for finding out more (i.e., to prepare for the next step in the cycle). For instance, in the calculus study example, immediate stakeholders would include the researchers on the team, calculus students at each university, those who offer calculus instruction and tutoring to students (e.g., instructors and grad students in the math department, staff at campus cultural centers, fraternities, sororities, publishers, and other sites that may offer in-person or online tutoring services), as well as informants from student services, the office of the Provost (who oversee instructional development), and the office of the Dean of Students (who are familiar with data on course/workload and student support practices that may shape goals).

***Research:*** Once a goal is set, the next step is to engage in background research. This is part of acknowledgement in that one learns about and acknowledges the nuances of issues surrounding a goal. While this step is revisited throughout an entire career, dwell time here depends on context. Do learn enough about the goal-related need, problem, or aim that any subsequent actions have a reasonable likelihood of positive outcomes. This step includes a literature review across stakeholder-relevant literature. The research step also involves learning what resources are available for this work. ***Consult with Stakeholders*** to discuss and negotiate priorities of the needs related to goals. Learn from people, organizations, and other venues what has been done before, what has worked or not, and why. In this step, relationships with critical friends for the project will continue to be negotiated. For example, individuals wanting to learn more about students may consult with someone with expertise in how to develop intercultural competence, a department incentivizing equitable research practices may want to research what other institutions have done in this area, and the RUME community might research the range of practices considered equitable by stakeholder groups. Resources that emerge from this step range from financial to conceptual frameworks. Through this research and stakeholder consulting step, a different or deeper understanding of the need or problem is made possible. Thus, folding back to update goals may be called for before going on to the development of strategies.

***Develop Strategies:*** Equipped with some idea of the nuances, considerations, and resources surrounding a goal, the next phase is to develop possible actions. This is the first step in the cycle directly related to action. It is important in this step to not focus on any one strategy but develop an array of strategies that may work. Also, in this phase are early plans for measuring effectiveness. Some strategies for attaining (and measuring progress towards attaining) goals may arise from research on approaches others have used. ***Consult with Stakeholders*** to develop, in conjunction with them, a pool of possible strategies and an awareness of the situated pros and cons. Several different types of relationships can be built in this step. New critical friends valuable to the project may emerge or need to be sought to answer key questions. Will there be surveys or interviews? Why? Which stakeholders are important to include in the measures? Will the work involve sensitive data? Who defines what is “sensitive” and why do they get to decide? Is permission needed? From whom? What, in context, is ethical? What do stakeholders feel are important measures? Why? The answers will inform prioritization of strategies.

***Select Strategy:*** Decide what action to take. Investigate each of the possible strategies and think about the merits and constraints of each. ***Consult with Stakeholders*** to determine which strategy best meets the need or goal while respecting all persons involved.

***Implement Strategy:*** Often, carrying out an intervention or program is seen as the most important or sole action. This step represents a large and complex set of activities! Something to keep in mind in this step are the importance of establishing and devotedly maintaining open and regular stakeholder communication. ***Consult with Stakeholders***, particularly those identified in earlier steps as (1) valuable critical friends and (2) brokers among stakeholder groups. Note that implementing a strategy is only one step of the cycle. If effective, it can be far-reaching. And, we know from existing research that effective implementations arise from strategies based in a robust theory and rooted in authentic educational settings.

***Measure and Evaluate:*** Measuring outcomes and evaluating effectiveness are central to accountability. Some of this must happen at the same time as implementing the strategy. Data collected throughout the cycle is used to measure progress to goals. ***Consult with Stakeholders*** to select and be guided by resources on formative and summative assessment for projects.

***Communicate Results:*** Once all data are collected for a cycle, analysis, reflection, and synthesis set up the next cycle. The definition of rigor in scholarship includes the expectation of peer review. In research that works towards equity, stakeholders are peers who review. ***Consult with Stakeholders*** by providing and seeking feedback on documentation of results. This is at the foundation of accountability. Keep in mind that "stakeholders" includes researchers! Thus, to "consult with stakeholders by providing and seeking feedback on results" includes submitting articles to peer-reviewed outlets.

Democratic systems function when stakeholders are involved and shape the work. Involving representative stakeholders in every step of the process is essential for developing equitable systems for research. Stakeholder communications holds research efforts accountable at each and every step of the cycle.

**Tool for Sharpening Awareness: Feeling Safe, Comfortable, and/or Brave**

Discerning difference, recognizing pattern, and anchoring new knowledge in those already noted differences and patterns are at the core of all human cognition. In other words, examining and making sense with our experiences are the essentials that allow humans to think, know, and learn. For researchers to engage in equitable, inclusive, and just efforts, means successfully noticing and navigating similarities and differences when collaborating with colleagues and communicating with stakeholders (including those who may identify with culture(s) different from their own). One resource we have found helpful in building awareness and responsiveness to the in-the-moment experiences is the Venn diagram shown in Figure 5 (next page).

The diagram can support self-aware communication about how people experience intellectual, personal, and professional risk. For example, a person may not feel safe having a conversation with (or about) people from other races but can be brave and handle the discomfort in order to stay engaged in a conversation (region *b*). For more on this tool, see Hauk, Toney, Judd, & Salguero (2017; in preparation).

We note here that the terms in Figure 5, and the meanings of the overlaps, must be negotiated between any two people who attempt to use the diagram as a tool in communication. In particular, safety is a characteristic of a person's experience, not a feature of the space. This differs from common (and often inequitable) assertions like, "the classroom should be a safe space." In general, such a statement is made without an accompanying expectation to *consult* *with stakeholders* (e.g., students and teachers) to validate it is possible for every person in the classroom to simultaneously experience the classroom as safe. Unaddressed is what is considered "safe" and who gets to decide (Nasir, 2016).

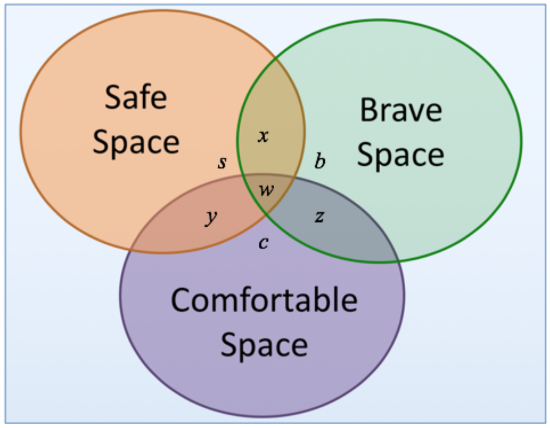


*Figure 5. Juxtaposition of three types of experience related to taking risk.*

As a tool for articulating self-awareness, the Venn diagram in Figure 5 is meant as a way for a person to identify the nature of individual experience in a particular moment. We are not claiming that a thing (e.g., a whole conversation, a research paper, a research design) might be a member of one of the sets or in a region. In fact, in the course of a 3-minute exchange during a research development meeting, one person could experience each of the seven regions in the diagram! To illustrate, a short example (from Hauk, Toney, Judd, & Salguero, in preparation):

***Researcher example.*** Suppose I am a white research faculty member talking with two white graduate students who have been vocal in their deficit views of non-whites as learners. The topic is a new research project they are to work on and my goal is to guide them to readings about how to fruitfully include attention to race and gender in their research. This is familiar and routine advice on my part, I feel comfortable in my role and what I am saying. I am not feeling the need to be brave (yet), because this is an initial conversation (region *y*); continuing conversation after the students have done some reading probably will involve being brave. At the same time, I realize the conversation is becoming professionally risky for me (region *z*) when one of the students says that if I really assign the readings, then he plans to complain to the chair about the assignment. Unsure of my relationship with the chair, I suspect there is a professional risk. At this moment, I would point to region *c* to identify my experience (still comfortable in my role and what I am saying, but not safe and not brave). Meanwhile, the graduate student who asserts, "students whose parents are from Mexico cannot do well because their parents do not value education" may experience the initial conversation from region *y* and rapidly move to region *b*, when his view is not met with the approval he expected.

## Orientations

Equity-aware research is shaped by what a researcher knows or anticipates about others' experiences and how that is communicated. Do we anticipate great risk? Do we have the privilege of assuming safety? Interactions with other people are shaped by the approach we take to the interaction and our orientation to noticing and engaging with difference in context. In this section we offer a framework for each of these. The first is an approach to being self- and other-aware during interactions. The second framework provides language and a developmental model for cross-cultural competence growth.

**Approach to Courageous Conversations**

Over the past 10 years, Singleton and colleagues' (2006, 2008) courageous conversations framework has become a cornerstone in the professional development of teachers. It is built on four agreements made by participants in a conversation before the conversation begins. These agreements contradict some tightly held cultural norms common to many in the U.S. when it comes to talking about interculturally challenging topics. To participate in a "courageous conversation" you agree to:

- stay engaged
- expect to experience discomfort
- speak your truth
- expect and accept a lack of closure.

The details for each of the four agreements are non-trivial. Staying engaged is different from being verbal; staying engaged means persisting as a listener and contributor, making conscious decisions about when to share thoughts aloud and when to reflect or write them down for further personal consideration before sharing. Expecting to experience discomfort also means being aware of one's own behavior and how that may ramp up discomfort for others - self-regulation, particularly for those unaccustomed to exercising it, can be challenging. The aim of a courageous conversation is *not* to convince or persuade others. Rather, the goals are determined (or at least negotiated) by the discussants. "Speak your truth" also means *listening to* the truth of others - this is a different activity from *listening for* echoes of one's own truth in what others say (or do, or value). Finally, the work of a courageous conversation may achieve milestones of agreement of understanding but rarely is the conversation tied off neatly. This is to be expected in a complex situation.

**Framework for Intercultural Orientation Development**

The ways people are aware of and respond to others is a consequence of their intercultural orientation. This is neither a reference to one's beliefs about culture or race nor about views on researching in mathematics education. Rather, intercultural orientation is the perspective about human difference each person brings to interacting with others, in context. For researchers, it includes perceptions about the differences between personal and professional views and values around various types of work in mathematics education, the views of colleagues, and the views of stakeholders in research. Intercultural competence is the capability to shift perspective and appropriately adapt behavior to differences and commonalities (Hammer, 2009).

To build skill at establishing and maintaining relationships in, and exercising judgment relative to, cross-cultural situation requires the development of intercultural sensitivity (Bennett 2004). The developmental continuum for intercultural sensitivity has five milestone orientations to noticing, making sense of, and responding to difference: denial, polarization, minimization, acceptance, and adaptation.

With mindful experience a person can develop from ethnocentric ignoring or denial of differences, moving through an equally ethnocentric polarization orientation of an us-versus-them mindset. With growing awareness of commonality, a person enters the less ethnocentric orientation of minimization of difference, which may over-generalize commonalities. From there, development leads to an ethnorelative acceptance of the existence of intra- and intercultural differences. Further development aims at a highly ethnorelative adaptation orientation in which differences are anticipated and responses to them readily come to mind for effectively brokering cross-cultural interaction. Figure 6 offers a visual summary.

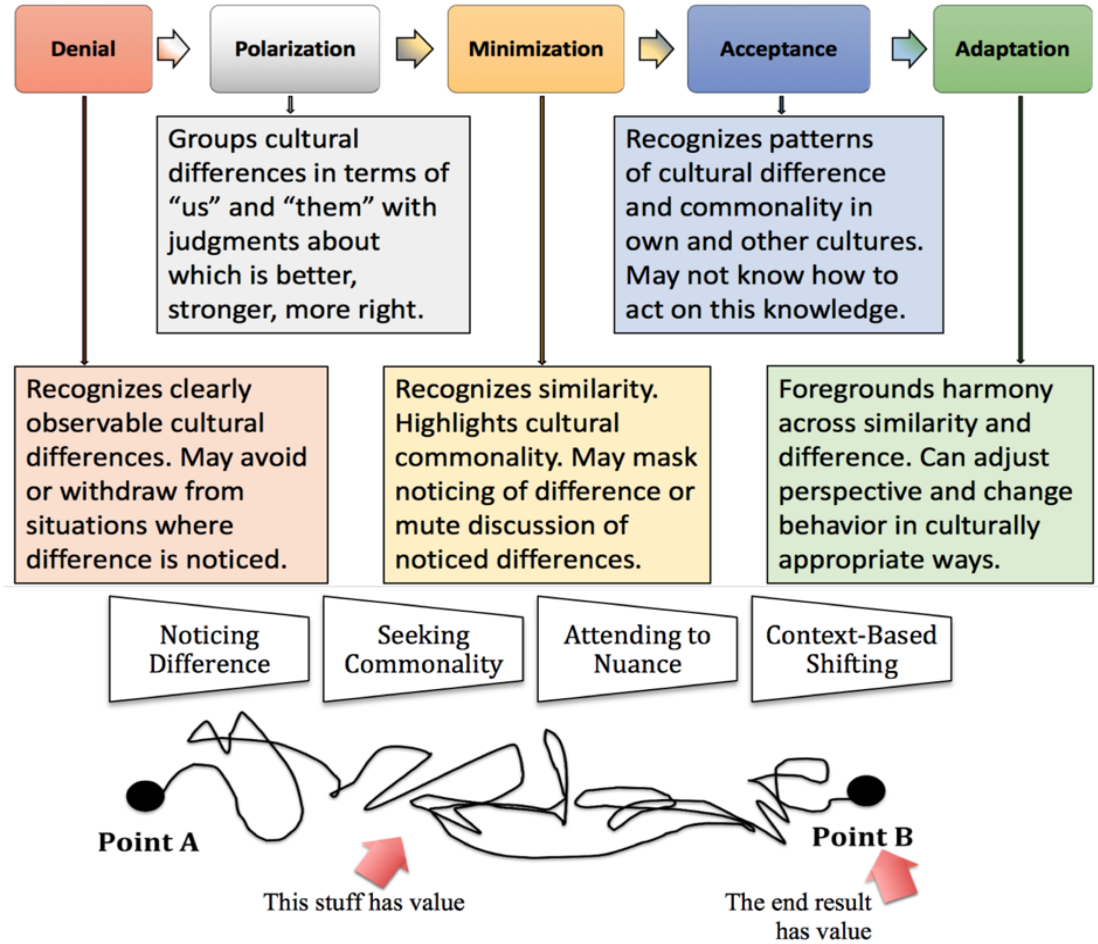


*Figure 6. Diagram summarizing intercultural orientation developmental continuum. One may start at point A and eventually develop to point B, but the path is not linear.*

To meet the goal for this report of synthesizing information that can support researchers in working towards equity as a central feature of RUME, here we have only briefly summarized some of the language and structure for intercultural orientation. We have discussed and illustrated use of this framework at length, particularly in RUME contexts, elsewhere (e.g., Hauk & Toney, 2016; Hauk, at al., 2017; Hauk, Toney, Jackson, Tsay, & Nair, 2014; Hauk, Toney, Nair, Yestness, & Troudt, 2015). Though not yet a common perspective in RUME, understanding oneself and others through the lens of intercultural orientation has proven transformational for the authors in our efforts at acknowledgement, action, and accountability.

The takeaway message about orientations: all the consultative activity at the center of the design cycle involves challenging conversations with people who are different from each other and may have widely varying orientations to those differences (and to research). In particular, an explicit action that can be taken by researchers is seeking agreement to Singleton et al.'s four foundations by participants in challenging conversations (this includes researcher and non-researcher stakeholders; American Evaluation Association, 2011).

## Conclusion

Regular engagement in courageous conversations is required for the kind of stakeholder-communication-centered design cycle described in this report. It takes practice to ensure such conversations are productive and worthwhile for discussants. Encounters with stakeholders, particularly when topics are challenging, requires carefully considered orchestration. Intentional effort is needed to learn to do this. As a community we are still in the early stages of building the necessary knowledge and skills for an equity lens on research. Recently, in their study of successful collaborations between education faculty and science, technology, engineering, and mathematics (STEM) faculty, Bouwma-Gearheart, Perry, and Presley (2014) noted that for experienced *brokers* (i.e., facilitators between two groups) it took multiple semesters of mindful effort to support development away from a polarizing stance (i.e., each group recognizing difference between the two cultures but being overly critical of the other) towards an adaptive orientation in which each accepted the other culture and used differences effectually in collaborative work.

The Venn diagram (and negotiated definitions) for safe, comfortable, and brave space can function as either a personal reflective tool or an explicit, shared, tool for mindful conversation. Use of the tool might help identify how conversations and research designs are courageous (and challenging enough) to meet goals for equity and justice. For example, personal monitoring during a challenging conversation can provide feedback on how and when seeking safety and comfort may interfere with attaining equity goals.

We have illustrated a cyclic framework for setting goals, leveraging resources, and reaching beyond personal orientations for a new and inclusive RUME. A RUME that dismantles inequity. A RUME in which the very definitions of rigor and quality in research unpack the current value set, the *status quo*, in research and make plans for how to rebuild. To make progress, we must embrace the humanity and fallibility of ourselves and of those our research is meant to serve. The interaction between the authors and you, the reader, was an acknowledgment, an action, and an opening for accountability.

**Acknowledgement**

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When "Negation" Impedes Argumentation: The Case of Dawn

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*Abstract: This study investigates one student's meanings for negations of complex mathematical statements. One student from a Transition-to-Proof course participated in two clinical interviews. The student was first presented with several statements with either one quantifier or one logical connective and asked to negate these statements. Then, the student was presented with statements containing a combination of quantifiers and logical connectives and was asked to negate these statements. Lastly, the student was also presented with several complex Calculus statements and asked to determine if these statements were true or false on a case-by-case basis using a series of graphs. The results reveal that the student used the same rule for negation in both simple and complex mathematical statements when she was asked to negate each statement. However, when the student was asked to determine if statements were true or false, she relied on her meaning for the mathematical statement and formed a mathematically convincing argument.*

*Key words:* Negation, Argumentation, Complex Mathematical Statements, Calculus, Transition-to-Proof

Previous literature shows that students often interpret logical connectives (such as "and" and "or") and quantifiers (such as "for all" and "there exists") in mathematical statements in ways contrary to mathematical convention (Dawkins & Cook, 2017; Dawkins & Roh, 2016; Dubinsky & Yiparaki, 2000; Epp, 2003; Shipman, 2013). Researchers have recently called for others to also focus on the logical structure found within Calculus theorems and definitions (Case, 2015; Sellers, Roh, & David, 2017). Even in introductory Calculus courses, students must reason with logical components, such as quantifiers and logical connectives, in order to verify or refute mathematical claims. However, introductory Calculus textbooks do not discuss the distinctions among different connectives nor do they have a focus on the meaning of quantifiers or logical structure in algebraic expressions, formulas, and equations, even though these components are used in definitions and problem sets (Bittinger, 1996; Larson, 1998; Stewart, 2003).

In undergraduate mathematics courses, including Calculus, students are frequently asked to evaluate the validity (i.e. determine the truth-values) of mathematical conjectures that are complex in nature. By *complex mathematical statements*, I mean statements that have two or more quantifiers and/or logical connectives. For example, the Extreme Value Theorem (EVT) is an example of a complex mathematical statement. This theorem may be stated as follows:

| |
|---|
| If a function is continuous on $[a, b]$, then there exists a $c$ in $[a, b]$ such that for all $x$ in $[a, b]$, $f(c) \geq f(x)$, and there exists a $d$ in $[a, b]$, such that for all $z$ in $[a, b]$, $f(d) \leq f(z)$. |

*Figure 1.* The Extreme Value Theorem (EVT).

The EVT is a complex Calculus statement because it contains multiple quantifiers ("for all" ($\forall$) and "there exists" ($\exists$)) and logical connectives ("and" ($\wedge$) and "if…then…" ($\rightarrow$)).

In order for students to properly justify why complex Calculus statements are true or false, and for students to develop logical proofs, they must be able to describe the relationship between these statements and their negations (Barnard, 1995; Epp 2003). For example, for students to build strong meanings for the EVT, they need to be able to determine when graphs *do* or *do not* have an absolute maximum or minimum. Similarly, students in Calculus, Transition-to-Proof, or

Advanced Calculus may be asked to determine if sequences are convergent *or* divergent, or determine if functions or sequences are bounded *or* unbounded. If students are in a Transition-to-Proof or Advanced Calculus course, they may also need to utilize proof by contradiction, which relies on the ability to assume the negation of a statement is true (Lin, Lee, & Wu, 2003). Thus, we also must explore student meanings for negation in mathematical contexts that they will encounter in their undergraduate courses. Several studies have investigated student meanings for negation (Barnard, 1995; Dubinsky, 1988; Lin et al., 2003), but these studies do not explicitly address complex Calculus statements. In this paper, I will investigate one student's meanings for the negation of various types of mathematical statements as well as how these negation meanings affect her justifications for several Calculus statements. Thus, I seek to investigate the following research questions for this student:

1. *As mathematical statements become increasingly complex, will the student keep the same negation meanings? If some or all of her negation meanings change, which meanings change and how do they change?*
2. *How do the student's meanings for negation affect her evaluations of complex Calculus statements and her justification for these truth-values?*

## Literature Review

In order to better understand students' negations of entire complex mathematical statements, we must first analyze their understanding of quantifiers and logical connectives in given statements before analyzing their negations of the entire complex statement. This manuscript includes a synthesis of the literature on quantifiers and logical connectives, including literature that pertains to negating statements with these logical components.

### Student Treatment and Negations of Quantified Statements

Quantified statements, like statements with logical connectives, elicit a plethora of student meanings that differ from mathematical convention and are found throughout many mathematics courses. Some studies focus on a single quantifier, while others involve statements with multiple quantifiers. I focus attention on each of these trends in literature before summarizing what is known about students' negations of quantified statements.

**Single quantifiers.** Some statements only have one quantifier such as "for all" ($\forall$) or "there exists" ($\exists$). Both universally-quantified statements and existentially-quantified statements may elicit student interpretations that differ from conventional norms. Students often suggest that multiple examples are sufficient justification for universal statements (Healy and Hoyles, 2000). On the other hand, when proving a universally quantified statement false, students often don't believe that only one counterexample is sufficient to prove a universally quantified statement false (Balacheff, 1986; Galbraith, 1981). Some students also reject the notion that one example suffices for proving an existential statement is true (Tirosh & Vinner, 2004). Sellers, Roh, and David (2017) analyzed students' meanings for the individual quantified variables in complex mathematical statements. They found that Calculus students often do not distinguish a difference between "for all" and "there exists" when analyzing the validity of mathematical statements and instead may interchange meanings for "for all" and "there exists." Sellers et al. (ibid) noted that some students appeared to skip over quantifier words in the mathematical statements given, but added their own quantifications based on their meanings for the phrase "$f(c)=N$."

**Multiple quantifiers.** Previous studies also show that students often do not problematize the distinction between "for all… there exists…" ($\forall\exists$) and "there exists… for all…" ($\exists\forall$)

statements, and that ∃∀ statements are frequently misinterpreted as ∀∃ statements (Dubinsky & Yiparaki, 2000; Sellers, et al., 2017). Sellers et al. (2017) also found that one Transition-to-Proof student in their study treated ∀∃ statements like ∃∀ statements. Other studies also note that students may reorder the variables attached to the quantifiers in their explanation of a complex mathematical statement because they do not recognize the independence of the first variable and dependence of the second variable (Dawkins & Roh, 2016; Roh & Lee, 2011). For example, in the Extreme Value Theorem provided in Figure 1, one must consider each value of $c$ before finding the associated $x$ that depends on each $c$.

**Negations of quantified statements.** Since students often interpret quantified statements in unconventional ways, one may not be surprised that they also often negate these statements in unconventional ways. Lin, Lee, and Wu (2003) gave students statements that either had no quantifiers, a universal quantifier, an existential quantifier, or a unique existential quantifier. Students in this study were most successful with negating statements with no quantifiers, followed by the existential quantifier (∃), then the universal quantifier (∀), and finally the unique existential quantifier (∃!). Approximately half of the students correctly negated the universally quantified statements, but less than 20% of the students in the study correctly negated statements with a unique existential quantifier. Barnard (1995) presented students with both colloquial and mathematical statements and asked them to negate the statements. Even the more advanced students negated correctly less than 75% of the time for the seven provided mathematical statements. All student success rates were lower for more logically complex statements, and only 1% of the students answered every one of his 21 negation items correctly. The low success rates on the more complex mathematical statements may be partially attributed to students' emphasis on the negation of one of two quantifiers from the original statement (Barnard, 1995; Dubinsky, Elterman, & Gong, 1988).

These results suggest that students need formal training to properly negate quantified statements. Yet, Dubinsky, Elterman, and Gong (1988) suggest that formal training should not be equated to the teaching and memorization of rules for negation. Dubinsky et al. (ibid) found that students often memorize rules for negation incorrectly, and then proceed to use their own rules when negating statements. In their study, they presented students with two statements with multiple quantifiers. They analyzed students' reasoning about the statements, with a specific aim to analyze their negations of the statements. They mention three types of negation with quantification: *negation by rules* (memorization of rules), *negation by recursion* (parsing a statement and then negating each part appropriately), and *negating meaning* of a statement. Dubinsky et al. claim that negating the meaning of the statement is the most difficult of the three, but that every student that used negation by recursion achieved a correct negation. While students who negated recursively also negated correctly, students who used negation by their own memorized rules often made errors in their negations. Dubinsky et al. claim that these students either did not remember the rules correctly, or in the process of negation, would forget the rule.

### Student Treatment and Negation of Logical Connectives

Complex mathematical statements involve both quantifiers and logical connectives, such as conjunctions (and) and disjunctions (or), which are involved in the complex statements in this study. Both students' interpretations and negations of statements with logical connectives may not follow mathematical convention.

**Conjunctions and disjunctions**. In Dawkins and Cook's (2017) study of student's meanings for statements with disjunctions ("or" statements), teaching experiments were conducted with Calculus 3 students. Students in this study sometimes treated the disjunction (or) as if it were a conjunction (and), and claimed that both parts of the statement would need to be true for the entire statement to be true. Some students would often think that if one part of an "or" statement was false, then the entire statement was false.

Another common issue with disjunctions that has been noted in the literature is students' use of "or" as the *exclusive or* (Dawkins & Cook, 2017; Epp, 2003). Mathematical logic utilizes the *inclusive or,* which means that "or" *includes* the conjunctive case. Students, however, use an *exclusive or* meaning for disjunctions because they often interpret "A or B" as "either A or B," but *exclude* the conjunctive case, "A and B," from their consideration. For example, consider the statement, "All rectangles are parallelograms or have four right angles." Students using the *exclusive or* may consider the statement false because rectangles are *both* parallelograms and have right angles, and students may think that only one of the properties may be true of rectangles for the statement to be true.

**Negations of compound sentences.** Just as students often use unconventional logic such as the *exclusive or* in their interpretation of compound sentences (i.e. statements with more than one subject or predicate), their negations for these compound sentences often follow unconventional logic patterns as well. Students often negate the parts of the statement before and after the logical connective, but retain the logical connective itself (Epp, 2003; Macbeth, Razumiejczyk, del Carmen Crivello, Fioramonti, & Pereyra Girardi, 2013). Epp (2003) claims that in her classes, students often negate "John is tall and John is thin" with the statement "John is not tall and John is not thin" (p. 890). Macbeth et al. (2013) also investigated how students negate statements with conjunctions and disjunctions. Students often retained the conjunction or disjunction in the original statement in their negations, as expected. Macbeth et al. (2013) conjecture that this tendency is due to the brain's attempt to reduce cognitive load. This conjecture aligns with Khemlani, Orenes, and Johnson-Laird's (2012) study, where subjects provided more accurate responses for the denials of disjunctions than conjunctions. Khemlani et al. (ibid) hypothesize that statements with conjunctions are easier to cognitively process and interpret than disjunctions because conjunctions are true if both propositions are true, whereas disjunctions include three combinations for the propositions (TT, TF, and FT) to imply that the overall statement is true. On the other hand, using this theory, negations of conjunctive statements are more cognitively demanding than negations of statements containing a disjunction.

### The Effect of Context on Students' Logic

Some suggest that students may have difficulty with interpreting quantified statements or statements with logical connectives because the use of these phrases may change from colloquial to mathematical contexts. For example, when I colloquially state, "I'll get Chinese or Italian for dinner" I do not intend that I may pick up *both* Italian and Chinese, and thus, use the *exclusive or*. Shipman (2013) discovered that students thought "unique" meant "unequaled" instead of "sole," which also may be a result of colloquial use of the word "unique." Dawkins and Cook (2017) also noted that several of their students also made *semantic substitutions* to sentences. For example, a student may substitute "increasing" with "is not decreasing." However, this substitution is not mathematically valid because one should also consider the case of neither increasing nor decreasing (constant)[1].

Even within mathematical statements, students' logic is often guided by the mathematical content of the statement rather than the logical structure of the statement. Dawkins and Cook (2017) found that students changed their interpretation of statements with a disjunction depending on the mathematical context of the statement. For example, some students deemed "Given an integer number $x$, $x$ is even or odd" true, but "The integer 15 is even or odd" false because they stated that they already knew that 15 was odd.

**Discussion of Literature**

I have detailed issues associated with students' meanings for quantifiers and logical connectives and their negations. However, more work needs to be done to investigate how students may think about statements in the Calculus context that combine quantifiers and logical connectives and how they might negate these statements. In this study, I investigate how one student negates statements with quantifiers and logical connectives, and analyze how her meanings for negation are related to her evaluations of given statements.

## Theoretical Perspective

My perspective throughout this paper is that students construct their own meanings for quantified variables and logical connectives, and they construct their own meanings for the negation of statements with these logical components. These meanings are schemes. A *scheme* is a mental structure that "organize[s] actions[2], operations, images, or other schemes" (Thompson et al., 2014, p. 11). I cannot see a student's schemes, but can only do my best to create a model of students' negation schemes by attending to their words and actions throughout the clinical interview process. Schemes are tools for reasoning that have been built in the mind of the student over time.

My goal in this paper is to describe my best perception of one student's meanings for negations of mathematical statements at different moments. I use the phrase "student meaning" throughout this paper the same way in which Thompson (2013) explains that, from a constructivist viewpoint, an individual constructs his own meanings by assimilation to schemes (Thompson, 2013). When the student *assimilates* a mental object to a scheme, she is "extending scheme[s] to new objects" (Montangero, 1997, p. 71).

Different meanings, which are types of schemes, may be elicited in different moments due to the nature of a given task or question. Thompson et al. (2014) distinguish between *stable meanings* and *meanings in the moment*. Thompson et al. (ibid) describe a *meaning in the moment* as "the space of implications existing at the moment of understanding" (p. 13). Student meanings may change from one moment to the next because particular tasks could elicit different schemes, or the student could be assimilating information in the moment by making changes to their current scheme(s). Thus, I consider several different moments of interaction for each student because different moments of interaction may result in different types of student negation. I aim to describe my view of this particular student's negation schemes for different tasks and to describe how her meanings for negation impact her justifications for her evaluations of various

[1] As one anonymous reviewer astutely noted, this particular semantic substitution may be due to students (or even textbooks) confounding monotonically increasing with strictly increasing.

[2] By action, Thompson is referring to Piaget's meaning for action: "any thought, emotion, or movement that satisfies a need" (Piaget, 1968, p.6).

types of mathematical statements. I also analyze the student across different moments to determine if her meaning for the negation of quantified statements and her meaning for the negation of statements with logical connectives appear to be stable meanings.

While a student's meaning may consist of what the student does with a particular task, a student's *way of thinking* (Harel & Sowder, 2005; Thompson et al., 2014) may be a student's problem-solving process that is used across investigation of many different types of problems. If the student anticipates how she will reason through a negation task before even being provided with a specific statement, this may also be evidence that the student has a way of thinking about negation across different tasks. Dubinsky et al.'s (1988) categories for negation, such as negation by rules, may be characterized as students' ways of thinking about negation, as these categories describe students' general problem-solving strategies for the negations of many types of mathematical statements.

## Methods

This study is part of a larger study that will seek to answer these research questions with undergraduate students from various mathematical levels. For this particular study, I conducted clinical interviews (Clement, 2000) with one student, Dawn, who was currently enrolled in a Transition-to-Proof course at the time of the interview. Dawn completed two clinical interviews that were each two hours long. Both interviews were video-recorded. One camera captured her written work, while another camera captured her gestures. I chose different levels of tasks to determine if Dawn's negations stayed the same or changed across different levels of mathematical complexity.

### Interview Tasks

I first presented Dawn with thirteen statements with either one quantifier or logical connective to address my first research question. Two examples of these statements are shown in Figure 2 (left).

| Statements with One Logical Component | Statement with Two Logical Components |
|---|---|
| 1. Every integer is a real number.<br>2. 12 is even and 12 is prime. | There exists a real number $b$ such that $b$ is odd and negative. |

*Figure 2*. Selected items with either one logical component or two logical components.

Dawn was asked to evaluate (i.e. provide a truth-value) and negate each statement, as well as evaluate her negations. After she completed these tasks, I presented her with a list of possible alternative student negations. I created these hypothetical negations by changing different parts of each statement. These hypothetical negations allowed me to test a wider range of possible negations that Dawn might accept as valid negations.

In the second clinical interview, I first asked Dawn to negate complex mathematical statements with both a quantifier and a logical connective in an attempt to test whether her meanings for negations were stable across different tasks. I presented Dawn with two statements, like the one shown in Figure 2 (right), which involves two logical components (an existential quantifier and either a conjunction or disjunction). I asked Dawn to evaluate and negate these statements in the same manner as she did in the first interview. I used Dawn's negations of the more complex statements along with her negations for the statements with one logical component in the first interview to try to answer my first research question.

During this second interview I also presented Dawn with three complex mathematical statements from Calculus (shown in Figure 3) in order to address my second research question.

Statements 1 and 2 are based on the conclusion of the definition of a bounded function[3] and Statement 3 is based on the conclusion of the Extreme Value Theorem. For the statements and graphs shown in Figure 3, I asked Dawn to evaluate each graph with each statement and determine if a given statement was true or false for each function. The EVT only holds for continuous functions. Since I omitted the hypothesis of the EVT, there are cases where this statement I present is false. These graphs were selected in hopes that the student would use some of these graphs to show that the statements are false in some cases. Then, I used her justifications of why the statements are false to determine how she negated complex statements in the context of her justifications. Finally, I was able to compare these justifications with her previous negations in the first clinical interview.

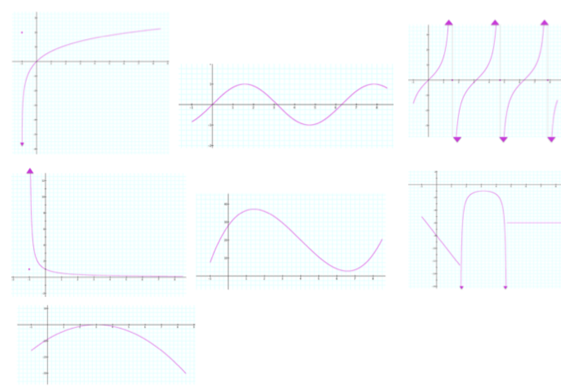
| **Example Complex Mathematical Statements** | **Graphs** |
|---|---|
| **<u>Statement 1.</u>** There exists a real number $m$ and a real number $M$ such that for all $x$ in [-1, 8.5], $m \le f(x) \le M$.<br><br>**<u>Statement 2.</u>** There exists a real number $m$ such that for all $x$ in [-1, 8.5], $m \le f(x)$, and there exists a real number $M$ such that for all $x$ in [-1, 8.5], $f(x) \le M$.<br><br>**<u>Statement 3.</u>** There exists a $c$ in [-1, 8.5], such that for all $x$ in [-1, 8.5], $f(c) \ge f(x)$, and there exists a $d$ in [-1, 8.5], such that for all $z$ in [-1, 8.5], $f(d) \le f(z)$. |  |

*Figure 3.* Complex statements and graphs used in follow-up interview.

The graphs that I presented and the $\exists\forall$ statements shown in Figure 3 provided opportunities for Dawn's meanings for negation to be elicited with statements from a Calculus context, in hopes of beginning to answer my second research question.

**Data Analysis**

I used grounded theory (Strauss & Corbin, 1998) to analyze videos from Dawn's interview as well as her written work. Hence, findings about Dawn's negation meanings emerged from the data. I identified moments where distinctions could be made about Dawn's negation meanings. A new moment began when Dawn was presented with a new question or task, she changed her evaluation or interpretation of a given statement, or if she changed her argument or negation of a statement in any way. After identifying these moments of interest, I compared Dawn's negations of statements with one quantifier or one logical connective with her negations of statements with both a quantifier and a logical connective. Finally, I compared her negations in the context of her justification for each Calculus statement with all previous negations.

## Results

I noticed a consistent pattern in how Dawn chose to negate the statements when I directly asked her to provide negations. However, when I presented the last three complex mathematical statements along with graphs, and only asked for her to justify why the statement was true or false for each graph, Dawn's negations in her argumentation did not always match her previous

[3] Statements 1 and 2 are mathematically equivalent, but were both given to students to determine if the way in which these statements were structured would change students' interpretations and negations of the statements.

patterns of negation. The mathematical content of the tasks and questions appeared to influence Dawn's patterns for the negation of statements with quantifiers and logical connectives.

## Consistent Negation Patterns

There were patterns in Dawn's thinking about quantifiers and logical connectives and patterns in her negations of statements containing quantifiers and logical connectives. I will highlight general patterns that emerged from my data analysis and show how these patterns remained consistent even with a more complex statement.

**Patterns across all negations.** Dawn stated that all negations were basically the "opposite statements" or the "reverse of what we originally said," but in practice, she pondered what would constitute an opposite meaning of the original statement. For example, when interpreting the statement "Every integer is a real number" she stated that the "opposite of every is none," and wrote the negations, "Every integer is not a real number" and "There is no integer that is a real number." However, in a later moment, she considered that the opposite of "every" may be "some."

While Dawn did not exhibit a consistent use of the word "opposite," she did have a consistent method for negating all the statements given in the first interview. She frequently stated that negations could negate one part of the statement, but not both parts of the statement. Dawn often wrote multiple negations as a result of this way of thinking about negation, changing one part of a given statement, but not both parts. I asked Dawn to explain why she believed she should only change one side of a statement for its negation. She stated, "In general, it's just some kind of rule that I follow, like you only negate one side."

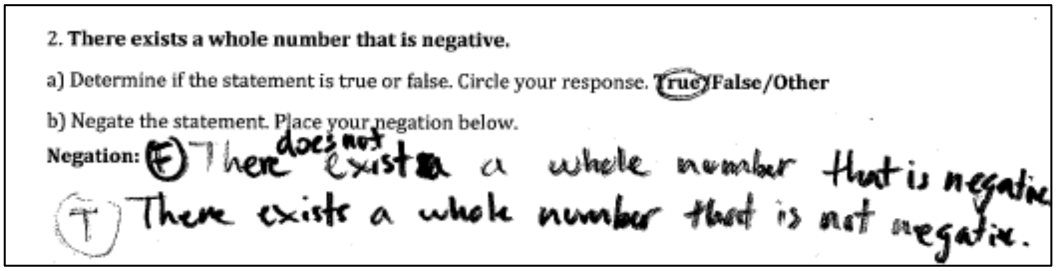
When evaluating the negations she constructed, she often concluded that there could be multiple negations of the same statement, and these negations could have different truth-values. I asked Dawn if two negations could have different truth-values. She replied, "I think it is okay for them to have different truth-values." For one statement where Dawn wrote two negations with different truth-values she also stated, "I don't think they [the two different negations] have the same meaning, but I think they're both valid."

Since Dawn stated that negations could have a variety of different truth-values, and she stated that negation involves changing one part of the statement, the evidence thus far suggests that, for Dawn, "negation" is related to a constructed *procedure* rather than a means for proof or disproof. Thus, the meaning of a statement and *its truth-value had no effect on Dawn's decision about the validity of a negation*; the validity of a negation for Dawn was assessed based on whether or not it followed her procedure.

Dawn's reliance on a rule to change one part of a statement for a negation appeared in statements with both one logical component and with multiple logical components. I will now show how Dawn used her same way of thinking about negation with statements containing an existential quantifier, a conjunction, or both a quantifier and a conjunction.

**Dawn's meaning for "there exists" and its negation**. Dawn consistently interpreted "there exists" as "there is at least one," but her negations for existential statements of the form "There exists an $x$ such that $P(x)$" did not always follow mathematical convention. For statements with an existential quantifier of the form "There exists an $x$ such that $P(x)$," she referred to "There exists $x$" as one part and "such that $P(x)$" as another part of the statement, and claimed that she "could only negate one part of the statement."

For any statement of the form "There exists an $x$ such that $P(x)$," Dawn preferred to start with the negation of the form "There does not exist an $x$ such that $P(x)$," which is a valid negation. However, she also stated that statements of the form "There exists an $x$ such that *not* $P(x)$" were valid negations because this also changed only one part of the statement. For example, for the statement, "There exists a whole number that is negative," Dawn referred to "There exists a whole number" and "that is negative" as two separate parts, and claimed that she could only negate one part of the statement. In Figure 4, Dawn's negations for this statement are provided.


2. **There exists a whole number that is negative.**

a) Determine if the statement is true or false. Circle your response. True/False/Other

b) Negate the statement. Place your negation below.

**Negation:** F There does not exist a whole number that is negative

T There exists a whole number that is not negative.

*Figure 4.* Dawn's negation for a statement with 'there exists.'

I asked Dawn to explain the meaning of each of her negations. She stated that her first negation, "There does not exist a whole number that is negative," means that "there is no whole number that is negative, so every whole number is going to be positive," and that her second negation, "There exists a whole number that is not negative," means that "there is a whole number that is not negative, so there is a whole number that is positive." She later based her evaluations of the statements on these meanings: "For the second one, I would say it is true, but the first one is false, because there does exist a whole number that is negative." I next presented Dawn with hypothetical negations. Even though she said that the meaning of the first negation is the same as "every whole number is going to be positive," she did not select this as a hypothetical negation. Dawn usually did not accept negations of the form $\forall x, \sim P(x)$ for statements with an existential quantifier because she said that changing the "there exists" to "for all" would be "changing too much."

In this example, Dawn evaluated each statement and each of her negations based on her interpretation of each statement and each negation. However, even though Dawn recognized that the meanings of her negations were different, and their truth-values were different, these differences were inconsequential to Dawn. Since these negations followed her rule to change one part of the original statement, she believed that the negations were valid. Her rejection of the hypothetical negation "Every whole number is going to be positive" indicates that she did not find alternative negations with equivalent meanings. Instead, the evidence indicates that she is choosing to follow her negation rule. Additional evidence that Dawn was relying on a rule for negation was found when Dawn negated another quantified statement where she provided two negations. In this moment she stated, "The only thing different [in the two negations] is the quantity [the quantifiers were different], and I don't think that matters, how many" elements satisfy a proposition. Since Dawn used a memorized rule for negation rather than viewing it as a way to prove or disprove the original statement, the number of elements she was referring to in each of her negations became inconsequential to Dawn in light of her rule.

**Dawn's meaning for "and" and its negation**. Dawn's algorithm for negating one part of a statement was also consistent with her negation of statements with a conjunction or disjunction because she still claimed that she could negate one part of a statement, but not both parts. She also continued to write and select negations regardless of their meaning or truth-value. For the statement, "12 is even and 12 is prime," Dawn wrote the negations shown in Figure 5:

8. **12 is even and 12 is prime.**

a) Determine if the statement is true or false. Circle your response. **True/False/Other**

b) Negate the statement. Place your negation below.

**Negation:** F 12 is odd and 12 is prime.

T 12 is even and 12 is not prime.

*Figure 5.* Dawn's negation of a statement with 'and.'

Yet again, Dawn changed one part of the statement for each negation, and again, she had two different negations with varying truth-values. The following dialogue between the interviewer (I) and Dawn (D) emerged after Dawn had created these two negations:

> *I*: Could you say that 12 is odd and 12 is not prime [is a valid negation]?
> *D*: I don't think so.
> *I*: Okay. Why not?
> *D*: You'd be negating both sides of the statement.
> *I*: Okay, and why is it not okay to negate both sides of an "and" statement?
> *D*: I just feel like you could only do one.
> *I*: Do you feel like that's something that you memorized, or are you thinking about this particular statement about evens and primes?
> *D*: In general, it's just some kind of rule that I follow, you can't negate both sides...I just took the original statement and did the opposite of the first part. It's not a true statement, but it *is* a possible negation.

For both of these negations, Dawn retained the logical connective and changed one part of the original statement in each negation. One may notice that Dawn retaining the logical connective is characteristic of shallow processing (Khemlani et al., 2012; Macbeth et al., 2013). However, her rule for changing one part of the statement does not align with shallow-processing expectations of retaining the conjunction and negating both propositions (i.e. accepting 12 is odd and 12 is not prime). Dawn never explained where she learned that she could only change part of the statement, but instead simply stated that she "felt like [she] could only [negate] one" part and that it was a "rule that [she] follow[s]." Even though she noticed that her other negation is true, and the original statement is false, she stated that this "*is* a possible negation," which indicates that, again, the truth-value was inconsequential to her determination about the validity of the negation.

**A combination of negation meanings**. The statement "There exists a real number *b* such that *b* is odd and negative," has two logical components. Dawn interpreted the negation of both the quantifier and the conjunction in this statement in a similar manner as her earlier negations, as seen in her two negations in Figure 6:

There exists a real number ***b*** such that ***b*** is odd and negative. True

**Negation:**

There does not exist a real number b such that b is odd and negative.
False

There exists a real number b such that b is ~~not odd~~ even and negative.
True

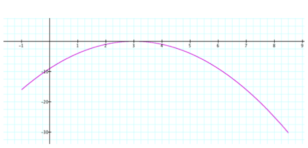
*Figure 6.* Dawn's negation for a statement containing both "there exists" & "and."

These negations are similar to the negations Dawn preferred for "there exists" statements in the first interview, as they are also of the form "There does not exist an $x$ such that $P(x)$," and "There exists an $x$ such that *not* $P(x)$" (even though her negation of $P(x)$ is incorrect). Yet again, she did not consider the use of a universal quantifier in her negations and only changed one part of the statement. Dawn also negated the proposition within the statement that contained a conjunction in the same manner that she did with the first set of statements. The phrase "*b* is odd and negative" has its own parts that Dawn also considered. She negated "*b* is odd and negative" as "*b* is even and negative." She verbalized that she could have also used "b is odd and positive" for this part of her second negation. I asked her to consider explaining to a friend why her negation for the first complex statement was valid, to which she replied, "I would tell them that [my negation is correct] because I changed the second half of the statement." This reply suggests that Dawn was assessing the validity of her negation based on her rule for negating one part of the statement, rather than comparing the *meaning* of the negation with the original statement.

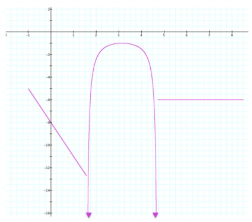
From all of the previous example moments, there appears to be sufficient evidence that Dawn's meaning for the negation of statements with quantifiers and her meaning for the negation of statements with a logical connective are both stable meanings. Dawn also appears to have a way of thinking about negation that ties together both of these meanings for negation. She appears to have a way of thinking about negation that is a rule for negation (Dubinsky et al., 1988). This rule appears to be that she can alter one part of a compound phrase or one quantifier, but not more than one part of a given statement.

**Negation in Argumentation: When Negation Isn't Viewed as Negation**

I have already detailed Dawn's treatment of negation for the previous examples where I asked her to provide a negation. In the last set of tasks, I did not ask her to negate, but rather asked her only to determine if the statements were true or false on a case-by-case basis and to justify her claims. For Statement 3, Dawn interpreted the original statement as intended. Dawn explained why the statement shown below is true for the given graph:

| Statement & Graph Presented | Transcript |
|---|---|
| There exists a $c$ in [-1, 8.5], such that for all $x$ in [-1, 8.5], $f(c) \geq f(x)$, and there exists a $d$ in [-1, 8.5], such that for all $z$ in [-1, 8.5], $f(d) \leq f(z)$.<br> | *D*: There is a maximum $y$-value at 3 [$x$=3] and a minimum $y$-value here (points to (8.5, $f$(8.5)). So no matter what $x$ is, this [$f$(8.5)] is going to be the least $y$-value.<br>*I*: So what part of the statement tells you [that] you need to focus on the least $y$-value and the largest $y$-value?<br>*D*: Because we want to pick values for $c$ and $d$ strategically so that they are going to be the maximum and minimum $y$-value.<br>*I*: What part tells us we're going to pick the max and min?<br>*D*: Here, for all $x$, you want it to, no matter what the value of $x$, the value of $f(x)$ is going to change. And you want this statement here, this inequality, to hold true, and there's only one instance where that can be true—at the max or min. |

Dawn exhibited a conventional interpretation for this statement. She expressed that she needed to choose *the* maximum or minimum that works *for all x*. Dawn's meaning for this statement and its negation was also revealed in her explanation of when the statement is not true. In the following example, Dawn claimed that the same statement is false for the given case.

| Statement & Graph Presented | Transcript |
|---|---|
| There exists a $c$ in [-1, 8.5], such that for all $x$ in [-1, 8.5], $f(c) \geq f(x)$, and there exists a $d$ in [-1, 8.5], such that for all $z$ in [-1, 8.5], $f(d) \leq f(z)$.  | *D*: The minimum *y*-value is $^{-}\infty$, so you couldn't pick a value for… *d* that would always make this inequality true (points to $f(d) \leq f(z)$).<br>*I*: Let's say your friend said, "For all the values that I look at, for all the *y*-values that I look at, if I chose *any* value for *d*, then I can always find a smaller value [than *f*(*d*)]..." Would you agree with your friend's argument?<br>*D*: Yeah, I would agree with his argument.<br>*I*: Would you say that his argument is the same as your argument?<br>*D*: Yeah, because I said there isn't a value for *d* where there's the smallest *y*-value. I think that's kind of the same thing. It isn't the smallest because you could always find one smaller. |

Dawn said that she *could not pick* a value for *d* such that this value of *d* would *always* satisfy the inequality. This response is similar to the negation "there *does not exist a d* such that $f(d) \leq f(z)$." Dawn's response was consistent with her prior approach to negate one part of a statement in her negation. I responded by asking Dawn to consider an alternative negation that used a universal quantifier and changed more than one part of the statement to test her meaning for negation of Statement 3 against her previous negations.

In the context of this statement where Dawn was asked about her argument rather than for a negation specifically, she accepted a negation that involved changing more than one part of a statement and she did not mention having an issue with the universal quantifier changing too much of the statement. Her original denial aligns with the argument, "there *does not exist* an *x* with a corresponding minimum *y*-value," but she also recognized that my proposed argument, "for *any x*-value, *d*, a smaller *y*-value than *f*(*d*) can be found," was equivalent to her original denial. Thus, she accepted the argument that aligned with the negation "for *any* value of *d,* there exists a *z* such that $f(z) \leq f(d)$" by stating that this argument was "kind of the same thing" as her argument. She even explained why the logic for the two negations is equivalent: the *y*-value "isn't the smallest because you could always find [a *y*-value] smaller." Even though she had previously rejected alternate negations in the first interview that involved a universal quantifier, in the context of justification for this Calculus statement, she recognized that an alternate negation with a universal quantifier was valid for showing that the statement was false for this graph.

In instances when my question or request omitted the word "negation," Dawn considered the *meaning* of the statement rather than her memorized rule to negate one part of the statement. Dawn considered the truth-value of the graph in relation to the statement first during this portion of the interview, in contrast to previous moments where Dawn used her rule first, and then evaluated the given statements. Her interpretation of a statement and her negation for that statement varied based on the context of my question. These moments in the second interview were characterized by the question, "Is this statement true or false for this graph?" rather than the command "Negate this statement." The word "negation" appeared to alert Dawn to negate only one part of the statement. However, when asked to think about the validity of a statement in a particular context, Dawn's approach was to use her reasoning about the truth of a statement, and apply logical argument to justify her evaluation.

## Conclusion & Discussion

When responding to negation tasks in the first interview, Dawn negated one part of a given statement, but not both parts of a given statement. Her meanings for the negation of quantifiers and logical connectives appeared to be consistent when she analyzed statements with multiple logical components for tasks that also asked her to provide a negation. This finding is similar to Dubinsky et al.'s (1988) finding that students tend to use rules (which may or may not be correct) to negate a statement. Dawn viewed that no matter what type of statement was given, the command to "negate" implied changing one part of the given statement. Her procedural approach for negating a statement could help explain why some students only negate one of two quantifiers when statements with multiple quantifiers (Barnard, 1995; Dubinsky, 1988) and why some often retain disjunctions and conjunctions in their negations (Epp, 2003; Macbeth et al., 2013). In Dawn's words, changing two quantifiers or changing a logical connective might be "changing too much" in the student's view.

Dawn's negations are also consistent with other literature that has claimed that students' logic can change across different tasks (Dawkins & Cook, 2017; Durand-Gurrier, 2003). In this study, I found that Dawn negated by rules when she was commanded to negate. In the second interview, complex statements with quantifiers and logical connectives were given, and Dawn was asked to provide reasons for why a graph made the statement true or false for a given graph. The directive to classify these statements as true or false did not include the word "negation." Dawn accepted different negations as valid in tasks that used the word "negation" than she did when she was asked to justify why a statement was false.

For many students, the word "negation" may be associated with a *procedure* rather than using logical arguments based upon their own *reasoning*. Dubinsky et al. (1988) noted that students in their study often negated by a set of memorized rules that may or may not be correct. This study sheds light on one specific way of thinking about negation, that is, one specific general rule for negation. More importantly, this study also reveals that some students who negate by memorized rules may not negate by rules if the task is changed from "negate" to "justify why the following statement is false." Whenever Dawn was asked to "negate," she appeared to have stable meanings for the negations of quantified statements and statements with logical connectives, even as statements became increasingly complex. However, the command to determine if a statement was true or false actually led her to negate according to mathematical convention. Similarly, her evaluations were more aligned with mathematical convention when she focused on forming arguments based on her evaluations. Students who apply a memorized rule to negate a mathematical statement may have the ability to negate appropriately if the word "negation" does not hinder their argumentation. This procedural emphasis means that students may not even connect the truth-value of a negation as opposite of the original statement.

Instructional considerations should be made as a result of this new information. Phrases like "opposite of the original statement" or "not the original statement" may be vague for students not attending to the elements of a set for which we are considering a proposition. The treatment of negation as a procedure rather than an argumentation tool may cause students to lose sight of the usefulness of a negation. If students are presented with rules for negation but do not have opportunities to construct these rules, they may view these rules as something to memorize for the command "negation" rather than logically derived properties for expressing refutation of a given statement. Students may benefit from tasks that use the command, "Explain why the following statement is true or false" and from negating quantifiers and logical connectives in different mathematical contexts. Then, the students may be asked questions that may help them construct their own rules for negation that are consistent with their argumentation. This study suggests that the word "negation" may be more appropriate to use *after* students have constructed formal rules for negation based on their justifications for why statements are false.

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Cognitive Consistency and Its Relationships to
Knowledge of Logical Equivalence and Mathematical Validity

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*The purpose of this study is to explore how cognitive consistency in logical thinking is related to knowledge of logic and knowledge of mathematical validity. We developed a logic instrument and administered it to forty-seven (47) undergraduate students who enrolled in various sections of a transition-to-proof course. The analysis of the students' scores on the logic instrument indicated that students' knowledge of logical equivalence and their knowledge of mathematical validity were somewhat related to one another. On the other hand, cognitive consistency was not closely related to either student knowledge of logic or knowledge of mathematical validity. Based on these findings, we address the importance of cognitive consistency in logical thinking and discuss implications for the teaching and learning of logic in mathematical contexts.*

*Keywords:* cognitive consistency, logical equivalence, mathematical validity, transition-to-proof

Our society expects people to have ability to make decisions in their workplaces more efficiently by deducing valid inferences from a tremendous amount of information and resources. In fact, a person's logical thinking plays a crucial role in generating valid arguments from the given information as well as in evaluating the validity of others' arguments in workplaces. Hence, training our students as logical thinkers has been a central component in education (NCTM, 2000; NGAC & CCSSO, 2010; NRC, 2005). Many universities offer mathematics courses to introduce logic and various proof structures for valid arguments in mathematical contexts. However, research in mathematics education reports that undergraduate students have weak knowledge of logic and mathematical validity (e.g., Dubinsky, Elterman, & Gong, 1988; Epp, 2003; Inglis & Simpson, 2007; Morris, 2002; Martin & Harel, 1989). Such a deficiency of student knowledge of logic would entail serious difficulties with using logical thinking and deductive reasoning, in particular, to construct valid arguments (Bell, 1976; Coe & Ruthven, 1994; Hanna & Barbeau, 2008; Healy & Hoyles, 2000; Holyes & Küchemann, 2002; Ko & Knuth, 2009; Moore, 1994; Recio & Godino, 2001; Senk, 1989; Weber, 2001), to comprehend or interpret arguments of their teachers or in textbooks (Alcock & Weber, 2005b, Hazzan & Zazkis, 2003; Mamona-Downs & Downs, 2005; Selden & Selden, 1995), or to evaluate the validity and the soundness of someone's arguments (Alcock & Weber, 2005; Ko & Knuth, 2013; Martin & Harel, 1989; Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; Selden & Selden, 2003).

With the importance of student knowledge of logic, we also consider *cognitive consistency* as an essential component in logical thinking. Generally, cognitive consistency refers to "an intra-individual psychological pressure to self-organize one's beliefs and identities in a balanced fashion" (Cvencek, Meltzoff, & Kapur, 2014, p.73). Cognitive psychologists explain such a tendency as people behave in ways that maintain cognitive consistency or minimize cognitive dissonance among their interpersonal relations, intrapersonal cognitions, beliefs, feelings, or actions (Bateson, 1972, Festinger, 1957; McGuire, 1966). Similarly, constructivists' theory of learning also posits students' recognition of cognitive inconsistencies as a necessary condition for their learning. Once a student recognizes inconsistencies in his way of thinking of a mathematical problem, he would be situated with a cognitive conflict and get perturbed because he is no longer able to assimilate to his existing scheme. Learning would then occur to the student by his attempt to modify his existing scheme or to construct a new scheme that resolves the cognitive conflicts (Piaget, 1967). For instance, based on his own knowledge and scheme, a student might deduce two statements such as '$x$ is an integer' and '$x$ is not an integer' from given information. Logically speaking, each of these statements contradicts one another, thus two statements cannot be accepted simultaneously. Such a logical contradiction is a fatal flaw that makes the student's entire argument meaningless. Once a student recognizes such a logical contradiction in his argument, he would attempt to find a way to remove it from his argument. On the other hand, if the student does not recognize the contradiction in his argument, he would be still in cognitive inconsistency in his logical thinking.

One's recognition of cognitive inconsistency in his own thinking will be the first step in self-regulating one's own cognition. However, if a student does not recognize cognitive inconsistency in his own belief, attitude, or knowledge structures, the student would not take any effort to change or modify his existing scheme or knowledge structure. Thus, it is very important to train students not only to gain more knowledge of logic but also to maintain cognitive consistency in logical thinking.

At this point, one might expect that the more knowledge of logic students has, they would less likely deduce logical contradictions from given information; or they would recognize logical contradictions if they happen to deduce logical contradictions from given information. It might also be expected that students who do not recognize logical contradictions in their arguments would not be knowledgeable in logic. This study explores *how students' cognitive consistency in logical thinking is related to their knowledge of logic and mathematical validity.* To be more specific, as an exploratory study, we address the following research questions:

1) Do students with more knowledge of logical equivalence tend to have stronger cognitive consistency in logical thinking?
2) Do students with more knowledge of mathematical validity of arguments tend to have stronger cognitive consistency in logical thinking?

For the purpose of this study with the specific research questions, we developed the logic instrument to systematically measure three components of students' logical thinking from student responses to the logic instrument as follows: knowledge of logical equivalence between two statements, knowledge of mathematical validity of arguments, and cognitive consistency in logical thinking. We use the term *Knowledge of logical equivalence* (KoLE) between two statement, or shortly, knowledge of logical equivalence, to refer to knowing a relationship between two statements that have the same truth value in every possible case. *Knowledge of mathematical validity* (KoMV) of arguments, or shortly knowledge of mathematical validity in this paper, refers to knowing that the truth of the premises of an argument in mathematics logically guarantees the truth of the conclusion of the argument. By cognitive consistency in logical thinking (CC), or shortly by *cognitive consistency* in this paper, we mean an individual's psychological pressure to self-organize his/her thinking to have no logical contradiction.

In this study, we shortly use the term cognitive consistency to refer to cognitive consistency *in logical thinking*. While we hope this study provides new insights into the theories of cognitive consistency, our foci are distinct to previous ones from two aspects. First, in exploring the role of cognitive consistency, this study pays more attention to *contexts of mathematical logic* such as logical connectives and quantifiers, rather than focusing on personal or interpersonal attitudes and behaviors in social contexts (c.f., Cooper, 1998; Gawronski & Strack, 2004; Gawronski, Walther, & Blank, 2005; Stone & Cooper, 2001). Second, this study focuses on whether students recognize cognitive inconsistencies in their logical thinking rather than how students reconcile cognitive inconsistencies after recognizing them in their reasoning (c.f., Dawkins & Roh, 2016; Ely, 2010; Oehrtman et al., 2014; Roh & Lee, 2011).

## Research Methodology

This study was conducted in the spring semester of 2014 at a large public university in the United States. Among 137 undergraduate students who were taking an introductory proof course at the semester, forty-seven (47) students voluntarily participated this study to complete the logic instrument that we designed to explore undergraduate students' logical thinking in mathematical contexts. Due to the pre-requisite for the introductory proof course at the university, the participants had already completed at least the first semester calculus course. In addition, as the logic instrument was administered at the last week of the semester when the participants enrolled in the introductory proof course, the participants of this study were those who had already been exposed to the terms used in the questions of the logic instrument, such as equivalent statements, implications, negation, and valid arguments. Twenty-three participants (48%) were mathematics majors whereas twelve participants (26%) were mathematics education majors. The rest of the participants (twelve students, 26%), labeled as others, were students whose major areas of study were neither mathematics nor mathematics education.

### The Logic Instrument

For this study, we developed a logic instrument consisting of twelve questions in total. The first part (seven questions) was designed to test students' knowledge of logical equivalence between two statements. On the other hand, the second part of the logic instrument (five questions) was designed to test students' knowledge of mathematical validity of arguments as well as cognitive consistency. We describe each part of the logic instrument in detail.

**Part 1 of the Logic Instrument** The first part consists of seven multiple choice questions. These questions were designed to assess students' knowledge of logical equivalence amongst conditional statements with/without quantifiers. All questions in Part 1 present one or a pair of statements whose logical forms are frequently found in undergraduate mathematics textbooks from calculus and beyond. See Table 1 for the logical forms of the statements given and the nature of each question in Part 1 of the logic instrument.

*Table 1 Summary of the questions in Part 1 of the logic instrument*

| | LOGICAL FORM OF THE GIVEN STATEMENTS | NATURE OF THE QUESTIONS |
|---|---|---|
| Q1 & Q3 | $P(x) \rightarrow Q(x)$ | Mark off all logically equivalent instances to the given statement |
| Q2 & Q4 | A pair of statements in the forms of $\forall x \exists y P(x, y)$ & $\exists y \forall x P(x, y)$ | Mark off the best description about the logical relationship between the given statements |
| Q5, Q6, & Q7 | $\forall x, P(x) \rightarrow Q(x)$ | Mark off all logically equivalent instances to the negation of the given statements |

Several instances are also presented with the statement(s) in each question and students are asked to mark off all relevant ones among the given instances. For instance, Q1 and Q3 (see Figure 1 for Question 1) presents a conditional statement and asks to mark off O for all its equivalent statements to the conditional statement among the given instances. The main difference between Q1 and Q3 is in the contexts: The statement given in Q1 is from a non-mathematical context (a person SAM) whereas the statement in Q3 is from a mathematical context (real numbers and inequalities between two functions).

*Q1. Consider the following statement* (*) *about a person, SAM:*
(*) *If $f$ is a quadratic function, SAM can solve the equation $f(x) = 0$.*
*For each of the following statements, mark (O) if the statement is equivalent to the statement* (*)*; otherwise mark (X).*
_________ $f$ is a quadratic function and SAM can solve the equation $f(x) = 0$.
(1) _________ $f$ is a quadratic function or SAM cannot solve the equation $f(x) = 0$.
(2) _________ $f$ is not a quadratic function or SAM can solve the equation $f(x) = 0$.
(3) _________ If SAM can solve the equation $f(x) = 0$, $f$ is a quadratic function.
(4) _________ If SAM cannot solve the equation $f(x) = 0$, $f$ is not a quadratic function.
(5) _________ If $f$ is not a quadratic function, SAM cannot solve the equation $f(x) = 0$.

*Figure 1. Q1 in the logic instrument*

Questions 2 and 4 (see Figure 2 for Question 4) both present a pair of statements involving two quantifiers ∀ and ∃ in which the order of quantifiers in one statement is ∀∃ and that in the other statement is ∃∀ while the predicates are the same. These questions ask to mark off the most relevant logical implication between the statements in a pair. The contextual difference between Q2 and Q4 is similar to that between Q1 and Q3.

Q4. *Let $f$ be any function from $\mathbb{R}$ to $\mathbb{R}$. Consider the following two statements* (i) *and* (ii).

(i) *For any $x \in \mathbb{R}$, there exists* $\mathrm{M} > 0$ *such that* $|f(x)| < \mathrm{M}$.
(ii) *There exists* $\mathrm{M} > 0$ *such that for any $x \in \mathbb{R}$,* $|f(x)| < \mathrm{M}$.

*Check the most accurate description about the two statements.*
(a) _________ (i) is equivalent to (ii).
(b) _________ (i) implies (ii) but (ii) does not imply (i).
(c) _________ (ii) implies (i) but (i) does not imply (ii).
(d) _________ None of the above is correct.

*Figure 2. Q4 in the logic instrument*

Q5, Q6, and Q7 present the same conditional statement with a universal quantifier, "for any $x > 0$, if $f(x) \geq g(x)$, then $g(x) < h(x)$" and ask to mark off all statements that are equivalent to the *negation* of the given statement. The difference among these three questions is mainly in the use of quantifiers and the sets of the discourse in given instances. To be more specific, the instances for Q5 remain the universal quantifier 'for any' whereas the instances for both Q6 and Q7 use the existential quantifier 'there exist'; in addition, the instances for both Q5 and Q6 remain the same set of discourse '$x > 0$' to that of the given statement whereas Q7 uses the compliment '$x \leq 0$' of the set of discourse of the given statement. See Figure 3 or the statement and some instances given in these three questions.

All statements given in the questions in Part 1 of the logic instrument are *open* statements involving at least one free variable so that the truth-value of each statement cannot be determined. We purposely created and included *only* open statements to the questions in Part 1 in order to avoid the cases of students who answer to the questions based on their determination of the truth-value of a statement. It is because otherwise students might focus not on the logical structures, but on the truth-values, of the given statements. For instance, the statement (*) given in Q1 "if $f$ is a quadratic function, then SAM can solve the equation $f(x) = 0$" is an open statement with two free variables $f$ and SAM. Since this statement is an open statement, one cannot determine if the statement is true or false unless plugging in specific values for $f$ and SAM. Nonetheless, the logical equivalence between the statement (*) and any of the instances given in Question 1 can be evaluated independently from the determination of the truth-value of the statement or the truth-values of any instances given in Question 1.

Q5.~Q7. Let $f$, $g$, and $h$ be functions from $\mathbb{R}$ to $\mathbb{R}$. Consider the following statement (***):
(***) For any $x > 0$, if $f(x) \geq g(x)$, then $g(x) < h(x)$.
For each of the following statements, mark (O) if the statement is equivalent to the **negation** of the statement (***): otherwise mark (X).

Examples of Instances from Q5:
(3) _______ For any $x > 0$, if $f(x) < g(x)$, then $g(x) \geq h(x)$.
(4) _______ For any $x > 0$, if $g(x) < h(x)$, then $f(x) \geq g(x)$.
(7) _______ For any $x > 0$, $f(x) \geq g(x)$ and $g(x) \geq h(x)$.

Examples of Instances from Q6:
(3) _______ There exists $x > 0$ such that if $f(x) < g(x)$, then $g(x) \geq h(x)$.
(4) _______ There exists $x > 0$ such that if $g(x) < h(x)$, then $f(x) \geq g(x)$.
(7) _______ There exists $x > 0$ such that $f(x) \geq g(x)$ and $g(x) \geq h(x)$.

Examples of Instances from Q7:
(3) _______ There exists $x \leq 0$ such that if $f(x) < g(x)$, then $g(x) \geq h(x)$.
(4) _______ There exists $x \leq 0$ such that if $g(x) < h(x)$, then $f(x) \geq g(x)$.
(7) _______ There exists $x \leq 0$ such that $f(x) \geq g(x)$ and $g(x) \geq h(x)$.

*Figure 3. The statement presented in Questions 5, 6, and 7 and some given instances*

**Part 2 of the Logic Instrument** The second part of the logic instrument (Q8 – Q12) was designed to assess students' knowledge of mathematical validity as well as cognitive consistency. Each question in Part 2 contains a statement and one or two arguments about the statement. Table 2 summarizes the statements and arguments given in the questions in Part 2 in terms of the truth-values of the given statements as well as the structures and validity of the given arguments. Whereas statements presented in the questions in Part 1 of the logic instrument are all open statements, the statements presented in the questions in Part 2 are either true or false statements. In addition, some questions in Part 2 present invalid arguments and the other question presents a valid argument. These arguments are structured by one of the four proof frames, all of which are frequently found in mathematical proofs: direct proof, proof by contradiction, proof by contrapositive, and proof by mathematical induction.

*Table 2.Summary of questions in Part 2of the logic instrument*

| | Statement | Argument Structure | Validity |
|---|---|---|---|
| Q8 | True | Example/counterexample | invalid |
| Q9 | True | Direct proof | invalid |
| Q10 | False | Proof by contradiction | valid |
| Q11 | True | Proof by contrapositive | invalid |
| Q12 | True | Proof by mathematical induction | invalid |

All five questions in Part 2 are set up similarly in the sense that each question asks to (1) determine the truth-value of the given statement; (2) determine if the given argument is either an attempt to prove or an attempt to disprove the given statement; and (3) evaluate if the given argument is valid. See Figure 5 in the next section (Data Analysis) for Question 9 as an example of questions in Part 2 of the logic instrument.

**Data Analysis**

The logic instrument described in the previous section was used in this study to measure students' logical thinking in terms of their knowledge of logical equivalence (KoLE), knowledge of mathematical validity (KoMV), and cognitive consistency (CC). We first generated the coding scheme in order to score students' mark-offs to the questions in the logic instrument. Different weights were applied to questions in Part 1 and questions in Part 2 as each examined different aspects of students' logical thinking. After coding student responses in terms of the scoring rubric, we finally generated KoLE, KoMV, and CC scores for each student.

**Scoring Rubric for Knowledge of Logical Equivalence (KoLE).** Student knowledge of logical equivalence between two statements was measured from student responses to the questions in Part 1 of the logic instrument**.** As each of the seven questions (Q1 ~ Q7) in Part 1 was scored between 0 and 2 as described in Table 3, total score of students' KoLE could be possible ranged from 0 to 14.

*Table 3 Scoring rubric for Knowledge of Logical Equivalence (KoLE)*

| Question | Scoring Rubric | | | Score Range |
|---|---|---|---|---|
| | Sub-question score | | Scoring Formula | Final score |
| | Correct / Incorrect | score | | |
| Q1, Q3, Q5~Q7 | Correct | 0 | S = max{2+∑(sub-question score), 0} | S |
| | Incorrect | −1 | | |
| | Correct / Incorrect | | score | Final score |
| Q2, Q4 | Correct | | 2 | 2 |
| | Incorrect | | 0 | 0 |

Questions 1, 3, 5, 6 and 7 in Part 1 of the logic instrument present a statement and a set of six to seven instances. For each of these questions, sub-question scores were first generated based on students' mark-off to the instances as follows: Students' mark-off of O or X to each instance was scored either 0 (for the correct response) or −1 (for the incorrect response). The final score for each of these questions was then formulated as the maximum value between 0 and 2+∑(sub-question score). Using this scoring rubric, the scores for Q1, Q3, Q5, Q6, and Q7 were ranged from 0 to 2. For instance, Q1 (Figure 1) consists of a statement (*) and six instances of statements, but only two instances (3) and (5) are statements equivalent to the statement (*). If a student marked off O for exactly these two instances (3)(5) and marked off X for the rest of the instances, 2 was given to the student response to Q1 since every sub-question was scored 0 and thus the maximum value between 2+∑(sub-question score) and 0 is 2. On the other hand, if a student marked off incorrectly for only one instance (e.g., marking off O for (1)(3)(5) or (5), and X for all other instances), then 1 was given to the student response to Q1 as one sub-question was scored −1 and all other sub-questions were scored 0; thus 2+∑(sub-question score) is 1. In this case, 1 was given to the student response to Q1. As another example, if a student marked off incorrectly two or more instances (e.g., marking off O for (1)(2)(3)(5)(6) or (1)(4)), then two or more sub-questions were scored $-1$ and all other sub-questions were scored 0; then 2+∑(sub-question score) is less than or equal to 0, and thus 0 was given to the student response to Q1.

On the other hand, Questions 2 and 4 present a pair of statements (i) and (ii) and a set of four instances (a) ~ (d) describing relationships between the pair of statements. For each of these questions, students' check of one of the four relationships was scored either 2 (for the correct response) or 0 (for the incorrect response). For instance, in Q4 (Figure 4), the statement (ii) implies the statement (i) but the statement (i) does not implies the statement (ii). Thus, if a student checked (c) as the most accurate description for the relationship between these statements, the student's response to Q4 was scored 2 whereas the other responses were scored 0.

**Scoring Rubric for Knowledge of Mathematical Validity (KoMV).** Student knowledge of mathematical validity was measured from student responses to the second and third sub-questions to each of the five question in Part 2 of the logic instrument. Each question (Q8 ~ Q12) in Part 2 of the logic instrument was scored between 0 and 2 as described in Table 4.

*Table 4 Scoring rubric for Knowledge of Mathematical Validity (KoMV)*

| Question | Scoring Rubric | | | | Score Range |
|---|---|---|---|---|---|
| | (2) Validity (Argument) | | (3) Validity (Argument) | | Final score |
| | Correct/Incorrect | score | Correct/Incorrect | score | |
| Q8 | Correct | 1 | Correct | 1 | 2 |
| | | | Incorrect | 0 | 1 |
| | Incorrect | 0 | Correct | 1 | 1 |
| | | | Incorrect | 0 | 0 |
| | (2) Prove/Disprove (Argument) | | (3) Validity (Argument) | | Final score |
| | Correct/Incorrect | score | Correct/Incorrect | score | |
| Q9~Q12 | Correct | 2 | Correct | 0 | 2 |
| | | | Incorrect | $-1$ | 1 |
| | Incorrect | 0 | - | - | 0 |

Q8. Consider the following statement (★):

(★) For any real numbers $x$ and $y$ with $y > 0$, there exists a positive integer $n$ such that $x < ny$.

The followings are Alan's and Bob's arguments about the statement (★):

*Alan's argument. The statement (★) is false because for $x = 4$, $y = 2$, and $n = 1$, $x > ny$.*

*Bob's argument. The statement (★) is true because for $x = 4$, $y = 2$, and $n = 5$, $x < ny$.*

(1) Check the most appropriate one about the statement (★):
  a. ______ The statement (★) is true.
  b. ______ The statement (★) is false.
  c. ______ We cannot determine if the statement (★) is true or false.

(2) Check the most appropriate one to describe if Alan's argument is valid.
  a. ______ Alan's argument is valid as a proof of the statement (★).
  b. ______ Alan's argument is invalid as a proof of the statement (★).
  c. ______ We cannot determine if Alan's argument is valid or invalid.

(3) Check the most appropriate one to describe if Bob's argument is valid.
  a. ______ Bob's argument is valid as a proof of the statement (★).
  b. ______ Bob's argument is invalid as a proof of the statement (★).
  c. ______ We cannot determine if Bob's argument is valid or invalid.

*Figure 4. Q4 in the logic instrument*

For Q8 (Figure 4), we scored student responses to the two sub-questions, the evaluation of the validity of the given arguments: 1 was given for the correct response to the validity of the given argument; otherwise 0 was given. For Q9 ~ Q12, we first reviewed student responses to the second sub-question asking to determine if the given argument is an attempt to prove or an attempt to disprove the statement: 2 was given to the correct mark-off to the second sub-question; otherwise, 0 was given. Next, we scored student responses to the third sub-question asking to evaluate the validity of the given argument. Among those who marked-off correctly to the second sub-question (proof or disproof), if the student also responded correctly to the third sub-question (valid or invalid), we scored 0 for the response to the third sub-question; otherwise, $-1$ was given. On the other hand, if the student response to the second sub-question (proof/disproof) was incorrect, we scored 0 to any response to the third sub-question regardless of its correctness.

For KoMV, we scored student responses to the second sub-question first, and then student responses to the third sub-question. However, we did not use student responses to the first sub-question (evaluation of the truth value of the given statement) when examining students' KoMV. Indeed, regardless of knowing whether a statement is true or false, students would be able to evaluate if someone else' argument about the statement is valid. On the other hand, we examined student responses to the second sub-question before examining their responses to the third sub-question because students' evaluation of the validity of an argument (the third sub-question) would be affected by their identification of the type of attempts for the argument (the second sub-question). For instance, if a student fails to identify correctly whether the given argument is an attempt to prove or to disprove a statement, the student' validation of the given argument would not be correct.

Q9. An integer $a$ is said to be odd if and only if there exists $n \in \mathbb{Z}$ such that $a = 2n + 1$. Tim was asked to prove or disprove:

(♣) For any positive integers $x$ and $y$, if $x$ and $y$ are odd, then $xy$ is odd.

The following is Tim's argument.

$$x = 2n + 1, n \in \mathbb{Z}$$
$$y = 2n + 1, n \in \mathbb{Z}$$
Therefore, $xy = (2n + 1)(2n + 1) = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$ is odd.

(1) Check the most appropriate one about the statement (♣).

a. _______ The statement (♣) is true.

b. _______ The statement (♣) is false.

c. _______ We cannot determine if the statement (♣) is true or false.

(2) Check the most appropriate one to describe what Tim attempted to prove.

a. _______ Tim attempted to prove the statement (♣) is true.

b. _______ Tim attempted to prove statement (♣) is false.

c. _______ We cannot determine if Tim attempted to prove the statement (♣) is true or he attempted to prove the statement (♣) is false.

(3) Check the most appropriate one to describe if Tim's argument is valid.

a. _______ Tim's argument is valid as a proof of the statement (♣).

b. _______ Tim's argument is invalid as a proof of the statement (♣).

c. _______ We cannot determine if Tim's argument is valid or invalid.

*Figure 5. Q5 in the logic instrument*

**Scoring Rubric for Cognitive Consistency (CC).** Cognitive consistency was measured from student responses to the questions in Part 2 of the logic instrument. We first identified cognitive inconsistencies only when student responses to sub-questions of a question imply any logical contradiction. For instance, in the case of Q9 (Figure 5), if a student were to mark off that (1) the statement (♣) is true, (2) Tim's argument is an attempt to prove the statement (♣) is false, and (3) Tim's argument is valid, then the student's responses contain a logical contradiction since an attempt to prove that a true statement is false cannot be valid. Similarly, if a student responds to Q9 that (1) the statement (♣) is false, and (2) Tim's argument is an attempt to prove the statement (♣) is true, and (3) Tim's argument is valid, then the student also appears to have cognitive inconsistency. Table 5 describes all instances evidently determined to have cognitive inconsistency from student responses.

We measured students' cognitive consistency by assigning either $-1$ or 0 to each of the questions (Q8~Q12) in Part 2 of the logic instrument. Specifically, we scored $-1$ whenever there is evidence of cognitive inconsistency, i.e., a logical contradiction from student responses to its sub-questions. On the other hand, we scored 0 in all other cases but the instances described in Table 5 since there is no evidence of logical contradictions from the cases. As there were five questions in Part 2, the total score on cognitive consistency could be possibly ranged from $-5$ to 0.

*Table 5 All instances of cognitive inconsistency*

| Question | Sub-Questions | | | |
|---|---|---|---|---|
| Q8 | | (1) True/False (Statement) | (2) Validity (Argument) | (3) Validity (Argument) |
| | Cognitive Inconsistency | (a) True or (c) Cannot determine | (a) Valid as a proof for false | - |
| | | (b) False or (c) Cannot determine | - | (a) Valid as a proof for true |
| Q9~Q12 | | (1) True/False (Statement) | (2) Prove/Disprove (Argument) | (3) Validity (Argument) |
| | Cognitive Inconsistency | (a) True or (c) Cannot determine | (b) Prove False | (a) Valid |
| | | (b) False or (c) Cannot determine | (c) Prove True | (a) Valid |

Obviously, if a student marks off correctly to all sub-questions to a question in Part 2, the student does not appear to have a cognitive inconsistency in his response to the question. On the other hand, although the student responses to some sub-questions are not correct, the student's cognitive consistency score to the question could still be 0 in the case when there is no evidence of logical contradiction within the student's responses.

## Results

Figure 6 summarizes the distributions of student scores in terms of KoLE score, KoMV score, and CC score. KoLE scores were ranged from 0 to 14 while the median of the KoLE scores was 5 (out of 14 points) and 50% of student KoLE scores were between 2 and 9. KoMV scores were ranged from 0 to 10 with the median 5 (out of 10 points) while 50 % of KoMV scores were distributed between 3 and 8. Finally, scores were ranged from $-2$ to 0, and about

21% of the participants showed at least once cognitive inconsistencies in their responses to the logic instrument.

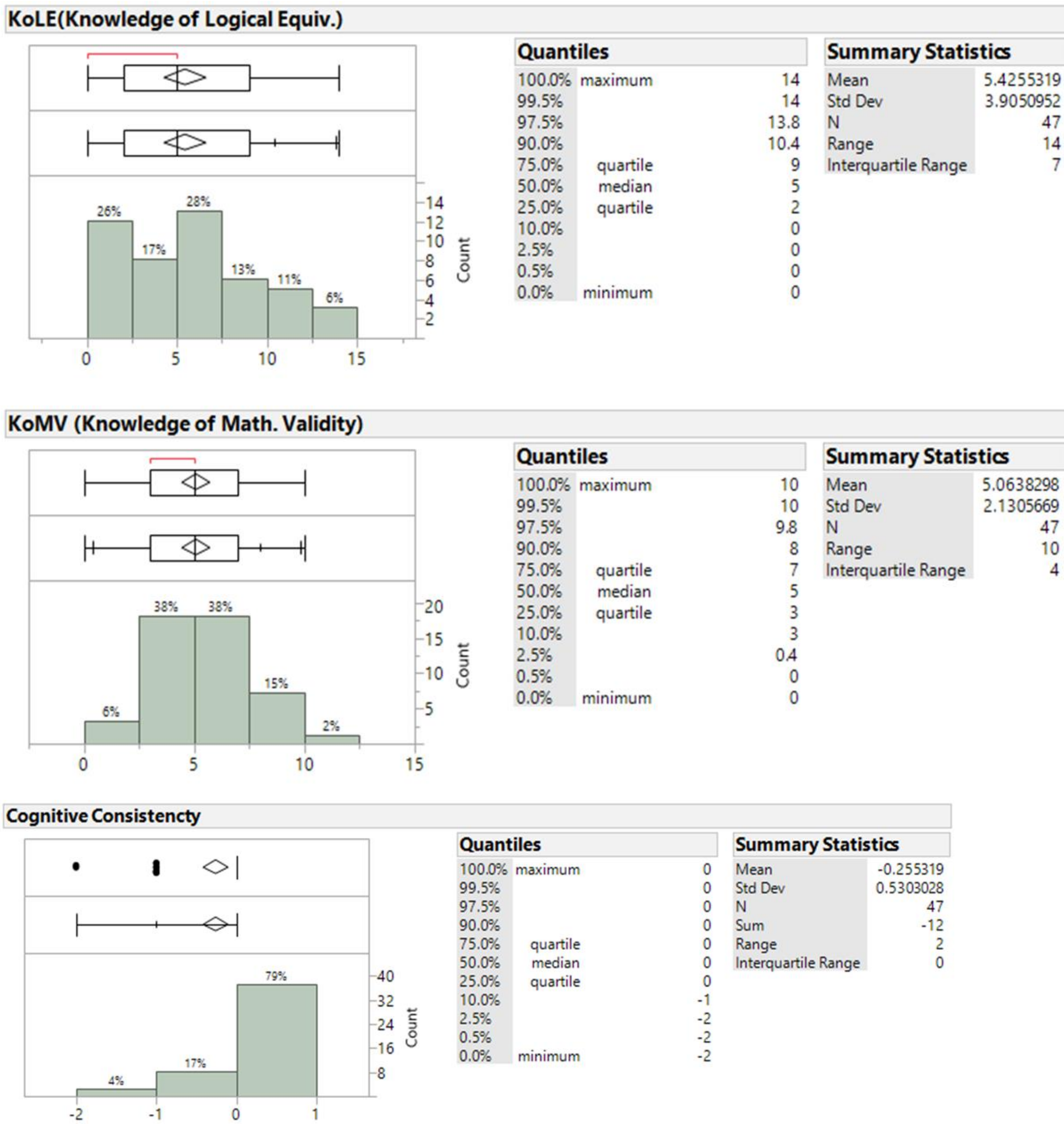


*Figure 6. Distributions of three parts of the overall logical thinking scores*

The scatter-density plot in Figure 7 further shows that students' knowledge of logical equivalence (KoLE) and students' knowledge of mathematical validity (KoMV) were somewhat related to one another.

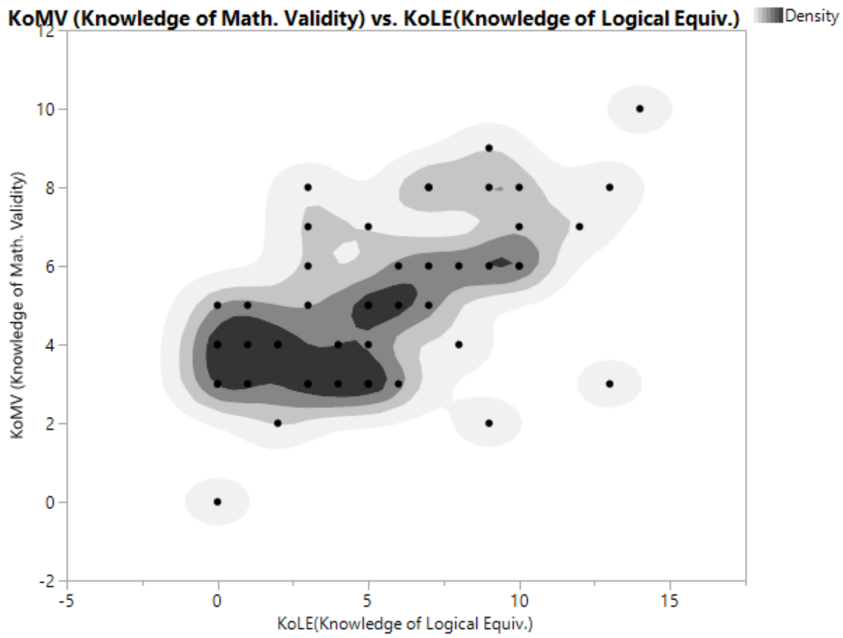


*Figure 7. Scatter-Density plot: KoMV vs. KoLE*

On the other hand, cognitive consistency (CC) was *not* closely related to either knowledge of logical equivalence or knowledge of mathematical validity. According to the scatter-density plots in Figure 8 students scored the cognitive consistency score $-2$ did not have higher scores than the median of each KoLE and KoMV scores.

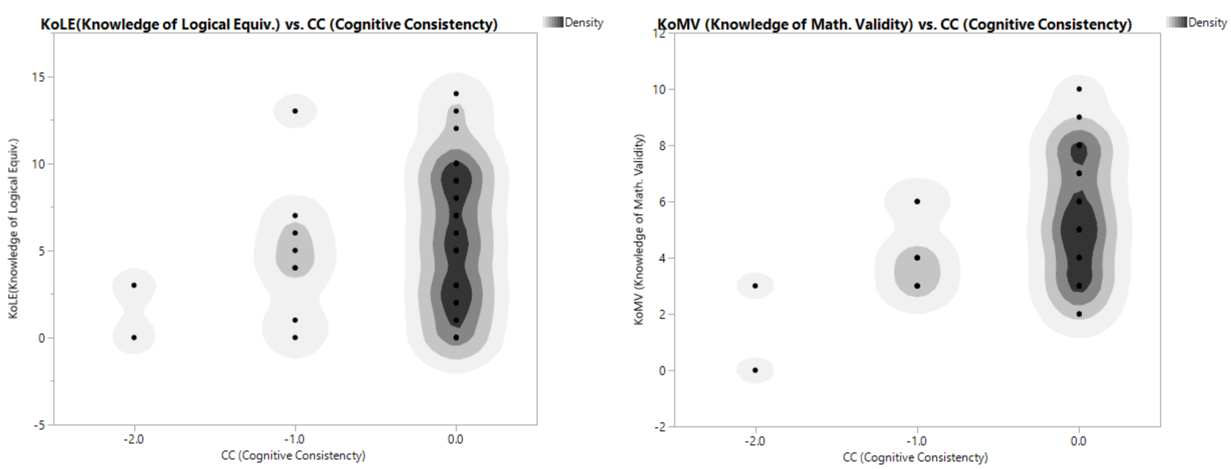


*Figure 8. Scatter-Density plots: KoLE vs. CC (left), KoMV vs. CC (right)*

On the other hand, in the case that the cognitive consistency score was $-1$, students' KoLE scores or KoMV scores were distributed with relatively wide range containing higher scores than the median (see the shaded cells in Table 6 and Table 7). There was one student who received a very high score on KoLE (13 out of 14) but scored $-1$ on the cognitive consistency (see Table 6). These findings indicate that students might have cognitive inconsistencies even though they attained high scores on knowledge of logical equivalence and knowledge of mathematical validity, respectively.

*Table 6 Contingency Table: Cognitive Consistency Score by KoLE Score (Median = 5)*

| | KoLE (Knowledge of Logical Equivalence) | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 13 | 14 | Total |
| Cognitive Consistency | | | | | | | | | | | | | | | |
| −2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| −1 | 1 | 1 | 0 | 0 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 8 |
| 0 | 4 | 2 | 3 | 5 | 0 | 5 | 2 | 3 | 2 | 4 | 4 | 1 | 1 | 1 | 37 |
| Total | 6 | 3 | 3 | 6 | 2 | 6 | 3 | 4 | 2 | 4 | 4 | 1 | 2 | 1 | 47 |

*Table 7 Contingency Table: Cognitive Consistency Score by KoMV Score (Median = 5)*

| | KoMV (Knowledge of Mathematical Validity) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total |
| Cognitive Consistency | | | | | | | | | | | | |
| −2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| −1 | 0 | 0 | 0 | 3 | 3 | 0 | 2 | 0 | 0 | 0 | 0 | 8 |
| 0 | 0 | 0 | 2 | 6 | 5 | 7 | 5 | 4 | 6 | 1 | 1 | 37 |
| Total | 1 | 0 | 2 | 10 | 8 | 7 | 7 | 4 | 6 | 1 | 1 | 47 |

## Conclusion & Discussion

In this study, we explored undergraduate students' cognitive consistency and its relation to their knowledge of logical equivalence and mathematical validity. The findings of this study indicate that students' cognitive consistency was not closely related to either their knowledge of logical equivalence or their knowledge of mathematical validity. Indeed, some students who received high scores on knowledge of logical equivalence or on knowledge of mathematical validity still had cognitive inconsistencies. Furthermore, these students already took a course for logic and mathematical proofs for about at least fifteen weeks. Thus, it might be an unreasonable expectation that students with more knowledge on logical equivalence and mathematical validity would not have cognitive inconsistencies.

The findings of this study also suggest some significant implications for the teaching and learning of logic and mathematical proofs. In particular, although undergraduate students received formal instruction for logic from a logic and mathematical proof course, they could still have cognitive inconsistencies. Furthermore, this implication of the results of this study is critical in the sense that having a cognitive inconsistency means that the student does not recognize a logical contradiction in his or her argument. Thus, we contend that cognitive consistency must be treated as a crucial component of logical thinking. Designing special tasks or instructional interventions would be needed to reveal students' cognitive inconsistencies and to help students recognize logical contradiction in their arguments if they have any. The structure of sub-questions in Part 2 of the logic instrument in this study could be an example of reference to reveal students' cognitive inconsistency what might have been.

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# Figurative Thought and a Student's Reasoning About "Amounts" of Change

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*This paper discusses a student coordinating changes in covarying quantities. We adopt Piaget's constructs of figurative and operative thought to describe her partitioning activity (i.e., mental and physical actions associated with constructing incremental changes) in terms of the extent that it is constrained to carrying out particular sensorimotor actions on perceptually available material, and we relate such descriptions to her thinking about quantitative amounts of change. We conclude the paper by discussing how characterizing these nuances in her partitioning activity contributes to current literature on covariational reasoning and concept construction.*

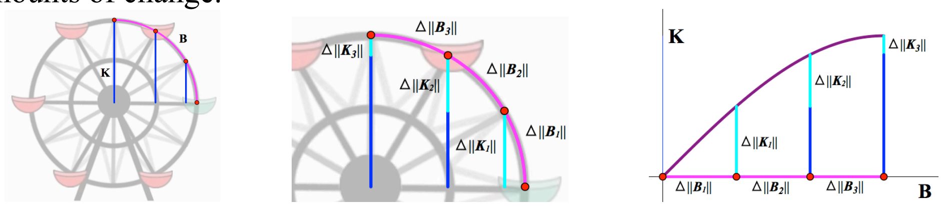
*Keywords:* Cognition, Piaget, Covariational Reasoning, Amount of Change

Researchers have shown that students' *quantitative* and *covariational reasoning*—the mental actions involved in conceiving measurable attributes changing in tandem (Carlson, Jacobs, Coe, & Hsu, 2002; Thompson, 2011)—are critical for their learning of function and rate of change (Ellis, 2011; Johnson, 2015; Thompson & Carlson, 2017). Stemming from the complexities of students' thinking, these researchers have called for investigations that identify nuances in students' covariational reasoning. We answer these researchers' call by using Piaget's (1976, 2001) notions of *figurative* and *operative* thought to explain the extent a student's reasoning of amounts of change of covarying quantities is constrained to sensorimotor actions and the produced results of those actions. Drawing on these distinctions, we discuss the importance of our findings with respect to students' concept construction.

## Quantitative Reasoning, Covariational Reasoning, and Partitioning Activity

Thompson (2011) described that the mental construction of a *quantity* involves "conceptualizing an object and an attribute of it so that the attribute has a unit of measure" (p. 37). Although Thompson used the term "measure," he emphasized that reasoning quantitatively does not require reasoning about a specified quantity's value; sophisticated conceptions of quantity entail reasoning about a quantity's magnitude (i.e., amount-ness) while anticipating that it has an infinite number of measure-unit pairs (Thompson, Carlson, Byerley, & Hatfield, 2014). A distinction between a quantity's magnitude and its measure enables us to account for reasoning about covarying quantities that is not constrained to the availability of values; importantly, our focus on a quantity's magnitude affords characterizing mental activity in terms of perceptual material associated with a quantity's amount-ness (e.g., a perceived segment and its length).

An individual imagining variations in a quantity's magnitude (and hence value) is positioned to reason covariationally. When reasoning covariationlly, "a person holds in mind a sustained image of two quantities' values (or magnitudes) simultaneously…one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (Saldanha & Thompson, 1998, p. 299). Building on Saldanha and Thompson's (1998) covariation, Carlson et al. (2002) specified mental actions involved in coordinating quantities, among which students' coordination of amounts of change of one quantity with respect to changes in another (Mental Action 3 in their framework) is central to our work here. An individual coordinating amounts of change imagines quantities' magnitudes accumulating in successive states (and possibly anticipates continuous covariation between these states; see Thompson and Carlson (2017)). To illustrate, a student reasoning about covarying quantities B and K can envision the magnitude ||B|| accumulating in equal accruals, construct the magnitude ||K|| accumulating in terms of corresponding accruals, and coordinate those accruals in ||K|| to conceive ||K|| increasing by decreasing amounts with respect to ||B|| (see Figure 1a-c for an illustration with respect to the Taking a Ride task in Figure 3a). Because coordinating amounts of change involves an activity of constructing a magnitude's accumulation in terms of accruals, we use *partitioning activity* to refer to students' mental and sensorimotor actions associated with their constructing and reasoning about these incremental changes that may represent amounts of change.




(a) (b) (c)

*Figure 1. As ||B||(denoted in pink) increases by equal amounts, ||K|| increases (denoted in (a) dark blue and in (b)-(c) light blue plus dark blue) by decreasing amounts (denoted in light blue), which can be represented in (c) a Cartesian coordinate system.*

In addition to research on covariation, we draw on research into students' partitioning activity (i.e., "the process of dividing a unit into equal-sized parts, either solely mentally or also materially"; see Hackenberg and Tillema (2009, p. 2)) primarily in the area of whole number, fraction, multiplication and unit coordination (Izsák, Tillema, & Tunç-Pekkan, 2008; Steffe, 2003; Steffe & Olive, 2010). These studies have illustrated that although many students are capable of carrying out partitioning actions, their understandings of fractions and number lines are not necessarily productive with respect to such activity. As it relates to the current study, we are interested in characterizing the extent that students' meanings of quantities are constrained to particular partitioning actions for envisioning or visualizing amounts of change magnitudes across different representations.

## Figurative and Operative Partitioning Activity

### Figurative and Operative Thought

We have found the theoretical distinction between *figurative* and *operative* thought (Piaget, 1976, 2001; Steffe, 1991; Thompson, 1985) useful in developing models of students' partitioning activity. Piaget (1976, 2001) characterized figurative thought as based in and constrained to sensorimotor actions and perception, and he described operative thought as the coordination of mental operations so that these coordinations dominate figurative material (i.e., sensorimotor and perceptual entailments). We emphasize that characterizing a student's thinking as operative does not imply her thinking does not entail fragments of figurative material. Likewise, characterizing a student's thinking as figurative does not imply that her thinking does not entail operative schemes. A researcher's sensitivity to these distinctions is an issue of "figure to ground" (Thompson, 1985, p. 195). When a student's thinking foregrounds carrying out repeatable (mental or sensorimotor) actions and the results of those actions, it is figurative; when a student's thinking foregrounds the coordination of actions and transformations of those actions and their results, it is operative. As we illustrate at the end of this section, the issue of foregrounding is important for describing students' partitioning activity because such activity necessarily entails figurative aspects and material (e.g., drawing graphs and partitions) and likely entails operative schemes (e.g., understanding a coordinate system in terms of directed distances).

### Anticipation and Re-presentation

Regarding students' capacity to imagine or repeat previously carried out partitioning activity, we find von Glasersfeld's (1998) notions of *anticipation* and *re-presentation* useful. Von Glasersfeld (1998) elaborated on the construct of *anticipation* to operationalize how humans make predictions through reflecting on past experience and abstracting regularities from it. A critical and advanced form of anticipation of a desired event, situation, or goal involves an individual recognizing a situation (i.e., perceiving a situation in experience via assimilating it to existing cognitive structures), expecting specific results to occur regarding that situation, and attempting to attain those results by generating their cause (e.g., carrying out associated activity). For example, if a student has experienced partitioning activity to represent decreasing amounts of change in height on a Ferris wheel (see Figure 1b), when she conceives a circle in another occasion she may recognize it as similar to the Ferris wheel, and she may attempt to carry out similar partitioning activity to reconstruct partitions on the circle expecting to obtain decreasing changes in height.

What is critical to this form of anticipation is the individual's ability to generate the cause of expected results; namely, the individual needs to "mentally run through the actions she might take in order to produce a particular result of goal, and in this process [she] adapts or refines her planned actions in relation to the effects she imagines they will generate" (Hackenberg, 2010, pp. 387-388). This requirement aligns with von Glasersfeld's (1995) notion of *re-presentation* as re-playing or reconstructing something that *was* present in a subject's experiential world *at some other time*. Importantly, it's "a mental act that brings a prior experience to an individual's consciousness" (von Glasersfeld, 1995, p. 95). Re-presentation and recognition are similar in a sense that they both require memory; there must be something experienced previously that has remained in a person's mind so that she can re-present or recognize it in another occasion (von Glasersfeld, 1991). However, recognition is less effortful than re-representation. Recognition requires the sensory material to be available (e.g., recognize an English word when hearing or reading it), while re-presentation is wholly self-generated (e.g., re-present an English word when writing or speaking it) (von Glasersfeld, 1991). As von Glasersfeld (1995) stated, re-presentation is "the recollection of the figurative material that constituted the experience" (p. 95), which requires the subject to mentally generate some substitute for the sensory material that was present in prior experience. In the current study, since we attempt to gain insights into students' ability to re-represent, we choose to use re-presentation in a general sense in that we partially supply students figurative material (e.g., circles, segments, coordinate systems with defined axes, etc.) for them to reconstruct their partitioning activity. In this case, re-presentation of previously constructed partitioning activity necessarily requires recognition of the supplied figurative material to be relevant or similar to a prior situation, and thus it is not strictly self-generated.

### Figurative and Operative Partitioning Activity

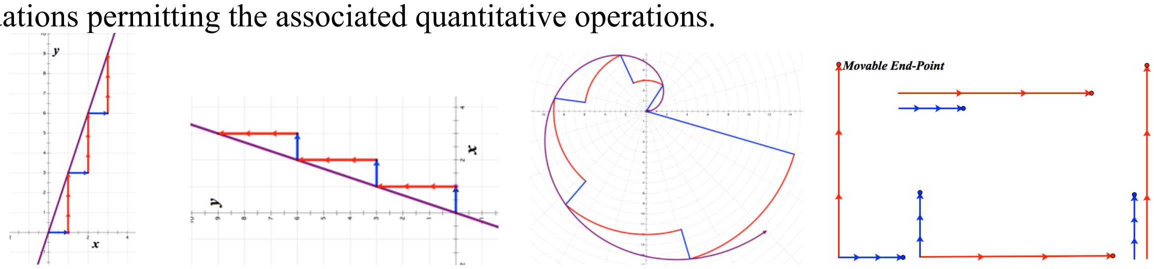
Combining these constructs to characterize students' partitioning activity, we make the distinctions of figurative and operative partitioning activity in Table 1. *Figurative partitioning activity* refers to an individual's re-presentation of partitioning activity being constrained to having available some perceptual material permitting the same sensorimotor actions of partitioning and, thus, repeating the same sensorimotor actions (i.e., in-complete re-presentation; see von Glasersfeld (1995)). Consequently, when the perceptual material constitutes a prior situation is unavailable in a new situation, she may not perceive the new situation to be relevant or similar to the prior (i.e., not recognize the current situation), or she may not be able to carry out associated partitioning activity due to the absence of available perceptual material. She thus does not anticipate re-presenting the activity (see an illustration in the first theme under the Results section). In another case, when an individual does recognize the current situation due to perceiving perceptually similar elements, she may re-present her partitioning activity by repeating the same sensorimotor actions in order to produce similar perceptual results regardless of the differences in those two situations (see an illustration in the second theme under the Results section). In contrast, *operative partitioning activity* refers to an individual's re-presentation of partitioning activity being based in coordinated mental structures and transformation of those structures. When she conceives a new situation to be relevant, she can anticipate re-presenting a relationship constituting another situation by transforming her mental activity to account for the new situation (see Figure 1b and 1c, where a student traces an arc on the circle and transforms this action to trace a horizontal increment on the horizontal axis; also see an illustration in the third theme under the Results section). She can also wholly self-construct her partitioning activity without any perceptual material given (i.e., complete re-presentation).

*Table 1. Figurative and Operative Partitioning Activity*

| Partitioning Activity | Foregrounded Actions of Partitioning Activity |
|---|---|
| *Figurative Partitioning Activity* | Repeating sensorimotor actions of partitioning tied to particular perceptual material and results;<br>Potentially constrained to re-presenting partitioning activity on available perceptual material permitting the same sensorimotor actions;<br>Conceived invariance among situations is with respect to sensorimotor actions and their perceptual results. |
| *Operative Partitioning Activity* | Sensorimotor actions subordinate to mental actions (e.g., quantitative and covariational reasoning);<br>Potentially can re-present partitioning activity on unavailable or novel perceptual material;<br>Conceived invariance among situations is with respect to coordinated mental actions and their transformations. |

To illustrate these distinctions, consider a student determining if the graphs in Figure 2a and Figure 2b represent the linear relationship $y = 3x$. With respect to Figure 2a, a student who engages in figurative partitioning activity could imagine the graph in terms of successive movements of one axes mark to the right (denoted in blue) and then three axes marks up (denoted in red), and associate such movements with a positive slope of 3 (Paoletti, Stevens, & Moore, 2017). With respect to Figure 2b (a rotated graph of Figure 2a), the student could conceive movements up the graph (denoted in blue) as corresponding to movements to the left along the graph (denoted in red), and associate such movements with a negative slope of $-\frac{1}{3}$. In each case, the student's thought is dominated by carrying out particular sensorimotor actions to the extent that associations (e.g., a line falling left-to-right necessarily has a negative slope) are tied to that activity and its perceptual results. Hence, the student concludes that the two graphs are different.

In comparison, a student who engages in operative partitioning activity could conceive that both graphs are such that any directed change in $x$ corresponds to a directed change in $y$ three times as large as that in $x$. The student's partitioning activity is operative because she can coordinate and transform activity specific to each graph to conceive an underlying invariance that dominates figurative differences in activity. The student might also anticipate re-presenting invariant partitioning activity in other coordinate systems (e.g., polar coordinates, Figure 2c) or contexts (Figure 2d), as well. Figure 2d illustrates four bar pairs (two are orthogonally-oriented, two are parallel), with each bar being able to be manipulated in length. Each pair provides different perceptual material than that of Figure 2a, 2b, and 2c, and none of the pairs provide a perceptually available trace or graph. But, a student might anticipate re-presenting the same relationship associated with Figure 2a, 2b, and 2c on these bar pairs by manipulating the end-points of those bars so that each successive increment in red is always three times as large as a corresponding blue increment. The anticipation of re-presenting partitioning activity aligns with Moore and Silverman's (2015) *abstracted quantitative structure*: a structure of related quantities a student has internalized as if it is independent of specific figurative material (i.e., representation free) so that they can re-present this structure to accommodate novel contexts or situations permitting the associated quantitative operations.




(a) (b) (c) (d)

*Figure 2. (a) A graph that represents the relationship of $y=3x$ in a Cartesian coordinate system, (b) a rotated graph of (a), (c) a graph that represents the relationship of $r=3\theta$ in a polar coordinate system, and (d) four bar pairs that represent the same relationship.*

## Methodology

We take the epistemological stance of radical constructivism (von Glasersfeld, 1995) to approach the current study. We consider knowledge as actively constructed through interaction with environment and in ways idiosyncratic to a knower. Knowledge is not a true representation of an objective ontological "reality"; rather, it is adaptive so that it functions and organizes viably at the moment in the course of an individual's experience. This implies that as researchers or mathematics educators, we should respect our students' mathematical knowledge and consider their mathematics as "legitimate mathematics to the extent that we can find relational grounds for what students say and do" (Steffe & Thompson, 2000, p. 269). Also, because we have no access to students' knowledge, we can only attempt to construct hypothetical models of their knowledge that viably explain our observations of their observable behaviors (Steffe & Thompson, 2000).

Under these theoretical assumptions, we conducted a semester-long teaching experiment (Steffe & Thompson, 2000) with prospective secondary mathematics teachers (Lydia, Emma, and Brian) to develop viable models of their thinking. They were in their first semester of a four-semester secondary mathematics education program at a large university in the southeast United States. Each student had completed at least one course past an undergraduate calculus sequence at the time of the study, and we chose them on a voluntary basis from a secondary mathematics content course. We conducted 10-11 teaching sessions (1 to 2 hours each) with each student. The project principal investigator (the second author) served as the teacher-researcher (TR) at every teaching session. At least one other research team member was present as the observer(s). Each session was videotaped and digitized for analysis. In both ongoing and retrospective analyses efforts, we conducted conceptual analysis (Thompson, 2008) to develop models of students' mathematics. Specifically, our iterative analyses efforts involved constructing hypothetical mental actions that viably explained the students' observable and audible behaviors. We continually searched the data for instances that the models could not account for, and we modified our models or we attempted to explain developmental shifts in a student's meanings. In this paper, we focus on the case of Lydia because of particular aspects of her partitioning activity that were consistent throughout the teaching experiment. We consider it important to characterize her ways of thinking in order to add nuances to our prior conceptualizations of students' quantitative and covariational reasoning.

## Task Design

We describe Lydia's activity on four related tasks: (1) Taking a Ride, (2) Which One, (3) Circle, and (4) Blue-Red-Green. The Taking a Ride task included an animation of a Ferris wheel (Desmos, 2016) (see Figure 3a). We designed this task to focus students on constructing the covariational relationship between the *height* of the green rider above the horizontal diameter of the wheel and its *arc length* traveled (the sine relationship; Moore (2014)). We then presented the Which One task (Figure 3b) after students' first attempt on the Taking a Ride task. It included a simplified version of a Ferris wheel (left) with the position of a rider indicated by a dynamic point. The topmost line segment (shown in blue, right) represented the arc length the rider had traveled counterclockwise from the 3 o'clock position. Students could vary the topmost segment length by dragging its endpoint or by clicking the "Vary" button, with the dynamic point on the circle (i.e., the rider) moving correspondingly. We asked students to determine which of the six red segments, if any, could accurately represent the rider's height above the horizontal diameter as the rider's arc length varied. Segment 1 is a normative solution and segments 2-6 vary with either different directional variation or rates. In students' initial attempt on these two tasks, we did not prompt them to construct a graph in order to gain insights into their reasoning with displayed magnitudes while minimizing the influence of their graphing meanings. For the Circle task (Figure 3c), we asked students to graph the relationship between the *horizontal distance* and the *arc length* associated with a dynamic point (i.e., the cosine relationship).

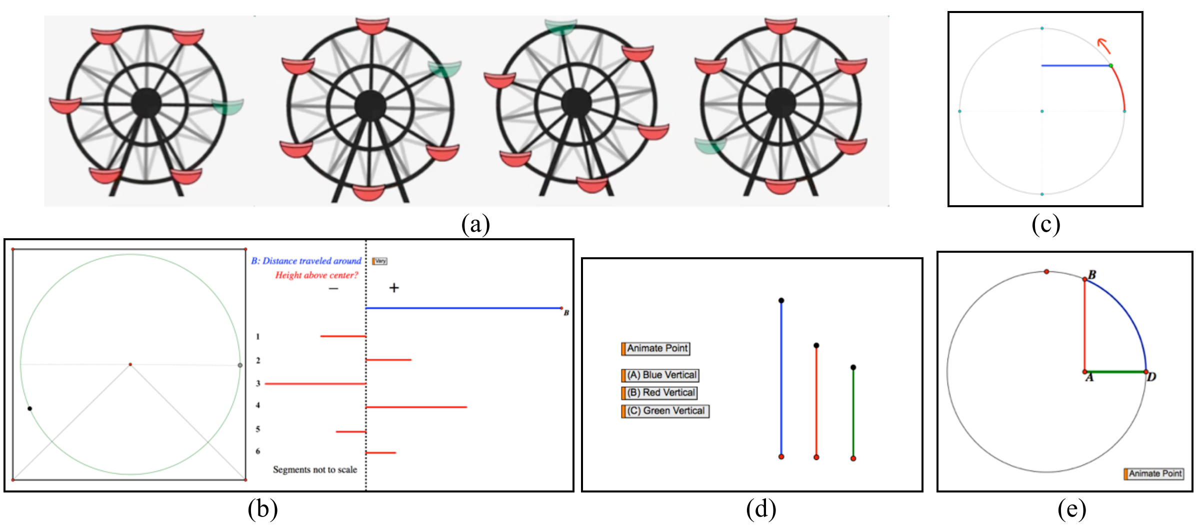



*Figure 3. (a) Animation snapshots of Taking a Ride, (b) Which One (numbering of segments is labeled for readers), (c) Circle, (d) Blue-Red-Green, and (4) the quantitative meanings of the three bars in the Red-Blue-Green task.*

The Blue-Red-Green task (Figure 3d) included an animation in which three vertically-oriented bars (shown in blue, red, and green) simultaneously varied. These three bars entailed the same variations and relationships as the three segments shown in Figure 3e, with their colors matching each other. Namely, with respect to the blue bar (arc) increasing (Point B is draggable along the circle), the red bar (sine) increased at a decreasing rate while the green bar (versine) increased at an increasing rate. We then asked students to describe how any of the two bars varied in tandem and to construct graphs to represent paired relationships. We intentionally chose the red and green quantities on the circle in an attempt to engage our participants in reasoning with quantities' magnitudes that entailed the same directional change (i.e., both increased or decreased) but different rates of change with respect to the blue bar. In students' initial attempt to this task, we presented the interface shown in Figure 3d but not Figure 3e. We were interested in understanding how reasoning with quantities' magnitudes independent on the circle context affected students' covariational reasoning and graphing activity.

We draw attention to a few common task design principles that underscore the focus of this paper, with additional design decisions reported elsewhere (Stevens, Paoletti, Moore, Liang, & Hardison, 2017). First, we designed the tasks to entail (what we perceive to be) figurative material representing the quantities' magnitudes that we asked the students to conceive, coordinate, and act upon; each task involves (varying) segments because a student can conceive a varying segment as figurative material associated with a distance magnitude (i.e., lengthiness). We consider it important to engage students in reasoning with quantities' magnitudes independent of numerical values (unless introduced by students) because written numbers are inscriptions that do not (naturally) permit quantitative activity. Second, we designed the series of tasks to provide different contexts and representations, but similar or identical covariational relationships, in order to tease apart differences in students' re-presentation and recognition activity including the extent it was dominated by figurative or operative thought. By supplying different figurative material (e.g., a circular Ferris wheel, circles, arcs, parallel bars, horizontal and vertical segments) and asking students to construct coordinate graphs, we attempted to determine if a student's partitioning activity reflects an abstracted quantitative structure that students can re-present and recognize across different figurative material (i.e., operative partitioning activity) or if their activity is constrained to re-presenting particular sensorimotor actions on available or similar perceptual material (i.e., figurative partitioning activity). For instance, we were interested in how (and if) the students would attempt to re-present the relationship they constructed during Taking a Ride when choosing a segment on Which One. Third, although we focused a majority of the tasks on the sine relationship, we included several tasks that involved other (but similar) relationships (e.g., cosine in the Circle task and versine in the Blue-Red-Green task) for the purpose of comparing students' partitioning activity across different covariational relationships.

## Results

In this section, we illustrate Lydia's partitioning activity with a focus on her attempts to re-present her partitioning activity, particularly as she considered a variety of representations.

### Partitioning Activity Constrained to Available Perceptual Material

In the first teaching session, we worked with Lydia on Taking a Ride (Figure 3a). She initially described, "the arc length has increased to this [*drawing an arc on the first quarter of the circumference of the wheel*] while the distance from the center has increased to that [*drawing a vertical segment from the top position to the center of the wheel*]." Eventually, with much effort, Lydia made use of the spokes of the Ferris wheel to partition traveled distance equally and construct what we perceive to be successive amounts of change of height for successive, equal changes in arc length (see her construction in Figure 4a-c). Noticing that the blue segments (in Figure 4c) decreased in magnitude, Lydia concluded that, "[A]s the arc length is increasing... [the] vertical distance from the center is increasing ... but the value that we're increasing by is decreasing." Suggesting she was excited that she had identified this relationship, she explained with enthusiasm, "I just discovered this by myself." This revealed that her activity of drawing partitions and identifying amounts of change was novel to her at the time.

(a) (b) (c)

*Figure 4. Lydia (a) used the four spokes to partition distance traveled into three equal increments, (b) identified height of the green rider in each successive state, and (c) identified amounts of change in height.*

Immediately following this task, we presented the Which One task (Figure 3b). After some exploration, Lydia claimed that she desired to choose a red segment that is moving at a constant rate. She eliminated four of the six segments and then had difficulty deciding which of the other two segments was moving constantly (see the top two red segments in Figure 5a). She then decided to orient one of the segments (the normatively correct solution) vertically, and placed it inside the circle (Figure 5b). She confirmed that the length of that segment matched the height of the dynamic point for different states (Figure 5c). When asked if that segment entailed the amounts of change relationship constructed in the initial Taking a Ride task, she responded:

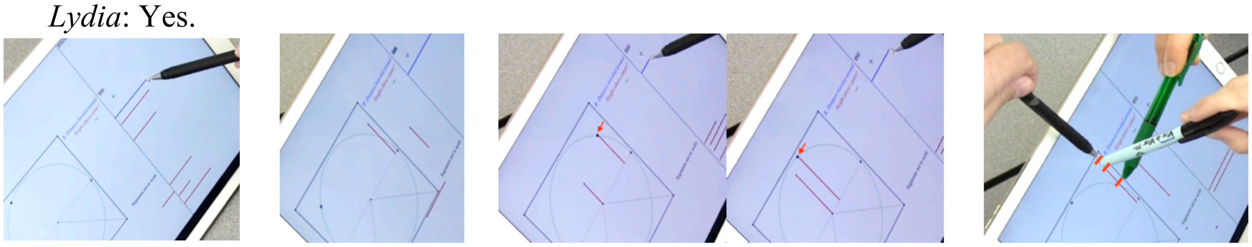
*Lydia*: Not really…Um, I don't know. [*laughs*] Because that was just like something that I had seen for the first time, so I don't know if that will like show in every other case…Well, for a theory to hold true, it like – it needs to be true in other occasions, um, unless defined to one occasion.

*TR*: So is what we're looking at right now different than what we were looking at with the Ferris wheel?

*Lydia*: No. It's – No…Because I saw what I saw, and I saw that difference in the Ferris wheel, but I don't see it here, and so –

*TR*: And by you "don't see it here," you mean you don't see it in that red segment?

*Lydia*: Yes.



(a) (b) (c) (d)

*Figure 5. (a) Lydia was limiting her solutions to two red segments, (b) checking a red segment point-wisely, and (c) we were assisting Lydia to identify amounts of change in height on a red segment.*

We find it noteworthy that Lydia described height increasing by decreasing amounts as a "theory" to be tested in this new situation despite her having identified that the red segment worked point-wise with respect to traversed arc length; her knowing that the red segment worked for each state did not imply by necessity that the red and blue segments existed in a covariational relationship consistent with that between height and arc in Taking a Ride. Furthermore, Lydia was unable to re-present her previous partitioning activity with respect to the red and blue segments in the Which One task ("I saw that difference in the Ferris wheel, but I don't see it here"). It was only after this exchange, and when the researchers intervened to create perceptually available material using pens to denote amounts of change of the red segment (Figure 5d), that Lydia responded (in surprise) that her "theory" held true.

We characterize Lydia's partitioning activity as figurative due to her difficulty re-presenting such activity from one context to another. Although she identified successive accruals in height on the Ferris wheel (Figure 4c), her understandings of amounts of change (or her "theory") were rooted in carrying out particular activity of partitioning and creating perceptually available increments for comparison in that specific context. When moved to a context in which there were several bars changing continuously and the spokes of the Ferris wheel were not perceptually available, she did not anticipate re-presenting her partitioning activity. As she considered successive red segment states in Which One (Figure 5c), she was unable to hold in mind the red segment associated with a prior state to compare it to a current state and thus was unable to mentally construct and re-present those incremental changes in height. In addition, she was engaged in a different sensorimotor action (e.g., point-wise checking of each red segment) and such action did not result in her producing similar perceptual results (i.e., incremental curves or vertical segments) to those on the Ferris wheel. Due to her conceived invariance of partitioning activity being limited to particular sensorimotor actions and their perceptual results in the Taking a Ride context (as opposed to abstracted quantitative structures), she had difficulty with conceiving the invariant relationship of the Ferris wheel in the Which One context and anticipating transforming her partitioning activity.

**Repeating Sensorimotor Actions of Partitioning Activity**

After the first teaching session, Lydia worked with two other students on the Taking a Ride task (Figure 3a) and the Circle task (Figure 3c). During these two group sessions, the students constructed graphs to re-present relationships they constructed during these tasks (i.e., sine and cosine), and Lydia primarily observed the other students. The TR began the fourth session by asking Lydia what she recalled from the previous sessions. She first drew a quarter of a circle (Figure 6a) and discussed the relationship between arc length and horizontal distance:

> "So we kind of said as the arc length is increasing in the first quadrant that our X distance is decreasing [*drawing the horizontal segments within the circle from bottom to top in Figure 6a*], and then…distance will decrease more in the same amount of space. So like from here to here [*highlighting the bottom blue arc*], then we'll say these are the same arc length [*highlighting the top blue arc*]…so we're going to take this point here [*marking a point at the top of the far-right pink segment*] and then drag it down [*drawing the far-right pink segment*], we've only lost this much [*highlighting the shorter red segment*]. And then from here [*drawing the middle pink segment*] to here [*tracing the far-left pink segment*] we lost this distance [*highlighting the longer red segment*], but we're saying those are the same arc length [*pointing to the two blue arcs*], so it's a lot more distance."

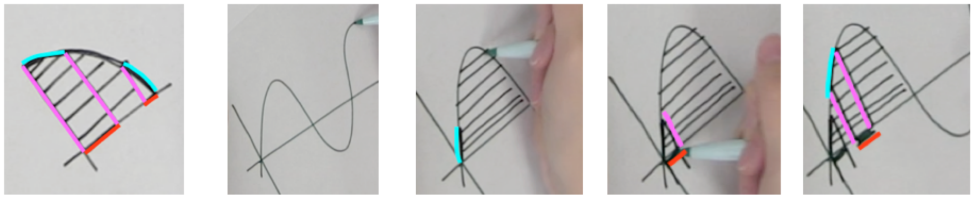

(a) (b) (c) (d) (e)

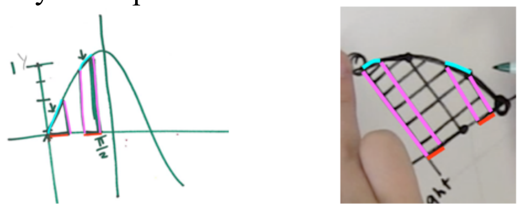
*Figure 6. (a) Lydia's drawing of the circle situation, (b) Lydia's graph, and (c)-(e) her partitioning activity.*

Lydia's re-presented partitioning activity appeared compatible with that from previous sessions, and thus the TR asked Lydia how such activity related to graphing the relevant relationship. The TR did not provide a graph in order to allow Lydia to re-present the activity (as she perceived it) from the previous sessions. Lydia drew a graph (Figure 6b; what we perceive to be a sine graph) and explained how the graph related to her partitioning activity in Figure 6a:

> "As we go up in arc length [*highlighting the blue curve in Figure 6c*]…that distance is decreasing [*drawing the horizontal segments from bottom to top in Figure 6c*], and so we see that here [*drawing the pink segment in Figure 6d*] is like this [*highlighting the red segment in Figure 6d*], and then [*highlighting the blue curve and drawing the pink segments in Figure 6e*] ... here is this [*drawing the red segment in Figure 6e*]. So that's the same conclusion we had gotten from the circle, so then we can say that this circle relates to this graph."

Lydia's partitioning activity across the situation and graph included: (a) drawing horizontal segments emanating from the circle and curve (see Figure 6a and 6c), (b) tracing arcs from lower end points to higher end points on the circle (denoted in blue, see Figure 6a) and tracing the curve in the same manner (denoted in blue, see Figure 6c and 6e), (c) drawing vertical segments from the end points produced by the arcs or curves to a horizontal segment or line (denoted in pink, see Figure 6a, 6d and 6e), and (d) drawing horizontal segments between two pink segments and comparing their lengths (denoted in red, see Figure 6a, 6d, and 6e). Although we infer that Lydia's activity did entail some operative schemes (e.g., making quantitative comparisons between lengths, denoted in red), we characterize her partitioning activity as figurative due to it foregrounding repeated sensorimotor actions that produce similar perceptual results (e.g., partitioning along something curved, drawing vertical segments, and drawing and comparing horizontal segments). We specifically note that her constructing partitions along the curve of her graph and not maintaining a fixed reference point for her horizontal segments are contradictions that her partitioning activity was operative (Lee, Moore, & Tasova, submitted).

Providing additional evidence that Lydia's partitioning activity was figurative, later in the teaching session, Lydia drew a similar graph (Figure 7a) in order to discuss the relationship between "height" and "arc length". Her activity included tracing from left to right two equal horizontal segments (denoted in red, Figure 7a), drawing vertical segments from end points of the horizontal segments up to the curve (denoted in pink, Figure 7a), and tracing two corresponding curves (denoted in blue, Figure 7a). She compared the lengths of these curves and concluded that the increases in height decreased for equal changes in arc length. Similarly, on a circle, she traced two horizontal segments (denoted in red, Figure 7b), drew vertical segments (denoted in pink, Figure 7b), and traced and compared two arcs on the circle (denoted with blue, Figure 7b). Again, Lydia's figurative partitioning activity involved her carrying out same sequence of sensorimotor actions on her curve and circle (e.g., the sequence of drawing horizontal and vertical segments, and curves), the elements of which entailed similar perceptual results. We draw attention to her constructing partitions along the horizontal diameter of the circle and constructing and comparing lengths of curves and arcs to refer to changes in height; these are contradictions that she constructed abstracted quantitative structures of the situation and graph or her partitioning activity was operative.



(a) (b)

*Figure 7. Lydia's (a) new graph and (b) circle with drawn partitions.*

**Re-presenting and Transforming Partitioning Activity**

As the teaching experiment proceeded, we provided Lydia with additional opportunities to engage in partitioning activity. During the tenth teaching session we presented Lydia with the Red-Blue-Green task. Differing from previous, her activity suggested her re-presenting partitioning activity operatively across multiple representations.

After watching the animation (Figure 3d), Lydia claimed that as the blue bar was increasing at a constant rate, the red bar was increasing at a decreasing rate. Here, she was making claims about each bar's "rate" based on fastness of its movement experientially. She then constructed a graph, carried out partitioning activity on her graph, and concluded from her orange highlighted segments that, "because that amount of change [in red] is getting smaller and smaller, it's increasing at a decreasing rate" (Figure 8). Here, we interpret that she was making claims about "rate" by parameterizing amounts of change in red with respect to implicit time.

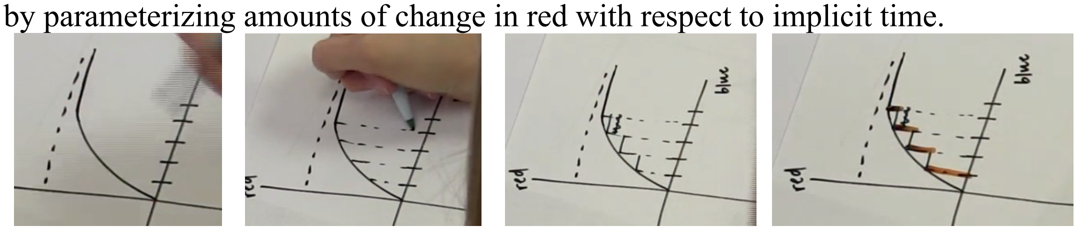
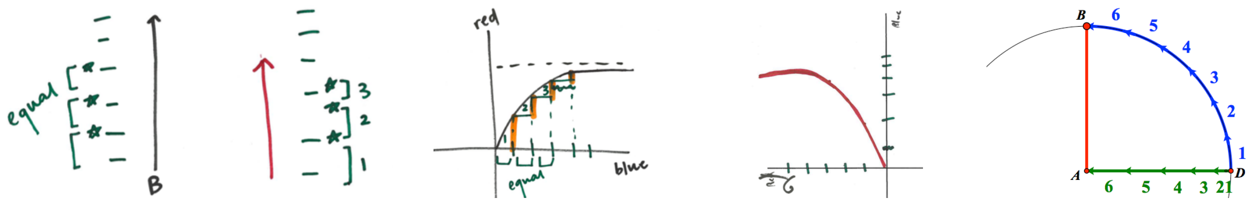



*Figure 8. Lydia's construction process of partitioning activity on her graph.*

The TR then asked Lydia to draw a picture of the situation with dynamic bars and show how she would manipulate the bars in ways that are consistent with those partitions on her graph. Lydia first drew two vertical lines and a collection of little horizontal segments to indicate landmarks of equal increments (Figure 9a). She then simulated how she imagined each bar first increasing to the respective bottom star symbol, then to the middle, then to the top correspondingly. She explained that she intentionally drew the three stars respective to the left segment at equal partitions but not the red segment on the right. She explained, "the red is already like started to slow down, so then it hasn't reached the next partition, so then we can see they're not traveling at equal paces… it's not reaching the next partition when the black is." The TR then asked her to return to her graph and talk about how her two drawings were related to each other. She wrote down labels of "equal," "1", "2," and "3" on both drawings to indicate how each increment was corresponding to each other (see Figure 9a and 9b). In contrast to her work on the Which One task, Lydia was able to mentally envision these two continuously varying bars as varying by successive increments and to re-present her partitioning activity on novel figurative material. Her representational activity also involved her transforming her partitioning activity between the graphical representation and the bar situation such that they both entailed the same quantitative structure.




(a) (b) (c) (d)

*Figure 9. (a) Lydia was re-presenting her partitioning activity on two bars, (b) her final graph of red versus blue, (c) her graph of blue versus green, and (d) annotated recreation of Lydia's partitioning activity on the circle as she dragged Point B from the 3 o'clock position to the 12 o'clock position.*

After the discussion on the blue and red bars, the TR drew her attention to the relationship between the blue and green in Figure 3d. Observing from Lydia's prior activity that she was able to re-present her partitioning activity on two bars, the TR decided to provide her another version of the sketch in which the same three bars were presented but were not growing and shrinking continuously; rather, the endpoint of each bar was movable and she could drag the endpoint to manipulate the length of each bar. By manipulating the blue bar to increase by equal increments, Lydia claimed that there was "hardly a change in green" for an initial increment in blue and there was a "decent jump in the value of green" for a subsequent equal increment in the blue. She then concluded that the green bar was increasing at an increasing rate with respect to the blue bar.

Turning to a graphical representation, TR decided to ask her to create a graph on a coordinate system with the horizontal axis labeled as green and the vertical axis labeled as blue (defined left and up as positive; see Figure 9c) in order to see whether she could re-present her partitioning activity in a non-conventional Cartesian coordinate system. Lydia then claimed she could transform her prior statement of the relationship between green and blue to a statement of "blue is increasing at a decreasing rate as the green increasing at a constant rate." She then created a drawing shown in Figure 9c where she drew equally-spaced partitions on the horizontal axis, drew partitions with decreasing amounts of space on the vertical axis (starting from the origin), and drew a graph that represented the uniting of corresponding partitions. She explained later that this graph could also represent the green bar increasing at an increasing rate with respect to the blue while she motioned her hand as if she was drawing equal partitions along the blue (vertical) axis and anticipating the space between successive green (horizontal) partitions increasing in size. We claim this activity indicated operative partitioning activity because she was able to mentally anticipate and re-present partitioning activity without physically carrying out specific sensorimotor actions to make their results perceptually available. Also, the novel coordinate system did not constrain her re-presenting invariant covariational relationship; rather, she transformed her partitioning activity on the two parallel blue and green bars to accommodate the given Cartesian coordinate system with novel axes orientations.

As further evidence of operative partitioning activity, approaching the end of the teaching session, we presented Lydia with the circle animation shown in Figure 3e and asked her to determine if the variation of the green segment on the circle corresponded to that of the green bar she discussed. She anticipated that, in order for the circle segments to have the same relationship as the bars, "the change in the arc length is like, when it changes 1 unit, then like the change in green is very small, but then as the green value increases, the change is also increasing." She then confirmed this relationship by dragging Point B for six successive equal increments along the circle circumference and envisioning the green segment increasing by increasing amounts (see her activity in Figure 9d).

## Discussion

Characterizing a student's thinking of amounts of change in terms of figurative or operative partitioning activity is significant in that it allows us to describe nuances in Carlson et al. (2002)'s covariation framework and, more generally, mental actions involved in quantitative reasoning (Thompson, 2011). A student's amounts of change meanings can differ in the extent that her partitioning activity is restricted to particular sensorimotor actions and the perceptual results of these actions. In this paper, we illustrated that a student's partitioning activity was initially figurative because it involved her seeking to repeat sensorimotor actions in a particular order across various situations. Furthermore, her partitioning activity was constrained to having perceptually available material. Consequently, when confronted with a novel situation in which these figurative elements were absent or carrying out the same sensorimotor actions failed (e.g., Which One), she had difficulty re-presenting partitioning activity.

We infer that Lydia's later success in anticipating and re-presenting partitioning activity in various contexts (e.g., circles, bars, and graphs) was partially due to her repeated experience of partitioning in previous teaching sessions. Throughout the teaching experiment, we intentionally prompted her to switch back and forth among various representational systems to re-present her partitioning activity, which included circle situations (e.g., Taking a Ride, Which One, and Circle), dynamic bars (e.g., Which One and Blue-Red-Green), and graphs with different axes orientations (e.g., Figure 9b and 9c). Towards the end, we interpret that her partitioning activity was not constrained to carrying our particular sensorimotor actions to produce similar perceptual results in a particular context as before; rather, she was able to sustain quantitative meanings of quantities across different representations and re-present and transform her partitioning activity to perceive invariant relationship among these representations.

von Glaserseld (1982) defined *concept* as "any structure that has been abstracted from the process of experiential construction as recurrently usable…must be stable enough to be re-presented in the absence of perceptual "input" (p. 194). Characterizing partitioning activity as we have enables us to extend and apply this definition in the context of students' reasoning about relationships between covarying quantities. When a student abstracts her partitioning activity so that it is not tied to particular figurative material, thus mentally anticipating transformations of such (e.g., changing orientations or representations), she has constructed a *concept* related to this relationship (e.g., the concept of sine or rate of change). The case of Lydia implies that researchers should be more careful about making claims about students' covariational reasoning and meanings before gaining insights into their activities among a variety of contexts. Students' mathematical meanings that involve carrying out particular actions in one specific context does not necessarily imply that their meanings are operative or they construct a concept related to those meanings. Moving forward, we call for continued explorations into how students reflect upon their sensorimotor actions related to some mathematical ideas (e.g., partitioning actions) and abstract quantitative relationships and structures (e.g., rate of change).

Moreover, we conjecture that engaging students in reasoning with varying bars that represent quantities' magnitudes can support their construction of graphs (Moore & Thompson, 2015). A potential explanation for Lydia's success in the Blue-Red-Green task is that she started with reasoning about continuous variations in the two parallel bars, and then imagined the two bars (including their partitions and variations) being oriented orthogonally to construct a graph, and finally *embedded* the bars back to the circle to construct their quantitative meanings. In contrast, in the Which One task, she had much difficulty with *disembedding* the height and arc segments from the Ferris wheel and anticipating variations of two corresponding bars independent of that.

The complex figurative features of the Ferris wheel situation might have constrained her from re-presenting her activity in another context.

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Geometric and Algebraic Reasoning in Adults on the Autism Spectrum: Excerpts from Case Studies

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*In this report, I examine the unusually precise geometric reasoning of a student on the autism spectrum in linear algebra given the beginning of the Magic Carpet sequence outside of their normal curriculum, as well as strong tendencies toward geometric interpretation of other problems and its effects. I contrast this with a similarly strong tendency toward algebraic reasoning in two other adults on the autism spectrum. Analysis of possible reasons for taking these approaches and implications for teaching are presented.*

*Keywords:* autism, linear algebra, geometric reasoning

My research attends to mathematical problem solving by adults on the autism spectrum (with a formal diagnosis), particularly those with a relatively strong background in mathematics. In this report, I focus particularly on the case of one student's work on one of the Magic Carpet problems of Wawro, Rasmussen, Zandieh, Sweeney, & Larson (2012).

There is a wide range of conceptions of what being on the autism spectrum means, including various academic and clinical definitions. The Autistic Self Advocacy Network (2014), the leading autism advocacy group run by people who are themselves autistic (and identify as such) states that autism is a neurological difference with certain characteristics, each of which is not necessarily present in any given individual on the autism spectrum. These include differences in sensory sensitivity and experience, atypical movement, a need for particular routines, and difficulties in typical language use and social interaction. They also list "different ways of learning" and particular focused interests (often referred to as 'special interests'), which are especially relevant for research in education. Of those characteristics, it is primarily the existence of special interests and the differences in language use and social interaction that are used as diagnostic criteria by the fifth edition of the Diagnostic and Statistical Manual of Mental Disorders (DSM-5).

Much of the research currently done on mathematics learning in people on the autism spectrum is focused on young children (e.g. Klin, Danovitch, Mers & Volkmar, 2010; Simpson, Gaus, Biggs & Williams, 2010; Iuculano et al., 2014) or looks at mostly arithmetic. There is also a notable strain of work done on the population of research mathematicians (e.g. James, 2003; Baron-Cohen, Wheelwright, Burtenshaw & Hobson, 2007), but very little attention is paid to groups in the middle (mainly high school and college students, or adults other than career mathematicians). This is a gap which I have sought to help fill with my own research, including the particular selection which I present here.

## Theoretical Framework

The theoretical framework that guides my research starts with the work of Vygotsky. In Vygotsky's writing, there is some work that directly addresses the study of "defectology". At the time, this was used to refer to studies involving children with certain disabilities (of a narrower scope than we might consider today) (Gindis, 2003). One of the main characteristics of Vygotsky's (1929/1993) conception of 'defectology' was the idea of overcompensation. Vygotsky explained this initially in a framework of physical overcompensation, such as a kidney or lung necessarily strengthening when the other one is missing or by analogy to vaccination. He argued that overcompensation also occurred in psychological development, both in its general course and in particular in the presence of various disabilities (concentrating primarily on those who were blind or deaf, as with most related efforts at the time). Based on this, he criticized the education of children with disabilities of the time as inappropriately focusing on only the weaknesses, not the strengths, of their students. Thus, Vygotsky's emphasis on the social reasons for psychological differences among people with disabilities also has much in common with modern social constructionist views of disability and the perspective of neurodiversity. Jones (1996) contrasts the social construction model (crediting its introduction to a paper by Asch in 1984) with "functional limitations" and "minority group" conceptions. Vygotsky (1929/1993) directly stated that "a handicapped condition is only a social concept" (p. 83), and Asch's (1984) criticisms of social attitudes toward people with disabilities are remarkably similar to statements by Vygotsky (1993) such as "the task is not so much the education of blind children as it is the reeducation of the sighted" (p. 86). In terms for specific to autism, the Autistic Self Advocacy Network and others work from a perspective in support of neurodiversity, a term coined by Judy Singer in the 1990s, and generally referring to a positive and inclusive perspective on not only autism, but also other neurological differences (Silberman, 2015). It is these more positive perspectives that I work from in my research and analysis.

While the diagnosis of autism did not exist when Vygotsky wrote, he viewed impacts on communication as a particularly important aspect of the effects of disability (with the issue coming up primarily in comparison between the blind and the deaf). Since differences in communication based on language are a major component of autism, it would be expected to have a strong impact on the shape of individual development in the Vygotskian framework. However, it should also be noted that all of the criteria for the autism diagnosis in the DSM-5 are entirely in the deficit-focused model which Vygotsky criticized. Additionally, since Vygotsky's perspective has a social-first model of child development (as opposed to the individual-first model of Piaget and others), the developmental differences linked to autism should be expected to be more significant and far-reaching in a Vygotskian model than in other models.

## Methodology

Given my interest in focusing in-depth on interviews with a small number of people, a method of case studies was a natural fit for my work. Case study focuses on in-depth understanding of the case in question, and only secondarily on generalizations from that understanding. Additionally, while generalization is possible, it is not of the same nature as generalization in other types of research (Stake, 1995). These are sometimes divided between embedded and holistic case studies, where an embedded case study is interpreted as examining a particular feature or subset of the case in question, while a holistic case study does not use such subdivisions (Yin, 2009). In this case, my decision for a holistic case study naturally follows from my neurodiversity-informed view that the nature of being autistic is not a discrete part of the person that can be separated, and thus an embedded design does not apply.

While my views are informed by the Vygotskian framework, there are some issues with using it directly. Some parts that are particularly relevant in autistic people, such as the ideas about atypical development and concept formation, particularly concern things that have already must have occurred far before starting university coursework, and thus cannot be observed in my interview subjects. The examination of inner speech also has difficulties; Vygotsky himself used children whose inner speech had not yet fully developed in his clinical experimentation on the subject. Thus, while those ideas from Vygotsky inform my views, additional constructs were required for the data analysis, and are elaborated upon below.

## Participants, Tasks, and Data Collection

Joshua (a pseudonym) was recruited from my university's center for students with disabilities. He received an Autism Spectrum Disorder diagnosis at age 18 (changed from a previous diagnosis of Obsessive-Compulsive Disorder), and was in his early twenties at the time of interview. He reported a strong interest in chemistry (which he was majoring in) as well as a particularly low level of interest in subjects unrelated to the sciences and a strong inclination to work alone. He was taking integral calculus and linear algebra courses during the time he participated in interviews.

Cyrus was recruited in the community outside of the university, received an ASD diagnosis at the age of 13, and was in his thirties at the time of interview. His mathematical background included a bachelor's degree in mathematics, and he was working in computer programming at the time of interview. In contrast to Joshua, none of his special interests were strongly apparent in the interviews (although mathematics or computing in general may be an exception).

Mark was also recruited in the broader community, received a diagnosis of Asperger syndrome at age 21 (this occurred before the release of the DSM-V), and was in his mid-twenties at the time of interview. Like Cyrus, no strong special interests appeared in Mark's interviews. His mathematical background included a bachelor's and master's degree in mathematics.

The data for my study comes from a series of clinical interviews with each of these participants that I conducted, focusing on a variety of problems (with most given to at least two participants as appropriate). The interviews were audio recorded on a password-protected device. In this report, I focus on results from three tasks. The first of these is the first of several Magic Carpet tasks, introduced by Wawro, Rasmussen, Zandieh, Sweeney, & Larson (2012). The task used is formulated as follows.

| |
|---|
| You have two modes of transportation: a hoverboard and a magic carpet. The hoverboard moves along the vector (3,1) and the magic carpet moves along the vector (1,2). Can you get to a cabin at (107, 64) using these modes of transportation? If so, how? If not, why is that the case? |

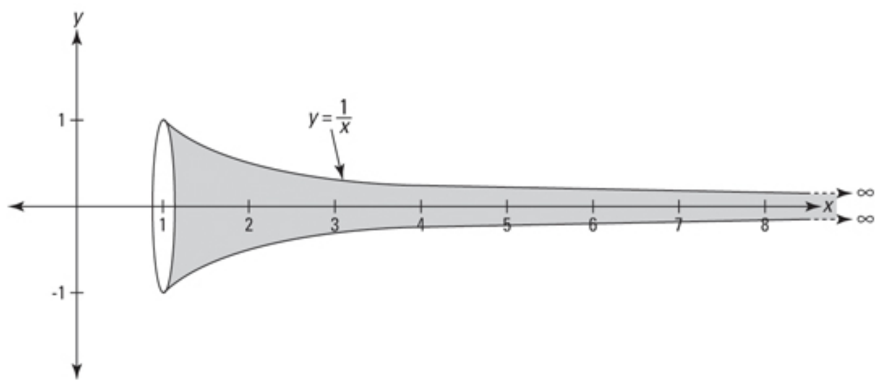
*Figure 1. Magic Carpet Problem Formulation*

For this problem, a possible (and expected) solution in the context of a course in linear algebra is to find values $a$ and $b$ such that $a\begin{bmatrix}3\\1\end{bmatrix} + b\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}107\\64\end{bmatrix}$; in this case, those are $a = 30, b = 17$.

The Magic Carpet tasks as a whole were designed for and used with students in a linear algebra course, who had completed at least two semesters of calculus. Since they were used at the beginning of their course, the students had not been previously exposed to the standard linear algebra solution above, although all had been introduced to the idea of a vector in some capacity. Instructionally, the intent of the problem in the context of a linear algebra context was primarily to introduce the idea of linear combinations, and to lead into other problems in the setting which introduce span and linear independence of vectors. For additional context, the second Magic Carpet task asks if there are any points that one cannot reach with a combination of the magic carpet and hoverboard from the first problem. This demonstrates part of the overall intent, to guide students toward ideas of linear independence and span as they find that there is no such point and justify their answer.

The second task I consider in this report is the classic Gabriel's Horn or Painter's Paradox.

Gabriel's Horn is the object created by rotating the graph of the function 1/x around the x-axis (starting from x=1), as shown below.




Its surface area and volume can be calculated as follows:

$$A = \lim_{b\to\infty} 2\pi \int_1^b \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx \geq \lim_{b\to\infty} 2\pi \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} 2\pi \ln b = \infty$$

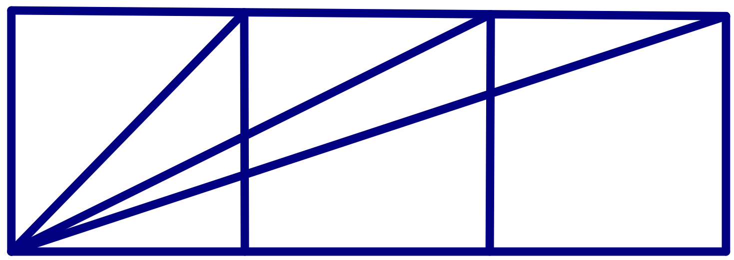
$$V = \lim_{b\to\infty} \pi \int_1^b \left(\frac{1}{x}\right)^2 dx = \lim_{b\to\infty} \pi\left(\frac{-1}{b}+1\right) = \pi$$

Since the surface area is infinite, it should require an infinite amount of paint to cover the surface. However, since the volume is finite, the horn could be painted by filling it with that amount of paint. Is this possible? How can this be resolved?

*Figure 2. Gabriel's Horn Problem Formulation*

This result was discovered by Torricelli in 1641, and regarded as paradoxical by its discoverer and many later mathematicians, who used the result in discussions about the philosophical status of mathematics (Wijeratne and Zazkis, 2015). Wijeratne and Zazkis gave the problem to twelve undergraduates in a calculus course who had been presented with the relevant integration techniques in their course, and conducted interviews about the task afterward. One of the more common responses, reported from a majority of participants, was something with regard to the physical or contextual considerations of the problem (such as paint 'getting stuck' at an atomic level once the horn is small enough); because of this, this problem is another case in which I saw possibilities in considering geometric approaches against the more standard approach, with added complications in the paradox with infinity and the related opposition of the geometric intuition and standard solution.

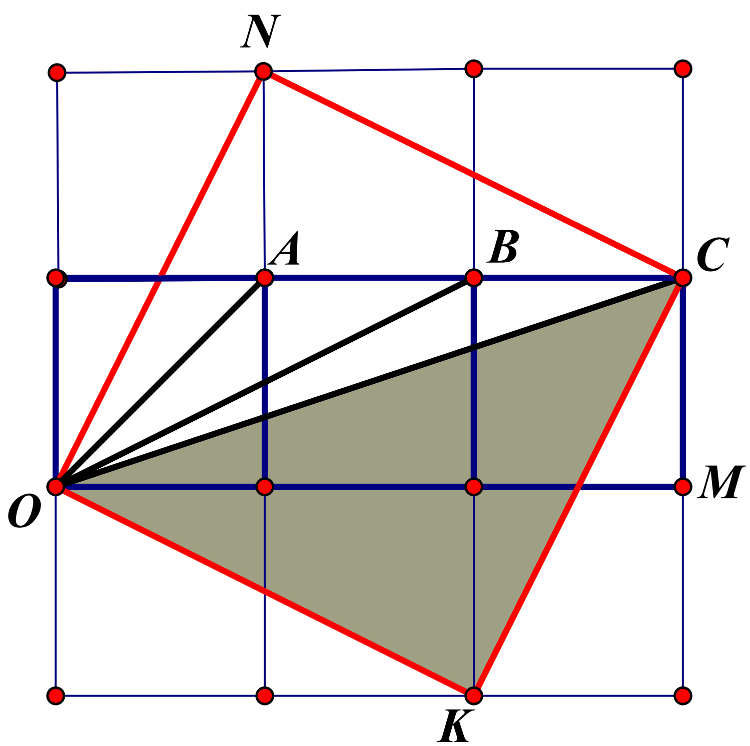
The third task is a more geometrically-focused one, starting with the figure below.



What is the sum of the three angles formed between the bottom line of the figure and the hypotenuses of the three right triangles in the figure?

*Figure 3. Squares and Triangles Problem Diagram*

The intended solution to this task is geometric, and can be seen from this second figure:




Here, it is easy to show that the angle ∠BOM is equal to the angle ∠MOK (for example, by considering congruent triangles), and that the sum of the measures of angles ∠COM and ∠MOK is 45 degrees (as this sum is ∠COK and OC is a diagonal in the square OKCN). Since ∠AOM is clearly also a 45 degree angle, the sum of the measures of the three angles is 90 degrees.

*Figure 3. Squares and Triangles Solution Diagram*

The inclusion of this problem was guided by the desire to introduce a counterpart to the first problem, one where the geometric interpretation is ultimately the intended solution. This occurred after some findings with Joshua related to geometric tendencies, and allowed a broader range of testing possibilities for or against such geometric tendencies.

### Interviews with Joshua

When presented with the Magic Carpet problem, Joshua's first response focused on drawing a sketch:

*Joshua*: Here's how I would do it. Draw a sketch. [drawing] So, I don't know if this has to be explicitly done mathematically or whether it can be done by drawing a sketch but in physics I know that we always drew sketches. That way, the person knows what is going on in our heads.

*Interviewer*: Okay, well, that's what I want to know, so that's good.

*Joshua*: Okay, so here's what's going on in my head. [more drawing]

*Joshua*: So now we've got the vector that we wanted to, which... look like so... 1, 2 we could scale up to 10, 20, so that's what I did here, and we have the vector 3, 1, which we could scale up to 30, 10. Oh, that's right, and so now, how we could approach this is we could follow this vector here, ...we could follow the vector, this is the vector 30, 10, and this is the vector 10, 20, I just scaled it by 10,

*Joshua*: Each vector scaled by 10, then from this point here, what we could do, is we could slide this vector up, slide it up all the way up to here and then, so what we could do, is from this point here, we could draw, so if we have 64, 107, we could- go down 20, 20 units, and 10 units to the left, like so, and then, we have the same vector, we just literally transformed the vector. I guess not [transformed], we moved the vector, the point is that it's the same thing, 10, 20, and then, this point of intersection, is where we would change what instrument we were using, so you want to look at that.

*Interviewer*: Okay. Interesting.

*Joshua*: So that's how I would do it. I'd approach it literally geometrically. Yeah, there you go.

*Joshua*: And so we've got two lines there, and that little point of intersection is where, and you can find that quite easily, on the x-axis, and that point of intersection is where you would switch your instruments, ...and again, we have the vector 1, 2, we can find the slope of that vector, and then we can move it over to the point 107, 64, and then, you know, it wouldn't really be that hard to find that point of intersection but that's how I'd do it, literally just play around with those vectors.

Joshua starts off by drawing the destination point, using a ruler to measure precisely where the point should be to get a drawing that is properly scaled. For the first step in the sketch, he draws the given vectors for the magic carpet and the hoverboard, scaled up by a factor of 10 (given the rest of the scaling, they would be barely visible otherwise). After doing this, he moves the vector (20,10) so that its tip touches the destination point, and then extends the two vectors so that they intersect, concluding that the intersection point is where you should change from one mode to the other. The end result sketch is shown in Figure 2 below.

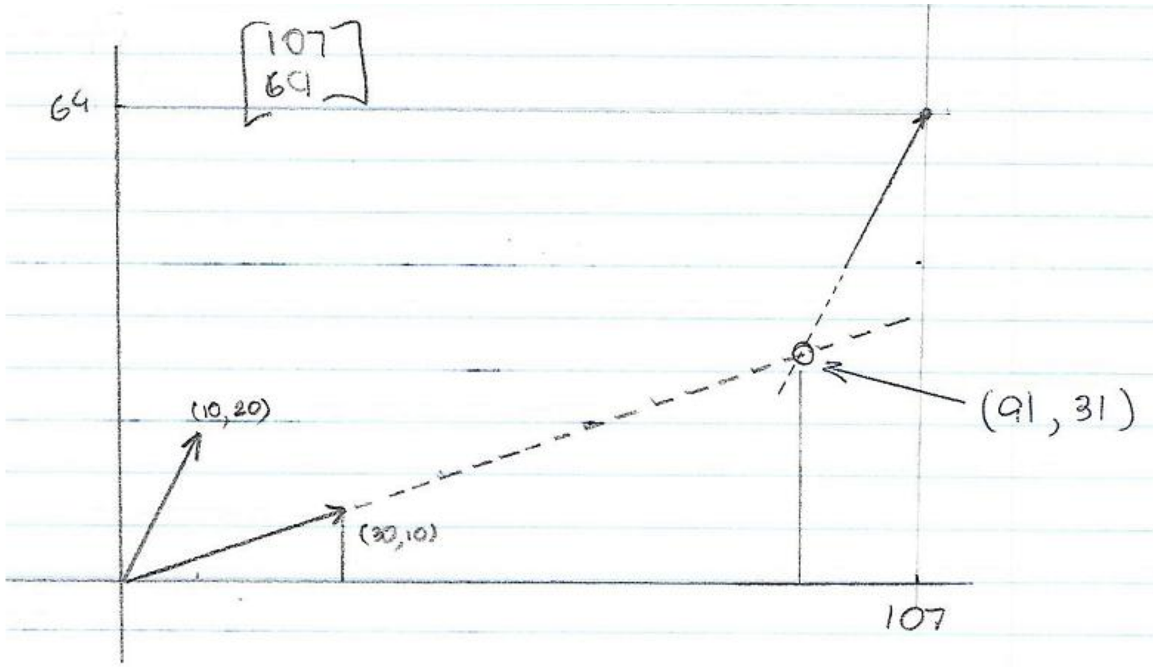



*Figure 2. Joshua's Magic Carpet Solution*

This solution is unlike those of any of the students observed by Wawro et al. (2012) in their use of this task. The drawings used by Joshua were produced using a ruler and were very precise, enough to give a correct solution (note that this was done on lined paper, not a square grid). However, it is notable that Joshua terms this a "sketch" (possibly ignoring some of that word's connotations) and seems to not regard this solution as "mathematical". The drawings were measured after they were produced, and the point of intersection found by the drawings was the correct point (though the coordinates written above are very slightly off). The interpretation as the location where the person in the problem changes from one mode to the other is also correct. Thus, this solution accomplishes the stated goal of the problem (to find a way to get to the cabin) perfectly well, although by approaching the problem this way, Joshua avoids the intent to push the student toward a standard linear algebra solution. Joshua's solution is less directly related to linear combinations of vectors, though a geometric version of the idea can be brought out from the drawing used for the solution. In particular, vector scaling is used to arrive at the solution, as well as vector addition (which is geometrically accomplished by placing the start of one vector at the end of another).

As Joshua acknowledges, he had been presented with the linear algebra material that one could use to solve it in the intended linear algebra way. Thus, it is particularly noteworthy that not only did he gravitate toward the geometric solution in Figure 1 first, but that even when asked for an algebraic solution, he adapted his geometric solution to its algebraic counterpart rather than produce a more standard solution:

*Joshua*: Well, like I said, I'd probably find the slope of the vector 1, 2, make that, once I have the slope, make a line with slope 1, 2, so literally it would be y is equal to m x plus b, so, our y and our x values would have been 64 and 107, we have 64, 107 with some slope, you find the b, you find the y-intercept, you'd have an equation y equals m x plus some y-intercept, you'd know the slope, you'd find the other slope, you'd find the other equation, and that point of intersection is where the x and the y values are the same. Do you know what I mean?

*Joshua*: You find two equations for both lines, so I found two lines there, find equations for both lines and then find the common solution for both lines.

*Interviewer*: So you're finding an equation that's sloped on one of the vectors and hits this point, and the other equation that's sloped on the other vector hits the origin.

*Joshua*: Yeah. Exactly. And then find a common solution to those. And that's where you would switch your implements [indicating the modes of transportation].

This version of the algebraic solution suggests that Joshua is connecting his extensions of the given vectors to ideas about lines, but not the algebraic presentation of vectors which he had learned. After Joshua presents his algebraic solution, he does ask if there is an easier way to do the problem. After being shown the standard algebraic solution, he responds:

*Joshua*: Oh, that's a lot easier. See, I didn't think of that application. It's easier for me to just literally draw it out. Yeah no, that didn't even come to mind. Goes to show you what I'm getting out of this class, [laugh]. Which I'm not saying is his fault, it's just the way it is.

For comparison, Wawro et al. (2012) state in their research that the student solution attempts they observed fell into three categories of "guess and check", "system of equations first", and "vector equation first". The third category fits most closely with the standard linear algebra solution, and its presence for the original study's students highlights the differences in Joshua's approach (which fits into none of the three): some students with no prior linear algebra instruction presented an algebraically vector-based solution, while Joshua did not, despite linear algebra instruction from the course he was taking at the time which included a discussion of linear combinations similar to what this problem is building toward.

When Joshua was presented with the Gabriel's Horn paradox and the mathematical solutions for the integrals (which he unfortunately did not arrive at himself), this was his response:

*Joshua*: Well, it does seem kind of strange, because, like I said [?] the integral test, we learned that if the volume underneath the series is divergent, then the series itself is divergent, but here you've clearly shown that, you know, we've got a convergent volume, but the series itself must be divergent, because you need an infinite amount of paint to paint the trumpet, the horn, whatever you want to call it. So, that's how I would think about it. Is there something that you would add?

*Interviewer*: Well, I'm just curious as to whether you feel this is a conflict or not.

*Joshua*: I do actually feel like it's a conflict. Because we've, again, when I work this out in my head, it would seem kind of weird that the volume is smaller than the surface area, and the surface area, if we take surface area to be in $\mathbb{R}^2$, should be smaller than the volume, which we take as being in $\mathbb{R}^3$, so yes, it does seem kind of conflicting. Do you know what I mean?

*Interviewer*: Okay.

*Joshua*: How can a volume be smaller than an area? For the same- and, I, you know, now I'm thinking about that orange paint thing, and it seems kind of bizarre. So yeah, I see what you're saying.

Unlike the first and other interviews, there was not as much of a geometric tendency shown (possibly due to Joshua being unable to produce his own solution), but the main objection above is one considering dimensions. In a subsequent interview, Joshua was pushed further, and ultimately asked which of the two interpretations he viewed as being in conflict was more correct. This was his response:

*Joshua*: The volume being defined. Um, it feels more correct because when... okay, I visualize a horn in real life, and I visualize literally filling the horn, tipping the horn vertically and filling it with paint, and, to me, um, if I filled the horn, which went all the way, you know, the little thing got- the tip went all the way to infinity, to me, you know, in real life that's not possible, and so the horn has to end somewhere, and so, to me, the volume having some defined volume makes more sense than, you know, the surface area being infinite. So I'm literally going with a real-life interpretation of this. Literally, you can't fill the horn with more paint than when the tip of the horn reaches the diameter of the paint molecule. You know, once the tip of the horn moves further out to space than the diameter of a paint molecule, you can't fit any more. You can't fit any more paint, and so it's got to have some defined volume.

*Interviewer*: Okay. What if we were to simply consider the abstract mathematical volume without putting any paint in it?

*Joshua*: I wouldn't really say that any of them are correct or incorrect, because mathematically they're, you know, that's... the law, I don't know if you want to call it the law, but the equations show what they show, and if they show what they show, you know, if they show that the surface area is infinite but the volume is defined, and not infinite, then that's what they show, you know, there's nothing we can really do about it, I mean, I'm sorry. So, graphically speaking, you know, neither of them seem really, you know, it's just like one over infinity. You know, infinity isn't a number, it's a concept, so how do you take the inverse of infinity? Or, how do you take the inverse of a concept? Well, it would seem kind of not correct to me, but I think it's perfectly fine to do that, and so I accept one over infinity as being zero, considering that if I plug in a really big number into my calculator, and take the inverse, it spits out some really tiny number that I more or less could consider zero. So I can't say that either [?] correct or not correct.

Overall, Joshua demonstrates a particularly high level of trust in mathematics. He also shows a lower level of trust in intuition, although it is unclear whether this is from having less of a sense of intuition or simply less trust or value put on it. The results here are quite different from those of Wijeratne and Zazkis (2015); most of the students in their research did not readily separate physical or 'real-life' considerations (typically expressed geometrically) from the mathematics. Here, while Joshua does produce some contextual geometric analogies as expected, he does not resist separating them from the abstract mathematical result, and again voices acceptance of its validity. This suggests that Joshua's strong preference for using geometric solutions does not translate into a belief in those geometric solutions as more true or more valid than more standard mathematical solutions.

## Interview with Cyrus

Cyrus was also presented with the Magic Carpet and Gabriel's Horn tasks. By contrast to Joshua, his approach to the first Magic Carpet problem was an entirely algebraic one, setting up and solving a system of equations using matrices. However, the solution he arrives at is not correct, having negative values for both coefficients. He recognizes that this is not reasonable, and suggests that this means reaching the point is not possible. When asked what kind of solution would indicate that reaching the point was possible, this is his response:

*Cyrus*: Okay, that's fine. Okay, so, yeah, that's pretty much what I make of this problem. If we were able to find- there would be two cases. If these had both been positive numbers and there was a unique solution, then definitely there is- this is- the problem was asking if you can- you can reach this point. Using either the carpet or the hover-thingy.

*Interviewer*: Not either-or. We can use part of one then part of the other.

*Cyrus*: But can we reach this as our destination and, if there was either one unique solution and it must be positive, or if there was infinitely many solutions, in either of those cases it would be possible to do this.

Here, we see that even approaching the problem from a hypothetical standpoint, the interpretations provided are still entirely algebraic. This approach fits with the majority of students in the study from Wawro et al. (2012), although those students likely did not have previous exposure to the method that Cyrus is recalling here. Cyrus' reaction to the negative solution he obtained, combined with his correct interpretation of the hypotheticals that he gives, shows that he is not approaching the algebraic problem in a way entirely detached from the original problem's context. However, he did not bring up the geometric interpretation of that context independently, so it is uncertain whether he is more inclined to translate the solution into terms of the geometric context, or to mentally translate the context into algebraic terms.

When Cyrus was presented with the Gabriel's Horn paradox, the interviewer presented each of the formulas present in the solution, similar to Joshua's case. Here, Cyrus is first given the description of painting the horn, and says (before seeing the calculation) that the surface area's amount should be sufficient to paint the horn, and the volume amount would be excessive but would also work. He is then presented with the solution and asked for his thoughts:

*Cyrus*: So, does it seem reasonable that this object would have a finite volume and an infinite surface area?

*Interviewer*: So, does it seem reasonable that this object would have a finite volume and an infinite surface area?

*Cyrus*: Does it seem reasonable, um,

*Interviewer*: Does it make sense?

*Cyrus*: Not at first. But it seems like it is mathematically sound, so, it does seem like the right answer. So I would say yes, it makes sense overall, when you find the limits. But it still kind of contradicts intuition.

*Interviewer*: Okay. So, that's an interesting thing you said there, ah, is contradicting intuition a particular problem? How do you feel about that?

*Cyrus*: It isn't a problem when you're dealing with mathematics. It happens, I don't want to say frequently, but often enough.

Given the results of the integrals, Cyrus does not show any signs of doubt, but instead immediately states that the volume is sufficient to paint the horn. While he recognizes it as a paradox, he doesn't seem to have any problem with this, and concludes that he would accept the mathematical answer. From the discussion of using volume or surface area, we saw that Cyrus started off with an intuitive idea that the volume should be larger than the surface area, an idea which Joshua also held. Compared to Joshua's case, however, Cyrus seems to be less attached to the idea that a volume should be larger than a surface area, referencing it only once and making no reference to physical impossibility.

Cyrus' responses when asked about the role of intuition suggest that he uses it to some extent, but considers the formal result more significant. In particular, he does not consider "contradict[ing] intuition" to be a barrier to "mak[ing] sense overall". He also suggests that this is particularly true in the area of mathematics; this belief may be simply stronger there, or may be restricted to certain fields only.

**Interview with Mark**

When Mark was presented with the sum of angles problem, his initial (rejected) thought was to consider it as a sum of arcsines.

*Mark*: I'd like to keep track of what I'm doing, so I'll call that theta one, theta two, theta three, and I want to find the sum, and okay, this is a square, so I'll just note that theta one is pi over four. Okay, so, there's probably a more elegant way to do this than oh, calculate all the angles and what they are, and these two aren't nice, anyway. Are they? Mm... no.

*Mark*: So, wait, what kind of thing am I supposed to be getting for the sum? I mean, I could say, arcsine of this plus arcsine of this plus arcsine of this but that seems to sort of go against the spirit of the problem. That seems to sort of 'not count', if you will.

After rejecting this first idea, Mark considers that in this context the sum should be a "nice number", and works on some ideas from there.

*Mark*: What is the ratio of these angles to... this. Let's see. I had this sort of intuitive idea that maybe this angle is half of this one, and I'm not sure if that's actually true. But, well, it would be pretty nice if it were true, and I could definitely use it. So, could I try and show if that were true in some way? Let's see. This is a slope of one, and this is a slope of one half, so this is halfway up. Uh, yeah, so this thing is, we've got one half, one, theta two, and, wait, no, if that were true, then sine would look like a bunch of jumping lines or something. Would it? Because sine is the ratio, of these. And this is staying the same, so yeah, that's probably not true. But I think this should come out to be something nice.

*Mark*: Wait, is this ratio something useful and I forgot? No. Hmm. Well, I could say that, these angles up here are also theta one, theta two, theta three and then, well, I'll just call these complementary angles phi one, phi two, phi three, and the sums of the theta plus phis are all ninety, or pi over two, hmm, does that help anything? I don't see how that's going to help anything.

*Mark*: Okay, I feel like I don't have anything else to say, but there's got to be something. I can do, do something like this, um, hmm. What useful facts about geometry could I use here? There's some trig identities about sums of angles, no, hmm. What if I think of this in a sort of polar coordinate radial thing? Like, what would these points be in polar coordinates and would that give me anything useful? Well, if I assume that the square side length is, one, might as well, then this is root 2, pi over 4 well, okay, this one is, this is, root 5, something, this is, root 10, something else, well, no, that doesn't seem to be a useful fact. Hmm. It must be one of those things that's simple when you see it. But I'm never quite sure what to do with those. Not some kind of, orderly procedure or sequence of building blocks of smaller facts or something. [Mark continues to make attempts at solving the problem for about 15 minutes, without success.]

After Mark's lack of success at solving the problem, the interviewer presents the solution detailed previously. After this explanation, Mark is asked if the solution is clear:

*Mark*: Yes and no, I mean, it's clear in that, I see that all of these things are true and they produce the result, but it's one of those, okay how am I supposed to think of doing this?

*Interviewer*: Okay. So you think that it's just a trick out of the pocket, or…

*Mark*: Well, there must be some sort of way that someone thinks of doing this, but it doesn't quite mesh with the sort of way I usually think about things.

Overall, we can see two characteristics demonstrated in Mark: an inclination toward algebraic solutions, and a tendency to think about the problem at least in part via considering the hypothetical problem designer's intent. In this particular case, the second leads to the quick dismissal of the first, though it is still mentioned again. The reoccurrence of mentioning the rejected solution suggests there may be some difficulty in switching away from the preferred method, even once it is recognized as necessary. This is somewhat demonstrated in the next few steps, as Mark puts some known facts about the problem into algebraic terms and searches for geometric facts with algebraic forms of expression. Outside of algebra specifically, there is also a desire to see things in a systematic way, or to fit them inside of an existing system. There is little to no mention of intuition in this consideration of the problem.

That this problem is ultimately not solved independently is a piece of evidence for the supposition that problems that are strongly skewed toward solution methods the participant disfavors can pose a particular challenge, which is most likely reinforced by that preference's leading to having had less prior practice in using those methods successfully. This could be particularly prominent in problems which require particular cognitive jumps, or have unusual solutions in a way that particularly draws on the problem solver's weak points.

Although Mark does not have confidence in understanding the rationale behind the solution's construction himself, for him this does not translate into a lack of confidence that such a rationale exists. This continues the trend shown with Joshua where trust and confidence in mathematics is generally high and not cast into doubt by individual problems with a task.

## Analysis

As mentioned above, while Vygotsky's ideas guided my design, additional theoretical constructs informed my analysis. In this report, I use two additional constructs: Fischbein's notion of intuition (1979, 1982) and Grandin's work on geometric reasoning and autism (1995).

In Fischbein's (1979) use of the idea, intuition is separated into different categories, particularly "primary intuition" (developed outside of a systematic instructional setting) as opposed to "secondary intuition" (developed in a systematic instructional setting). The division of categories here has similarities to Vygotsky's distinction between everyday and scientific concepts, and I find it reasonable to consider the primary and secondary intuition used by Fischbein as identifying intuitive reasoning related to everyday or scientific concepts, respectively. Further exploration of intuition by Fischbein (1982) uses a similar division between "affirmatory intuitions" and "anticipatory intuitions", focusing primarily on the former. In this division, affirmatory intuitions are those that are "self-evident [and] intrinsically meaningful", which again stands outside the systematic instructional context. In the context of other works, it is this definition that is closest to what is typically meant when 'intuition' is named but not explicitly defined, which is useful for situating other work which mentions intuition but does not focus on it.

The precision and completeness of Joshua's geometric solution suggests some possibility for a tendency toward thoroughness instead of skipping steps, which is consistent with other observations of more systematic and less intuitive reasoning. In particular, it may suggest that the geometric solution in his mind has more in common with a scientific concept and less in common with everyday concepts or primary intuitions than is generally expected from students using geometric/visual solutions (as seen in the geometric attempts of students observed by Wawro and colleagues). In fact, Joshua explicitly situates his geometric work in a classroom context (although one in physics rather than mathematics), also pointing to the realm of secondary rather than primary intuition. From Joshua's response to the Gabriel's Horn paradox, he demonstrates a valuing of results from systematic reasoning over results from intuition, further supporting this difference in his view. Cyrus' response to this paradox reflects similar values, and Mark also explicitly expressed a desire for more systematic forms of reasoning when thinking about the geometric problem he was given. For some students on the autism spectrum, a general problem with or mistrust of intuitive reasoning (particularly primary intuition) may lead to a kind of compensation where knowledge that is learned explicitly as a scientific concept, instead of formed as an everyday concept, is emphasized. Additionally, these results and Mark's confidence in a systematic reason for the construction in the solution presented for his problem (although he did not know it himself) support a trend of high trust and confidence in the structure of mathematics.

Joshua's tendency toward geometric thinking about the problem may result from an instance of overcompensation, as defined by Vygotsky (1929/1993). Joshua may have particular strengths in areas related to the geometric reasoning he uses here, which he is using to compensate for weaknesses in areas related to the algebraic reasoning that would be involved in the 'standard' solution to the problem. Cyrus and Mark's tendency toward algebraic thinking may be from a similarly structured instance of overcompensation based on different individual strengths and weaknesses. In their solutions, both Joshua's neglect of the standard linear algebra solution from his course and Mark's returns to the arcsine solution are examples of the strength of these tendencies even when working against them in different ways. However, since overcompensation is an idea defined in relation to individuals' development, the observations in interviews with adult students will most likely be of the end result of the compensation process that Vygotsky described (and not show the process itself). The more common tendency toward some more systematic form of reasoning may be an avenue for further investigation along these lines.

The use of a geometric/visual approach in Joshua also notably fits with what we see in other sources, such as a description of the thought process in Temple Grandin's work. Grandin (1995) describes her own memory as being based on remembering static or moving images, and being able to both understand others' information and express her own better in writing than verbally (which may suggest an issue with the interview process). She also describes thinking of abstract ideas in terms of images or sequences of images. However, the range of variation in autism as well as other interview experiences (as seen here) lead me to believe that the underlying principle for differences in problem-solving methods is more complex and does not always push toward a geometric approach; it may be more common, but it was not with the sample in my study. The adaptation of the geometric solution for an algebraic solution, staying rooted in the original geometric thinking, further highlights the strength of Joshua's inclinations toward geometric methods as well as suggesting a possibility for drawing connections. More broadly, while Mark's attempt at an algebraic solution was not successful in this instance, in my overall data there are examples of both success and failure with both participants' preferred approaches. In terms of the case study method, Joshua can be considered representative of the subgroup with geometric problem-solving tendencies, while Cyrus and Mark may be representative of algebraic or abstract problem-solving tendencies.

## Conclusions

One possible effect of the unusual tendency seen here is that it may pose a difficulty for an instructor's plan to confront students with a problem that would ordinarily necessitate a particular approach (the introduction of which is the goal of the activity), as happened with this problem. The intent seen in the problem design by Wawro et al. (2012) was to create a need for the use of a vector-based approach, which notably did not occur here. The purpose of this seen in the context of the full Magic Carpet sequence is to move toward an understanding of linear combinations and linear independence of vectors. By avoiding the use of vectors entirely, Joshua would not get the intended effect of a logical transition to the following topics without further input. The resignation in the expression of this by Joshua above suggests that this is something that has occurred in classes before.

Since mathematics coursework in a traditional academic setting uses scientific concepts explicitly, and the role of everyday concepts is often viewed as interference to be minimized, an inclination away from intuitive reasoning would most likely be helpful in the context of a standard mathematics classroom. By contrast, ideas for mathematics instruction involving the use of students' pre-existing real-world concepts or other ideas, such as the Magic Carpet problem used in this case study, may be less helpful for students on the autism spectrum without additional attention to their particular differences.

The concluding remarks by Joshua – "It's easier for me to just literally draw it out…that didn't even come to mind. Goes to show you what I'm getting out of this class" – not only point to a need for instructional attention in the linear algebra class, but also suggest that there could be a more general pattern across multiple courses of using unexpected approaches that may avoid (or appear to avoid) the general intent of the lesson. Although the other tasks are less explicitly instructional, a similar possibility can be observed with Mark's attempted solution. I suggest that while this can certainly be a problem if it goes unnoticed, with a well-tuned approach it could be turned to an advantage. This is much like Vygotsky's concept of compensation, although strictly speaking Vygotsky's original conception of compensation was for development of more general reasoning abilities as well as child development. While the strict conception would not apply to adults or to specific linear algebra skills, I think that the general idea of using strengths to reinforce weaknesses, possibly in ways that have a different form than the expected one, is useful here. The construction of connections between a student's unusual approach and the standard approach can lead to a deeper understanding, particularly if it allows the student to make use of their particular strengths. Additionally, taking advantage of the opportunity to demonstrate the connections between standard and unusual approaches in a classroom setting has the potential to enrich learning for the class as a whole.

These results as well as others in my line of research show that there is not necessarily a single approach that students on the autism spectrum can be expected to use, and that a variety of forms which may be considered unusual can produce successful results. This highlights the importance of being able to see validity in unusual student work and interacting with students without deficit-based preconceptions, something which holds particular importance across a variety of forms of disability-related education research.

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## Math Help Centers: Factors that Impact Student Perceptions and Attendance

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*Mathematics help centers have become common fixtures in post-secondary education, where undergraduate students can go for more assistance on typically first and second-year courses. However, there is scant research on them. In this study, we report on existing literature concerning Math Centers. Then, we use data collected at one university in the southwestern United States from 1088 students over six academic semesters and grounded theory analysis techniques to study and draw initial conclusions. For example, of the 14% of students who did not attend, 45% stated that they did not feel they needed any help. Roughly half of the 67% of students that went to the math center more than once a month felt as if the tutors were responsive to their needs and willing to help. We claim that more work needs to be done, specifically inter-institutionally, on math centers in order to corroborate many of our results.*

*Keywords:* tutoring, university mathematics, support services, math center

Mathematics help centers, also known as mathematics learning centers, mathematics support centers, mathematics tutoring centers, or simply "math centers" (the term that will be used in this paper) typically aim to provide support to undergraduate students in their mathematics courses. Math centers often focus on the mathematics courses students take during their freshman and sophomore years of study. At most of these facilities, students typically receive tutoring services from peer tutors (Rickard & Mills, 2018) who are often advanced undergraduates or, in some cases, graduate students. This tutoring is usually provided on a drop-in basis.

Math centers have become common fixtures in postsecondary education. This is evidenced in a recent study of calculus conducted by the Mathematical Association of America, where over 70% of the course coordinators surveyed reported that their institutions had a math center (Johnson & Hanson, 2015). There has also been an increased focus on math centers as noted by a recent handbook for math center directors (Coulombe, O'Neill & Shuckers, 2016). The handbook, which had contributors from 31 institutions ranging from two-year community colleges to liberal arts institutions to large research universities, demonstrated that a variety of institutions have given math centers a permanent position in their academic-support service structure. The increase in math centers is not just a U.S. phenomenon; efforts to implement math centers are found in the United Kingdom as well (Gill, Mac an Bhaird, & Ní Fhloinn, 2010; Matthews, Croft, Lawson, & Waller, 2013).

With a rise in the numbers of math centers and increased attention on them, there has also been an increase in educational research related to math centers. This can be noted by the new working group in RUME (i.e., the *Research Opportunities for RUME Researchers in the Context of Mathematics Resource Centers* working group) dedicated to this line of research. We aim to add to this small, but growing, research on math centers.

## Research on Math Centers

While there is a sizable body of research on tutoring at the collegiate level across subject areas, Cooper (2010) has suggested that tutoring at math centers with drop-in attendance differs significantly from traditional tutoring and has called for more research on the impact of this kind of tutoring. Some educational researchers (e.g., Matthews, Croft, Lawson &Waller, 2013; Rickard & Mills, 2018) have answered this call. In the most recently published study related to this line of inquiry, Rickard and Mills (2018) reported that attending math center tutoring had more of an impact on lower-achieving students' grades than the grades of other students for their first calculus course, even when other variables were considered. Their model predicted that students' final course grades increase by one percent with every three visits to the math center.

Rickard and Mills' findings are in line with those of Cohen, Kulik and Kulik (1982). In their meta-analysis that considered tutoring for a variety of content areas, they found a positive correlation between students' attendance in tutoring programs and their course outcomes. For the more than 60 studies they considered, Cohen and colleagues reported that students who received tutoring consistently outperform students who did not. In addition, they reported that tutors also benefitted from the interactions with the students. Moreover, the noted benefit of attending tutoring centers was reported as being more pronounced with mathematics tutoring than other content areas.

While many of the math centers provide drop-in tutoring, it should be noted that math centers may also offer other services to students. Some services include access to print or digital resources, guidance on use of digital devices and platforms used in mathematics courses, and review sessions prior to mid-semester and final examinations (Coulombe, O'Neill & Shuckers, 2016). In one of the few studies that considered a mathematics support service other than tutoring, White, O'Connor, and Hamilton (2011) investigated the reasons students in a statistics class gave for attending peer-led review sessions. The authors reported that there was increased attendance when the students had positive attitudes of the review sessions. The results of this study also support the theory of planned behaviors, saying the intention to perform a behavior (e.g., attend a review session) is related to the rate of carrying out that behavior (e.g., actually attending the review session).

Many studies related to attendance often focus on data associated with sign-in information that many math centers collect. These data are often gathered from students' academic files or students' self-reports. Bannier (2007) used correlational analyses to examine which students attended math centers based on a variety of variables, such as the student's age, prior college experience, confidence level in mathematics, perceived importance of mathematics, current course enrollment, and enrollment history. She found that academic experience (i.e., years in college) and life experience (i.e., years since high school graduation) both positively correlated with math center attendance, while confidence in mathematics had a negative correlation with attendance. Bannier concluded that young, inexperienced students might be the least likely population to visit a math center. This finding is in line with Hodges and White's (2001) study with high-risk students in a university setting. Their design featured four groups of students, one control group and three treatment groups (one of which involved explicit encouragement for students to attend tutoring). None of treatments produced any increase in tutoring attendance. In a third study, Rogers (2010) came to a similar conclusion and reported that underprepared students were less likely to seek out tutoring than other students.

Mac an Bhaird, Morgan, and O'Shea (2009) and Halcrow and Iiams (2011) reached similar conclusions as the studies previously mentioned. Mac an Bhaird and colleagues (2009) reported that attending a math center had a positive effect on students' grades, and this was particularly beneficial for students whose mathematical backgrounds were weaker. Halcrow and Iiams (2011) found that lower ability students were less likely to attend a math center, and that there was a correlation between the time spent in a math center and course grades. They also reported that once students overcame their fears of interacting with tutors, they generally found them to be helpful. According to the authors, students felt that tutoring helped contribute to their mathematical success. These studies add on to the growing number of studies regarding math centers that have taken place in Ireland (Dowling & Nolan, 2006; Gill & O'Donoghue, 2007; Mac an Bhaird & O'Shea, 2009; Ní Fhloinn, 2010). Most of these studies have considered either the contributions of math centers or ways to evaluate the services offered by math centers.

## Research Questions

The benefits of attending a math center are somewhat documented, particularly that those who would benefit most from attending a math center are often the least likely to make use of it. However, little is known regarding students' perceptions of and reasons for attending a math center. For that reason, the following open research questions guided the current study: *What do students expect from a math center? What are their perceptions of a math center? What impacts students' attendance at a math center?*

## Context

This study was conducted at a large, public, research university in southwestern United States. The university has a math center that serves students in freshman- and sophomore-level mathematics classes. The math center primarily offers drop-in mathematics tutoring. Undergraduate tutors offer just over half of the tutoring, and mathematics graduate students working as graduate assistants for the mathematics department offer the rest. The math center also offers other support services, such as access to print resources, guidance on use of digital devices and platforms used in mathematics courses, and review sessions before exams.

Students who attend the math center are in a wide range of courses including a) the general mathematics course typically taken by arts and humanities majors; b) algebra through calculus courses typically taken by business, life science, and social science majors; and c) algebra through multivariable calculus courses typically taken by science, technology, engineering, and mathematics majors. Each fall semester approximately 5000 students are enrolled in courses that are served by the math center. Of all individual students eligible to attend the math center, well over 40% typically visit the center at least once during the fall semester.

The research team consisted of four individuals with a variety of backgrounds. They included the math center director, who has served in this position for five years; a mathematics professor with RUME interests, who helped create the survey; the first-year mathematics director with RUME interests, who started in her position in fall 2017; and a Ph.D. mathematics education graduate student, who has served as a tutor in the math center.

## Data Collection

A voluntary, online survey was used to collect data for this study. All instructors for courses served by the university's math center were asked to share the link for this survey with their students. For the last three semesters of data collection, the math center director also sent the survey link out directly to all students in courses served by the math center. In addition, signs were posted in and around the math center with information regarding the survey link.

Data from student responses are obviously self-reported. Students participating in the survey were not asked to share any identifying information to encourage honest participation from the students. A total of 1,088 students participated in the survey. Of the participants, 63% were freshman; 24% were sophomores; 8% were juniors; and 5% were seniors, while the rest did not respond. The participants' attendance is given in Table 1 below.

*Table 1: Self-Reported Attendance of Participants to Math Center (Note: Sum is not 100% due to rounding.)*

| | Respondents | Percent of Respondents |
|---|---|---|
| Never | 154 | 14% |
| Once or Twice | 207 | 19% |
| Once a Month | 190 | 17% |
| Once a Week | 352 | 32% |
| More | 177 | 16% |
| No Response | 8 | 1% |

The survey consisted of multiple-choice, Likert-scale and free-response items. Individual items will be discussed in the findings section; however, we first outline the manner in which the coding system, which was used for the free response items, was created.

## Data Coding

Similar to grounded theory analysis, the research team used general inductive techniques and constant comparison to study and draw initial conclusions on students' perception of and reasons for attending the math center. According to this method, the researcher does not begin with a preconceived structure but allows categories to emerge from the data. The researcher utilizes these categories to make sense of observed activity or phenomena (Thomas, 2006).

Two members of the research team (the first and third authors) individually read through the data individually using theoretical memoing (Glaser, 1998) to record and classify ideas evident in the data set. They then discussed their findings together combing back through the data until categories emerged. As the two discussed the data, new categories were examined as they emerged to determine if they were unique or could be subsumed under, or merged, with other categories. Once it was determined that all coding categories had been developed, they went through and coded a subset (10%) of the data with over 90% agreement. They discussed discrepancies and decided that more clarification on some of the subcategories was needed. The following categories and subcategories, which are presented in Table 2, were used for coding. These were shared with the other two research team members to verify that they were reasonable. Note that responses not related to the math center, such as comments on issues related to courses or instructors (e.g., desire for a high grade, perception of instructor deficiencies) were not coded.

Two members of the research team, the first and second authors, coded all of the responses using a single response to a free-response item as the unit of analysis. When there was a discrepancy, they discussed this between themselves and with the third author, until there was a resolution. For each coding category, the research team recorded the comment as being either positive (i.e., agreement that the math center under study was doing well in this area) or as being negative (i.e., comment related that the math center under study needed improvement in this area). For the current study, the number of positive and negative responses are not addressed; instead, both positive and negative responses were calculated in a total sum since either type of response indicated that the issue was of sufficient importance for the student to make the comment.

*Table 2: Categories Used for Coding*

| Category | Subcategory | Code |
|---|---|---|
| *Facilities* (should have) | Sufficient space, tables, chairs | F1 |
| | Good environment/atmosphere for studying | F2 |
| | Outlets for digital devices | F3 |
| | Food in or nearby area | F4 |
| | Quiet areas, no noise distractions | F5 |
| *Tutors* (should be) | Attentive, "on point", not caught up in their own work, not distracted | T1 |
| | Knowledgeable, know material, not sharing incorrect methods | T2 |
| | Able to teach or explain, able to show different ways to do problems | T3 |
| | Able to help with digital devices and platforms (e.g., WebWork) | T4 |
| | Supportive, encouraging, not condescending | T5 |
| | Patient | T6 |
| | Friendly, pleasant, kind, approachable, not rude | T7 |
| | Responsive to needs, willing to help/helpful | T8 |
| | Proactive in seeing if students need help | T9 |
| | Able to communicate in understandable English | T10 |
| | Able to help students learn how to work independently; don't take over | T11 |
| | Good hygiene (e.g., cover mouth when coughing) | T12 |
| | Able to admit not knowing, willing to get help if needed from others | T13 |
| *Systems & Procedures* (should ensure) | Sufficient number of tutors available, scheduling enough tutors | S1 |
| | Ability to make appointments with tutors | S2 |
| | Access to tutor schedules (for specific tutors) | S3 |
| | Access to support materials, do more than tutoring (general) | S4 |
| | Math center is staffed with course instructors rather than tutors | S5 |
| | Students from a variety of academic majors serve as tutors | S6 |
| | Tutors are easily identifiable | S7 |
| | System is in place to efficiently get a tutor's attention | S8 |
| | Organization of students by class at same table | S9 |
| *Access* (to the following should be provided) | Solutions from the textbook, with photographing options | A1 |
| | Solutions to homework problems, specifically related to the digital platform being used | A2 |
| | Additional practice problems, not assigned as homework | A3 |
| | Exam review sessions | A4 |
| | Exam answer keys | A5 |
| | Tutoring for all math courses, not just those in the first two years | A6 |
| | Books to check out | A7 |
| | Online tutoring | A8 |
| Hours | Extended evening hours (i.e., stay open late) | H1 |

| | | |
|---|---|---|
| (should include) | Extended morning hours (i.e., open early) | H2 |
| | Extended weekend hours | H3 |
| Location (should be) | Specific cite (e.g., central, in dorms, close to math classes) | L1 |
| | Be more than one place; have multiple locations | L2 |

## Findings

In the first non-demographic item, 82% of students responded that they felt like they were encouraged to attend the math center by their instructors. Students were then asked about their math center attendance, which was reported in Table 1 above. Students who never attended the math center (n=154) were given a multiple-choice item that asked why they did not attend and allowed more than one response to this item. The results are in Figure 1: Reasons students did not attend the math center below.

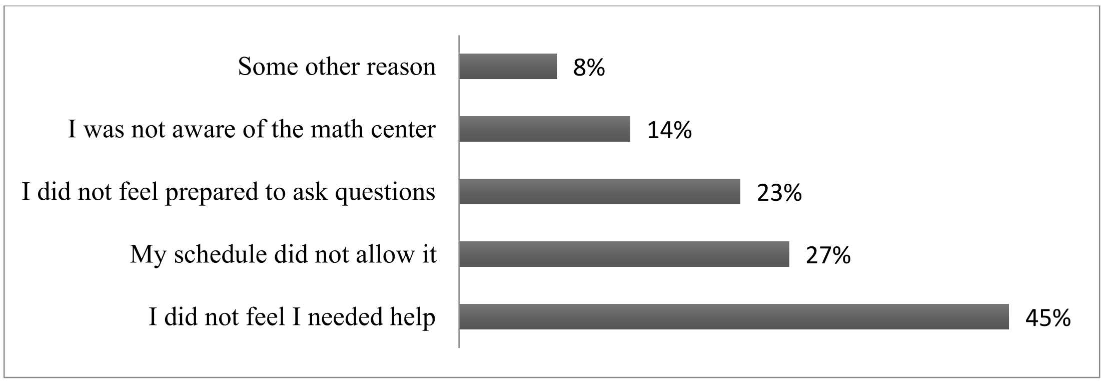


*Figure 1: Reasons students did not attend the math center*

*Students who attended the math center once a month or more (n=530) were given a multiple-choice item that asked why they did attend and allowed more than one response to this item. The results are in*

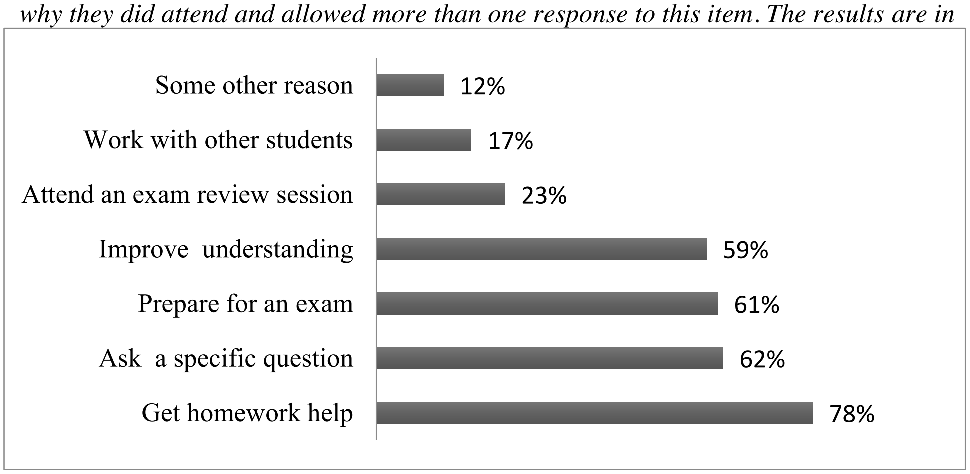


Figure 2: Reasons students did attend the math center below. Note that this population was intentionally used to eliminate those students who only attend the math center immediately before (often the same day of) an exam.

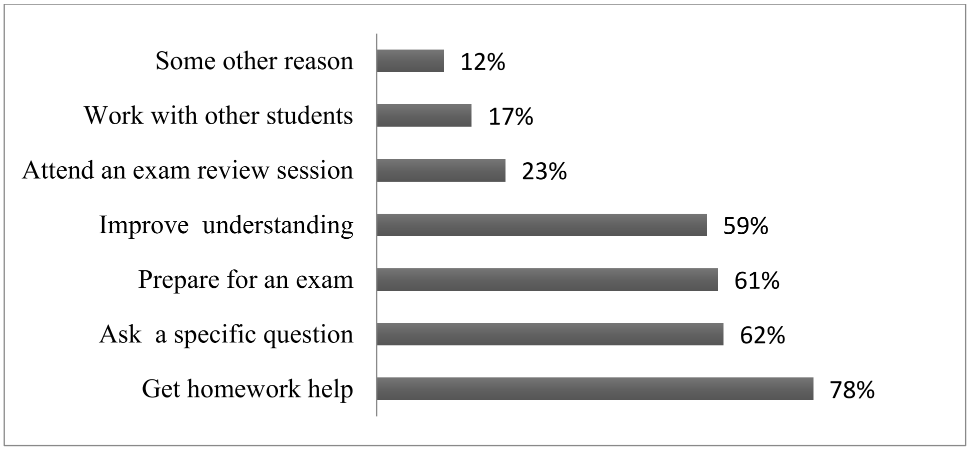


*Figure 2: Reasons students did attend the math center*

Students who had attended the math center at least once a month were then given four different Likert Scale items related to their impressions of the math center. The results are reported in Table 3.

*Table 3: Responses to Likert-Scale Items Related to the Math Center*

| The math center... | Agree to Strongly Agree | Neutral | Disagree to Strongly Disagree | No Response |
|---|---|---|---|---|
| Was helpful to me | 83% | 7% | 7% | 3% |
| Was dedicated to my success | 82% | 6% | 2% | 10% |
| Improved my math performance | 77% | 9% | 10% | 3% |
| Created a positive learning atmosphere | 75% | 11% | 9% | 10% |

These same students also responded to four different Likert Scale items related to their impressions of the math center tutors. The results are reported below in Table 4.

*Table 4: Responses to Likert-Scale Items Related to Math Center Tutors*

| The math center tutors... | Agree to Strongly Agree | Neutral | Disagree to Strongly Disagree | No Response |
|---|---|---|---|---|
| Were responsive and patient | 78% | 8% | 6% | 9% |
| Helped me feel at ease | 73% | 12% | 6% | 9% |
| Were knowledgeable | 75% | 11% | 5% | 9% |
| Explain in ways that I understood | 72% | 13% | 6% | 9% |
| Encouraged me to work independently | 65% | 22% | 5% | 9% |

Those students who had never attended the math center, or only once or twice a semester, were given two opportunities to respond to free-response items that asked for comments on and recommendations to improve the math center. Students who had attended the math center at least once a month were given the same two items as well as a third item that asked for comments on the math center tutors. Responses from both groups to these free-response items were coded with the subcategories listed in Table 1.

Even though most (52%) of the responses to free-response items were coded as pertaining to the *Tutor* category, there are two other categories that merit mention. The most noted subcategory of the *Systems & Procedures* category was “sufficient number of tutors available, scheduling enough tutors” with 82.6% of the responses recorded in this category mentioning how important this was to them. A number of students commented this was an issue especially right before examinations. The *Facilities* category’s most noted subcategory was with regard to providing “sufficient space, tables, chairs” (71.1% of the responses recorded). The responses coded in the *Tutor* category, for subcategories noted in 20 or more responses, are in Table 5.

*Table 5: Responses to Related to Tutors Coding Category (n=719)*

| Tutors are... | Number of Responses Coded |
|---|---|
| Responsive to needs, willing to help/helpful | 346 |
| Knowledgeable, know material, not sharing incorrect methods | 147 |
| Able to teach or explain, able to show different approaches to do problems | 124 |
| Friendly, pleasant, kind, approachable, not rude | 103 |
| Patient, supportive, encouraging, not condescending | 84 |
| Able to help students learn how to work independently; don’t take over | 23 |
| Attentive, “on point”, not caught up in their own work, not distracted | 22 |
| Able to admit not knowing, willing to get help if needed from others | 20 |

## Discussion and Future Work

We will now consider each of the research questions and attempt to answer them using the most apparent findings. The first research question was, “*What do students expect from a math center?”* When looking at the responses across the survey items, the results suggest that respondents focused on the tutoring provided by the math center and the capabilities of the tutors. Students expect the math center to serve as a place where they can: ask specific questions, receive help with homework, and prepare for upcoming exams. Moreover, students had a number of certain expectations of math center tutors. Primarily, students expect the tutors to be responsive to their needs and willing to help them. Students also expected tutors to be both knowledgeable and approachable with an ability to explain concepts, sometimes using multiple approaches. Finally, students want tutors who will be both patient and encouraging when they are helping them. Some of the findings of this study correspond with those reported by Johnson (2014); however, more studies are needed to determine how student-tutor interactions in math centers play a role in students’ perceptions of math centers and their attendance in math centers.

The second research question asked, “*What are students’ perceptions of a math center?”* Those respondents who attended a math center regularly (once per month or more) tended to find the math center helpful, feeling it was dedicated to their mathematical success. They also felt it helped improve their performance in mathematics and provided a positive learning environment. All of the items received at least 75% ratings of “agree” or “strongly agree.” This might be, however, an artifact of those students who received this question. The online survey directed students to different questions based on their responses. The intention of the original survey design was to provide feedback from those students who attended the math center for more than exam preparation. Yet, this might artificially inflate the agreement ratings. For this reason, work is currently underway on this corpus of data to consider all of the student responses, be they positive or negative, to the free-response items to consider student perceptions from all students regardless of the number of times they attended the math center.

Finally, we consider the third research question, "*What impacts students' attendance at a math center?*" While 82% of the respondents felt like they were encouraged to attend the math center, 14% of the respondents never attended. The data shows that out of that 14%, only 45% did not attend because they did not feel they needed help. The results of this study suggest that future research is needed that focuses on the students who are not coming to the math center and the reasons for their lack of attendance. An interesting follow-up study would focus on those who did feel they needed help but still didn't attend, especially since 23% of those who never attended said the reason for this was because they did not feel prepared to ask questions.

## Limitations and Contributions

There are definite limitations to this study. The most obvious is the poor response rate (which was about 1%). This is not surprising considering it was an optional, anonymous, online survey. Another limitation is that the survey data was primarily collected to benefit the institution and department, and the participants came from a single institution. However, there are still elements of this study that might be beneficial to others studying math centers. The data provides insight as to what students want and expect from a math center, especially related to the tutoring provided there. It also provides information as to why students might not attend the math center.

As the body of research literature on math centers expands, we hope to see three different, yet potentially overlapping, types of studies related to math centers. First, we would like to see more statistically rigorous studies, such as those recently published by Mills and Rickard (2018) and Byerly, Campbell, and Rickard (2018). There is also room to do qualitative studies to explore a number of math center issues, including some of those that have been raised above regarding future work.

Finally, a third line of research that would be assist the field of mathematics education would be more inter-institutional studies. One of the contributions this study provides is the coding system for students' math center expectations which was developed for the free-response items. Others studying math centers might want to use, revise, or expand on the categories and subcategories used in this system to study data that has been collected. In this manner, there would be a mechanism to compare the student expectations of math centers across institutions.

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# Generalizing in Combinatorics Through Categorization

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*Basic counting formulas, such as the combination $\binom{n}{k}$ and permutation $n!/(n-k)!$, constitute students' initial exposure to the structure of combinatorial processes. Understanding the structure and nuance of these counting formulas is important, as more complicated processes rely on the foundation these formulas set. In this paper we describe a study in which we used a categorization task to have students focus on salient aspects of counting processes, ses of outcomes, and formulas/expressions. We describe their generalizing activity and present an instructional theory for the production of four basic counting formulas.*

Key words: Generalization, Combinatorics, Design Experiment, Instructional Design

## Introduction

The activity of generalization is integral to mathematical thinking and affects all levels of education (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005). While there is a growing body of literature on student generalization, we still have much to learn about fostering productive generalizing activity in various contexts. Through a multi-phase study, we sought to better understand students' generalizing activity in a combinatorial setting. Combinatorics provides a natural setting for generalization, as counting problems are often accessible yet challenging (Kapur, 1970; Tucker, 2002), and these accessible problems provide a natural structure from which students may generalize. In this report, we discuss the results of student engagement with a categorization task designed to facilitate reflection on prior activity in which they solved various counting problems. The students collectively produced sophisticated generalizations while individually maintaining unique combinatorial understandings. We discuss the various nuances of their understandings as well as some affordances of attending to certain combinatorial structures.

We will discuss the students' generalizing activity in accordance with Lockwood's (2013) model of combinatorial thinking. This framing provides insight into the potential source material for students' generalizations in combinatorics. Leveraging generalization as a guiding mechanism for learning, we conducted two iterations of a design experiment in which students reinvented the basic counting formulas $\binom{n}{k}$, $n!/(n-k)!$, $n!$, and $n^k$. One goal of these experiments was to investigate how students might develop sophisticated understandings of these formulas, including the underlying counting processes and sets of outcomes associated with these problem types. To facilitate reflection on prior activity, we prompted students to solve multiple counting problems of each type and then later categorize the problems they solved. Using data from these experiments, we present an initial *domain-specific instructional theory* (Cobb and Gravemeijer, 2008) for the learning of these four basic counting formulas. We will discuss students' generalizing activity (in the sense of Ellis, Lockwood, Tillema & Moore, 2017) as well as the combinatorial nature of students' generalizations (according to Lockwood's 2013 model) as theoretical underpinnings of the instructional theory. We seek to answer the following research questions: *What do students attend to combinatorially as they generalize? How can generalizing activity be leveraged to help students understand key combinatorial ideas?*

## Literature Review

### Generalization

Generalization has been recognized as a key aspect of mathematical activity by both researchers (Amit & Klass-Tsirulnikov, 2005; Ellis, 2007b) and policymakers (Council of Chief state School Officers, 2010). While much of the literature on student generalization focuses on algebraic contexts (Amit & Neria, 2008; Carpenter, Franke & Levi, 2003; Ellis, 2007a/2007b; Radford, 2008; Rivera, 2010; Rivera & Becker, 2007/2008), there has been some history of examining generalization at the undergraduate level (e.g., Dubinsky, 1991), and more recently studies have looked at undergraduate student generalizations in calculus (Fisher, 2007; Jones and Dorko, 2015; Kabael, 2011) and combinatorics (Lockwood & Reed, 2016). Combinatorics provides a natural setting to investigate student generalization, as problems are often both accessible and challenging (Kapur, 1970) and have regular structure that can be naturally generalized. This report contributes an account of student generalization as the foundation for an initial *combinatorics-specific instructional theory*.

### Student Reasoning about Basic Counting Formulas

Though combinatorics provides accessible and deep tasks (Kapur, 1970; Tucker, 2002), students struggle with reasoning combinatorially (Batanero, Navarro-Pelayo, & Godino, 1997; Eizenberg & Zaslavsky, 2004). Our hope is that through investigating how students reason combinatorially, we may discover ways to foster productive thinking in combinatorics. One such productive way of thinking that emerged from research is a set-oriented perspective (Lockwood, 2014), where students consider the set of outcomes as integral to the solving of counting problems. Our study contributes to this literature base by leveraging generalization as a means to develop deep understanding of basic counting phenomena. By analyzing students' combinatorial understandings as they generalize, we learn more about the nature of students' combinatorial thinking about fundamental combinatorial ideas.

Previously, reinvention studies (e.g., Lockwood, Swinyard, & Caughman, 2015) have revealed ways in which students come to discover basic counting formulas. For instance, Lockwood, et al. (2015) conducted a teaching experiment in which two students numerically generated four basic counting formulas. This experiment is particularly noteworthy because the students constructed the counting formulas by attending to numerical regularity rather than attention to combinatorial structure. While these results provide an instance of students coming to know the basic counting formulas, the students did not attend to the combinatorial processes at play. There is a need for more work that tries to help students focus on processes and structure and not just on numerical results. Our current report builds off of this previous study by examining students' reinvention of the basic counting formulas while engaging with the combinatorial processes the formulas count. Further, research has identified ways in which students might confuse basic counting formulas, such as the combination and permutation, by appealing to a problem's wording as a means of differentiation (Batanero, Navarro-Pelayo, & Godino, 1997; Lockwood, 2013; 2014). This report contributes to the literature by exploring how generalization might facilitate alternative ways in which students might differentiate basic counting problems.

## Theoretical Perspectives

### Generalization

For purposes of describing students' activity as they generalize, we adopt Ellis, Lockwood, Tillema and Moore's (2017) *Relating-Forming-Extending* (R-F-E) framework of generalizing activity. Ellis et al. draw from Ellis' (2007) taxonomy of generalizing activity to describe ways in which students across mathematical disciplines generalize their knowledge. We use the term *generalizing* to refer to the activities in which students engage, and *generalization* refers to the product of their activity. Each category in the framework represents a different manifestation of generalizing activity in student work. *Relating* occurs when students establish "relations of similarity across problems or contexts" (p. 680). This form of generalizing involves the construction of meaningful relationships across mathematical contexts. *Forming* involves student attention to regularity within the context of a particular mathematical task. In particular, while forming, students "[search] for and [identify] similar elements, patterns, and relationships" within a single task (p. 680). Finally, *Extending* involves the application of established patterns and regularities to new cases (p. 680).

We will primarily see students *relating situations*, which involves creating "a relation of similarity across contexts, problems, or situations" (p. 680). Note that careful analysis does attend to the students' consideration of context in their activity. Our instructional theory leverages *relating* to establish relationships between similar counting problems that students have already solved. Moreover, the relationships formed can then be *extended* to constructs such as general formulas or general problem statements. Specific *extending* activities relevant to the task used in this study include *continuing*, *transforming*, and *removing particulars*. *Continuing* involves application of a regularity, pattern or activity to a new case (p 682). *Transforming* occurs when students "[extend] a generalization and, in doing so, changing the generalization that is being extended". (p 682). Finally, *removing particulars* involves extending "a specific relationship, pattern, or regularity by removing particular details to express the relationship more generally" (p 682). Student engagement in *extending* will result general statements of the counting problems.

**Combinatorial Reasoning**

To frame students' combinatorial reasoning, we use Lockwood's (2013) model for the different kinds of reasoning in combinatorics. Lockwood describes three components of students' combinatorial reasoning: formulas/expressions, counting processes, and sets of outcomes. *Formulas/expressions* refer to the expressions (involving numbers, variables, and/or operations) that symbolically express the answer to the counting problem. For example, some ways to express the solution to a problem like *How many arrangements are there of the letters in the word MATH?* include $4 \cdot 3 \cdot 2 \cdot 1$ and $4!$. These expressions have underlying *counting processes*, which refer to the enumeration procedures or steps that someone engages in when they solve a counting problem. For example, $4 \cdot 3 \cdot 2 \cdot 1$ reflects a process of first choosing which of the four available letters can go in the first position, then which of the three remaining letters can go in the second position, then which of the two remaining can go in the third position, and then placing the last remaining choice in the last position. *Sets of outcomes* refer to the set of objects (concrete or not) being counted. The counting processes generates the set of outcomes and imposes a particular structure on the set of outcomes. In our example, the process we described would created a lexicographic list of the set of outcomes. Lockwood (2013) maintains that there are important relationships between these components, and students would do well to understand and reinvorce the relaitonship between counting processes and sets of outcomes especially. Lockwood (2014) further emphasized this point by introducing a *set-oriented perspective*, where the set of outcomes becomes a cornerstone of reasoning about any particular counting problem. Students reason in this way by viewing "atten[tion] to sets of outcomes as an intrinsic component of solving counting problems" (p.31). We refer to these components of the model as we describe and characterize students' combinatorial reasoning in this paper.

**Domain-Specific Instructional Theory**

We follow Cobb and Gravemeijer (2008) in viewing a *domain-specific instructional theory* as complementary to an instructional sequence of tasks designed to foster a particular form of learning. Cobb and Gravemeijer consider a *domain-specific instructional theory* to be "a substantiated learning process that culminates with the achievement of significant learning goals as well as the demonstrated means of supporting that learning process" (p. 77). Our *instructional theory* is specific to the domain of combinatorics, counting in particular. Cobb and Gravemeijer speak to the usefulness of *domain-specific instructional theories* as their justification "offers the possibility that other researchers will be able to adapt, test, and modify the activities and resources as they work in different settings" (p. 77). Thus, the "underlying rationale" of the *instructional theory* allows for further refinement of the instrucitonal sequence in research settings.

We leverage student generalizing in combinatorics as the rationale for the learning process supported by our instructional sequence. In particular, the Categorization Task leverages prior activity as a means of abstracting meaningful relationships between counting problems with similar combinatorial structure, resulting in the learning of the basic counting formulas. This instructional theory has three key elements: 1) facilitating student engagement with novel counting problems to establish meaningful activity to draw from, 2) facilitate explicit *relating* of solved counting problems so that combinatorial relationships can be established across various structually similar contexts, and 3) facilitate *extension* of the constructed relationships and reflections on activity to general counting formulas abstracted from previous activity. Applying the R-F-E framework, the initial categorization of their previously solved problems allows the students to construct meaningful relationships amid present cognitive material through *relating*. Our analysis using Lockwood's model to the students' combinatorial reasoning will reveal qualitatively different nuances of the combinatorial relationships being constructed during categorization. In particular, students will need to leverage all three components of the model to co-construct both the general counting formulas as well as the combinatorial problems that generate them. The relationships formed through categorizing then constitute abstractable cognitive material to be *extended*. *Removing particulars* plays a key role in facilitating the construction of the general counting formulas, as the combinatorial quantities are *extended* from specific problem solutions. Moreover, the activities of the students reveal combinatorial understandings (in the sense of Lockwood's model) that persist throughout the generalizing. The data will further demonstrate the ways in which the R-F-E framework and Lockwood's (2013) model for combinatorial understanding provide a rationale for the learning process facilitated by the instructional sequence.

**Mathematical Discussion**

We are specifically concerned with the combinatorial operations of arrangement with unlimited repetition, arrangement without repetition, permutation, and selection. We characterize these four problem types because they tend to be some of the four basic problems to which students are introduced (in courses or in textbooks, for example). Thus, we feel that, while this is not the only way to characterize basic counting problems, this is a useful distinction to make for students who are first learning counting problems.

One problem type that involves iterative multiplication is arrangement with unlimited repetition. This involves constructing an outcome by making $k$ choices repeatedly from the same set of $n$ objects, where repetition of an element is allowed. An example problem of this type is *How many length 5 ternary sequences exist?* An expression for a solution to this problem is $3^5$, which represents the process of considering three choices (0, 1, and 2) for each of 5 positions in the sequence. As each choice made is from $n$ distinct objects, the solution to these problems involves iterative multiplication of the number of choices for each selection by the number of choices for the next selection until all choices have been made. As the number of choices is constant in this case, the solution to these problems is $n^k$, as $k$ selections are being made.

Another problem type is permutation, which involves the ordering of $k$ distinct objects from a potentially larger collection of $n$ distinct objects, as seen in the example problem *10 horses run in a race. How many ways can a gold, silver and bronze medal winner?* We focus on two interpretations of the permutation operation. The first primarily considers the multiplication principle and warrants a solution method identical to that of arrangement (in fact, arrangement problems are permutations of $n$ objects from a set of $n$ objects). Specifically, the outcomes of a permutation process form $k$ ordered objects which can be selected from the set of $n$ objects one at a time. This construction warrants iterative multiplication resulting in the solution $n \cdot (n-1) \cdot ... \cdot (n-k+1) = n!/(n-k)!$. The final form of the answer warrants a different interpretation of the outcome structure which was leveraged in the design experiment. Specifically, the $n$ objects can first be arranged in $n!$ ways. As the first $k$ objects in each arrangement are desired, this method produces an extra $(n-k)!$ outcomes for each desired outcome of $k$ ordered objects. Thus, dividing $n!$ by $(n-k)!$ produces the desired arrangements of $k$ objects as the $(n-k)!$ further arrangements of each $k$-object arrangement are each reduced to a single outcome.

A special case of permutations occurs when we arrange all of the $n$ objects, which is arrangement without repetition. An example problem of this type is *How many words can be formed from the letters in MATH?*, which we briefly described in terms of Lockwood's (2013) model above. The expression of $4 \cdot 3 \cdot 2 \cdot 1$ represents a four-stage counting process.

Finally, selection problems (also sometimes called combination problems) make up the final problem type. These problems involve the selection of $k$ objects from $n$ distinct objects, an example problem being *How many ways are there to hand out three identical lollipops to eight children?* Here, we can select 3 of the 8 children to get lollipops, yielding an expression of $\binom{8}{3}$. The solution method achieved by both student groups builds off of the permutation process. Specifically, a permutation $n!/(n-k)!$of $k$ objects from $n$ objects results outcomes of $k$ ordered objects taken from a set of $n$ objects. A selection concerns only the collection of $k$ objects chosen from the set of $n$ objects. Orderings of those $k$ objects produces $k!$ outcomes for each of the desired selected outcome. Thus, division of the permutation by $k!$ results the solution as the $k!$ differently ordered selections are reduced to a single outcome, resulting in the solution $\binom{n}{k} = n!/[(n-k)!k!]$.

## Methods

The work in this paper draws on two iterative design experiments conducted as part of a larger study in which we investigated the nature of student generalization in combinatorial settings. We first conducted a paired teaching experiment (Steffe & Thompson, 2000) consisting of fifteen hour-long sessions followed by a small group design experiment consisting of nine ninety-minute sessions with four students. The students from both studies were recruited from vector calculus courses, and they were selected from an initial set of applicants based on a selection interview process. Each of the students in these studies had not taken a discrete or combinatorics course before so that their activity and generalizations were indeed based on their activity rather than trying to recall formulas or implement extant schemes.

This study reports on the first three sessions of each experiment, wherein the students reinvented the basic counting formulas $\binom{n}{k}$, $n!/(n-k)!$, $n!$ and $n^k$ (discussed above in the Mathematical Discussion section). The goal of these sessions was to facilitate reflection on activity with basic counting problems, culminating in the construction of the four general formulas as well as general statements of problems that yield those formulas. Below we describe the students in both experiments and present their activities in the teaching experiments and typical interventions implemented. We then conclude with a description of the sequence of activities in the Categorization Task that demonstrate our initial domain-specific instructional theory.

**Paired Teaching Experiment**

We recruited two vector calculus, Rose and Sanjeev (all names are pseudonyms), to participate in our study. As noted above, because of their activity during their selection interviews, we felt confident that Rose and Sanjeev were novice counters adept at communicating their thinking and working together. The fifteen hour-long sessions occurred over the course of six weeks and the students were monetarily compensated for their time. During each session the students primarily worked on a chalk board, solving counting problems either together or individually followed by discussion. The students had no exposure to any of the general counting formulas before the sessions so that their reinvention was a result of their generalizing activity rather than recollection of a previously encountered general formula. The students were given specific counting problems to work through together, and were often prompted to explain their thinking as they worked. The interviewer occasionally prompted students to further their thinking, explain their reasoning, or to reflect on a particular representation of the combinatorial setting. The sessions were each video taped, and a retrospective analysis (Steffe & Thompson, 2000) was performed on the video records. Transcripts of each session were made and then enhanced with notes and figures representing student activity. Episodes of student generalizing were cataloged and coded using the R-F-E generalization framework and students' combinatorial reasoning was coded using Lockwood's (2013) model for student combinatorial thought. Further, models of students' combinatorial reasoning were constructed (in the sense of Steffe & Thompson, 2000) so that productive ways of reasoning combinatorially could be identified and used to refine both the sequence of activities as well as our instructional theory. The data analysis then informed the sequence of tasks and interventions implemented in the small group design experiment conducted the following year.

**Small Group Design Experiment**

The purpose of the design experiment was to observe student generalization in a small group setting that more realistically represents the classroom environment. Much like the paired teaching experiment, the small group design experiment consisted of students recruited from vector calculus courses that also engaged in selection interviews. Our selection criteria was slightly different for the design experiment, as we sought a group of four that demonstrated a more realistic collection of students in a classroom small group. Specifically, while we still pursued novice counters who were willing to engage with the material and communicate their ideas willingly, we sought to have students that demonstrated different levels of adeptness with counting. Specifically, we sought to balance students who demonstrated productive counting strategies in the selection interview with students who might perhaps struggle with more complicated counting processes and arguments. Our small group thus consisted of a student who performed well in the selection interview, two students who performed moderately and one student who still engaged with the material but struggled at times to reason combinatorially. This experiment consisted of nine ninety-minute sessions in which the students again solved problems of the same type as solved in the teaching experiment. The sessions and activities largely imitated those of the paired-teaching experiment, with a few added interventions aimed at facilitating productive student thinking as identified in the teaching experiment. For instance, one such intervention involved prompting reflection on the relationship between division and the structure of the set of outcomes. Also during the design experiment, the researcher more freely communicated summative ideas to the students. For instance, a frequently discussed topic in the initial interview was the generation of outcomes as reflected in multiplication. Once the students had a few discussions of generating a set of outcomes to reflect multiplication, and the students had some exposure to reasoning about outcomes based on the positioning of elements within the outcome, the researcher gave a brief summative discussion of the ways in which outcomes might be generated specifically through iterative multiplication. Two video cameras were used to record these sessions. During individual group work, the cameras each focused on two separate students to capture their activity. During group discussions, one camera would focus on referenced written work and the other camera would capture the social interaction between students. The videos were then spliced together so that both records could be viewed at once during data analysis. Data analysis was then conducted in a similar manner as that of the teaching experiment, allowing us to again refine the instructional theory for production of the basic counting formulas.

**Categorization Task**

We now present the instructional sequence from the experiments that culminated in the reinvention of the basic counting formulas $\binom{n}{k}$, $n!/(n-k)!$, $n!$ and $n^k$. Throughout the first two sessions, we gave students the problems in Figure 1. Their activity with these problems constitutes the first component of our instructional theory, namely that the students meaningfully engage in novel counting to establish patterns of activity to later be leveraged through *relating* and *extending*.

| **Arrangement Problems** |
| --- |
| *The Recess Problem*: How many ways are there for 27 kids to line up for recess? |
| *The Projects Problem*: How many ways are there to assign 8 projects to 8 different students? |
| *The FAMILY Problem*: How many ways can you arrange the letters in the word FAMILY? |
| *The MATH Problem*: How many ways can you arrange the letters in the word MATH? |
| **Permutation Problems** |
| *The Restaurant Problem*: Corvallis has 25 restaurants. How many ways are there to pick your 5 favorite Corvallis restaurants? |
| *The Collar Problem*: You have a red, blue, yellow and purple collar. How many ways can |

| |
|---|
| you distribute the collars among 7 cats so that each cat will wear only 1 collar? |
| *The Race Problem*: 10 horses run a race. How many ways can the horses finish in 1st, 2nd and 3rd place? |
| **Selection Problems** |
| *The Subset Problem*: How many 4-element subsets are there of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$? |
| *The Lollipop Problem*: How many ways are there to distribute 3 identical lollipops to 8 children? |
| **Selection with Repetition Problems** |
| *The License Problem*: How many ways are there to make a 6-character license plate consisting of the letters A-Z and the numbers 0-9? |
| *The Sequences Problem*: How many 3-letter sequences are there consisting of the letters a-f? |
| *The Ternary Problem*: How many 3-digit ternary sequences are there? |
| *The Houses Problem*: How many ways can you paint 6 houses if you have 3 available paint colors? |
| *The Quiz Problem*: How many ways are there to complete an 8-question multiple choice quiz if there are four possible answers to each question? |

Figure 1: The Counting Problems of the Categorization Task

After solving various problems of each type, we gave the students the following broad prompt to engage them in a categorization activity: *You have previously solved these fourteen problems, we want you to do now is as a group is categorize them in some way*. The goal was to facilitate generalization between the problems through reflection on their prior activity, resulting in the production of the basic counting formulas. Our broad stating of the task was so that the students created categories that were meaningful to them rather than by their interpretations of categories meaningful to the researchers. This constitutes the second component of our instructional theory wherein students engage in *relating* to construct meaningful relationships between each of the four problem types based on their previous activity with the problems. Once the students agreed on the categories for the problems, they were asked to describe a general formulation of each category, and then to construct a general formula for the solution to each problem type. This culiminated in the production of the four counting formulas described above, as well as general statements of problems and conditions in which they might be applied. This step also concludes our instructional theory by facilitating student *extending* the relationships constructed through the *relating* activity. In particular, combinatorial strategies, activities and structures are abstracted from the established relationships to form general counting processes and formulas, thus marking the completion of the instructional sequence.

## Results and Discussion

We present episodes of student activity from both the paired teaching experiment and small group design experiment as examples of the ways in which students learn the basic counting formulas according to our domain-specific instructional theory. For the purposes of this report we will focus primarily on the Categorization activity itself and demonstrate the ways in which students can create the general relationships that then abstract to general versions of the basic counting formulas. We will first present an episode of student novel combinatorial activity within the first two sessions to represent the first component of the instructional theory and to demonstrate the kinds of activities that then later generalize to the students' constructs of the basic counting formulas. We then present data from the Categorization activity itself to represent the second component of our instructional theory and to demonstrate the combinatorial nuances of the students' generalizations as well as the formation of some productive understandings as influenced by their generalizing. This then culminates in the records of the students' generalizations of the basic counting formulas from both experiments as representations of the final component of our instructional theory. Specifically, in these results we use data as examples of the three key elements of our domain-specific instructional theory described in our theoretical framework.

**Initial Activity with Selection Problems**

This first section gives a brief episode from the small group design experiment when the students first engaged in a selection problem (The Lollipop Problem). When first solving selection problems, the students engaged in *set-oriented* reasoning, which they would then later generalize as the primary distinction between permutations and combinations. We see the students' *set-oriented perspective* as a productive way to understand the distinction between permutations and combinations. We will later highlight the students' generalizing activity that facilitated the *continuing* of this *set-oriented* thinking, thus demonstrating the ways in which their generalizing facilitated productive understandings in this instructional sequence. The students initially produced the solution $8 \cdot 7 \cdot 6$ to The Lollipop Problem. The researcher then intervened and asked the students to discuss the distinction between identical lollipops and distinct lollipops. As The Lollipop Problem involves counting identical lollipops, exploration of the same problem set up but with distinct lollipops was meant to facilitate reflection on the ordering implied by their initial solution, thus prompting a new solution accounting for the uniformity of the lollipop. Responding to this prompt, Josh and Carson had a nice discussion about how they would solve this problem. Josh first realized that they needed to account for identical possibilities by incorporating the $3!$ arrangements of unique lollipops. This then prompted the following solution from Carson:

*Carson*: That makes sense, yeah. So, for any given three kids, the lollipops can be arranged - So, if this solution [their $8 \cdot 7 \cdot 6$] is treating the lollipops as distinct from one another and we're just trying to find distinct groups of three kids that could be given a lollipop, then we need to find the number of ways that any number of lollipops can be distributed to any number of kids. So, I think that we need to divide again [the first division being for a permutation] because - rather than subtract, we need to divide because this [$3!$] is our distinct number of outcomes and then we need to divide it by . . . Because that's going to happen for all the different groups of kids rather than just one group of kids. Right? So, this makes sense but you need to divide rather than subtract. I'm having a hard time putting into words why, but it's because this is all the possible arrangements - all the possible combinations of three kids - are included, not just one possible arrangement of three kids. Yeah, so it should be $8!/(5! \cdot 3!)$ which just comes out to 8 times 7.

This discussion demonstrates that Carson was focused on *outcomes*, which would later be *continued* for other selection problems and then referenced during the categorization. In particular, note that Carson's justification for division considers that a given outcome of three kids has $3!$ orderings, and this ordering happens for each different outcome. His reasoning specifically attends to the entire set of outcomes and the multiplicative effect of ordering each outcome. This focus on *outcomes* reasoning provides an example of the kinds of activities the students engaged in while solving counting problems. This also provides an example of the combinatorial relationships that were later generalized through categorization. To summarize this section, this episode provides an example of the kind of thinking, activity, and conversation in which the students engaged in this initial activity. Such activity with novel counting situations provides meaningful and nuanced cognitive material to then be generalized through categorization. Thus, this activity is a representative how the first component of our instructional theory might unfold.

**Categorization Task**

We now detail the generalizing activity of the students during the Categorization Task. We will discuss various moments of generalization during the categorization, including the *continuing* of their focus on *outcomes* discussed on the Lollipops problem. This will then culminate with the students' final generalized statements of the counting formulas.

Throughout the categorization of their prior solved problems, we saw students engage in *relating* by creating meaningful relationships between the problems they categorized. Much of the combinatorial meaning can be characterized in terms of Lockwood's (2013) model.

For instance, Carson provided the following description of two arrangement with unlimited repetition problems he grouped during the first few moments of the categorization task:

> *Carson:* This [The Quiz and Sequence Problems] is independent events. So, [The Quiz Problem] there are eight questions but the outcome of one doesn't affect the others. [Points to The Sequence Problem] There are six characters, but the outcome of one doesn't affect the others.

Here he was relating that each question described a combinatorial structure in which there was no dependence between selections. Indeed, while his language was in terms of outcomes, he described the outcomes not affecting other outcomes in the process. From this we infer Carson attended to the combinatorial *process*. Similarly, Josh then identified two more arrangement with unlimited repetition problems that had still not been categorized:

*Instructor*: . . . and why did those two go with those [the original collection Carson grouped together]?
*Josh*: Those two also deal with independent events and finding all the possibilities in those events depend on something raised to some power.
*Instructor*: Okay, okay, good.
*Josh*: Like the number of choices that you have raised to the number of choices that you make.

Again, we see Josh combinatorially *relating* different selection with repetition problems. Specifically, Josh identified his new problems as similar according to the formula for the answer. Thus, Josh attended to *formulas/expressions* while Carson is attended to *counting processes*. This variety in what aspects of the model students attended to was common during these discussions. Indeed, Josh and Anne-Marie often attended to the *formulas/expressions* of the problems as category indicators while Carson and Aaron indicated more frequent attention to the combinatorial *process* generated by the problems. We see this as a strength of the Categorization Task, as students often collaboratively generated the categories while appealing to individually different combinatorial details. The students be explored the problems through all components of Lockwood's model, which seemed to promote well-rounded combinatorial understandings.

There were similar instances in the teaching experiment. For example, while categorizing the same type of problem, the students in the teaching experiment had the following exchange:

*Sanjeev*: And then you want to paint 6 different houses on your block and there are 3 acceptable paint colors you can pick —
*Rose*: Would that one come down here? Because that would be —
*Sanjeev*: You have 6 houses and —
*Rose*: 3 to the power of 6?
*Sanjeev*: you have 3 different paint colors for each, yeah. So this [The Houses Problem] would be this one [referring to the collection of arrangement with unlimited repetition problems]?

Notice that Sanjeev and Rose were attending to different components of Lockwood's (2013) model during this exchange. Sanjeev attended to *processes* by focusing on the process of picking paint colors. Rose, in turn, attended to the *expression* involving exponentiation as a means of relating the houses problem to other arrangement with unlimited repetition problems she experienced. This further demonstrated the students' abilities to communicate and generalize across varying combinatorial language.

While there was, as noted, variety in the students' generalizations and combinatorial understandings throughout the task, we found a surprising uniformity of language pertaining to combination problems in particular (that is, selection problems). Indeed, all students demonstrated attention to the structure of the *sets of outcomes* when discussing combinations. The discussions about differentiating combinations from other combinatorial processes revolved around taking care not to count two similar outcomes as different. For instance, when separating the permutations and the combinations, Rose and Sanjeev said the following:

*Rose*: It's [referring to their collection of permutations]— it's how many — it's basically how many ways to put certain amount of items into fewer spots where 1, 2, 3 and 3, 2, 1 are different. And this [their collection of combinations] is how many ways you put a certain amount of things into fewer spots where 1, 2, 3 and 3, 2, 1 are the same.
*Sanjeev*: On these ones [permutation problems] you've got combinations [not referring to the combinatorial sense of the word]. So 1, 2, 3 - 3, 2, 1 would be different combinations. With this one [The Lollipop Problem], for example, if you have identical lollipops you can label them 1, 2, 3 or you can just label them 1, 1, 1. So 1, 2, 3 and 3, 2, 1 would be the exact same thing, because 1, 2 and 3 are all the same.

The students in the teaching experiment had also initially solved combination problems by considering sets of outcomes much in the same way as the students in the design experiment. The distinction of making $\{1, 2, 3\}$ and $\{3, 2, 1\}$ the same indeed explicitly involves attending to which *outcomes* should or should not be counted the same. We find attention to outcomes in this way as productive, as it allows for careful consideration of what is being counted. Indeed, this form of differentiation meaningfully examines the permutation/combination distinction as opposed to other means of differentiation present in the literature such as attending to montras and phrases in problems. Further, this highlights that differentiation between combinatorial objects is also a means of students engaging in *relating*. Indeed, understanding of the permutation and combination is occurring through the creation of similarity-based relationships and also differentiations between combinatorial structures.

As another example, we saw similar discussions of combinations emerge from the design experiment. Initially, when describing the difference between combinations and permutations, Ann-Marie remarked:

*Anne-Marie*: Yeah, so in those two problems [a pair of combination problems] you divide by two factorials to cancel out the duplicate answers whereas in the other ones you don't have to do that.

Notice that her response also included a focus on a *formula/expression*. Indeed, Ann-Marie confessed that she primarily thought of the formula representation when thinking of the problem types. Ann-Marie made the distinction of "two factorials" in this case to contrast division by "one factorial" in the group of permutation problems. What we see here is that within her formula driven remarks, she also used outcome-based language to describe the need for the extra division by a factorial. Also, later when explaining why the formula for $\binom{n}{k}$ includes an addiitonala division by $k!$, Aaron explained:

*Aaron*: Well, because you're trying to get rid of all the combinations that you're not looking for that you can make out of those three slots because they're all the same. So, that just accounts for it.

Indeed, most descriptions of combinations involved *outcome* - based language so that they could be differentiated from permutations. Often, the design experiment students described "dividing by redundancies" when performing combinations. It is interesting that among the students we worked with, combinations were uniformly a source of *outcome* - based language. Returning to the teaching experiment, we see Rose also using *outcome* - based language when describing why a subset selection problem is grouped with other combinations. After negotiating the particulars of the problem involving finding subsets of a set of numbers, Rose said the following:

*Rose*: and if that was the case then we'd want to put it over into this group [the collection of combination problems].

*Int*: Okay. And how come?

*Rose*: Because now you don't want — you just want unique combinations. And if you're getting rid of all the — the repeated subsets, then you're just finding the unique combinations.

Here, we see Rose clarified that the desired outcomes were indeed "unique combinations". The uniqueness was generated by getting rid of repeated subsets, which indeed would emerge from a standard permutation. Thus, we see that Rose diverged from her typical focus on *formula/expressions* to attend to unique outcomes.

The above examples illustrate the kinds of generalizing that is facilitated through categorization. Indeed, these excerpts represent the second component of our instructional theory.

**Generalizaing the Basic Counting Formulas**

While much of the above discussions centered around the activity of *relating*, the overal Categorization Activity culminated with the students *extending* the relationships they had formed to produce general counting formulas and statements. To facilitate this, we gave the students the prompt to characterize what each group of problems was counting (and each set of students also produced a general formula). Our goal was to have them engage in generalizing and to focus on the various aspects of the model. To give examples of possible final generalizations, we present the students' general of each category and formula. The design experiment students gave the following characterizations of the arrangement and permutation problem types: *1) Limited slots and limited objects. Unique arrangements of unique objects: $x$ objects $\rightarrow x!$ 2) Selection from* *arrangement - order matters:* $a!/(a-b)!$*,* $a$ *=total amount of objects,* $b$ *=how many objects you choose*. For the third group (combinations), the students *continued* their earlier activity of first permuting the combinatorial objects and then accounting for repeated outcomes. Indeed, while categorizing Carson described the combination problem as"an arrangement one as well just with the caveat that there are only four of them". Carson is referring to a permutation problem when he talked about arrangement. In this dialogue, he described this combination problem as a permutation problem with an extra condition. Indeed, this type of language was common when the students would describe combinations. Often when describing the different categories, the students would reference combinations as permutations with added conditions. While writing the general statement of the combination category, the students had the following exchange:

*Carson*: So, the same as above [their general permutation statement], with another caveat that some of the elements are —

*Josh*: Arrangement doesn't matter.

*Carson*: Right, that arrangement doesn't matter.

*Aaron*: I believe for this one [the combination category], you'd be dividing by three factorial in addition to the same process here [their permutation category].

*Ann-Marie*: Mm — hmm.

*Josh*: Yeah, I think that you're also going to — it's basically this [a permutation] except you're also dividing by the number of things that you're picking from because order doesn't matter. So, it would be I guess —

*Aaron*: B factorial on the side [as in $a!/[(a-b)! \cdot b!]$?

*Josh*: Yeah.

Through this discussion the students thus *extended* their understanding of combinations by *transforming* their generalization of a permutation via division by $b!$. This indeed is the same activity they initially engaged in when solving the lollipop problem and thus constitutes *continuing* of their activity to a general setting. Moreover, we consider division by $b!$ as *transforming* because they specifically reference that the division operation is an intentional alteration of the permutation $a!/(a-b)!$ to *extend* to this new case by accounting for the ordering of the outcomes. Thus, *extending* in this way was the mechanism that produced their third category: *3) Arrangement does not matter. Divide out duplicates, similar to 2):* $a!/[(a-b)! \cdot b!]$. Their final category took the form *4) Independent events with given number of elements:* $a^b$*,* $a$ *=possibilities of each choice,* $b$ *= total number of choices you make*. Note that the students indeed produced the four distinct counting formulas as a result of their generalizing activity from prior work.

The teaching experiment students wrote out the following general statements of the problem types accompanied by the general counting formulas: *1) How many ways to arrange a given number of elements w/o reusing:* $n!$*. 2) How many ways to arrange a given number of elements into a given number of spots without reusing any elements:* $n_e!/(n_e - n_s!)$*. 3)* $\uparrow \ldots$ *and divide by the factorial of the given spots to delete repeated sequences:* $n_e!/(n_e - n_s)!/n_s!$*. 4) How many ways to put a certain amount of things into a certain amount of places assuming you can put the same thing into more than one place:* $n_c^{n_e}$. Note that the general statements of the problems reflect various components of Lockwood's model, and that again the combination is stated in term of a *transformation* of the permutation group. Indeed, these generalizations represent the final product of the instructional sequence and the culmination of the students' *extending* in the final component of our instructional theory.

## Conclusions and Future Directions

We see that the categorization task allowed the students to generalize their prior work on individual counting problems into more general contexts in which different combinatorial structures could be illuminated. In particular, the students productively engaged in *relating* and *extending*, both activities underpinned by the nuances of the combinatorial settings, as described by Lockwood's (2013) model. This constitutes a *domain-specific, instructional theory* to accompany the instructional sequence of activities provided by the Categorization Task. In particular, the Categorization task provides an instructional approach to introducing the counting formulas that relies on student generalizing to generate the formulas from meaningful activity. This provides evidence of student reinvention of the basic counting formulas from combinatorial understandings rather than the numerically-based reinvention as seen in Lockwood, Swinyard and Caughman's (2015b) study. Further, the students' distinctions between permutations and combinations demonstrated productive understandings of the combinatorial objects through a *set-oriented perspective*. In particular, this type of reasoning provides productive understandings of the combinatorial objects rather than previously documented problematic strategies such as attending to the wording of problems or appealing to "montras" such as "and means multiply" (Batanero, Navarro-Pelayo, & Godino, 1997; Lockwood, 2013; 2014). The Categorization Task thus constitutes an instructional sequence supported by a *domain-specific, instructional theory*. Future research will see further refinements of our *instrucitonal theory* moving towards classroom implementation.

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# Examining the Effectiveness of a Support Model for Introductory Statistics

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*This paper describes a pilot program, aimed at improving outcomes in Introductory Statistics, in which undergraduate peer coaches led teams of students in activities designed to address common misconceptions about statistics during weekly sessions. Preliminary analysis suggests that introducing these sessions may reduce the percentage of students that finish Introductory Statistics with a grade of D, F or W, although the small number of students in the pilot program did not provide sufficient power to detect statistical significance. We also observe that the population of students who attend most of the optional sessions seems to be a mixture of high performing students and lower performing students. Participants in the program reported mainly positive perceptions of the program's usefulness. We intend to continue investigating these observations in future iterations of the program where we hope to improve participation and refine the session activities.*

*Keywords:* Introductory Statistics, Peer-Led Team Learning, Collaborative Learning, Statistics

An ongoing concern in undergraduate mathematics education is students' struggle in "gateway" mathematics courses, such as College Algebra or Basic Statistics. Research suggests that student success in these gateway courses strongly correlates with retention and degree completion (Adelman, 2006). Thus, these courses offer a significant opportunity to boost overall retention. In this paper, we describe one attempt to increase course success in such a gateway course: Introductory Statistics. The choice of Introductory Statistics was motivated by several factors. First, increasing numbers of students are taking an introductory Statistics course to satisfy either a major or general education requirement (Blair, Kirkman, & Maxwell, 2013). This increasing enrollment is in response to calls to increase access to college level courses (e.g. Treisman, 2015), and boosted by evidence that success in a first Statistics course does not require algebra-intensive preparation (Charles A. Dana Center, 2015). However, Introductory Statistics can only fulfill the vision of improving access to college mathematics if students master course material and successfully complete the course. At our own institution, historical failure rates (including students earning D, F, and W) range between 0-65% by section, with a mean failure rate of 24%. Because we offer a large number of sections (~ 20 per semester), reducing this failure rate (even by a modest amount) could meaningfully improve overall retention.

With the goal of improving student success in this course, we implemented a Peer-Led Team Learning (PLTL) model, in which students attend collaborative sessions to improve their prerequisite skills and conceptual understanding. Chemistry programs have used this model extensively (see, e.g., Grosser et al., 2008), and a recent meta-analysis indicated significant evidence that this model is effective across the STEM disciplines (Wilson & Varma-Nelson, 2016). Indeed, PLTL programs have demonstrated success in improving mathematics students' understanding of content (Merkel et al., 2015), course passing rates (Hooker, 2011), retention (Quitadamo et al., 2009), and attitudes toward mathematics (Curran, Carlson, & Celotta, 2013).

In this paper, we aim to describe our methodology, including a detailed description of program structure, in order that other institutions might learn from our successes and challenges. We also present data on the effectiveness of the program, focusing on three central questions:

A) Who attended the PLTL sessions?

B) What effect did the PLTL sessions have on course outcomes?
C) How did participants perceive the PLTL sessions?

In our presentation, we aim to begin a conversation, soliciting ideas from others about how the PLTL program might be improved, as well as how such complex interventions can be studied and replicated.

## Methodology

The implementation and study of our PLTL program was informed in part by descriptions of similar programs that reported some success (e.g. Carlson et al., 2016). In order to test the effectiveness of the program, both in terms of course outcomes and participants' perceptions of the program, we used a quasi-experimental, control group design with matched pairs of course sections.

### Participants and Measures

Study participants consisted of students enrolled in ten sections of introductory statistics, chosen among a total of 20 sections offered at a large, comprehensive, regional public university in the Mid-Atlantic. We selected five pairs of sections, such that a common instructor taught each pair of sections. From each pair of sections, we randomly selected a treatment section. Thus, each of the five treatment sections had a corresponding control section taught by the same instructor. Total enrollment was 134 students in the treatment sections and 136 students in the control section. To investigate the effectiveness of the program, we examined students' final course grades, as well as performance data (e.g. exam scores) provided to us by the course instructors.

In order to gauge students' thoughts about the recitation sessions (and answer Research Question C), we created an online survey and invited all treatment group students to participate in the survey. The survey was administered after the end of semester. The participation was voluntary and anonymous. Students were able to leave any question blank. The survey consisted of both open-ended and multiple-choice items students' perception of the sessions in terms of content, instructional techniques, and schedule. Students were also asked to give feedback for further improvements of the program. Further details about the survey are shared alongside the results.

### Program Structure and Content

Students from the five treatment sections (each of which enrolled approximately 28 students) of Introductory Statistics were invited to attend weekly 2-hour collaborative problem-solving sessions, facilitated by paid peer coaches. A total of 72 students could be accommodated each week, choosing from among six different sessions, each with a different coach and at a different day and time. PLTL sessions began about one week prior to the halfway point of the semester and one or two weeks after students' first midterm exam was returned. The hope was for struggling students to attend voluntarily, encouraged by a modest grade incentive.

Coaches were undergraduate students who had received an "A" in the same course two semesters prior and who were hired based on an interview to ascertain their relatability and confidence to lead sessions. Coaches attended Friday training meetings at which they went through and provided feedback on the following week's session activities. Two of the authors, one a mathematician and one a mathematics educator, both faculty in the Department of Mathematics, devised the weekly session activities and ran the training meetings. The goal was to prepare the coaches to facilitate the students' work through the activities and especially to stimulate discussion of the focal ideas. Thus, besides becoming familiar with the specific activities, coaches were given lesson plans that included discussion-generating questions to help focus attention on the most crucial concepts.

The intent of the program was to deepen students' conceptual understanding of key ideas in statistical thinking outlined in the GAISE College Report (2016). Additionally, although the sessions were constructed so as not to feel like another "lecture class," our hope was for students to engage with those ideas through activities recommended by the report as much as possible. Finally, we wanted weekly topics to be timely relative to recently covered class material. That was accomplished to the extent possible, given that individual instructors have some flexibility in the sequence and timing of topics on the course outline. Ultimately, a schedule of 8 sessions was constructed to best meet the aforementioned goals. Brief descriptions of the content of each session follow.

**Session 1: Histograms, standard deviations, and normality**. Students created and examined histograms for six different large sample data sets, matching each to its respective population description based on shape. Students revisited standard deviation calculation, then determined the percentage of data falling within one, two, and three standard deviations of the mean.

**Session 2: Normal distribution and the empirical rule.** Students examined a uniform distribution as a comparison to the normal. They used the "empirical rule" on a contextualized normal distribution to identify scores associated with different relative frequencies and vice versa, and to compare normal curves on same axes, given means and standard deviations.

**Session 3: Meaning of z-scores and associated probability statements for a random score from a normal population.** Students developed meaning of a z-score and reasoned out the formula based on its meaning. They standardized scores from a contextualized distribution, locating them along the x-axis using the empirical rule as a reference. They determined probabilities associated with population values falling in various intervals and vice versa.

**Session 4: Sampling distributions 1 – sample proportions**. Students physically tossed coins to first predict, then examine the distribution of proportions of heads from samples of size five and then 10, comparing results. Next, they observed computer simulations for proportions of heads for samples of size 10, then 30, then 100, where dot plots for 1000 samples were generated. Discussion focused on effect of sample size on variability.

**Session 5: Sampling distributions 2 – sample means**. Students physically rolled dice to first predict, then examine the distribution of mean rolls from samples of size one (uniform distribution) and size three (more normal). Key features of the distributions were discussed. Next, they observed computer simulations for mean dice rolls for samples of size 10, then 30, then 100, where dot plots for 1000 samples were generated. Students reviewed z-scores in light of CLT and answered questions, relating questions and answers to confidence intervals.

**Session 6: Confidence intervals and hypothesis tests on real data**. Students gathered data on Hershey's Easter candy to check the reasonableness of company claims found on product packaging using confidence intervals and hypothesis tests on means and proportions.

**Session 7: Connecting concepts and representations**. Students played matching games and completed sorting tasks that had them reviewing big ideas of the course, particularly those central to the inferential techniques learned thus far.

**Session 8: Correlation and linear regression**. Students played matching games and used computer applets to combine their intuition and knowledge to answer questions about how two numerical variables were linearly related.

## Results

Our results indicate that the PLTL intervention had somewhat complicated and mixed results; overall, the small number of participants limits our ability to detect statistically significant relationships. We present our results in terms of each of our initial research questions.

### Research Question A: Who attended the PLTL Sessions?

We invited all 134 students enrolled in treatment sections to participate in the eight PLTL sessions. The number of students who chose to participate is presented in Table 1. Out of the 134 students enrolled in a treatment section, 40 (29.9%) participated in at least one session, 27 (20.1%) participated in three or more sessions, and 15 (11.2%) participated in six or more sessions. The percentage of students participating in one or more sessions ranged from 18.5% to 42.3% in treatment sections. Similarly, the percentage of students participating in three or more sessions ranged from 10.7% to 34.6%. Although Table 1 indicates that many students chose to participate in the PLTL sessions, we plan to make attempts in future semesters to both recruit students to attend a session and retain those students to additional sessions.

*Table 1. Number of students participating in one or more and six or more sessions organized by instructor.*

| Instructor | Enrollment: Control Section | Enrollment: Treatment Section | Students attending one or more sessions | Students attending three or more sessions |
|---|---|---|---|---|
| 1 | 24 | 28 | 6 (21.4%) | 3 (10.7%) |
| 2 | 29 | 26 | 9 (34.6%) | 8 (30.8%) |
| 3 | 27 | 27 | 9 (33.3%) | 3 (11.1%) |
| 4 | 28 | 27 | 5 (18.5%) | 4 (14.8%) |
| 5 | 28 | 26 | 11 (42.3%) | 9 (34.6%) |
| Totals | 136 | 134 | 40 (29.9%) | 27 (20.1%) |

We were also interested in the achievement levels of the students who chose to participate in the PLTL sessions. We recorded the overall GPA (for all collegiate courses that students had completed at the end of the semester they enrolled in Introductory Statistics) for students in the control group, and for those attending different number of PLTL sessions. These results are presented in Figure 1. Examining this figure, we can observe from the first graph that the entire treatment group and control group had similar distributions of GPA. In the second graph, we notice that the distribution of GPA for students attending three or more sessions is slightly bimodal, with higher frequencies of both students with overall grades in the A and C range. This bimodal tendency is even stronger for students attending six or more sessions.

We are encouraged by the fact that a mix of students of different overall achievement levels chose to participate in the PLTL sessions. In future semesters, our goal is to maintain a diversity of achievement levels. However, given our goal of reducing course failure (DFW) rates, it might be prudent to target recruitment toward students performing at the overall level of "C" or lower, and/or scoring below average on the first course exam.

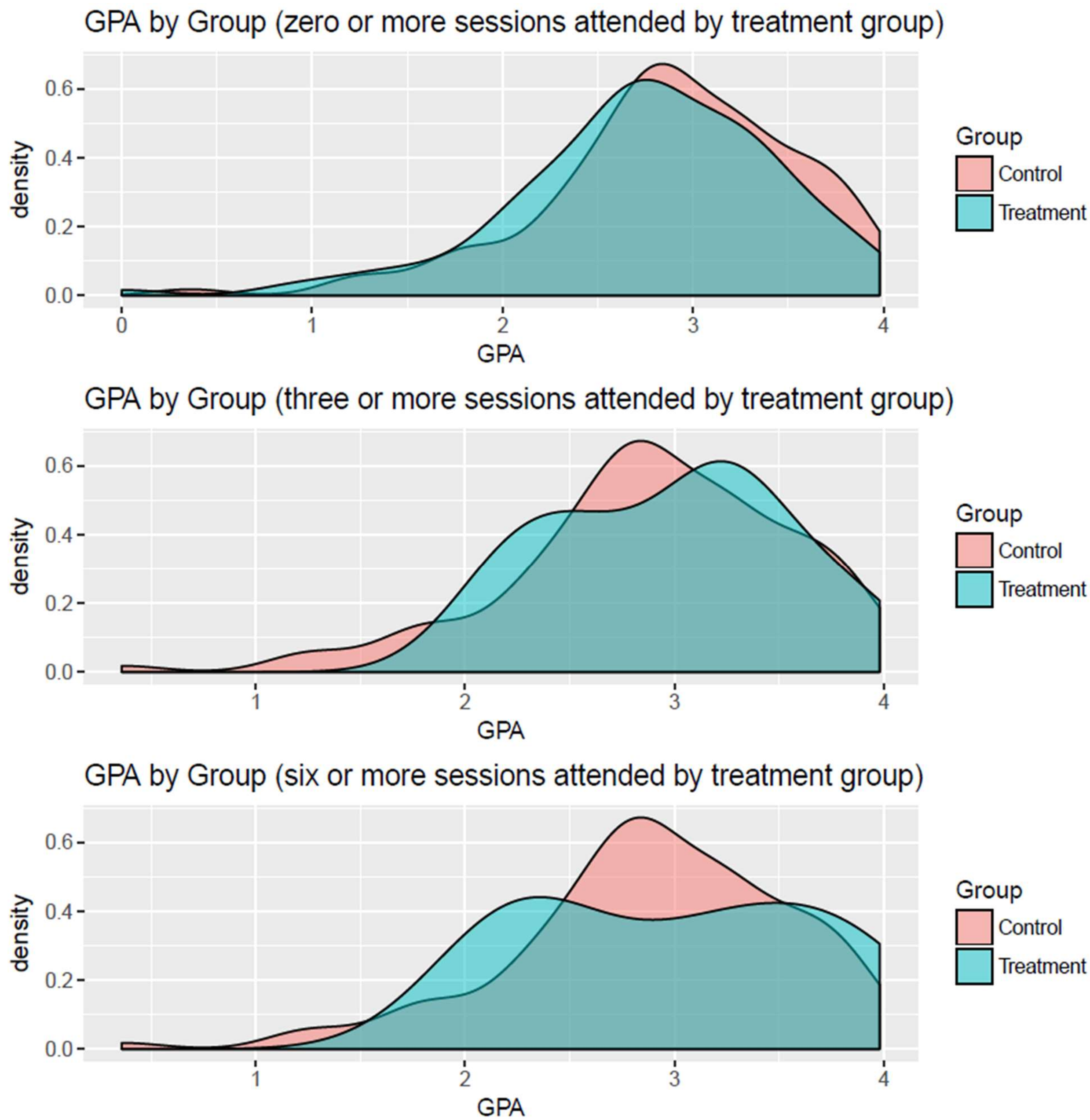


*Figure 1. Density plots of overall college GPA for students in the control group and attending PLTL sessions.*

**Research Question B: What effect did the PLTL sessions have on course outcomes?**

Because our primary goal in instituting the program was to increase student success rates, we began by analyzing its effects on students' course passage. We recorded the number of students obtaining a grade of D, F or W (that is, students not completing the course successfully) and compared the percentage of DFW's in the control group to students who participated in three or more sessions, and to students who participated in six or more sessions. The students who participated in three or more sessions were well matched to students in the control group based on first exam performance (Hedge's g=0.007). This indicates that, on average, students who chose to participate in three or more sessions and students in the control group did not perform differently on the first exam, which occurred before sessions began. We observe that the percentages of DFWs in the participating groups (32.5% and 33.3%) are approximately 5% lower than the percentage of DFWs in the control group (37.5%), but this difference is not statistically significant ($p = 0.347$ for students who attended one or more sessions, and $p = 0.49$ for students who participated in six or more sessions).

We also examined the effects of the intervention on course grade. Figure 2 presents a histogram displaying the course grades for the control group (in order to provide a comparison) and students who attended three or more PLTL sessions. The mean course grade (with A=4, B=3, etc, and W grades removed) was 2.20 (SD = 1.30) for the control group and 2.11 (SD = 1.16) for students attending three or more sessions; this difference was not significant. This finding, along with the data in Figure 2, demonstrates that attending three or more sessions did not appear to have a strong influence on students' success in the course.

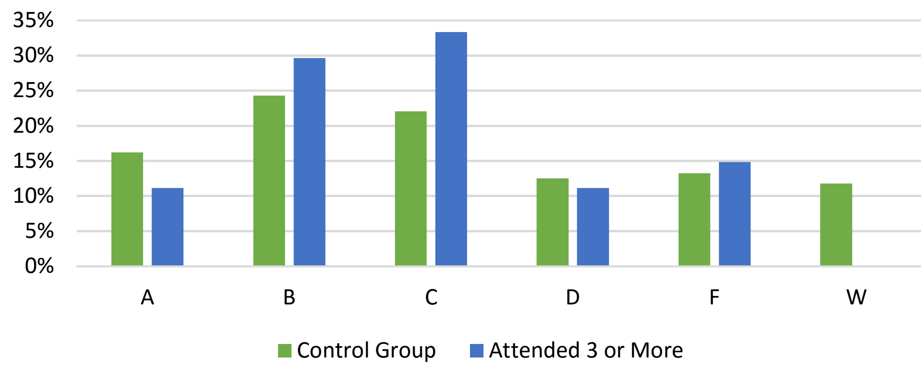


*Figure 2. Course grades for control group and students attending three or more sessions.*

While the observed data does not allow us to draw any firm conclusions about the effectiveness of the PLTL at lowering the DFW rate and increasing course success in introductory statistics, we plan to collect data during the Fall 2017 and Spring 2017 semesters. Program improvements might enhance the effects on student achievement, while increased participation may allow us additional power to observe such effects.

**Research Question C: How did participants perceive the PLTL sessions?**

All students who attended one or more PLTL sessions (N=40) were invited to complete a survey about their perception of the PLTL intervention; 20 students responded (a response rate of 50%). Table 2 shows the number of sessions attended by each of the 20 respondents. We asked why participants attended sessions and how useful they found the sessions to be. We further asked about the most helpful characteristics of the sessions and which topics they found most valuable. Finally, we asked a variety of questions to ascertain participants' perceptions of the coaches, including their helpfulness, preparedness, and enthusiasm.

*Table 2. Number of sessions each of the 20 survey respondents attended*

| Number of sessions attended | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Number of students responding | 1 | 1 | 2 | 3 | 2 | 2 | 2 | 7 |

The most common reasons for attendance were "to earn the incentive" (nine students) and "to hopefully do better in the course by learning the content better" (nine students). When asked "If you didn't come to the sessions regularly or if you stopped coming, what was the reason?", six out of nine students chose “Session content did not match up with my class,” and three students stated that “Two hours for sessions was too long.”

For the most helpful and valuable characteristics of the sessions, common responses included “The sessions reviewed ideas and gave me practice on material I needed” (nine students) and “The sessions taught me something I hadn't understood before” (five students). According to the responses, coaches were viewed positively overall. On a 5-point scale that included “neutral,” 80% or more of the responding attendees agreed or strongly agreed that coaches explained concepts clearly, seemed knowledgeable, and were helpful, organized, enthusiastic, and encouraged discussion.

For each specific topic we covered (see Program Structure, above), we asked students: “How much did the session improve your understanding on the following topics?” 60% or more respondents considered sessions on the following topics at least somewhat helpful: standard deviation, z-score, probabilities associated with the normal curve, hypothesis testing, and meaning of distribution shapes. Eleven students (58%) found half or more of the sessions to be useful. Moreover, the majority of the students stated that going to the sessions was a positive experience and they would recommend these sessions to other students.

Responding attendees made several suggestions for improving the program. More than one student expressed the wish: 1) that sessions started earlier in the semester; 2) for more problems from the textbook and 3) for more hands-on activities. An unsolicited email came to us after the semester in which the student expressed that whereas the tutoring center was not a helpful support service for the course, these “extra classes” were. The student elaborated, “[Coach’s name] ran the classes I went to, and I got a good bit from them. I really got the Empirical Rule, and could estimate some answers just based on my understanding of it. Based on my experience and talking to the other students in [Coach’s name]'s group, I recommend the program.”

## Discussion

In this pilot project, we were unable to document statistically significant changes to students’ success in Introductory Statistics. However, some data, such as the positive perception that students had of the program and the anecdotal finding that shows a small drop in DFW rates, motivate us to offer this program again. Looking forward, we aim to conduct further research on Peer-Led Team Learning structures by conducting a second iteration of this program in a future semester. We plan to make some important adjustments, including starting earlier in the semester and redesigning activities to make them more engaging. Our goal is to increase the program’s impact on student learning and course success. In addition, we aim to offer the intervention to more course sections and increase recruitment efforts. Additional participants will allow a more robust statistical analysis of the effect of sessions on student outcomes.

Implications for undergraduate mathematics education point to the importance and difficulty of creating scalable extra-curricular programs that can support student success in “gateway” mathematics courses. Using knowledgeable peers (as we did in our PLTL program) requires less faculty involvement, but introduces the possibility of misconceptions being passed from coaches to students. We also experienced tensions between our goal of deepening students’ conceptual understanding and students’ goals of improving procedural fluency in answering homework and exam questions, as well as difficulty in providing content that was applicable to students across different sections and instructors. Addressing these challenges will take collaboration and conversation across Mathematics Departments nationally; we aim to add to that conversation through our presentation and this paper.

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Students' Strategies for Setting up Differential Equations in Engineering Contexts

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*Ordinary differential equations (ODEs) comprise an important tool for mathematical modelling in science and engineering. This study focuses on how students in an engineering system dynamics course organized the act of setting up ODEs for complex engineering contexts. Through the lens of ODEs as a "coordination class" concept, we examined the strategies that seemed to guide the students' interpretations of problem tasks and their activation of knowledge elements during the tasks, as the students worked to produce ODEs for those tasks. This led to our uncovering of three main strategies guiding the students' work, and the finding that being able to flexibly draw on all of these strategies may be beneficial for student success.*

*Keywords:* differential equations, mathematics in engineering, system dynamics

Ordinary differential equations (ODEs) comprise a branch of mathematics that is extremely useful for mathematical modelling in a range of STEM (science, technology, engineering, and mathematics) fields. For example, it can be used in biology to model population dynamics, in engineering to model the evolution of mechanical system, and in physics to model changing quantities. A growing body of research has been examining how students understand, solve, and interpret ODEs in mathematics. Most of this work has focused on how students understand solution processes and the solutions themselves (Arslan, 2010; Camacho-Machín, Perdomo-Díaz, & Santos-Trigo, 2012; Habre, 2000; Rasmussen, 2001; Rasmussen & Blumenfeld, 2007). From this we know that students may struggle with the idea of a *function* being a solution (Rasmussen, 2001), that students may be hesitant about using graphical solution procedures (Camacho-Machín et al., 2012; Habre, 2000), and that equilibrium solutions are not well understood by students (Rasmussen, 2001; Zandieh & McDonald, 1999).

There is much less we know about how students organize their work for *setting up* ODEs for given contexts. Rowland and Jovanoski (2004) and Camacho-Machín and Guerrero-Ortiz (2015) each examined the setting-up process and interpretation of simple ODEs and found that students struggled to use "rate of change" thinking when doing so. They often thought of constants in ODEs as representing *constant amounts* rather than *constant rates of change*. While these studies provide useful results, the contexts used in the tasks were fairly simple and all the needed information was provided in the task. By contrast, in engineering, students encounter quite complex situations for which not all of the information is directly presented. This type of situation implies more challenges for the students as they attempt to organize their work to produce an ODE. We believe it important to extend the research on ODEs by examining how students go about the process of setting up differential equations for tasks that involve complicated systems. In summary, this report is meant to investigate the research question: What strategies do students use when setting up ODEs for complex engineering tasks?

## Coordination Class Concepts

For this study, we used the lens of coordination classes from the knowledge-in-pieces paradigm (diSessa & Sherin, 1998). Coordination classes are useful for describing concepts whose purpose is "getting information" (p. 1171). In the context of system dynamics, the information regarding the system is obtained through an ODE. A coordination class concept involves readout strategies and causal nets. *Readout strategies* are the "means of seeing things that relate to the target information" (diSessa, 2004, p. 141). For our purposes, we see readout strategies in terms of how one interprets external stimuli. The *causal net* is "the set of all possible inferences that lead to determining the relevant information" (diSessa, 2004, p. 141). That is, once a person has interpreted external things, those interpretations can then be linked with other pieces of knowledge so as to progress toward the desired information. Causal net elements might consist of known relationships, formulas, informal ideas, beliefs, and so on.

Next, diSessa and Wagner (2005) define a *concept projection* as the collection of knowledge elements related to that concept, as well as the guiding strategies, that a person uses in a particular context. The strategy used to obtain the information impacts the readouts and causal net elements that are activated. For example, suppose one wants to find the volume of a three-dimensional object (the desired information). One might first use a readout strategy to identify whether the object's shape is a typical geometric shape, or not. If it is, such as a box, the person might activate the causal net knowledge element $V=lwh$. Using this geometric formula, they could obtain the object's volume. We could call this a "geometric strategy," because the person used the geometric regularity of the object to determine how to find the information. On the other hand, if the object is an irregular shape, the person might instead activate a causal net knowledge element of Archimedes principle, which states that the volume displaced by water is equal to the volume of the submerged object. The person could then use that inference to determine the object's volume. We could call this a "experiment strategy," because the person would enact an experiment based on a known principle to determine the information.

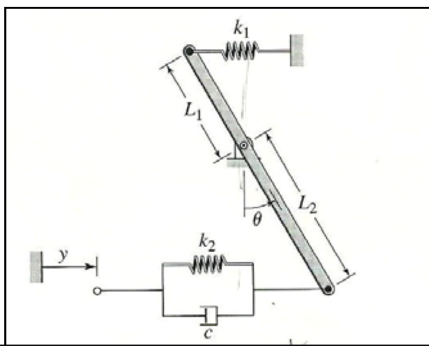
## Data Collection and Analysis

In order to provide insight into the strategies students use to set up ODEs for complex tasks, we recruited students for interviews who were taking an engineering "system dynamics" course. We chose a system dynamics course because (1) taking an ODE course is a prerequisite for the system dynamics course, meaning all the students had experience with ODEs; and (2) the system dynamics course is designed entirely around the idea of setting up and solving ODEs for different engineering systems. Thus, the students were in the process of learning to set up ODEs for complex contexts, though the instructor mostly just lectured on how to set up ODEs. To recruit students from the system dynamics class, we first administered a survey to the entire class to obtain background information on how the students interpreted a generic ODE. The survey displayed the equation $ay''+by'+cy = 0$ and asked the students to describe what this equation meant and what the various symbols in it represented. We chose two students who provided strong responses regarding the equation (Rebecca and Zane), two students who provided moderate responses about the equation (Harry and Josh), and one student who showed some weaknesses in their understanding of the equation (Kira). These five students participated in two interview sessions where they were asked to set up an ODE for a total of three different tasks.

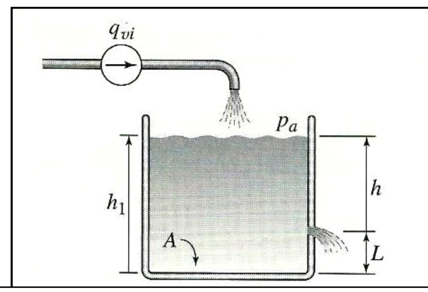
We designed the interviews to focus on contexts that matched those seen in the students' system dynamics course. The three tasks consisted of a mechanical context, an electrical context, and a fluid context. For the purposes of this abbreviated conference report, we focus on the mechanical task (task 1) and the fluid task (task 3), shown in Figure 1, as they suffice for describing the main strategies the students used for organizing their work of setting up an ODE. For the interview, the students worked out the tasks, explaining their thinking aloud, and the interviewer asked follow-up and clarifying questions while the student worked.

The interview data were analyzed in two separate phases. In the first phase, which was essentially a preliminary phase in terms of this paper's research question, we identified readouts and causal net elements the students used while working on the tasks. Operationally, readouts were defined as any place in the data where a student appeared to make a direct interpretation of any part of the given interview task, whether symbols, words, or parts of the figure. The apparent interpretation was recorded as the "readout." Causal net elements were operationally defined as any time a student mentioned, wrote, or suggested an idea that was not a direct interpretation of a part of the task. The substance of the causal net element, as well as what other piece(s) of information may have triggered its activation, was recorded as a "causal net link."

In the second phase, which allowed a more direct answer to this paper's research question, we used the resulting readouts and causal net links recorded in phase one to examine the overall flow of the students' work. This allowed us to infer strategies the students appeared to be using to set up the ODE. We did not have pre-set notions of what the strategies would consist of, but rather let the nature of the strategies emerge from the student's documented process. This led to the identification of three main strategies, described in the next section. Lastly, we determined whether the strategies were productive for the students, by observing (1) whether a particular strategy helped the students produce a solution, (2) whether that solution was correct, and (3) whether the student had to revise their solution because the approach led to a "dead end."




Task 1: Assuming that $\theta$ is small, derive the equation of motion of the pendulum shown in the figure. The input is $y(t)$ and the output is $\theta$. The equilibrium corresponds to $y = \theta = 0$.




Task 3: The cylindrical tank shown in the figure has a circular bottom of area $A$. The volume inflow rate from the flow source is $q_{vi}(t)$, a given function of time. The orifice in the side wall has an area $A_0$ and discharges to atmospheric pressure $p_a$. Develop a model of the liquid height $h$.

*Figure 1. The mechanical and fluid tasks from the interviews (taken from Palm, 2005, p. 244 and 397).*

## Results

In this section, we describe, one by one, the three main strategies used by these five students to guide their work setting up ODEs for the tasks. We do this by providing a single illustrative case from the data for each of the three main strategies. We then provide a summary about which students used each strategy, and end by discussing possible benefits of the three strategies.

### Strategy #1: Diagram-based Approach

To illustrate the first strategy, we describe the case of Zane working on task 1. While his complete work is too lengthy to describe here in its entirety, we highlight enough of his work to hopefully demonstrate his main guiding strategy. An important early readout in Zane's work was to view the bar in the figure as the main object about which to reason. This led Zane to draw what, in engineering, is called a free-body diagram. His diagram consisted of the bar, by itself, which he continually annotated and revisited throughout his work (see Figure 2). The diagram helped him focus on two other readouts, namely the top of the bar and the bottom of the bar (arrows at the top and bottom of his diagram). He then inferred that forces and horizontal displacement were both relevant attributes of the top and bottom of the bar. The top arrow was linked to a single force from the upper spring, while the bottom arrow was linked to two forces from the lower spring and the damper. His diagram also helped him visualize the implicit presence of two right triangles each having $\theta$ as an angle (added to Figure 2). He used the triangles, together with Hooke's Law ($F=k\cdot\Delta x$) and the fact that $\sin(\theta) \approx \theta$ (inferred from the "small angle"), to describe the three forces in terms of $L_1$, $L_2$, and $\theta$ (seen in Figure 2 as $k_1L_1\theta$ and $k_2L_2\theta$ for the springs, and $cL_2\dot{\theta}$ for the damper). Causal net links between *velocity* and *first derivative* and *rotational acceleration* and *second derivative* allowed him to invoke the $\dot{y}$, $\dot{\theta}$, and $\ddot{\theta}$ seen around the "cloud" in Figure 2. Zane used the standard engineering "dot" notation for time-derivatives, as did the other students. While there are additional science-based knowledge elements Zane used along the way, such as a "moment" being force times the distance from the center of rotation, and $\sum moments = I\ddot{\theta}$, we can see that most of his readouts and causal net links were scaffolded by his free-body diagram. Putting all of these elements together, Zane was successful at producing a correctly set up ODE for this task, shown in Figure 3.

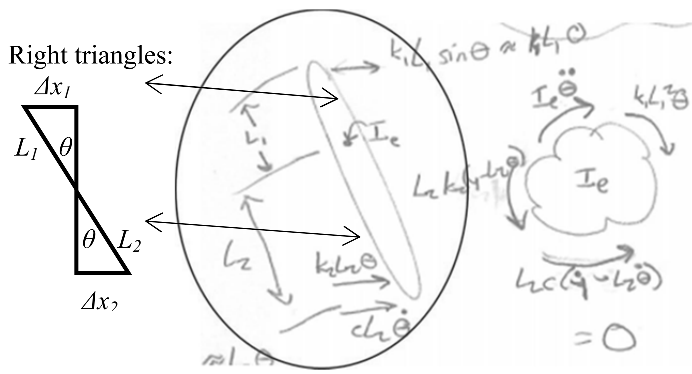



*Figure 2. Zane's initial free-body diagram (circled) and his work based off the diagram*

$$I_e\ddot{\theta} + k_1L_1^2\theta - L_2c(\dot{y} - L_2\dot{\theta}) - L_2k_2(y - L_2\theta) = 0$$

*Figure 3. Zane's correctly set up ODE for task 1.*

Having briefly recapped Zane's work, we can see that his overall strategy consisted of using a single diagram to organize most of his readouts and causal net links. Thus, we call Zane's strategy the "diagram-based" approach. In general, we can think of the diagram approach as a *general-to-specific* method that initially focuses on the entire system. Then, within that system, the student can attend to individual parts that have relevance to the system. The diagram approach is not limited to mechanical free-body diagrams, but can also be seen in "schematics" for electrical contexts and "control volumes" for fluid contexts. In fact, the simple existence of names for these types of diagrams in various engineering contexts suggests its generalizable usefulness as a strategy for setting up equations, which apparently extends to ODEs as well.

**Strategy #2: Components-based Approach**

The second strategy we describe contrasts with the diagram-based approach in that it could be considered a *specific-to-general* strategy. To illustrate it, we describe the case of Rebecca also working on task 1. Unlike Zane, who first "read" the bar in isolation, Rebecca's initial readouts were to scan the task to locate and identify various individual elements and to begin to keep track of them. She immediately identified three elements, $k_1$, $k_2$, and $c$, each as representing forces. This is different from Zane, who initially began with only two elements, namely the top and bottom of the bar. Thus, we can see a distinction in what these strategies might focus on. She then made the causal net link that each force multiplied to its distance from the center of rotation gives the moment at that point. Using these individual components, and the fact that the sum of moments equals $I\ddot{\theta}$, she wrote an early version of the ODE, shown in Figure 4.

$$F_{k_1}L_1 + F_{k_2}L_2 + F_c L_2 + I\ddot{\theta} = 0$$

*Figure 4. Rebecca's initial equation, focused on compiling individual elements of the system.*

Rebecca then returned to each individual element in order to flesh each one out, which resembled Zane's work at this point. She similarly inferred $\sin(\theta) \approx \theta$ from the "small angle" in order to elaborate on $F_{k1}$, $F_{k2}$, and $F_c$. This approach is visible in her work shown in Figure 5, where she used a string of causal net links to establish how each element was related to $y$, $\dot{y}$, $\theta$, $\dot{\theta}$, and $\ddot{\theta}$. After finding each element in Figure 4 in terms of these variables, she combined them into a correctly set up ODE, shown in Figure 6.

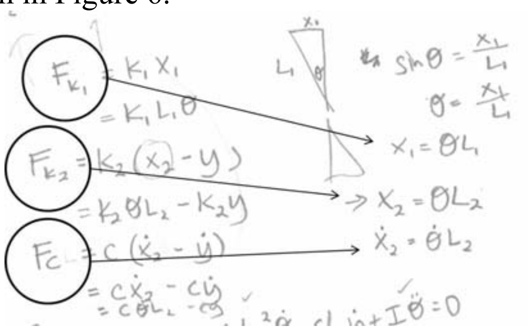



*Figure 5. Rebecca's work of focusing on each element and how it could be represented in terms of θ.*

$$I\ddot{\theta} + cL_2^2\dot{\theta} + k_1L_1^2\theta + k_2L_2^2\theta = cL_2\dot{y} + k_2L_2y$$

*Figure 6. Rebecca's correctly set up ODE for task 1.*

In Rebecca's work, rather than beginning with a diagram, we can see the strategy of reading out specific elements *first* and *then* subsequently trying to piece them together. Of course, there were many overlapping readouts and causal net inferences with Zane's work, once she performed her initial organization of the task. Also, it is certainly true that Rebecca *did* employ holistic thinking in her work, evidenced by when she put the various components together, like in Figure 4. However, what is different and noteworthy is that her guiding initial strategy was "reading" the task through the identification of each individual element and then figuring out how to compile them. For Rebecca, the individual elements seem to have come first, and then knowledge pieces were used to organize the elements into a coherent whole. We can see that, for Rebecca, this strategy was just as successful as Zane's diagram approach, since it provided a direct path toward creating a correct ODE for this context.

**Strategy #3: Equation-based Approach**

For the third strategy, we again describe Rebecca's work, but this time with task 3. An important initial readout for Rebecca in task 3 was simply to attend to the general *fluid flow* context of the problem, as opposed to any individual element within it. Her recognition of this type of context seemed to immediately activate a causal net link that an adaptation of Bernoulli's equation governs these types of fluid contexts. This link allowed Rebecca to immediately invoke an entire equation as a single knowledge resource, $q_{in} - q_{out} = \rho\dot{V} + \dot{\rho}V$ (where $q$ is a flow rate, $\rho$ is the fluid's density, and $V$ is the fluid's volume in the container). That is, rather than *piecing together* an equation, as Zane and Rebecca (and other students) did for task 1, in this case an entire equation was *recalled* from memory because of its relevance to the context. The remaining work for Rebecca in this task was then to manipulate this equation by making substitutions or cancelations that would produce the desired ODE.

To do so, Rebecca first used the readout of "water" to infer incompressibility, meaning that the density would not change and $\dot{\rho} = 0$ (see Figure 7, and note the scribbled out "$\dot{\rho}V$" above the last term). Next, she used the facts that $V = Ah$, and that the cross-sectional area was constant, to substitute $\rho A\dot{h}$ in place of $\rho\dot{V}$. Lastly, she used Toricelli's Law, $q_{out} = c\sqrt{2gh}$, to make a substitution for $q_{out}$ (where $h$ is the distance between the fluid surface and the outflow). Notice that in her final equation (Figure 8), she could not recall exactly what was supposed to be "inside" the square root, and so her expression diverges a little from a "correct" solution. However, had she had access to a book or sheet of equations, she could have easily corrected this and thus we still consider her final ODE to essentially be "correct." Also, for clarification, her "sgn" term is the "sign function" for whether the argument is positive or negative.

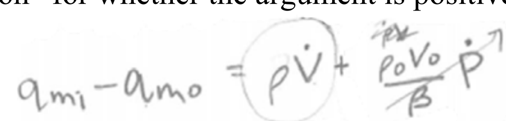



*Figure 7. Rebecca's direct invocation of an entire equation governing fluid flow.*

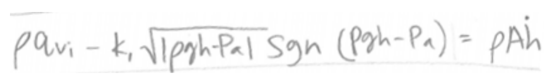



*Figure 8. Rebecca's essentially "correct" final ODE for task 3.*

In general, Rebecca's equation approach seemed to rely on the fact that there was a single main equation governing that particular class of systems. From that equation, Rebecca centered all her efforts to obtain the ODE by manipulating the equation through substitutions or cancellations. Thus, the use of this strategy would first require the perception (i.e. causal net link) that there is, in fact, such an equation that can be used for a given system. This strategy was also successful in that it provided Rebecca a clear path toward an essentially correct ODE.

**Approaches Used by All of the Students**

We now provide a brief summary of all five students in terms of which strategies they used (diagram, component, or equation) and whether they were successful, partially successful, or unsuccessful at setting up an ODE (see Table 1). We note that we allowed "successful" set ups to include equations where there was a simple recall mistake, like Rebecca's in task 3. We considered "partially successful" set ups to be those that had one or two significant flaws (beyond simple recall) but that still contained many correct elements in the equation. An "unsuccessful" set-up was one where the student never produced a final equation, or one in which the equation had multiple major flaws.

We see in Table 1 that not all students confined themselves to a single strategy for a given task. Harry, Josh, and Kira each used multiple strategies for at least one task in order to help them progress in their work. In some ways, because these students struggled more than Zane and Rebecca, who each only used one strategy per task, one might conclude that using more strategies is a sign of weakness. However, we do note that using different strategies actually allowed Harry, Josh, and Kira to each make more progress than they would have otherwise made with a single strategy alone. That is, once they were stuck, switching modes to a different strategy often seemed to unlock additional causal net links that may have been hidden from them while using the other strategy, even if they did not fully reach a completed ODE.

*Table 1. Strategies used and whether the student was successful (S), partially successful (PS), or unsuccessful (UN)*

| | Rebecca | Zane | Harry | Josh | Kira |
|---|---|---|---|---|---|
| Task 1 | Component (S) | Diagram (S) | Component (PS) | Diag/Comp (PS) | Diag/Comp/Eq (UN) |
| Task 3 | Equation (S) | Equation (S) | Diag/Eq (S) | Diag/Eq (PS) | Diag/Eq (S) |

**Discussion of the Three Strategies**

We believe the empirical documentation of these three strategies provides some insight into how students might identify and use information relevant to ODEs in complex contexts through readouts and causal nets. While experts might see the strategies as equivalent, we believe there are nuances to each, and this report may be seen as an "unpacking" of possible ways to reason about ODEs for complex engineering tasks. In fact, our study suggests these strategies could be important in *developing* expertise. All three approaches were used by students to correctly set up ODEs for these complex tasks, or at least to construct partially correct ODEs, as seen in Table 1.

We can see that there is not necessarily "one correct strategy" for a given problem. Rebecca and Harry both used the component approach to productive ends for task 1, but Zane and Josh both used the diagram approach instead to make progress on that task. While all of the students used the equation approach on task 3, Harry, Josh, and Kira also used the diagram approach (in the form of a "control volume") to help further their work. Yet, while there may not be one correct strategy, we observe that the trends in Table 1 suggest some strategies being more easily invoked for some tasks than others. The equation approach was hardly used at all for task 1, but was used extensively for task 3. This gives evidence that some problems may lend themselves better to bringing in an overarching governing equation. For example, task 1 could be considered to have the governing equation $\sum moments = I\ddot{\theta}$, but this is not where these students tended to start. Rather, this equation emerged as a causal net link further down the line, once the students were ready to organize the elements into a whole. By contrast, the fluid flow equation seemed readily available as an immediate starting place for task 3. Thus, developing expertise in setting up ODEs in these types of contexts may have something to do with being able to recognize when it may be best to start with a diagram, start with individual components, or start with an equation.

Since using multiple strategies helped the weaker students in this study make more progress than they otherwise would have with only a single strategy, evolving expertise could also partially deal with being able to switch strategy modes if a roadblock is reached within a task. Perhaps it is true that greater expertise may lead to better identification of a single productive approach, as with Rebecca and Zane. However, as students mature *toward* that point, becoming aware of which of the three strategies they are using may help them see the value in switching between strategies for a given task. This may help them develop better flexibility in which strategies they use, and to begin to see connections between certain problem types and certain strategies that are useful for that type. We see this exploratory study as a useful step, in that it could be expanded into a teaching experiment to confirm, refute, or nuance these results.

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Student Status in Peer Conferences

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*This paper provides an analysis of students' peer assessment conversations in introductory college calculus. In particular, it explores gender differences in the types of feedback and word choices used by students. Using computer-aided textual analysis, it draws connections between the types of words that students use and their relative status in the class. Surprisingly, the use of pronouns based on gender did not follow what one would predict based on prior studies. Possible explanations and implications for future research are discussed.*

*Keywords:* peer assessment; status; feedback; equity.

## Introduction

Imagine one hundred calculus students trading papers amongst themselves and providing constructive feedback. You overhear the following in a peer conference:

> It said in the beginning of the problem that each statement below is true sometimes, and it says give an example of a function when it's true and when it's not true. For yours you only put when it's true, but when it's not true you didn't really put anything there.

As you continue walking, you overhear another student:

> I think the main thing is that for part A, I think there should be just one equation dV/dt and you would incorporate both dh/dt and dr/dt. I think just somewhere in your differentiation you didn't do something…I'm not sure what. I think there should just be one equation for dV/dt.

What could you learn about these students and this classroom based on the words you overheard? In the first excerpt, the student gives process-focused feedback, telling their peer what is required to answer this type of question, but does not give the answer itself. In contrast, the second excerpt is product-focused, describing the correctness of an equation. While both of these types of feedback can promote learning, process-focused feedback tends to promote better learning (Hattie & Timperley, 2007; Reinholz, 2015a).

What else could you infer? In the first excerpt, few first-person pronouns are used. Rather than saying what they *think* about their peer's solution, the student simply makes statements about what their solution *should be*. In contrast, the second excerpt features a large number of "I" statements. The feedback is clearly coming from the perspective of *that student*. This difference in pronoun usage is often indicative of status hierarchies (Pennebaker, 2011). The first student, who uses many more second-person pronouns, is likely of higher status than the student who uses primarily first-person pronouns. Thus, simply by looking at the types of pronouns students use in the conversations, it may be possible to uncover subtle status hierarchies in the classroom.

This paper responds to recent calls to focus on issues of equity in undergraduate mathematics (Adiredja & Andrews-Larson, 2017). By studying the language use of students in calculus, it provides insight into status hierarchies, which are of consequence for understanding how students may have access to differential opportunities to learn (e.g., Cohen & Lotan, 1997). This article focuses on the following research questions: *How does participation differ for individual students, or groups of students (e.g., by race, gender)?* In light of this question, the implications of using peer assessment as a tool to create more equitable learning opportunities are discussed.

## Theoretical Framing

A large body of literature connects classroom discourse and learning (Bransford, Brown, & Cocking, 2000; Lampert, 1990; Sfard, 2008). This literature emphasizes opportunities to participate in meaningful discourse constitute opportunities to learn (Hufferd-Ackles, Fuson, & Sherin, 2004; Michaels, O'Connor, Hall, & Resnick, 2010). Simply speaking during class, for instance, in low-level, Initiate-Response-Evaluate (IRE) sequences (cf. Cazden, 2001; Mehan, 1979), is insufficient to promote deep learning. Rather, students need opportunities to engage with mathematics in cognitively demanding ways that push them to engage in mathematical sense making (cf. Stein, Grover, & Henningsen, 1996).

Simultaneously, a growing literature has examined issues of equity in classroom discourse (Esmonde & Langer-Osuna, 2013; e.g., Herbel-Eisenmann, Choppin, Wagner, & Pimm, 2012). This literature highlights how subtle inequities can emerge, particularly in terms of gender, race, and other social markers. For instance, some groups tend to receive lower-level participation opportunities, based on their gender (Sadker, Sadker, & Zittleman, 2009), race (McAfee, 2014), and immigration status (Planas & Gorgorió, 2004). While these patterns of marginalization often emerge unintentionally, they are nonetheless problematic and require attention.

For the purposes of this paper, equality is taken as a necessary but insufficient baseline for equity (Secada, 1989). While it may be impossible to decide exactly what instruction is required to provide *equitable* opportunities for all students, it is clear that if students from historically-marginalized groups receive proportionally less opportunities to participate than their historically-dominant peers (which literature shows is often the case), it is highly problematic. In other words, if all students receive at least *equal* opportunities to participate, it is a positive (yet insufficient) step in the right direction.

Accompanying the wealth of literature describing inequity in discourse, there are also valiant efforts to reduce such inequity. One well-known example is the set of techniques associated with Complex Instruction. These instructional moves (e.g., the multiple ability treatment, assigning status) help mitigate status hierarchies in heterogeneous classrooms, leading to more equitable outcomes for all students (Cohen & Lotan, 1997; Nasir, Cabana, Shreve, Woodbury, & Louie, 2014). In other words, power imbalances (e.g., who is perceived as an authority) lead to less equitable outcomes (Engle, Langer-Osuna, & Royston, 2014; Langer-Osuna, 2016), but when these imbalances can be addressed, learning becomes more equitable (Cohen & Lotan, 1997).

The above literature highlights how issues of inequity arise across a variety of mathematics classroom settings. This paper focuses particularly on calculus, which is known to significantly decrease student confidence, enjoyment, and interest in mathematics (Bressoud, Carlson, Mesa, & Rasmussen, 2013). While these effects impact all students, they differentially impact non-dominant students. For instance, women with the same grades as men are 1.5 times as likely to leave the calculus sequence (Ellis, Fosdick, & Rasmussen, 2016). Moreover, there are salient societal narratives about who can and cannot do mathematics, which can have a negative impact on students (e.g., Nasir & Shah, 2011; Stinson, 2008). In other words, the status quo for calculus is severe inequity. A classroom that were to achieve *equality* would be a considerable step in the right direction.

From this backdrop, the present study focuses on issues of equity and status in peer conferences. Peer conferences are an important feature of peer assessment (cf. Falchikov & Goldfinch, 2000; Topping, 2009), and offer unique opportunities for addressing issues of inequity in discourse. In particular, peer conferences generally involve only two students, so the complexities of promoting participation from all students in a small group or a whole class are reduced. Moreover, peer conferences position students as competent authorities, because they must critically judge the work of their peers, which provides them with space in the classroom to act as experts (cf. Engle & Conant, 2002; Reinholz, 2015b).

## Method

### Context

The present study took place in calculus I at a relatively large (over 30,000 students), racially diverse (e.g., ~65% students of color) research-extensive university in the US. The course consisted of a combination of large lectures (100-200 students) taught by full-time instructors and smaller breakout recitation sections (30-40 students) taught by Graduate Teaching Assistants. This paper focuses on a single large-lecture section (N=124), which met three times weekly for 50 minutes at a time. In addition, students in the lecture met twice weekly for 50 minutes for their recitation sessions, but those sessions are not a focus of this paper.

### Design

Each week students engaged in a peer assessment learning activity called Peer-Assisted Reflection, or PAR (Reinholz, 2015b; Reinholz & Dounas-Frazer, 2016). The goal of PAR is for students to develop self-assessment skills as they assess the work of their peers (Black, Harrison, & Lee, 2003; Reinholz, 2015c). Specifically, PAR consists of a four-part cycle through which students: (1) complete a draft solution to a conceptual mathematics problem for homework, (2) reflect on their solution by identifying which aspects of their solution they would like to receive feedback on, (3) trade papers with a peer in class and exchange peer feedback, and (4) revise their work before turning in their solution. Students receive homework credit both for the correctness of their solution and for completing the PAR process, which encourages students to revise their work (in practice nearly all students do so). Prior studies show that PAR has a significant impact on student learning (Reinholz, 2015b, 2016), but the learning impact of PAR is not the focus of the present study.

This implementation of PAR differed from prior iterations (in nearly 20 courses), because it took place a large-lecture course, which imposed different logistical constraints. In terms of the actual PAR process, students were able to engage productively during their large lecture sessions: they simply turned to a peer, traded papers, and conferenced about their work. Yet, prior research suggested that PAR was most effective when students chose their partners randomly, as this allowed them to get a variety of perspectives and it changed the dynamics of peer relationships (Reinholz, 2015b). While students were encouraged to choose new partners each week, in practice, this was difficult to enforce because of the large-lecture environment. Moreover, the large number of students enrolled in the course (N = 124) meant that students received little feedback from the instructor on the quality of their PAR solutions and the feedback that they provided to their peers, in contrast to prior implementations of PAR. Because students received less feedback about the quality of their feedback, it was assumed that the learning impact of PAR would be lessened somewhat.

During the feedback exchange component of PAR (step 3), students read each other's work silently for five minutes and write comments, and then have five minutes to discuss their feedback. Forcing students to engage silently with each other's work before the discussion helps ensure that students actually talk about their peers' solutions, not just the problem. Moreover, PAR positions both students as competent, as they *both* provide feedback to one another, rather than creating an asymmetric relationship in which only one student provides feedback to the other. This was a feature designed to promote student authority (cf. Engle & Conant, 2002). In the context of a large-lecture course, this was intended to provide *all* students with opportunities to engage in meaningful talk around mathematics, which can otherwise be difficult to facilitate in whole-class conversations.

### Participants and Data

A total of 84 students participated in the study (in a class of N=124). Demographic information was collected from the university's office of Institutional Research (see Table 1).

*Table 1. Participant demographics (N = 84).*

| | Women | Men | **Total** |
|---|---|---|---|
| African American | 0 | 3 | **3** |
| Asian/Pacific Islander | 4 | 11 | **15** |
| Hispanic | 8 | 16 | **24** |
| International | 0 | 1 | **1** |
| Multiple Ethnicities | 1 | 4 | **5** |
| White | 9 | 19 | **28** |
| Unknown | 4 | 4 | **8** |
| **Total** | **26** | **58** | **84** |

The primary data source for this article was students' peer conversations. During their PAR conferences students recorded their conversations (the second part of their feedback exchange) using audio recorders on their cellular phones. A total of 172 conversations were recorded in this way. While the majority of conversations consisted of student dyads, some of the conversations involved three students at a time. To account for a variable number of students in certain conversations, a unit of 'participant-conversations' was used for analysis, which represents how many times some student from a particular group participated in some conversation. This paper focuses on gender of students. There were 116 participant-conversations for women, and 254 for men.

### Analytic Methods

To support data analysis, all conversations were transcribed and linked to student names. This allowed for demographic information to be attached to each individual student contribution. Cleaning of the dataset and data analysis was completed in R statistics, using a variety of text processing packages (e.g., stringi, lsr, lexicon). Student conferences were analyzed for the type of feedback provided using a prior coding scheme that focused on process, product, and person feedback (Reinholz, 2015a). Student conferences were also analyzed for their pronoun usage. In particular, when students have higher status, they tend to use fewer first-person singular pronouns, more first-person plural pronouns, and more second-person pronouns (Pennebaker, 2011). Thus, by looking at the relative use of pronouns in these three categories, it was possible to explore issues of authority in peer conferences. One would expect that historically dominant students (e.g., White/Asian men) would speak as though they had more status.

All analyses must be interpreted with some caution. For instance, a wealth of literature highlights differences in word usage based on gender (Argamon, Koppel, Fine, & Shimoni, 2006), task characteristics (Newman, Groom, Handelman, & Pennebaker, 2008), topic (Bamman, Eisenstein, & Schnoebelen, 2014), and age (Huffaker & Calvert, 2005). Despite this level of nuance, some commonalities exist across settings. For instance, women tend to use first-person singular, cognitive, and social words more, while men use more articles, and there are no meaningful differences for first-person plural or positive emotion words (Pennebaker, 2011).

In sum, one can expect that there will be differences in word usage by different groups of students in the peer assessment process, simply by virtue of their membership in particular gender, racial, or other demographic groups. Simultaneously, it will be difficult to predict in advance what these differences may be. Nevertheless, as others continue to look at such patterns of word usage in other educational contexts, this paper will provide a baseline to compare to.

Finally, it is recognized that reducing socially-constructed identities (e.g., based on gender, race) can be potentially problematic, as it obscures that positioning individuals is a power-laden process (Davies & Harré, 1990). While it can potentially be problematic to essentialize such characteristics, it can also be 'strategic,' as a tool to highlight or address inequities (Gutierrez, 2002). In other words, this strategic essentialism makes it possible to illuminate subtle patterns of inequity (e.g., men speaking more than women), which are problematic and need to be addressed. As such, essentialism can be used as step towards greater equity, while acknowledging the need for complementary approaches that treat social markers more fluidly (e.g., Nasir, McLaughlin, & Jones, 2009).

**Results**

Table 3 summarizes students' word usage. To contextualize these results, they are compared to two prior iterations of PAR (Reinholz, 2015a). The table shows that in the present study each student contributed an average of 149.09 words to each conversation. These conversations were shorter than those in prior iterations. Also, at an absolute level, the Phase II conversations contained more feedback in these three categories than during the current study. Yet, when looking at density of feedback, the amount of feedback based on how many words were spoken is highest for the current study. In other words, it seems that students were saying more with fewer words, and likely there was less off-topic talk. Given differences in the implementation of PAR and student populations, it is difficult to identify exactly the source of these differences.

*Table 3. Average word usage (by participant conversation)*

| | Present (N=370) | Phase I (N=116) | Phase II (N=184) |
|---|---|---|---|
| Total Words | 149.09 | 163.17 | 295.19 |
| Process Words | 6.04 | 4.86 | 9.00 |
| Product Words | 1.32 | 0.70 | 1.07 |
| Person Words | 3.54 | 2.43 | 2.48 |

Figure 1 shows feedback types by gender. The data are expressed as an "equity ratio" (Reinholz & Shah, in press). This ratio describes the actual participation by students (in number of words), divided by what one would expect based on demographics alone. Thus, a ratio greater than one means that students contributed more than expected, a ratio of one is what would be expected, and a ratio less than one means that they contributed less than expected. Here we see that men contributed more than one would expect for total words $\chi^2(1, N = 55284) = 102.67$, p = $3.9 * 10^{-24}$, Cramer's V = 0.53 (large effect size). Women used more person-focused feedback (i.e. praise), $\chi^2(1, N = 1387) = 39.18$, p = $3.85 * 10^{-10}$, Cramer's V = 0.33 (medium effect size). Men used more product feedback (i.e. describing right or wrong), $\chi^2(1, N = 484) = 19.2$, p = $1.2 * 10^{-5}$, Cramer's V = 0.22. There were no significant differences for process words.

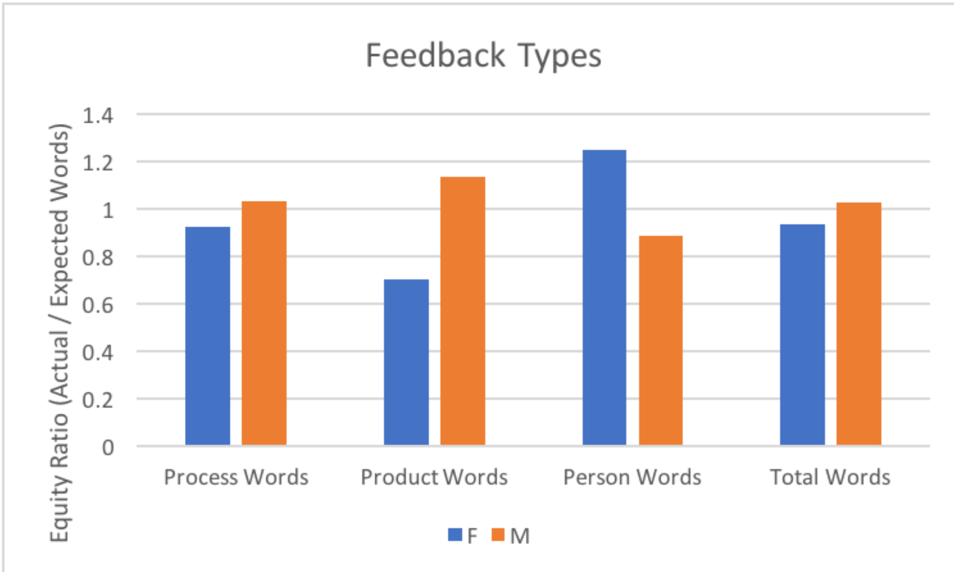


*Figure 1. Feedback types by gender*

What can be inferred from these results? Figure 1 indicates that men and women behaved the same when it came to giving process-focused feedback, which is the most valuable type for learning. Yet, there were also stylistic differences with men focusing more on correctness of the solution and women offering more praise. This provides contrast to some other settings, where women tended to use more positive emotion words (Pennebaker, 2011). On the whole, men did talk more, but the equity ratio for total words was near one. Thus, this statistically significant difference may have less practical significance.

Figure 2 shows pronoun usage by gender. Men used significantly more first-person singular pronouns $\chi^2(1, N = 3751) = 71.34$, p = $3 * 10^{-17}$, Cramer's V = 0.44 (medium-large effect size), and more second-person pronouns, $\chi^2(1, N = 5121) = 24.25$, p = $8.4 * 10^{-7}$, Cramer's V = 0.26 (medium effect size). There were no significant differences for first-person plural pronouns.

These results related to pronoun usage add a second layer of understanding. Across prior studies, women tended to use more first-person singular pronouns (Pennebaker, 2011), but the opposite was true here. This would generally indicate that the women were of higher status in these conversations, which is possible, but would contradict one would expect based on prior studies. Men used more second-person pronouns, which does provide some indication of higher status, but these equity ratios were much closer to one than for first-person singular.

To interpret such results, I considered the pairings of students based on gender. Of the conversations that included women, 42 of them were entirely women groups, and 29 of them were mixed gender. This indicates that women were mostly talking with their women peers, which may have had an impact on status differences. Moreover, in the cases where there were mixed-gender groups, these were chosen by the students, and they were typically friends in the class. This may have also had an impact. Finally, one must also consider that peer conferences themselves could have been effective in mitigating some power differentials in the classroom, because they are highly-structured and generally take place between two students.

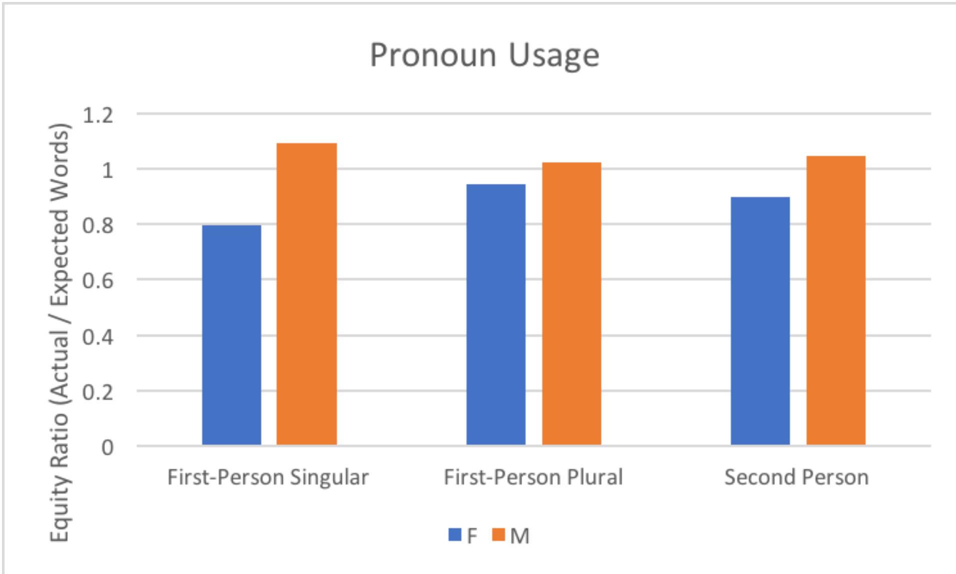


*Figure 2. Pronoun usage by gender*

**Discussion**

Promoting equity in classroom interactions is a challenge and an ongoing concern (e.g., Adiredja & Andrews-Larson, 2017). This paper offers a new method for studying such issues: analyzing pronoun usage in a peer discussion to study status differences. In future studies, this methodology could be used in conjunction with qualitative analyses to provide a deeper picture of such equity issues.

The paper also contributes a baseline understanding for gender differences in talk in undergraduate mathematics. The results are somewhat surprising, suggesting that the women in the class were actually of higher status in the peer conversations. Of course, further study is required, but this result suggests that peer conferences could be a powerful tool for promoting equity in the classroom space. They have a number of affordances that support more equitable interactions: they position students as competent, they are generally between two students, and they are highly structured. This structured nature of the activity makes it more likely that students from different groups will have an equal opportunity to contribute, rather than allowing historically dominant students to dominate. These are all issues for further study.

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Extending Prospective Secondary Teachers' Example Spaces for Functions

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*The focus of this study is on secondary school teachers' example spaces for the concept of a function. This is examined via participants' responses to a scripting task – a task in which participants are presented with the beginning of a dialogue between a teacher and students, and are asked to write a script in which this dialogue is extended. The examples for a function under certain constraints provide a lens for examining participants' concept images of a function, as well as what they perceive to be concept images of their students. These scripts are then used as a springboard for extending participants' example spaces.*

*Keywords:* function, script writing, example space

A function is a fundamental concept in mathematics. A large amount of literature in mathematics education has attended to this concept over the past 50 years (e.g., Dubinsky & Wilson, 2013). Researchers have identified conceptual difficulties associated with the concept, such as recognizing what a function is (or is not) (e.g., Breidenbach, et al., 1992, Clement, 2001) or recognizing a function in its various representations (e.g., Thompson, 1994). This study is focused on the examples of functions generated by a group of prospective secondary school teachers in an imagined instructional situation.

## Example Spaces and their Features – Theoretical Underpinning

Watson and Mason (2005) introduced the notion of example spaces, which are collections of examples that fulfill a specific function. They argued that learner generated examples (LGEs) are valuable pedagogical tools. Zazkis and Leikin (2007) extended this argument, noting that LGEs are also a valuable research tool, because the examples individuals generate provide researchers with a lens into their cognitive structures.

Watson and Mason (2005) distinguished between personal example spaces, which are triggered by a task, and collective and situated example spaces, which are local to a classroom or other group at a particular time. In a follow up study, Sinclair, Watson, Zazkis and Mason (2011) described features of personal example spaces:

> *Population*: refers to the scarcity or density of available examples.
> *Generativity*: refers to the possibility of generating new examples within the space using given examples and their associated construction tools.
> *Connectedness*: refers to whether examples are disconnected, loosely connected, or well-connected.
> *Generality*: refers to the extent to which the given example is specific or whether it is representative of a class of related examples. (pp. 301-302).

This study stems from the assumption that examples generated by the participants illustrate their concept images (Vinner, 1983) as well as features of their personal example spaces.

## The Study

Twenty prospective secondary school teachers participated in the study. They held degrees in mathematics or science, with extended coursework in mathematics (usually a minor) required for teaching certification. At the time of data collection they were in the final term of their teacher education program, enrolled in a course titled "Investigations in Mathematics". Extending teaches' knowledge of school mathematics while drawing explicit connections to disciplinary mathematics was an explicit goal of the course.

During the course the participants completed a series of scripting tasks (e.g., Zazkis & Kontorovich, 2016). In each scripting task participants are presented with a prompt, which is the beginning of a dialogue between a teacher and students. They are asked to continue the dialogue in a way that reflects how they imagine the instructional interaction may progress. In addition to writing a script (Part-A), they are asked to explain their choice of instructional approach (Part-B) and to note if their personal understanding of the mathematics involved in the task differs from what they chose to include in a conversation with students (Part-C).

Participants' responses to the 'Table of Values' scripting task comprise the data corpus for this study. The associated prompt is presented in Figure 1.

Teacher: Consider the following table of values. What function can this describe?
Alex: y = 3x
Teacher: And why do you say so?
Alex: Because you see numbers on the right are 3 times numbers on the left
Jamie: I agree with Alex, but is this the only way?
Teacher: …

| x | y |
|---|---|
| 1 | 3 |
| 2 | 6 |
| 3 | 9 |
| 4 | 12 |
| 5 | |
| 6 | |

*Figure 1: A prompt for the Table of Values scripting task*

The scripts developed by participants were analyzed with a focus on the particular examples of functions considered in the dialogues. The following research question guided the analysis: What are the participating teachers' example spaces for a function that contains the four given points? Part-C of the task is of major importance as it helped distinguish between instructional choices, in which examples can be purposefully limited, and the participants' personal example spaces triggered by the task.

## Results and Analysis

The analysis is presented by the main themes that were identified in the scripts. In designing the prompt, Jamie's question, "is this the only way" was intended to direct the script-writers to consider alternative functions. However, in several scripts the question led to alternative ways of describing the identified relationship, either by giving alternative algebraic expressions or by considering the relationship recursively. In what follows, after presenting "the other ways" to describe y=3x, we highlight several repeated features in the script-writers' example spaces.

### Different Ways of Describing y= 3x

In four scripts, exemplified in the excerpt by Charlie below, the examples of a different way to represent the y=3x function included various expressions that can be simplified to get 3x. The example spaces of these script-writers appears limited by the suggested linear relationship concluded from the table of values.

*Teacher*: Do you have another way Jamie?
*Jamie*: When I did it I came up with an equation $= x^2 + (3 - x)x$ .
*Alex*: No, the relationship is clearly linear you can't be right.

*Teacher*: Well Alex and Jamie, why don't you two find some board space with the rest of your classmates and investigate the different approaches together. First just try to confirm if substituting the values gives you what you expect.

*Alex*: I can't believe it, and I really don't understand how Jamie's equation can be right because it looks so wrong, but when we substitute the values for y we get the expected values for x.

*Teacher*: Well this is the beauty of mathematics; things can be represented different ways. I challenge you two to come up with a way to show the two equations will or will not always yield the same response.

**[…]**

*Alex*: I think I see the teachers point. Although it may be the same equation, we can represent it differently. Watch… $y = 3x$ … $x = \frac{y}{3}$

*Jamie*: Then I guess we could also just say… $0 = 3\mathrm{x} - \mathrm{y}$

*Alex*: Ohhh right I didn't even consider that!

Teacher: This is excellent work; Jamie, do you have any idea what the equation you just came up with is called?

The dialogue in Charlie's script continued to consider how to determine whether the presented relationship is linear. The commentary that Charlie provided (in Parts B and C of the task) did not mention alternative functions, rather, it focused on the value of different representations, reconfirming the teacher-character's claim that "things can be represented different ways". This leads to the conclusion that the population feature of Charlie's personal example space triggered by the prompt was limited to different representations of 3x, rather than resulted in the consideration of different functions.

**Identifying the Relationship Recursively**

In a search for an additional way to describe y=3x, three script-writers focused on describing the identified pattern recursively. An excerpt from Angel's script exemplifies this approach:

*Teacher*: What was the pattern we were seeing?

*Alex*: We just add 3 to get the next value.

*Teacher*: Perfect, so we can rewrite our y values so they show that relationship. $y_2 = y_1 + 3$ and $y_3 = y_2 + 3$. What would $y_4$ be?

*Jamie*: It should be $y_3 + 3$ right?

*Teacher*: Exactly right. And what would $y_{10}$ be?

*Alex*: $y_9 + 3$

*Teacher*: And what about $y_n$ ?

*Jamie*: $y_{n-1} + 3$ ?

In this excerpt, the identified linear relationship is explicitly linked to its recursive description, pointing to the connectedness of (this part of) Angel's example space.

**Focusing on Domains**

Eight scripts included an example of the function y=3x, in which the domain was restricted to integers or to natural numbers. In five scripts out of these eight the issue of domain was explicitly mentioned (as is demonstrated in the excerpt from Jill's script), where in others it was implied graphically, by plotting the dots, but not connecting them.

*Teacher*: You plotted the points in the table of values, totally correct. Then you connected the dots using a straight line, what is the assumption here?

*Alex*: Assumption? ……
*Teacher*: The table of values only gives you the natural numbers, 1, 2, 3, and so on.
*Alex*: Oh, I guess I assumed that all the points in between follow the same pattern.
*Jamie*: Well, I guess so too. But now that the teacher mentioned it, maybe the points in between don't have to follow the same pattern?
*Alex*: I guess so… because they are not in the table of values anyways.
*Teacher*: That's right! So what other functions can you have?
*[Alex and Jamie look at the graph and think.]*
*Alex*: Can we just have those points in the table of values?
*Jamie*: Like this?
*Alex*: Yah. It looks a little wired. But it is still a function, right?
*Jamie*: Right, because it passes the vertical test. It is a function. How do we write the equations then?
[*Alex and Jamie feel stuck here.]*
*Teacher*: What is the difference between graph 1[line] and graph 2 [only discrete points]?
*Jamie*: Graph 1 has all the x values, and graph 2 only has natural numbers.
*Teacher*: Can you describe this difference in more mathematical terms?
*Alex*: They have different domains?
*Teacher*: Right, now, can you write the domains for both functions?
*Alex*: The first one is all real numbers.
*Jamie*: The second one is all natural numbers.
*Teacher*: Exactly, when you write the equations, you need to specify domains. By restricting the domains, you have different functions.

In this excerpt we note connectedness of Jill's examples, highlighting their different attributes. We further note that student-characters consider "vertical line test" as the main identifying criterion for a function, which points to Jill's awareness of this tendency.

**Connecting the points and "covering" the real numbers**

While in the above excerpt from Jill's script the teacher confronts students' tendency to connect the points, in other scripts "connecting the points" appears to be the convention that is either supported or invited by the teacher. Taylor exemplifies this tendency:

*Teacher*: Excellent question Jamie [.… ] Why don't we start by plotting these points. And by we I mean you. [Student plots the points]
*Teacher*: Good, so how would it look if we used Alex's function?
*Jamie*: It would have a straight line through all the points.
*Teacher*: Yes, but how else can we connect these points?
*Jamie*: I suppose we could do a zig zag line.
*Teacher*: Sure, that would work. But we want this to be a function, so what rule do we need to follow?
*Jamie*: The vertical line test.
*Teacher*: Which is the easy way of remembering what?
*Jamie*: Each output can only have 1 input.
*Teacher*: Correct, so how can we connect these points then?
*Jamie*: Any way we want as long as we don't break the vertical line test.

In this excerpt "how else can we connect these points" is the teacher's question, which leads students to explore various connections, in addition to the straight line, implicitly restricting the domain to all real numbers. While connecting the points limits the population feature of example spaces, various ways of connecting the points "anyway we want" indicate generativity, as well as the generality, of Taylor's example space.

**Acknowledging a Polynomial**

A possibility of a polynomial function that can be generated from the given table of values was mentioned by three participants. It is exemplified by Logan below:

*Teacher*: Well in all of these cases we have assumed something subtle. If we filled the table of values what would we get for the remaining y entries?

*Alex*: 15 and 18

*Teacher*: Does it have to be those values? What if I put 16 and 23?

*Jamie*: … Can you do that?

*Teacher*: Why not? The points could be modeling anything! There is nothing there that says it has to be a line.

*Jamie*: Can we find an equation for that though?

*Teacher*: Certainly, but I need to talk about degrees of freedom. In our table of values we could make up 6 values of y and therefore we have 6 degrees of freedom. Simple enough?

*Jamie*: Mhmm.

*Teacher*: So we need to find a polynomial with at least 6 degrees of freedom to describe it, that is a polynomial with at least 6 terms.

*Alex*: So a $5^{th}$ order polynomial?

*Teacher*: Exactly Alex, we could find a polynomial of the form $y=ax^5+bx^4+cx^3+dx^2+ex+f$ that fits the table of values.

*Jamie*: But how can we ever assume that any patterns we see in a table of values continues?

*Teacher*: An excellent question, short answer is we don't. When we make these equations we are assuming that the trend we observe will continue. When making this assumption we need to look for reasons to explain the trend and then ask if we expect those factors to stay the same. Maybe the data was showing the population of a species but at x=5 more food is introduced or a predator is removed and the species can grow at a faster rate.

While general solutions are often considered in mathematics as more valuable than specific ones, Zazkis and Leikin (2008) noted that often general examples point to an individual's inability to generate a specific one. In this case, the possibility of a polynomial function can be seen as a generality of Logan's personal example space, while it may also point to inability to generate a specific polynomial function.

While Logan noted the existence of a polynomial function, Corey provided such a function "out of the blue" and left it for students to verify that it is consistent with the entries on the table of values.

*Jamie*: It's kind of obvious that it's y = 3x. What are we learning here?

*Alex*: I guess it's making us think outside the box a little, but yeah, our other answers are kind of lousy. […]

*Teacher*: Then let me give you an extension. Check out this function

$$y = x^4 - 10x^3 + 35x^2 - 47x + 24.$$

*Alex*: Where did you get that from?

[…]

*Alex*: But it's not a line!

*Jamie*: Who cares? It's a function. And I guess it takes going to the power of four to hit all four points.

*Teacher*: I'll leave you to it. Figure out how to derive that equation! I didn't just pull it out of thin air.

In his commentary in Part-C, Corey added that the polynomial was generated by a computer program, using matrices to solve systems of equations. He felt, however, that this material was inappropriate for his students. Corey wrote: "The level of math needed to determine the final function is beyond what I consider high school level math. After being given the function the answer can be easily revealed, but it still is not easy."

**"Shield"**

A *shield* is the term used by Koichu and Zazkis (2013) to describe a situation in which a script writer elaborates on related concepts or pedagogical strategies to avoid dealing with the mathematical core of the task. In the Table of Values task, the intended mathematical core involved responding to a student's question, "is this the only way", and in exemplifying a variety of alternative functions that result in values consistent with the table provided in the task.

However, in three scripts the student question was not addressed and no additional example for a function that fits the table of values was provided. Instead, the dialogues focused on either revisiting the notions of slope and intercept (review as a shield) or creating additional tables of values from which a unique formula was to be determined (extension as a shield). From the commentary that accompanied these scripts we conclude that these participants have not considered any examples of functions that satisfy the given table of values, beyond y=3x.

**Features of the Examples**

The scripts provided a lens on what instructional examples teachers plan to use and how they imagine students' ideas about functions. Since most of the script writers did not provide additional examples in Part-C of the task (how their self-explanation differs from what they chose to discuss with students) the examples mentioned in the scripts, either by student-characters or by the teacher-characters, reflect participants' personal example spaces. While the population feature of participants' example spaces was not extensive, generativity was a feature in scripts that involved multiple examples.

In addition, the examples clearly address the expected students' belief that a function should be described by a single formula (e.g., Vinner and Dreyfus, 1989). Script-writers either relied on their experience with students or their personal former confusion, and addressed this expected student belief via creating student characters, who believe that "other points follow the same pattern" (see excerpts from Angel, Logan and Jill above). Moreover, students' reliance on the "vertical number test" (see Wilson, 1994) was clearly present in scripts as an identifying feature for a function (see excerpts from Taylor and Jill above).

The chosen examples of participants reported herein also point to teachers' concept image of a function that was not elaborated upon in prior research: that the domain of a function is infinite and unbound. Focusing on the domain, Bubp (2016) noted that in an attempt to prove mathematical statements students often used "implicit, unwarranted assumption that the domain of the function $f$ was $\mathbb{R}$ "(p. 592) and that "a function cannot have a restricted domain" (p.593). While such assumptions were not apparent in our data, even in the examples where the domain was restricted to integers, it still included infinitely many points. No example of a finite domain or of a function on a bound interval was provided. In addition, there was a repeated tendency to consider continuous functions.

## Follow up Discussion

The scripts provide not only a valuable lens for a researcher for investigating participants' understanding of a concept, they also provide valuable information for a teacher educator for orchestrating a follow up discussion. The examples provided in the scripts constitute the collective example space of the participants. Once the scope of the collective example space is understood, there is an opportunity to extend the participants' example spaces via presenting alternative prompts or in class discussion.

An alternative prompt, in which there was no "easily determined" polynomial function, was presented to the participants (see Figure 2) after the original Table of Values task. This served as a scaffold for ideas and led to the consideration of "split domain" and multiple examples of functions defined piecewise – a family of examples that was not explicitly featured in the responses to the original prompt.

| | |
|---|---|
| Teacher: | Consider the following table of values. What function can this describe? |
| Alex: | It doesn't fit a function. |
| Teacher | And why do you say so? |
| Alex: | For 1, 2 and 3 you clearly see that this is x-squared. But for 4 it isn't. |
| Teacher: | And what about other values? |
| Janie: | I see that there is another pattern. 11= 4×3-1; 14=5×3-1, so 3x-1, same for 6. |
| T: | Interesting… |

| x | y |
|---|---|
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |
| 4 | 11 |
| 5 | 14 |
| 6 | 17 |

*Figure 2: A prompt for the New Table of Values scripting task*

The ensuing in-class conversation focused on considering potential student misconceptions related to the concept of a function as well as on further extending teachers' personal example space. For example, consistent with Clement (2001), most teachers' expressed the belief that a table of values or set of ordered pairs do not identify a function, but are derived from a function.

Furthermore, the in-class discussion focused on how to identify a polynomial non-linear function consistent with the given table of values. While teachers easily generated a function that has zeros in 1,2,3 and 4 [$f(x) = (x-1)(x-2)(x-3)(x-4)$], it was a conceptual leap to combine it with the $g(x) = 3x$ (suggested by the table of values in Fig. 1) to generate a family of polynomial functions $h(x) = kf(x) + g(x)$. Of note, for $k = 1$, $h(x)$ is simplified to $x^4 - 10x^3 + 35x^2 - 47x + 24$, which is the function that a computer program generated for Corey.

## Conclusion

It was noted in prior research that scripts generated by teachers provide a lens – for researchers and teacher educators – for examining teachers' mathematical knowledge and their instructional choices (e.g., Zazkis, Sinclair & Liljedahl, 2013). Extending this observation, the presented study demonstrates that scripts provide a lens for studying participants' personal example spaces, as well as the perceived limited example spaces of their imagined students. Most scripts attempted to confront a student idea that a function should be represented by a single formula. In addition, the chosen examples pointed to specific identifying features of teachers' example spaces of functions: unbound domain and continuity. Further research will investigate whether these features point to ideas broadly held, or were specific to the given task.

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**Katlyn's Inverse Dilemma: School Mathematics Versus Quantitative Reasoning**

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*In this report, I examine the interplay between Katlyn's (an undergraduate student's) inverse relation (and function) meanings developed through her continued school experiences and her reasoning about relationships between quantities. I first summarize the literature on students' inverse function meanings and then provide my theoretical perspective, including a description of a quantitative approach to inverse relations (and functions). I then present Katlyn's activities in a teaching experiment designed to support her in reasoning about a relation and its inverse relation as representing an invariant relationship. Although she engaged in such reasoning, her continued school mathematics experiences constrained her in reorganizing her inverse function meanings. I conclude with a discussion and areas for future research.*

Keywords: Inverse Function, Inverse Relations, Preservice Teacher Education

Researchers examining students' quantitative reasoning (Thompson, 2011) have found that students can develop foundational meanings for various concepts such as linear (Johnson, 2012) and exponential functions (Ellis, Ozgur, Kulow, Williams, & Amidon, 2012) by reasoning about relationships between quantities *before* developing more formal mathematical understandings. In contrast, examinations of students' inverse function understandings have found students often maintain disconnected inverse function meanings after they have received instruction (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Vidakovic, 1996). I conjectured quantitative reasoning could potentially support undergraduate students relating and connecting their inverse function meanings developed through their school experiences. Working with a pre-service teacher, Katlyn, who had K-14 school experiences with inverse function, I investigated how she could potentially re-construct her inverse function meanings via her reasoning quantitatively. In this report, I present Katlyn's progress in a semester-long teaching experiment intended to investigate the question: How does a student's quantitative reasoning interplay with her inverse function meanings developed through her continued school mathematics experiences?

## Research on Inverse Function

Vidakovic (1996) proposed that students develop inverse function schemas in the following order: (a) function, (b) composition of functions, then (c) inverse function through a coordination of (a) and (b). Whether implicitly or explicitly, many researchers (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Vidakovic, 1996) examining students' inverse function meanings have emphasized composition of functions and the formal mathematical definition (i.e. $f$ and $f^{-1}$ are inverse functions if $f(f^{-1}(x)) = f^{-1}(f(x)) = x$) as paramount to students developing productive inverse function meanings. However, these and other researchers have found students often maintain disconnected (from the researcher's perspective) inverse function meanings, often related to executing certain activity in analytic rule or graphing representations (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Paoletti, Stevens, Hobson, LaForest, & Moore, 2015). For instance, students often use a "switching-and-solving" technique when determining the inverse function of a given function represented by an analytic rule (i.e., given $y = x - 2$ they switch the variables and solve for $y$ to obtain $y = x + 2$) but are experience difficulties interpreting the results of this activity for a contextualized analytic rule. The extent to which students relate their switching-and-solving technique or their other activities to function composition is an open question. The current body of research indicates that current approaches to teaching inverse function have been ineffective in supporting students in developing interrelated inverse function meanings. In this report, along with Paoletti (2015), which I elaborate on below, I begin to address the evident need to re-conceptualize ways to support students developing sophisticated inverse function and inverse relation meanings.

## Theoretical Framing

I examined the possibility of supporting students developing inverse relation (and function) meanings via their reasoning about relationships between quantities. A *quantity* is a conceptual entity an individual constructs as a measurable attribute of an object or phenomena (Thompson, 1994, 2011). An individual constructs *quantitative relationships* as she associates two varying (or non-varying) quantities (Johnson, 2012; Thompson, 1994). As an individual constructs and analyzes these relationships, she engages in *quantitative reasoning* (Thompson, 1994).

Specific to inverse relations, I conjectured if a student constructed a (non-causal) quantitative relationship between two quantities (e.g., quantities A and B), then she could decide to consider one quantity as the input of a relation (e.g. B input, A output) whilst anticipating the other quantity would be the input of the inverse relation (e.g., A input, B output). By focusing on the underlying quantitative relationship, the 'function-ness' of a relation and its inverse falls to the background; a student can describe and represent a relation and its inverse without (necessarily) being concerned if either represents a function. Further, the student maintains an understanding that choosing input-output quantities does not influence the underlying relationship that the associated relations or functions describe.

I conjectured a student maintaining such understandings can interpret a single analytic rule or graph as simultaneously representing a relation and its inverse relation. With respect to graphing, the student anticipates that either axis can represent the input quantity of a relation. Although this reasoning may seem insignificant, Moore, Silverman, Paoletti, & LaForest (2014) illustrated that students' graphing meanings are often restricted to reasoning about the input quantity exclusively represented on the horizontal axis.

In Paoletti (2015), I demonstrated the feasibility of this way of thinking by presenting one undergraduate student's (Arya's) activities as she reorganized her inverse function meanings compatible with this description. When addressing inverse function tasks in a pre-interview Arya relied on switching techniques (e.g., switching-and-solving) and understood a function and its inverse represented different relationships. Throughout the teaching experiment Arya experienced several prolonged perturbations. Resolving these perturbations supported her in reorganizing her inverse function meanings as well as her meanings for variables and graphs. Specifically, at the conclusion of the study, Arya understood that a relation and its inverse relation, regardless of 'function-ness', represented an invariant relationship. Keeping this invariant relationship in mind, Arya understood a single graph could be interpreted as either a relation or its inverse by choosing either quantity represented on either axis as a relations input. In context, she made sense of the switching-and-solving procedure by changing the quantitative referent of each variable when switching variables (i.e. if $V$ represented volume and $s$ represented side length in the original analytic rule then $V$ represented side length and $s$ represented volume in the inverse analytic rule). Arya's meanings at the conclusion of the study demonstrate both the viability of the ways of thinking described above and one way students can relate their quantitative reasoning to the switching-and-solving procedure.

## Methods and Task Design

I conducted a semester-long teaching experiment (Steffe & Thompson, 2000) with Katlyn and Arya (pseudonyms), two undergraduate students enrolled in a secondary mathematics teacher education program. The students were juniors who had successfully completed at least two courses beyond a calculus sequence. I engaged the students in three individual semi-structured clinical interviews (Clement, 2000) and 15 paired teaching episodes. Clinical interviews and teaching episodes provided flexibility to create and adapt tasks to explore how students might develop meanings compatible with those I described. Specifically, I used clinical interviews as one pre and two post interviews to develop models of Katlyn's mathematics (Steffe & Thompson, 2000), including her quantitative reasoning and her inverse function meanings, without intending to create shifts in her meanings. I used teaching episodes to examine the viability of my hypothesized models and to pose tasks I conjectured might create perturbations for Katlyn, possibly leading her to make accommodations to her meanings.

I analyzed the data using open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998) in combination with conceptual analysis (Thompson, 2008). I developed and refined models of Katlyn's mathematics by initially analyzing the videos identifying episodes of Katlyn's activity that provided insights into her meanings. These instances supported my generating tentative models of her mathematics that I tested by searching for corroborating or refuting activity. When Katlyn exhibited novel activity, I adjusted my models to explain this activity including the possibility that this activity indicated fundamental shifts in her meanings. Through this iterative process of creating and adjusting hypotheses of Katlyn's mathematics, I was able both to characterize her thinking at a specific time and to explain shifts in Katlyn's meanings throughout the teaching experiment.

I first raised the notion of inverse function in the *Graphing Sine/Arcsine Task* (Figure 1). The research team designed this task to support students in developing inverse relation meanings compatible with those described above. The first two prompts ask students to create graphs of the sine (Graph 1) and arcsine, or inverse sine, (Graph 2) functions. The third prompt asks the students to consider how they could interpret Graph 1 as representing the arcsine function. This prompt also asks the students to consider if Graphs 1 and 2 represent "the same relationship." I conjectured asking the students to foreground the "relationship" represented by the graphs might support them in conceiving either quantity, on either axis, could represent the input of a relation in order to conceive Graph 1 as representing both the sine and arcsine functions or relations.

| | |
|---|---|
| Graph 1: | Create a graph of the sine function with a domain of all real numbers. What is the range? |
| Graph 2: | Using **covariation talk**, create and justify a graph of the arcsine (or inverse sine) function. |
| Prompt 3: | Can you alter (**do not draw a new graph**) Graph 1 such that it represents the graph of the arcsine function? Does this graph convey the same relationship as the second graph? How so or how not? |

*Figure 1*. The Graphing Sine/Arcsine Task.

## Results

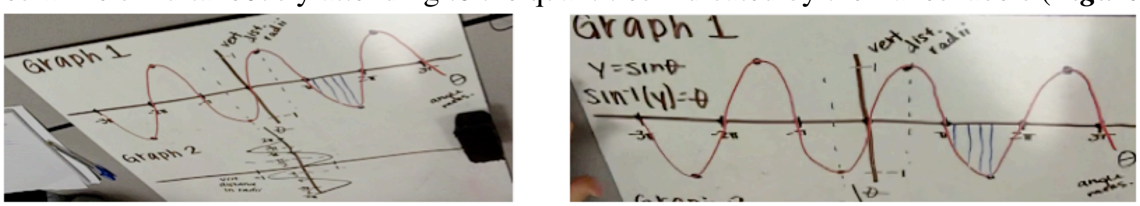
I first present analysis from the initial clinical interview that provides insights into Katlyn's inverse function meanings relevant to this report. I then present her activity addressing the prompts in the Graphing Sine/Arcsine Task. I conclude with Katlyn's activity in the final clinical interview, highlighting the interplay between her quantitative reasoning and her inverse function understandings developed through her continued school mathematics experiences.

### Results from the Initial Clinical Interview

During the initial interview Katlyn's predominate meaning for inverse functions involved "switching." For example, given the equation $C(F) = (5/9)(F - 32)$ defining the relationship between degrees Celsius and degrees Fahrenheit, Katlyn switched $C$ and $F$ and solved for $C$ determining the inverse rule $C^{-1}(F) = (9/5)F + 32$. Given a line representing the relationship between temperature measures, Katlyn estimated values of several coordinate points then switched abscissa and ordinate values to determine points on a line she drew to represent the inverse function. In both cases, Katlyn was uncertain how to interpret the results of her activity in relation to temperature measures indicating she did not attend to the underlying quantities when engaging in these techniques. For example, Katlyn identified that the point (10, 50) on the given line represented that 10 degrees Fahrenheit corresponds to 50 degrees Celsius but when interpreting the point (50, 10) on her constructed line Katlyn said, "My whole reasoning in this entire process… is switching $x$ and $y$, is switching $C$ and $F$ which is how I came up with this graph. So I don't necessarily know what... the new graph would stand for." Katlyn did not assign any meaning in relation to temperature measures to the point (50, 10) on her constructed graph representing the inverse function.

**Reasoning about the Sine and Arcsine Relationships**

In the first four teaching episodes the students represented the relationship between angle measure and vertical distance above the horizontal diameter in a circular motion context and understood this relationship was defined by the sine function, compatible with the descriptions of Moore (2014). After these episodes, I prompted the students with the Graphing Sine/Arcsine Task. They carefully attended to the quantitative relationship between angle measure and vertical distance as they created Graph 1, then constructed Graph 2 by switching abscissa and ordinate values while simultaneously attending to the quantities indicated by their axes labels (***Figure 2***a).



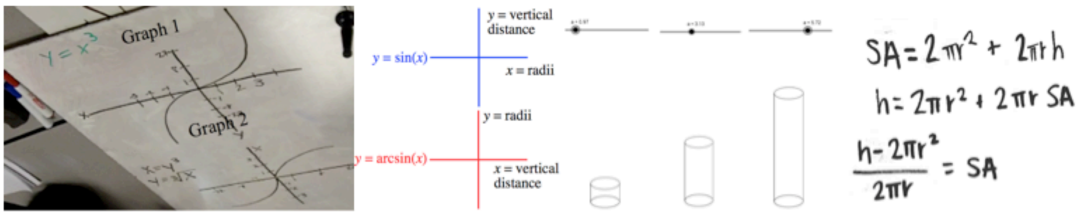

(a) (b)

*Figure 2*. The pair's (a) Graph 1 and Graph 2 and (b) Graph 1 with added equations.

Having drawn both graphs, the pair set out to address Prompt 3. Katlyn wrote $y = \sin(\theta)$ next to Graph 1, indicating this was the equation they initially represented with Graph 1. She then added $\sin^{-1}(y) = \theta$ below $y = \sin(\theta)$ (see added labels in ***Figure 2***b). Katlyn anticipated considering vertical distance as her input, represented on Graph 1's vertical axis, stating, "We're looking at the $y$ [*pointing to y in* $\sin^{-1}(y) = \theta$], so we go to one [*motioning to 1 on the vertical axis*] and then we're like okay well… which angle's sine is one?" Katlyn motioned horizontally to the three points on Graph 1 with a vertical distance value of one. Continuing to explain her reasoning, Katlyn said, "If we're switching the input and output… So we want theta to be our answer, 'cause then originally theta was our input but now we want it to be our output." As in the initial clinical interview, Katlyn referred to "switching" but in this episode she maintained her focus on the invariant relationship between vertical distance and angle measure as she considered how to interpret Graph 1 as representing a relation with vertical distance as the input quantity. Katlyn reasoned quantitatively to consider a relation and its inverse relation as representing an invariant relationship but with different chosen input and output quantities.

**Considering a Decontextualized then Contextualized Relationship**

Because the students never referenced switching-and-solving when working with the sine and arcsine relations, two teaching episodes later, I asked the pair to address the prompts in Figure 1 for a decontextualized function ($y = x^3$) to investigate if, and if so how, their activity would be different for a decontextualized function. The students drew Graphs 1 and 2 (see Figure 3a) by maintaining the relationship between $x$ and $y$ (e.g., they described that for $x > 0$ with $x$ represented on the horizontal and vertical axis in Graph 1 and Graph 2, respectively, $y$ increased at an increasing rate with respect to $x$). However, the students experienced a perturbation as this graph, which they understood was defined by $x = y^{1/3}$, was not defined by the analytic rule, $y = x^{1/3}$, they had determined by switching-and-solving. This perturbation led the students to question their prior activity with the sine and arcsine relations in which they did not switch-and-solve.




(a) (b) (c) (d)

*Figure 3*. (a) The pairs decontextualized graphs, (b) the color-coded axes with Katlyn's added labels, (c) the cylinder animation, and (d) Katlyn's work.

Intending to maintain the students' focus on quantitative relationships, I contextualized this function as representing the volume and side length of a cube ($V = s^3$) as I conjectured they would not switch the variables to represent the inverse rule in a context. I asked the pair what the inverse rule would be and Katlyn immediately responded "cube root of $V$ equals $s$." I repeated, "cube root of $V$ equals $s$," to which Katlyn refuted, "No, but that's not right." Katlyn experienced a perturbation as she oscillated between her switching technique and maintaining the relationship between volume and side length while maintaining the quantitative referent of each variable.

I asked Katlyn if she knew why she switched variables and she responded, "No, I just remember doing that, that's just our definition… you like switched $x$ and $y$ and solved for $y$ again because in standard position $y$ is [on the vertical axis] and $x$ is [on the horizontal axis]." I considered her argument of "standard position" of $x$ and $y$ as a way to support Katlyn in relating her switching technique and maintaining the underlying quantitative relationship. Drawing attention to the possibility of using variables to arbitrarily represent quantities values, I wrote $y = \sin(x)$ in blue and $y = \arcsin(x)$ in red along with Cartesian coordinate axes next to each (Figure 3b). For each graph, I asked Katlyn to identify the variable and quantity each axis would represent if she were going to graph each rule. Responding to this, Katlyn used the variables $x$ and $y$ arbitrarily to define angle measure and vertical distance to represent the input with the variable $x$ on the horizontal axis in each graph (see black labels in Figure 3b).

After this Katlyn described her reasoning about the inverse function in the side length-volume context, arguing, "We've just been saying like we need to switch them in the equation [*pointing to* $y = x^{1/3}$] but like, we're like switching them in real life." Katlyn then reasoned she had to reassign the quantitative referents of the variables when switching-and-solving (i.e. $s$ represented volume and $V$ represented side length in $V = s^{1/3}$). In the moment, Katlyn understood that in a contextualized situation (e.g., sine and arcsine, volume and side length) a relation and its inverse represented the same quantitative relationship but with different input quantities; she reassigned the quantitative referents of each variable when switching in order to maintain this relationship.

**Results from the Final Clinical Interview**

Based on the described teaching sessions, which spanned two weeks, I conjectured Katlyn potentially reorganized her meanings such that she understood a relation and its inverse represented an invariant relationship with the difference being which quantity she chose to represent the input. I intended to test this conjecture in an interview two months after the last teaching episode addressing inverse relations. I showed Katlyn an applet displaying a cylinder with varying height and a constant radius (Figure 3c) and asked her to determine a relationship between the cylinder's surface area and height. Reasoning quantitatively, Katlyn described imagining the net of the cylinder composed of two circles with constant area and a rectangle with varying area (i.e. $h$ varies and $r$ is constant) and determined the analytic rule $SA = 2\pi r^2 + 2\pi rh$. She drew a linear graph and described the relationship stating, "As like the height is increasing, surface area is also increasing." Conjecturing Katlyn was capable of considering surface area as the input, I asked, "Is there another way to read [the graph]?" Katlyn responded, "As surface area increases, height increases.... whatever happens to one is like happening to the other one." Although Katlyn chose to consider height first, she anticipated this was only one of the options; from my perspective Katlyn reasoned about a relation and its inverse relation as she anticipated coordinating either quantity varying first.

I asked Katlyn to determine the inverse analytic rule conjecturing she would maintain the relationship she had described. However, Katlyn switched-and-solved (Figure 3d). I asked Katlyn to "talk me through what you did there", and she responded:

*Katlyn:* It's funny that you say that 'cause I'm tutoring two girls and we were doing inverses yesterday. And I don't, and I still can't explain why we do this. I was trying to think of a way to explain it to them, and I didn't know the answer. Um [*pause*]. Because that's what I've been told to do for six years…

*TP:* Okay. So you said you were just tutoring someone on this?

*Katlyn:* Yeah, and… they were just like, 'well how do I do it?' And so I told them, like you have to make sure the… function is one-to-one so like for every… input there's only one output and for every output there is only one input. All that nonsense that doesn't, I don't really know why we do that. But that's what has to happen before you can switch your input and output and then solve. So, why do we do this? I don't know. But I know this is what the answer is and I. Yeah, I don't know.

*TP:* Okay and so this is the answer [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]?

*Katlyn:* Yes. Yeah, yeah.

*TP:* But, you're sort of also acting like there's something you're not comfortable with about it.

*Katlyn:* I just don't know what it means, like I don't, why do I care about this [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]?

*TP:* So say a little bit more what do you mean you don't know what this [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$] means?

*Katlyn:* I don't know what it means. I know [$SA = (h - 2\pi r^2)/(2\pi r)$] is the inverse, for surface area of a cylinder. That is all I know. Why is it the surface area? What does it, what does the inverse for surface area mean? I guess I'm thinking like. [*pause*] Okay, it reminds me of that time that we were doing like volume of a cube being like side-squared and then we switched the two and then I was like, okay so now, $s$ means volume and $V$ means side[length]. So now does here, [*pause*] surface area mean height and height mean surface area? Or did we just not finish the problem in class to conclude about what, I don't, I don't remember. I have no idea why we do this.

*TP:* So, you're starting to say here [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]. If, if $SA$… represented height, and $h$ represented surface area?

*Katlyn:* Well, it wouldn't make any sense. Because then it would just be the same. Like if you multiplied [$SA = (h - 2\pi r^2)/(2\pi r)$] all back out you would get [$SA = 2\pi r^2 + 2\pi rh$], I guess. And so like I'm attributing [$SA = (h - 2\pi r^2)/(2\pi r)$] to be the same thing where this is now height [*pointing to SA in* $SA = (h - 2\pi r^2)/(2\pi r)$] and this is now surface area [*pointing to h in* $SA = (h - 2\pi r^2)/(2\pi r)$]. That doesn't make any sense. We might as well have kept it that way [*indicating* $SA = 2\pi r^2 + 2\pi rh$]. [*pause*] That's probably not right then cause it has to mean, it has to mean something different.

From my perspective Katlyn described the relation and its inverse prior to the term "inverse" being raised but reverted to switching-and-solving when asked about the "inverse". She engaged in this technique, which she learned as a student and was reinforced as a tutor, despite her being reflectively aware that she did not know *why* she engaged in this activity (e.g., "So, why do we do this? I don't know") or *how to interpret* the activity's results (e.g., "*I just don't know what it means*… why do I care about this"). Katlyn recalled the volume-side length situation from months earlier and considered switching the quantitative referent of each variable. However, she rejected this as the inverse rule would represent the same relationship as the original rule leading her to conclude a function and its inverse function must represent different relationships (e.g., "That doesn't make any sense. We might as well have kept it that way").

## Discussion and Concluding Remarks

Katlyn's story exhibits difficulties students may encounter when attempting to reason about relationships between quantities by leveraging their non-quantitative mathematical meanings. Compatible with Arya (Paoletti, 2015), Katlyn reorganized several of her meanings during the teaching experiment (i.e., using variables arbitrarily to represent quantities). However, these reorganized meanings did not lead to shifts in her inverse function meanings. One possible explanation is that Katlyn engaged in in-the-moment activity (potentially both in the study and in her tutoring) to assuage a perturbation without reflecting on if her activity was related to other contexts or situations.

Despite not reorganizing her meanings, Katlyn's inverse function meanings at the end of the study were not significantly different than other students' meanings researchers have characterized (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Paoletti, Stevens, Hobson, LaForest, & Moore, 2015; Vidakovic, 1996). Thompson, Phillip, Thompson & Boyd (1994) distinguished between teachers maintaining calculational and conceptual orientations, noting the latter "focus students' attention away from thoughtless application of procedures and toward a rich conception of situations, ideas and relationships among ideas" (p. 86). If a teacher maintains inverse function meanings similar to Katlyn's, she will be unable to support her students in developing a rich conception of relationships among ideas and instead will have to focus on a thoughtless application of the switching-and-solving technique (i.e. "I was trying to think of a way to explain it to them, and I didn't know the answer…"). Hence, future researchers should continue to address calls (Thompson, Phillip, Thompson & Boyd, 1994; Thompson, 2008) for increased focus on ways of reasoning that support future teachers development of rich conceptions of ideas and relationships among ideas that they can call on in their teaching.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# Planning to Succeed in a Computer-Centered Mathematics Classroom

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*Most developmental mathematics students in community colleges, particularly those of color, are unsuccessful and fail to reap the benefits of higher education. In-class computer-centered (ICCC) classes are a possible solution to this issue because students work independently at computers during class time while instructors facilitate learning by answering students' questions. This case study focuses on one student's ICCC classroom experience by focusing on how the student's plan to pass the course were validated by the classroom environment. Ultimately, that plan was insufficient to address the needs of the student.*

*Keywords:* Developmental mathematics, computer-centered learning, student agency

Many high school graduates who want a postsecondary education turn to community colleges to further their academic development or acquire credentials and job training leading to hitherto inaccessible opportunities. Yet their mathematical background obstructs this path when they place into developmental, not college-level, mathematics courses. Mathematics departments have introduced in-class computer-centered (ICCC) classes to support these students, providing a flexible environment to help struggling students proceed at their own pace, meeting their academic and personal needs. Whether or how students use this flexibility is not yet clear.

Research examining the ICCC class compares achievement outcomes and result vary. What has not yet been fully explored is the set of actions students take to learn mathematics in this setting. Plans and actions are key to student learning – and in a computer environment in which students have significant independence – it is important to consider the role of agency in students' engagement in the course. In this study, I focus on the plans students make to achieve their goals, how the course structure validates a plan, and what interferes with a plan's execution. Ultimately, this study sheds light on how a classroom setting designed to be flexible around the needs of as-yet-unsuccessful students supports or hinders their mathematics learning.

## Addressing Remediation in the Community College

Many students do not complete postsecondary degrees because they are not successful in required developmental courses. Of students who entered a public two-year postsecondary institution in 2003 – 2004, 68% took at least one developmental course, with mathematics being the most common (Chen, 2016). Furthermore, passing rates in developmental classes are low and generally disfavor students of color (Bahr, 2008, 2010). Students who need support in such courses are not an anomaly; those who succeed in them are. Furthermore, forty percent of students enrolled in postsecondary institutions and who required remediation did not complete a Certificate, Associates, or Bachelor's within six years (Green & Radwin, 2012). Enrolling in developmental courses reduces the likelihood that a student will achieve their academic goal.

Computer-centered instruction has grown in the United States and is being utilized in many postsecondary institutions to help students improve their mathematics proficiency (Allen & Seaman, 2011). These courses combine student individualization and flexibility with instructor support. In ICCC classes, at least 80% of the content is delivered via stand-alone software during scheduled class time using prerecorded video lectures. Students work online answering questions to test their learning, advancing at their own pace. They can also work outside scheduled class times. Instructors monitor student progress, offering assistance by answering questions. The individualization and autonomy offered by these courses gives students opportunities for agency.

Overall few studies on ICCC mathematics classes focus on the student experience (Webel, Krupa, & McManus, 2016). However, there is no shortage of quantitative studies on computer-centered classes which compare student achievement in a computerized class with another format, usually lecture-based, instructor-centered classes. Results vary as to whether ICCC mathematics classes have higher achievement rates than traditional lecture courses (Bishop, 2010; Carrejo & Robertson, 2011; Herron, Gandy, Ningjun, & Syed, 2012). Many of these evaluative studies do not provide a pedagogical rationale for incorporating technology into the classroom or consider the pedagogical differences between computerized learning and traditional courses (Tallent-Runnels et al., 2006). Such studies for example do not differentiate the role the computer plays over that of a traditional instructor.

One way to theorize the relationship between technology and learning is to consider the student experience rather than achievement since these courses provide options like the flexibility to work at convenient hours and the ability to rewatch lectures, options heretofore nonexistent in traditional classes. This research is needed since this course structure harkens to Earlwanger's (1973) Benny (Webel, Krupa, & McManus, 2015). Recent research in this area showed that computer-centered courses better serve students who do not require significant remediation and improves students' ability to answer familiar problem sets (Webel et al., 2016). Short, focused computerized interventions have also been successful (Li & Ma, 2010; Wladis, Offenholley, & George, 2014) Students in ICCC classes also felt that course requirements necessitated a significant time commitment (Ariovich & Walker, 2014), supporting other findings that time on task related to successful completion (Fay, 2017). Overall, more research is needed to better understand the student experience in this realm.

## Student Agency and ICCC

In ICCC courses, students can choose from a variety of learning activities. This freedom provides students with extensive agency, defined as the set of actions students take to achieve their goal. For example, students have the flexibility to work at their own pace in and out of class (Aichele, Francisco, Utley, & Wescoatt, 2011; McClendon & McArdle, 2002; Vassiliou, 2012; Xu, Meyer, & Morgan, 2009). For example, a single parent can make up for days missed or can work ahead or at home to account for unexpected absences to care for their child.

The agency available in the ICCC classroom is not infinite. The software's didactic approach limits mathematical agency, the ability to develop mathematical conjectures or explore mathematical concepts, by only accepting specific answers or methods. Thus, while students have significant student agency by being able to study when, where, and as much as they want, WHAT they must do to succeed and HOW they demonstrate knowledge is narrowly defined. This paradox of limited mathematical agency, and unlimited student agency provides a tension likely to yield findings on student actions.

In this study, I sought to understand a part of student agency in an ICCC mathematics classroom. More specifically, this study focuses on the intention a student sets, a future goal, action or purposeful outcome, and the basic plan for achieving said goal (Bandura, 2001, 2006, 2008). Without intention to establish purpose, a person's actions could not be considered agentive since they cannot be distinguished from unintended outcomes. Intention is one of four characteristics of agency which Bandura places in the realm of Social Cognitive Theory (Bandura, 1986). According to Bandura, a person's agentive acts are a part of his or her behavior and both affects and is affected by environmental and personal factors.

## Methods

The data presented here come from a larger body of work researching four cases of student agency in an ICCC developmental mathematics classroom. This study reports on only one case, Eduardo, a 24-year-old Hispanic student entering college for the first time. Three research questions were considered: (1) What are the student's intentions in an ICCC developmental mathematics class, (2) what portions of the course structure validate this plan, and (3) what challenges does the student encounter when attempting to fulfill his intention? These questions consider Eduardo's intention while taking into account social-cognitive factors that may contribute to success. The site of this study was a developmental-level mathematics class in a community college in the Southwestern United States, a designated Hispanic Serving Institution where at least 25% of the student body is of Hispanic origin. In the ICCC course, students move from developmental coursework to college-level content using Pearson's MyMathLab software to complete modules, similar to a chapter in a mathematics textbook. Students demonstrate mastery of a module by answering questions on assignments and exams. They are expected to complete twelve modules per semester, completing the course in approximately three semesters.

Over the course of one academic semester, I collected four main sources of data. The first was Eduardo's classroom activity to understand his actions with respect to his intentions. This included over eleven hours (nine classes) of video recordings covering what Eduardo did in class, corresponding recordings of his computer screen, interactions with his instructor, photographs of his written work, and supplemental field notes. Two interviews with Eduardo comprised the second and third sources. They addressed Eduardo's mathematics background, his study habits, his intentions in the class, and clarifying questions to understand his actions. Interviews with the instructor, Shaun, was the fourth source of data and asked about his philosophy when teaching this course and discussed Eduardo's progress.

Eduardo's intentions were identified and coded using template analysis (Ray, 2009). Actions were intentive if a second datum (action or utterance) supported such. In other words, a second source of data must support the determination that a given act had intention. Actions and utterances were also coded based on Bandura's remaining characteristics of agency, forethought, reflection, and reaction. Codes were sorted and counted and explored for code co-occurrences. A second round of descriptive open coding allowed other themes to emerge.

I wrote analytic memos to make sense of the data as they were coded. Analytic memos clarified my reflections on the coding, overall inquiry process, and emergent patterns and themes (Saldaña, 2009). These analytic memos asked and addressed questions of the data. An example of such a question is "What did the participant do after answering a question incorrectly?" Answering these questions helped sort through the data so themes could emerge. These memos were shared with peers to check analysis and findings.

Since multiple types of data were recorded, findings and interpretations were triangulated. Data were collected in multiple class sessions giving long-term and repeated observations that allowed for the development of accurate findings.

## Findings

In ICCC classes, the plans students develop to achieve a specific goal may be validated by the classroom environment. However, these assumptions may have fundamental flaws which could adversely affect whether the goal, passing the class, is achieved. Eduardo is an example of such a case. Eduardo's plan for success centered around working outside of class and relying on the computer, rather than his instructor, to learn mathematics. This plan was based on assumptions that were insufficient because of Eduardo's weak mathematical skills.

**Eduardo's Intentions**

Mathematics courses were required for Eduardo to receive a degree in Business Administration and Management. Results of his placement exam placed him in developmental mathematics, MAT075. Eduardo had a specific goal for massing MAT075. "I'll get my 12 modules done in 5 months" (20:170). Eduardo planned to achieve this goal by working extensively on the course material outside of class and watching videos repeatedly. When Eduardo was asked, "What's your game plan for getting through your 12 modules?" He replied, "Definitely doing 'Homework'. As far as doing that outside of the classroom. Definitely doing that." (20:175). Eduardo intended to and was certain that he could get an A in the course by working outside of class. "I think I'm gonna be doing this more out of class because you get better in math if you practice and practice and practice and I don't think like an hour and a half is much time you know to finally get it" (22:1). Eduardo recognized how important it was to work on mathematics class outside of his class. He also had a computer and high-speed internet at home giving him the ability to work at home at his discretion. The other main component of Eduardo's plan was his decision to extensively use video lectures, which he favored over conventional class lectures. This was demonstrated several times, where Eduardo would replay videos or assert his intention to rewatch videos.

> I'm gonna go over this [video] again at home and then I'll do the concept check. . . . then I'll do it again [watch the video] like probably two more times until [I] master this small piece and then move on to the next one. (5:1)

Eduardo began his next class reviewing his previous work. "I started with the first page [of the corresponding text] so I could refresh my memory on it because I wanted to do that. I want to learn it" (20:75). By reviewing, Eduardo reaffirmed the importance of repetition.

**Factors supporting Eduardo's plan**

Eduardo's plan to pass MATH075 was validated through the design of the software and course structure. These factors supported Eduardo's plan to spend adequate time working through the course and consistently review the content.

The course encouraged students to work as often as possible, placing a stronger emphasis on seat time rather than conceptual understanding. The classroom was available for over 40 hours per week, and was designed so that computers were available to students who were not scheduled to attend, so students could feel free to come in when their schedule allowed. The online nature of the course also allowed students to work outside of class whenever they wanted. Students were also able to work ahead one module, encouraging them to keep working.

Shaun, Eduardo's instructor, expressed how critical it was to maximize seat time and reinforced the importance of working as often as possible. If they were not discussing procedural questions related to the course structure, Shaun and Eduardo's interactions were centered around the idea that spending time outside of class was essential to passing the course.

> [You] might think about what your time is like and can you be in here outside of class. Is there time between classes? Is there time after classes where you don't have to be somewhere right away or before? Can you come in early, you know? You think about your own personal circumstances and see if there's more time that you can squeeze. Any time you can be here, you're welcome here, right. (23:20)

Shaun attempted to help Eduardo with his time-management skills and help Eduardo see multiple opportunities during the day where Eduardo could work on the class.

Eduardo reflected on how the course structure allowed repetition and on the importance of replaying videos and to help him learn.

> I love the structure of it. I think this works out better for me because I could keep - go back and back and back, you know. Reread the video or replay the video over and over again. And sometimes, like, well, the way I learn, you have to, like, tell me a lot of times for me to, like, learn something new until I really get it. So I love it. (5:4)

The feature of the MyMathLab software that he used most often, the videos, was a feature that aligned with his belief in how he learned best. With the course being on computer, Eduardo had complete control over his learning the material, answering questions, and his ability to rewatch videos so he could advance at a comfortable pace.

Eduardo also believed the software was fully contained, in that all answers to his questions could be found in the video lectures or another part of the software. When I asked what he would do if he had a question about the content, Eduardo was very direct. "The way he [the narrator] explains it there is no questions; well at least for me. You just have to read it. I mean he explains everything. If I did [have a question], maybe I missed it when he was talking (30:3)". Eduardo believed that the computer was the source to be trusted and if he was unclear about a specific concept, it was his fault. This assumption supported his belief that by spending more time engaged in the software and by reviewing material, he could pass a module.

The assumption that MyMathLab was designed to be a fully contained program was supported by the software. The program had no surprises in that questions presented to students on exams are of the exact form given in "Homework" assignments. There are no advanced, conceptual questions or questions in forms students have not seen before. This allowed Eduardo to work through challenges and answer questions on his own.

Shaun, throughout his discussions with Eduardo, supported Eduardo's belief that seat time was essential for success in MATH075. The course structure and environment also emphasized seat time over understanding. Thus, there was no indication that Eduardo's plan was not reasonable. However, relying exclusively on the computer's features to review and answer his questions and focusing on the time spent in front of a computer did not meet his academic needs.

**Intention Thwarted: Eduardo's Plan Did Not Work**

Eduardo's plan to pass the course did not work. Before the middle of the semester, Eduardo had stopped attending the course and ultimately failed the class. Eduardo's plan assumed that his arithmetic skills were sufficient to succeed and that he only needed the software to be successful.

Like all students new to MATH075, Eduardo began with Module one, which introduced whole numbers, rounding, the arithmetic operations, and orders of operations with whole numbers. Shaun strongly encouraged all new students to pretest this module, taking the Module 1 test without working through the content, saving the time of working through "Homework" problems that students could presumably do. Eduardo chose to work through the module.

Shaun's attempts to have Eduardo finish this module demonstrated the extent to which Shaun considered this material rudimentary. Shaun tried to encourage Eduardo to come into class ready to test Module one. "Do you think you can do the topics, finish the 'Homework', over the weekend, and test on Monday?" (33:6). In this interaction, Eduardo was expected to complete units 1.7 – 1.11 so he could test. Shaun's tone with Eduardo was more imperative than curious, attempting to motivate Eduardo rather than inquire if doing that much work was possible. This statement implied that Eduardo could complete these modules if he put in the time to work through the questions. There was no consideration as to the academic challenges these units may have posed for Eduardo. Unfortunately, Eduardo had significant difficulty with these units.

In an example of how weak his arithmetic skills were, Eduardo was asked to solve the division problem $7\overline{)469}$. Eduardo relied heavily on the calculator and the video lectures to help him answer this question. He did not know the mechanics of dividing a three-digit number by a single-digit number until he watched the video. Eduardo relied heavily on his calculator to assist him through the intermediate steps. When the computer indicated that he had the wrong answer, Eduardo replayed the video on division and followed his extensive notes on how to perform long division. This single problem, including re-watching the video, took over thirty-five minutes to complete. The time and effort demonstrated here stand as a testament of Eduardo's dedication and resiliency, and as an indication of the extent of his mathematical deficiencies and the amount of effort necessary to overcome them. It was also an indication as to how challenging this course could be for someone with Eduardo's level of content knowledge when they relied exclusively on the computer software to advance.

Eduardo's progress through this division problem shows the extent to which he had difficulty with and needed mathematical support. His assumptions that the program was self-contained was demonstrated when Eduardo did not seek Shaun's help on solving the problem, even when Shaun interrupted Eduardo to discuss his progress while Eduardo was working on said problem. Furthermore, Shaun's classroom statements that the first module should be skipped along with his focus on how little time Eduardo was spending working at home may have prevented Shaun from recognizing Eduardo's challenges and intervening to help him.

The idea that the software was self-contained, that all questions could be answered in some way using MyMathLab, was incomplete. Although all questions could be answered based on definitions, examples, or lectures in the software, students were expected to have a certain amount of prerequisite knowledge, namely a command of addition, subtraction, multiplication, and division facts up to thirteen. This basic knowledge could have helped Eduardo with the aforementioned division problem and other problems in this module. In addition, without this basic knowledge, Eduardo could not pass the exam which did not allow use of a calculator.

MyMathLab was able to support Eduardo in working through the problem by allowing him to watch online videos as often as necessary to understand concepts. However, it did not address that he did not understand how division is a grouping operation or recognize that he did not know his basic multiplication / division facts. Eduardo may have successfully completed one problem, but this did not ensure he could do similar problems without the same extensive support from the computer and calculator. Although the features in MyMathLab helped Eduardo work independently through confusing questions, because of Eduardo's weak arithmetic skills, the amount of time it would have taken him to work through the course would have been prohibitive.

Eduardo and Shaun's focus on overall time spent working may have deflected attention in the wrong direction. Shaun was consistently focused on whether Eduardo worked outside of class and did not realize the extent to which Eduardo was having difficulty with the content. Whenever Shaun initiated a conversation with Eduardo, it was always about Eduardo's pace or progress and Eduardo would indicate he was not working outside of class. It would then be reasonable for Shaun focus on Eduardo's pacing, rather than focus on any challenges with content since Eduardo spent minimal time working and not asking for help. Shaun assumed Eduardo's lack of progress was due to his sparse seat-time and directed his energy towards this area of need.

At no time did Eduardo's reflections on his progress in the course focus on factors other than whether he was spending enough time with the software. He in fact was not working outside of class, and was not implementing that portion of his plan due to transportation difficulties and other external factors. Yet it cannot be denied that Eduardo also had to overcome multiple hurdles due to his extensive arithmetic weaknesses. However, he did not look for help beyond the computer. Instead, he leveraged multiple electronic avenues to work through the immediate question such as using a calculator and replaying a video to further understand the mathematical procedures. However, these avenues did not address his underlying arithmetic weaknesses, which were necessary for him to advance.

## Discussion

In the MATH075 classroom Eduardo did not achieve his goals. He did not ask for help with difficult problems or concepts because the software was designed to be fully contained, in that no outside help was needed to work through the material. In addition, Eduardo had extensive deficiencies in his arithmetic which made the likelihood of success remote. Focusing on Eduardo's seat time became a distracting influence, preventing both Eduardo and Shaun from recognizing and addressing Eduardo's actual challenges with mathematics. Shaun never asked Eduardo whether he was having difficulty with the content and Eduardo did not reflect on his mathematical skills and how that may affect the assumption that seat time was sufficient for him to succeed. His plan also did not consider or account for the conceptual challenges he had with foundational topics in mathematics.

This case demonstrates how assumptions can be insufficient for a plan to successfully achieve a goal, consequently leading to intentions not being fulfilled. In this case, Eduardo established his plan, which was validated through the course structure. This plan turned out to be problematic and insufficient for student success. Eduardo's assumptions, that the computer alone was a reliable instructor, ultimately doomed his success in MATH075. Furthermore, no part of his plan accounted for how much help Eduardo needed to understand and work through basic arithmetic concepts. Eduardo's focus on the computer being the ultimate authority on learning did not take into account that the computer did not give more nuanced feedback to him. At no point did the computer indicate that he needed to, for example, learn his multiplication facts. Eduardo may have trusted the computer, but the computer was not providing him with the support he needed to succeed. Furthermore, both Eduardo and the instructor were preoccupied with seat-time rather than challenges with content.

Overall, Eduardo was not made conscious of his mathematical challenges. Furthermore, Eduardo believed strongly that MyMathLab was sufficient for him to learn, but MyMathLab did not provide the type of feedback that his instructor could. Likewise, Shaun did not recognize that Eduardo was held back mathematically. MyMathLab consequently created a wedge, preventing the instructor from diagnosing the student's challenges and discouraging the student from looking to the instructor for support. As each trusted in the software, the expertise in the course instructor was marginalized to the detriment of the student.

Students' failed plans may have devastating consequences. In Eduardo's case, as a student on financial aid, if he cannot succeed in his courses, he will be left with student loan payments, taking on a new financial burden without the added economic benefits of a college degree. Community colleges, and postsecondary institutions in general, must address students' assumptions about learning, the knowledge base they bring, and how success is achieved.

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Performance and Participation Differences for In-Class and Online Administration of Low-Stakes Research-Based Assessments

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*Research-based assessments (RBAs), such as the Calculus Concept Inventory, have played central roles in many course transformations from traditional lecture-based instruction to research-based teaching methods. In order to support instructors in assessing their courses, the online Learning About STEM Student Outcomes (LASSO) platform simplifies administering, scoring, and interpreting RBAs. Reducing barriers to using RBAs will support more instructors assessing the efficacy of their courses and transforming their courses to improve student outcomes. The purpose of this study was to investigate the extent to which RBAs administered online and outside of class with the LASSO platform provided equivalent quantity and quality of data to traditional paper and pencil tests administered in class for both student performance and participation. We used an experimental design to investigate the differences between these two test modes. Results indicated that the LASSO platform can provide equivalent quantity and quality of data to paper and pencil tests.*

*Keywords:* Assessment, quantitative methods, technology

## Introduction

Research-Based Assessments (RBAs), such as the Calculus Concept Inventory (Epstein, 2007), are often used to both develop and disseminate research-based teaching methods that improve student outcomes. Subsequently, RBAs are the focus of many influential publications in physics education research, such as Hake's (1998) comparison of traditional and interactive-engagement courses. The large increase in the number of RBAs in physics education research coincided with a dramatic increase in the collaboration in the PER community (Sayre et al., 2017). Because of these successes, many educators are interested in using RBAs. Madsen et al. (2016), however, found that many instructors want support in choosing appropriate assessments, administering and scoring the assessments, and interpreting the results of their assessments. To address these needs the Learning Assistant Alliance developed the LASSO platform to host and administer RBAs online (LA Alliance, 2017). Hosting the RBAs online meets instructors' needs by allowing for the tests to be administered outside of class, to be promptly and automatically scored, and for instructors to be provided with a summary report to help interpret the results.

Extensive research has investigated the differences between computer based tests (CBTs) and pencil and paper tests (PPTs). Meta-analysis of the literature has revealed that there is no systematic difference in scores between these two modes of administering tests (Wang et al., 2007). However, the studies in these meta analyses were conducted using high-stakes standardized tests at the K-12 level, and most had the CBT being administered in class. Because the LASSO platform is designed to administer RBAs outside of class in order to free up class time, the results of this earlier work may not apply to the LASSO platform.

In a similar study to this one, Bonham (2008) conducted research in college astronomy courses and administered assessments online outside of class. Bonham and colleagues had students complete both a locally-made concept inventory and a research-based attitude survey.

The students were randomly assigned to two conditions with either the concept inventory done in class and the attitude survey done outside of class via an online system or the reverse. A matched sample was then drawn from the students who completed the surveys. They concluded that there was no significant difference between CBT and PPT data collection. In contrast to their findings, a close analysis of their results revealed that there was a small but meaningful difference in the data and that the study did not have a sufficient sample size to rule out any meaningful differences; their study was underpowered. Their results indicated that the online concept inventory scores were 6% higher than the in class scores, which was an effect size of approximately 0.30. While this is a small difference, lecture-based courses often have raw gains below 20% and a 6% difference would skew comparisons between data collected with CBT and PPT modes. Therefore, it is not clear from the prior literature that low-stakes tests provide similar data when collected in class with PPTs compared to outside of class with CBTs.

## Research Questions

The purpose of the present study was to inform if data collected with LASSO is consistently different than data collected with paper tests. In pursuit of this purpose we asked:

(1) To what extent does the online administration of RBAs outside of class using the LASSO platform provide comparable data to the in-class administration of RBAs using PPTs? (2) How do instructor administration practices impact participation rates for low-stakes RBAs, if at all? (3) How are student course grades related to participation rates for low-stakes RBAs, if at all?

If the LASSO platform provided equivalent data to paper based administration, then the LASSO platform represents a much simpler entry point for instructors to begin assessing and transforming their own courses because it addresses many of the instructors' needs that Madsen et al. (2016) identified. A second major benefit of the widespread use of the LASSO system is that it automatically aggregates all of the data and makes this data available for research. The size and variety of this data allows for investigations that would have been underpowered if conducted at only a few institutions or lacking generalizability if only conducted in a few courses at a single institution.

## Methods

The data was collected at a large regional Hispanic-serving university across two semesters in three different introductory physics courses: algebra-based mechanics, calculus-based mechanics, and calculus-based electricity and magnetism (E&M).

The study used a between-groups experimental design (Figure 1). Stratified random sampling created two groups within each section with similar representations across student gender, race, and honors status. One group completed a concept inventory (either the Force Concept Inventory [FCI] or Conceptual Survey of Electricity and Magnetism [CSEM]) online outside of class using the LASSO platform and an attitudinal survey (the Colorado Learning Attitudes about Science Survey [CLASS]) in class using paper and pencil. The other group completed the concept inventory in class and the attitude survey online outside of class. Both conditions were repeated at the beginning and end of the semester. Paper and pencil assessments were collected by the instructors, scanned using automated equipment, and uploaded to the LASSO platform. Student assessment data was downloaded from the LASSO platform and combined with student grades and demographic data provided by the university. The data analysis did not include students who joined the class late, dropped, or withdrew, leaving a total sample of 1,310 students in 25 course sections.

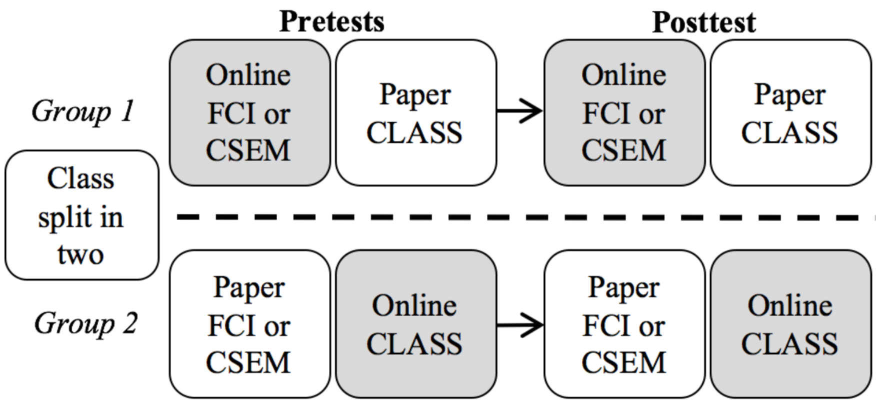


*Figure 1. Design of the research conditions.*

At the end of each semester of data collection participating faculty were interviewed to identify how the faculty motivated their students to complete the CBT. Four different practices were identified that we will refer to as *recommended practices*: 1) email reminders, 2) in class announcements, 3) participation credit for the pretest, and 4) participation credit for the posttest.

We used the HLM 7 software package to create multi-level models to analyze the performance and participation data. We analyzed the performance data for the concept inventories using 2-level Hierarchical Linear Models: test conditions (level 1) were nested within course types (level 2), no covariates were used. We analyzed the participation data using 3-level Hierarchical Generalized Linear Models: assessments (Level 1) were nested within students (level 2) nested within either course sections (level 3), the number of recommended practices and students grades in the courses were used as covariates.

The final models for performance and participation consisted of posttest score or participation as the outcome variables. The models were built in 3 or 4 steps: (1) no predictors, (2) add level 1 predictors, (3) add level 2 predictors, (4) add level 3 predictors (if applicable). This four-step process informed how much additional information was being explained by the addition of the new predictors in each step as indicated by a reduction in the variance for that variable.

Completion rates for the PPT condition were 94% for the pretest and 74% for the posttest and for the CBT were 68% for the pretest and 54% for the posttest. For the performance analysis, missing concept inventory data (i.e. students who did not take either the pre or posttest) was replaced using Hierarchical Multiple Imputation (HMI) with the MICE package in R. HMI is a form of multiple imputation (MI) that takes into account the fact that students were nested in different courses and that their performance may have been related to the course they were in. MI addresses missing data by (1) imputing the missing data m times to create m complete data sets, (2) analyze each data set independently, and (3) combine the m results using standardized methods (Dong & Peng, 2013). Our MI produced m=10 complete data sets. Multiple imputation is preferable to list-wise deletion because it maximizes the statistical power of the study and has the same basic assumptions.

## Findings

### Performance

The model of student performance on concept inventories showed very little differences in either pretest or posttest performance across test conditions. The largest predicted effect of test condition on student performance was on posttest for E&M students (Figure 2). This predicted effect bordered on being large enough to be meaningful because it indicated a 2.2 points higher posttest score for students doing the CBT and the overall predicted gain for the E&M students was only 11.6 points. However, the pre- and posttest across the three courses created six total measurements of the predicted effect for test condition; in three of those measurements the effect was nearly zero, in one it was positive, and in two it was negative. In addition to these inconsistencies in all six comparisons across condition there was large overlap in the 95% confidence intervals, indicating that the differences were not statistically reliable. Examination of the model variances showed that the inclusion of test conditions led to larger variances, indicating that conditions were not a reliable predictor of student performance.

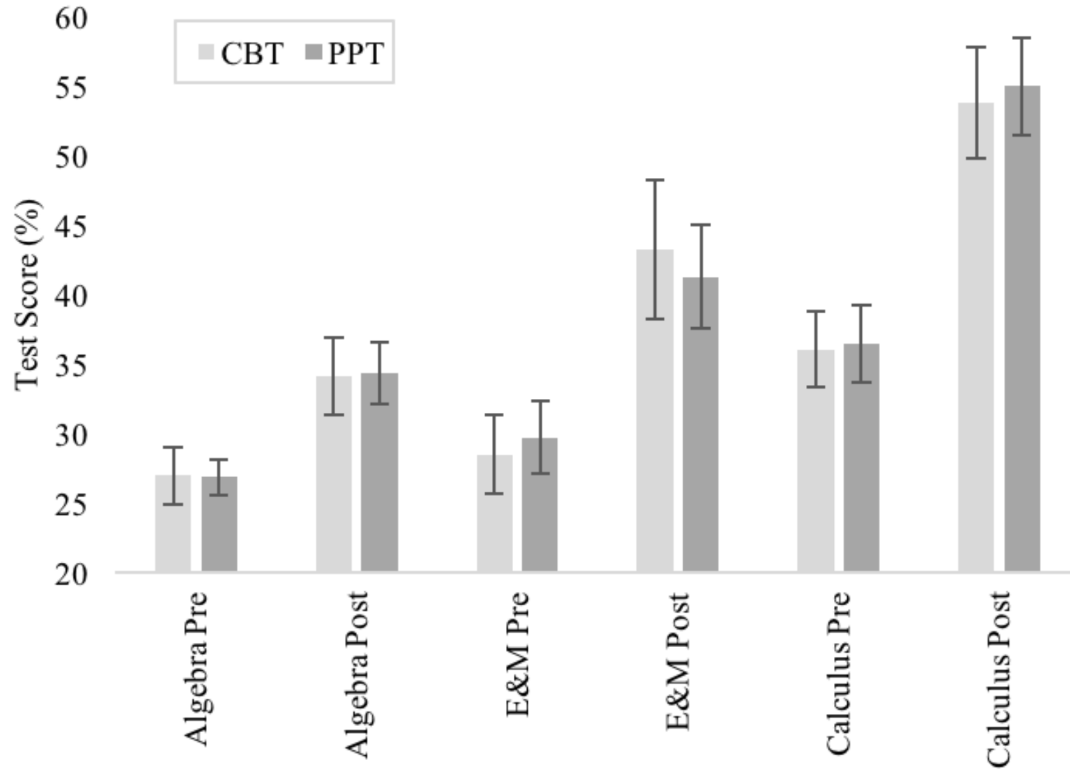


*Figure 2. Predicted Mean Scores with 95% CIs.*

**Participation**

The results of our HGLM model of the student data, indicate that the more recommended practices instructors used, the higher the participation rates were for their CBT assessments. Student course grades were also a statistically reliable predictor of student participation.

Figure 3 illustrates the predicted student participation rate based on student course grades and the number of recommended practices that instructors used. In terms of data collection, the posttests represented the limiting case as predicted participation rates on the posttests for both the PPT and CBT were lower than on the pretests. With the exception of the PPT pretest there was a large difference in predicted participation based on course grades. The number of recommended practices that instructors used dramatically increased predicted participation rates such that when instructors implemented all four recommended practices the participation rates of the CBT and PPT posttest were very similar. The impact of recommended instructor practices on predicted participation rates occurred for all students, but was largest for high achieving students. Relationships between student participation, grades, and instructor practices on the CBT pretest were similar to those on the CBT posttest. These results indicated that similar participation rates to those on PPT can be achieved via CBT when instructors use all four recommended practices.

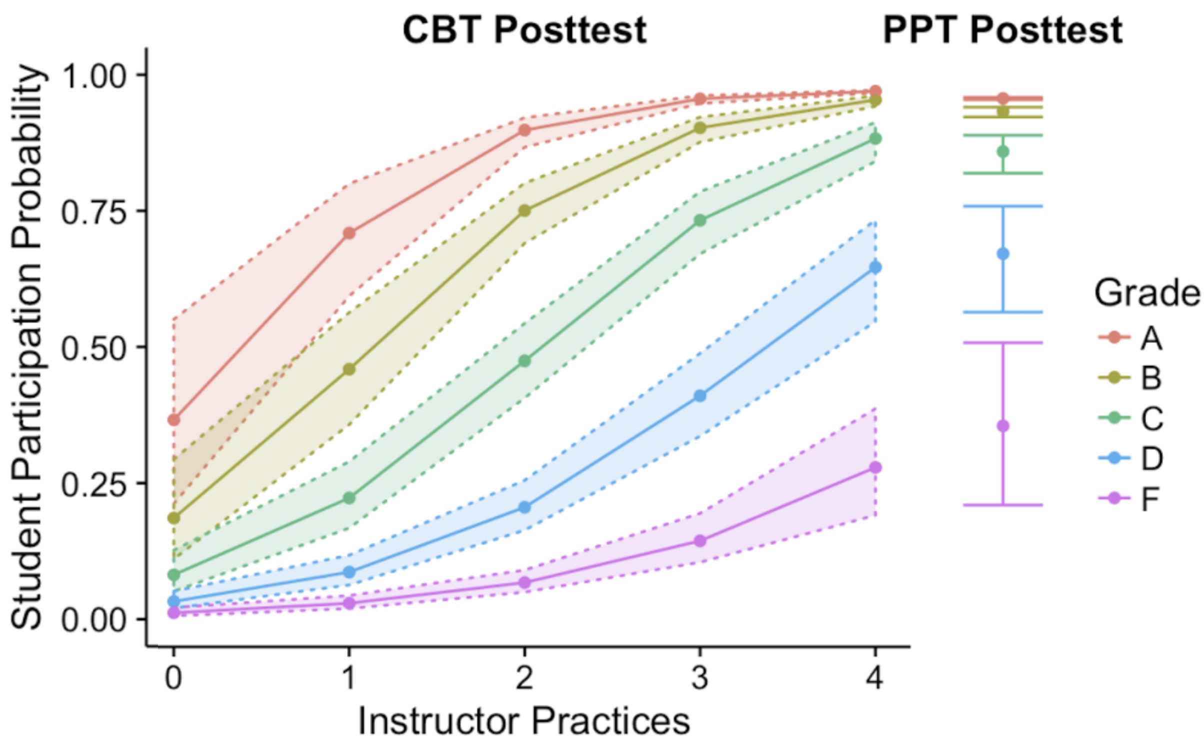


*Figure 3. Predicted student participation rates with 95% CIs. Only posttest predictions are shown as it is the test with the lower participation rates and is the primary limiter for data collection.*

## Conclusion and Implications

Our study shows that CBT and PPT administrations of low-stakes assessments can lead to similar student performance and participation. This similarity indicates that when our recommended practices are implemented instructors and researchers can use online systems, such as the LASSO platform, to collect valuable information about the impacts of their courses that is comparable to prior research that was collected with paper and pencil tests. Collecting data with the LASSO system can also greatly reduce the barriers to instructor's use of RBAs since instructors do not need to dedicate class time to collect the data or their own time to sort, scan, and analyze the data. It is important to note, however, that instructors do need to make some effort to motivate their students to complete the online assessments. We have found that by making announcements in class, sending out email reminders, and giving credit to students who complete the RBAs instructors can achieve similar participation rates on CBT assessments as on PPT assessments. Our hope is that reducing the barriers to using RBAs use will lead more instructors to assess the efficacy of their courses and, subsequently, to adopt research-based teaching practices that support student success.

In addition to promoting the use of RBA's developed by the DBDR community, the LASSO platform anonymizes, aggregates, and makes its database available to researchers with appropriate IRB protocols. The LASSO database has already provided multi-level large-scale data to examine questions of equity in student outcomes (Van Dusen & Nissen, in press), effects of near-peer mentors on student outcomes (White et al., 2016), and effects of instructor experience on their effectiveness (Caravez, in press). As the LASSO dataset grows, it will allow the DBER community the ability to quickly access a dataset designed to support the investigation of student outcomes from across the country.

While these findings are generally encouraging, there are several unexamined factors that could strengthen the conclusions and generalizability of the work. Useful areas for future research includes: (1) examining the associations between student demographics and student participation and performance in CBT and PPT conditions, (2) comparing student performance at the item-level (rather than total score) on CBT and PPT conditions, and (3) replicating the study in diverse institutional settings.

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# An Activity Theory Approach to Mediating the Development of Metacognitive Norms During Problem Solving

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*Metacognition has long been identified as an essential component of the problem-solving process. Research on metacognition and metacognitive training has historically adopted an acquisitionist view. This study takes a participaionist lens by considering metacognition as a habit of mind or dispositional tendency. Problem-solving habits of mind can be viewed as normative ways of thinking to which students become attuned by participating in authentic problem-solving situations. This study explored one such situation, in which portfolio problem-solving sessions and write-ups were used to mediate metacognitive thinking. Periodically, students worked together on non-routine problems and submitted individual write-ups documenting their judgement and decision-making processes. Analysis utilized Activity Theory, which operationalizes the participation structure of a classroom, to document the nonlinear development of classroom metacognitive norms during problem solving. Micro-analysis revealed a shift from product- to process-oriented metacognitive norms. Macro-analysis situated these results, highlighting social mediators of activity and contradictions as catalysts for change.*

*Keywords:* Metacognition, Problem Solving, Norm Development, Activity Theory

## Introduction and Motivation

The importance of problem-solving practices has been emphasized and studied extensively (NCTM, 2010). Although literature has identified metacognition as a key component of the problem-solving process (e.g. Schoenfeld, 1985), metacognition remains undertheorized and under-studied in its application to classroom communities (Carroll, 2008). While the importance of prolonged metacognitive instruction embedded in content matter has been emphasized (Lester, Garofalo, & Kroll, 1989; Veenman, Van Houte-Wolters, & Afflerbach, 2006), most metacognitive research has overlooked the crucial impact of sociocultural contexts and learning environments in its development (Larkin, 2015). Further, metacognitive research, with foundations in cognitive information processing theory, has taken an almost exclusively acquisitionist (Sfard, 1998) approach to metacognition, where metacognitive skills are decontextualizable commodities (products) to be transmitted to students. As such, research concerning the teaching and learning of metacognition has been limited in its practical classroom application by overlooking the process characteristics of metacognitive thinking. There is a difference between "*knowing-about*" and "*knowing-to act in the moment*" (Mason & Spence, 1999). Becoming a skillful problem solver means coming to know the nuanced ways of *doing* or acting (as opposed to *having*) in authentic mathematical problem-solving situations.

Rather than viewing metacognitive knowledge and skills as objects to be transmitted, this research study took a complementary approach by appealing to metacognition as a habit of mind (Costa & Kallick, 2000), a disposition toward certain ways of acting during the problem-solving process. Taking a participationist (Sfard, 1998) view of the teaching and learning of metacognition, a skilled problem solver must both "communicate in the language of the community and act according to its particular norms" (p. 6). Problem-solving habits of mind, such as metacognition, can be viewed as normative ways of thinking or acting within the "skilled problem solver" community of practice. Through legitimate peripheral participation (Lave & Wenger, 1991), students become attuned to these normative, habitual tendencies or dispositions, eventually transforming their own habits of mind as they become full participants (i.e., skilled problem solvers) in this community. This theoretical approach requires understanding the process of student participation in metacognitive thinking, with attention to the contexts that afford or constrain such dispositional transformations toward full participation.

Accordingly, the purpose of this study was to investigate the use of portfolio problems (defined below) as a mediator of participation in metacognitive thinking during problem solving, delivering a prolonged intervention embedded in mathematics content called for by Lester, Garofalo, and Kroll (1989). Students worked on six portfolio problems throughout the semester, each of which consisted of two main parts: small group problem-solving sessions in class and individual write-ups. Except for the first problem which only involved one session, groups worked on a given problem over two in-class sessions, with each session lasting roughly one-third of a class period. These non-routine problems were chosen to align with the NCTM's (2010) "worthwhile-problem criteria," and to increase the likelihood that a solution path was not immediately known to students. Further, problems were selected with key mathematical ideas directly related to the content unit in which the problem was presented. The instructional team encouraged students to record their work and observations or questions on scratch work. Students wrote in different colored pens to identify individual contributions. This scratch work was emailed to each group after class, and students were expected to continue working on the problem outside of class. Students then submitted individual write-ups documenting a revised solution that included mathematical justification and reasoning, as well as their judgement and decision-making processes during the entire problem-solving attempt, from initial thoughts to final result. For example, students might include questions they asked themselves, or a discussion of why they employed or abandoned a particular representation or problem-solving strategy.

## Theoretical Framing

In studying students' attunement to normative ways of thinking, one must consider that the natural, purposeful activity within a classroom creates a microculture of negotiated activities and interactions among students and the teacher (Lave & Wenger, 1991). Over time, normative behavior emerges, but the interpretation and function of these norms change through iterations of negotiation. Consequently, metacognitive norms may develop in a way so that the resulting activity is not necessarily identical to that intended by the teacher. Thus, the focus of investigation turns to the development of a classroom community's normative metacognitive activity during problem solving. This necessitates an appropriate framework to document the nonlinear development of classroom problem-solving norms.

Third-generation Activity Theory (Engeström, 1987), which is conducive to a participation metaphor for learning (Barab, Evans, & Baek, 2004), was used in the present study as an analytic framework for systematic investigation. Activity Theory accounts for the complex interaction between the individual and community by expanding Vygotsky's (1978, 1986) notion of mediated activity to include additional social mediators (Engeström, 1987) (Figure 1). Individuals or groups of individuals form a motivated, object-oriented activity system, where the entire activity system forms the unit of analysis. An activity system dynamically transforms, expanding or changing qualitatively over (relatively long periods of) time through adaptation to contradictions or tensions (Engeström, 1987). In the context of this study, students in a first-year mathematics content course for pre-service elementary teachers formed one activity system, while an instructional team, consisting of myself and the instructor of record, created a "culturally more advanced" (Engeström, 1987) activity system that interacted with the student activity system.

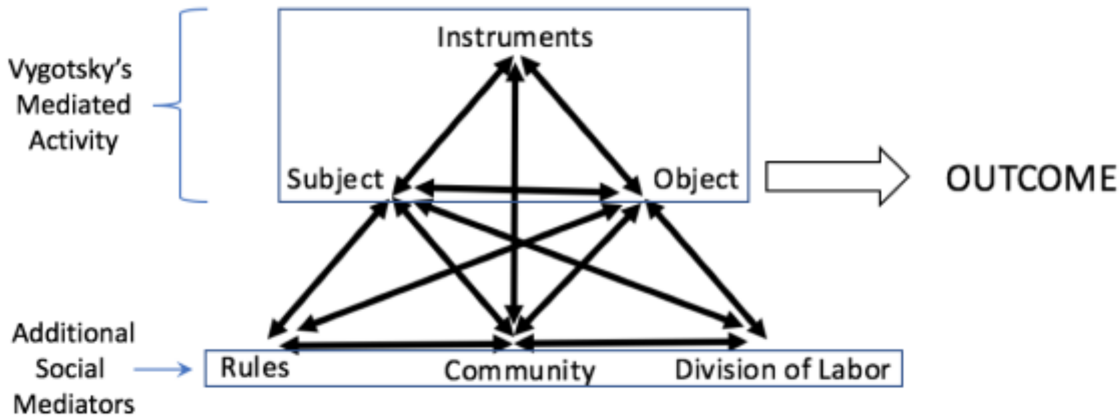


*Figure 1. Vygotsky's mediated activity embedded within the expanded activity triangle.*

In the context of documenting the process of classroom norm development, (Third-generation) Activity Theory is particularly advantageous. Specifically, Activity Theory provides:

1) *Operationalization*: Activity Theory operationalizes the participation structure of a classroom community for detailed, systematic investigation (described in the following Methods section).
2) *Attention to Reflexivity*. The classroom collective and individual students influence each other in a cycle of negotiation and influence (see Ernest, 2010). By framing students and the teacher as interacting activity systems, Activity Theory provides explicit language with which to document this nuanced, reflexive interaction over time.
3) *Expansion, and Horizontal Expansion.* Classroom norms are not pre-established, unchanging concepts, and their development can be influenced by students. Further, students' learning and development is not necessarily vertical, from "lower" to "higher" levels of competence. Activity Theory accounts for student growth and potentially non-vertical, or horizontal growth, through the process of expansive transformation, which occurs "when the object and motive of the activity are reconceptualized to embrace a radically wider horizon of possibilities than in the previous mode of the activity" (Engeström, 2001, p. 137).
4) *Process-Focused Interaction of Social Activity and Individual Actions*. Activity Theory focuses on the process of interaction over time, allowing for documentation of the transformational process of reflexive interaction between covert social activity affecting participation and individual actions of participation.
5) *Contradictions as Catalysts for Change*. Characterizing system dynamics through contradictions and tensions is a powerful means for interventionist-motivated design research. This study capitalized on portfolio problems as a mediating instrument to create a purposeful contradiction between the object/motive pair of the student activity system and the object/motive pair of a culturally more advanced form (the teacher activity system).

## Methods

This qualitative research study, grounded in the aforementioned theoretical perspective, was guided by the research question: What is the role of portfolio problems as a mediating instrument in the development of normative metacognitive activity during problem solving, within an undergraduate mathematics classroom community of pre-service elementary education majors? To address this question, sub-questions were posed: (a) What metacognitive actions during problem-solving become normative activity? (b) What contradictions or tensions exist within the classroom community that catalyze the development of such normative metacognitive activity during problem solving? and (c) What actions of the teacher influence, positively or negatively, the development of such normative metacognitive activity during problem solving?

Six qualitative data sources were collected in the 15-week semester: (1) video- and audio-recorded classroom sessions, (2) three videotaped, semi-structured individual interviews with 13 of the 23 students at the beginning, middle, and end of the course, (3) two audio-recorded interviews with the instructor of record, (4) students' written artifacts (assignments, exams, and portfolio-problem submissions and scratch work) collected before grading, (5) recorded planning sessions of the instructional team, and (6) journal reflections written by each member of the instructional team after each class session. The first data source, recorded classroom sessions, was utilized for micro-analysis, while all data sources were used for macro-analysis.

The first and second student interviews (data source (2)) had three parts. First, students were asked questions targeting their beliefs about mathematics, mathematical problem solving, and perceptions of the course. Students then worked, thinking aloud, on non-routine problems related to course content. This portion of the interview provided a reference point for students to discuss their problem-solving activity more generally. Finally, students compared their problem-solving attempts during the interview with their "typical" problem-solving activity in the course, as well as the problem-solving activity of course instructors, other students in the course, and other courses. The third interview did not include problem solving, but was a series of questions asking students to reflect on their experiences in the course.

Two levels of analysis were employed: a micro-analysis of language-mediated discourse [the upper boxed portion of the activity triangle in Figure 1], followed by macro-analysis using Engeström's expanded activity triangle to highlight tensions within the activity system (Jaworski & Potari, 2009). Micro-analysis served to address research sub-question (a) by identifying metacognitive actions (adapted from Carlson & Bloom, 2005) present during each of the in-class portfolio problem-solving sessions (Table 1). Recalling that a participation metaphor for learning was adopted in this study, the focus was on students' real-time *actions*. Thus, while written artifacts of students' judgement and decision-making processes were collected as part of the portfolio problems, only those actions demonstrated in situ were documented to evidence normative metacognitive activity.

*Table 1. Metacognitive actions identified during portfolio problem-solving sessions*

| | |
|---|---|
| MA 1. | Mathematical concepts, knowledge, tools, and facts are assessed and considered |
| MA 2. | Various solution approaches or strategies are assessed and considered |
| MA 3. | Validity/reasonableness of solution process is assessed/considered/tested |
| MA 4. | Results are assessed/tested/considered for their reasonableness/validity |
| MA 5. | Reflects on the efficiency and effectiveness of cognitive activities |
| MA 6. | Manages emotional responses to problem-solving situation |

Macro-analysis situated micro-analysis results, using a six-step method (Jonassen & Rohrer-Murphy, 1999) to describe various components of the student activity systems (Table 2). The student activity system was analyzed at multiple points throughout the semester to document potential change over time. Contradictions and tensions within the student activity system, as well as between the student and instructional team activity systems, were detected in the final step of macro-analysis, addressing research sub-question (b). As an additional part of both micro- and macro-analyses, actions taken and decisions made by the instructional team potentially impacting the development of metacognitive norms were identified, addressing research sub-question (c).

*Table 2. Six Steps for Analyzing an Activity System (Jonassen and Rohrer-Murphy (1999))*

| | |
|---|---|
| Step 1. | *Clarify the purpose of the activity system.*<br>Describe the motives and conscious goals of the activity system. |
| Step 2. | *Analyze the activity system.*<br>Define the subject, object, community, division of labor, and rules. |
| Step 3. | *Analyze the activity structure.*<br>Delineate the hierarchy of activity, concrete actions, and automatized operations. |
| Step 4. | *Analyze tools and mediators.*<br>Describe the tools, rules, and roles of participants that mediate activity within the system. |
| Step 5. | *Analyze the context.*<br>Characterize the internal, subject-driven and external, community driven contextual bounds. |
| Step 6. | *Analyze activity system dynamics.*<br>Step back from the delineated activity system to describe and assess how components affect each other. |

## Results and Discussion

Micro-analysis results, addressing research sub-question (a), indicated a shift in function of metacognitive thinking during problem solving. Over the course of the semester, the normative metacognitive activity employed during portfolio problem-solving sessions transformed from a retroactive focus on checking answers (*products*), to a proactive focus on the evaluation of the problem-solving *process*, especially the consideration of various solution approaches, tools, and strategies. Figure 2 broadly illustrates this change, where Metacognitive Action 4 (MA4) was prevalent at the beginning of the semester, but dissipated in use over time, with process-focused actions becoming dominant (e.g., MA2, MA3). By the end of the semester, students recognized this transition away from reflecting *only* at the end of a problem-solving attempt, as highlighted in the following student quote taken from the final interview:

> I've just been able to be actively engaged in the problem, realizing what I'm doing. Rather than just like, 'Well, this is the first step and second step,' and then afterwards I'm like, 'Oh, that was wrong, and that was wrong.'

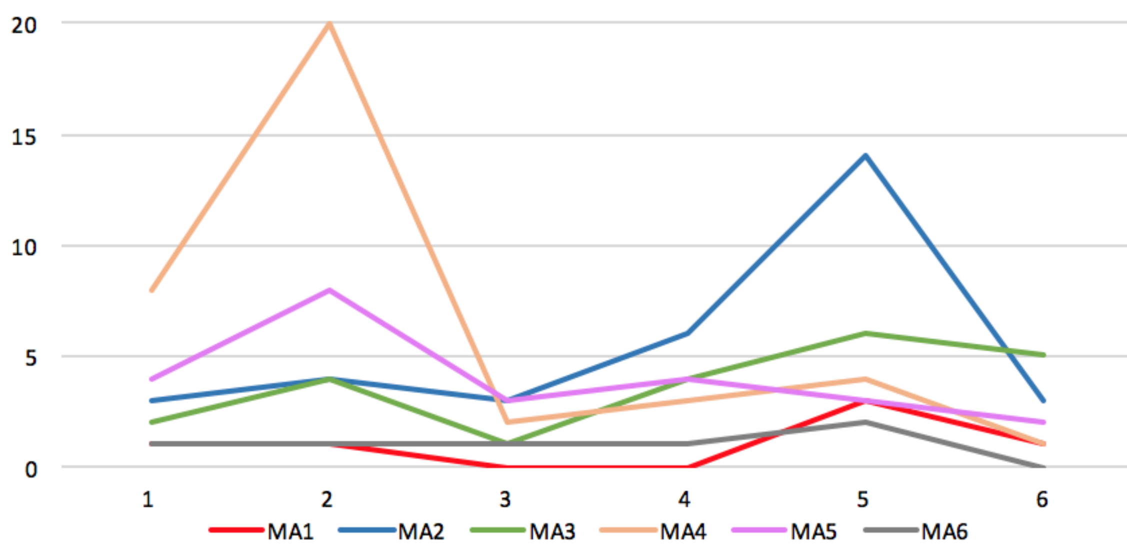


*Figure 2. Total occurrences of Metacognitive Actions during the six in-class portfolio problem-solving sessions.*

Macro-analysis situated these results, revealing contradictions that shaped the development of the normative metacognitive activity from micro-analysis, directly addressing research sub-question (b). Notably, the introduction of portfolio problems as a mediating instrument for problem solving constituted a significant catalyst for change. Students identified all aspects of the portfolio problems (in-class problem-solving sessions, scratch work, and submitted write-ups) as contributing to the awareness of their thinking during the process of problem solving. The portfolio problems also contradicted many students' motives and expectations for the course. While students anticipated learning to teach mathematics, the focus of the course was for students to improve as mathematical thinkers themselves. The non-routine, open nature of the portfolio problems brought this tension to the fore, encouraging students to adjust their course goals and embrace opportunities for personal development.

Additionally, instructor actions contributing to the development of metacognitive norms were identified during both micro- and macro-analyses, directly addressing research sub-question (c). The instructional team encouraged students to focus on process over product, generalize their problem-solving solutions and methods, and look for commonalities across problem contexts. Further, the team attempted to make overt the typically invisible mediators of mathematical problem solving. While this motive of the instructional team was consistent throughout the course, portfolio problems provided a rich setting within which to have these conversations, amplifying the influence of teacher actions. For example, the instructional team noticed students' increased frustration that they were not finding solutions to the portfolio problems. The team used this as an opportunity to discuss perseverance in the problem-solving process, using a video describing Andrew Wiles' lengthy process for generating a proof of Fermat's Last Theorem to aid in facilitating this discussion. This discussion allowed the class to both focus on the process of problem solving, and see that perseverance is an important aspect of this process.

In this study, portfolio problems contributed to students' shift in focus from reflection on the answers or outcomes (*products*) of a problem-solving attempt, to metacognitive thinking during the entire *process* of problem solving. While an inquiry-oriented course design and complementary teacher actions facilitated this shift, the portfolio problems accelerated the effects of these actions by creating contradictions or tensions as catalysts for student transformation. The intervention design in this study, with portfolio problems directly related to course content and used throughout the entirety of the course, supports the claim that metacognitive instruction must be embedded as part of the classroom culture. Additionally, Activity Theory proved useful as a framework for analysis, as it explicated the role of portfolio problems as facilitators of change through the creation of contradictions or tensions. This characterization is a powerful tool for intervention-based design research intended to create *purposeful* contradictions that can lead to productive beliefs (NCTM, 2014) about the teaching and learning of mathematics.

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Collective Argumentation Regarding Integration of Complex Functions Within Three Worlds of Mathematics

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*Although undergraduate complex variables courses often do not emphasize formal proofs, many widely-used integration theorems contain nuanced hypotheses. Accordingly, students invoking such theorems must verify and attend to these hypotheses via a blend of symbolic, embodied, and formal reasoning. This report explicates a study exploring student pairs' collective argumentation about integration of complex functions, with emphasis placed on students' attention to hypotheses of integration theorems. Data consisted of task-based, semistructured interviews with pairs of undergraduates, as well as classroom observations. Findings indicate that participants' explicit qualifiers and challenges to each other's assertions catalyzed new arguments allowing students to reach consensus or verify conjectures. Although participants occasionally conflated certain formal hypotheses, their arguments married traditional integral symbolism with dynamic gestures and clever embodied diagrams. Participants also took care to avoid invoking attributes of real numbers that no longer apply to the complex setting. Teaching and research implications are discussed as well.*

*Keywords:* Collective Argumentation, Complex Variables, Integration, Reasoning

## Introduction and Literature Review

Although the discipline of mathematics often rests on generalizing results from one domain to another, at times "mathematical thinking may involve a particular manner of working that is supportive in one context but becomes problematic in another" (Tall, 2013, p. xv). Such considerations can arise when studying the teaching and learning of complex analysis. For example, Danenhower (2000) discovered a theme of "thinking real, doing complex" (p. 101) wherein participants invoked attributes of real numbers that do not necessarily apply in the complex context. Troup (2015) additionally evidenced this phenomenon when undergraduates reasoned about complex differentiation. Within the setting of real-valued functions, the literature abounds with examples of students' struggles with integration (Grundmeier, Hansen, & Sousa, 2006; Judson & Nishimori, 2005; Mahir, 2009; Orton, 1983; Palmiter, 1991; Rasslan & Tall, 2002). However, most of these studies showcased the *product* of students' deficiencies and misconceptions rather than the *process* of students' reasoning. Accordingly, although students might sometimes draw incorrect conclusions regarding integration, their process of reasoning may healthily appeal to intuition or past experiences. When cultivated properly, such connections between experientially-based intuition and formal mathematics could benefit students' reasoning in courses such as complex variables (Soto-Johnson, Hancock, & Oehrtman, 2016).

Furthermore, according to Wawro (2015), by researching students' successful reasoning about undergraduate mathematics topics, we can document "what deep understanding and complex justifications are possible for students as they engage in mathematics" (p. 355). The subject of complex variables is particularly amenable to such an investigation, as the content in this course often lies between symbolic calculation and formal proof. Specifically, students that integrate complex functions tend to apply powerful theorems that rely on idiosyncratic hypotheses and draw on notions from real analysis and/or topology. Though formal proof is typically not emphasized in undergraduate courses in complex variables (Committee on the

Undergraduate Program in Mathematics, 2015), the application of such theorems requires that students at least recognize when these hypotheses apply. As such, students may invoke a blend of intuition, visualization, symbolic manipulation, and formal deduction when integrating complex functions. Accordingly, integration of complex functions lends itself to eliciting the rich student justifications called for by Wawro. Complex integration also has numerous practical applications for students, such as computing flux, potential, or certain real-valued integrals.

Despite these practical and theoretical assets, no existing educational research examines undergraduates' reasoning about integration of complex functions. In particular, researchers have not yet documented how students reason algebraically, geometrically, and formally when integrating such functions. This study served to ameliorate this gap in the literature and to inform the teaching and learning of complex variables by analyzing undergraduates' multifaceted argumentation about integration of complex functions. Using Tall's (2013) Three Worlds of Mathematics framework, my research sought to answer the following guiding questions:

1. How do pairs of undergraduate students attend to the idiosyncratic assumptions present in integration theorems, when evaluating specific integrals?
2. How do pairs of undergraduate students invoke the embodied, symbolic, and formal worlds during collective argumentation regarding integration of complex functions?

In this study, argumentation is defined according to Toulmin's (2003) model consisting of six components: data, warrant, backing, qualifier, rebuttal, and claim. Given that my study considers how pairs of students reason about integration tasks, it is additionally important that I consider how each individual contributes to an argument. Accordingly, I adopt Krummheuer's (1995) notion of *collective argumentation* in which multiple participants construct arguments through emergent social interaction. These interactions involve four speaker roles (author, relayer, ghostee, and spokesman), classified according to how syntactically and/or semantically responsible an individual is for the content of his or her statement. Readers unfamiliar with these speaker roles may consult Krummheuer (1995) or Levinson (1988) for more information.

The existing mathematics education literature implementing Toulmin's model manifests itself in several contexts. In the in-class setting, some researchers (Krummheuer, 1995; Krummheuer, 2007; Rasmussen et al., 2004; Stephan & Rasmussen, 2002) used a reduced Toulmin model omitting the qualifier and rebuttal, and rarely evidenced explicit backing. However, when more formal arguments such as proofs were concerned, researchers (Alcock & Weber, 2005; Inglis, Mejia-Ramos, & Simpson, 2007; Simpson, 2015) argued for the use of the full Toulmin model. These researchers also highlighted that simply reading the finished product of a purported proof is inherently difficult because backing and warrants are often implicit and cannot be elicited through real-time discourse with the proof author. Thus, my investigation into undergraduates' nuanced argumentation about integration of complex functions incorporated the full Toulmin model as well as opportunities for clarification in an interview setting.

## Theoretical Perspective

This work is theoretically oriented by Tall's (2013) Three Worlds of Mathematics as a lens through which to analyze undergraduates' reasoning pertaining to integration of complex functions. Tall's perspective situates mathematical knowledge within three distinct but interrelated forms of thought: *conceptual-embodied*, *operational-symbolic,* and *axiomatic-formal.* Conceptual embodiment begins with the study of objects and their properties, and can incorporate mental visualization. Operational symbolism grows out of actions on objects and can be symbolized flexibly as *procepts*, symbols operating dually as process and concept (Tall, 2008). The world of *axiomatic formalism* attends to “formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof” (p. 17). These three worlds can also combine to form, for example, *embodied-symbolic* or *formal-embodied* reasoning. As mentioned earlier, our prior experiences with mathematics can either support or clash with new and abstracted mathematical notions. Tall refers to the mental schemas predicated on these prior experiences as *met-befores*. He also argues that mathematical growth is afforded by three innate *set-befores* of recognition, repetition, and language. These set-befores foster categorization, encapsulation, and definition in order to compress knowledge into *crystalline* structures, which house various equivalent formulations of a mathematical object and can be unpacked in various worlds.

Moreover, “each world develops its own ‘warrants for truth’” (Tall, 2004, p. 287). In the embodied and symbolic worlds (respectively), truth derives from what is *seen* to be true by the learner visually, and from calculation. Yet in the formal world, a statement is either assumed as an axiom, or can be proven from axioms. Hence, Tall’s three-world perspective can complement the Toulmin analysis of a mathematical argument by adding specificity with regard to the types of backing and warrants used. As such, I classify participants’ Toulmin components as embodied, symbolic, formal, or various mixtures of these, as viewed through Tall’s three-world lens. Consequently, I define *reasoning* as mathematical argumentation within one or more of the three worlds. I also garner specificity by adopting Simpson’s (2015) three classifications of backing. Specifically, *backing for the warrant’s validity* explains why a warrant applies to a given argument. A second type serves to “highlight the logical field in which the warrants are acceptable,” which Simpson characterized as *backing for the warrant’s field* (p. 12). The third type, *backing for the warrant’s correctness*, demonstrates that a given warrant is actually correct.

## Methods

In order to rigorously address my research questions, I enlisted the help of two pairs of undergraduate students to partake in a videotaped, semistructured (Merriam, 2009), task-based interview comprised of two 90-minute portions and 13 tasks. Participants were selected from undergraduate students at a military academy in the United States, enrolled in a complex variables course during the spring 2015 semester. My first pair of participants consisted of Sean and Riley. Sean was a fourth-year physics and mathematics major and Riley was a second-year applied mathematics major with a cyberwarfare concentration. The second pair consisted of Dan, a third-year mathematics major, and Frank, a second-year applied mathematics major with an aero concentration. All participants’ names listed here are pseudonyms. A sample analysis of interview data is detailed in the next section.

To obtain a rich understanding of the context in which these participants learned about integration of complex functions, I also observed and videotaped six class sessions at participants’ undergraduate institution. These observations and ensuing field notes allowed me to document what mathematical content was introduced and emphasized during the integration unit in the complex variables course. They also allowed me to discern the nature of mathematical argumentation that was deemed appropriate for the complex variables course. For the sake of brevity, I restrict the presentation of results here to my interview findings. I also note here that I read tasks aloud verbally during the interviews so as not to overtly suggest any particular representation or world to participants.

## Results and Discussion

Due to my definition of reasoning in the context of this study as collective argumentation within one or more of Tall's (2013) three worlds, I format my results within each task according to argument. Included in my account of each collective argument are: pertinent excerpts of the participants' interview transcript; a Toulmin (2003) diagram summarizing the argument; and figures illustrating participants' gestures or inscriptions, often for the purpose of documenting embodied reasoning. Because of page constraints, this report showcases select results from Riley and Sean's interview. In particular, I present analysis of Riley and Sean's response to one task, in which participants evaluated the integral $\int_L \frac{1}{z} dz$, where $L$ denotes the unit circle $|z| = 1$ traversed counterclockwise. Afterwards, I allude to general findings from both pairs' interviews, and discuss various implications of my work.

### Sample Task Analysis

In illuminating Riley and Sean's reasoning about the task, I reference line numbers from their transcript excerpts and refer to various components of the Toulmin diagrams I constructed based on my interpretation of their responses. I also convey individual participants' speaker roles germane to each Toulmin component in the collective argument. Throughout the transcript pieces and Toulmin diagrams presented in this section, 'Int.' signals statements that I said aloud as the interviewer, while 'R' and 'S' stand for Riley and Sean, respectively. Bracketed phrases represent non-verbal events such as gestures or written inscriptions produced by the participants. In the Toulmin diagrams, italicized statements represent participants' exact verbiage from the transcript, while non-italicized statements more succinctly summarize participants' reasoning or deduce implicit Toulmin components based on their verbiage, gestures, and inscriptions, or lack thereof. Horizontal and vertical lines show how argumentation components are linked within a collective argument or subargument. Following the format of Wawro (2015), I represent shifts in the Toulmin categorization from one type of component to another (such as claim to data) in the figures by a diagonal line.

As I read the task aloud, Sean symbolically relayed the data comprised of the integral $\int_L \frac{1}{z} dz$ and the path $|z|= 1$ (line 4). He also authored an embodied datum by drawing the circular path on an Argand plane (see Fig. 1). As spokesman, Sean then symbolically rewrote $L$ as $C_1^+(0)$, and I acknowledged this alternate symbolism from their class (lines 4-5). Riley agreed, but Sean made sure to document that this was the professor's notation, as if indicating that he did not hold any agency when using it (lines 6-8).

Sean proceeded as spokesman, indicating that they could apply an antiderivative, as in the last task (lines 9-10). He also qualified this suggestion with the phrase, "I think I'm pretty sure that…" (line 9). However, Riley challenged Sean as she authored a warrant: "There's no branch we can choose […] so that [the integrand] is going to be analytic over the entire path" (lines 11-12). Invoking embodied reasoning, Riley also revised Sean's initial diagram of the circular path to include a positive orientation (see Fig. 1).

*Int:* Ok so more integration stuff. And I think given what we just discussed, I think this one shouldn't be too bad for you. So how would you find the integral of $\frac{1}{z}\,dz$, and this is over the path $L$, if $L$ represents the unit circle $|z| = 1$, traversed counter-clockwise?
*S:* [Writes $\int_L \frac{1}{z} dz$, |z|= 1, and draws the path on an Argand plane. Changes notation of $L$ to $C_1^+(0)$]
*Int:* Yeah, and I realize you guys have your own notation for that too, so—
*R:* Yeah.
*S:* Dr. X's notation. I'll document that [laughs].
*Int:* Right [laughs].
*S:* Ok so kind of like we did last time, I think I'm pretty sure that— I normally call it using the antiderivative, but—
*R:* [Draws in orientation on path] There's no branch we can choose, right, so that it's going to be analytic over the entire path?
*S:* Yeah, so I'd just say, $z = e^{i\theta}$ [writes this]. Therefore $z'(\theta) = ie^{i\theta}$ [writes this]. So we say the integral from 0 to $2\pi$, because theta is of course from these values [writes $0 \leq \theta \leq 2\pi$ ], 1 over $e^{i\theta}$, times $ie^{i\theta}\,d\theta$ [writes $\int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}\,d\theta$]. Which then you're going to get $i\int_0^{2\pi} d\theta$ [writes this], $2\pi i$ [writes $= 2\pi i$], which gives the well-known result of integral of 1 over $z$ minus a pole [points to $\int_L \frac{1}{z} dz$ inscription], a circle centered around it [points to diagram of path], gives you $2\pi i$ [points to $2\pi i$ inscription].

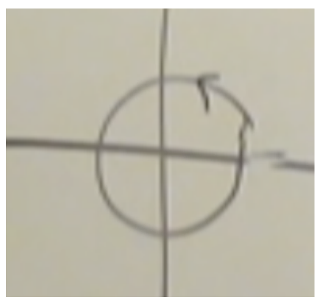

*Figure 1. Sean's diagram for the path L and Riley's counterclockwise orientation in Task 6.*

Sean conceded, and used their warrant to author an alternate approach implementing parametrization. Specifically, he first used embodied-symbolic reasoning to conclude that $z = e^{i\theta}$ is a parametrization of their path (line 13). Using this now as a datum, he further concluded that $z'(\theta) = ie^{i\theta}$, evidencing symbolic reasoning (line 13). As spokesman, Sean implemented embodied-symbolic reasoning to re-write the original integral, incorporating this new parametrization. The embodied aspect of this rewriting came from the decision to allow theta to vary from 0 to $2\pi$, a decision qualified by the phrase, "theta is of course from these values" (lines 13-15). Sean symbolically simplified this integral to obtain $i\int_0^{2\pi} d\theta$, and claimed that they obtained the "well-known result" of $2\pi i$ (lines 16-18). This sole argument for Task 6 is summarized in Figure 2.

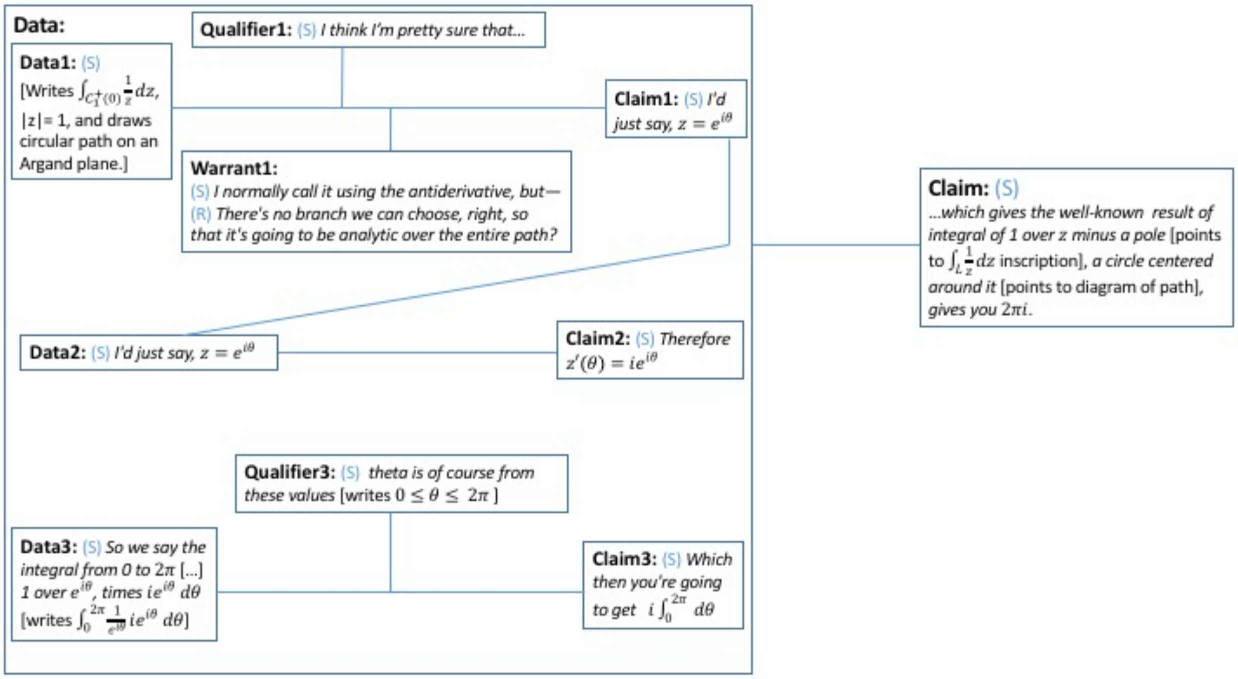


*Figure 2. Toulmin diagram for Riley and Sean, Task 6.*

## General Findings and Implications

With the above sample analysis in mind, I now elucidate some general findings that address my aforementioned research questions. I also discuss teaching and research implications associated with these results. Recall that my first research question regarded how undergraduate student pairs attended to the assumptions pertaining to integration theorems. In the present study, neither pair of participants appeared confident nor certain about the assumptions needed for employing certain tools, approaches, or theorems. For instance, Riley and Sean repeatedly questioned themselves in a previous task about whether the integrand function needs to be differentiable in order to employ parametrization. By explicitly qualifying such arguments, and in conjunction with my follow-up questioning, they eventually reached a consensus that the function only needs to be continuous. However, because they did not spend significant time in their course carefully justifying continuity arguments, the students exhibited substantial difficulty justifying why given functions, such as $\bar{z}$, are continuous or not. In particular, they pursued limit calculations to try to show this function was not continuous, but muddled their symbolic limit inscriptions.

Although Dan and Frank exhibited more confidence and decisiveness when deciding a function's continuity, they faltered a bit when justifying their application of Cauchy's Integral Formula in the above task. In particular, when Dan claimed they could produce a simply-connected domain containing the path $L$, Frank questioned the existence of such a domain, and his attempt at drawing one resulted in a domain that was not simply-connected. However, as with the above example, Dan and Frank's eventual consensus resulted from an explicit modal qualifier. The importance of such explicit qualifiers across the interviews was that they often led to follow-up arguments wherein the participants discussed assumptions in greater detail, including their applicability to the integral at hand. As such, my findings corroborate previous researchers' contention that one should consider the full Toulmin (2003) model when analyzing undergraduate level mathematical arguments.

My second research question inquired about the nature of students' invocation of Tall's (2013) three worlds during collective argumentation about complex integration. Quite unsurprisingly, my participants' formal reasoning dealt primarily with Cauchy's Integral Formula, the Cauchy-Goursat Theorem, the Cauchy-Riemann equations, and related results when evaluating specific integrals. However, more illuminating were the ways in which participants instantiated formal-symbolic, formal-embodied, or embodied-symbolic reasoning to justify the implementation of such theorems. For instance, Riley (and eventually Sean) explicitly instantiated embodied-symbolic reasoning by drawing arrows on the whiteboard between symbolic inscriptions and embodied paths of integration which were drawn on the board. Participants also expressed a symbolic answer next to a particular embodied path of integration by writing "= 0" next to a diagram of a closed path containing no singularities, for example. When discussing limits and path-independence, all four participants produced symbolic limit inscriptions, but also conveyed corresponding dynamic gestures embodying their chosen paths of approach. In one task, Riley and Sean demonstrated a purely embodied method for integrating the conjugate $\bar{z}$. The pair plotted tangent vectors along the circular path of integration and conjugates resulting from reflection transformations, and Riley and Sean also enacted visual vector addition.

Accordingly, the manners in which students intertwined embodied reasoning with symbolic and formal reasoning highlight the importance of visualization and geometry in the study of complex integration. Although complex variables courses tend to focus on symbolic computations and applications involving integration, the above examples point to an important consideration for teaching such a course. Specifically, they suggest that instructors might want to more explicitly highlight how the symbolism that abounds during the integration unit of a complex variables course can intertwine with the embodied and formal worlds. For instance, after providing a formal definition for a simply-connected domain or a simple curve, students could benefit from drawing numerous examples and counterexamples with one another. At times, my participants conflated some of these formal requirements, suggesting that additional care should be taken to produce examples that satisfy one requirement but not another. Despite participants' occasional struggles with formal hypotheses, both pairs were cognizant of the *thinking real, doing complex* (Danenhower, 2000) phenomenon, and avoided inappropriate applications of it. For instance, all participants voiced concerns such as "I'm tempted to think of this in terms of real numbers, but I know the analogy doesn't work" at various times during the interviews.

Finally, my study complements and extends the mathematics education literature regarding students' mathematical argumentation, particularly regarding how Toulmin's (2003) model is adopted to the context of *collective* argumentation. Specifically, not only did my participants' explicit qualifiers catalyze new arguments, but follow-up arguments also ensued when individuals challenged each other's assertions. According to Krummheuer (2007), individuals participate in collective argumentation in two ways: (1) the production of statements categorized according to Toulmin's model, and (2) an individual's speaker role (author, relayer, etc.). Notice that both of these forms of participation primarily serve to either introduce new ideas or support/re-voice existing ideas. However, they do not account for disagreement between parties or changing one's own mind following internal reflection. Accordingly, I contend that a third type of participation can drive collective argumentation, namely *challenging*.

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## Students' Usage of Visual Imagery to Reason about the Divergence, Integral, Direct Comparison, Limit Comparison, Ratio, and Root Convergence Tests

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*This study was motivated by practical issues we have encountered as second-semester calculus instructors, where students struggle to make sense of the various series convergence tests, including the divergence, integral, direct comparison, limit comparison, ratio, and root tests. To begin an exploration of how students might reason about these tests, we examined the visual imagery used by students when asked to describe what these tests are and why they provide the conclusions they do. It appeared that each test had certain types of visual imagery associated with it, which were at times productive and at times a hindrance. We describe how the visual imagery used by students seemed to impact their reasoning about the convergence tests.*

*Key words:* calculus, sequences, series, convergence tests, visual reasoning

This study was motivated by practical issues we have encountered as second-semester calculus instructors, where students work with the concepts of sequences ($a_n$), series ($\sum_{n=1}^{\infty} a_n$), and the notion of convergence. Students are typically supposed to learn several convergence tests that can be used to determine whether a given series will converge or diverge. Students struggle with these, both in terms of calculation and in terms of reasoning about why these convergence tests work (i.e. why the results are valid). In appealing to the research literature for insight into this topic, we found that while studies have examined student understanding and reasoning about sequences and series (e.g., Alcock & Simpson, 2004; Mamona-Downs, 2001; Martinez-Planell & Gonzales, 2012; McDonald, Mathews, & Strobel, 2000; Tall, 1992; Tall & Vinner, 1981), there is very little work done about how students reason about *convergence tests* specifically.

González-Martín, Nardi, and Biza (2011) noted that visual representations of convergence were mostly limited to depictions of the integral test. Earls and Demeke (2016) have found that students made many types of mistakes or errors when testing convergence, such as using an inappropriate convergence test or failing to check whether a series meets the criteria for a given convergence test. Earls (2017) also found that students may confuse sequence convergence with series convergence while performing these tests, sometimes imagining them as interchangeable. While these few studies have provided an initial foray into the topic of convergence tests, we feel there is much work to be done in this area. This study is meant to contribute by investigating the question: How do students reason about the convergence tests presented in second-semester calculus? Additional specificity about "reasoning" is given in the "Framework" section.

### Recap of the Convergence Tests

In our study, we examined the divergence, integral, direct comparison, limit comparison, ratio, and root tests. For our purposes, the "p-test" is considered a special case of the integral test and what we call "direct comparison" is often simply called the "comparison" test. The divergence test states that if $\lim_{n\to\infty} a_n \neq 0$, or does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges. The integral test states that if $f(x)$ is continuous, positive, and decreasing on $[1,\infty)$, and $a_n = f(n)$, then the series converges if and only if $\int_1^{\infty} f(x)dx$ converges. The direct comparison test begins with the assumption that $0 \leq a_n \leq b_n$ for all $n$. Then, if $\sum_{n=1}^{\infty} b_n$ is convergent, so is $\sum_{n=1}^{\infty} a_n$. If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$. The limit comparison states that for positive sequences

$a_n$ and $b_n$, if $\lim_{n\to\infty}(a_n/b_n) = c$ where $0 < c < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge. The ratio test begins with the assumption that $\lim_{n\to\infty}|a_{n+1}/a_n| = L$. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent. If $L = 1$, the test is inconclusive. The root test begins with the assumption that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent. If $L = 1$, the test is inconclusive.

## Framework: Visual Imagery

For this study, we narrowed our scope on reasoning by focusing on reasoning based on *visual* imagery. We chose a visual imagery perspective because it has been well-studied and advocated for in mathematics education research. Also, it has been applied specifically to sequences and series (Alcock & Simpson, 2004), and there is a small amount of information known about how textbooks use visual imagery for the integral test (Martinez-Planell & Gonzales, 2012). For our framework, we began with Presmeg's (1986, 2006) five visual imagery categories, shown in the first five lines of Table 1. During analysis, we also remained open to the possibility of additional visual imagery types being added to this framework. In fact, we identified two useful additional categories (the last two lines in Table 1). First, many students used the *visual appearance* of the symbols and expressions in reasoning. Second, students often used their hands to *spatially locate* conceptual objects, like a sequence or a series, in the physical space in front of them, similar to the use of a "signer's box" in sign language. These two categories were added to our framework.

*Table 1. Definitions and operationalization of the visual imagery categories we used in our study.*

| Imagery | Definition | Operationalization: The student… |
|---|---|---|
| Concrete | Static image or picture in the mind | …produces a non-moving/changing image, whether with pencil/pen/fingers or verbally described. |
| Pattern | Imagined relationship stripped of concrete details | … quickly recites a specific structure or pattern pertaining to a convergence test. |
| Memory of Formula | Image recall of a literal formula or expression | …writes, gestures, or verbalizes a generic symbolic template associated with a convergence test. |
| Kinesthetic | Imagery inherently using physical movement | …uses bodily movement as an inherent part of the initial image (not just for communication purposes). |
| Dynamic | A static image that is then moved or transformed | …begins with a static image and then describes it as transforming (including if gestures are used). |
| Symbol Appearance | Visual look of a symbol or symbolic expression | …uses the way a symbol or expression looks to make a conclusion, comparison, or connection. |
| Spatial Location | Physically locating a conceptual object in space | …uses gestures to "place" an object in physical space around them, or to references those objects. |

## Methods

We used semi-structured interviews with nine second-semester calculus students (labelled A–I) who had recently learned about series convergence tests in their calculus course. The students came from the same "large-lecture" calculus course (~250 students), which was taught in a fairly traditional manner. However, the instructor did attempt to incorporate discussions about what series convergence meant, typically centered on sequences terms becoming small "fast enough" for the series to converge. The students were selected based on their performance to three series convergence questions, with four correct on all three questions and five correct on two questions.

In the interviews, the students were asked to discuss the six convergence tests. For each test, the students were asked (a) to describe what that test is, (b) to explain why that test works for assessing convergence, and (c) to explain why the conclusions of each test are justified. The first part gave us information on the student's knowledge of each test's contents, and the second and third parts allowed us to examine the visual imagery and visual reasoning used by the students.

The analysis consisted of four phases. In the first phase, we reviewed the interviews to look for additional possible visual imagery categories, which produced the two described previously. In the second phase, we applied our visual imagery framework (Table 1) to the data, coding every instance in each interview that fit within any of those categories. Our coding was then independently checked by a separate research assistant. Any changes made by that assistant were reviewed again by us, to make final decisions. For each student, counts were made for how many instances of each type of imagery occurred during their discussion of the individual convergence tests. Aggregate counts were also tabulated across all nine students. In the third phase, we noted trends of how each type of visual imagery was typically used across the students. This was done by grouping together all instances of one type of imagery and looking for commonalities. Then, in the fourth phase, we examined how the different types of imagery influenced how students reasoned about each type of convergence test. This was done by looking at whether certain types of imagery proved helpful, or not, for making sense of the each convergence test.

## Results

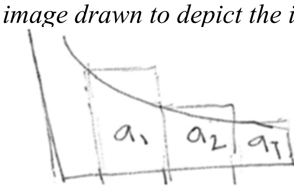
We organize the results as follows. First, we describe, generally, the trends we saw for how each type of visual imagery was used across the students. Then, we provide summary frequencies for each kind of visual imagery used for each convergence test. Last, we explain how these types of visual imagery affected how the students reasoned about each convergence test.

### How Students Generally Used Each Visual Imagery Type

To acquaint the reader with the overall trends in student imagery use, we start our results by first describing the common ways that each type of visual imagery was present in the data set.

**Concrete imagery.** Students frequently used concrete imagery. For the integral test, they drew the "typical" textbook image (Figure 1) of a continuous function passing through the "dots" representing the sequence, with rectangles having heights equal to the sequence values.

*Figure 1. Typical concrete image drawn to depict the integral test (from Student C).*



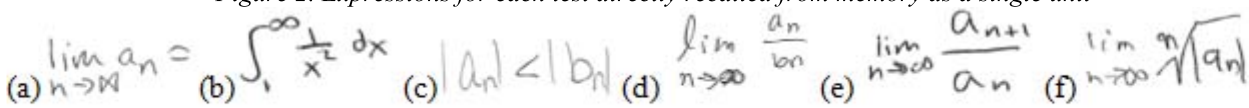

In fact, students drew many decreasing "functions," either on paper or in the air. Students also used concrete imagery to imagine sequences as an ordered list of numbers $a_1$, $a_2$, $a_3$,... . This was evidenced by gestures where students would "point" to the imagined successive numbers as they described a sequence. Concrete imagery was also frequently used by students to imagine "sizes" of numbers or sequence terms. For example, students used their index finger and thumb to make small or large "size" gestures, or they used the distance between their hands to show size.

**Pattern imagery.** Overall, there was less evidence for pattern imagery. The main instances of this imagery were students quickly reciting the pattern of a convergence test's results. For example, when describing the ratio or root test, many students quickly recited the $L > 1$, $L = 1$, and $L < 1$ cases. Students seemed to have this pattern laid out visually in their minds, as they would sometimes point "up" when referring to the $L > 1$ case and "down" when referring to the $L$

< 1 case. Pattern imagery was also used, but much less so, for limit comparison, in quickly stating that $0 < c < \infty$ implied one thing and that 0 or $\infty$ implied the opposite.

**Memory of formulae imagery.** Each convergence test had an expression type recalled from memory by the students directly in its symbolic form (see Figure 2). These expressions seemed invoked as a single visual unit when the students initially discussed a given test. Note that for the integral test, students typically *verbally* stated that the integrand was the function obtained by replacing "$n$" in the sequence with "$x$," but usually wrote an example rather than a generic $f(x)$. For some expressions, there were variations in how the students imagined it, such as the expression $a_{n+1}/a_n$ with no "$\lim_{n\to\infty}$" on it, or some expressions having absolute values or not.

*Figure 2. Expressions for each test directly recalled from memory as a single unit*



**Kinesthetic imagery.** This imagery was the most extensively used by the students, largely because of the kinesthetic nature of how the students seemed to think about convergence and divergence. This is likely related to the instructor's discussions of sequence terms becoming small "fast enough." Students frequently gestured off to the right when talking about these concepts, sometimes "downward" for convergence and "upward" for divergence (though not always). Another common place this type of imagery was used was when students thought of the series as summing up the terms of the sequence. The students sometimes made gestures like "collecting" terms together, or "grabbing" them one by one, implying *action* during summation.

**Dynamic imagery.** Dynamic imagery mostly showed up when students began with a *symbolic expression* and then imagined it transforming in some way. We highlight that this is different from symbolic manipulation, because it was the *imagery* associated with the expression itself that changed. For example, Student G, in describing how the root test is convenient for cancelling off $n$-th powers, stated, "If you look at just what's *inside* [student emphasis] the $n$-th power, you'll kind of have a better idea of what's going on." This statement was accompanied by a gesture where he put his rounded his hands next to each other and then shrunk the space between his hands as though "zooming." Instances like this one suggest that the students were implicitly imagining some transformation to the symbolic expression itself.

**Symbolic appearance imagery.** This type of imagery was less common, but was used across the tests. When discussing the integral test, the students generally used the appearance of the sequence to write the function for the integral. This was done by simply swapping any "$n$" in the sequence formula for an "$x$" in the integral. Next, several students discussed the comparison tests in terms of how related the symbolic expressions of the two sequences were. For example, Student A began explaining the direct comparison test by using the example of two series with sequences $1/(n^2 + 2)$ and $1/n^2$. She stated that she used these two because, "You're just comparing something that is similar." Also, the students frequently invoked sequences of the form $[]^n$ as examples for the root test, and sometimes used sequences with factorials for the ratio test. These suggest the usage of symbolic appearance to match some series with certain tests.

**Spatial location imagery.** Students frequently used a hand or finger to locate a number, a sequence, or a series in the space in front of them. While this type of visualization occurred throughout the interviews, it occurred more often when students discussed the comparability of sequences or series. For example, Student B described the limit comparison test by stating, "If it's not zero or infinity, you know they are related to each other enough that they're going to do the same thing." As she said this, she brought her fingers together at two locations on the table that were close to each other. Later, she stated, "If it goes to zero, if it goes to infinity, it gives you the same result, but you can't compare them because they're too different." In this case, she cupped her hands in two different locations that were much farther apart.

**Summaries of Visual Imagery Used**

We next provide summary counts for how often each type of imagery was used by the students while discussing each convergence test (Table 2). The percentages are out of the total number of instances coded for *that* particular convergence test. Each test had certain types of imagery used more frequently than other types, and we shaded in gray any having at least 20%. We note that within each test, there was a reasonable amount of consistency across the students in terms of which types of visual reasoning they employed for that convergence test. Thus, we consider it sufficient for this report to only show the aggregate data across all students.

*Table 2. Frequencies and percentages of each type of visual reasoning for each convergence test.*

| | Divergence ($n$ = 103) | Integral ($n$ = 97) | D. Comp ($n$ = 69) | L. Comp ($n$ = 79) | Ratio ($n$ = 160) | Root ($n$ = 86) |
|---|---|---|---|---|---|---|
| Concrete | 15 (15%) | 39 (40%) | 12 (17%) | 7 (9%) | 44 (28%) | 9 (10%) |
| Pattern | 0 (0%) | 0 (0%) | 2 (3%) | 4 (5%) | 13 (8%) | 7 (8%) |
| Formulae | 9 (9%) | 4 (4%) | 5 (7%) | 13 (16%) | 14 (9%) | 7 (8%) |
| Kinesthetic | 46 (45%) | 22 (23%) | 12 (17%) | 18 (23%) | 52 (33%) | 18 (21%) |
| Dynamic | 13 (13%) | 2 (2%) | 6 (9%) | 2 (3%) | 11 (7%) | 23 (27%) |
| Sym App | 3 (3%) | 14 (14%) | 9 (13%) | 10 (13%) | 7 (4%) | 18 (21%) |
| Spatial Loc | 17 (17%) | 16 (16%) | 23 (33%) | 25 (32%) | 19 (12%) | 4 (5%) |

**Using Visual Imagery to Reason about the Convergence Tests**

In this final results subsection, we now describe how these types of visual imagery appeared to influence the students' reasoning about each of the convergence tests.

**Divergence test.** Kinesthetic imagery seemed to help the students reason about the divergence test. Students B, C, E, F, and G visualized active summations of the sequence terms and gestured grabbing or gathering the terms into a total. Many also raised their hand up and to the right to talk about how adding up infinitely non-negligible terms would result in a total that diverged. However, the dichotomous nature of the "upward" divergence and "downward" convergence motions may have caused occasional confusion. Student I used this dichotomous visualization to conclude that *any* sequence with limit zero would have a convergent series. She stated that the purpose of all other tests was simply "making it easier to take the divergence test."

For this test, students also used the concrete imagery of a list of sequence terms. However, a problematic component of this imagery arose for Students D and I. These students envisioned sequences as having a "final term" that is the value of the limit (cf. Davis & Vinner, 1986). For example, Student D was discussing a sequence with limit "$e$," and said, "The last number that we're going to add is $e$. So, there is a possibility that all of the numbers before that might be small enough so that the whole thing doesn't go to infinity, or the series doesn't go to infinity."

**Integral test.** A significant portion of the students' imagery used for the integral test involved the type of concrete image shown in Figure 1. However, importantly, while all of the students were able to produce this type of image, most of them did not know how to reason about it. In fact, only Student C was able to describe how the areas of these rectangles could be used to show that $\sum_{n=2}^{\infty} a_n \leq \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} a_n$. Several students' reasoning simply focused on the visual aspect of how the heights of the rectangle "followed" the graph. For example, Student C stated, "It's drawing a line through all your little points, and if that line converges to something, then that means that your series will converge." Thus, it was the simple visual similarity between the curve and rectangle tops that became the salient feature, rather than the comparison between the area under the curve and the area of the rectangles. Student H even compared the rectangles to "a kind of histogram-like thing," which likely elicited ideas about how statistical histograms can sometimes be seen as an approximation to a population distribution curve.

**Direct comparison test.** The students were generally successful in reasoning about this test, mostly through identifying "comparable" series and sequences. The students often spatially located two series near each other when describing compatibility, or far from each other when describing incompatibility. Symbolic appearance was a key reasoning tool in identifying comparable series. The students often described two series as comparable if the sequences had the same "leading term" symbol, such as $1/n$ and $1/(n–1)$, or $1/n^2$ and $1/(n^2+1)$. One issue with reasoning for this test was a failure to distinguish between comparing sequences, $a_n \leq b_n$, versus comparing series, $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$ (cf. Earls, 2017). For example, Student F stated, "So, when you compare to a series that you know diverges, if it's greater than that series, then you know that this one also diverges." Student F compared the series, rather than the sequences as stipulated by the test. Focusing on symbolic appearance may have led some students to overlook the differences between sequences and series, as well as which one the condition of this test uses.

**Limit comparison test.** For this test, as with direct comparison, the students also frequently reasoned about "comparability." Symbolic appearance imagery again helped the students know what comparisons to make, with Student H even suggesting that the point was to find a "prettier" version of the given sequence. However, the students had more difficulty reasoning about why the test actually works. Some knew that a finite, non-zero result meant that the two series would do the same thing, but they were unsure how to justify it. Only Students B and C gave possible reasons. They loosely argued, based on kinesthetic imagery, that as the sequences "went" to infinity, their terms became more similar to each other. The spatial location imagery was also at times helpful, but at other times not. For example, Student F justified the result "infinity means inconclusive" by saying, "If you compare them and they go to infinity [places one hand off to the right and the other off to the left], then it kind of seems like they're not similar at all." Infinity seemed to have induced a spatial arrangement that led Student F to correctly believe the two sequences under consideration were not "similar." However, he then struggled to use the same type of reasoning for the result "zero means inconclusive." He said, "But zero, I guess, they're also not similar [hesitantly placing hands apart]… I don't know." It appeared that the use of the word "zero" felt inconsistent with placing two objects far apart, which interfered with Student F's reasoning and led him to doubt his conclusion.

**Ratio test.** All of the students struggled to reason about the ratio test. Most struggled to do more than describe the computational process of using the test. Four students (B, E, F, and G) made incorrect connections to the limit comparison test, based on the similar symbolic appearance of $a_n/b_n$ and $a_{n+1}/a_n$. Problematically, this led Student E to claim that the ratio test results "should be the same as the limit comparison test." Based on the analogy to limit comparison, Students E and G used dynamic imagery to incorrectly imagine the term $a_{n+1}$ as representing some sort of graphical shift of the $a_n$ sequence. Student E imagined $a_{n+1}$ as an "upward" shift of $a_n$, and Student G described $a_{n+1}$ as, "If you add one, you shift the graph, basically, to the left." On the other hand, computationally, students tended to use factorial examples, suggesting that that symbolic appearance was at least beneficial in determining one type of series for which it is helpful to *use* this test. Lastly, two students (D and H) used a problematic concrete image of a graph with a horizontal line at $y = 1$ to reason about the ratio test results. They incorrectly imagined that if the sequence converged to anything below that line, the series converged, which seemed to conflate sequences and series convergence (cf. Earls, 2017).

**Root test.** The root test was also quite difficult for the students to reason about. Students B and C, however, made meaningful comparisons to the geometric series based on the symbolic similarity of the $[]^n$ format. This reasoning is used in one proof of the root test. Thus, in this case, symbolic appearance played a useful role in these students' reasoning. Other students attempted to use dynamic imagery to help provide some rationale for the test. The students saw the root test as getting rid of the exponent, *n*, allowing them to focus in on the "inside" of the sequence. They were not able to provide a rationale for why the test had the results that it did, though. Yet, Student D did use dynamic imagery in a different way, explaining that *n*-th roots of numbers less than one produce larger values. He described a sequence converging to zero and imagined a term for a very large *n*. "That would be a really small value. And then when we take the *n*-th root of that, that will increase the value [spreads hands apart]. If that increased value is between 0 and lesser than 1, that means that it went to 0 fast enough." Lastly, pattern imagery also played a useful role for most students by assisting in the quick recall of the appropriate test results.

## Discussion

In this study, we saw differences in the type of imagery students relied on to reason about each test, such as kinesthetic imagery for the divergence test, concrete imagery for the integral and ratio tests, and spatial location imagery for the two comparison tests. Further, we saw that some tests were easier to reason about, like the divergence and direct comparison tests, and others were more difficult to reason about, like the integral, ratio, and root tests. We identified particular aspects of visualization that helped or hindered the students' reasoning. For example, symbolic appearance helped students reason about the direct comparison and root tests, but then seemed problematic for reasoning about the ratio test. Dynamic imagery helped students focus their attention on the relevant parts of a symbolic expression. Concrete imagery was useful in imagining "sizes" or sequence terms, but was problematic when that imagery was not well understood, like incorrectly "completing" the image of a sequence by including a "last term." The common picture used for the integral test also turned out to be poorly understood by the students. This result has implications for how instructors may wish to use concrete imagery for convergence tests, such as using formative assessment to ensure that students understand what those images are meant to be representations of. Lastly, spatial location imagery may have been helpful for students in cognitively managing the various objects being referred to in a test. This lesser-studied type of visualization may need more attention in future work.

Our study has many connections to research on sequences, series, and convergence. The sequence list imagery is closely related to the SEQLIST conception described by McDonald et al. (2000). Martinez-Planell and Gonzales (2012) argued that a SEQLIST conception is less productive for understanding series convergence than if the sequence is understood to be a function from the natural numbers to the reals. Our results in some ways agrees with this claim, but in other ways disagrees, since this imagery was at times helpful for the students. However, it is true that the sequence image with a "last term" is certainly problematic (see Davis & Vinner, 1986). For this misconception, our results may actually indicate a possible *imagery*-based origin.

The concrete imagery of a continuous graph for the integral test also shows that students seemed to have internalized this common image (see González-Martín et al., 2011). However, the students' understanding of this image seemed far from the intention of the picture. Finally, we can see that series of the form $\sum_{n=1}^{\infty} 1/n^m$ for $m = 1, 2$, or 3 were commonly used and might indeed be "prototypical" examples of series, as mentioned by Alcock and Simpson (2002).

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# A Study of Calculus Students' Solution Strategies when Solving Related Rates of Change Problems

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*Contributing to the growing body of research on students' understanding of related rates of change problems, this study reports on the analysis of solution strategies used by five calculus students when solving three related rates of change problems where the underlying independent variable in each problem was time. Contrary to findings of previous research on students' understanding of related rates of change problems, all the students in this study were able to translate prose to algebraic symbols. All the students had a common benchmark to guide their overall work in one of the tasks but no benchmark to guide their overall work in the other two tasks. Three students exhibited weaker calculational knowledge of the product rule of differentiation. Directions for future research and implications for instruction are included.*

*Key words:* related rates, implicit differentiation, problem solving, calculus education

Related rates of change problems form an integral part of any first-year calculus course. However, there have been relatively few studies that have examined students' reasoning about related rates of change problems. Engelke (2007) argued that there is a dearth of research that examines how students understand and solve related rates of change problems in introductory calculus. Findings of a comprehensive review of literature on students' understanding of various topics in college calculus by Speer and Kung (2016) indicate that studies on related rates of change are scarce. Of the few studies involving related rates problems, Piccolo and Code (2013) found that students had computational difficulties when calculating derivatives involving more than one time-dependent variables. Engelke (2007) described several beneficial components for successful solutions, including drawing diagrams, determining functional relationships (algebraic equations), and checking the answer. She also added that proficiency with the chain rule helped the students make sense of the problem context and the components of the functional relationship. Other studies have focused more on how students read and interpret the problem statement (Martin, 2000; White & Mitchelmore, 1996). White and Mitchelmore found that students struggled to "symbolize," or mathematize, the related rates problems, often being unsure as to how to use all the given information in a related rate problem. Martin also found that students struggled to convert the written prompt into a mathematical structure on which the students could operate.

While these studies have provided beneficial information about how students set up and solve related rates of change problems, there is still much to be explored about the specific difficulties that limit students' success when solving such problems, which is the motivation for this study. Thus, to build on these studies, we intend to explicitly examine students' understanding of the key steps that are generally involved in the process of solving related rates of change problems where the underlying independent variable is time. In particular, our study was guided by the following research question: What do calculus students' solution strategies when solving related rates of change problems reveal about the difficulties that limit students' success when solving such problems?

## Related Literature

For a working definition, a mathematical task is said to be a related rates of change problem (abbreviated as "related rates problem") if it involves at least two rates of change that can be related by an equation, function, or formula. As noted earlier, research on students' understanding of related rates of change is sparse. Piccolo and Code (2013) analyzed Calculus I students' written responses to related rates problems at a large research university. This analysis revealed that the students were successful in performing the early steps of solving the problems (e.g., identifying the quantities involved in the problem and finding an equation that relates these quantities). However, using implicit differentiation was a major issue for the students' success in the problems. More specifically, Piccolo and Code reported that implicitly differentiating functions with several time-dependent variables, with respect to time, was problematic for a majority of the students. Consequently, most of the students were unsuccessful in solving the problems they were given. Piccolo and Code asserted that students' difficulties with solving related rates problems stems from a lack of facility with the process of differentiation, rather than a misunderstanding of the physical context of such problems.

Engelke (2007) used a teaching experiment, consisting of six teaching episodes, to examine how calculus students understand and solve related rates problems. Engelke argued that knowledge of the chain rule appeared to help the students in solving the problems they were given. This researcher reported that the students had difficulty imagining each variable in the problems as a function of time especially when time was not explicitly mentioned in the problem statement. Engelke further proposed a framework for analyzing students' work when solving related rates problems. Details of this framework are provided in the next section.

Martin (2000) analyzed students' responses in a problem-solving test containing several items assessing the students' ability to solve related rates problems in an introductory calculus class. Martin reported that overall performance was poor, and claimed that "the poorest performance was on steps linked to conceptual understanding, specifically steps involving the translation of prose to geometric and symbolic representations" (p. 74). White and Mitchelmore (1996), who studied the conceptual knowledge of 40 undergraduate mathematics majors when solving four application problems (including two related rates problems) at the level of introductory calculus, reported similar results. White and Mitchelmore also found that the students tended to replace unfamiliar variables with either $x$ or $y$, an idea that has come to be known as the "$x, y$ syndrome" (p. 89).

## Conceptual Framework for Related Rates

This study uses Engelke's (2007) framework, which characterizes the phases involved in solving related rates problems in calculus. The framework emerged from interviews, using a think-aloud protocol, with three mathematics professors who were solving three related rates problems similar to the tasks we used in this study. The framework lists five phases that one follows when solving related rates problems. These phases are: (1) draw and label a diagram, (2) determine a meaningful functional relationship, (3) relate the rates of change, (4) solve for the unknown rate of change, and (5) check the answer for reasonability. However, in our study, we realized the need to add an additional "orienting" phase, because the students often spent time simply acquainting themselves with the problem context. We label this the "zero-th" phase, (0) orienting to the problem. The following is a description of each of these phases.

The orienting phase consists of the solver carefully reading the problem (aloud or silently), with the goal of identifying what the solver considers to be important information. More specifically, the solver identifies given quantities and the required quantity (the unknown rate).

Then, in the diagram phase, the solver draws and labels a diagram illustrating the relationship of the quantities in the problem. In Task 1 (methods section), for example, the solver may draw a right triangle where the horizontal leg of the triangle represents the distance between the westbound plane and the airport, the vertical leg of the triangle represents the distance between the northbound plane and the airport, and the hypotenuse of the triangle represents the distance between the two planes.

In the functional relationship phase, the solver constructs a meaningful relationship (algebraic equation) between the quantities he/she identified while orienting himself/herself with the problem or while drawing a diagram. In the case of Task 1, the solver may use the Pythagorean Theorem, $x^2 + y^2 = z^2$, where $x$ is the distance of the westbound plane from the airport at any point in time, $y$ is the distance of the northbound plane from the airport at any point in time, and $z$ is the distance between the two planes at any point in time. During the relate the rates phase, the solver implicitly differentiates with respect to a time variable the algebraic equation he/she identified as relating the quantities in the problem. The process of implicit differentiation results in the creation of a new equation that shows a relationship of the rates of change involved in the problem. In Task 1, the process of implicit differentiation with respect to a time variable $t$, would result in the equation, $2xdx/dt + 2ydy/dt = 2zdz/dt$, where $dx/dt$ is the speed of the westbound plane at any point in time, $dy/dt$ is the speed of the northbound plane at any point in time, and $dz/dt$ is the rate at which the distance between the two planes reduces at any point in time.

In the solve for the unknown rate phase, the solver substitutes all the given quantities in the new equation and then solves for the required rate of change. In Task 1, this means solving for the value of $z$ at a particular point in time when the quantities of $x$ and $y$ are known, using this value of $z$ together with the known values of the quantities $x$, $y$, $dx/dt$, and $dy/dt$ to solve for the unknown quantity, $dz/dt$. Finally, during the check the answer phase, the solver uses certain goals or benchmarks to guide their overall work. The goals or benchmarks include: (a) having a sense of knowing if the answer the solver found (the unknown rate) is higher or lower than would be expected, (b) expecting the answer to have a particular sign (positive or negative), and (c) expecting the answer to have particular units (e.g., miles per hour instead of miles). Checking the answer for reasonability in Task 1 may, for instance, mean having an awareness that the units of $dz/dt$ should be in miles per hour since it is a rate and that the sign of the value of $dz/dt$ should be negative as the distance between the two planes decreases over time.

## Methods

This qualitative study used task-based interviews (Goldin, 2000) with five students. The interviews lasted about 45 minutes, on average, and contained three tasks:

**Task 1:** Two small planes approach an airport, one flying due west at a speed of 100 miles per hour and the other flying due north at a speed of 120 miles per hour. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 miles from the airport and the northbound plane is 200 miles from the airport?

**Task 2:** A leak from the sink is creating a puddle that can be approximated by a circle, which is increasing at a rate of 12 $cm^2$ per second. How fast is the radius growing at the instant when the radius of the puddle equals 8 $cm$?

**Task 3:** For the next problem, let me give you a little background on a formula that we will use. Suppose a gas is inside of a container. Many gases under normal conditions follow the "ideal gas law," $PV = kT$, where $P$ is the pressure the gas exerts on the container, $V$ is the volume of the container, $T$ is the temperature of the gas, and $k$ is a constant. $P$ is measured in "atmospheres," $V$ is measured in cubic meters, and $T$ is measured in Kelvins. Kelvins is a lot like Celsius, except that it is scaled so that 0 means absolute zero (lowest possible temperature), which makes water's freezing point to be $273\ °K$. Do you have any question(s) about this formula, or any of the quantities [like temperature in Kelvins] before we proceed?

In a laboratory, an experiment is being done on a gas inside a large, flexible rubber balloon. For the experiment, the temperature of the gas is being heated at a rate of 8 degrees per second. At one point, when the temperature of the gas is $300\ °K$, the pressure is 1.5 atmospheres, the volume of the gas is one cubic meter, and the volume of the gas is growing at a rate of 0.01 $m^3$ per second. At that moment, is the pressure in the balloon increasing or decreasing? What is the rate of that increase/decrease?

The students worked through these tasks while the interviewer asked clarifying questions about their work. After the student concluded their work for each problem, the interviewer asked the following questions about the task and the content of their solutions: (a) Have you seen a problem like this before? (b) What did you need to do to solve this task? (c) What does your answer tell you? (d) What does the sign of your answer tell you? (e) What are the units for the rate you found? (f) What does each quantity throughout your solution [including amount quantities and rate of change quantities] mean? (g) What does each computational step mean in terms of the quantities? (h) Will your answer for this problem be the same for all points in time for this context? Most of the students' time was spent on Task 1 while the least amount of time was spent on Task 3. We remark that Task 3 was not a routine task to the students in that the students' prior experiences with related rates problems in course lectures and in the course textbook was limited to problems similar to Task 1 and Task 2.

**Setting, Participants, and Data Collection**

The study participants were five undergraduate students (pseudonyms Ben, Bill, Jake, Nick, and Tim) at a research university who were enrolled in a traditional calculus I course in the summer of 2017. The course met twice a week (each meeting lasted for 2 hours and 45 minutes) for a duration of 12 weeks. The students were recruited via an official class roster obtained from their professor. The students were chosen based on their willingness to participate in the study. The students were familiar with the key ideas examined in the three tasks (instantaneous rate of change and the process of implicit differentiation) from course lectures and the course textbook. Three of the participants (Ben, Nick, and Tim) were Business/Economics majors while the other two students (Bill and Jake) were Engineering majors. Two students (Ben and Tim) had taken a high school calculus course prior to participating in this study. At the time of the study, three of the students were sophomores, one student was a junior, and the other student was a senior. The cumulative grade point averages (GPAs) of the five students had a mean of 2.02 on a 4.0 scale and a standard deviation of 0.71, suggesting that these were low performing students. As we would discuss in the concluding section of the paper, analysis of interviews conducted with this sample of students were both surprising and interesting at the same time. Data for the study consisted of transcriptions of audio-recordings of the task-based interviews and work written by the five students during each task-based interview session. When transcribing the audio recordings of the interviews, we used video recordings of the interviews to check what students were referring to when they pointed at something during the interviews in cases when such information could not be easily obtained from work written by the students during the interviews.

## Data Analysis

Data analysis was done in two stages. In the first stage, we used a priori codes, consisting of the five phases from Engelke's (2007) framework, plus the additional "orienting" phase we included. More specifically, we carefully read through each interview transcript and coded instances where each student reasoned about: (0) how they interpreted the problem, (1) drawing diagrams illustrating what is happening in each task, (2) constructing algebraic equations relating the quantities in each task, (3) differentiating the equations, (4) solving for the unknown rate, and (5) checking the answer (unknown rate) for reasonability. In the second stage of analysis, we looked for patterns in each of the codes identified in the first stage of the analysis. These patterns included trends in the students' understandings, or difficulties they had in connection with each of the phases of the related rates framework. The common understandings or difficulties in students' reasoning found in the second stage of our analysis provided answers to our research question.

# Results

Analysis of the data revealed that: (1) all the students were able to translate prose to algebraic symbols, (2) all the students had a common benchmark to guide their overall work in the first task, (3) students typically had no benchmark to guide their overall work in Tasks 2 and 3, and (4) three students exhibited difficulties with the product rule of differentiation.

## Translating Prose to Algebraic Symbols

All of the students successfully identified the appropriate algebraic equation that shows how the quantities in each of Task 1 and Task 2 are related (Task 3 already had a formula given to the students). That is, translating the text of each task into algebraic symbols, and relating these symbols using the appropriate equation, was not problematic for the students. As previously noted in the methods section, we argue that this may have been because the students had been shown how to solve problems similar to Task 1 and Task 2 through examples that were given during course lectures. Four of the students provided reasonable rationales for using the Pythagorean Theorem in Task 1. Bill is representative of this group of four students. When asked how he moved from the "right triangle," which he claimed was a "picture" of what is happening in Task 1, to the equation $x^2 + y^2 = z^2$, Bill stated that the two planes "are heading due west and due north, it's a right triangle, so use the Pythagorean Theorem [pointing at the equation $x^2 + y^2 = z^2$]." When probed on what the Pythagorean Theorem meant in terms of $x$, $y$, and $z$, Bill indicated that it "relates all of them together." We remark that Bill correctly interpreted $x$, $y$, and $z$ as distances that are measured in miles. More specifically, Bill interpreted $x$ as the distance of the westbound plane from the airport, $y$ as the distance of the northbound plane from the airport, and $z$ as the distance between the two planes. When asked to elaborate on what the Pythagorean Theorem meant, Bill stated, "this side [pointing at one side of the triangle which he labelled $x$] squared plus this side [pointing at another side of the triangle which he labeled $y$] squared, equals this side [pointing at the hypotenuse of the triangle which he labeled $z$] squared." We argue that Bill recognized that by virtue of the two planes flying west and north respectively, the distance between the two planes is given by the hypotenuse of a right triangle. This is rightfully so because the two planes are assumed to be constantly flying at the same elevation in the description of the task.

One student, Ben, knew that he had to use the Pythagorean Theorem to relate the quantities in Task 1, because it was used in a similar problem that was solved in class. Ben, however, did not provide a convincing rationale for using the Pythagorean Theorem in this task. When asked on why he used the Pythagorean theorem in this task, Ben said "I don't know, I just wrote it up here [pointing at the equation, $a^2 + b^2 = c^2$] because I figured that the problem might have something to do with it." Ben added, "I don't know, it's a habit I have had since I started learning calc, or geometry for a long time, and so I actually implemented it to the solution right here [pointing at the equation, $x^2 + y^2 = z^2$ which was part of his solution]." All the students provided reasonable rationales for using the formula for the area of a circle as an equation that relates the quantities in Task 2. Common among these rationales was that the puddle is circular, as described in the task description.

**Benchmarks used by Students to Guide their Overall Work**

None of the students used any goals or benchmarks to guide their overall work in Task 2 and Task 3. More specifically, they did not mention any specific way of having a sense about whether or not their answers to these tasks were correct. However, in Task 1, all the students had an expectation that the required unknown rate had to be negative regardless of whether or not their work leading to the answer is correct. The following excerpt, illustrates how Nick, for example, determined the benchmark for his solution to Task 1.

*Researcher:* I noticed that $c'$ came out as positive 160mph [Nick's Answer], what does that mean?

*Nick*: That it [$c'$] is increasing over time, like going up by increments by units so like if it's a hundred and sixty miles per hour now, over time it will go up to a hundred and seventy, a hundred and eighty, and it will get faster and faster. If it [$c'$] were negative, it would decrease. So in this case, I would assume it will be negative [changing 160mph to -160mph] because they [the two planes] are coming closer and closer [to each other], and they are getting near to the airport, so it [$c'$] will be decreasing.

We argue that Nick's claim that $c'$ should be negative may have been a result of two things. First, Nick may have correctly determined that $c'$ should be negative by reasoning about the context of the task, that is, the distance between the two planes is getting smaller as the planes approach the airport hence $c'$ should be negative. Second, Nick could simply be recalling a justification for $c'$ to be negative that was given by his calculus professor when he did a problem similar to Task 1 during course lectures.

**Difficulties with the Product Rule**

Three students (Tim, Bill, and Ben) incorrectly applied the product rule when differentiating the equation, $PV = kT$, in Task 3. Figure 1 shows how Tim used the product rule to differentiate $PV = kT$.

$$P\frac{dP}{ds}V + PV\frac{dV}{ds} = t\frac{dt}{ds}$$

*Figure 1. Tim's derivative of PV=kT.*

By incorrectly applying the product rule, Tim concluded that the derivative of $PV = kT$ with respect to a time variable $s$ (which he stated would be measured in seconds) would be

$PVdP/ds + PVdV/ds = TdT/ds$ instead of $PdV/ds + VdP/ds = kdT/ds$. Tim did not use the stated conditions in the task to determine the value of the constant $k$. Instead, he either discarded the constant $k$ or substituted a "1" for it when taking the derivative of $PV = kT$. Bill and Ben treated $k$ in a similar way when finding the derivative of $PV = kT$. Bill's work (Figure 2) is representative of how these two students differentiated the equation, $PV = kT$.

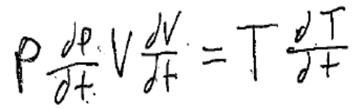

*Figure 2. Bill's derivative of PV=kT.*

Bill differentiated the equation, $PV = kT$, implicitly with respect to the time variable $t$ which he said would be measured in "seconds". As can be seen in the solution in Figure 2, Bill recognized that he had to use the product rule to find the derivative of $PV = kT$. He, however, incorrectly applied the product rule. More specifically, Bill's derivative of $PV$ is $PdP/dt * VdV/dt$ instead of $PdV/dt + VdP/dt$. His derivative of $kT$ is $TdT/dt$ instead of $kdT/dt$. The other two students (Nick and Jake) did not make any attempt of finding the derivative of $PV = kT$ or let alone mention that they have to find one. Instead, they systematically guessed the answer to the question of whether or not the pressure of the gas inside the balloon is increasing or decreasing.

## Discussion and Conclusions

Contrary to findings of previous research (Martin, 2000; White & Mitchelmore, 1996) on students' understanding of related rates problems, findings of this study indicate that translating prose to algebraic symbols was not the problematic part of the process for the students in this study. More specifically, the students in our study did not have any difficulty with symbolizing, algebraically, the quantities in each task in addition to finding an appropriate equation that relates the quantities mentioned in each task. This result is more striking given that the students in our sample were weaker overall, meaning that even these weaker students did not have issues with these parts of the process.

Interestingly, the students in this study expressed a common benchmark to guide their overall work in one of the tasks but did not express any benchmark to guide their overall work in the other two tasks. In particular, all the students indicated that the unknown rate they were trying to find in Task 1 (how fast the distance between the two planes is changing) had to be negative since the distance between the two planes decreased as the planes approach the airport. Future research might examine the role of problem context in students' use of benchmarks to guide their overall work when solving related rates problems. Finally, students' reasoning about the non-routine task (Task 3) revealed that most of the students in this study had difficulty using the product rule. As a recommendation, calculus instructors may need to help students develop greater facility with procedures such as the chain rule, product rule, and quotient rule prior to applying these rules when solving contextualized related rates problems (cf. Engelke, 2007). As a limitation of our study, we note that because the participants are five students from the same class, the results may not generalize.

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Peer Mentoring Mathematics Graduate Student Instructors: Discussion Topics and Concerns

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*We developed and implemented a peer-mentoring program at two US universities whereby nine experienced mathematics graduate student instructors (GSIs) each mentored three or four first- and second-year GSIs (novices). Mentors facilitated bi-weekly small group meetings with context-specific support to help novices use active-learning techniques and augment productive discourse (Smith & Stein, 2011). Meeting discussion topics were informed by novices' interests, concerns raised by both mentors and novices, and ideas from other small groups. We examined what topics from small-group peer-mentoring meetings novices valued and timing of the topics that mentors suggested for future cycles. We qualitatively coded meeting topics and analyzed novices' ratings of topics discussed. Results indicate specific topics novices valued and the importance of timing some topics appropriately, informing future professional development for GSIs. These results offer insight and synergy between educating GSIs and improving undergraduate mathematics teacher pedagogy.*

*Keywords:* Teaching Assistants (TAs), Professional Development, Peer-Mentoring

Mathematics graduate student instructors (GSIs)[1] teach hundreds of thousands of undergraduate mathematics students each semester, yet typically lack guidance and support to teach undergraduate students effectively (Rogers & Steele, 2016; Speer & Murphy, 2009). GSIs' initial teaching experiences represent a crossroad between how they teach in the short term in graduate school and in the long term as potential future faculty members (Lortie, 1975). Moreover, in this context, GSIs participate in professional development (PD) concurrent with their first couple semesters as instructors of record, responsible for the day-to-day interactions and content-delivery and assessments in undergraduate mathematics classrooms. That is, GSIs are uniquely positioned as a population of instructors who are simultaneously receiving and applying strategies and theories learned in PD seminars, courses, and other such opportunities. As researchers have documented (e.g., Belnap & Allred, 2009; Bressoud, Mesa, Rasmussen, 2015; Ellis, Deshler, & Speer 2016a; 2016b), PD opportunities and teaching assignments for GSIs vary significantly in mathematics departments and universities across the US, which makes it challenging to determine what and how PD for GSIs can be the most impactful and effective for improving student learning outcomes in undergraduate mathematics courses. In light of this challenge, we developed and implemented a peer-mentorship program to provide additional support for novice GSIs' learning to teach.[2] We saw this as a prime opportunity to study this undergraduate instructor population and find what pedagogical topics are perceived as valuable from GSIs' perspectives as novices discuss their teaching and concerns with one another and with peer-mentors in small-group meetings. Therefore, these topics, practically, inform the next iteration of the peer-mentorship program, but more broadly inform GSI education because it highlights topics of value and significance from the novices' perspective. We investigate the following two research questions:

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[1] GSI was used instead of TA (Teaching Assistant) because GSI targets the specific set of graduate students who are instructors of record.
[2] Supported by a Collaborative IUSE NSF grant (Awards #1544342 & 1544346).

1. What value do novice mathematics GSIs place on pedagogical topics from peer-mentoring small group meetings?
2. What pedagogical topics, and in what order, do experienced, mentor GSIs suggest for future cycles of peer-mentoring small groups?

## Related Literature and Framework

### Secondary Teacher Education and GSI Teacher Education

To support mathematics GSIs' development as instructors, we draw upon and learn from the history of teacher education. Initially, in the U.S., novice secondary teachers were given a short three-to-five-day orientation and then thrown into the classroom with a sink-or-swim mentality (Portner, 2005). Educational policies (e.g., No Child Left Behind and emergency teacher certification) provide multiple avenues to teacher certification, limiting uniformity in teacher preparation programs (Ganser, 2005). Thus, secondary mathematics education researchers suggest that mentoring allows secondary schools to align teachers' prior experiences with their cultural and professional expectations increasing teacher support and retention (Portner, 2005).

Similarly, mathematics graduate students arrive at universities with diverse teaching backgrounds and teaching experience with many novices having no-prior-teaching experience and some may have extensive undergraduate teaching experience if they previously taught while completing a different graduate degree (Rogers & Yee, 2017). Morover, in a similar vein as noted in the secondary education setting, collegiate institutions vary regarding how they prepare and support novice undergraduate instructors, including brief orientations or seminars for PD or required PD courses. Although these forms of PD can be useful in helping novices recognize important issues within teaching, they do not provide one-on-one support that a mentor can to address each individual's needs (Yee & Rogers, 2017). In this research project, therefore, we incorporated mentoring with the intention being to help GSIs navigate the new teaching expectations at their new university and mathematics department.

When applying a pedagogical model (teacher mentoring) to a new audience (from primary and secondary preservice teachers to mathematics GSIs), it is critical to justify the framework through empirical research (e.g., Speer King, & Howell, 2015) to make sure one does not overgeneralize. Empirical research in GSI teaching practices is also needed due to the limited research of undergraduate mathematics education (Speer, Smith, & Horvath, 2010). Therefore, we look to teacher mentoring research to justify our framework and study design.

### Peer-Mentoring for GSIs Drawing from Teacher Education Research

Emerging trends in K-12 mentoring indicates that workshops, classroom visits, and meetings are vital to provide improvement and develop sustainability in leadership where novices eventually become mentors (Ganser, 2005). Overlapping these results with Boyle and Boice's (1998) empirical research on university teacher mentoring, further emphasizes the importance of systematic meetings among mentors and novices. Therefore, the mentoring structure used in our project includes these identified key components for teacher mentoring: systematic small group meetings, observations, and post-observation discussions (Ganser, 2005; Rogers & Yee, 2017).

We focus specifically on peer-mentoring (instead of faculty-mentoring) because faculty's relationships with doctoral GSIs can become ethically complicated since the faculty member can also take on different positions of power; i.e., advisors, qualifying exam evaluators, and course instructors (Johnson & Nelson, 1999). Furthermore, when a mentor is a peer, they are more likely to be genuinely aware of the individualized pedagogical decisions and needs associated with a novice's current experiences (Yee & Rogers, 2017). In this study, experienced GSIs who apply to be mentors are selected and serve as guides and resources for novice GSIs. Prior to mentoring, we provided a research- and practice-based PD seminar for mentors, where mentors met with the mentor facilitators (authors) for 1hr/wk for 15 weeks to learn the roles and expectations of mentors (Portner, 2005; Yee & Rogers, 2017).

Following Speer et al.'s (2010) call for increased research in undergraduate teaching practices, this peer-mentorship program provided a unique and credible lens for examining GSIs' pedagogical needs. As Speer et al. point out, undergraduate teaching practices for GSIs are still in their infancy and also lack significant empirical research about how and what to teach GSIs in PD courses. In our research study, mentor small-group meetings had topics chosen by the mentors and novices as critical and time-sensitive to their current work as new instructors of record. Thus, mentor and novice GSI topics of discussion could offer the field important insight into what teaching topics are critical for GSI development from their point of view.

## Method

### Peer-Mentoring Program Participants

Experienced GSIs at two universities applied and were selected to be mentors by the researchers based on their teaching experiences (aptitude for implementing student-centered techniques), their pedagogical accolades (teaching awards and student evaluations), and most importantly their desire to help novices to improve teaching at their university (essay responses were required). A total of nine mentors, who were mathematics and statistics doctoral candidates, were involved in this study across the two universities.

The number of participants was determined by the average size of each university's mathematics GSI program. Novices who were teaching an undergraduate mathematics course for the first time were required to have a peer mentor as an aspect of the mandatory PD seminar for new GSIs in both universities' mathematics departments. Novices who were teaching these courses for the second time or who already took the PD seminar but did not previously have a teaching assignment were invited to participate. The peer-mentoring program continues over an entire academic year, but we focus on the 32 novices who participated during a single semester because of the timing of when we collected survey data about their experiences.

### Pedagogical Topics Data Collection and Analysis

On a written survey that listed all topics discussed in small groups during the semester for each of the nine mentors, 23 novices. Novices to provided feedback on how valuable they found topics that were discussed in their peer-mentoring, small-group meetings in two ways:

a. From the list of topics from meetings this semester, they found their mentor's name and rated the topics listed on a scale from 1-10 (1=not valued, 10=highly valued).
b. They looked at the topics listed under the other mentors' names and circled those that they believed could have been valuable to discuss in small-group meetings.

To address RQ1, we analyzed the survey responses by considering frequencies of ratings novices provided for part (a) about how valuable they considered the topics they personally discussed with their mentor and small group. Using clustering analysis (Willig & Stainton-Rogers, 2007), we categorized the specific topics small groups discussed by grouping them by themes of participatory structure and educational context (Table 1). Additionally, since different mentors discussed different topics, we analyzed novices' responses to part (b) on the survey by tallying the number of topics circled and determined frequencies for each topic.

*Table 1. Topic Categories Discussed During Peer-Mentoring Small Group Meetings*

| **Categories** | **Examples** | **No. of Mentors** |
|---|---|---|
| (A) Facilitating Collaborative Learning | Strategies to enhance student interaction (e.g., pro and cons of group work, anonymizing questions, and giving students a voice and a choice) | 5 |
| (B) Facilitating Student Engagement | Encouraging student participation; Motivating students; Teaching students with varying levels of background knowledge | 3 |
| (C) Facilitating Reflection | Reflecting on the semester thus far; Things to try next time you teach; Video reflection | 5 |
| (D) Facilitating Constructive Criticism About Teaching | Mock lessons; Discussing strengths the mentor observed in novice's lessons | 6 |
| (E) Creating & Using Formative Assessment During Class | Using formative assessments (e.g., minute papers, polling); Incorporating assessments during class time; How to monitor student learning in class | 5 |
| (F) Creating & Using Effective Summative Assessments | Writing exams, quizzes, or homework assignments | 5 |
| (G) Grading Assessments | How to grade (incl., consistency, remaining objective, & group grading) | 6 |
| (H) Managing Students in Class | Addressing aggressive/overbearing students; Helping with submissive/quiet students; Addressing mathphobia | 3 |
| (I) Managing Students Outside Class | Communicating with students (via emails or in office hours). Communicating about grades | 5 |
| (J) Negotiating GSI Small Group Meeting Behavior | Determining expectations for small group meetings | 6 |
| (K) Managing Time Outside Class | Work-life-school balance | 2 |
| (L) Creating & Modifying Lesson Plans | Ways to save time lesson planning; Creating emergency lesson plans; Modifying lesson plans | 3 |
| (M) Brainstorming Course-Specific Advice | Advice about teaching MATH X; Difficulties about teaching MATH X in the first Y weeks | 2 |

To investigate RQ2, we solicited input from mentors after they completed their first year mentoring. We asked mentors to provide a likely timeline for the topics from Table 1 that they would most likely use in small group meetings if they mentored again. Since small-group meetings were typically bi-weekly, we broke a fifteen-week semester out into two-week blocks and listed the 15$^{th}$ week alone. Mentors specified at most two categories from Table 1 for each 2-week span. They could add additional topics or exclude any of the thirteen categories from their timeline, as they saw fit. Seven of the nine mentors suggested timelines in this way. From these data, we determined the frequencies and timing for each topic, focusing on topics that were suggested by multiple mentors for a similar time period of the semester.

## Results

We first present results relevant to RQ1, focused on what small-group meeting topics novices valued. Based on the mean across all topics by all novices was a 7.85 out of 10 with a standard deviation of 1.46, providing a viable striation of the data into thirds with the partitions of 1-5, 6-8, and 9-10. Thus, when recording responses to part (a) in the survey, we considered their perceived value of a given topic to be reported as: *High* with a 9 or 10 rating, *Medium* with a 6-8 rating, or *Low* with a 1-5 rating. Frequencies for how valuable novices rated each topic are displayed in Figure 1. Since novices only attended small-group meetings with their peer-mentor, they had the opportunity to rate how valuable they would find topics that other small groups discussed. The frequencies of circled topics are displayed in Figure 2, where the percentages are out of the total number of circled items.

We can see there are some topics, that were reported as highly valued (Figure 1) that were also rarely circled (Figure 2). For instance, *grading* (Topic G) was discussed in six mentor's small group meetings, and it was ranked highly valued, but it was only circled 1% of the time.

We interpret this to mean that for topics such as these, novices tended to perceive them as initially helpful, but they did not consider them as necessary to discuss multiple times or in a subsequent semester of peer-mentoring with the same group of novices.

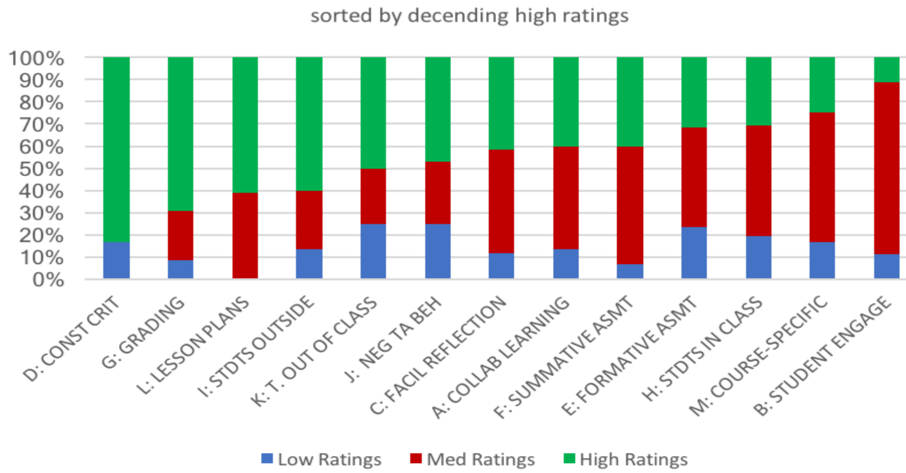


*Figure 1. Small-group meeting topics, sorted by novices' ratings for how they valued that topic.*

Other topics, however, were highly valued (Figure 1) and highly requested for future small-group meetings (Figure 2), even though they were addressed by many of the mentors. For instance, the most highly rated and most frequently circled topic *facilitating constructive criticism about teaching* (Topic D) was also discussed by six of the nine mentors (Table 1).

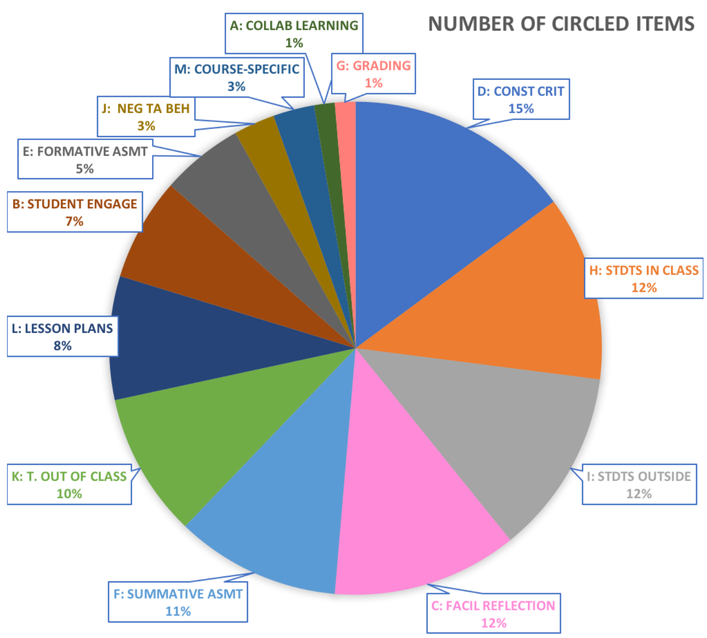


*Figure 2. Frequencies of topics novices circled in answering part (b) of the survey*

This topic often included opportunities for novices to participate in *mock lessons* during the small group meeting. This strategy was suggested when mentors were meeting with the peer-mentoring program facilitator (one author) and brainstorming ways to address one another's concerns. Specifically, one mentor was trying to figure out how to help a novice develop the ability to respond to students' questions with confidence and precise mathematical language during class. Another mentor suggested having the novice present a portion of a prepared lesson to the small group so the rest of the group, and the mentor, could pretend to be undergraduate students and then provide feedback and suggestions for the novice to improve. The rest of the mentors were excited about this strategy and decided to call it a "mock lesson." There was concern, however, that a novice might feel singled out if asked to present a mock lesson, so the mentors further brainstormed about ways to mitigate that possibility (e.g., having the mentor present a mock lesson first or soliciting volunteers from the small group initially then asking the remaining novices to select a week to do the same). This strategy was so well received at the university where it was first discussed that mentors at the other university involved on this project implemented this strategy a few weeks later. The survey results suggest, therefore, that there are topics that are popular among novices despite being addressed by many of the mentors, and should therefore continue to be incorporated into novices' small group meetings.

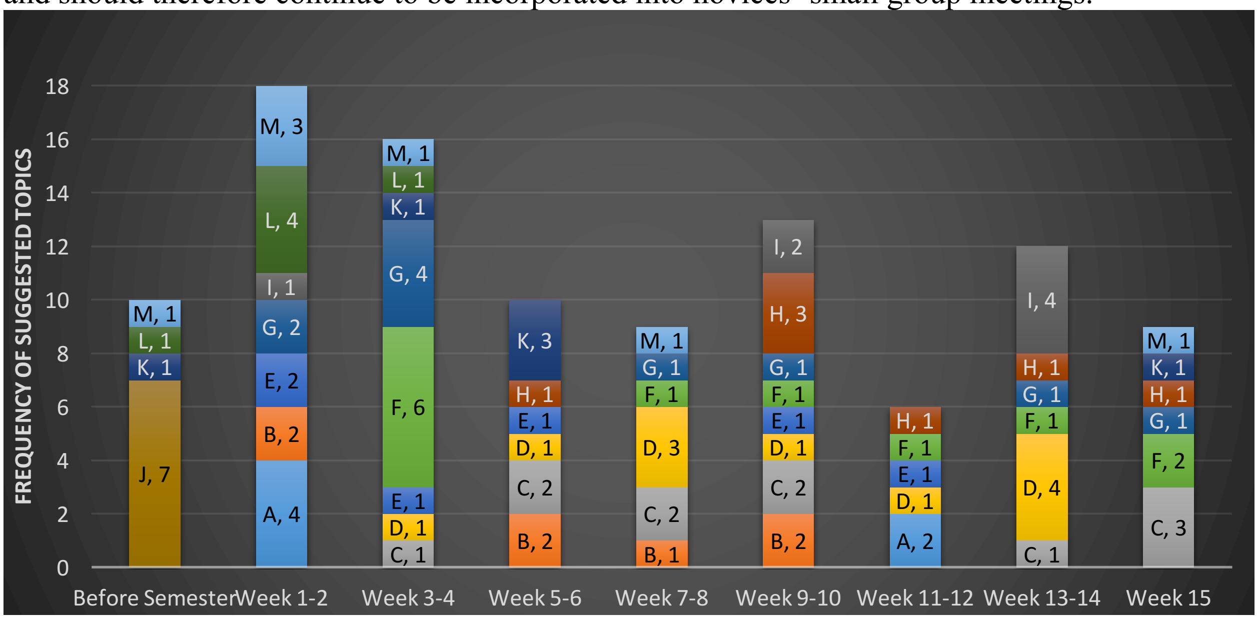

*Figure 3. Mentors suggested timeline of pedagogical topics relative to weeks of a 15-week semester*

Results relevant to RQ2 stem from seven suggested timelines that mentors created after mentoring for two semesters. The frequencies and timing for each topic, focusing on topics that were suggested by multiple mentors for a similar time period of the semester, are presented in Figure 3. The total number of suggested topics on the vertical axis could indicate that there were certain weeks of the semester that mentors considered more critical for sharing issues than others. That is, Weeks 1-2 (18) and Weeks 3-4 (16) had the greatest frequency of topic suggestions, which could suggest mentors saw these as crucial times to work with novices. Later in the semester, however, Weeks 11-12 received the fewest suggested topics.

Figure 3 also shows that before the semester began, all the mentors (7) preferred to discuss small-group meeting behavior (Topic J) rather than letting it unfold throughout the semester. During the first two weeks of the semester, mentors frequently suggested that collaborative learning (Topic A, 4), lesson plans (L, 4), and course-specific advice (M, 3) should be discussed while a majority of mentors suggested summative assessments (F, 6) and grading (G, 4) during Weeks 3-4. Moreover, nearing the end of the semester, mentors frequently suggested different topics: mock lessons (D, 4) and outside of class interactions (I, 4) during weeks 13-14 and reflection (C, 3) during the final week of the semester. By tallying each topic's frequency across the semester, we see that Topics C (11 total), D (11 total), F (12 total), and G (10 total) were the most popular throughout the entire semester with certain weeks where some topics may have been more frequently suggested (Figure 3). If we take the most frequented topic from each two-week timeframe, we see one possible timeline for the small-group meetings to be J, A & L, F & G, K, D, H, A, D & I, and C.

## Discussion

This study investigated what pedagogical topics novice GSIs perceived as valuable when participating in small group meetings during their first or second semester teaching collegiate mathematics and how experienced mentor GSIs suggested (re-)ordering these topics in a semester timeline. For RQ1, we found there were four meeting topics highly valued by at least 50% of novice: mock lessons, grading, interactions outside class, time management, and lesson plans (D, G, I, K, & L, Figure 1). Cross referencing these results with topic preferences novices circled (Figure 2) suggests that novices desired additional future group meetings to be centered around four of these five highly-valued topics (D, I, K and L). These results, coupled with the fact that the number of mentors who addressed these topics varied (6 mentors vs. 2 or 3 mentors; Table 1), suggests novices valued and desired more discussion of these critical topics. Considering the low-ratings (Figure 1), we note that time management (K) and small group behavior (J) had very similar ratings overall, but only two mentors discussed K while a majority discussed J. Cross referencing these results with Figure 2 suggests these topics are ones that novices would like to discuss further, but mentors may need additional, explicit support to facilitate effective discussions about them.

For RQ2, certain pedagogical topics were suggested at certain times with higher frequency than other topics (Figure 3). The most frequented topics throughout a semester were reflection, mock lessons, summative assessments, and grading (C, D, F, & G). Overlapping these results from RQ1, we see that only topic D, facilitating constructive criticism about teaching especially using mock lessons, is pervasive throughout the timeline for the mentors and a highly valued topic of the novices. This suggests that both the novices and mentors valued this topic for peer mentorship. This also supports the need for peer observations (see Yee & Rogers, 2017), another aspect of this peer-mentorship program, because themes from these observations contributed to mentors' use of mock lessons and these observations were designed to provide constructive criticism of novices' teaching.

Findings from this study provide empirical data that can inform undergraduate mathematics education. Our research design allowed us to capture some information about novices' perspectives of pedagogical topics discussed and mentors preferred timeline of those topics while novices were developing as instructors. Together, this data answered our research questions and provided empirical results, desperately needed by the field (Speer et al., 2015), on which pedagogical topics are specifically valued by novice and experienced GSIs of undergraduate mathematics education. Moreover, our work build's on Portner's (2005) work by expanding the field's understanding of teacher mentoring to GSIs of undergraduate mathematics courses.

The valued topics, and their suggested order, directly informed the next iteration of the peer-mentoring program; specifically, before the next mentoring cycle began mentors generated a draft of meeting topics informed by these results. More broadly, these findings can also provide structure for other universities designing teaching seminars, pedagogical courses, and teacher development. For example, an undergraduate teacher educator or course coordinator could use the results from Figure 3 to determine an ordering of certain course material relative to the timeline of struggles relevant to novice undergraduate instructors.

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What are Conveyed Meanings from a Teacher to Students?

Author 1 Author 2

*In this paper we provide a new lens to explain conceptual connections between what a teacher knows, what a teacher does in the classrooms, and what his or her students learn. We observed three teachers' lessons and interviewed the three teachers and their students. By examining our data, we see that teachers' meanings and their assumptions about what students already understand have an impact on the ways they expressed their meanings during instruction. Then, students developed their meanings in trying to understand what the teacher said and did. Our analyses suggest that teachers need to think about how students might understand their instructional actions so that they can convey what they intend to their students.*

*Keywords:* Slope, Mathematical meanings, Conveyance of meaning

Teachers' mathematical meanings play a significant role in student learning. It is plausible that the more coherent mathematical understanding a teacher holds, students will have greater opportunities to construct robust understanding students have. In order to see how teachers' mathematical understanding influences student understanding researchers have investigated relationships between teacher understanding and student learning. They have tried to measure teachers' mathematical knowledge, the quality of instruction, and student performance. Studies have demonstrated that there is a positive relationship between teacher knowledge and student performance as well as teacher knowledge and their instructional quality (Baumert et al., 2010; Hill, Ball, Blunk, Goffney, & Rowan, 2007). However, researchers used different instruments to measure teachers' mathematical knowledge, the quality of instruction, and student performance: an assessment for teachers' performance, measure for teachers' performance in actual instruction, and a test for student achievement. Thus, what the researchers measured as teacher knowledge is not connected with what teachers did in instruction and what students understood because they used different frameworks for viewing them.

This study is designed to provide a new lens to explain conceptual connections between what teachers know, what teachers do in the classrooms, and what students learn. We studied these connections by using the constructs of *meaning* and *conveyance of meaning* to guide observations and analyses of classroom observations and interviews with teachers and students. We present a subset of our data by focusing on three teachers who taught the concept of slope and investigate teachers' understanding of slope, their instructional actions, and what students understood from the lessons. We seek to answer the following research question: *What relationships are there between teachers' mathematical meanings for slope, teachers' instructional actions, and meanings that their students construct regarding slope?*

## Literature Review

Deborah Ball and colleagues in Michigan and have developed assessments for mathematical knowledge for teaching (MKT) targeting elementary or middle school teachers in the U.S. (Charalambous & Hill, 2012; Schilling, Blunk, & Hill, 2007). COACTIV (Professional Competence of Teachers, **Co**gnitively **Activ**ating Instruction, and the Development of Students' Mathematical Literacy) also developed assessments for secondary mathematics teachers in Germany (Baumert et al., 2010; Kunter et al., 2007). The two research groups agreed that teacher knowledge influences students' mathematical achievement and the quality of instruction. Both the MKT group and the COACTIV group investigated two relationships: (1) the relationship between teachers' knowledge and students' performance, (2) the relationship between teachers' performance and the quality of their instruction. They used three different assessments to measure teacher knowledge, the quality of instruction, and student achievement.

The MKT group found that there is a positive relationship between teachers' knowledge and students' performance by showing that teachers' higher scores on the MKT assessment did predict gains in students' scores over the course of one year. They also demonstrated that teachers' higher scores on MKT assessment predict higher quality of instruction by reporting that teachers with lower scores showed more errors in their lesson whereas teachers with higher scores provided students with rich examples and representations (Hill et al., 2007). Similarly, the COACTIV group demonstrated that there is a positive relationship between teachers' higher scores on an assessment and students' higher scores on a single test (Baumert et al., 2010). They also found empirical evidence to confirm that there is a positive relationship between the quality of instruction and teachers' knowledge (Baumert et al., 2010; Kunter et al., 2007).

Both the MKT and COACTIV groups tried to find statistical relationships between what teachers know and what students performed as well as between what teachers know and what teachers teach. However, the groups focused on whether or not students solved mathematical problems correctly rather than focusing on the concepts students formed from teachers' instruction. Moreover, the two groups did not try to explain how teachers' knowledge influences student performance through their instruction because they did not ask what a student understands of what his or her teacher said. Thus, the MKT and COACTIV groups did not explain why what teachers know led them to do what they did in their instruction, nor how their instruction led students to learn what they learned. In the following, we present a new perspective to investigate what students learn from their teachers and the mechanisms of teacher influence.

## Theoretical Framework

Coherent mathematical meanings serve as a foundation for future learning, so it is important that students build useful and robust meanings. One way students develop meanings is by trying to make sense of what their teacher says and does in the classroom. Before discussing how meanings are conveyed from a teacher to students in the classroom, we explain what we mean by *meanings*. According to Piaget, to understand is to assimilate to a scheme (Thompson, 2013) and "assimilation is the source of schemes" (Piaget, 1977, p.70 cited in Thompson, 2013). A scheme is an organization of images, meanings and schemes. For example, a student can understand slope as a coefficient of $x$ because she learned "$m$" is slope in $y = mx + n$. This is her understanding of slope in the moment. Then, she could think about slope as "1" when first looking at $x = 1$ because 1 is the coefficient of $x$. This is an implication of her understanding in the moment. The students' meaning in the moment of understanding is the space of implications of that understanding. In this sense, what Thompson (2013) meant by *meaning* is the space of implications of an understanding.

Consider a teacher who teaches mathematical ideas to his students. A teacher has his meanings for the mathematical ideas. The teacher intends to convey the mathematical ideas to his students. In doing so, the teacher and his students are interacting and making an attempt to interpret others in class. Thompson (2013) proposed a theory to explain how two persons (person A and person B) attempt to have a conversation.

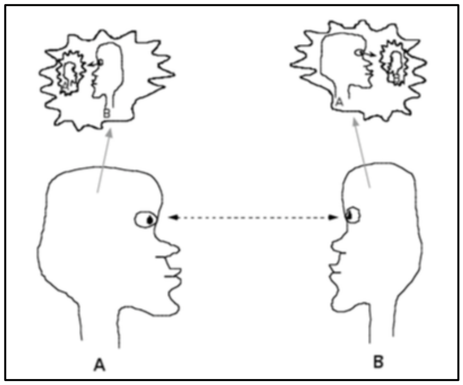


*Figure 1. Person A and B attempting to have a conversation (Thompson, 2013)*

According to Thompson (2013), person A in Figure 1 holds something in mind that he intends Person B to understand. Person A considers not only how to express what he intends to convey but also how person B might hear person A. In doing so, person A constructs his model of how he thinks person B might interpret him. Person B does the same thing in the conversation. Person B constructs her understanding of what person A said by thinking of what she would have meant if she were to say that. Thus, person B's understanding of what person A said comes from what she knows about person A's meanings, thereby person B's understanding of person A's utterance does not have to be the same as what person A meant.

Thompson's (2013) theory of conveyance of meaning is useful to explain what occurs in class. A teacher will express his meanings to his students by saying or doing something. Then, his students try to understand what the teacher says and does. Whatever meanings his students construct by attempting to understand what the teacher intends is the meaning that the teacher conveyed to the students. The conveyed meaning might or might not be the same as the teacher's meaning, and most likely is not.

Lew, Fukawa-Connelly, Mejía-Ramos, and Weber (2016) showed a discrepancy between what an instructor intended and what his students understood. The theory of conveyance of meaning explains why students might not understand what the teacher tried to convey. The conveyed meaning, whatever senses students made of, is rarely consistent with teachers' meaning or intended meaning.

## Methodology

Our research team developed the Mathematical Meanings for Teaching Secondary Mathematics (MMTsm), a diagnostic instrument designed to investigate mathematical meanings with which teachers operate. We administered the *MMTsm* to 513 U.S. and Korean high school teachers in 2013, 2014 and 2015. Eight Korean teachers agreed to classroom observations. Three high school teachers taught lessons on linear equations.

We used items in the *MMTsm* to see the three teachers' meanings for slope. One of items used is in Figure 2.

| |
|---|
| Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.<br>Convey to Mrs. Samber's students what 3.04 means. |

*Figure 2. A slope item in the MMTsm © 2015 Arizona Board of Regents. Used with permission*

The first author asked each teacher to select two students who, in their judgment, have attended during lessons. For each classroom observation we conducted separate pre-lesson interviews and post-lesson interviews with the teacher and two students. The process of this study for one teacher is shown in Figure 3.

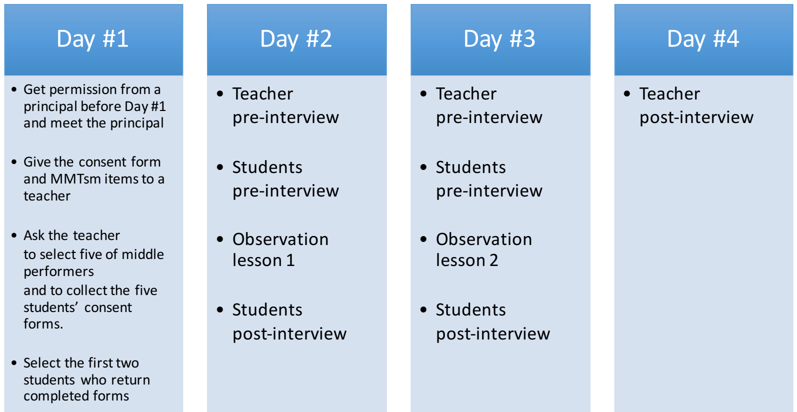


*Figure 3. The process of classroom observations for one teacher*

Before observing the first lesson, the first author conducted a pre-lesson interview with the teacher to investigate what the teacher intended students to learn. One of pre-lesson interview questions was, "Do you think your students might understand slope differently than what you intend?" We asked this to discern what the teachers think about student thinking before the lesson. After the pre-lesson interview with the teacher, the first author met the two students selected for interviews. The purpose of the pre-lesson interviews for students was to see their understanding of the topic covered in the upcoming lesson. We compared students' meanings demonstrated in pre-lesson interviews to their meanings demonstrated in post-lesson interviews to conclude what they understood from the lesson. After the lesson, the first author asked a student to describe what he learned from the lesson. The purpose of the post-lesson interviews for teachers was to give them the opportunity to reflect on their teaching and their meanings by showing excerpts from the two student interviews that reveal how they understood central ideas of the lesson.

We audio recorded pre-lesson interviews with teachers and students and post-lesson interviews with teachers. However, we video recorded the lessons and post-lesson interviews with students because we showed video clips of students' post-lesson interviews to their teachers.

## Results

Prior to pre-lesson interviews with the three teachers the first author reviewed their responses to the MMTsm item on meanings for slope. Table 1 summarizes three teachers' meanings for slope. The interviewer asked the three teachers in pre-lesson interviews to see their intentions for the lesson and what they think about student thinking.

*Table 1. Teachers' meanings in the slope item (see Figure 2)*

| | What the teacher wrote in the slope item |
|---|---|
| Jessica | Slope 3.04 means if $x$ increases by 2.7 (or 1), $y$ increases by 8.2 (or 3.04). |
| Katie | Rate of change in $y$ (8.2) when $x$ changes by 2.7 |

| | | |
|---|---|---|
| Liam | $\frac{\text{change in } y \text{ (rate of change)=8.2}}{\text{change in } x \text{ (rate of change)=2.7}} = 3.04$ |

The three teachers' meanings for slope focused on the relationship between a change in $x$ and the associated change in $y$. Additionally, teachers' thoughts on student thinking and their assumptions about what students already knew were similar. Each of the teachers seemed to be unaware that students might understand the concept of slope differently than they did. Although all of the teachers assumed students already knew the definition of slope before the lesson, their intentions for the lesson were different. Jessica and Liam wanted students to understand the idea of slope conceptually. In particular, Jessica said she would introduce advanced mathematical terms such as infinitesimal to help students understand slope conceptually. On the other hand, Katie wanted her students to understand slope as a part of formula. Katie wanted her students to use the slope formula algorithmically. The teachers taught the concept of slope differently, according to their intentions. What they said and did in class is in Table 2.

*Table 2. Teachers' lessons*

| Teacher | Lesson |
|---|---|
| Jessica | She said that slope is defined as "change in $y$/change in $x$" and "$\Delta y/\Delta x$". She drew $y=y_1$ graph, and said the slope is zero (without explanation). When teaching $x=x_1$ graph, she said the slope of $x=x_1$ is not defined because the denominator is zero in the slope formula that is $\Delta y/\Delta x$. She also said $\Delta y/0$ is not the same as neither infinite nor infinitesimal because infinite or infinitesimal is not a number but a process that is close to infinity or a very small number. |
| Katie | She said that when there are two points given, slope is $\frac{y_2-y_1}{x_2-x_1}$. Then, she said that the denominator of $\frac{y_2-y_1}{x_2-x_1}$ would be zero when $x_1=x_2$, so we cannot use $\frac{y_2-y_1}{x_2-x_1}$. She asked students to think about a case where slope is zero. Then, she kept saying $x_1=x_2$. One student asked "what is the slope of $y_1=y_2$?" She answered slope is zero by using $y=\frac{y_2-y_1}{x_2-x_1}(x-x_1)+y_1$. |
| Liam | He taught slope is tanθ and the coefficient $a$ of $x$ while pointing $y=ax+b$. Then, he compared $y=ax+b$ with $ax+by+c=0$. |

Jessica introduced infinite and infinitesimal to explain why the slope of $x=x_1$ is undefined. Katie emphasized the formula for slope during the lesson. During her lesson, Katie frequently said $x_1=x_2$ after discussion cases where the slooe was zero. Students in Katie's class might have understood Katie's claims to mean that the slope of $x_1=x_2$ is zero. In Liam's class, he said slope is tanθ because he assumed students already knew slope is $\Delta y/\Delta x$. He also mentioned slope is "$a$" while pointing at the equation $y=ax+b$.

After the lessons, the first author interviewed two students in each teacher's class to determine the conveyed meanings from the lesson, that is, to determine what the students understood. By comparing students' pre and post-lesson interviews we can witness meanings that students developed in the lesson. One student's meanings per each teacher in pre and post-lesson interviews are in Table 3.

*Table 3. Students' meanings in pre and post-lesson interviews*

| Teacher | Student | Meaning in pre-lesson interview | Meaning in post-lesson interview |
|---|---|---|---|
| Jessica | Justin | Slope is $\Delta y/\Delta x$ and slantiness. So, if $\lvert m \rvert$ (when $m$ is slope) is increasing, the line would be closer to $y$-axis. | Slope is $\frac{y_1 - y}{x_1 - x}$. He realized $\Delta y/\Delta x$ is connected to $\frac{y_1 - y}{x_1 - x}$. Regarding the slope of $x=x_1$ he said "the teacher said the denominator cannot be zero, but I cannot remember why she explained it." $x=x_1$ doesn't have slantiness, so the slope is zero. That means there is no slope of $x=x_1$. |
| Katie | Kim | Slope is slantiness and the length of hypotenuse (of a triangle), so she can measure slope by using Pythagorean Theorem. Slopes of $x$=2 and $y$=2 are all zero. | She still tried to use Pythagorean Theorem to find slope, then wrote $y - x_1 = \frac{x_1}{x_2}(x - x_1)$. Then, she found $\Delta y/\Delta x$ is different from $\frac{x_1}{x_2}$. After trying (3,4) and (2,6), she arrived at $\frac{y_2 - y_1}{x_2 - x_1}$. Slopes of $x$=3 and $y$=3 are zero. There is no slope of $y$=3. The slope of $y$=3 cannot be represented because it's parallel to $x$-axis. "No slope", "not being represented", and "zero slope" are same. |
| Liam | Lori | Slope is change in $y$ divided by change in $x$ and something related to an angle. Slope of $y$=2 is zero according to $\Delta y/\Delta x$. Additionally, Slope of $y$=2 is zero because the coefficient of $x$ is zero in $y=ax+b$. Slope of $x$=2 is 1 because the coefficient of $x$ is 1 in $y=ax+b$. | Slope is "$a$" in $y=ax+b$. Slope of $y$=2 is zero because the coefficient of $x$ is zero in $y=ax+b$. Slope of $x$=2 is 1 because the coefficient of $x$ is 1 in $y=ax+b$. |

Jessica's student, Justin, did not remember why she mentioned infinite and infinitesimal during the lesson. Justin thought the slope of $x=x_1$ is zero because $x=x_1$ does not have slantiness, which means no slope of $x=x_1$. Slantiness seemed to dominate Justin's meaning for slope even after the lesson. Kim's meanings for slope were changing during her post-lesson interview. Although she still made an attempt to use Pythagorean Theorem after the lesson, she came up with $\Delta y/\Delta x$. However, three statements, "the slope of $y$=3 cannot be represented," "the slope of $y$=3 is zero," and "there is no slope of $y$=3," were equivalent to Kim. In the case of Lori, her meanings for slope had not changed after the lesson. She kept saying that slope is the coefficient of $x$, so slope of $y$=2 is zero and slope of $x$=2 is 1. Lori's meaning for slope as the coefficient of $x$ was consistent with what Liam said and wrote in the lesson.

## Conclusion

The results show that teachers' meanings for slope and their assumptions about what students already understood influenced their instructional decisions. Three teachers assumed that students already learned the idea of slope, which led them to explain the basic concept of slope briefly. Jessica introduced difficult terms such as infinitesimal and Liam taught slope as tanθ and a coefficient of *x*.

The ways teachers expressed their meanings for slope in the lessons have an effect on meanings that students developed in the lessons. In Katie's lesson, she asked about a case where slope is zero and kept saying $x_1 = x_2$, which led her student to think slope of *x*=3 is zero. In the case of Liam, he said slope is a coefficient of *x* while pointing to the equation *y*=*ax*+*b*. What Liam said in class influenced his student's meaning as demonstrated in her claim that the slope of *x*=2 is 1 because the coefficient of *x* is 1 in *y*=*ax*+*b*.

In this study, we present a subset of our data as an illustration of the method for exploring teachers' meanings for the ideas they teach, the ways they express their meanings in instruction, and how students' meanings are affected by their attempts to understand what their teacher intends to convey. To do so, we created a method to explain how what teachers know led them to do what they did in the instruction, which affected what students learned. This study necessitates a conceptually coherent framework to investigate the relationships between what teachers know, what teachers do in the classrooms, and what their students learn. Suppose that we had scored what teachers wrote in the slope item and what students expressed during interviews in terms of correctness. Every teacher would have received a perfect score whereas every student would have received a low score because they expressed mathematically incorrect reasoning. Thus, we could not have demonstrated why students developed their meanings in the lessons. Additionally, if we had used different frameworks for teachers' meanings, their instructional actions, and meanings that their students developed we could not have explained what students understood from their teacher's instruction as well as why teachers made their instructional decisions.

Our results also suggest that teachers' high level of meaning does not guarantee that they convey their meanings to students in class. In post-lesson interviews, all three teachers said the conveyed meanings to their students were not what they intended to convey after watching students' video clips. This points to a breakdown in the conveyance of meaning from teacher to student when the teacher has no image of how students might understand his or her statements and actions. In this sense, this study indicates that teachers need to think about how students might understand their statements and actions when preparing for lessons in order to convey what they intend.

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The Creation of a Humanistic Educational Framework for the Nature of Pure Mathematics

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*Within the field of mathematics education research, scholars have found that students often have naïve views about the nature of mathematics. Mathematics is seen as an impersonal and uncreative subject. What can educators do to challenge such views, and support students in developing richer understandings of the nature of mathematics? In this paper, I describe my dissertation study, the goal of which was to identify humanistic characteristics of pure mathematics which may be of benefit for undergraduate students in a transition-to-proof course to know and understand. Using the methodological framework of heuristic inquiry, which leverages the researcher as instrument in qualitative research, I identified humanistic characteristics of mathematics by reviewing relevant literature, collaborating with a professional mathematician, co-teaching an undergraduate transition-to-proof course, and being open to mathematics wherever it appeared in life. The main result is the IDEA Framework for the Nature of Pure Mathematics.*

*Keywords:* Nature of Mathematics, Identity, Dynamic Knowledge, Exploration, Argumentation

Students rarely have an opportunity to reflect on the nature of mathematics. Many have naïve views of mathematics, perhaps believing that mathematics is a static body of knowledge consisting of arbitrary rules and procedures (Beswick, 2012; Erlwanger, 1973; Muis, Trevors, Duffy, Ranellucci, & Foy, 2016; Presmeg, 2007; Solomon & Croft, 2016; Thompson, 1992). These naïve views may negatively affect the learning of mathematics (Erlwanger, 1973; Maciejewski, 2016). As Maciejewski (2016) claimed, "A deeper, connected view of the subject correlates to a deeper approach to study [...] Fragmented, superficial perspectives often result in less desirable outcomes" (p. 1). Many mathematics education scholars view and describe mathematical knowledge as a dynamic human product (Boaler, 2016), and emphasize the human aspects of mathematical work such as creativity (Burton, 1999) and fallibility (Ernest, 1991). These modern views are influenced by cultural approaches to mathematics (Bishop, 1988), theories of embodied cognition (Lakoff & Nuñez, 2000), humanistic philosophy of mathematics (Ernest, 1991), or perhaps scholars' own experiences doing mathematical work (e.g. Hersh, 1997). The gap between the views of mathematics held by students and the perspectives held by scholars needs to be addressed within mathematics education research.

## Purpose of the Study

While scholars in science education have done significant research aimed at understanding the teaching and learning of the nature of science (Lederman & Lederman, 2014), including undergraduate research (e.g. Abd-El-Khalick & Lederman, 2000; Schalk, 2012; Willoughby & Johnson, 2017), relatively little research has been done on this subject within mathematics education (Kean, 2012; Jankvist, 2015; White-Fredette, 2010). Research on the teaching and learning of the nature of science (NOS) is guided by frameworks or lists that explicitly outline goals for students' understanding of NOS (Lederman & Lederman, 2014). For instance, a goal is for students to understand that "Scientific knowledge is open to revision in light of new evidence" (NGSS, 2013, p. 4). These lists aid researchers in assessing whether instruction is effective in teaching students about the nature of science.

Researchers in mathematics education have not systematically studied the nature of mathematics to the extent that science education researchers have studied NOS (Kean, 2012). Our field has lists that outline important mathematical practices (e.g. CCSSI, 2010; NCTM, 2000) and mathematical habits of mind (e.g. Cuoco, Goldenberg, & Mark, 1996), but we do not have lists that outline goals for students' understanding of the nature of mathematics. Such a list would provide university instructors a guide for teaching the nature of mathematics to undergraduates mathematics students, including pre-service teachers. Alba Thompson (1992) noted, "Very few cases of teachers with an informed historical and philosophical perspective of mathematics have been documented in the literature" (p. 141). School teachers will not have informed views until the university, the place where teachers are educated, makes the nature of mathematics a subject of study for its students.

The purpose of this research project was to produce a humanistic framework for the nature of mathematics outlining characteristics of mathematics that may serve as goals for undergraduates' understandings. Two broad questions, "What is the nature of pure mathematics?" and "What should students understand about the nature of pure mathematics?" guided this study. Moreover, I focused on undergraduate students' understanding of the nature of pure mathematics within a transition-to-proof course. I sought to understand, "What should undergraduate students in a transition-to-proof course understand about the nature of pure mathematics?"

Felix Browder (1976) defined pure mathematics to be "that part of mathematical activity that is done without explicit or immediate consideration of direct application to other intellectual domains or domains of human practice" (p. 542). Undergraduate mathematics majors and minors experience pure mathematics in courses such as abstract algebra, topology, analysis, and transition-to-proof. Within transition-to-proof courses, students are expected to pick up the terminology of pure mathematics (e.g. theorem, conjecture, proof), learn to write proofs, and develop an understanding of selected pure mathematics content (e.g. set theory, functions and relations). To meet these learning goals, it may be necessary for instructors to discuss pure mathematics' particular nature, because what is valued in transition-to-proof may be different than what has been valued in students' prior mathematics courses.

## Methodology

### Theoretical and Methodological Frameworks

I sought to understand what is the nature of pure mathematics? But of course, pure mathematics is what mathematicians do. Courant and Robbins (1941) wrote, "For scholars and laymen alike it is not philosophy but active experience in mathematics itself that can alone answer the question: What is mathematics?" (p. xix). I reasoned that if I really wanted to understand the nature of mathematics, then I must have experience doing mathematics. I thus decided that a core feature of my study would be the documentation of and reflection on my collaboration with a research mathematician. Patton (2015) wrote that the core question of heuristic inquiry is "What is my experience of this phenomenon and the essential experience of others who also experience this phenomenon intensely?" (p. 118). In this light, heuristic inquiry seemed to be a perfect fit to study my experience doing pure mathematics for the purposes of developing a humanistic educational framework for the nature of mathematics. Heuristic inquiry is a self-study, and Douglass and Moustakas (1985) noted that, "It is the focus on the human person in experience and that person's reflective search, awareness, and discovery that constitutes the essential core of heuristic investigation" (p. 42). The ultimate end of heuristic inquiry is what Moustakas (1990) called the creative synthesis, in which

> The researcher creates an original integration of the material that reflects the researcher's intuition, imagination, and personal knowledge of meanings and essences of the experience. The creative synthesis may take the form of a lyric poem, a song, a narrative description, a story, or a metaphoric tale. In this way, the experience as a whole is presented, and, unlike most research studies, the individual persons remain intact. (p. 51)

Narratives play an important role in the mathematics education research (e.g. Ball, 1993; Erlwanger, 1973; Lampert, 1990), as stories can provide context for discussing and reflecting on ideas. In addition to a humanistic framework for the nature of mathematics (presented in the results of this paper), my dissertation also features ten stories that illuminate the characteristics of mathematics that comprise the framework. These are stories of my collaboration with a professional mathematician, events that took place in a transition-to-proof classroom I co-taught, or perhaps meaningful stories of my own family's interaction with mathematics. Each of these stories features direct quotations from the data that I collected.

The methodological framework of heuristic inquiry, which has roots in humanistic psychology, meshes well with the theoretical stance of humanism which I also take in this study in regards to the nature of mathematics. Humanistic philosophers of mathematics (e.g. Lakatos, 1976; Tymoczko, 1988) are frequently cited in mathematics education literature (e.g. Ball, 1988; Boaler, 2016; Komatsu, 2016; Lampert, 1990; Larsen & Zandieh, 2008; Weber, Inglis, Mejia-Ramos, 2014). Humanistic approaches are unique in that they take as foundational the notion that mathematical knowledge is a human product. As Hersh (1997) wrote, "To the humanist, mathematics is *ours*—our tool, our plaything" (p. 60). I sought to create a humanistic educational framework for the nature of mathematics that may guide the teaching and learning of the nature of mathematics and challenge naïve views. Humanistic philosophy of mathematics (e.g. Ernest, 1991; Hersh, 1997; Lakatos, 1976) and relevant mathematics education literature (e.g. Lampert, 1990, Thompson, 1992, White-Fredette, 2010) informed an initial review of the literature in which I identified several possible goals for student understanding of the nature of mathematics (Author, 2017). After the completion of this literature review, I continued my dissertation study using the methodological framework of heuristic inquiry.

**Data Sources**

In efforts to understand the nature of pure mathematics, I sought collaboration with a graph theorist, a full professor and active research mathematician, whom I refer to as Dr. Combinatorial. Dr. Combinatorial and I worked together in efforts to prove one of his unsolved conjectures related to the chromatic number of a graph. I recorded all of our conversations in which we discussed the conjecture, and kept hard copies or photos of all of our mathematical work. Throughout the process of working on the conjecture, I was not only doing mathematics, but I was constantly reflecting on my own experience and the nature of pure mathematics.

In order to reflect on what undergraduates should understand about the nature of pure mathematics, I also collected data in a transition-to-proof course required of undergraduate mathematics majors at a large Southeastern university. The course is called "Foundations of Higher Mathematics" and is meant to serve as a transition course as students proceed from lower-level to upper-level mathematics coursework. The transition represents a shift from the traditional procedurally-based school mathematics to the work that more closely resembles that of pure mathematicians. I co-taught this course with another mathematics education scholar, Dr. Amicable, who had designed the course and taught it for seven prior semesters. I fully took over teaching the last month of the semester as she took a planned leave of absence. The course was inquiry-based in nature, and students were constantly working together to draft arguments, critique arguments, and discuss and debate proof writing techniques. Twenty-three students from the course agreed to participate in the study. Dr. Amicable asked all of the students to choose a number type that best captured their own personalities. I have chosen these number types (e.g. Binary, Whole, Natural) to be their pseudonyms in this paper. I chose the number type Surreal as my own pseudonym. The data I gathered from this course included audio recordings of discussions I had with the co-instructor, audio of whole-class discussions, student homework, classwork, exit tickets, and all other class materials.

Another crucial piece of data for this self-study was a personal journal that I kept in order to write and reflect about my experiences doing and teaching mathematics. My writings were particularly focused on documenting and reflecting on my experiences relevant to the nature of mathematics (NOM) and its teaching and learning. Another source of data came from audio recordings of informal coffee-shop style interviews that I conducted with persons whom I was interested in speaking to about NOM (e.g. mathematicians). These interviews generally consisted of conversations about NOM and interviewees' opinions about what students should understand about NOM. Six people agreed to such interviews, and in some cases multiple interviews were conducted. Most notably among these were two mathematicians. Speaking to these mathematicians, I was able to get feedback on my ideas about possible goals for students' understanding of the nature of mathematics. See Table 1 for a list of all the data that was collected for this study.

*Table 1. Data Sources*

| |
|---|
| <u>Mathematics Collaboration Data</u> |
| Audio-recordings of discussions with mathematician |
| Hard copies of mathematical work (whiteboard photos and personal notebooks) |
| <u>Mathematics Course Data</u> |
| Class materials (e.g. handouts, PowerPoint slides) |
| Audio recordings of whole class discussions |
| Audio recordings of discussions with co-instructor |
| Student homework, classwork, and exit tickets |
| <u>Journal Data</u> |
| Journal in which the researcher reflected on his experiences doing mathematics, teaching mathematics, discussing NOM, and reading NOM literature |
| <u>Other Data</u> |
| Informal Interviews |
| Personal Audio / Other Photos / Documents / Notes |

**Data Analysis**

Moustakas (1990) wrote that heuristic analysis is on-going from the beginning to the end of an inquiry. Throughout the data collection process, I had in mind the inquiry questions, "What is the nature of pure mathematics?" and "What should students understand about the nature of pure mathematics (NOM)?" Whenever I had an idea for a possible NOM goal (for student understanding), I wrote it out and then saved it into a single word document. At the end of data collection I had a list of fifteen possible candidates for a NOM framework in addition to the initial characteristics identified in the literature review for a total of nineteen characteristics.

Often these characteristics were the topics of conversation during the informal interviews, as I asked mathematicians and others if they considered these characteristics to be worthy goals for student understanding of the nature of mathematics.

After the data was collected, I received feedback on preliminary results at research conferences and job presentations. I then transcribed all of the data (frequently making reflective notes pertaining to the nature of mathematics), and coded the entire set of data using the qualitative software Atlas-ti according to the potential NOM characteristics, which were used as deductive codes (Patton, 2015). Based on the collected data quotations associated with each code, I drafted stories of my experience to illuminate key features of the nature of mathematics. I sought to identify features of the nature of mathematics for which I could tell clear and compelling stories; characteristics that were not only grounded in the data, but also representative of my experience.

## Results

### The IDEA Framework for the Nature of Mathematics

The main result of this study is the IDEA Framework for the Nature of Pure Mathematics which consists of four characteristics: 1) Our mathematical ideas and practices are part of our ***i**dentity*; 2) Mathematical knowledge and practices are ***d**ynamic* and forever refined; 3) Pure mathematical inquiry is an emotional ***e**xploration* of ideas; and 4) Mathematical ideas and knowledge are socially vetted through ***a**rgumentation*. Note that IDEA corresponds to the key concepts of each of the four characteristics: I-Identity, D-Dynamic, E-Exploration, and A-Argumentation. I also tell ten stories to illuminate these characteristics of the nature of mathematics, but due to space limitations I will only present two abbreviated stories in this paper, *Tension* and *We are the Future*. In terms of the IDEA framework, these stories primarily illustrate the *E* and *D* characteristics of the framework. The first narrative, *Tension*, highlights the notion that pure mathematical inquiry is an Exploration of ideas. The second narrative, *We are the Future*, highlights the idea that mathematical practices (particularly standards of proof) are Dynamic, negotiated through Argumentation. The notion that our mathematical ideas are part of our Identity will be explored in-depth in another paper presented at the conference on RUME 2018. I tell the two stories now, followed by discussion and conclusions.

### Tension: Pure Mathematical Inquiry is an Emotional Exploration of Ideas

One of the first significant realizations I had during my inquiry into pure mathematics was that engaging with pure mathematics involves an emotional exploration of ideas. One night I began to work on Dr. Combinatorial's conjecture, and I wanted to summarize the important theorems I had just begun to understand. I wished to solidify them in my own mind so that I could make progress on finding a proof for the conjecture. I sat on my bed, writing theorems and proofs in my notebook. Upon writing a proof for a simple result, I noticed a tension. In at least one line, it is clear that I was writing the proof as I would write a proof in my graduate mathematics courses, as if I expected it to be read and graded. I labeled a 7-cycle as $u_1 - u_2 - u_3 - u_4 - u_5 - u_6 - u_7 - u_1$, but I did not use this symbolization elsewhere in the proof. Rather, I convinced myself of the truth of the conjecture through informal methods—drawing a diagram and counting possible chords. I could have written a formal argument, but it did not seem necessary. The tension is that on the one hand, I was working for personal understanding and on the other I was writing with the standards of rigor I believed to be expected in mathematical writing. The conflict is between a personal exploration and understanding of ideas versus the crafting of a communicative proof that satisfies perceived norms of rigor and symbolization.

After proving that theorem I moved onto another one, which involved a proof by induction. I wrote out minute details of the basis step for the n=0 and n=1 cases that were already clear in my own mind (but may not have been clear to a reader). I then wrote, "I find myself realizing this proof is more for me than another. I don't need to communicate all the details. The magic of mathematics is in the ideas one experiences when proving." Essentially I was giving myself permission, with those words, to drop any unnecessary symbolism and tedious explication, and just explore the mathematical ideas (and document that exploration). The very next thing I wrote was, "Out of curiosity, can I show [the n=2 case]?" I already knew a proof by induction could prove for all cases, but I decided to look at a specific case so I could better understand the general argument. I worked through this case myself, drawing several interesting figures. Then I wanted to keep going. I went on to prove the n=3 case. I was enjoying looking at the individual cases, and gaining insight through my work on them. I found the ideas involved in these types of proofs intellectually stimulating. As I began exploring the mathematical ideas related to this conjecture, I found deep satisfaction. Pure mathematics is an enjoyable exploration of ideas. The mathematics came alive through the proving process. Consider this journal entry:

> It is interesting how I see the problem forming. The proof of the problem is different in nature than the class of graphs the proof refers to. The proof has its own concept imagery in my mind—different mathematical processes and procedures disjoint from the class of graphs itself. … The mathematics is alive within the proof. When I imagine the truth of the conjecture, it is some sad lonely objective reality. But the proof is where the magic is. It is where my mind is. It is where the structure can be seen.

**We are the Future: Mathematics is Dynamic and Forever Changing**

One day near the end of a transition-to-proof class session, students were debating how much detail they needed to put into their proofs. If $k$ is an integer and $j$ is an integer, do you have to write "$k + j$ is an integer" if you use the fact within a proof? And do you have to justify this step by mentioning the closure property of the integers under addition? Some of the students say yes. Others say no. Others want to know if they will be "docked for points" if they do not.

Dr. Amicable says that the students should do whatever the classroom community agrees is best for communication. She asks me what I am thinking and I mention that in professional mathematics papers, there will often be gaps. I say, "It is assumed the mathematician audience knows these things. This sometimes makes the papers difficult for me to read—for someone like me who is not a super mathematician. So I would maybe appreciate some clarity sometimes."

Infinitely Repeating Decimal asks if he, or any other member of the class, were going to write up something for publication, "Would it be viewed in a negative light if it was too expository in areas in which it over explains?" I explain that it is a difference of opinion:

> Surreal: When I wrote my thesis, my professor said, "If we are going to publish this you will have to cut a bunch of stuff." But to me the papers are so hard to read. I would welcome someone coming into the mathematics community who was very explanatory. I just wish more mathematicians could really clearly convey their ideas. But it is just a difference of opinion. There is another mathematician I know who says, "that is the fun of it. You have to go check everything yourself and make sure you do all the side work."

That class laughs about this comment. Another student, Odd, recommends footnotes as a "happy medium" and Infinitely Repeating Decimal agrees. Then Dr. Amicable poses an interesting question taking the discussion to a different place: "You know who the next generation of mathematicians are, right?" There is silence until someone hesitantly says, "us."

Dr. Amicable: Yes! Right? So you are the community. And you will be able to determine those things. What counts as proof is really determined by who is in the community. So that's what's really neat. So if you all go out there and say I'm going to become a mathematician, and I'm going to change this. Just like Surreal. He is going to be right along with you. I want to change it so that it is a little bit easier to understand these arguments. Right?

Infinitely Repeating Decimal: We are going to change the world. I am going to change the entire mathematics community just for you.

## Conclusion and Discussion

The purpose of the IDEA framework is to be a list of goals for students' understanding of the nature of pure mathematics. I presented two stories: *Tension,* which touched upon my experience of pure mathematical inquiry as an exploration of ideas, and *We are the Future*, which focused upon a classroom discussion about the dynamic nature of mathematics in regard to standards of proof. Dr. Amicable and I tried to paint a dynamic picture of mathematics for our students. We told them they were the future of the discipline. We taught them that what counts as a proof is negotiated amongst mathematicians, and gave them the opportunity to debate what makes a good proof themselves. We encouraged them to see the value of mistakes in revising their knowledge.

Although students did have the opportunity to reflect on the dynamic nature of proof standards, I was unable to identify a time when students had the opportunity to experience pure mathematical inquiry as an exploration of ideas (as I did during my work on Dr. Combinatorial's conjecture). While Dr. Amicable and I encouraged students to make meaning of statements before proving, perhaps by constructing examples, what was ultimately deemed credit-worthy in the course was a valid deductive proof. I believe students frequently engaged in a syntactical proof production process like that defined by Weber and Alcock (2004):

> We define a syntactic proof production as one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. […] In the mathematics community, a syntactic proof production can be colloquially defined as a proof in which all one does is 'unwrap the definitions' and 'push symbols'. (p. 210)

In the transition-to-proof course, students' ability to write deductive proofs was prioritized over the ability to explore mathematical ideas. Perhaps it is a sign of the times, a result of the culture. According to Hersh (1997),

> Mathematics as an abstract deductive system is associated with our culture. But people created mathematical ideas long before there were abstract deductive systems. Perhaps mathematical ideas will be here after abstract deductive systems have had their day and passed on. (p. 232)

Are we satisfied to be part of a culture in which students spend less time exploring the ideas behind a theorem than on producing a valid deduction? We must put serious thought into how we structure pure mathematics courses so students develop healthy and productive conceptions of the nature of mathematics. To renew the culture of pure mathematics instruction will require a commitment from instructors and scholars to make choices that promote the values and vision expressed by humanistic philosophers of mathematics, ideas which are represented in the IDEA framework. To bring about changes in students' conceptions of mathematics they must be provided with opportunities to explicitly reflect on their own beliefs about mathematics while also being confronted with positions that challenge those beliefs.

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# Sparky the Saguaro: A Teaching Experiment Examining a Student's Development of the Concept of Logarithms

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*A number of studies have examined students' difficulties in understanding the idea of logarithm and the effectiveness of non-traditional interventions. However, few studies have examined the understandings students develop when completing conceptually oriented exponential and logarithmic lessons that build off prior research and understandings. This study explores one undergraduate precalculus student's understandings of concepts foundational to the idea of logarithm as she works through an exploratory lesson on exponential and logarithmic functions. Over the course of a few weeks, the student participated in a teaching experiment that focused on Sparky – a mystical saguaro that doubled in height every week. The lesson was centered on growth factors and tupling periods in an effort to support the student in developing the understandings necessary to discuss logarithms and logarithmic properties meaningfully. This paper discusses an essential component that students must conceptualize in order to hold a productive meaning for logarithms and logarithmic properties.*

*Key words:* Exponent, Growth factor, Tupling-period, Logarithm, Exponential

The idea of logarithms is useful both in mathematics (e.g., number theory – primes, statistics – regression, chaos theory – fractal dimension, calculus – differential equations) and in modeling real-world relationships (e.g., Richter scale, Decibel scale, population growth, radioactive decay). Therefore, a goal for mathematics educators should be to assist students in developing coherent meanings for the idea of logarithms. How does one achieve this goal? One hypothesis is to research the aspects of the idea of logarithm students have difficulties with. In particular, studies have shown that students have difficulty with logarithmic notation, logarithmic properties and logarithmic functions (Kenney, 2005; Strom, 2006; Weber, 2002; Gol Tabaghi, 2007). Another hypothesis is to develop and test the efficiency of interventions relative to standard curriculum (Weber, 2002; Panagiotou, 2010). Although these methods may shed light on epistemological obstacles students encounter or how successful a non-traditional approach was, neither examine the reasoning abilities needed to coherently understand and utilize the idea of logarithms. In fact, relatively few studies have examined what meanings students have for the idea of logarithms, and fewer have examined how students come to conceptualize the idea of logarithms.

This study investigated one undergraduate precalculus student's understandings of the idea of logarithm and concepts foundational to the idea of logarithm as she worked through an exploratory lesson on exponential and logarithmic functions. The research questions informing this study were:

1. What understandings are foundational to understanding the idea of logarithm?
2. What understandings of logarithmic functions do students develop during an exponential and logarithmic instructional sequence that emphasizes quantitative and covariational reasoning?

The findings of this study revealed an essential component that students must conceptualize in order to hold a productive meaning for the idea of logarithms. That is, in order to reason through tasks involving logarithmic expressions, logarithmic properties, and logarithmic functions in a way that both builds off prior meanings and is useful for more complex tasks, students must first conceptualize that multiplying by $A$ and then by $B$ is equivalent to multiplying by $AB$. In this study we modeled a student's thinking as she participated in an exponential and logarithmic instructional sequence that included cognitively scaffolded tasks designed to support students in constructing coherent meanings for the idea of logarithm.

## Literature Review

### Quantitative and Covariational Reasoning

Quantitative reasoning involves conceptualizing measureable attributes of objects and assigning these observations to a quantitative structure (Thompson, 1988, 1990, 1993, 1994, 2011). This way of thinking is critical for developing a coherent understanding of the idea of logarithm. For example, if one conceptualizes $\log_b(x)$ to represent the number of $b$-tupling periods necessary to $x$-tuple, then one could reason that $\log_b(b)$, the number of $b$-tupling periods necessary to $b$-tuple, should equal 1. The ability to conceptualize the expression $\log_b(b)$ in this way is foundational for their understanding the logarithmic properties and for using logarithms in applied settings. Smith and Thompson (2007) argue that students' ideas and reasoning (with quantities) must become sophisticated enough to warrant the use of algebraic notation and to reason productively with such tools. This investigation was designed to emphasize quantitative reasoning in the context of an exponential situation to motivate students to reason productively with the expressions, equations and functions they define.

The purpose of this study was to uncover the understandings of logarithmic functions students develop when working through an instructional sequence informed by the construct of covariational reasoning. Covariational reasoning is when a student conceptualizes two quantities' values varying in tandem while considering how they are varying together (Thompson & Carlson, 2017). Thompson and Carlson (2017) argue that being able to reason covariationally is crucial for students' mathematical development, especially when constructing meaningful expressions, formulas and graphs. Our lesson begins by attending to two varying quantities individually and then together to influence student thinking as they begin to construct exponential and logarithmic models. Students who are able to reason covariationally may find it easier to coordinate additive changes in one quantity with exponential changes in another quantity (Ellis et al., 2012).

### Research Literature on Students' Understandings of Exponents and Exponential Functions

Viewing exponentiation as repeated multiplication is a primitive, yet insufficient interpretation. While some researchers advocate a repeated multiplication approach (e.g. Goldin & Herscovics, 1991; Weber, 2002), others believe this approach limits students (e.g. Ellis, Ozgur, Kulow, Williams & Amidon, 2015; Davis, 2009; Confrey & Smith, 1995). In particular, Confrey and Smith (1995) argue that the standard way of teaching multiplication through repeated addition is inadequate for describing a variety of situations. Weber (2002) proposed that students first understand exponentiation as a process before viewing exponential and logarithmic expressions as results of applying the process. Once this is achieved, the student should be able to generalize the understanding to cases in which the exponent is a non-natural number. Specifically, Weber defined $b^x$ to represent "the number that is the product of $x$ many factors of $b$" and $\log_b(m)$ to be "the number of factors of $b$ there are in $m$." If a coherent understanding of exponential functions (and later logarithmic functions) is desired of our students, it is imperative that they have productive meanings for exponents.

Ellis et al. (2015) conducted a small-scale teaching experiment, informed by Smith and Confrey's (Smith, 2003; Smith & Confrey, 1994) covariation approach to functional thinking, with three middle school students that examined continuously covarying quantities. The students were asked to consider a scenario of a cactus named Jactus whose height doubled every week. The authors noticed three significant shifts in the students' thinking over the course of the study: (1) from repeated multiplication to coordinating $x$ and $y$, (2) from coordinating $x$ and $y$ to coordinated constant ratios, and (3) generalizing to non-natural exponents. The authors noted that a student's ability to coordinate the growth factor (or ratio of height values) with the changes in elapsed time contributed to the student successfully defining the relationship between the elapsed time and Jactus' height. This study leveraged findings from Ellis et al.'s study of Jactus the Cactus to promote more meaningful discussions on logarithms.

**Research Literature on Students' Understandings of Logarithms**

The topics of logarithmic notation and logarithmic functions often pose a variety of challenges to students (Kenney, 2005; Weber, 2002). Similar to the complexities present in function notation, logarithmic notation consists of multiple parts each with their own dual nature (Kenney, 2005). In the equation $\log_b(x) = y$, $b$, $x$, and $y$ take on a variety of meanings (i.e. parameters, variable). Kenney (2005) noted that because function names are often one letter, students do not naturally view log($x$) as representing an output to a function. In addition to these unavoidable complexities, Kenney's (2005) study discovered other difficulties students have with understanding logarithmic notation. The data revealed that students displayed mixed understandings of the bases in the expressions. For example, the students appeared to think that different bases always meant the logarithmic expressions were not equivalent (with the inputs being the same). However, when the expression involved the sum of logarithms, some students claimed equivalence because the bases would cancel out. Students also claimed that ln was equivalent to $\log_{10}$. The study also revealed that students would disregard or "cancel out" the word "log" when simplifying equations involving logarithms and solving for $x$. Despite the aforementioned difficulties, a few of the students were successful in arriving at the correct answer. However, Weber (2002) found that this was an unlikely result of traditionally taught students.

Weber (2002) conducted a pilot study that compared a traditional approach to teaching logarithmic functions with a more conceptual approach that introduced $\log_b(m)$ as the number of factors of $b$ there are in $m$. Weber's way of discussing the meaning of a logarithmic expression more clearly describes what the multiple parts of the notation represent - therefore addressing the issues Kenney observed in her study. In his study, Weber found that the students who received more conceptually based instruction were more likely to catch their mistakes when it came to identifying and justifying properties of logarithms and exponents. This data emphasizes the importance and need for more coherent and conceptually taught lessons for exponents, logarithmic expressions and logarithmic functions.

## Theoretical Perspective and Methodology

The theoretical framework of genetic epistemology (Piaget, 2001) and the theoretical perspective of radical constructivism (Glasersfeld, 1995) form the foundation of this study. A key assertion of radical constructivism is that knowledge is constructed in the mind of an individual and is not directly accessible to anyone else. Steffe and Thompson (2000) label the mathematical constructions made in the mind of a student as "student's mathematics." At best, researchers can develop models of student thinking based on the student's utterances, movements, written work, and essential mistakes. Such models of student's mathematics are referred to as "mathematics of students" (Steffe & Thompson, 2000). A model is considered reliable when the student acts in a way that remains consistent with the model. The process of developing the mathematics of students is one of scrutiny. Models are formed, tested, revised, and tested again until a viable model is developed. However, to say that a model is reliable is not the same as claiming the model directly represents the student's thinking – that is an impossible objective. Genetic epistemology focuses on both "what knowledge consists of [cognitive structures - schemes] and the ways in which knowledge develops [what those structures do]" (Piaget, 2001, p. 2). Piaget believed that knowledge is not static, but is always in a stage of development (1977). Therefore, for example, in order to discuss the ways in which students come to understand that $\log_b(m)$ represents the number of b-tupling periods needed to m-tuple, we must develop a model of students' cognitive structures and a roadmap of what happens to those cognitive structures as students' knowledge progresses from point A to point B. In this study, we attempt to model the participants' knowledge development of the ideas foundational to the idea of logarithm.

For this study, we conducted a teaching experiment (Steffe & Thompson, 2000) over the course of a three-week period in an effort to gain insight into student thinking and to develop the mathematics of students regarding logarithms and logarithmic functions. This study consisted of four 1.5-hour sessions with Lexi, a precalculus student, covering the topics of exponential and logarithmic functions in the context of a saguaro cactus that grows exponentially with respect to time (specifically doubling in height each week). Lexi, worked through a packet of questions while referring to a premade Geogebra applet to guide her thinking. As we conducted this teaching experiment, the lesson used was modified as needed during the stages of retrospective analysis. Lexi did not complete any additional assignments between teaching episodes.

## Results

This study's findings identified understandings foundational to the concept of logarithms. The section that follows reports findings that revealed foundational weakness that prevented Lexi from constructing targeted meanings in the lesson. Our findings are supported in our analysis of the discussions between Lexi and me as she completed the tasks.

### Foundational Understanding: Multiplying by *A* then multiplying by *B*, has the same effect as multiplying by *AB*

In this section, we present and discuss clips from the teaching episodes that suggest Lexi did not distinguish multiplying by *A,* then multiplying by *B* as having the same effect as multiplying by *AB*. This understanding, or lack thereof, reoccurred throughout the teaching experiment when discussing the meaning of percentages, growth factors and logarithmic ideas. We realized this crucial issue during the retrospective analysis of the third teaching episode and developed a task to allow Lexi an opportunity for reflective abstraction (Piaget, 2001; Thompson, 1985, pg. 196). We conclude this section by discussing the intervention and noting changes in Lexi's thinking.

The first two episodes focused mainly on percentages, percent change, growth factors and an exponential function. Throughout the first lesson, it became apparent that Lexi had two ways of acting on tasks involving percentages – one more dominant than the next. At first, Lexi associated percentages with a repositioning of the decimal place, but remained in a state of disequilibrium as she proposed a variety of values to represent the percent in decimal form. Lexi resorted to what ended up being her most dominant actions for percent problems. This action entailed Lexi first finding 1% of a value by dividing that value by 100 and then scaling this value to find the desired percent value. For example, to find 73% of $27, Lexi divided the $27 by 100 and took the result, $0.27, and multiplied it by 73 to get $19.71. When Lexi was presented with a percentage task involving multiples of 10%, she acted on the task in a different way. This action involved moving the decimal place of the value she was trying to find the percent of to the left one place (finding 10% of the value) and scaling up to find the multiple of 10. For example, the first author asked Lexi to determine 20% of $27, she moved the decimal place over one place to get $2.7 (10% of $27) and multiplied this value by 2 to get $5.40 (20% of $27).

Although Lexi's dominant action for percentages worked for her, her approach it is not the most productive way to approach tasks involving calculating a percent of a value. To address this observation in the second teaching session, we presented Lexi with the following two questions:

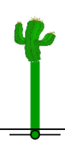
1. Suppose the division button on your calculator wasn't working. How would you determine 1% of $45.67?
2. Suppose the division button on your calculator wasn't working. How would you determine 73% of $45.67?

The purpose of this task was to help Lexi make the abstraction that to determine $n$% of a number, one can multiply by the decimal representation of $n/100$. She began by stating she could divide $45.67 by 100 to calculate 1% of $45.67. We then reminded her that she should assume the division button on the calculator was broken and that she needed to come up with a different way to calculate 1% of $45.67. Lexi's next response was to multiply $45.67 by 1/100. However, we noted that in order to enter 1/100 in the calculator, she would still need to utilize the division button. We followed that statement by asking her, "What is another way to represent 1/100?" and she responded, "0.2? 0.1? 0.01?" – eventually settling on 0.01. When attempting the second problem, Lexi stated, "Don't we just do the same thing?" and said she could determine 73% of $45.67 by multiplying $45.67 by 0.73. Lexi's attention to the results of her actions for the first problem suggests that she developed a new action in her scheme for percentages via a pseudo abstraction (Piaget, 2001). We asked Lexi how she might calculate the same value by using her answer in part (1). She explained that she would just have to multiply the 1% value by 73 to calculate 73% of $45.67. We attempted to draw Lexi's attention to the actions she performed in hopes that she would reflect on her work and abstract that multiplying by 0.73 has the same effect as multiplying by 0.01 and then by 73. That is, multiplying a value by 0.73 finds 73 1/100ths of that value, therefore calculating 73% of the value. Instead, Lexi claimed that the first method uses the 1% and the other (multiplying by 0.73) doesn't "necessarily need the 1% to find (the output)." Lexi's description of the two methods suggests that she viewed them as disjoint from one another. In other words, Lexi's actions suggest she viewed multiplying by 0.01 and then by 73 as being quantitatively different than multiplying by 0.73.

During the remaining portion of the second teaching episode, Lexi worked on a lesson that prompted her to determine different growth factors to represent Sparky the Saguaro's growth. In an attempt to determine the 3-week growth factor, Lexi began by noting Sparky's initial height of one foot at week zero and then claimed, "three time(s)– no, every week it's doubling, or times two for the height. So to get to week three, you'd say it's like, you wouldn't say 6 times as large – that wouldn't make sense. I feel like you would say 3 times as large – that doesn't make sense either." This quote suggests that Lexi first considered multiplying the 1-week growth factor (2) by the number of elapsed weeks (3) to calculate the 3-week growth factor. However, she quickly ruled out that option and looked to other values appearing in the situation. Lexi then appeared to observe the height of the cactus three weeks after its purchase and eventually concluded that the week 3 Sparky would be 8 times as large as the initial Sparky. However, there was no evidence to suggest that Lexi had reflected on the relationship between the 1-week growth factor (2) and the number of weeks that have elapsed (3) relative to the 3-week growth factor (8). In particular, although Lexi noted that Sparky was doubling in height every week, her responses and attention to the heights of the cacti suggest she had not yet abstracted that if Sparky doubles in height three weeks in a row, that will have the same effect as growing by a factor of $2^3$, or 8.

During the third lesson, we introduced the biconditional nature between statements involving growth factors and tupling periods. For example, we say the n-unit growth factor is b if and only if the b-tupling period is n-units. In the Sparky context, since the 1-week growth factor is 2, the 2-tupling period is 1 week. Lexi struggled with *n*-tupling periods when *n* was not a power of 2. For example, when we asked Lexi to approximate the 3-tupling period, she claimed it should be 1.5 weeks (so that the three foot Sparky would lie halfway between the 2 foot and 4 foot Sparky). Under the assumption that Sparky was three feet tall after 1.5 weeks, we asked Lexi to determine the number of weeks it would take Sparky to 9-tuple (or to determine the total amount of elapsed time if Sparky 3-tupled in height again). At this point in the teaching experiment, Lexi and the first author had already discussed and concluded that for equal changes in elapsed time, Sparky's height would grow by a constant factor. Therefore, if it took 1.5 weeks to triple, it should take 3 weeks to 9-tuple (but this is impossible since 3 weeks is the 8-tupling period). However, despite our conversations, Lexi's initial response to the 9-tupling question did not appear to rely on her statement that the 3-tupling period was 1.5 weeks. Instead, Lexi claimed the 9-tupling period would be 3.5 weeks and then modified her response to be 3.25 weeks (so that the 9 foot tall Sparky would lie closer to the 8 foot tall Sparky). Again, there was no evidence to suggest that Lexi had reflected on the relationship between the 9-tupling period and the 3-tupling period. In particular, Lexi's response suggests she did not have the understanding that in order to Sparky to 9-tuple in height, he must 3-tuple in height twice. For the remaining portion of the teaching session, Lexi continued to struggle with the idea that if Sparky first *m*-tupled and then *n*-tupled, we could describe his total growth as growing by a factor of *mn*.

After analyzing the third teaching episode and recognizing Lexi's main difficulty, we began the fourth teaching episode with an activity (Figure 1) to allow Lexi opportunities to engage in reflective abstraction on this topic before we introduced logarithmic notation.



| (A) At some point in time, Sparky was this tall. | (B) After some time, Sparky 2-tupled in height. Draw the resulting Sparky. | (C) After some more time, Sparky then 4-tupled in height. Draw the resulting Sparky. |
|---|---|---|

*Figure 1: Task to address foundational understanding*

Lexi drew Sparky (B) and Sparky (C) using a straightedge, documenting the initial height of the intervals and constructing a length that is 2 times as tall and 4 times as tall respectively. Lexi and the first author then had the following discussion:

INT: Sparky (C) is how many times as large as Sparky (A)?

Lexi: Um, wouldn't it be like 6 times as large?
INT: OK, can you verify that?
Lexi: Sure (reaching for straightedge)
INT: And as you are marking that off, can you explain how you concluded it should be 6?
Lexi: Um, well I figured that it would be 6 times as tall because right here this is two times so then that 2 plus that 4 would be 6. (Uses the straightedge to measure how many Sparky (A)'s fit into Sparky (C)) Oh so maybe I was wrong. OK, wait, so it's 8 because is it because it's 4 times 2? Would you multiply those instead of adding them?
INT: Mhmm
Lexi: OK
INT: But can you, can you think about, um, instead of just saying "We're going to multiply instead of add," can you think about why it is multiplication?
Lexi: Um, I guess that would make sense because right here, if you're like doubling it in height, you're multiplying it by two. And then if you're 4-tupling it I guess you are going to increase it by like another factor of 4. So instead of adding the factors you would need to multiply them.

Following this first activity, Lexi correctly completed and interpreted two similar tasks – one where Sparky tripled and then doubled in height, and another where Sparky tripled in height twice in a row. Lexi reasoned with the quantities and was able to conclude that if it took Sparky one week to 2-tuple and approximately 1.58 weeks to 3-tuple, then it should take 1+1.58=2.58 weeks to 6-tuple. In other words, the number of 2-tupling periods (weeks) needed to 2-tuple plus the number of 2-tupling periods (weeks) needed to 3-tuple is equal to the number of 2-tupling periods (weeks) needed to 6-tuple. Symbolically, $\log_2(2)+\log_2(3)=\log_2(6)$ - a specific case of a logarithmic property!

## Conclusion

Many studies have examined aspects of logarithms that present difficulties for students, while others have investigated the effectiveness of interventions. In this study, however, we examined the subject's thinking as she participated in a conceptually based lesson on exponential and logarithmic functions. Our findings revealed that the understanding that multiplying by $A$ and then multiplying by $B$ has the same effect as multiplying by $AB$ is crucial throughout a lesson on exponential and logarithmic functions. Types of problems that involve such reasoning include: calculating percentages of values (as witnessed in Lexi's interpretation of finding 73%), determining partial and $n$-unit growth factors (as witnessed in Lexi's struggle with determining the 9-tupling period), representing, interpreting and calculating logarithmic values (in this case, we measure one tupling period using another tupling period), and working with and explaining logarithmic properties (as witnessed with Lexi's interactions in the fourth episode). A student who does not hold this understanding can be successful in answering questions to determine percentages of values, as when Lexi first calculated 1% of a value and then scaled her answer to find a different percent. If our goal is for students to develop coherent understandings of exponential and logarithmic functions, then we must ensure that this foundational understanding is also developed. This finding will be used to improve the Sparky the Saguaro lesson for future research in an effort to provide students more opportunities to develop these foundational understandings at the beginning of the intervention. The Geogebra applet utilized in this study can be requested at egkuper@asu.edu.

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An Initial Exploration of Students' Reasoning about Combinatorial Proof

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*Combinatorial proof involves proving relationships among expressions by arguing that the two expressions count sets with the same cardinality. It is an important topic because it is a kind of proof that has not been studied extensively, yet it represents an aspect of combinatorial reasoning that students should develop. In this paper, we report on data from two students who participated in a paired teaching experiment during which they solved tasks involving combinatorial proof. We highlight some productive aspects of their conceptions of combinatorial proof, and we also report on some pedagogical interventions that seemed to help students progress with successful combinatorial proving. We also argue that combinatorial proofs may naturally tend to be semantic rather than syntactic proof constructions (Weber & Alcock, 2004).*

*Keywords:* Combinatorics, Discrete mathematics, Combinatorial Proof, Proof, Student Thinking

## Introduction and Motivation

Binomial identities are equalities that describe relationships between binomial coefficients, such as $\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$. These identities are important because they establish relationships that can be leveraged in a variety of combinatorial settings. While there are often multiple ways to prove such equalities (such as through algebraic equivalences or proofs by induction), a common way to establish binomial identities is through combinatorial proof. Through this technique, we prove that an equality holds by arguing that both sides of the equation count the same set of outcomes. Combinatorial proof tends to be introduced in discrete mathematics or combinatorics classes, and the mathematics community has established the fascinating and valuable nature of this method (e.g., Benjamin & Quinn, 2003). In addition to its use in combinatorial settings, combinatorial proof also provides an interesting setting for students to gain experience with proof and justification. Combinatorics is an accessible mathematical domain, and researchers have made the case that this makes it a useful context for mathematical exploration (Kapur, 1970). Similarly, combinatorial proof could provide an accessible context in which students can gain experience justifying and proving mathematical ideas. In particular, as we will argue, combinatorial proof may naturally provide students experience with semantic (rather than syntactic) proof productions (Weber & Alcock, 2004).

In light of the fact that combinatorial proof is useful for developing both students' combinatorial competency and their proving and justifying, we argue that it is a topic worth studying. However, to date, not much has been explored about this interesting topic. In this paper, we present results from an initial exploration into undergraduate students' reasoning about combinatorial proof. We seek to answer the following research question: *What are key elements of students' conceptions of combinatorial proof that facilitate success with combinatorial proof?*

## Literature Review

### Literature on Combinatorial Proof

A handful of studies have focused on students' reasoning and activity related to binomial coefficients. For instance, in their longitudinal study, Maher, Powell, and Uptegrove (2011) describe several instances in which students made meaningful connections between binomial coefficients, particular counting problems, and Pascal's Triangle. More specifically, Maher and Speiser (2002) documented student's reasoning about problems involving block towers, which can be solved using binomial coefficients. In a similar vein, Tarlow (2011) reported on eight 11th grade students who could make sense of a well-known binomial identity using both pizza and towers contexts. These studies provide examples of students reasoning about binomial coefficients and identities and show students forming (and in some cases justifying) relationships using combinatorial arguments.

There is another way to think about combinatorial proof, in which each side of an identity counts a different set, and the identity is proved by establishing a bijection between the sets (Mamona-Downs & Downs, 2004; Spira, 2008). The establishment of a bijection is not our emphasis in this study; rather we focus on proofs that count the same set in two different ways.

**A Model of Students' Combinatorial Thinking**

We draw on Lockwood's (2013) model of students' combinatorial thinking in order to frame our discussion of combinatorial proof; indeed, the model was an integral aspect of the design and analysis of the teaching experiment and design experiment. Lockwood (2013) describes three different components of her model: formulas/expressions, counting processes, and sets of outcomes. Formulas/expressions are the "mathematical expressions that yield some numerical value" (p. 252). Counting processes are "the enumeration process (or series of processes) in which a counter engages (either mentally or physically) as they solve a counting problem. These processes consist of the steps or procedures the counter does, or imagines doing, in order to complete a combinatorial task" (p. 253). Sets of outcomes are "the collection of objects being counted – those sets of elements that one can imagine being generated or enumerated by a counting process" (p. 253). The relationships between these components can help to articulate phenomena that occur when solving counting problems.

To see how this model applies to combinatorial proof, consider the binomial identity $\binom{n}{k}\binom{k}{m}=\binom{n}{m}\binom{n-m}{k-m}$, which we call Identity 1. We could establish this identity algebraically by using the definition of $\binom{n}{k}=\frac{n!}{(n-k)!k!}$. However, to prove this identity combinatorially the goal is to demonstrate that both sides of the equation are counting the same set of outcomes. We must first identify the counting process that is represented by each respective expression. Then, we argue that those two counting processes are counting the same set of outcomes. Since that set of outcomes has a certain cardinality, the two expressions will be equal.

For Identity 1, the expression on the left-hand side can be thought of a two-stage process of first selecting a $k$-person committee from $n$ people, and then selecting $m$ of those $k$ people to be on a subcommittee. Thus, the left counts all possible subcommittees of size $m$, which were chosen from committees of size $k$ (from a total group of size of $n$). Alternatively, the two-stage process that reflects the expression on the right-hand side can be thought of as first picking subcommittees of size $m$ from $n$ people, and then picking $k - m$ people from the remaining $n - m$ people to fill out the rest of the subcommittee. The right hand thus also counts the same set, and we can conclude that the identity holds.

In terms of the model, we view this combinatorial proof as being represented by the flow of arrows in Figure 1. Given a relationship between formulas/expressions, we identify two counting processes that reflect the respective formulas/expressions but count the same set of outcomes.

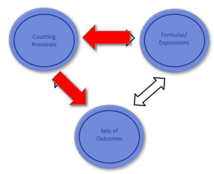


*Figure 1. Lockwood's (2013) model of students' combinatorial thinking*

We also note that while this direction (formulas/expressions → counting processes → sets of outcomes) reflects how proving combinatorial identities is typically introduced, there are other ways to potentially think about combinatorial proof in terms of the model. In particular, one way to introduce combinatorial identities is through leveraging the fact that there may be more than one way to solve a problem. So, following the example of Identity 1, we could consider trying to answer the question "How many ways can you choose committees of size *k* from *n* people, each of which has a subcommittee of size *m*?" If we tried to solve this problem in two different ways, two natural solutions would be first to pick the committees and then pick the subcommittees, or first to pick the subcommittees and then to pick the committees around them. In this way, we start with the set of outcomes, then we build up two counting processes, ultimately determining two respective formulas/expressions to reflect those processes.

## Theoretical Perspective

Harel and Sowder (1998) define *proving* as "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). We adopt this definition of proving and consider a proof to be the product of the proving process. Broadly, we consider *combinatorial proving* to be this process of removing doubts about the truth of an observation about a combinatorial relationship, and a combinatorial proof is the product of that process. Specifically, *combinatorial proof* is the result of a certain process of counting the same set of outcomes in two different ways. Thus, to be a combinatorial proof the student must leverage some counting argument in order to establish the relationship. As we have documented above, this involves articulating counting processes that count the same set of outcomes (or, counting the same set of outcomes via two different processes that can be reflected in two expressions). In this paper, the observations that are being proven are always binomial identities.

Weber and Alcock (2004) identified two qualitatively different ways in which someone might produce a correct proof, and we use this distinction as a way of conceptualizing combinatorial proof. They define a *syntactic proof production* as "one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. In a syntactic proof production, the prover does not make use of diagrams or other intuitive and non-formal representations of mathematical concepts" (p. 210). In contrast, they define a *semantic proof production* to be "a proof of a statement in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (p. 210). We interpret that semantic proof productions describe proof productions in which students meaningfully draw on some instantiation of a mathematical object or idea that may be external from the situation at hand. By emphasizing meaning, they highlight the importance of this instantiation providing some meaning that the symbolic proof normally would not. Although this distinction was introduced in terms of formal proofs (specifically in algebra and analysis context), we argue that these terms could still be a useful lens through which to think about combinatorial proof. We will argue that to prove a binomial identity, a combinatorial proof typically reflects a semantic proof production, whereas an algebraic or inductive proof might naturally be representative of a syntactic proof production.

## Methods

Our investigation of combinatorial proof is situated within a broader study investigating generalization in combinatorial contexts. For this paper, we present data from a paired teaching experiment, and we focus on those sessions in which we had students engage in tasks related to combinatorial proof. We conducted a teaching experiment (in the sense of Steffe & Thompson, 2000), during which we interviewed two students over 15 hour-long videotaped sessions. The sessions occurred over approximately 6 weeks during the school year, and the participants were monetarily compensated for their time. We sought students who satisfied three criteria. We wanted them a) to be novice counters, without having formal experience with counting in college, b) to demonstrate a disposition inclined toward problem solving, and c) to be able to articulate their thought process. With these criteria in mind, we chose students based on individual hour-long selection interviews during which they solved counting problems. Two students who fit the criteria (Rose and Sanjeev, pseudonyms) were engineering majors enrolled in a vector calculus class. During the interviews the two students worked together at a chalkboard, and they both regularly contributed to the conversation. The interviewer posed tasks and occasionally asked clarifying questions. We describe the tasks below.

In choosing tasks, we were motivated by the idea that it might be productive to have students first gain experience going from sets of outcomes to counting processes to formulas/expressions by essentially asking students to solve counting problems in two different ways. We also thought that students would benefit from considering a concrete problem (involving specific numerical values) instead of starting with a general statement involving variables. Binomial identities are typically stated as general statements (involving variables like $n$, $k$, and $r$), but we felt it would be useful for students to consider specific instances of those relationships. Due to space, we provide only a partial list of tasks in Table 1.

*Table 1. Tasks in the teaching experiment*

| **Activity** | **Task** |
|---|---|
| Starting with a specific problem, solving it in two different ways, then moving toward generalization<br><br>Formulas/expressions → Counting Processes → Sets of Outcomes | Task 1a: How many 15 person committees are there from 25 people? … Can you solve that in two different ways?<br>Task 1b: What about $n$ people and $k$ people committees? How would you count them in two different ways?<br>Task 2a: There are 10 people, and I want a committee of size six, there is one appointed chairperson. How many such committees are there, and can you solve it in two different ways?<br>Task 2b: Now what if there are $n$ people with committees of size $k$ and a chairperson? Can you solve it in two different ways? |
| Giving students the binomial identity and having them argue they count the same set of outcomes. | Task 3: $\binom{n}{k}\binom{k}{m}=\binom{n}{m}\binom{n-m}{k-m}$<br>Task 4: $\binom{n}{k}=\binom{n}{k-1}+\binom{n-1}{k-1}$ |

| | |
|---|---|
| Sets of Outcomes $\rightarrow$ Counting Processes $\rightarrow$ Formulas/Expressions | Task 5: $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}=\binom{m+n}{r}$ |

The interview sessions were all transcribed. We created enhanced transcripts, which involved inserting relevant images and gesture descriptions into the transcript. We reviewed the enhanced transcripts of the teaching experiment first and wrote down interesting phenomena about students' reasoning about combinatorial proof. Once we had broad themes we then used the qualitative data analysis software MAXQDA to identify and code relevant data segments. We then synthesized and coordinated our themes into a coherent narrative. We also went through and identified proof productions that we determined to be syntactic or semantic. Due to space, we highlight only a couple of salient findings.

## Results – Students' Conceptions of Combinatorial Proof that Facilitate Success

### Students Should Understand What It Means to Prove a Relationship "Combinatorially"

Not surprisingly, it was not trivial for students to reason about what was entailed in a combinatorial proof, but we argue that it is important for them to develop this understanding. One noteworthy phenomenon is that the students had to reckon with what a proof is and why counting the same set might actually constitute a mathematical proof.

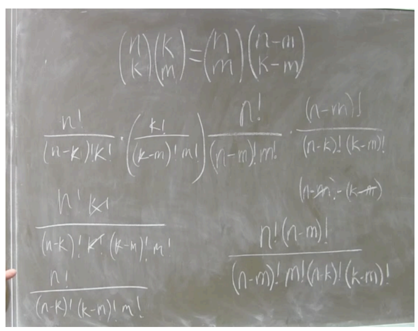

*Figure 2. The students establish an algebraic proof*

We asked Rose and Sanjeev to prove Task 3 (written at the top of Figure 2), and they asked if they could write it out in a different way. They then immediately started to write out the expressions and work toward an algebraic proof (Figure 2). Although the students were able to make combinatorial arguments, which they had demonstrated by correctly solving previous tasks combinatorially, their instinct was to use an algebraic justification. They subsequently went on to solve the problem combinatorially, but interestingly, even after providing combinatorial arguments, the students seemed more convinced by algebraic arguments.

*Interviewer*: You proved it algebraically, but suppose you hadn't. Would you be convinced then by your argument that that equation has to be true? Like, are you pretty convinced that that equation is true?

*Rose*: Uh-huh.

*Sanjeev*: If we didn't see algebra?

*Interviewer*: Yep.

*Sanjeev*: Probably not.

*Rose*: No.

On Task 4, the exact same phenomenon occurred, where the students immediately tried to prove it algebraically even after they had just combinatorially proved Task 3. This phenomenon is not necessarily surprising. It is important to note that these students were novice provers. As vector calculus students, they had not taken a course involving mathematical proving, and they likely had not been previously confronted with the question of what it means to prove a relationship (they may have seen 2-column geometry proofs in high school, but they had not taken a proof-based undergraduate mathematics course). Thus, it makes sense that perhaps the students' only way of understanding how to establish the equation would be to demonstrate equality through algebra. Nonetheless, even though these students were new provers, we do gain some insight from their work. In particular, their work suggests that when students are introduced to combinatorial proof and combinatorial identities, it may be worthwhile to have a discussion of what it might mean to prove an identity combinatorially.

This data suggests to us that developing a combinatorial proof is understandably nuanced. This implies that students may need to be explicitly taught what combinatorial proof is, both in terms of why it is a valid form of mathematical proof and what is entailed in making a combinatorial argument. Differences between combinatorial versus algebraic arguments might need to be addressed directly if we expect students to understand how to combinatorially prove an identity. Again, this is not surprising, but we have overwhelming evidence that even with very successful and consistent counters, this was a mysterious, new, and challenging idea for them.

**Students should develop a particular combinatorial context**

Our data also suggests an important aspect of combinatorial proof is for students to be able to reason within a particular context. This is how combinatorial proof tends to be taught, and we are not claiming to offer some new mathematical insight. However, what is noteworthy is that we see evidence of students establishing and leveraging particular contexts, which give them something concrete to count from which they can then generalize. For example, we gave students Task 1 and asked them to count it in two different ways, and their response is seen in the excerpt below (their work is seen in Figure 3).

*Interviewer*: So are those two things counting the same committees?

*Sanjeev:* Yeah.

*Rose*: Yeah.

*Sanjeev*: The remaining is the number of 15 people committees. In this case, you're making the committees, whereas in that case, you're making them not committees.

*Rose*: Making them leave.

*Interviewer*: Okay, but both are giving you the 15 people.

*Sanjeev*: I think so.

*Rose*: Yeah, because if you make these ten people leave, then you'll just be stuck with 15 people.

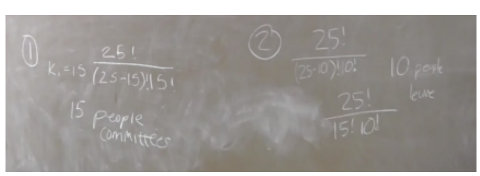


*Figure 3. Two different expressions for counting 15-person committees*

When we then asked them to generalize we could ask in terms of the same context, and students used the committees context to correctly establish the identity that in Task 1. We contend that being able to reason about and contextualize a problem is instrumental in supporting the combinatorial argument – without it, the formulas/expressions have no combinatorial meaning

## Discussion

### Combinatorial Proof as Semantic Proof Production

Combinatorial proof is a very specific kind of proof technique. However, although it is narrow, it can also be useful and important for a couple of different reasons. First, it is a specific combinatorial topic that reinforces other important combinatorial ideas like emphasizing sets of outcomes and the relationships between the components of Lockwood's (2013) model. Second, we also argue that it offers a different perspective on mathematical justification and proof. In particular, we propose that combinatorial proof naturally lends itself to semantic proof production. Weber and Alcock (2004) identify several aspects of knowledge required to produce semantic proofs, and we highlight a couple of them as being similar to aspects of knowledge required to produce combinatorial proofs. They emphasize instantiation and say that "One should be able to instantiate relevant mathematical objects. These instantiations should be rich enough that they suggest inferences that one can draw" (p. 229). They also note that "One should be able to connect the formal definition of the concept to the instantiations with which they reason" (p. 229). We interpret that the contextualized combinatorial arguments represent domain-specific instantiations that allow for meaningful proving and justifying. While it may be possible for a student to produce a combinatorial proof that does not involve knowledge that Weber and Alcock describe, we suggest that the kinds of context-based combinatorial justifications required for combinatorial proof are generally indicative of such knowledge.

## Conclusion and Implications

Combinatorial proof is a fascinating topic that is relevant both to the teaching and learning of combinatorics and to students' proving activity. In an initial exploration of students' engaging with combinatorial proof we have identified some key conceptions that may help students productively engage in combinatorial proof. We conclude with a couple of potential pedagogical implications of this work. First, students should focus on specific contexts and concrete problems. Then, teachers should give students opportunities to generalize from these particular cases. Our trajectory of concrete to general problems seemed productive in helping students gain familiarity with a context before generalizing using variables. Then, once students have established relationships in the concrete cases, they can attempt the more traditional combinatorial proofs of binomial identities. Overall, teachers should try to make sure students understand how their counting processes relate to the formulas/expressions and the set of outcomes, as discussed in Lockwood's (2013) model.

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# Computing as a Mathematical Disciplinary Practice

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*The role of computation continues to be prominent in the STEM fields, and the activity of computing has become an important mathematical disciplinary practice. Given the importance of computational fluency in science and mathematics, we are curious about the nature of such activity in mathematics. To study this, we interviewed six mathematicians about the role of computation in their work, and we identified several aspects of computation that sheds light on the nature of computing as a mathematical disciplinary practice. In this paper, we present examples and applications of computation for these mathematicians, highlight types of computation, provide specific examples of computation in their work, and emphasize how computation relates to mathematics in particular.*

Keywords: Computation, Mathematical disciplinary practices, Mathematicians

## Introduction and Motivation

What is the role of computation for doing mathematics? What does computation mean, given the broad range of settings in which mathematics is applied? How could one justify the teaching and learning of computation, given the national focus on reasoning, problem solving, and abstract thinking at all levels of mathematics? Our research is driven by questions of this type, especially in light of the technological advancements that continually blur the lines that define what counts as doing mathematics among those in the profession. The content and practices of different levels of mathematics have traditionally been aligned to varying degrees with the discipline of mathematics (Moschkovich, 2007; Rasmussen, Wawro, & Zandieh, 2015). Thus, with current efforts to incorporate the use of technology, and to see mathematics as a setting to explore relationships with computer science (Grover & Pea, 2013), a natural question is how such work is conducted and perceived by professionals within the field of mathematics. Such perspectives can help to inform answers to questions about the meaning and purpose of computation at post-secondary levels.

With this study, we explore the relationship between activities that have come to be described as *computation* and *computational thinking* and the practice of doing mathematics. We used the lens of *disciplinary practices* (Rasmussen et al., 2015) to consider ways in which mathematicians view computation as an element of their professional work. To do this, we interviewed mathematicians about computation in their research and teaching, specifically, according to the following research question: *How do mathematicians characterize and use the disciplinary practice of computing in their work?* The results of our analysis indicate that members of the mathematical community value computing as a distinct practice, and that it may be beneficial to foster computational fluency among students. These findings give disciplinary support to efforts at incorporating computing and computer science into mathematics, and they begin to suggest some of the ways in which that integration may naturally surface.

## Background Literature

In computer science education research, there is a construct called *computational thinking* (CT) (Grover & Pea, 2013; Wing, 2006, 2008), which Wing initially described as "taking an approach to solving problems, designing systems and understanding human behaviour that draws on concepts fundamental to computer science" (2006, p. 33). Wing went on to characterize CT broadly and as encompassing many kinds of thinking and activity, such as "thinking recursively" (p. 33), "using abstraction and decomposition when attacking a large complex task or designing a large complex system" (p. 33), "using heuristic reasoning to discover a solution" (p. 34), and "making trade-offs between time and space and between processing power and storage capacity" (p. 34). Wing did not intend for computational thinking to be neatly defined, and indeed the broad characterization makes it difficult to pin down a precise definition. However, describing a notion of computational thinking provides a starting point for identifying common threads among computational activity. While we do not explicitly examine computational thinking in this paper, we acknowledge the role that this idea played in the design of our study.

In exploring the idea of defining computational thinking, Weintrop et al. (2016) developed a "taxonomy of practices focusing on the application of computational thinking to mathematics and science" (p. 128). For each of these practices, Weintrop et al. (2016) elaborated certain activities that the practice may entail. For example, they said that *Programming* "consists of understanding and modifying programs written by others, as well as composing new programs or scripts from scratch" (p. 139). For *Troubleshooting and Debugging*, they explained that there are "a number of strategies one can employ while troubleshooting a problem, including clearly identifying the issue, systematically testing the system to isolate the source of the error, and reproducing the problem so that potential solutions can be tested reliably" (p. 139).

In developing their taxonomy, Weintrop et al. (2016) started with activities that elicited computational activity and refined that framework through interviews with experts (including school teachers and scientists). Notably, though, while they interviewed scientists (biochemists, physicists, material engineers, astrophysicists, computer scientists, and biomedical engineers), they did not interview research mathematicians. Thus, even though our study bears similarities to this work – namely asking experts about the nature of computational activity – we highlight two key differences. First, rather than beginning with a set of computational activities, we begin with mathematicians' descriptions of their work, forming categories and types of computation based on their experiences and responses. Second, by interviewing research mathematicians, we focus specifically on the role of computation within discipline of mathematics and not on STEM more widely. This attention to research mathematicians is closely related and relevant to undergraduate mathematics education in ways that broader STEM and K-12 emphases are not.

## Theoretical Perspective

As we will see in the results, it is not trivial to define computation, and there are many ways in which computation can be characterized and framed. However, to clarify, in this paper we provide the following broad characterization as our working definition of computation: Computing is the practice of using mathematical calculations, processes, or algorithms, often to generate products that can be interpreted, investigated, or implemented in other contexts and problems. Computing often involves the aid of technology but can also be performed by hand.

Rasmussen, Wawro, and Zandieh (2015) defined disciplinary practices as "the ways in which mathematicians go about their profession" (p. 264), which they viewed as related to Moschkovich's (2007) notion of "professional discourse practices" (p. 264). These are the practices in which mathematicians engage in their professional spheres. Examples of disciplinary practices include algorithmatizing, theoremizing, defining, and symbolizing (Rasmussen et al., 2015). In this study, we are conceiving of computing as a disciplinary practice, something that mathematicians now do. Indeed, both Rasmussen et al. and Moschkovich argued that such practices are culturally and historically situated, and "socially, culturally, and historically produced practices that have become normative" (p. 25). We feel that this is an apt way to characterize computing, because computing seems like a particularly important disciplinary practice in our increasingly computerized society. That is, in light of increasing computational requirements for mathematics majors and computational methods in mathematical research (e.g., Bagley & Rabin, 2016), we feel that computing is becoming a relevant practice that is increasingly becoming an integral part of "being a mathematician." We thus consider computing to be a disciplinary practice and use this lens in framing our study. While the term computation could refer the product of the activity of computing, we use the terms interchangeably (as the participants used the term interchangeably during the interviews).

## Methods

To answer our research question, we interviewed six mathematicians in single 60-90 minute semi-structured interviews. The mathematicians were professors in mathematics departments, all holding PhDs in mathematics (see Table 1; all names are gender-preserving pseudonyms). It was a convenience sample (mathematicians to whom the authors had access and proximity), but we sought to maintain a balance of sub-disciplines of mathematics (especially pure versus applied).

*Table 1. Information about the Interviewees*

| Mathematician | Area of specialty | Years in field | Programming Language(s) |
|---|---|---|---|
| Michael | Mathematical Biology | 4 years | Mathematica, Matlab |
| Liliana | Applied Mathematics | 30 years | Matlab, Tecplot |
| Paul | Numerical Analysis | 12 years | Matlab, Comsol, Maple |
| Carter | Geometry | 35 years | Mathematica |
| Peter | Algebraic Combinatorics | 18 years | Maple |
| Andrea | Applied Mathematics | 7 years | Matlab, Python |

All interviews were audio-recorded. We began the interviews by asking the mathematicians to reflect upon various aspects of computation, including computation in their own work, the value of computing for themselves and for students, and how they might teach computing. For example, after asking some preliminary demographic questions, we asked the following: *Do you use computation in the work that you do? How so? How are you defining 'computation'?* and *What are some specific ways (or contexts) in which you use computation in your work? Could you provide an example or two?* This enabled us to get a sense of how they viewed and might use computation. We also asked whether (and why) they thought computation is important for students to learn. We concluded with discussions about whether and how they had taught computation before, and for them to weigh in on how students might learn computation.

We used a combined process of open and axial coding (Strauss & Corbin, 1998) to describe the concepts, perspectives, and processes that characterized the mathematicians' ideas about computation in mathematics. In the first phase, the first author studied the transcripts and coded them with descriptions of the core ideas or themes from the mathematicians' comments. Examples of codes that emerged through this round of analysis include "computation is related to proving," "computation is used for generating examples," and "computation requires the compartmentalization of steps."

In the second phase of analysis, the second and third authors applied the generated list of codes to the entire set of interviews. All three authors met regularly throughout this process, during which time we compared our coding of the transcripts to resolve any discrepancies, refined the meanings of the different codes, and began to articulate a set of themes according to which the codes could be organized. We returned to the interview transcripts during these meetings looking for evidence for and against the common perspectives we saw within each theme. The results of our analysis include a set of themes that can be used, broadly, to categorize our participants' comments about computation, as well as examples from the data to support the variety of viewpoints that surfaced within each of these themes.

## Results

We describe three main themes that characterize the mathematicians' comments about computation. First, we offer insights about how mathematicians characterized computation in their work, including similarities to and differences from programming. Second, we discuss practical applications of computation, including particular examples of how computation arises for these mathematicians. Third, we present mathematicians' views about the relationship between computing and mathematical problem solving. Through these results we seek to paint a more complete picture of how mathematicians think about and use computation in their work. Because of space we cannot highlight every point or make every part of it clear, but we can emphasize the main findings and provide evidence from a number of the mathematicians.

### Types of Computation

In order to get a sense of how mathematicians defined or exemplified computation, we asked all participants a variation of the question, "what is the computation involved in your work?" Their responses indicated that the definition of computation, even within the field of mathematics, is difficult to articulate and is context dependent. In particular, the mathematicians made a distinction between numerical computations, and what might be considered algebraic computation, as exemplified by Michael's comments below:

*Michael:* There's computation, for instance, like if you're proving some theorem and you need a technical lemma and you've got to work out this computation just to show that that lemma is true. So that's one way. The other way is sort of like numerical computation. Computations that you're not going to do by hand, so you get a computer to do it.

Michael gave as an example of the first type of computation the case of showing that a particular function is Lipschitz – which involves verifying a string of inequalities – in order to use that property toward proving a more involved theorem. Numerical computation itself, according to Michael, could be further broken down into two different types: a tedious calculation that might best be done by a computer (e.g., a binomial probability with a large number of events) or computations akin to mathematical modeling, for which a set of data needs to be analyzed with no predetermined algorithm or formula.

The types of computation that mathematicians saw as most relevant to their work corresponded to the specifics of different sub-disciplines of mathematics. Peter, an algebraic combinatorialist, described his use of computation primarily in terms of algebraic computation (e.g., factoring complex expressions) and numerical calculations (e.g., calculating the determinant of a matrix). Mathematicians in more applied fields described computation in terms of solving models (e.g., solving partial differential equations numerically) or using computation to analyze data or approximate solutions. It was clear from our interviews that there was no consensus on a single definition of computation, although computation can be characterized broadly by a few different types of activities.

To summarize, in response to the question of how mathematicians use computation in their work, we saw that what constitutes computation varies according to the types of problems that are relevant within different subfields of mathematics. Computation as a practice occurs at different scales, from performing symbolic manipulations and numerical calculations, to creating and implementing mathematical models.

### Examples and Practical Applications of Computation

The mathematicians articulated a number of ways in which they use computation in their work, and this provides insight into how and why computation can be so useful. Table 2 shows instances of what we coded as a practical application of computing, each with a supporting and exemplifying quote from a mathematician. This gives a set of concrete examples and evidence of the variety of ways in which mathematicians use computation in their everyday work.

*Table 2. Examples and quotes of practical applications of computation*

| Practical Application | Specific Example(s) and Supporting Quote(s) |
|---|---|
| Testing conjectures | *Peter*: I use software to enumerate combinatorial objects that satisfy certain constraints where I have, say a conjecture – a prediction of how many there should be or a predicted bound on how many there should be and I'll collect numerical data to test my results. |
| Visualizing | *Liliana*: And sometimes I use computers to illustrate, um, some salient features of a problem that are otherwise hard to just understand. You can formulate them using proper algebra, calculus, whatever. But, you know, how common is it for someone to understand a complicated feature of a problem, um, using just, um, a formal, a very formal statement that involves, I don't know, derivatives or something like this? We can do that, but people really—if you have a finite amount of time to describe a problem, um, visualization is an important component. Uh, I think of what it means to convey some ideas. It does take a lot of time, and it is frequently unappreciated as part of that, um, research. But I think it's a very important part. |
| Communicating | *Paul*: Yes, definitely. I mean, you wouldn't be able to do proofs if you couldn't do that and you wouldn't be able to do computation if you couldn't do that and you couldn't make clear arguments to convince people of other types of conclusions that you're trying to make if you didn't take your arguments through logical steps. So, if you're trying to convince anybody of something and you need to tell them that your solution or your idea does what you think it does and nothing else. And that's exactly what a code's supposed to do too.<br>*Interviewer*: Okay. Nice. And so, getting experience with that kind of coding could basically model that kind of experience of being able to give an argument and show that logical process of it.<br>*Paul*: Right. I'm saying that it not only helps with doing math, it also helps with communicating math. |

| | |
|---|---|
| Recovering from mistakes | *Liliana*: And so that's the other ability. That, and, well, of course, and there is the other ability of being able to recover from mistakes. Which, in computing, is fundamental. And not being too frustrated and just keep going back and forth. And trying to morph something that you know worked to something you know should work. |
| Using Computation in Teaching | *Carter*: I use the computer to check exam solutions when I teach calculus. I use the computer to draw graphics that I use both in my own textbooks and in my own teaching. |

Table 2 gives a sense of the variety of ways that mathematicians use computing in their work. Some of these applications (such as *using computation in teaching* or for *visualization* of ideas in research) help the mathematicians accomplish specific, practical goals. Other applications (such as *communicating* and *recovering from mistakes*) facilitate the development of other essential, broader practices and habits of mind. These examples inform why computation is a valuable and useful practice for mathematicians and suggest why it should be developed among students.

**Relationship between Computation and Mathematics**

The mathematicians also talked about the relationship between computation and additional mathematical practices. For example, some articulated that computation was related to proof, problem solving, and other aspects of mathematical thinking and research. In this section, we highlight some excerpts that raise some salient points of discussion about the role of computing in the field of mathematics specifically.

In the following exchange, Liliana describes the back and forth relationship between the mathematical and theoretical analysis she does and the computing in which she engages. We see that Liliana describes how she applies computational activity to mathematical analysis of a problem, which suggests that she is relating her computational activity back to the mathematical processes on which she is working.

*Liliana*: So, the other part, the theoretical part, is different. It can be supported by experimentation, so let's say you're not quite sure how the solution to this equation is going to behave. I don't know, nonlinear equation depending on the parameter, so you're not quite sure. You can do some analysis, and that can be tedious, but you can also explore it computationally. Which will suggest what tools you would use to analyze that. Or which would sort of verify some of the intuitions you had which you can later use here. Um, but generally—or you're deriving some kind of a theoretical result, which actually typically now we're, uh, research—you should verify, you should show some experiments that verify that indeed the convergence rates are this or that. If you can't get it to confirm, that's bad.

*Interviewer*: Mmhmm, because it suggests something is wrong your analysis.

*Liliana*: Something is either wrong with the implementation, or with the analysis.

*Interviewer*: Yeah.

*Liliana*: If you get a better result, then it's okay. I mean, if experimentally you're getting a higher rate than the one you did, then it means you didn't get sharp results. But if it was the other way, then you should go back to the drawing board because there was an unrealistic assumption you made or something like this.

Peter also emphasizes this relationship between the computation and the theoretical mathematical research he does. It is worth noting that Peter viewed computation as a way to generate examples and explore conjectures (as noted in the first row of Table 2 above). The following quote highlights that he views computational activity as providing necessary content that he can then use and apply in his mathematical research.

*Peter*: The act of doing the computations and the results that are obtained by computations are the content of mathematics. And so, the theorems and relationships we're describing are only good to the extent that they reflect something that either is calculated or that you're evaluating by not calculating directly or – you know, yeah. It's like if a poet has no life experience they can't write good poems. If a mathematician doesn't do computation or have the results of computations, they don't really have the subject that they're supposed to be reasoning about in writing their theorems?

Andrea also spoke to the important relationship between computational activity and theoretical mathematical work. In discussing teaching computing to students and what she wants them to learn through the practice of computing, she indicated that she would want them to be able to relate the programming they do to the mathematical ideas.

*Andrea*: Like, knowing a program, or knowing how to, say, code in Python is a skill. I think it's not really useful unless you can – like, by itself I don't think is useful. So, I would hope that the student also knows how to do mathematics on paper.

*Interviewer:* So, what else do they need besides just that ability to program in Python?

*Andrew:* I guess it's what I would say any mathematician or math student would need to have, and it's the ability to – I'm not going to say this right because this is not my area, but – to be mathematically skilled. But I don't really know what I would say that is. It's like, the ability to solve any type of problem. Maybe the ability to approach a math problem with some insight into which direction to go in.

In this section, we see that the mathematicians understand that computation in their work must be connected to the theoretical mathematics, and we gain insight into how this practice of computing can complement and enrich the mathematical work that they do.

## Conclusions and Avenues for Future Research

In this paper, we have reported on interviews in which we asked mathematicians about the role of computation in their work and in the field. We discussed their characterizations of computing, examples and practical applications of computing, and the relationship between computing and the theoretical mathematics they do in their research. Together these findings paint a picture of the varied ways that computation arises in mathematicians' work, and they highlight the important role that computation plays. In this way, we feel that our findings make a case for computing as a key mathematical disciplinary practice, helping us to justify the importance of developing this practice in undergraduate mathematics classrooms. Indeed, the fact that mathematicians use computation in a number of ways underscores that it is a practice that deserves more attention in mathematics education research.

In light of these findings, we have several ideas for further research. First, we feel that we should broaden our set of mathematicians, perhaps interviewing or surveying greater numbers of mathematicians and mathematicians in industry. Second, but we are interested in exploring the notion of computational thinking in mathematics. We contend that there may be certain ways of thinking that facilitate computation in mathematical contexts, and we want to investigate what such a way of thinking might entail. Third, we would like to investigate undergraduate students' characterizations and uses of computation in mathematics.

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# Finite Mathematics Students' Use of Counting Techniques in Probability Applications

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*In this study we seek to better understand how students are using counting techniques within the context of the probability application. To do so we investigate three semesters of finite mathematics students' use of enumeration, Venn diagrams, and counting formulas on probability free-response exam questions at a large public university in the mid-south. The study found that appropriate use of enumeration techniques and Venn diagrams both statistically significantly increased a student's likelihood of arriving at a correct answer, while there is statistically significant evidence that the use of counting formulas decreased a student's likelihood of arriving at a correct answer. We conclude with a discussion of the implications of this study for the practice.*

*Keywords:* Combinatorics Education, Probability, Enumeration, Venn Diagrams, Combinations

## Introduction and Motivation

The original finite mathematics course debuted in 1957 at Dartmouth College (Kemeny, Snell, & Thompson, 1957). While the course has changed some in the intervening span, it has remained relatively unaltered in the last forty years. This era of stability began when the business major became popular in the United States and business faculty recognized the importance of the finite mathematics course. This change also increased the popularity of the course. The Conference Board of Mathematical Sciences (CBMS) has not yet released their 2015 mathematics programs census reports, the 2010 report gives enough information to conservatively estimate that 120,000 students were enrolled in an introductory finite mathematics course in the United States per annum (Blair, Kirkman, & Maxwell, 2013). Despite having such a long, established history and substantial enrollment, the research literature regarding finite mathematics courses remains limited.

The counting and probability unit taught in Introduction to Finite Mathematics courses is full of versatile topics that matter to the population of students taking the course. This unit has been included in one form or another as a part of the finite mathematics from the courses' beginning in the late 1950s and are not likely to be excluded while business majors are still required to take the course. As Tucker (2013) states, "measuring and counting things [has] interested business-minded Americans from the republic's founding" (p. 692). Despite the interest in these topics and the importance put on them in the CCSSM, they remain largely unstudied at the undergraduate level. Yet, many people utilize counting and probability in their daily decision making without ever being conscious of it.

While it may be that finite mathematics courses and the way instructors teach counting and probability at the undergraduate level are maximally effective, educators cannot know for sure until the topic is fully explored. To date, Elise Lockwood has been the primary contributor to the field. Her studies have been qualitative in nature and have largely focused on students' association of counting with sets (Lockwood, 2011a, 2011b, 2012, 2013, 2014, 2015; Lockwood & Gibson, 2016; Lockwood, Reed, & Caughman, 2016). However, she has not yet addressed the application of counting techniques to probability. Counting and probability are interrelated topics and educators do not know if students are making the necessary connections. The current study allows educators to better understand how students are using counting techniques within the context of the probability application in a finite mathematics course.

As a part of a broader project (Blyman, 2017), this study seeks to address the following research question: *How successfully are students using the counting techniques of enumeration, Venn diagrams, and counting formulas when completing free response probability exam questions?* The results of this study provide insights to improve instruction in courses that include introductory counting and probability.

## Literature Review and Framework

All of Lockwood's work has contributed to a better understanding of students' thought processes which can be applied in the classroom. Lockwood (2011a, 2013) posited a model of students' combinatorial thinking where she explored connections students make between counting processes, formulas and expressions, and sets of outcomes. This model paired with mathematical theory relating probability to counting provides a framework for this study.

Lockwood has published various qualitative studies in undergraduate combinatorics education. Throughout her work, she primarily uses student interviews as a tool for gaining insight into the thought processes of students. Investigating counting techniques used by students has led to significant evidence that students struggle to solve counting problems (Lockwood, & Gibson, 2016). More specifically, students "struggle to detect common structures and identify models of underlying problems" (Lockwood, 2011b, p. 307) when solving counting problems. However, the roots of these struggles and ways to mitigate them have not yet been thoroughly studied (Lockwood, 2015).

Particularly relevant to this study are those studies which focus on listing sets of outcomes when working to solve counting problems (Lockwood, 2012, 2014; Lockwood & Gibson, 2016). These studies have resulted in evidence that students understand counting problems best when they enumerate sets of outcomes (Lockwood 2012, 2014; Lockwood & Gibson, 2016). Consequently, Lockwood (2012) encourages students and instructors alike not to be tempted to skip over the crucial step of listing outcomes when learning and teaching students to do counting problems. The multiplication principle connects counting processes with sets of outcomes and, consequently, deserves to be studied in and of itself (Lockwood, Reed, & Caughman, 2016). To begin studying it, Lockwood, Reed, and Caughman (2016) examined many finite mathematics textbooks' treatments of the multiplication principle in counting. They found there were many ways textbooks covered the multiplication principle and hypothesized this could have significant impacts on students' combinatorial thinking (Lockwood, Reed, & Caughman, 2016).

## Methodology

### Assumptions and Delimitations

This study assumes the course was taught identically by all lecturers and recitation instructors involved as collaboration and sharing of materials was common; however, they were not required to conduct lectures or recitations in the same manner. While informal conversations with lecturers and recitation leaders each semester have made it clear all classes look quite similar, lecturers and recitation leaders each bring their own style and varied experiences into the classroom, so some variance in instruction from section to section likely occurred.

Additionally, data were collected from students enrolled in an introductory finite mathematics course at a large public university in the mid-south for the Spring 2015, Fall 2015, and Spring 2016 semesters. Due to the structure of the course and homework, demographic information was not collected from participants. Consequently, demographics could not be used as covariates in the analyses of this study.

Finally, the format of the exams collected for qualitative data analysis changed between the Spring 2015 and Fall 2015 semesters. The Spring 2015 semester exams consisted of only free response problems while the Fall 2015 and Spring 2016 semester exams each only had 3 free response problems with the rest of the problems being multiple choice. As only the free response questions were considered in this study, the Spring 2015 exams provided much more data for each participant than did the Fall 2015 and Spring 2016 exams.

**Data Analysis**

To determine how successfully students are using counting techniques on free-response probability exam questions, a stratified random sample of the exams were coded and categorized using the provisional coding method (Saldaña, 2016). Provisional coding makes use of a list of *a priori* codes. For this study the list of codes was created by consulting answer keys to exams which were produced by instructors, considering mathematical connections, and by considering previous research in the field. Since the answer keys to exams from previous semesters are made available online to students as a study tool and the counting techniques which they were exposed to as a part of the course are rather limited, provisional coding using these *a priori* codes was an appropriate choice. While the textbook includes a section on the multiplication principle, the lecturers who wrote the answer keys chose to make use of combination notation over use of the multiplication principle. After scanning several exams and noting how much they resembled the answer keys from previous semester, it was clear that the set of *a priori* codes developed was sufficient to determine how successfully students were using enumeration, Venn diagrams, and counting formulas.

The strata for the sample were formed by recitation leader and the semester the data were collected in order to best form a truly representative sample. For each full-time recitation leader (leading *4* sections) *18* exams were selected and for each half-time recitation leader (leading *2* sections) *9* exams were selected. Sampling was used in order to make the data set more manageable. The numbers *18* and *9* were chosen because it resulted in over one quarter of the exams being coded. In total, *208* exams (*31.3%*) were coded. We scanned over the remaining exams to ensure the stratified random sample was a reasonable representation of all the exams collected. When coding exams, we considered any listing of elements of sets of outcomes to be enumeration, any attempt at using a Venn diagram to be using a Venn diagram, and any attempt at using a counting formula – even simply using combination notation – as making use of a counting formula. The exam answer keys provided examples of what each of the codes represented for reference throughout the coding process. Each part of each free response probability question was treated as an individual problem to be coded. During the coding process, neither coder found any student exams that warranted the addition of a new code.

Both authors coded the exams. Ten exams were coded together to establish consistency. To further guarantee consistency, both intra-rater and inter-rater reliability studies were conducted (Huck, 2012). Each grader reanalyzed a random sample of five exams to measure intra-rater reliability. Intra-rater reliabilities were *86.3%* and *91.5%*. We both analyzed a random sample of ten exams to establish inter-rater reliability. Inter-rater reliability was *86.8%*. In addition to categorizing techniques students were using, the study made use of frequency of codes in a quantitization process (Miles, Huberman, & Saldaña, 2014). Quantitization was used so that it could be determined how successfully students were using each of the counting techniques. Quantitizing the data allowed us to objectively determine if students were answering questions correctly more often or not when they appropriately used enumeration, Venn diagrams, or counting formulas. Using the quantitized data, three chi-square tests of association were conducted pairing each counting technique with correctness of the student's response. Based upon previous studies in the field and by recognizing Venn diagrams as a way of semi-enumerating a larger population by segmenting it, we hypothesized that enumeration and Venn diagrams would be positively associated with correctness on exam problems, while counting formulas would be negatively associated with correctness on exam problems.

## Results

Each of the chi-square tests was conducted with the null hypothesis that correctness is not associated with the use, or lack of use, of the given counting technique.

### Enumeration

The first of these tests examined the relationship between correctness and enumeration of possible outcomes. The questions included in this test were those on which the instructor chose to utilize enumeration on the published answer key for the exam and those questions which have an easily enumerated set of possible outcomes which the instructor chose not to list on the answer key. Namely, these were Spring 2015 (*n=81* coded exams) questions 1a-c and 7a-b; and Spring 2016 (*n=90* coded exams) questions 14b-c. This yielded a total of $5 \times 81 + 2 \times 90 = 585$ cases where enumeration was an appropriate technique for students to employ. The results of this chi-square test were $\chi^2(1{,}585) = 34.293, p < .001$. Therefore, this result is statistically significant with a small-medium effect size ($\varphi = .242$) since *.1* is considered a small effect and *.3* is considered a medium effect (Cohen, 1988). While students who chose not to enumerate the set of possible outcomes when solving the exam questions were split relatively evenly between those who got the question correct or not, students who used the enumeration strategy were more than twice as likely to get the question correct as incorrect (see Table 1). This result rejects the null hypothesis and confirms the hypothesis that enumeration is positively associated with correctness.

*Table 1. Chi-Square Test Results for Enumeration with Correctness.*

| | No Enumeration | Enumeration | Total |
|---|---|---|---|
| Incorrect | 154 | 78 | 232 |
| Correct | 147 | 206 | 353 |
| Total | 301 | 284 | 585 |

### Venn Diagrams

The second test examined the relationship between correctness and the usage of Venn diagrams. The questions included in this test were those on which the instructor chose to utilize a Venn diagram on the published answer key for the exam. Namely, these were Spring 2015 (*n=81* coded exams) question 5c; Fall 2015 (*n=27* coded exams) question 15a; and Spring 2016 (*n=90* coded exams) question 15a. This yielded a total of $81 + 27 + 90 = 198$ cases where using a Venn diagram was an appropriate technique for students to employ. The results of this chi-square test were $\chi^2(1{,}198) = 5.942, p < .05$. Therefore, this result is statistically significant with a small- medium effect size ($\varphi = .173$) since *.1* is considered a small effect and *.3* is considered a medium effect (Cohen, 1988). While students who chose not to use a Venn diagram when solving the exam questions were approximately two and a half times more likely to get the questions correct as incorrect, students who used Venn diagrams were almost six and a half times as likely to get the question correct as incorrect (see Table 2). This result rejects the null hypothesis and confirms the hypothesis that the use of Venn diagrams is positively associated with correctness.

Venn diagram problems were decidedly easier for students to correctly answer than their enumeration counterparts. Perhaps this is because Venn diagram problems require students to distinguish between at most three distinguishing traits and to classify portions of the population accordingly so there are at most eight numbers which the student is required to determine, while enumeration problems could require students to list as many as *36* possible outcomes to an experiment. On a timed test, students are more likely to not persist and not take the required time to make a list of *36* possible outcomes in such a way as to be able to successfully complete the exam problem.

*Table 2. Chi-Square Test Results for Venn Diagram with Correctness.*

| | No Venn Diagram | Venn Diagram | Total |
|---|---|---|---|
| Incorrect | 16 | 19 | 35 |
| Correct | 41 | 122 | 163 |
| Total | 57 | 141 | 198 |

**Counting Formulas**

The final test examined the relationship between correctness and the usage of counting formulas. The questions included in this test were those on which the instructor chose to utilize a counting formula on the published answer key for the exam. Namely, these were Spring 2015 (*n=81* coded exams) questions 1b, 2a-b, 3a-d, 7a-b, 8a-c, and 9a-b; Fall 2015 (*n=27* coded exams) questions 13a-c, and 14c; and Spring 2016 (*n=90* coded exams) questions 13a-c, and 14b-c. This yielded a total of $12 \times 81 + 4 \times 27 + 5 \times 90 = 1692$ cases where using counting formulas was an appropriate technique for students to employ. Students were counted as having used a counting formula if they made any use at all of a formula, even simply using combination notation in their answer. The results of this chi-square test were $\chi^2(1{,}1692) = 22.636, p < .001$. Therefore, this result is statistically significant with a small-medium effect size ($\varphi = .116$) since *.1* is considered a small effect and *.3* is considered a medium effect (Cohen, 1988). While students who chose to use a counting formula on exam questions were approximately equally likely to get the questions correct or incorrect, students who did not use counting formulas were approximately one and a half times as likely to get the question correct as incorrect (see Table 3). This result rejects the null hypothesis and confirms the hypothesis that counting formulas are negatively associated with correctness.

*Table 3. Chi-Square Test Results for Counting Formula with Correctness.*

| | No Counting Formula | Counting Formula | Total |
|---|---|---|---|
| Incorrect | 288 | 480 | 768 |
| Correct | 453 | 471 | 924 |
| Total | 741 | 951 | 1692 |

## Conclusion

The results of the chi-square tests show students were most successful solving probability problems when using enumeration and Venn diagrams. Students who enumerated sets of outcomes on problems where it was appropriate were more likely to correctly solve the problem than those who chose not to enumerate the set of outcomes on the same problems. Moreover, students who used Venn diagrams on the problems where Venn diagrams were used on the instructor-provided answer key, were much more likely to correctly solve the problem than those who chose not to use a Venn diagram on the same problems. However, students who used counting formulas on the problems where counting formulas were used on the instructor-provided answer key, were more likely to incorrectly solve the problem than those who chose not to use accounting formula on the same problems.

When attempting to solve probability problems set within the context of an inclusion-exclusion counting problem, Venn diagrams were highly effective as a problem-solving technique. A much higher percentage of students correctly responded to the Venn diagram questions than the enumeration or counting formula questions regardless of whether or not a Venn diagram was used. Therefore, students found the Venn diagram problems easier than their enumeration and counting formula counterparts. However, the likelihood of getting a probability question situated in an inclusion-exclusion setting correct increased quite dramatically when a Venn diagram was used.

These results confirm the hypotheses and Lockwood's findings that students understand counting problems best with enumeration (2012, 2014; Lockwood & Gibson, 2016). The chi-square test considering enumeration paired with Lockwood's work make it clear enumeration is an effective strategy for students to use when solving probability problems involving small sets of possible outcomes. Therefore, this study provided quantitative evidence for the findings of Lockwood's previous qualitative studies and extends those findings to the probability application of counting. Additionally, the study provided evidence that students using Venn diagrams have a relatively strong understanding of the problem and are equipped to take steps beyond the construction of their Venn diagram to answer a probability application question related to the Venn diagram that they have created.

Finally, the results of this study make it clear that counting formulas did not help students solve probability problems correctly. In fact, the study revealed students who use counting formulas have a diminished chance of correctly solving the probability problem. At the site of this study, students are offered the option of presenting their final answers as a quotient of two values written in combination notation rather than as the standard proper fraction, decimal, or percentage form of a probability. Through this policy and the presentation of solutions to past exam problems using combination notation, students are not only offered the chance to use this notation, they are strongly encouraged to use it rather than any other counting technique. While, the problems in which students are encouraged to use counting formulas are sometimes more difficult than the problems where they are encouraged to use other counting methods; however, not all the counting formula problems are more difficult. Whether the formula itself was the hindrance or there was some underlying factor at work, between their choice not to explicitly use a counting formula in their work and their misuse of said formula when they chose to use one, the majority of students made it clear they do not understand counting formulas.

**Implications for the Practice**

The implications of this study for undergraduate mathematics instructors are extensive. These implications stem from the struggles students are having using combination formulas to correctly solve probability problems. Instructors should be encouraging students to use those counting methods which best help them to understand the probability problem which they are attempting to solve. By allowing students to leave their answers in combination notation rather than requiring them to arrive at a proper presentation of a probability, instructors are allowing students to use a counting method without requiring the students know how to use it. Not only are they allowing this phenomenon to occur, they are actually encouraging it by providing students with answer keys to previous semesters' exams in which the solutions are only given in combination notation rather than as a decimal or fraction which the student could use to check an answer arrived at in a different way.

Additionally, since students are largely succeeding at enumeration and are struggling with properly applying the combination formula, it would be advantageous for instructors to give more attention to the multiplication principle – a known intermediary. In fact, given the nature of the counting and probability problems presented at the introduction to finite mathematics level, instructors should be reconsidering if the presentation of combination and permutation formulas is appropriate for the audience. With the limited time allotted for the counting and probability unit, perhaps students would be better served given a conceptual understanding of combinations and permutations while the methods for solving combination and permutation counting and probability problems are restricted to applications of the multiplication principle.

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## Analyzing Narratives About Limits Involving Infinity in Calculus Textbooks

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*We analyze Calculus textbooks to determine to what extent narratives about limits at infinity and infinite limits align with research in mathematics education. As reasoning about limits falls within the domain of advanced mathematical thinking (AMT), we looked for evidence of appropriate treatment of, and support for, AMT: clear and precise narratives, deductive and rigorous reasoning, intuitive development that does not create or enhance students' misconceptions, opportunities for "personal reconstruction" (Tall, 1991), adequate representations, and the appropriate use of definitions. In conclusion, both high school and university Calculus textbook narratives do not place infinity in a precise, well-defined context, thus possibly creating or strengthening (novice) students' misconceptions. We identified very little evidence of the type of support for AMT that we were looking for. This paper concludes with several suggestions for possible modifications of narratives which involve infinity.*

*Keywords:* Narratives, Limits at Infinity, Infinite Limits, Advanced Mathematical Thinking, Mathematics Textbooks

This study reports on an analysis of presentations of the concept of infinity in textbooks. To be more specific—our aim is to determine if, or to what extent, and how, research in mathematics education has informed, and possibly affected, narratives about infinity in Calculus textbooks. We focus on infinity in the context of limits: limits at infinity (i.e., independent variable "approaches infinity") and infinite limits (i.e., dependent variable "approaches infinity"). This study falls within attempts at bringing research in mathematics education closer to teaching practice.

Discussing ways of improving the quality of mathematics instruction, Artigue (2001) writes: "existing research can greatly help us today, if we make its results accessible to a large audience and make the necessary efforts to better link research and practice" (p. 207). Burkhardt and Schoenfeld (2003) are not optimistic: "In general, education research does not have much credibility—even among its intended clients, teachers and administrators. When they have problems, they rarely turn to research" (p. 3).

In general, mathematics education research rests on well-developed theoretical foundations, and contains constructive information, suggestions and insights for teaching; however, these rarely go far enough and do not touch upon practical aspects of teaching—for instance, by providing content-specific teaching ideas, or by suggesting a rough lesson plan.

Case in point: numerous papers (including almost all cited throughout this paper) address challenges, problems and misconceptions related to teaching and learning infinity at secondary and/or tertiary levels —and yet none gives specific guidelines and suggestions which a textbook writer (or a course instructor) could readily learn from and use. Burkhardt and Schoenfeld (2003) echo this view:

> "The research-based development of tools and processes for use by practitioners, common in other applied fields, is largely missing in education. Such "engineering research" is essential to building strong linkages between research-based insights and improved practice. It will

> also result in a much higher incidence of robust evidence-based recommendations for practice." (p. 3)

There are exceptions. Kajander and Lovric (2009) examine the ways in which the concept of the line tangent to the graph of a function is presented in high school and university textbooks. Their analysis points at exact locations within the narratives that could be problematic (i.e., could lead to the development or strengthening of students' misconceptions), and concludes with specific alternative approaches.

We asked ourselves whether the views presented in Artigue (2001), Burkhardt and Schoenfeld (2003), and others—such as Ball (2000)—accurately portray textbook development of the concept of the limit, in particular when limits involve infinity. The fact that mathematics education researchers—almost as a rule—do not author mathematics textbooks, did not give us much hope that theoretical developments about teaching and learning of limits and infinity found their way into Calculus textbooks.

## Limits as Advanced Mathematical Thinking

Tall (1981), Davis and Vinner (1986), Tall (2001), Fischbein (2001), Jones (2015), as well as other researchers (some mentioned in this section, or later in this paper) agree that reasoning about limits falls within the domain of advanced mathematical thinking (AMT). Edwards, Dubinski, and McDonald (2005) write: "AMT is thinking that requires deductive and rigorous reasoning about mathematical notions" (p.17). AMT operates with abstract concepts which require serious mathematical treatment, usually reserved for advanced mathematics courses. Dynamic approaches to introducing the limit, usually discussed in introductory calculus courses, need to be rethought and modified to accommodate for AMT, as otherwise they lead to a variety of misconceptions (Nagle, 2013).

Plaza, Rico, and Ruiz-Hidalgo (2013) assert the importance of definitions as a characteristic that distinguishes elementary from advanced mathematics. Vinner (1991) argues that teaching and learning definitions is a serious problem, and states that a definition "represents, perhaps, more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition" (p.65). Edwards and Ward (2004) echo this view: "many students do not *use* definitions the way mathematicians do, *even when the students can correctly state and explain the definitions*" (p. 416).

In Tall (1991), we read: "Advanced mathematics, *by its very nature*, includes concepts which are subtly at variance with naïve experience. Such ideas require an immense personal reconstruction to build the cognitive apparatus to handle them effectively" (p. 252). Edwards, Dubinski, and McDonald (2005) concur, and state that

> "In dealing with limits, students often struggle with the human need to make sense of things by attempting to carry out a process that is impossible to see to the end. Students who view the concept of limit as a dynamic process (meaning a process of getting closer and closer to a limit, but not the object that is the limit) or an unreachable bound, for example, are demonstrating in this instance a failure to use AMT as they are not transcending the finite physical models available to them." (p. 21)

Some researchers identify "process and object components" (Cotrill et al., 1996; Jones, 2015) of the numeric, algorithmic, or theoretical calculation of a limit, with process being equivalent to the notion of "dynamic" in the quote above. For some, "dynamic reasoning" about limits includes both components (Jones, 2015).

Further challenges to creating narratives about limits lie in the language. It is well known that the differences between everyday language and the mathematics language contribute to students' difficulties in understanding (Cornu, 1991; Monaghan, 1991; Kim, Sfard, & Ferrini-Mundy, 2005). Using colloquial phrases such as "to reach," "to exceed," "to approach" in articulating understanding of limits negatively affects students' understanding (Plaza, Ruiz-Hidalgo, & Romero, 2012). Kajander and Lovric (2009) show that this colloquial, "reader-friendly" language leads to the development of misconceptions in textbook presentations of the concept of a tangent line.

Probing narratives can further profit from awareness of the distinction between "transparent" and "opaque" representations in the sense of Lesh, Post, and Behr (1987). As a way of summarizing, Zazkis (2005) writes: "A transparent representation has no more and no less meaning that the represented idea(s) or structure(s). An opaque representation emphasizes some aspects of the ideas or structures and de-emphasizes others" (p. 209).

In conclusion, in our analysis of textbooks, we look for evidence of appropriate treatment of, and support for, AMT: clear and precise narratives, deductive and rigorous reasoning, intuitive development that does not create or enhance students' misconceptions, opportunities for "personal reconstruction" (Tall, 1991), adequate (transparent or opaque) representations, and the appropriate use of definitions.

## Infinity in a High School Textbook

We examined one textbook (Dunkley, Carli, & Scoins, 2002), which has been used in grade 12 classrooms in Ontario, Canada. As it accurately reflects the expectations of Ontario high school grade 12 mathematics curriculum (Ontario Ministry of Education, 2007), we believe that this textbook is a likely representative of other textbooks in use.

High school students hear about infinity in a variety of contexts, including: (1) there are infinitely many real numbers; (2) an irrational number is an infinite decimal; (3) limits involving infinity and asymptotes; (4) notation for an infinite interval. (In (1), (2) and (4), we purposely used phrases found in textbooks, to hint at, and to illustrate potential problems.)

Ontario curriculum document (Ontario Ministry of Education, 2007) does not require a clear conceptualization of infinity, for instance by suggesting that infinity be discussed in a *precise, well-defined context.* For instance, the phrase "infinitely *many*" in (1) might suggest a counting approach (*potential infinity*) for a concept that is an instance of *actual infinity*. In (2), it is not clear what "infinite" means—an irrational number has an infinite non-repeating decimal interpretation (i.e., the number is not infinite, but its decimal representation does not terminate). In *Glossary* in Dunkley, Carli, & Scoins (2002), we read that the basis of natural logarithms *e* is a "non-repeating, infinite decimal" (p. 457). Not aware of the subtleties involved, some students think of irrational numbers as infinite (and yet having a finite value). In (4), it is not clear what is infinite about the "infinite interval" (we discuss this further later in this section).

Dunkley, Carli, & Scoins (2002) define infinity as "something that is not finite, that is countable or measurable" (p.459), without explaining what the terms "countable" or "measurable" mean (these two terms do not appear in *Glossary*, nor elsewhere in the text). Not only is this notion confusing, but there is no indication how it relates to the infinity in the context of infinite limits which are discussed in the textbook.

Often, infinity is qualified by what it is not. For example: "We say that the function values approach $+\infty$ (positive infinity) or $-\infty$ (negative infinity). These are not numbers" (Dunkley, Carli, & Scoins, 2002, p. 353). Stating that it is not a number does not clearly articulate what infinity is; the authors continue: "They are symbols that represent the value [plural needed] of a function that increases or decreases without limits" (p.353). Furthermore, the quoted sentences define an infinite limit using the word limit (which makes it a circular definition); or, they are just confusing, as they mix up different meanings of the term "limit" (Jones, 2015). As well, the phrase "without limits" suggests the meaning of the limit as something that bounds, which is a common misconception that students have about limits (Tall, 1991).

Routinely, the same symbol $\infty$ is used both in interval notation, such as $(1,\infty)$, and in the context of limits. In this case, an adequate narrative needs to resolve this conflict between static and dynamic interpretations of $\infty$. For instance, treating $\infty$ in $(1,\infty)$ as transparent (Lesh, Post, & Behr, 1987; Zazkis, 2005), a textbook author could say that the symbol $\infty$ in $(1,\infty)$ is there for convenience, and could be replaced by some other symbol; all it means ("no more, no less") is that the interval $(1,\infty)$ represents the set of all real numbers larger than 1. We did find such an explanation: Stewart, Davison, and Ferroni (1989) write: "This does not mean that $\infty$ is a real number. The notation $(a,\infty)$ stands for the set of all numbers that are greater than *a*" (p.162).

The use of the term "infinite interval" for intervals such as $(1,\infty)$ is ambiguous, as it is not clear what is infinite about it—the number of points it contains, or its infinite size, or something else? A correct term "unbounded interval," as in advanced calculus/ analysis books, should be used.

## Infinity in University Textbooks

Calculus textbooks published in North America since 1980s have been influenced by the so-called calculus reform, or reform-based learning. Besides precise definitions and statements of theorems we find metaphors and explanations which are supposed to help students develop deeper (often intuitive) understanding of concepts. We found out that, in the sections about limits involving infinity, it is these "aids" to building an understanding that are often, due to their authors' disregard for research in math education, worrisome, ineffective and sometimes make no sense. A possible reason for inclusion of narratives (in the case of limits) is an attempt to strike a balance between rigorous (theoretical) development of the limit concept (which is, however, rarely covered in year 1 calculus classrooms) and the need to provide some opportunities for students to deepen their understanding. As well, these narratives could support theoretical understandings, and thus enrich the classroom coverage, often heavily biased toward technical (algorithmic) aspects of limits. Liang (2016) writes: "Calculus teachers usually focus on the calculation of limit, sometimes on graphical illustration of limit, rarely on theoretical aspect (or definition) of limit" (p. 37).

We do not argue against using narratives to enhance understanding, but suggest that they be constructed with care, and with awareness of situations which could lead to the development of students' misconceptions (or complete misunderstandings). In this section we outline a small selection of common narratives that we found in Calculus textbooks. With *novice learners* in mind, we aim to alert instructors to potential issues that these students might face. Of course, certain narratives that we identify as problematic for novices have become an integral part of a language that experts, as well as senior mathematics students, use routinely, and with appropriate and accurate understandings.

All narratives that we discussed in the previous section are found in university textbooks as well. Of the many university Calculus textbooks available to us, we looked deeper into six only, realizing that many have almost identical narratives and identical features. (This important problem of an almost complete absence of variety in presentations, content, and design of Calculus textbooks will not be discussed here.)

Common phrases found in describing an infinite limit of a function "$f(x)$ *becomes* [emphasis added] infinitely large," and "*becomes* [emphasis added] a negative number of large magnitude" (Edwards & Penney, 2008, p. 281) suggest that infinity is "reachable" (an object) as the end of the process of calculating a limit. Instead, a dynamic (process) representation (Cotrill et al., 1996; Jones, 2015) such as "the values of $f(x)$ *surpass* any real number," or similar, should be used.

The phrase "$f(x)$ grows larger and larger" (Jones, 2015) is even more problematic: it suggests that a function has a *size*; however, $f(x)$ is a real number, and has a value, but not a size. The use of "size" as in this example should be avoided; we use "size" when we refer to the set of real numbers, and say that the set of real numbers is infinite (*actual infinity*). As well, "growing larger and larger" does not suffice to guarantee that the limit of a function is $\infty$.

Another notable feature of textbooks is the absence of a rigorous treatment of infinite limits that would parallel the development of limit laws in the case when all limits involved are real numbers. This absence is truly perplexing, as students are expected to routinely argue about, and use, formulas such as $\infty+1=\infty$, $3\cdot\infty=\infty$, and $\infty+\infty=\infty$. For instance, they need all three of these to compute the limit $\lim\limits_{x\to\infty}\left(3x+\ln x+1\right)$. Stretching their intuition further, students might (and do!) erroneously conclude that the indeterminate forms $\infty-\infty$ and $\infty\cdot 0$ are both equal to zero (Lovric, 2012).

More of an exception, we find *Limit Laws for Infinite Limits* explicitly stated in Adler and Lovric (2015, p. 218).

Textbooks often use narratives about infinite limits in the section on L'Hopital's rule. However, almost all that we found are inadequate, or do not contribute to understanding. In Anton, Bivens & Davis (2009), we read:

> "[…] a limit involving $+\infty$ and $-\infty$ is called an indeterminate form of type $\infty-\infty$. Such limits are indeterminate because the two terms exert conflicting influences on the expression; one pushes it in the positive direction and one pushes it in the negative direction." (p. 225)

It is unclear what students are supposed to make of "conflicting influences," i.e., why conflicting influences make an expression indeterminate. For example, in the expression $\lim\limits_{x\to 10}\left(100x-x^3\right)$ the two terms $100x$ and $x^3$ exert conflicting influences (in the sense of the quote above); however, this limit is not an indeterminate form.

In the same book, the authors discuss the indeterminate form $1^\infty$ coming from the expression $\lim\limits_{x\to 0^+}\left(1+x\right)^{1/x}$. They state that $1^\infty$ is indeterminate because "expressions $1+x$ and $1/x$ exert two conflicting influences: the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$" (p. 225). Apart from other issues in this narrative, it is completely unclear why approaching 1 and approaching $+\infty$ are "conflicting influences." The two are certainly not conflicting, if we consider $\lim\limits_{x\to 0^+}\left(1+x\right)\cdot\frac{1}{x}$, i.e., in this case the limit is not an indeterminate form (we replaced exponentiation with multiplication).

Indeterminate forms have also been articulated as "competing forces." When describing $\infty/\infty$, Stewart (2016) uses a somewhat successful metaphor of a "struggle"

"There is a struggle between numerator and denominator. If the numerator wins, the limit will be $\infty$ […] if the denominator wins, the limit will be 0. Or, there might be some compromise, in which case the answer might be some finite positive number." (p. 305)

Later in the text, the author uses "contest" (p. 309). However, both metaphors break in the case of exponents, and the textbook offers no explanation as to why $1^{\infty}$ is indeterminate.

Hass, Weir, and Thomas (2016) call the indeterminate limit $0/0$ (equivalent to $\infty/\infty$) a "meaningless expression, which we cannot evaluate" (p. 242), without supplying any rationale as to what makes it "meaningless." Hass, Weir and Thomas (2007) use the term "ambiguous expression" (p. 285) when talking about other indeterminate forms; this phrase disappeared from Hass, Weir, and Thomas (2016).

Smith and Minton (2012) are a bit more explicit, when they state that "mathematically meaningless" means that "we'll need to dig deeper to find the value of the limit" (p. 223). However, there is no narrative explaining why these forms are indeterminate (is it just because we need to "dig deeper"?), or suggesting an approach that would help to understand them. Edwards and Penney (2008, p. 296) use inappropriate term "order of magnitude" in discussing the functions in the numerator and the denominator of the indeterminate form $\infty/\infty$ (one possible correct term is "leading behaviour").

Due to space limitations, we presented only a small sample of Calculus textbook narratives about infinity in the context of limits. However, we attempted to select narratives which are more common, and representative of issues and problems that could emerge when students try to read and understand them.

## Conclusion

Understanding of, and working with, the concept of the limit requires "an upgrade from intuitive concrete understanding to abstract recognition" (Merenluoto & Lehtinen, 2000). Such an upgrade requires an "immense personal reconstruction" (Tall 1991, p. 252), which includes "deductive and rigorous reasoning" (Edwards, Dubinski, & McDonald, 2005, p.17) and needs to be supported by adequate teaching and resources. Examining a sample of university Calculus textbooks for their treatment of infinite limits and limits at infinity, we have not identified much evidence of this, much needed, support.

Merenluoto & Lehtinen (2000) claim that before students learn about the (mathematical) concept of the limit, they already have experiences about limits. "Their understanding is mainly based on everyday experiences rather than mathematical understandings" (p. 37). As limits are "subtly at variance with naïve experience" (Tall, 1991, p. 252), it is important that textbooks address these experiences head-on to avoid creating or enforcing students' misconceptions about limits. Our examination shows that such narratives are missing from textbooks.

Although all textbooks we examined do cover theoretical aspects of the development of the concept of the limit, they do not dedicate sufficient attention to it. These "theoretical" sections *look* different compared to other sections in the textbooks: with dense presentations, terse language, abundance of symbols and a small number of examples, they seem to have been borrowed from advanced mathematical texts. They are organized in such a way that it is easy for an instructor to skip the material, or to assign it as optional reading. Perhaps we should not be too critical: these "theoretical" sections are an awkward compromise—textbook writers, under pressure from their editors, are forced to include "theory," although they know that many instructors will just skip it. This situation is in line with the general trend of moving theoretical material in Calculus textbooks from dominant to marginal locations (Bokhari & Yushau, 2006).

Nearing the end of this paper, in order to bring our analysis closer to teaching practice, we outline several suggestions, with textbook writers, as well as Calculus instructors, in mind.

Discussing clarity and transparence in teaching infinity, Lovric (2012) writes:

> "We need to make sure that the concepts are precisely *defined*. The necessity for, and a power of a mathematical definition now become obvious. Students will see how the precise and clear language of a definition eliminates multitudes of meanings, inappropriate metaphors and ambiguities in their understanding." (p. 141)

This demands that textbooks, as well as course instructors, bring certain theoretical considerations about limits back to their dominant position. For instance, a precise articulation (definition, together with appropriate illustrations) of the fact that $f(x) \to \infty$ should be accompanied by carefully crafted, transparent, narratives which alert the reader to possible misconceptions and misinterpretations (Monaghan, 1991; Jones, 2015).

All textbooks examined use the phrase "limit does not exist," but mostly do not clearly state that its precise (transparent, "nothing more, nothing less") meaning is "limit is not a real number." As illustration of a possible narrative that attempts to shed some light on this, but is not explicit enough, we quote Smith and Minton (2012): "It is important to note that while the limits […] *do not exist*, we say that they "equal" $\infty$ and $-\infty$, respectively, only to be specific as to *why* they do not exist" (p.97). Common misconception that "limit does not exist" means that the limit is $\infty$ or $-\infty$ leads students to conclude, confused, that "infinity does not exist." By the way—besides including infinite limits, "limit does not exist" refers to the case when left and right limits (which could be real numbers) are not equal.

Indeterminate forms should not be referred to as a "meaningless expressions" or "ambiguous expressions." For instance, they could be qualified in the following way: indeterminate forms are algebraic expressions which appear in the context of limits only; they include: division of zero by zero, the cases which are not covered by the limit laws for infinite limits, as well as certain exponential forms involving zero and $\infty$. (Next, the seven indeterminate forms are listed.) These expressions are called "indeterminate" because their values depend on the limits that generated them; in other words, just by looking at the limits of the form $\infty - \infty$, $\infty/\infty$, or $0 \cdot \infty$, we cannot tell what their values are. For instance, the following four expressions are all of the same indeterminate form $\infty/\infty$, yet the limits are 0, 1, 7, and $\infty$ respectively:

$$\lim_{x\to\infty} \frac{\ln x}{x}, \quad \lim_{x\to\infty} \frac{x+4}{x-1}, \quad \lim_{x\to\infty} \frac{7x+4}{x+3}, \quad \lim_{x\to\infty} \frac{x^2+1}{x}$$

An introduction to a discussion of infinite limits could start by stating that "'infinity' is really an extrapolation of our finite world, meaning that it is purely a mental construct that we do not encounter in our daily lives" (Tall, 1981, as quoted in Jones, 2015, p. 107). As this *mental construct* appears in various (sometimes incompatible) forms in different contexts, textbooks must be explicit about the context, and then keep their focus. A presentation about infinite limits and limits at infinity (*potential infinity*, *dynamic* nature of infinity) should avoid using terms and phrases such as "size," or "reaches infinity," or any narrative that would suggest objectification of infinity "as a sort of 'generalized large number'"(Tall, 1992, as quoted in Jones, 2015, p. 108).

In conclusion, these are initial, perhaps rough, findings of our analysis. As we probe deeper, we hope to come up with further insights into narratives related to limits involving infinity.

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Cognitive Consistency and Its Relationships to
Knowledge of Logical Equivalence and Mathematical Validity

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*The purpose of this study is to explore how cognitive consistency is related to knowledge of logic and mathematical proofs. We developed a logic instrument and administered it to forty-seven (47) undergraduate students who enrolled in various sections of a transition-to-proof course. The analysis of the students' scores on the logic instrument indicated that students' knowledge of logical equivalence and their knowledge of mathematical validity were somewhat related to one another. On the other hand, cognitive consistency was not closely related to either student knowledge of logic or knowledge of mathematical validity. Based on these findings, we address the importance of cognitive consistency in logical thinking and discuss implications for the teaching and learning of logic in mathematical contexts*.

*Keywords:* cognitive consistency, logical equivalence, mathematical validity, transition-to-proof

Our society expects people to have ability to make decisions in their workplaces more efficiently by deducing valid inferences from a tremendous amount of information and resources. In fact, a person's logical thinking plays a crucial role in generating valid arguments from the given information as well as in evaluating the validity of others' arguments. Hence, training our students as logical thinkers is a central component in education (NGAC & CCSSO, 2010; NRC, 2005). On the other hand, research in mathematics education reports that undergraduate students have weak knowledge on logic (e.g., Dubinsky, Elterman, & Gong, 1988; Epp, 2003; Inglis & Simpson, 2007). Such a deficiency of knowledge of logic would entail difficulties with using logic in deducing valid inferences to construct mathematical proofs (e.g., Moore, 1994), to comprehend mathematical proofs or interpret mathematical statements (e.g.,Mejia-Ramos et al., 2012; Selden & Selden, 1995), or to evaluate the validity and the soundness of someone's proofs (e.g., Selden & Selden, 2003).

With the importance of student knowledge of logic, we also consider *cognitive consistency* as an essential component in logical thinking. By cognitive consistency, we refer to "an intra-individual psychological pressure to self-organize one's beliefs and identities in a balanced fashion" (Cvencek, Meltzoff, & Kapur, 2014, p.73). Cognitive psychologists explain such a tendency as people behave in ways that maintain cognitive consistency or minimize cognitive dissonance among their interpersonal relations, intrapersonal cognitions, beliefs, feelings, or actions (Festinger, 1957; McGuire, 1966). For instance, a student might deduce two statements such as '$x$ is an integer' and '$x$ is not an integer' from given information as well as based on his own content knowledge of mathematics. Logically speaking, each of these statements contradicts one another, thus two statements cannot be accepted simultaneously. Since such a logical contradiction is a fatal flaw that makes the student's entire argument meaningless, it must be excluded from the argument. Once a student recognizes a logical contradiction in his argument, he would attempt to remove it from his argument. However, if the student does not recognize such a contradiction in his argument, he would be in cognitive inconsistency. One's recognition of cognitive inconsistency in his own reasoning or thinking will be the first step in self-regulating one's own cognition. However, if a student does not recognize cognitive inconsistency in his or her own knowledge structures, the student may not take any effort to change or modify his knowledge structure. Thus, it is very important to train students not only to gain more knowledge of logic but also to maintain cognitive consistency.

One might expect that the more knowledge of logic students has, they would unlikely deduce logical contradictions from given information or they would recognize logical contradictions if they happen to deduce them from given information. It might also be expected that students who do not recognize logical contradictions in their arguments would not be knowledgeable in logic. This study explores how students' cognitive consistency is related to their knowledge of logic and knowledge of mathematical validity, addressing the following research questions:

1. Do students with more knowledge of logical equivalence tend to have stronger cognitive consistency?
2. Do students with more knowledge of mathematical validity tend to have stronger cognitive consistency?

We developed the logic instrument to systematically measure three components of students' logical thinking: knowledge of logical equivalence between two statements, knowledge of mathematical validity of the arguments, and cognitive consistency. While we hope that this study provides new insights into the theories of cognitive consistency, our foci are distinct to previous ones from two aspects. First, in exploring the role of cognitive consistency, this study pays more attention to mathematical contexts such as logical equivalence of mathematical statements and mathematical validity, rather than focusing on personal or interpersonal attitudes and behaviors in social contexts (c.f., Cooper, 1998; Gawronski & Strack, 2004; Gawronski, Walther, & Blank, 2005; Stone & Cooper, 2001). Second, this study explores whether students recognize cognitive inconsistencies in their logical thinking rather than how students reconcile cognitive inconsistencies after recognizing them in their reasoning (c.f., Dawkins & Roh, 2016; Ely, 2010; Roh & Lee, 2011).

## Research Methodology

This study was conducted in the spring semester of 2014 at a large public university in the United States. Among 137 undergraduate students who enrolled in a transition-to-proof course various instructors, forty-seven (47) students voluntarily participated this study to complete the logic instrument. Due to the pre-requisite for the transition-to-proof course at the university, the participants had already completed at least the first semester calculus course. In addition, as the logic instrument was administered at the last week of the semester when the participants enrolled in the transition-to-proof course, the participants of this study had already been exposed to the terms used in the questions of the logic instrument, such as equivalent statements, logical connectives, quantifiers, negation, and valid arguments. Twenty-three participants (48%) were mathematics majors whereas twelve participants (26%) were mathematics education majors. The rest of the participants (twelve students, 26%) whose major areas of study were neither mathematics nor mathematics education were labeld as others.

### The Logic Instrument

The logic instrument we developed for this study consists of two parts with twelve questions in total. The first part (seven questions) was designed to test students' knowledge on logical equivalence between two statements. On the other hand, the second part of the logic instrument (five questions) was designed to test students' knowledge of mathematical validity.

**Part 1 of the logic instrument.** All questions in Part 1 present one or a pair of statements. We chose logical forms for these questions in Part 1 of the logic instrument among those that are frequently found in undergraduate mathematics textbooks from calculus and beyond. Several instances are also presented with the statement(s) in each question and students are asked to mark off all relevant ones among the given instances (See Table 1). All statements given in the questions in Part 1 of the logic instrument are *open* statements involving at least one free variable so that the truth-value of each statement cannot be determined. We purposely created and included such open statements to the questions in Part 1 in order to avoid the cases of students who answer to the questions based on their determination of the truth-value of a statement.

*Table 1 Summary of seven Questions in Part 1 of the logic instrument*

| | **Logical form of the given statements** | **Nature of the Questions** |
|---|---|---|
| Q1 & Q3 | $P(x) \to Q(x)$ | Mark off all logically equivalent instances to the given statement |
| Q2 & Q4 | A pair of statements in the forms of $\forall x \exists y P(x, y, z)$ & $\exists y \forall x P(x, y, z)$ | Mark off the best description about the logical relationship between the given statements |
| Q5, Q6, & Q7 | $\forall x, P(x, y) \to Q(x, y)$ | Mark off all logically equivalent instances to the negation of the given statements |

**Part 2 of the logic instrument.** All five questions in Part 2 are set up similarly in the sense that each question asks to (1) determine the truth-value of the given statement; (2) determine if the person whose argument is given in the question attempts to either prove or disprove the statement; and (3) evaluate if the person's argument is valid. See Figure 1 for Q9 as an example of questions in Part 2 of the logic instrument.

Q9. An integer *a* is said to be odd if and only if there exists $n \in \mathbb{Z}$ such that $a = 2n + 1$. Tim was asked to prove or disprove:

(♣) For any positive integers *x* and *y*, if *x* and *y* are odd, then $xy$ is odd.

The following is Tim's argument.

$$x = 2n + 1, n \in \mathbb{Z}$$
$$y = 2n + 1, n \in \mathbb{Z}$$

Therefore, $xy = (2n + 1)(2n + 1) = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$ is odd.

(1) Check the most appropriate one about the statement (♣).

a. _______ The statement (♣) is true.

b. _______ The statement (♣) is false.

c. _______ We cannot determine if the statement (♣) is true or false.

(2) Check the most appropriate one to describe what Tim attempted to prove.

a. _______ Tim attempted to prove the statement (♣) is true.

b. _______ Tim attempted to prove statement (♣) is false.

c. _______ We cannot determine if Tim attempted to prove the statement (♣) is true or he attempted to prove the statement (♣) is false.

(3) Check the most appropriate one to describe if Tim's argument is valid.

a. _______ Tim's argument is valid as a proof of the statement (♣).

b. _______ Tim's argument is invalid as a proof of the statement (♣).

c. _______ We cannot determine if Tim's argument is valid or invalid.

*Figure 1 Q9 in the logic instrument*

## Data Analysis

The logic instrument described in the previous section was used in this study to measure students' logical thinking in terms of *knowledge of logical equivalence* (KoLE), *knowledge of* *mathematical validity* (KoMV), and *cognitive consistency* (CC). We first generated the coding scheme to score students' mark-offs to the questions in the logic instrument. Different weights were applied to different questions as each question was used to examine different aspects of students' logical thinking. After coding student responses in terms of the scoring rubric, we also generated *the overall logical thinking* (OLT) scores as the sum of the three scores: KoLE, KoMV, and CC scores.

**Scoring rubric for knowledge of logical equivalence (KoLE).** Student knowledge of logical equivalence was measured from student responses to the questions in Part 1 of the logic instrument (see Table 2). Questions 1, 3, 5, 6 and 7 in Part 1 present a statement and a set of six to seven instances. For each of these questions, sub-question scores were first generated based on students' mark-off to the instances as follows: Students' mark-off to each instance was scored either 0 (for the correct response) or -1 (for the incorrect response). The final score for each of these questions was then formulated as the maximum value between 0 and 2+∑(sub-question score). Using this scoring rubric, the scores for Q1, Q3, Q5, Q6, and Q7 were ranged from 0 to 2. On the other hand, Questions 2 and 4 present a pair of statements (i) and (ii) and a set of four instances (a) ~ (d) describing relationships between the pair of statements. For each of these questions, students' check of one of the four relationships was scored either 2 (for the correct response) or 0 (for the incorrect response). KoLE score was then given as the sum of the scores on these seven questions in Part 1, which could be possible ranged from 0 to 14.

*Table 2 Scoring rubric for Part 1 of the logic instrument (Q1 ~ Q7)*

<table>
<tr><th>Question</th><th colspan="3">Scoring Rubric</th><th>Score Range</th></tr>
<tr><td rowspan="2"></td><td colspan="2">Sub-question score</td><td rowspan="2">Scoring Formula</td><td rowspan="2">Final score</td></tr>
<tr><td>Correct / Incorrect</td><td>score</td></tr>
<tr><td rowspan="2">Q1, Q3, Q5~Q7</td><td>Correct</td><td>0</td><td rowspan="2">S = max{2+∑(sub-question score), 0}</td><td rowspan="2">S</td></tr>
<tr><td>Incorrect</td><td>-1</td></tr>
<tr><td></td><td>Correct / Incorrect</td><td colspan="2">score</td><td>Final score</td></tr>
<tr><td rowspan="2">Q2, Q4</td><td>Correct</td><td colspan="2">2</td><td>2</td></tr>
<tr><td>Incorrect</td><td colspan="2">0</td><td>0</td></tr>
</table>

**Scoring rubric for knowledge of mathematical validity (KoMV).** Student knowledge of mathematical validity was measured from student responses to the second and third sub-questions to the questions in Part 2 of the logic instrument (see Table 3). First, we evaluated students' student responses to the second sub-question (asking to determine if the given argument is an attempt to prove or an attempt to disprove the statement); and then evaluated student responses to the third sub-question (asking to evaluate the validity of the given argument). To be more specific, for Q8, 1 was given for the correct response to the validity of each argument in the second and third sub-questions, respectively; otherwise 0 was given. For Q9 ~ Q12, 2 was given to the correct mark-off to the second sub-question; otherwise, 0 was given. Next, among those who marked-off correctly to the second sub-question (proof or disproof), if the student also responded correctly to the third sub-question (valid or invalid), we scored 0 for the response to the third sub-question; otherwise, $-\mathbf{1}$ was given. On the other hand, if the student response to the second sub-question (proof/disproof) was incorrect, we scored 0 to any response to the third sub-question regardless of its correctness. We then added the scores on its second and third sub-questions according to the scoring rubric described above. For instance, Q9 (Figure 1) presents an argument (2) attempting to prove the statement is true where (3) the argument is invalid. If a student were to mark off that (2) the given argument in Q9 is an attempt to prove that the statement is false (incorrect), and (3) the given argument is invalid (correct), then 0 was given to this response as the response to the second sub-question is incorrect. On the other hand, if a student were to mark off that (2) the given argument in Q9 is an attempt to prove that the statement is true (correct) and (3) Tim's argument is valid (incorrect), then 1 is given to the student response to Q9 as the correct response to the second sub-question is scored to 2 and an incorrect response to the third sub-question is scored to $-1$ while the correct response to the first sub-question is neglected due to the incorrect response to the third sub-question. The KoMV score was then given as the sum of the scores on these five questions in Part 2, which could be possibly ranged from 0 to 10.

*Table 3 Scoring Rubric for Part 2 of the Logic Instrument (Q8 ~ Q12)*

<table>
<tr><th>QUESTION</th><th colspan="4">SCORING RUBRIC</th><th>SCORE RANGE</th></tr>
<tr><td rowspan="2"></td><td colspan="2">(2) Validity (Argument)</td><td colspan="2">(3) Validity (Argument)</td><td rowspan="2">Final score</td></tr>
<tr><td>Correct/Incorrect</td><td>score</td><td>Correct/Incorrect</td><td>score</td></tr>
<tr><td rowspan="4">Q8</td><td rowspan="2">Correct</td><td rowspan="2">1</td><td>Correct</td><td>1</td><td>2</td></tr>
<tr><td>Incorrect</td><td>0</td><td>1</td></tr>
<tr><td rowspan="2">Incorrect</td><td rowspan="2">0</td><td>Correct</td><td>1</td><td>1</td></tr>
<tr><td>Incorrect</td><td>0</td><td>0</td></tr>
<tr><td rowspan="2"></td><td colspan="2">(2) Prove/Disprove (Argument)</td><td colspan="2">(3) Validity (Argument)</td><td rowspan="2">Final score</td></tr>
<tr><td>Correct/Incorrect</td><td>score</td><td>Correct/Incorrect</td><td>score</td></tr>
<tr><td rowspan="3">Q9~Q12</td><td rowspan="2">Correct</td><td rowspan="2">2</td><td>Correct</td><td>0</td><td>2</td></tr>
<tr><td>Incorrect</td><td>$-1$</td><td>1</td></tr>
<tr><td>Incorrect</td><td>0</td><td>-</td><td>-</td><td>0</td></tr>
</table>

**Scoring rubric for cognitive consistency (CC).** For cognitive consistency scores, we first identified cognitive inconsistencies when student responses to sub-questions of a question imply any logical contradiction. For instance, suppose a student marks off to Q9 (Figure 1) as follows: (2) Tim's argument is an attempt to prove the statement (♣) is false, and (3) Tim's argument is valid. This student's responses contain a logical contradiction since an attempt to prove that a true statement is false cannot be valid. Similarly, if another student responds to Q9 that (2) Tim's argument is an attempt to prove the statement (♣) is true, and (3) Tim's argument is valid, then the student also appears to have cognitive inconsistency. Table 4 describes all instances of cognitive inconsistencies to be evidently found from student responses to the questions.

*Table 4 All instances of cognitive inconsistency*

<table>
<tr><th colspan="2">Question</th><th colspan="3">Sub-Questions</th></tr>
<tr><td rowspan="3">Q8</td><td rowspan="3">Cognitive Inconsistency</td><td>(1) True/False (Statement)</td><td>(2) Validity (Argument)</td><td>(3) Validity (Argument)</td></tr>
<tr><td>(a) True or<br>(c) Cannot determine</td><td>(a) Valid as a proof for false</td><td>-</td></tr>
<tr><td>(b) False or<br>(c) Cannot determine</td><td>-</td><td>(a) Valid as a proof for true</td></tr>
<tr><td rowspan="3">Q9~Q12</td><td rowspan="3">Cognitive Inconsistency</td><td>(1) True/False (Statement)</td><td>(2) Prove/Disprove (Argument)</td><td>(3) Validity (Argument)</td></tr>
<tr><td>(a) True or<br>(c) Cannot determine</td><td>(b) Prove False</td><td>(a) Valid</td></tr>
<tr><td>(b) False or<br>(c) Cannot determine</td><td>(c) Prove True</td><td>(a) Valid</td></tr>
</table>

Cognitive consistency *was* measured from student responses to all three sub-questions of the questions in Part 2 of the logic instrument. We measured students' cognitive consistency by assigning either $-1$ or 0 to each of the questions (Q8~Q12) as follows: $-1$ was assigned whenever there is evidence of cognitive inconsistency, i.e., a logical contradiction from student responses to its sub-questions. On the other hand, we scored 0 in all other cases but the instances in Table 4 since there is no evidence of logical contradictions from the cases. As there were five questions in Part 2, the total score on cognitive consistency could be possibly ranged from $-5$ to 0. Obviously, if a student marks off correctly to all sub-questions to a question in Part 2, the student does not appear to have a cognitive inconsistency in his response to the question. On the other hand, although the student responses to some sub-questions are not correct, the student's cognitive consistency score to the question could still be 0 in the case when there is no evidence of logical contradiction within the student's responses.

## Results

The overall logical thinking (OLT) scores were distributed between $-2$ and 24 with the interquartile range between 6 and 15. In addition, the mean of the OLT scores was about 10 and the highest OLT score was 24 which was the possible maximum for the OLT score with only one student receiving the highest score. On the other hand, there was one student who received $-2$ on the OLT scores due to negative values on the cognitive consistency score, which will be discussed later more in detail when analyzing the cognitive consistency scores. Furthermore, KoLE scores were ranged from 0 to 14 while the median was 5 (out of 14 points) and 50% of student KoLE scores were between 2 and 9. KoMV scores were ranged from 0 to 10 with the median 5 (out of 10 points) while 50 % of KoMV scores were distributed between 3 and 8. Finally, the CC scores were ranged from $-2$ to 0 and about 21% of the CC scores were negative.

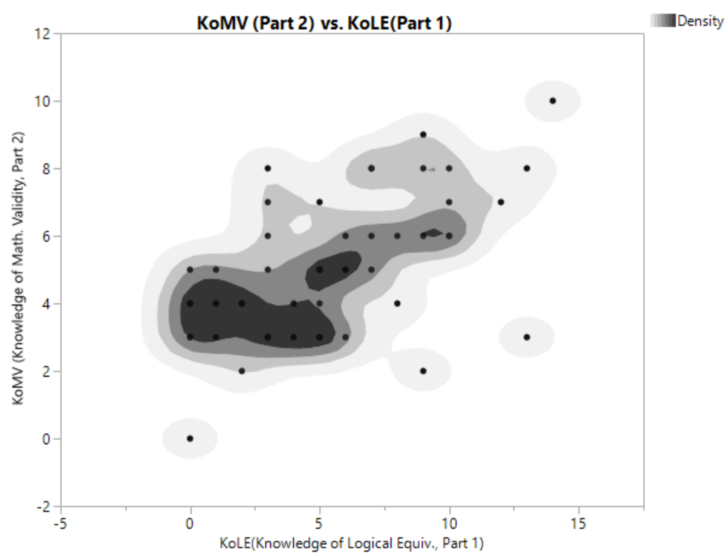


*Figure 2 Scatter-Density plot: KoMV vs. KoLE*

The scatter-density plot in Figure 2 further shows that students' knowledge of logical equivalence (KoLE) and students' knowledge of mathematical validity (KoMV) are somewhat related to one another. On the other hand, cognitive consistency (CC) was *not* closely related to either KoLE or KoMV. According to the scatter-density plots in Figure 3 and Figure 4, students whose cognitive consistency score was $-2$ did not have higher scores on KoLE and KoMV than the median of each score. However, in the case that the cognitive consistency score was $-1$, students’ KoLE scores or KoMV scores were distributed with relatively wide range containing higher scores than the median. There was one student who received a very high score on KoLE (13 out of 14) but scored $-1$ on the cognitive consistency. These findings indicate that students might have cognitive inconsistencies even though they attained high scores on knowledge of logical equivalence and knowledge of mathematical validity, respectively.

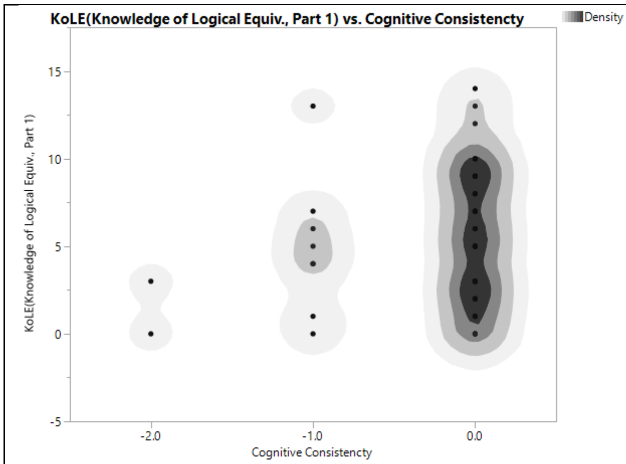


*Figure 3 Scatter-Density plots: KoLE vs. CC*

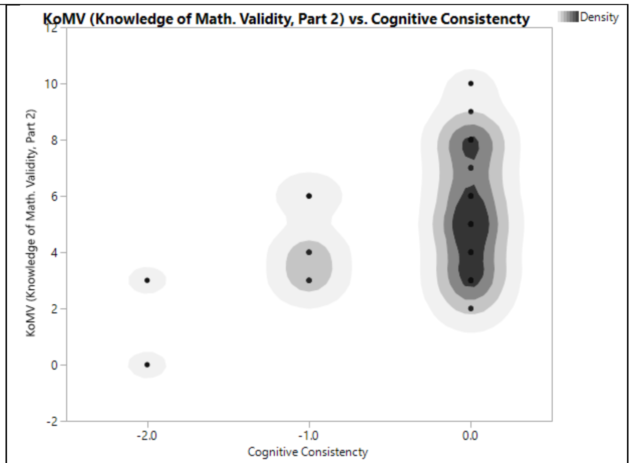


*Figure 4 Scatter-Density plots: KoMV vs. CC*

## Conclusion & Discussion

In this study, we explored undergraduate students’ cognitive consistency and its relation to their knowledge of logical equivalence and mathematical validity. The findings of this study indicate that students’ cognitive consistency was not closely related to either their knowledge of logical equivalence or their knowledge of mathematical validity. Indeed, some students who received high scores on knowledge of logical equivalence or on knowledge of mathematical validity still had cognitive inconsistencies. Furthermore, these students already took a course for logic and mathematical proofs for about at least fifteen weeks. Thus, it might be an unreasonable expectation that students with more knowledge on logical equivalence and mathematical validity would not have cognitive inconsistencies.

The findings of this study also suggest some significant implications for the teaching and learning of logic and mathematical proofs. Although undergraduate students received formal instruction for logic from a logic and mathematical proof course, they may not recognize a logical contradiction in his or her argument. Thus, we contend that cognitive consistency must be treated as a crucial component of logical thinking. Designing special tasks or instructional interventions would be needed to reveal students’ cognitive inconsistencies and to help students recognize logical contradiction in their arguments if they have any. The structure of sub-questions in Part 2 of the logic instrument in this study could be an example of reference to reveal students' cognitive inconsistency what might have been.

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Stepping Through the Proof Door:
Undergraduates' Experience One Year After an Introduction to Proof Course

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*Navigating the transition from computing to proof writing remains a key challenge for mathematics departments and undergraduate students. Numerous departments have developed courses to introduce students to the nature of proof and effective argument (David & Zazkis, 2017), but research assessing the impact of these courses has just begun. This paper reports the experience of four introduction to proof "graduates" after they completed a semester of real analysis. Each had participated in our prior study of students' experience in the introduction to proof course. Results indicate that students' success in real analysis was supported by their work in the introduction to proof course. Two students exploited the structure common to many proofs in real analysis; the other two relied on extensive practice with example problems. For both pairs, we see linkages between students' work in real analysis and their prior procedurally-oriented work in mathematics.*

*Keywords*: transition to proof, proof reasoning, students' experience, qualitative analysis

This paper extends our prior research that examined undergraduate students' experience in one introduction to proof course taught at a research-intensive university (Smith, Levin, Bae, Satyam, & Voogt, 2017). Most of the N = 14 participants in that study clearly indicated that they found the work to write proofs different from their prior work to compute numerical or symbolic "answers". Where the majority found proof writing challenging, most were relatively successful in the course, as judged by final grades and self-reports. But the success of courses designed to introduce students to proof and proof-writing cannot be judged "locally". As the warrant for such courses is to increase learning and achievement in upper-division mathematics, the "success" of these courses depends on how well students perform in subsequent proof-focused courses.

Here we report on the experience of four "graduates" of an introduction to proof course in their first semester of real analysis. All were successful in that course, as judged by both grades and self-reports. But their descriptions of their work in real analysis, offered in comparison to the introduction to proof course and prior work in mathematics through calculus, reveal a more complex pattern of similarities and differences in how students see and carry out mathematical work. For some students in real analysis, the differences between following procedures to compute answers and writing effective proofs may be less stark than we initially conjectured (and than they experienced in their introduction to proof course). If so, characterizing the transition to proof may need to embrace important continuities as well as discontinuities with prior mathematical work.

## The Transition to Proof and Proving

Understanding the challenges that undergraduate students face in learning to prove mathematical statements and designing courses and experiences that support their efforts to address those challenges have become major foci of research in undergraduate mathematics education. Recent work has focused on the nature and diversity of courses intended to introduce students to proof and proving (David & Zazkis, 2017; Selden, 2012), specific cognitive challenges in understanding and writing proofs (Sellers, Roh, David, & D'Amours, 2017), and following students' proving work and reasoning after an initial introduction to proof (Benkhalti, Selden, & Selden, 2017).

As one contribution to this growing literature, we interviewed N = 14 undergraduate students after they completed a one-semester introduction to proof course. Our analysis focused on four issues—how they saw the course as different from prior courses, the activities they undertook to learn the course content, how they characterized their thinking during work on proof (proof reasoning), and their sense of success in the course (Smith et al., 2017). None reported any significant prior work on proof. Most were clear that the course made new and different demands than prior courses, and in response, many initiated different patterns of work. Despite their reported struggle, most completed the task being and feeling relatively successful, leading us to conclude that the course had been successful in bringing students to and through the doorway to proof. In particular, the course had placed students into the work of solving mathematical problems and supported their adjustment to that work.

But the merit and impact of introduction to proof courses lies as much, if not more in how students perform after they complete such courses as it does in how well they perform in the courses. These courses typically survey the major domains of algebra, analysis, and number theory without exploring the content area in any depth (David & Zazkis, 2017). The main task of repeatedly addressing proof tasks in one content area for an extended period and thereby coming to understand more about that mathematics via proof lies ahead of them. If the gap between carrying out known procedures to compute single answers and proving statements is wide and deep (Selden & Selden, 2013), the transition to proof and proving will not be accomplished in a single semester. So it makes sense to ask about successful "graduates" of introduction to proof courses: Where does reasonable success in that course lead? How do they experience their first proof-based course situated in a particular content area? How do they compare their experience in that course to their "preparation" in the introduction to proof course and to their prior mathematical experience? Is it possible, at a reasonable level of precision, to chart students' experience and work from computing to proving?

## Conceptual Framework

Our analysis was informed by the main concepts that had oriented our prior work (Smith et al., 2017). Oriented by work to understand pre-college and college students' experience of work in "reform" and "traditional" courses (e.g., Smith & Star, 2007), we see the shift from computing single answers to proving statements to set the stage for major transitions in students' experience of mathematics, where their understanding of the nature of their work, how they feel about their experience and their abilities, and what they do to carry out that work change in quite substantial ways. Where mathematical transitions are not determined by the learning environment, some "external" structures make them more likely (e.g., fundamental changes in curriculum and pedagogy and new courses focused on proof). Our prior study conceptualized students' experiences in terms of (a) the differences they saw between the work in their introduction to proof course and their prior mathematics, (b) their sense of the task of writing proofs, (c) their learning activity, in and outside of class, and (d) their subjective "sense of success" in the course. As specified in these four foci, our task was to understand the participants' perspectives and judgments in their own terms.

In the present study, we focused on students' experience in real analysis in relation to their work in the introduction to proof class a year earlier. The above four foci again informed the development of our interview questions and the direction of our analysis. For the first focus (differences with prior courses), we were particularly interested in how students compared the introduction to proof course to real analysis. Our overarching theoretical stance remains constructivist: Students bring forward mathematical "resources" (knowledge, skills, learning practices) developed in prior courses and attempt to use them to address the tasks of their present courses. New challenges, at any scale, mean that some resources will work well, some must be adapted, and some are developed, more or less *de novo*, in the new setting. The view of the student as an agent in her own learning is also central to our perspective, especially with respect to learning activities outside of class.

## The Program, Courses, and Participants

In the university where our research is situated, students—both mathematics majors and minors—complete a calculus sequence, the introduction to proof course, a linear algebra course, and a number of proof-focused content courses. After linear algebra, the first semester of real analysis and the first semester of abstract algebra are the two most common sites where students experience proof-intensive work in a specific content area. Both courses are required for majors and minors, and both first require completion of the introduction to proof course. Majors are required to complete additional proof-based courses, including the second semesters of both real analysis and abstract algebra, as well as other courses. In the semester of our study, two sections of real analysis were taught by different instructors, but both used the same textbook (Ross, *Elementary analysis: The theory of calculus*, 2013). They differed somewhat in in-class activities, homework, and assessments. In this department, real analysis is widely seen by students, instructors, and support staff as among the most, if not the most challenging undergraduate course.

In Spring 2017, six of the 14 students from our previous study responded to our invitation to participate in a follow-up study. All the six were mathematics majors or minors and had taken real analysis and/or abstract algebra in 2016-17. The other eight initial participants either did not respond or indicated they had not taken either course, had changed majors, or left the university. Two respondents took both real analysis and abstract algebra; the other four took only one. For those who had taken both courses, our interview focused on the most recent course to reduce concerns about constructed memory. With four participants, the interview focused on real analysis; with the other two, it focused on abstract algebra. In this presentation, we will focus on the former group, who are described in Table 1 below.

*Table 1. Overview of participants*

| Student | Gender | Standing | Home | Major | Career Obj. | Other proof-based courses |
|---|---|---|---|---|---|---|
| S1 | Female | 3 | US | Mathematics | Teaching | Higher geometry (F16) |
| S2 | Female | 3 | US | Mathematics | Actuary | None |
| S3 | Female | 3 | US | Mathematics | Uncertain | None |
| S4 | Male | 4 | Int. | Mathematics | Grad school | Abstract algebra I (F16)<br>Abstract algebra II (Sp17) |

All four participants took the course in the same semester (Spring 2017), and we interviewed them just after they completed it. S1, S2, and S3 had the same instructor; S4 was taught by the other instructor. Though we did not directly observe either instructor's teaching as we had in the previous study, S1, S2, and S3 provided consistent descriptions of the course, their instructor's teaching, the assigned homework, the use of the text, and the course assessments.

The interviews were semi-structured around focal questions, about an hour in duration, and conducted either face-to-face or via video conference. In two cases (S1 and S3), follow-up interviews were used to clarify their responses from the first interview. Our central goal was to understand students' experience in real analysis relative to their experience in the introduction to proof course—with particular attention to the task of writing effective proofs. After checking basic information (e.g., major/minor, standing, career plan, other math courses), we asked about their sense of how well the introduction to proof course prepared them for the real analysis course (and any other proof-based courses they had taken). Making no assumptions about how participants saw the mathematics courses they took that year (e.g., linear algebra), we asked how they viewed each relative to its focus on proof (very little, somewhat, strongly). All four participants indicated that real analysis was strongly proof-based. For the course(s) characterized as somewhat or strongly proof-based, we asked participants to compare the difficulty of that course(s) to the introduction to proof course. Then we explored their experience in each course, but with greater attention to real analysis, including assignments and instruction, learning activities in and outside of class, and their view of proof tasks and work to produce acceptable proofs. The interviews also provided opportunities to return to participants' experience in the introduction to proof course, affording us the chance to check for consistency in their characterizations. Toward the end, we asked them to draw a Confidence Graph to represent the dynamics of their confidence across the semester (Smith et al., 2017). As before, these helped us understand the challenges participants faced at different points in the semester and how they addressed the challenges.

Figure 1 below represents the comparisons between the different mathematical experience that were supported in the two studies, the present (Phase 2) and the previous (Phase 1). The interviews from the present study supported comparisons between real analysis and the introduction to proof course, but also with participants' experience prior to any focus on proof.

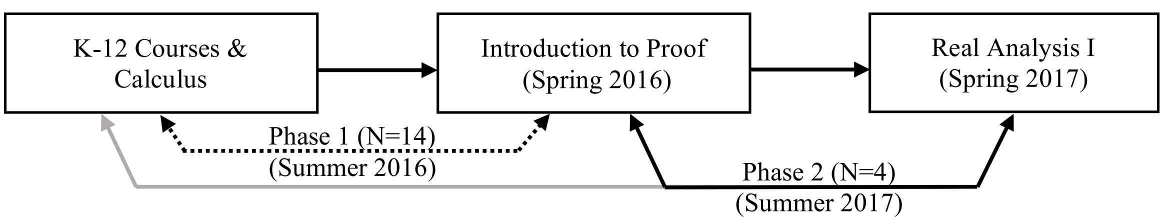


*Figure 1. The previous study and the present study across the sequence of the courses*

## Results

All four students reported success in real analysis, with both final grades (all received 4.0) and sense of mastery of the content. As indicated in Table 2 below, all four appreciated and valued the preparation for real analysis they received in the introduction to proof course, citing (a) "getting their feet wet" with proof, (b) learning specific proof methods (e.g., mathematical induction), and (c) gaining some introduction to real analysis content. However, S1 and S3 made a stronger case for their preparation in the introduction to proof course, where S2 and S4 indicated they did not learn some things that would have been useful in real analysis. S2 stated that she was not required and taught how to build up the structure of a proof; S4 mentioned that some methods (e.g., epsilon-delta proofs) were not taught in detail enough in the introduction to proof course. All four participants noted that the introduction to proof course moved frequently between content areas (making the course more difficult in the process), where real analysis focused on one set of related ideas. S1, S2, and S3 each indicated that they appreciated learning in real analysis why theorems and rules they learned in calculus were justified.

*Table 2. Summary of students' sense of preparation of the introduction to proof course for Real Analysis I*

| Student | Preparation | Relative Difficulty |
|---|---|---|
| S1 | Very well | Introduction to proof > Real analysis |
| S2 | Good | Real analysis > Introduction to proof |
| S3 | Very well | Introduction to proof > Real analysis |
| S4 | Good | Real analysis > Introduction to proof |

S1 and S3 found the introduction to proof course more difficult than real analysis, despite the fact that prior reports led both to expect that the latter would be very challenging. In contrast, S2 and S4 indicated that real analysis was more difficult than the introduction to proof but cited different reasons for their judgments. S4 indicated that the absence of sufficient example problems in his real analysis contributed significantly to its difficulty, where S2 found the concepts as well as proof construction more challenging in real analysis. Beyond these top-level judgments about "preparation," we found the two pairs of the participants (S1 & S3 and S2 & S4) provided two quite different narratives about the challenges of the course and how they had worked to address the challenges.

**S1 and S3: Work Together, and Exploit Similarity Across Tasks**

In explaining their success in real analysis, S1 and S3 both emphasized the quality of their instructor's teaching, citing four main similarities to instruction in their introduction to proof course: (a) group work in class, (b) regularly assigned and graded homework, (c) weekly quizzes, and (d) instructor encouragement. But this shared experience with instruction was coupled with changes in their learning activity. Where both S1 and S3 attended the Math Learning Center (MLC) at the university for the first time during their introduction to proof course and benefited from the activities and relationships supported there, neither attended in the MLC during real analysis. Instead, they worked remotely outside of class with the other members of their classroom small group that they maintained for the entire semester. When they got stuck on homework problems, they messaged with each other, sent pictures of the status of their solution attempts, and asked each other for suggestions. Both also reported they could reasonably predict the general nature of exam questions. They completed their homework each week, whose content predicted the weekly quizzes, which in turn predicted the content of exams. Their instructor also gave a practice final exam, described to resemble the actual final. S1 came to see a common structure among real analysis proofs (i.e., a standard way to develop and write epsilon-delta and epsilon-N arguments), whereas she could find no commonality among the proofs in her proof-based geometry course. When asked, S1 agreed that this common structure bore some similarity to her prior mathematics work, before the introduction to proof, of identifying known procedures and executing them on familiar tasks. S3 did not explicitly indicate an awareness of structural similarities among real analysis proofs but did strongly endorse the importance of collaboration with her peers, as complement to her own problem solving work on homework.

**S2 and S4: Work Independently on Lots of Examples**

In contrast, S2 and S4 emphasized the importance of repeated practice on numerous example problems for each course topic, as practice increased the likelihood of mastery and success on course assessments. In addition, both carried out this practice-focused work on their own. S4 expressed frustration that his instructor (different from S1-S3's) did not provide a sufficient number of examples comparable to his experience in his introduction to proof course. So he actively searched the internet for them, explaining that he looked for problems that were related to those worked in class and had complete solutions (proofs). S4 would then work the problem and compare his proof to the one provided. If he was unsure how to start, he reviewed the provided proof and then attempted to complete it on his own—comparing his proof to the one provided when he finished. S4 never went to the MLC during real analysis (in part because he did not think that Center personnel were prepared to help with that content), though he had done so regularly during his introduction to proof course. Also, he stated that he did not need to get help from MLC or office hours to complete the homework problems in real analysis, which are pretty much similar to what his instructor showed in class, whereas he was not able to even start some of the homework problems in the introduction to proof so he had to go to the MLC. S2 did not complain about the supply of example problems; she found the combination of problems worked in class, homework problems, and problems in the text not assigned for homework sufficient. Though she was part of the in-class group that S1 and S3 cited as important, S2 seldom contacted her group and solved most course problems on her own. She described her method of study for exams to involve "just doing lots of problems." Like her peers, S2 did not attend the MLC during real analysis, though she had done so repeatedly and productively during her introduction to proof course. She was also able to complete almost all the homework problems just from what her instructor showed her in class, whereas she reported that there were significant gaps between worked problems in class and homework, and between homework and exam problems in the introduction to proof.

**Common Structure Among Real Analysis Proofs**

One common thread in these results is the importance of noticing and abstracting a structure common to many real analysis proofs (what Selden & Selden [2013] have called a "proof framework"). Where S1 and S2 took different approaches to their work in real analysis, principally in how they engaged their peers, both spoke to the common structure they saw among the proofs their instructor produced and they produced in real analysis. S4 spoke to this issue in different terms, and S3 did so only obliquely and without emphasis. S1 saw the common structure among epsilon methods with some variation (e.g., epsilon-delta, epsilon-N) depending on the concepts involved in the statements (e.g., functions, sequence, and series). She was taught to always start with specific sentences in the structured way of proving the statements using the epsilon methods. She liked her instructor's practice of assigning similar problems using same approach/structure in homework and claimed her instructor's proof writing in class emphasized this pattern. Her perception of common structure contributed substantially her confidence going into major course assessments. S2 stated that the real analysis proofs were a lot more structured than the those in her introduction to proof course. She asserted this pattern ("the proof was basically the same for every type of like, every type of problem") and indicated that real analysis proofs had an "introduction" that stated an arbitrary epsilon, the body of the proof, and a "conclusion" that related the particular case to the definition. In contrast, S4 described the process of completing a real analysis proof after setting up its structure as "computation." He used that term to indicate the repeated process of determining appropriate values for delta or N in epsilon arguments. In his view, real analysis was 50% proof writing and 50% computation, where his abstract algebra experience was 80% of proof writing. Though he used different terms than S1 and S2, we interpret his assertion as similar to theirs: All three are citing structural regularities across many different real analysis proofs. This abstraction of common structure across many different proofs is significant for many reasons, not the least of which is that it narrows considerably the "problem solving space" students found themselves in during course assessments. None of the participants spoke to specific challenges in "filling in the blanks" of the common structure proof—Selden and Selden's (2013) "problem-centered part" of proof writing.

## Discussion

This study produced three main results; all concern "outcomes" from one introduction to proof course. First, the introduction to proof course prepared all four participants relatively well for proof-based work in real analysis, one major content area of advanced mathematics. If the goal of such courses is to increase students' achievement in upper-level coursework, this course succeeded, at least for some students. Note that the introduction to proof course covered the basics of proof and proving and situated students' work in three different content areas. As such it fell into David and Zazkis's (2017) category of "Standard + Sampler" introduction to proof courses. Only five of the 176 courses they reviewed across the R1/R2 institutions in the U.S. were of this type. Second, even in our small sample, we have examples of students pursuing and achieving success in real analysis in different ways, even after "the same" introduction to proof. In particular, these four students took up group-work from their introduction to proof course in quite different ways—from substantially to not at all. Third, returning to our opening metaphor, mathematical work on the other side of the "proof door" can be similar in important ways to the computationally focus of their prior work. Three of the four participants reported regularities across real analysis proofs that resemble in some ways the mathematical work that preceded the focus on proof—recognize problems and apply the appropriate procedure to produce answers without significant feature of problem-solving. Though their introduction to proof course regularly asked these students to solve real problems, the tasks in real analysis significantly reduced the problematic nature of their mathematical work, as noted by S1, S2, and S4.

One major limitation of this study is our small and "correlated" sample. Three of the four participants experienced real analysis with the same instructor and engaged each other in the same small group—though they indicated no knowledge of their joint participation in the study. Our two different approaches to mastery (engage one's peers vs. repeated individual practice) are likely not the only narratives of mastering real analysis. Variation among students (e.g., in prior mastery experiences) and among instructors both likely contribute to the diversity of students' experience in real analysis. A second limitation leads to our next steps in this research: Most of the "graduates" of the introduction to proof course in this study have thus far had only modest experience in proof-based courses. Their journey will continue into new content areas and under the direction of different instructors. In the next phases of the research, we intend to track their experience in these new contexts (e.g., abstract algebra, real analysis II) and extend the reflective comparison of present and past experiences that we initiated in this study. We also hope to increase our sample size as more participants in our previous study enroll in proof-based content courses.

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# The use(s) of ‘is’ in mathematics

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*This paper analyzes some of the ambiguities that arise among statements with the copular verb* is *in the mathematical language of textbooks as compared to day-to-day English language. We identify patterns in the construction and meaning of* is *statements using randomly selected sample statements from corpora representing the two linguistic registers. In particular, for the grammatical form "[subject] is [noun]," we compare the relative frequencies of the subcategories of semantic relations conveyed by that construction. Specifically, we find that this construction – in different situations – conveys a symmetric relation, an asymmetric relation, or an existential relation. The intended logical relation can only sometimes be inferred from the grammar of the statement itself. We discuss the pedagogical significance of these patterns in mathematical language and consider some strategies for helping students interpret the intended meaning of the mathematical text they hear or read.*

*Keywords:* mathematical language, corpus analysis, copular verbs

What does ‘is’ mean in mathematics? This is an important question because ‘is’ is used much more often in mathematical English than it is in day-to-day English. In both British and American English "is" represents around 1.01% of words (Davies, 2017), however in mathematics research papers the figure is 2.66% (Alcock, Inglis, Lew, Mejia-Ramos, Rago & Sangwin, 2017). The relative frequency of ‘is’ in mathematics can perhaps be explained by its ability to encode logical relationships. Linguists categorize ‘is’ as a copular verb, meaning it is used to join an adjective or noun to a subject. While copular verbs are known to be confusing in all languages – they can mean both predication (an asymmetric relation) and identity (a symmetric relation) (e.g., Geist, 2008; Russell, 1919) – they can be especially problematic in mathematics teaching and learning because of potential logical misinterpretations (e.g., Moschkovich, 1999; Schleppegrell, 2007).

Inspection of some ready examples suggests that ‘is’ can have at least three distinct logical meanings, as biconditional (↔), conditional (→), and existence (∃):

i. In "a square *is* a regular quadrilateral," *is* is intended to represent a biconditional (↔) relationship: an object is a square if and only if it is a regular quadrilateral;
ii. In "a square *is* a rectangle," *is* is intended to only represent a conditional (→) relationship: if an object is a square then it is a rectangle;
iii. In "there *is* a rectangle that's a square" *is* is intended to assert existence (∃): there exists a rectangle that is also a square.

The potential confusion between the biconditional (i) and conditional (ii) interpretations is especially challenging. From our experience, high school geometry students often object to the statement "a square is a rectangle"; however, it is unclear if they do so because they do not recognize the entire set of objects that fulfill the definition of a rectangle (i.e., their concept image of rectangle is at odds with the given definition), or because they interpret this statement as a biconditional, rendering it false. In other words, correctly interpreting a mathematical statement, at times, requires knowing the conveyed relationship prior to reading the statement.

Consider another example. The statement "an isosceles trapezoid *is* a quadrilateral with congruent diagonals" intends to assert a conditional relation and not a biconditional relation. We know this despite the fact that this sentence structure looks nearly identical to the biconditional in (i) above. Even though in many cases one communicates a biconditional by providing a narrowing clarification (e.g., a square is not just a quadrilateral but a *regular* quadrilateral), in this example, the narrowing of quadrilaterals to only those with congruent diagonals is still not narrow enough to be defining.

The major point is that *is* can be logically ambiguous, which means there may be important issues that arise in the teaching and learning of mathematics around use of this word that are worth further consideration. In this paper, we investigate the various uses of and grammatical constructions with the word 'is' in mathematics (in comparison with common English), as a means to reflect on communication in the teaching and learning of mathematics.

## A corpus approach

Corpus linguists study language by analyzing large collections of texts – corpora – intended to be representative samples of particular types of language. Our goal here was to compare the usage of *is* in day-to-day English and in mathematical English in pedagogical contexts. To this end we randomly sampled occurrences of *is* from two corpora. We used the Brown and LOB corpora (Kucera & Francis, 1967; Johansson, Leech, & Goodluck, 1978) to represent day-to-day written English and a corpus of mathematics textbooks compiled by Alcock et al. (2017).

Kucera & Francis (1967) compiled the Brown corpus in the 1960s. It contains 500 samples of American English text, totaling around 1 million words, from a balance of sources (e.g., newspaper articles, biographies, government documents and so on). Johansson, Leech and Goodluck (1978) compiled a British English version of the Brown corpus using texts taken from a similar range of sources, and in similar proportions. It too contains around 1 million words. We combined these two corpora to form a supercorpus of day-to-day English.

To study pedagogical language in mathematics, we used the textbook corpus constructed by Alcock et al. (2017). This consists of processed versions of language taken from undergraduate-level textbooks (Alcock et al. describe the process required to convert LaTeX source files into analyzable plain text). All the textbooks were taken from the Open Textbook Library, the College Open Textbooks site, or the American Institute of Mathematics Approved Textbook list. Topics included abstract algebra, analysis, linear algebra, complex analysis, and transition to proof. In total, 21 complete undergraduate textbooks are included in the pedagogical corpus, comprising of 1.5 million words. In order to conduct the analysis reported below, we randomly selected 250 instances of the word *is* from each corpus, together with the surrounding context.

## Analytic strategy

The rationale for our analysis strategy was the belief that the comparison of *is* in day-to-day language and pedagogical mathematical language would lead to insights about the kinds of mathematical statements likely to be difficult for students to interpret appropriately. Motivated by our examples of ambiguity described in the introduction, we initially began by coding the randomly selected sample of *is* statements as expressing *symmetric relations* (if and only if), *asymmetric relations* (if, then), or *existential relations* (there is). It became clear that we needed to distinguish an additional fourth category of *verb phrases* such as "is graphed" or "is rolling" since *is* operated as part of the conjugation of another verb rather than as a simple linking verb. Doing so, however, led to the realization that there was great variation among such structures.

One of the most problematic issues with this coding related to the role of verbs in past participle form. For example, in mathematics we use phrases like "is graphed" or "is connected" that consist of *is* followed by a past participle verb. However, the former is a verb phrase expressing the result of past action and the latter is a property attribution where *connected* acts as an adjective. Because mathematicians are careful to define terms like *connected*, this distinction can be made with some certainty. In the Brown and LOB corpora (representing American and British English respectively), we found more challenging statements like "Mrs. Lavaughn Huntley is accused of driving the getaway car used in a robbery of the Woodyard Bros. Grocery." In this case, *accused* could be an adjective describing Mrs. Huntley or a verb phrase describing ongoing action. This distinction appeared much more challenging to apprehend.

The fact that we had to rely on our understanding of mathematical content to recognize that *connected* acted as an adjective led us to develop a two-stage coding process that distinguished words' grammatical form from their operative role in the statements. Doing so helped us differentiate what information the grammatical form of a statement makes available from what information the reader's knowledge of semantic relations must provide. Using the TagAnt software package (Anthony, 2015) – which identifies parts of speech in a corpus – the first stage in our coding process involved determining the subject and object of each of the 500 *is* statements. While both the subject and object often constituted phrases, we identified one representative word as the object of *is* and then categorized each statement by the object word's part of speech (which we shall call the *grammatical category*). The object words were coded as: 1) nouns; 2) adjectives; 3) verb phrase, in gerund or infinitive form; 4) verb phrase, in past participle form; and 5) prepositions. Then, the second stage in our coding process involved analyzing the sentences within each grammatical category by determining the semantic role that the object word played in each *is* sentence (which we shall call the *semantic subcategory*). The semantic subcategory thus identifies the type of relation *is* is intended to convey.

In what follows, we elaborate on the first grammatical category, [Subject] *is* [noun], and its semantic subcategories. We deemed this category to be of particular interest because it involved both a broad range of *semantic variation*, as well as apparent differences between its uses in day-to-day and mathematical language – what we refer to as *register variation*. In other words, we were interested in *is* constructions in which students would have to use semantic cues to infer the logical relations conveyed in the statement because the grammatical cues are ambiguous. This seems more likely to be difficult if there are a variety of possible semantic subcategories and the frequencies of these subcategories differ between day-to-day and mathematical usage.

## [Subject] *is* [noun]

In this and the following sections we shall present our analysis of the statements coded in each grammatical category along with the frequency of each category in our sample. We shall begin our discussion with example *is* statements taken from the corpora.

- Example 1 (Ped): "Inlinemath *is* the standard basis for inlinemath."[1]
- Example 2 (Ped): "The definite integral of a constant times inlinemath *is* the constant times the definite integral of inlinemath."
- Example 3 (Ped): "A rational number *is* a fraction built out of integers."
- Example 4 (Ped): "This map *is* an isomorphism because it has an inverse."
- Example 5 (B/LOB): "a distinction must, however, be drawn between that which is traditional and enduring and that which *is* the result of current political necessity."
- Example 6 (Ped): "Show that there *is* one dimensionless product."

[1] The mathematical corpora replaced all mathematical symbols and expressions with "inlinemath" to facilitate search functions and word counts without having to account for the complexity of LaTeX code for mathematical notation (Alcock et al., 2017).

- Example 7 (B/LOB): "And there *is* enough truth in that to set you thinking."

We identified three semantic subcategories of statements of the form "[subject] *is* [noun]" that correspond closely to our original categories: symmetric relation (1-3), asymmetric relation (4-5), and existential statements (6-7).

**Symmetric relation**

When *is* conveys a symmetric relation, it indicates "is the same as." We present three cases of the symmetric relations because we observe there are subtle variations among them. In Example 1, the subject and object noun phrase both refer to the same mathematical object, so the two are being identified as the same. Here both are understood as singular, though if either involved variables the entire claim may be understood as implicitly quantified. Example 2 similarly conveys that both the subject and object phrases refer to the same object, though in this case that object is a number. In both of these cases, the article *the* before the object noun provides an explicit cue that *is* conveys a symmetric relation. This was common among our sample of statements in the symmetric relation subcategory, as displayed in Table 1. Example 3 portrays how statements conveying symmetric relations can nevertheless use *a* or *an* before the object word. Because the object phrase "a fraction built out of integers" can be taken to define the subject "rational number," the relation is symmetric.

*Table 1. Article choice within the symmetric and asymmetric relation subcategories.*

| Corpus | Ped | B/LOB | | Ped | B/LOB |
|---|---|---|---|---|---|
| Total symmetric statements with articles (SYM) | 31 | 19 | Total asymmetric statements with articles (ASM) | 59 | 32 |
| - SYM with *a/an* before object | 2 (7%) | 2 (11%) | - ASM with *a/an* before object | 53 (90%) | 25 (78%) |
| - SYM with *the* before object | 27 (87%) | 17 (89%) | - ASM with *the* before object | 0 (0%) | 3 (3%) |

**Asymmetric relation**

When *is* conveys an asymmetric relation, it signifies "is one of" or "is an element of the set of." Example 4 is a prototypical example of this form because the object noun is preceded by *a* or *an* (see Table 1), which cues that the subject noun is an example of the class specified by the object noun (and not *the* class itself). Example 5 portrays how statements in this subcategory can still use the article *the* before the object word. It uses *is* to say "that" is an example "result of political necessity," meaning *is* conveys an asymmetric relation. Thus, the article on the object noun usually provides a grammatical cue for whether *is* conveys a symmetric or asymmetric relation, but there are both symmetric and asymmetric constructions that use the alternative articles.

**Existential relation**

Though questions of existence may differ between day-to-day and mathematical contexts, we did not observe semantic ambiguity in statements of this form in either corpus. The phrase "there is" seems to clearly distinguish statements in this subcategory. However, we observed an interesting trend in the frequency of this semantic subcategory, as presented in the next subsection.

**Frequencies of this grammatical category and semantic subcategories**

Figure 1 presents the frequencies of "[subject] *is* [noun]" statements in our two samples and the relative frequency of each subcategory. This grammatical category was much more common in our sample of mathematical statements, which may reflect mathematicians' tendency to use nominalizations for concepts or processes (Morgan, 1996). There was a significant difference in the balance of subcategories found in each corpus (Fisher's exact test, $p$ = .001), symmetric relations occurred with about equal frequencies while mathematics text conveyed asymmetric relations more often and day-to-day text conveyed existence relations more often. The latter fact seems surprising, though we expect this is because mathematicians more often use the more formal "there exists" (instead of "there *is*"), or the symbol ∃, since existential claims are by no means scarce in mathematics text.

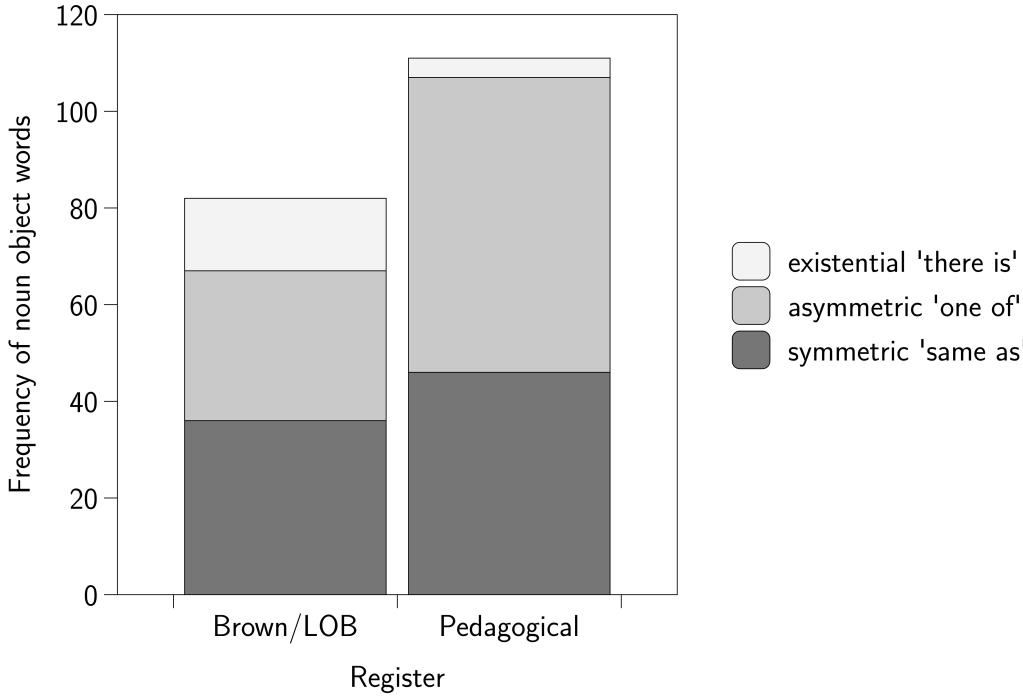


*Figure 1: Frequencies of noun object words and subcategories thereof.*

**Quantification in "A [Subject] is a [noun]" constructions**

The construction that began our investigation of *is* statements occurs when *is* links two nouns each with articles *a* or *an*. In this section, we explore further ambiguities that arise in this construction, particularly as they pertain to quantification and generalization, including a few more examples from the pedagogical corpus for discussion:

- Example 8 (Ped): "If inlinemath *is* a complete binary tree of height inlinemath, then…"
- Example 9 (Ped): "If inlinemath *is* a family of sets which covers inlinemath and inlinemath *is* a subfamily of inlinemath which also…"
- Example 10 (Ped): "The Cartesian product of two sets inlinemath and inlinemath, written inlinemath, *is* the set of all ordered pairs inlinemath, where inlinemath and inlinemath."
- Example 11 (Ped): "It can be shown that the best strategy is to pass over the first inlinemath candidates where inlinemath *is* the smallest integer for which inlinemath."
- Example 12 (Ped): "If inlinemath *is* a type 1 integer and inlinemath is a type 2 integer, then inlinemath is a type 2 integer."
- Example 13 (Ped): "If inlinemath, we say that inlinemath *is* a compact subset of inlinemath if, regarded as a subspace of inlinemath, it is a compact metric space."

As noted above, these *is* statements generally convey either a symmetric relation ("same as") or an asymmetric relation ("one of"). In most all cases the nouns on either side of *is* are singular with singular articles (*the, a, an*). However, given the value placed upon generalization in mathematics, these singulars are understood to represent entire classes through arbitrary selection (Durand-Guerrier, 2008). It is this implicit generalization that introduces so much of the ambiguity into statements of this grammatical form.

For instance, Example 1 seems to identify two singular objects. The subject of the sentence is the same as "the standard basis" for some other object. However, if this sentence is introducing a general notation for standard bases, it means to convey a universal relationship. Without recognizing whether *inlinemath* in that sentence represents a generic placeholder or some representation of a singular mathematical object (or a placeholder for some more specialized class), one cannot discern what relation *is* conveys. Example 2 conveys a general law of integrals, not merely a naming convention (despite being structurally similar to the definition in Example 10). However, one cannot tell from the grammatical form of the statement whether this sentence is stating the law in general or applying it to a particular case (Example 12 is similar in this regard). In Example 2, the article *the* is misleading. *The* marks the singularity of indefinite integrals, but the function being integrated should likely be understood as a placeholder representing any function. In other words there is one indefinite integral per function, but the statement almost certainly applies to a range of functions. Example 3 quite clearly means to convey a universal (defining) relationship, despite the singular article on both sides of *is*. The key point is that one cannot discern this merely grammatically – familiarity with the mathematical concepts is essential. In contrast, the grammatical cues in Example 4 convey more accurately that *is* relates a particular object ("this map") to a general class ("an isomorphism").

Our examples reveal other common grammatical cues that mathematicians use to convey the implicit generality behind nouns and noun phrases with singular articles. For instance, the *if*s at the beginning of Examples 8 and 9 are there to convey universal quantification of the subject of the *is* claim[2]. Example 13 presents an odd case where *if* is used in two slightly different ways in the same definition. The first *if* calls out an arbitrary metric space (a context assumption) while the second presents the defining condition for being a compact subset. In cases where *is* could relate an entire class represented by an arbitrary placeholder or a particular case, deciding whether the variable or name given to an object has appeared before or not (i.e. is already *bound*, Epp, 2009) provides a subtle cue. For instance, this would resolve some ambiguity in Examples 2 and 12. If the variable is not bound then the claim is likely universal; otherwise it may be an application of a warrant to a particular case or an introduction of cases within an argument. The mere grammar of the construction "If [subject] is a [noun]" does not distinguish between these uses. Furthermore, Example 11 shows how mathematicians sometimes compress the process of binding and using a variable by referring to a quantity before defining it in an appended clause.

What we gather from these examples is that the "[subject] is [noun]" grammatical structure entails semantic ambiguity that is only partially resolved by other grammatical cues (articles and conjunctions). In other words, one cannot infer the relationship between the subject and object nouns merely by the statement's construction. Mathematicians tend to state the general using arbitrary particulars, usually using placeholder variables or names with singular articles. This construction is not unique to mathematics (e.g. "The redeemed soul is a debtor to mercy alone"),

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[2] Indeed one of our philosopher colleagues argues that such claims are not really conditional at all, but rather universal (L. Clapp, personal communication, December, 2016; c.f. Durand-Guerrier, 1996).

but it appears from our samples to be much more common. This means students will likely need to be trained to properly interpret such common constructions in the mathematical register.

## Reflections

The goal of our grammatical analysis was to 1) identify differences between *is* usage in day-to-day and mathematical language and 2) to identify the semantically ambiguous *is* constructions in mathematical language. Due to space limitations, we have only presented our analysis of "[subject] is [noun]" constructions.

We proffer two tentative points from preceding analysis regarding the nature of the issue and what can be done to address it. First, we do not mean to belittle or demonize semantic ambiguity in mathematical discourse. We view it as inevitable, despite mathematicians' pursuit of precision and explicitness. However, we observe there is a tradeoff between simple statements that entail semantic ambiguity and complex statements that are grammatically hard to parse (c.f. Schleppegrell, 2004). Pedagogically speaking, we must create ways for students to be apprenticed into mathematical knowledge and language, requiring that we make it easier to parse and interpret. Simplifying language often incurs a cost in precision. In many cases, we judge that this price must be paid. However, problems arise when mathematics instructors treat dense constructions like "A square is a rectangle" or "A rational number is a fraction built out of integers" as completely unambiguous, without recognizing the role their expertise plays in rendering these claims interpretable.

Second, we comment on what might be done to maintain efficiency in pedagogical language while increasing the fidelity of communication. We recognize that empirical study must ultimately determine this, but we offer two ideas for consideration. One, it may help to alternate the grammatical cues we use to convey similar relationships. For instance, one could state and restate one of our first examples – "A square is a rectangle" – in multiple ways:

- "Each square has all the properties of a rectangle."
- "All squares are also rectangles."
- "Each square is also in the class of rectangles."

Similarly, statements conveying symmetric relations – "A square is a regular quadrilateral." – can be restated:

- "A regular quadrilateral is known as a square."
- "A square is the only kind of regular quadrilateral."

Alternating *a* and *an* with *any* and *each* or clearly designating defining actions with phrases such as *is called* and *is known as* can help cue students to the relations that *is* statements convey. We do not think any one of these is uniquely best. A longer statement is more explicit while "A square is a rectangle" is easy to recall. We recommend that instructors practice parallel articulations conveying the same relations to scaffold mathematical parlance. Some of our other work in mathematical logic demonstrates the importance of students associating mathematical properties with the sets of objects exhibiting the properties (what Dawkins, 2017, calls *reasoning with predicates*). Helping students to manage some ambiguities tied to implicit quantification aligns closely with developing a set-oriented way of thinking about mathematical claims. Two, we perceive that interpreting these mathematical statements is directly tied to understanding mathematical practices such as defining, representing, equating, stating general claims, and applying general warrants to particular cases. Future research on linguistic interpretation may benefit from integrating analysis of students' emergent interpretations of mathematical practices.

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**Reasoning about Quantities or Conventions: Investigating Shifts in In-service Teachers' Meanings after an On-line Graduate Course**


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*Although pervasive in school mathematics, few researchers have paid explicit attention to the impact graphing conventions have on teachers' meanings for function and rate of change. We examine the role conventions play in in-service teachers' (ISTs') meanings and ways to promote their developing more sophisticated meanings. We provided pre and post surveys to ISTs enrolled in an on-line graduate course specifically designed to promote their development of more sophisticated meanings for function and rate of change via reasoning quantitatively. We prompted them to consider hypothetical student responses about these ideas in unconventional representations. In this report, we characterize ISTs' meanings in relation to conventions commonly maintained in school mathematics and examine shifts in the ISTs' meanings.*



Keywords: Function; Rate of change; On-line education; In-service teachers


Whereas certain conventions (i.e., order of operations) impact the underlying mathematics at hand, other conventions are strictly representational choices (i.e., the input of a function is represented on the horizontal axis of a Cartesian coordinate system). Both types of conventions play an important role in mathematics but in this report we focus on the latter type of convention; although such conventions are pervasive in school mathematics (e.g., Hewitt, 1999), few researchers have examined the consequences for individuals' understandings of various ideas when particular conventions are strictly maintained. We are particularly interested in the extent to which teachers understand conventions as representational choices versus understanding these "conventions" as necessary features of particular mathematical ideas.

Thompson (1992) differentiated between a person using a "convention" unthinkingly and therefore being unaware of the "convention" as a convention versus understanding a convention as a particular choice that is customary (and often useful) while being aware that other choices may be equally correct or appropriate. Researchers have posited that students and teachers are hindered in making the latter distinction when they only have experiences in which particular conventions are maintained (e.g., Mamolo & Zazkis, 2012; Zazkis, 2008). Other researchers have noted that providing students opportunities to reason about relationships between quantities in non-canonical situations has the potential to support students in developing more sophisticated understandings that rely less on representational conventions and more on core mathematical ideas and understandings (e.g., Moore, Silverman, Paoletti, & LaForest, 2014).

In this report, we examine in-service teachers' (ISTs') function and rate of change understandings in relation to graphing conventions before and after an on-line course that was designed to support them in developing more sophisticated understandings of these ideas via reasoning about relationships between quantities (Thompson & Carlson, 2017). We address the questions: (a) To what extent do ISTs understand certain graphing conventions as choices or as mathematical rules that must be strictly maintained? (b) What impact does taking a graduate course focused on quantitative reasoning have on ISTs' meanings (and use of conventions)? As the intervention was on-line, we also seek to provide an existence proof that the impacts documented can be supported through carefully designed on-line professional development.

## Theoretical Perspective

The on-line course in which this study is situated was designed to leverage ISTs' quantitative and covariational reasoning to support their developing more sophisticated mathematical meanings. Quantitative reasoning consists of an individual conceiving of a situation, constructing quantities as measurable attributes of objects, and reasoning about relationships between quantities (Smith III & Thompson, 2008; Thompson, 2011, 2013). When an individual conceives and coordinates two quantities together, they engage in covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998). An increasing number of researchers have highlighted how students can leverage quantitative and covariational reasoning to develop understandings of various topical areas including function classes, rate of change, and the fundamental theorem of calculus (e.g., Confrey & Smith, 1995; Ellis, Ozgur, Kulow, Williams, & Amidon, 2015; Johnson, 2012; Thompson, 1994a, 1994b) and to enact important mental processes such as generalizing and modeling (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Larsen, & Lesh, 2003; Ellis, 2007).

Of relevance to this report, Moore et al. (2014) highlighted the extent to which engaging students in reasoning about relationships between quantities can support students in developing mathematical understandings that are not constrained by conventions commonly maintained in school mathematics (i.e., representing the input of a graphically represented function on the horizontal axis with the variable $x$). The researchers outlined several principles teacher educators can use to support pre-service teachers (PSTs) and ISTs developing more sophisticated meanings including (a) using tasks that intentionally break from conventional representational systems, (b) routinely using quantitatively rich situations (i.e., situations in which an individual can construct and reason about a variety of quantities in order to solve a problem), and (c) maintaining an explicit focus on quantities and their relationships in classroom discourse.

## Relevant Literature

### Students' and Teachers' Convention Understandings

Several researchers have noted that students and teachers can develop insufficient mathematical understandings if certain conventions are strictly maintained in school mathematics (Mamolo & Zazkis, 2012; Thompson 1992; Zazkis, 2008). For example, researchers who have investigated students' meanings for function and rate of change (e.g., Akkoc & Tall, 2005; Montiel, Vidakovic, & Kabael, 2008; Moore et al., 2014; Oehrtman, Carlson, & Thompson, 2008) have found that students often maintain meanings that require certain representational conventions to be followed. With respect to students' function meanings, Montiel, Vidakovic, and Kabael (2008) identified students applying the vertical line test, a common procedure included in U.S. curricula, to determine if a graph defined by $r = 4$ represented in the polar coordinate system represented a function. Breidenbach, Dubinsky, Hawks, and Nichols (1992) illustrated that only 11 of 59 undergraduate students in their study understood a graph we interpret as representing the function $x = f(y) = \sin(y)$ for $-4 < y < 4$ with $x$ and $y$ represented on the horizontal and vertical axis respectively as representing a function (i.e., $x$ as a function of $y$). In these examples, the researchers posed graphs they intended to represent functions but the students' meanings did not afford such interpretations; one possible explanation for this observation is that the students understood representational choices (e.g., graphs are unquestionably represented in the Cartesian coordinate system with the independent quantity represented by $x$ on the horizontal axis) as mathematical rules that must be followed.

Moore et al. (2013, in preparation) highlighted the extent to which PSTs in their study understood function and rate of change in relation to graphing conventions. The researchers noted less than 36% of PSTs interpreted hypothetical student work as unquestionably correct when these responses used unconventional, but mathematically viable graphs. Many PSTs indicated the hypothetical student would be correct if a certain feature of the graph was changed to maintain conventions but concluded that in the given orientation the hypothetical student was incorrect. We extend Moore and colleagues (2013, 2014, in preparation) work by examining a different population's, ISTs', function and rate of change understandings in relation to graphing conventions. We also examine the extent to which an on-line course focusing on reasoning quantitatively has the potential to promote shifts in ISTs' meanings.

**Teaching and learning mathematics on-line.** Online courses at the university level continue to grow as there is a belief that such courses can reduce expenditure and increase enrollment (Allen, Seaman, Poulin, & Straut, 2016). In this study, we employed an instructional environment grounded in design-based research that is referred to as Online Asynchronous Collaboration (OAC) in Mathematics Teacher Education (Silverman & Clay, 2010). At its core, the OAC model is grounded in the belief that replicating traditional teaching practices is not sufficient for online learning environments (Reeves, Herrington, & Oliver, 2004). The implementation of the OAC we report here consists of iterative cycles of three to four day "private" problem solving in an on-line discussion board (viewable only by the individual student and instructor), then three to four days of "public" discussion in which all students are given the opportunities to read, comment on and ask questions about each other's solutions. The last few days of each unit are designed to support students' synthesis and reflection on the ways of reasoning each problem set was designed to highlight. Researchers (Silverman, 2011; Silverman & Clay, 2010) have shown that this OAC model has the potential to support ISTs' development of pedagogical content knowledge and mathematics knowledge for teaching; we extend these results by examining how this model has the potential to support teachers' developing more sophisticated understandings in relation to graphing conventions.

## Methods and Analysis

### Participants and Settings

The ISTs who participated in the study were enrolled in a fully online graduate mathematics program designed specifically for ISTs. The ISTs were geographically distributed across the U.S. and each was, at the time of the study, a 6-12 grade mathematics teacher who was certified to teach mathematics in his/her home state. All of the ISTs had completed a minimum of three mathematics courses beyond Calculus III and had an undergraduate GPA of 3.0 or better. In total 34 ISTs took both the pre and post survey.

The on-line course was designed with the intention of leveraging the teachers' quantitative and covariational reasoning to develop more sophisticated understandings and followed the recommendations put forth by Moore et al. (2014) outlined above. The initial unit asked the ISTs to track and describe the behavior of various contextualized relationships (i.e., a car driving back and forth along a road as described by Saldanha and Thompson (1998)). There was a particular focus on supporting ISTs in identifying quantities from a given context, using variables to represent varying quantities, and analyzing relationships between relevant quantities verbally, numerically, and graphically. The remainder of the term asked ISTs to leverage these skills with a focus on exploring a variety of functional relationships (e.g., polynomial functions, trigonometry, related rates problems, modeling, and ideas from calculus) from a quantitative perspective. Table 1 presents an overview of the 10-week course.

*Table 1. 10-week Course Overview*

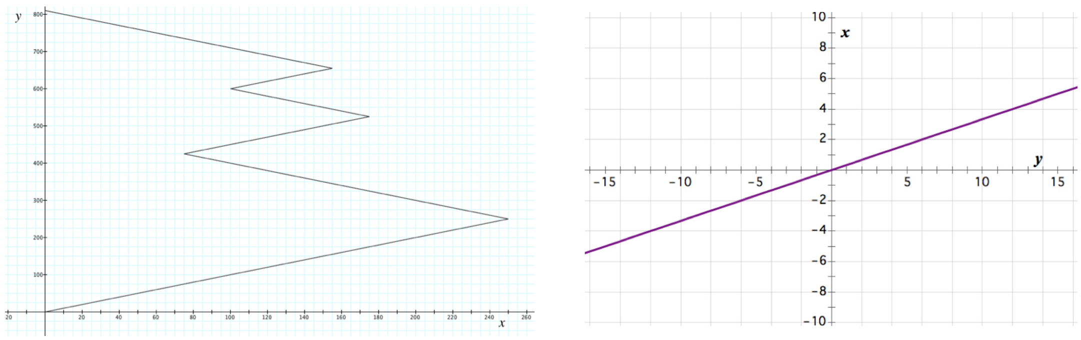
| Week | Focus |
|---|---|
| 1 | Covariation of Quantities |
| 2 | Trigonometry |
| 3 | Periodicity and Covariation: Trigonometric Functions |
| 4 | Functions as Relationships in Context |
| 5 | More Functions as Relationships/Functions as Actions and Processes |
| 6 | Families of Functions |
| 7 | Average Rate of Change |
| 8 | Rate of Change and Rate of Change Functions |
| 9/10 | Covariation in the Classroom |

## Analysis

We coded the ISTs' responses using open and axial approaches (Strauss & Corbin, 1998) and thematic analysis (Braun & Clarke, 2006). Throughout the coding process, the researchers did not know which IST's response they were coding or if the IST's response was part of the pre or post survey. A member of the research team read a subset of IST responses and we met to discuss our observations, identify commonalities across responses, and adapt or create new codes to capture more responses. We iterated this process four times as we refined our codes to accurately capture all responses; as the resulting codes are both methods and results, we present the codes themselves in the results. After we agreed on a final set of codes, a second researcher recoded approximately 65% of the data. We calculated inter-rater reliability by comparing the number of times both coders agreed on a code, achieving a high level of agreement on each problem (Sideways Mountain Task, Kappa = 0.78 and $y = 3x$ Task, Kappa= 0.85).

**Task design.** We adapted tasks used by Moore et al. (2013, submitted) to make inferences about PSTs' understanding of function and rate of change in relation to graphing conventions into items ISTs responded to in pre and post-course on-line surveys. Each task was designed with the intention of examining ISTs' understanding of mathematical ideas in relation to graphing conventions. In order to ensure the ISTs noticed the unconventional nature of the graphs, the tasks included hypothetical student responses that deviate from a particular convention but are mathematically viable (from the researchers' perspective). For example, the Sideways Mountain Task prompts an IST to respond to a student who stated for the graph in Figure 1a that "Sure, it can be a function… $x$ is a function of $y$." Whereas from the researchers' perspective the students' statement is mathematically correct, the graph, in its given orientation does not pass the vertical line test, which as described above, is often critical to students' and teachers' meanings for function in a graphing context. Hence, the tasks allow us to examine the extent to which an IST's function understandings are related to particular graphing conventions (i.e., a function's input must be represented by the variable $x$ or on the horizontal axis, or both).

Like the Sideways Mountain Task, the $y = 3x$ Task supports our examining ISTs' rate of change understandings in relation to graphing conventions. The task prompts ISTs to consider a student who graphed the relationship $y = 3x$ as shown in Figure 1b. Although the graph does represent the relationship defined by $y = 3x$, the hypothetical student's work deviates from the convention of representing $x$ and $y$ on the horizontal and vertical axes, respectively. Hence, the task provides insights into the extent to which ISTs' meanings for graphs and rate of change rely on representing particular variable quantities on particular axes versus accurately representing relationships between two quantities.




(a) (b)

*Figure 1.* (a) Sideways Mountain Task: Is $x$ a function of $y$? (b) The $y = 3x$ Task: A hypothetical student's work.

## Results

In this section we first describe the codes we created to capture the ISTs' responses. We then compare the ISTs' pre and post survey results for each of the two tasks described above. For both tasks, our final coding scheme categorized the extent to which the ISTs interpreted the hypothetical students' mathematical statement as viable. This analysis provides insights into the extent to which the ISTs' meanings for graphs, function, and rate of change are rooted in reasoning about relationships between quantities versus requiring graphing conventions to be maintained. Demonstrating a focus on understanding statements concerning rate of change and function to be statements about relationships between quantities, the first code was for responses that indicated the student's mathematical statement is correct notwithstanding the student breaking from conventions. Indicating a tension between reasoning about relationships between quantities and graphing conventions, the second category was for responses that specified the student's statement was mathematically true but, despite this, the student's solution was wrong because he or she did not follow conventions. Signifying the ISTs' meanings required certain conventions to be maintained, the final category was for responses that either indicated the student's mathematical statement was incorrect or did not address the student's statement.

Table 2 presents the code description, an example response to the Sideways Mountain Task and the counts for the pre and post survey. We first highlight that prior to the course, a majority of the ISTs interpreted the hypothetical student's solution as incorrect, despite the student's statement being mathematically viable from our perspective. Second, we note the trend of a positive shift in ISTs' responses towards interpreting the student's mathematical statement as correct despite the student breaking from conventions after taking the on-line course. We take this to indicate that the course supported many of the ISTs in developing more sophisticated meanings in regards to functions and their graphs. Finally, we note that despite this trend, nine ISTs still interpreted the student's mathematical statement as *incorrect* or did not address the students' mathematical statement in the post-survey. We return to this observation in the implications.

*Table 2. Code descriptions, sample responses, and counts for the pre and post survey for the Sideways Mountain Task.*

| Code description (value) | Example Responses to the Sideways Mountain Task | Pre | Post |
|---|---|---|---|
| The student's mathematical statement is correct despite breaking from conventions. (1) | That's great! I am so glad you were able to apply the "vertical line test" in a horizontal orientation and realize that you would have a function. You are correct in saying that $x$ is a function of $y$. | 11 | 19 |
| The student's mathematical statement is true but the student is incorrect because he/she broke from conventions. (2) | I think the student is understanding that $x$ can be a function of $y$ but they are not displaying it correctly through the graph. | 5 | 6 |
| The student's mathematical statement is incorrect or the IST did not address the student's mathematical statement. (3) | It was not a good explanation and $x$ is not a function of $y$, $y$ is a function of $x$. The value of $y$ depends on $x$. They also did not describe what would make it a function. | 18 | 9 |

Table 3 presents the code description, an example response to the $y = 3x$ Task and the counts for the pre and post survey. We again highlight that there is a general trend towards more ISTs' responses indicating that the hypothetical student's response is correct despite breaking from conventions. In contrast to the responses to the Sideways Mountain Task, we note that a majority of ISTs' pre-survey responses indicated that the student's statement was correct before the intervention. We take this finding to indicate that the ISTs' meanings for rate of change may be less reliant on certain conventions being maintained prior to taking the on-line course as compared to their function meanings.

Table 3. Code descriptions, sample responses, and counts for the pre and post survey to the $y = 3x$ Task.

| Code description (value) | Example Responses to the $y = 3x$ Task | Pre | Post |
|---|---|---|---|
| The student's mathematical statement is correct despite breaking from conventions. (1) | In this case, the student has graphed the relationship correctly given their choice of axis. Technically there is absolutely nothing wrong with this graph. | 20 | 24 |
| The student's mathematical statement is true but the student is incorrect because he/she broke from conventions. (2) | The student cannot receive full credit, as the graph is wrong, however it can easily be fixed by discussing the $y$ as the vertical axis and the $x$ as the horizontal axis. Once this discussion has ensued, I would ask the student to graph again but prompt them that they were correct in their understanding of $y$ being equal to 3 times the given $x$ value. | 6 | 8 |
| The student's mathematical statement is incorrect or the IST did not address the student's mathematical statement. (3) | The student did not graph the slope correctly, instead of a positive 3 they graphed a negative 3. They did label their $x$ and $y$-axis. Therefore, they are showing some correlation as to how the values of $x$ and $y$ vary and covary with each other. | 8 | 2 |

**Comparing surveys.** To further examine shifts in the ISTs' meanings from the pre to post survey, we assigned numerical values to each of the categories (shown in parentheses in the code description column). Table 4 presents the pre and post averages for each task; a score closer to 1 indicates that on average, the ISTs were attending more to the underlying quantitative relationships than to the student's response adhering to graphing conventions. In order to examine if there were statistically significant differences between the ISTs' responses pre and post course, we conducted one-tailed Wilcoxon Signed-Rank tests to examine if the mean scores differed significantly. We conducted one-tailed test because we expected the ISTs would exhibit a positive shift in their meanings based on the intervention and we conducted Wilcoxon Signed-Rank tests rather than *t*-tests as we cannot say if the population is normally distributed. Table 4 presents the *p*-values for each test. We note that there was a statistically significant result for the Sideways Mountain Task but not for the $y = 3x$ Task. We conjecture the latter observation may be due to the fact that the ISTs' initial responses indicate a tendency to evaluate the student's statement in the $y = 3x$ Task as correct prior to the on-line course.

Table 4. Average scores of pre and post survey for ISTs and *p*-values from a Wilcoxon Signed-Rank test.

| | Sideways Mountain Task | $y = 3x$ Task |
|---|---|---|
| Pre | 2.21 | 1.65 |
| Post | 1.71 | 1.35 |
| *p*-value | 0.0037* | 0.0618 |

## Discussion and Implications

In this report, we make several contributions to the research examining ISTs' understandings of mathematical ideas and ways to support ISTs' quantitative reasoning. We demonstrated many ISTs' initial meanings for function required certain graphing conventions to be maintained which is largely compatible with the PSTs reported by Moore and colleagues (2013, submitted). This finding underscores the importance of addressing such meanings in professional development and in PST training programs as teaching experience is not enough to support teachers in developing these meanings. We also highlight, and compatible with the PSTs reported by Moore and colleagues, the ISTs' responses to the $y = 3x$ Task differed from their responses to the Sideways Mountain Task. This finding highlights the extent to which an individual is constrained by a particular convention (i.e., a function's input is represented on the horizontal axis by the variable $x$) may be idiosyncratic to the particular mathematical idea at hand. Some may interpret this finding to indicate ISTs' and PSTs' meanings for rate of change are more focused on the underlying relationship between quantities (i.e., reason quantitatively) rather than maintaining particular conventions. Before we make such an argument, we believe there needs to be more research investigating teachers' understandings of rate of change in other non-canonical situations (i.e., polar coordinates).

Researchers (e.g., Mamolo & Zazkis, 2012; Moore et al., 2014; Paoletti, Stevens, & Moore, 2016; Thompson, 1992) have indicated that educators should provide students, PSTs, and ISTs with repeated opportunities to address unconventional situations in order to support them in expanding their meanings for various mathematical ideas such that they understand what aspects are conventional and what are required mathematically. Our data provides an existence proof that an on-line course can provide such opportunities for ISTs. This finding is especially important as on-line interventions have the potential to be scalable in ways that face-to-face courses typically are not. Future researchers may be interested in implementing and studying such scaling efforts to improve teachers' mathematical meanings.

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A Case of Community, Investment, and Doing in an Active-Learning Business Calculus Course

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*Presented here is a case study of Christina and her goals in an active-learning business calculus course. The larger project, from which this report emanates, involved an instructional redesign of a business calculus course intended to address two main student concerns: (seemingly) irrelevant content and a lack of opportunities to be active in class. Class size mediated Christina's access to community, which she perceived to be a necessary condition for her learning. Additionally, Christina drew a distinction between authentically situated mathematics and pseudo-situated problems that fail to invest her in the problem-solving process. She valued opportunities to do mathematics during class and receive feedback from her instructor and peers. This project has implications for the mathematics education of business students, active-learning in post-secondary mathematics, and situated mathematics problems.*

*Keywords:* business calculus, non-STEM, active-learning, situated mathematics, student goals

Student engagement is undoubtedly an important factor in the learning of post-secondary mathematics. As business calculus instructor, I especially struggled to develop and foster student engagement. Furthermore, students in this course considered the content to be irrelevant to their lives and futures. Business has long been the most popular post-secondary bachelor degree awarded in the United States (National Center for Education Statistics, 2015). Given the large student population associated with business calculus and the value of mathematics proficiency in society, addressing issues of student engagement and perceived irrelevant content in this course is critical. Aptitude in mathematics is a powerful societal tool regardless of students' trajectories. This project aims to contribute to existing research in the area of teaching business mathematics and hopes to emphasize the importance and value of the mathematics education of non-STEM students.

## Research Questions

The case presented in this report was part of a larger study which examined student agency in a business calculus course. A more detailed description of that study, its context, and methods is described elsewhere (Higgins, 2017). In the larger study, an operationalized understanding of an individual's achievement of agency was the resistive or supportive moves one makes in response to structure. Structure was defined as the set of forces that constrain or enable an individual's goal. Thus, to examine a student's achievement of agency in the context of this course, it was necessary to first determine that student's goals in this environment. This report focuses on the identification of a particular student's goals in this course. The goals of students in a mathematics course influence their engagement with mathematics and their behavior in the classroom. An instructional design informed by the intentions and ambitions of students in a mathematics course can leverage this information to better align the goals of students and instructors.

My instructional redesign of this course that was intended to address student concerns of passivity and irrelevant mathematics in this course. Given the time and curriculum constraints of this course at this institution, how could an instructional design address these student concerns?

1. What can an active-learning business calculus course designed to address student perceptions of irrelevant content and passivity look like?
2. What are the goals of a student in an active-learning business calculus course?
3. What elements of an active-learning instructional design facilitate or constrain these goals?

The results from this study inform the practice of teachers who value student goals in this context and have research implications for the mathematics education of students majoring in non-STEM fields.

## Theoretical Perspective and Implications for This Study

My theoretical perspective is rooted in sociocultural theory and draws on Lave and Wenger's (1991) model of situated learning. From this perspective, learning is conceived of as a process of apprenticeship. Experts model practices for novices and gradually include novices in increasingly legitimate participation in the community. Novices learn through both modeling and doing. Within a business calculus course, I am positioned as an expert in mathematics, however these students are explicitly not apprenticing into a mathematics community. In the larger and perhaps more relevant context of their academic majors, these students need to be apprenticing into a community of schooled businesspersons. This influenced my instructional design and motivated me to include more real-world problems and contexts that students might encounter in a business career.

## Literature Review

### Business Calculus

There is currently a striking gap in mathematics education literature regarding calculus courses for business students (Mills, 2015). Considering that business is the most popular undergraduate bachelor degree awarded in the United States (National Center for Education Statistics, 2015) and that many institutions require business majors to take courses through mathematics departments, this paucity of research is alarming. Due to the motivations and career goals of business students, there are many issues specific to this population that are not shared by students in a traditional calculus course. An intention of this project is to contribute to this area of research and highlight the importance of this student population and their mathematics education. In 2000, the Curriculum Renewal Across The First Two Years (CRAFTY) subcommittee of the MAA released a report that included recommendations for addressing needs unique to business students enrolled in mathematics courses (Lamoureux, 2000). Given the situated nature of businesspersons' actual use of mathematics, problems and associated decisions encountered in the real world are naturally coupled with significant ambiguity. This team recommended that this ambiguity be reflected in the problems included in mathematics courses for business students. Pedagogically, this report suggested including opportunities for student discussion of problems during class, invitations for students to present solutions and justifications to the rest of the class, and emphasized the importance of making relevant mathematics explicit. Given that these students are apprenticing into the world of business, a social industry, group work can help develop important social skills that are valued by employers. In this vein, student assignments should reflect the same material and scope that they might submit to a superior in the workplace. This report noted that mathematicians typically lack the resources to create valid and appropriate business-contextualized problems. Increased communication between mathematics and business faculty regarding mathematics courses for business majors would work toward solving this issue.

### Active-Learning Strategies

The literature on active-learning strategies is consistent and indisputable. Research indicates that these practices are linked to significant positive learning outcomes in students. Freeman and colleagues (2014) conducted a meta-analysis of research on active-learning and its effects on students in STEM courses and their learning. They found that for students who were in courses with at least some active-learning, student performance increased by nearly half a standard deviation when compared to student outcomes from a traditional lecture course. Additionally, students in lecture-style courses were more likely to fail than students taking courses that incorporated active-learning. Freeman and colleagues concluded that active-learning positively affects student performance in STEM courses at the post-secondary level. In 2016 the Conference Board of the Mathematical Sciences published a statement calling for an increase in active-learning strategies in the teaching of post-secondary mathematics (Braun et al., 2016). This statement advocates for the inclusion of active-learning techniques to provide students with meaningful mathematical experiences and as an avenue for modernizing the instruction of mathematics at the post-secondary level.

## Methodology

As stated previously, this report was part of a much larger project that involved a significant amount of pilot work including interviews with both business and mathematics faculty, interviews with students currently enrolled in the business calculus course at this institution, and a pilot version of my course re-design during Summer 2016, accompanied by interviews with students in this course. My course redesign included four intentional elements: daily reviews, in-class group activities, readings, and punctuated lecture style. Daily reviews were ungraded problem sets distributed at the beginning of each class that reviewed the content from the previous class and typically included a conceptual question and a few procedural problems. Student were encouraged to work on these problems with their peers. I circulated through the classroom answering both group and individual questions. After 10-15 minutes, we went over the solution as a class. Six in-class group activities coincided with application sections and foundational sections in the curriculum. Students split themselves into groups and worked through the problems during the class period. Each activity was also accompanied by a reading outside of class. These readings included motivation for the topic covered in the activity, an explanation of the concepts underlying the mathematics, the mathematics used to solve the problem, and a fully worked-out example problem. Students were required to digitally annotate each reading prior to the corresponding activity day. In order to allow for more time for students to be active, I was interested in eliminating lecture-style instruction as much as possible. The curriculum in this course covered a tremendous amount of content, which created challenging time constraints. Rather than resorting to direct instruction, I adopted a punctuated lecture style that encouraged students to participate routinely. I regularly asked students questions during instruction and included frequent opportunities for students to try problems on their own, work on problems with their classmates, or discuss questions with their classmates. These times generated ways for students to be actively engaged during instruction.

## Main Study

**Student make-up.** My business calculus course during Fall 2016 began with 51 students and ended with 50 students (one student withdrew from the course after approximately ten weeks). This was a reduced class size that required permission from the department chair. This course typically enrolls approximately 90 students in each section. There were 20 female students (19 after one student dropped) and 31 male students. There were nine freshman students, 20 sophomores, 18 juniors, three seniors, and one post-baccalaureate student. There were 43 students majoring within the college of business, four students majoring in economics, and four students whose majors were listed as either "exploring" or "undecided".

**Selection of participants.** This report focuses on one of my participants, Christina. She was initially selected to be part of the larger study because she appeared to be an exemplar of a very actively engaged student. Given her high level of engagement in the course, I was surprised to learn that Christina was retaking this course. She had previously taken business calculus in Fall 2015 and earned a D. Ultimately, Christina earned a B in my course. I was particularly interested in learning what her goals were in this course and the elements of my instructional design that facilitated or constrained these goals.

**Data Collection.** All class sessions were video and audio-recorded, save a few due to user error. The six videos of the in-class activities were completely transcribed (the first and second activity transcriptions contained less information due to missing recordings). In addition to the video and audio-recorded class sessions, a minimal amount of student work was collected for the larger study.

After the course ended, Christina participated in three interviews. The first interview involved her history with mathematics at the university level. This interview also helped me initially determine what her goals in this course were and what aspects of the design of our Fall 2016 business calculus course influenced those goals. The second interview was intended to validate things that I had inferred from the first interview. This involved confirming Christina's goals and the factors that influenced these goals. By our third interview, I had written a description of her goals in the course. This interview was meant to answer any remaining questions I had about her goals as a student in this course.

**Analysis.** To identify Christina's goals in this course, I began by writing a description of her experiences with post-secondary mathematics, primarily based on our first and second interviews. After the initial, general description of Christina and post-secondary mathematics, I composed a detailed summary of Christina as a student in our business calculus course, again mostly from our first and second interviews. These characterizations enabled me to both explicitly determine Christina's goals and identify characteristics of my instructional design that Christina perceived to be constraining or enabling. From these influential forces, I backward-inducted the goals that these affected. Once I had identified Christina's goals associated with this course, I coded all three of her interview transcriptions for these goals. These codes were used to perform a second coding pass, where I identified instances in our interviews when Christina referred to structural elements in our course. These codes were used to verify and justify claims made in her case. From the sets of goals and associated structural elements that I had identified, I described what Christina was a case of, her goals in the course, elements of our course that facilitated or constrained those goals, and the actions she took towards the progress of those goals.

## Christina: A Case of a Student Valuing Community, Investment, and Doing

### Classroom Community

Size of class was an important factor for Christina[1]. The first business calculus course in which she was enrolled consisted of approximately 80 students. During our first interview, she discussed her struggle to form working relationships with her classmates and connect with the instructor in that course. Christina uses her relationships with her classmates to secure feedback about her understanding of topics and to establish her belonging in a course: "It's just so not personal and I can't learn the same way. I feel like I'd be terrible in online classes because I need that interaction. And so, in some classes I am very involved, but other classes not." She explained in our interviews that she would have been much less likely to ask questions in our course if she had not formed a friendship with the student who sat next to her, Andy. Not only did she use this relationship to solicit feedback from Andy, she also explained that establishing a friendship in the class made her more comfortable asking questions and being honest about her misunderstanding. A sense of community is an important factor in the ways in which Christina advocates for her own learning. She is much more likely to act in ways that serve her other course goals if she has established relationships with other classmates and the instructor. Christina observed that the size of class affected her access to community, which is a mediating factor for advocating for her own learning. Regardless of instructor ability or effectiveness, in a large class, she perceives her learning to be inhibited. In this way, class size affects her ability to find community in a mathematics course, and a sense of community affects her successful achievement of agency.

### Contextualized Mathematics versus Pseudo-Situated Mathematics

Contextualized mathematics problems help invest Christina in the problem-solving process and connect mathematics to real-life issues, which motivates her to work to understand the solution process. When I asked her about the readings in our course, Christina drew a distinction between problems contextualized in real-life scenarios and problems that are pseudo-situated:

> Yeah, I think a lot of them had actual stories, like things that would actually happen in real life. And that's really helpful. Because a lot of math is a lot of numbers, so when you put words in there – I always hate math word problems. But if they're actually relevant and you're like, 'Ok, I'm understanding. Ok, this would happen in real life,' or like, 'This is something that does happen,' versus, I don't know, like apples and whatever. You know what I mean?

Being able to identify with a problem and recognize it as a valid real-life predicament invested Christina in the solution process and motivated her to focus and value the problem-solving process. Relevant problems motivated her to learn the material because she saw a potential future benefit for her in her career and she identified this connection between school mathematics and real life as a helpful force in her understanding of the material.

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[1] The quotes and claims in this section come from transcriptions of interviews conducted with Christina.

### Opportunities to do mathematics in class

Christina noted that our course offered frequent opportunities to do mathematics during class. She especially valued this so that she could self-assess her knowledge and receive appropriate feedback. In her words:

> So when we're in class and you're making us do what we would be doing for homework and then you're there to ask – to help us if we need questions, that was really helpful for me, because sometimes I don't know what I don't know until I'm already home. And then I can't – it's harder to email about a question than it is to just ask in person.

She observed that this characteristic of our course was unique in comparison to her previous post-secondary mathematics courses. Prior to our course, she had mostly spent class time watching mathematics happen, rather than doing mathematics herself. Working through problems during class gave Christina an opportunity to address misunderstandings during class time and with the instructor or her peers. Opportunities to try problems, discuss with her classmates, and receive feedback from her instructor were greatly valued and noticeably used by Christina to aid in her learning.

## Discussion and Implications

### Real-Life Versus Pseudo-Situations

By the time students in this course reach higher education in the United States, they have a somewhat defined career path. To only include problems that are inherently procedural in nature, such as the pseudo-situated problems to which Christina referred, is a failure to serve our students. Problems situated within a business context should be an inherent characteristic of a business calculus course. Rather than trying to find how business situations can fit into the calculus curriculum, instructors should be focused on how calculus can fit into business contexts.

### Business Education

As referenced in the MAA CRAFTY (Lamoureux, 2000) recommendations, business solutions in the real world are often messy, ambiguous, and fail to adhere to procedures prescribed by school mathematics. Given this phenomenon, instructional practice in a business calculus course should reflect this same ambiguity. The mathematics education for these students should model, in some regard, the situations they will encounter in their career. Mathematics instructors are not typically experts in real life business situations. In order to effectively incorporate these kinds of problems and data sets, mathematics instructors need to collaborate with business faculty and with businesspersons working in industries.

### Research and Teaching Implications

**Business calculus instruction and course design.** Christina's experiences in this course suggest that opportunities to be active in class are valued by students. Student populations are dynamic and it is the onus of instructors to evolve our practices to best meet their needs. Opportunities for interaction with other classmates and with the instructor were conspicuous in this course. Past post-secondary mathematics courses in which Christina was enrolled allowed for little student activity. Christina considered interaction with other people to be a necessary condition for her learning. This finding is consistent with current research on active-learning in post-secondary mathematics courses.

**Real-life business problems and data sets.** The case of Christina illustrates that students in this course value problems that are contextualized in real-life situations and that students develop a stronger investment in solution processes when problems are situated in relatable contexts. The students in a business calculus course are not typically STEM students and most do not enter this course with an intrinsic interest in mathematics. They are majoring in a social field and are aware that their careers will likely involve working with other businesspersons. This finding is consistent with the recommendations from the MAA CRAFTY subcommittee.

**Non-STEM post-secondary mathematics education.** Recently, mathematics education has included a strong focus on the mathematics education of STEM majors. Like Mills (2015) reported, there is a striking lack of literature on calculus for non-STEM majors. This project examined student experiences in a non-STEM post-secondary mathematics course and contributes to the existing literature on calculus as a client discipline, mathematics education for business majors, and post-secondary mathematics education for students majoring in non-STEM disciplines. Mathematics courses for students in non-STEM disciplines encompass issues of motivation, relevancy, and confidence. While mathematics education for STEM students is clearly a significant issue, the lack of literature on calculus for non-STEM majors might suggest that these are not important issues. Despite this dearth of literature, I imagine mathematics educators and researchers would all agree that these student groups and their mathematics education are as valuable as any other student group. However, it is indisputable that non-STEM students are underrepresented in post-secondary mathematics education research. This underrepresentation is especially troubling considering that business is the most prevalent bachelor degree awarded in the United States. While this project hopes to contribute to the existing literature, there is a desperate need for more research in this area.

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Individual and Situational Factors Related to Lecturing in Abstract Algebra

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*In this study, we report the results of a national survey of 219 abstract algebra instructors concerning their instructional practices and pedagogical decision-making. Organizing our respondents into groups (Alternative, Mixed, Traditional) based on proportion of class time lecturing, we investigated differences in the prevalence of specific pedagogical practices and the individual/situational factors influential therein. We used the reported teaching practices to generate profiles of the salient features of each instructional type and attempted to explain these differences through a combination of individual/situational factors. Results indicate that while significant differences in teaching practices exist, these are primarily explained by individual factors such as personal beliefs, level of experience, and interest in various scholarly activities. Situational characteristics, apart from institution type as identified by terminal degree, such as perceived departmental support and situation of abstract algebra in the broader mathematics curriculum did not appear to be related to instructional differences.*

*Keywords: pedagogy, abstract algebra, instructor decision-making*

In STEM higher education, and specifically in mathematics, lecture-based pedagogy is the norm. The most current report (2014) from the Higher Education Research Institute found that "more than two-thirds of faculty across STEM sub-fields utilize[d] extensive lecturing in all or most of their courses" (Eagan, 2016). This is despite the growing volume of literature, both from the learning sciences and professional organizations, urging teachers to adopt student-centered practices. Indeed, with the mounting body of evidence (see Freeman, et al., 2014 for a meta-analysis), the question is no longer "should we still be lecturing?" but is instead "why are we?"

Literature on instructional decision-making has long focused on individual characteristics, such as beliefs about teaching and learning (e.g., Calderhead, 1996), knowledge of research-based instructional practices (e.g., Henderson and Dancy, 2009), and professional development (e.g., Belnap, 2005; Speer, Gutman, & Murphy, 2005). While we do not discount the enormous influence of individual characteristics, we also acknowledge that a bevy of other external circumstances factor considerably. Research in science and mathematics education had identified a number of external influences, including expectations of content coverage, time expectations, promotion and tenure requirements, and class size (e.g., Henderson & Dancy, 2007; Hora & Ferrare, 2013; Hayward, Kogan, & Laursen, 2016; Johnson et al., 2013; Turpen, Dancy, & Henderson, 2016).

In this paper, our goal is to better understand the teaching practices of mathematicians and to begin to tease apart how both individual and situational characteristics relate to instructional decision-making. We conducted a national survey informed by findings from small-scale interview studies (e.g., Johnson et al., 2013; Roth McDuffe & Graeber, 2003) and from national surveys of undergraduate science educators (e.g., Henderson & Dancy, 2007). Our analysis of the responses provides a nuanced characterization of instruction, while also identifying individual/situational characteristics that are (not) associated with different instructional profiles.

## Literature Review and Theoretical Perspective

Case studies of mathematics instructors have uncovered multifaceted beliefs and goals that inform instructional practices (e.g. Johnson et al., 2013; Lew et al, 2016; Weber, 2004). As to be expected, beliefs about teaching and learning appear to be quiet varied. For instance, some are convinced of the benefit of non-lecture instruction as were the interview participants in a 2003 study who reported a desire to teach with “constructivist activities where the depth of knowledge is really greater” (Roth McDuffe & Graeber, p.336). On the other hand, there are those that are convinced of the strength of lecture, stating, “I believe students benefit from seeing education embodied in a master learner who teaches what she learned” (Burgan, 2006, p. 32). These beliefs do not exist in a vacuum and are likely informed by personal experiences – both as a student and a teacher – and influenced by activities such as attending workshops and conferences.

Regardless of how they developed, beliefs about teaching and learning alone do not appear to be deterministic of teaching style. For instance, in a previous report, we found that 64% of the respondents at research-focused institutions who think lecture is *not* the best way to teach lecture anyway (Johnson, Keller, Fukawa-Connelly, 2017). A common reason provided to explain the incongruity between how instructors want to teach and how they actually teach is a concern about content coverage (e.g., Johnson et al., 2013; Roth McDuffe & Graeber, 2003). Content concerns represent an interesting intersection of internal and external pressures. As discussed by Turpen, Dancy, and Henderson (2016), coverage concerns are influenced by personal beliefs about what should be included in the course, expectations from others in the department to cover a set curriculum, and more subtle indicators such as textbooks, their experiences as a student, and emotional attachments to topics. In this way, coverage concerns are to some extent self-imposed, yet attributed to external factors such as a common syllabus or need to prepare students for success in future courses. Such responsibilities to the discipline and the department illustrate that, while university instructors do control their own courses, “the influence of various external factors diminishes their perceived control over their teaching” (Lea & Callaghan, 2008, p.174). Similarly, time constraints are often cited as a strong factor influencing instructional practice. Turpen et al. (2016) noted interviewees discussing both “a broad sense of feeling overwhelmed with the responsibilities and demands on their time” and “specific aspects of their job description or institutional situation that led to them being stretched too thin” (p. 7).

This literature highlights the importance of instructors’ beliefs and experiences and institutional and departmental context on instructional decision-making – both of which are central to Henderson and Dancy’s (2007) model for predicting instructional behavior (Figure 1). This model provides a framework for considering instructional practices in light of the characteristics of both the instructor and his/her department. However, as acknowledged by Henderson and Dancy (2007), this is a “toy model” –

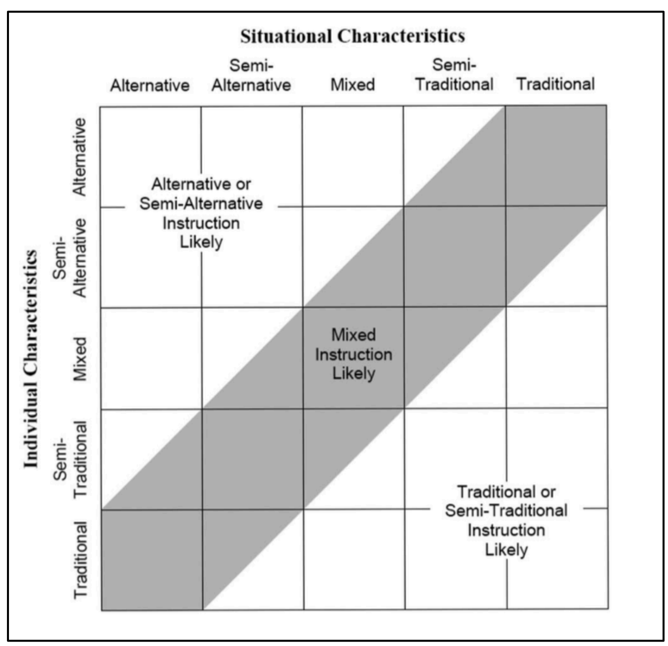


Figure 1. Toy Model

one that simplifies complex systems by highlighting dominant features, without detail about what is important within those domains. Better understanding the complex relationship between instructor characteristics (e.g., background, beliefs, knowledge, goals) and situational context (e.g., departmental norms, departmental supports, institution type) is the focus of this research. In this report we draw on data collected through a national survey of abstract algebra instructors in order to first characterize instruction and then to identify individual and situational factors that are associated with different teaching profiles. Specifically, we investigate three research questions: 1) What is the range and distribution of reported instructional practices in an upper division mathematics course (in this case, abstract algebra), as interpreted as traditional, mixed, and alternative instruction? 2) What individual characteristics (background, beliefs, knowledge and goals) are associated with instructors characterized as traditional, mixed, and alternative? 3) What situational characteristics are associated with instructors characterized as traditional, mixed, and alternative?

## Study Context, Data, and Methods

### Study Context

We focus on abstract algebra for the following three reasons. First, the major professional organizations have released a joint course-guide for abstract algebra calling for increased activity on the part of the students during the course meetings. Second, the research base in abstract algebra, including curricular innovations, is significant (at least in comparison to other proof-based courses). Finally, the course is often a small class taught by tenure-stream faculty. We have argued that these factors position abstract algebra, of all the required courses in the undergraduate mathematics plan of study, as the course best positioned to be taught with significant non-standard pedagogy.

For this report, we are drawing on two rounds of data collection, both of which used the same survey. In the first round of data collection the target population was instructors at universities that offer a graduate degree in mathematics. In the second round of data collection, we chose to target instructors at institutions not offering graduate degrees in mathematics. This decision was made upon considering the extant literature on teaching practices and observing that this research is primarily conducted by faculty at research-intensive universities about faculty teaching at such universities. Thus, even though the literature includes claims like: “lecture is overwhelmingly the dominant pedagogical technique both in terms of percentage of instructors claiming to use it and percentage of class time they report devoting to its use” (Fukawa-Connelly, Johnson, & Keller, 2016), it is possible that the lack of substantial representation of faculty at teaching-focused institutions may be a problem in terms of understanding the collected instructional practices of mathematicians in proof-based courses. For instance, the types of individuals who seek employment at research universities may have different beliefs about teaching and learning than their counterparts at teaching colleges. This, coupled with the disparate demands on time use, might influence the instructors’ willingness to adopt non-traditional pedagogies in different ways.

### Survey Design

The survey was designed to solicit information about the teaching practices, beliefs, and situational context of abstract algebra instructors. This survey was informed in part by both Henderson and Dancy’s physics-education survey (Henderson & Dancy, 2009) and the Characteristics of Successful Programs in College Calculus surveys. Our survey had sections to address: basic demographics and course context, teaching practices, beliefs and influences (including perceived supports and constraints), and knowledge of/openness to non-lecture practices. In the second round of data collection, the target population changed, but the information we were soliciting did not. For that reason, it was methodologically important that the items under investigation remain largely unmodified. While a few supplemental questions were added, the majority of the items were a subset from the previous survey with formatting intact. For the purposes of this paper, only those items unchanged by version were considered for analysis.

**Participants and Data**

The present data set is the result of two independent sampling attempts. The first data collection period was conducted in 2015. Survey requests were sent to departmental administrators at approximately 200 institutions. We received 126 responses, 91% of which represented instructors teaching abstract algebra at an institution offering at least a Master's degree in Mathematics. The second data collection period took place in summer 2016. In this follow-up, a random sample of 400 institutions was drawn from the IPEDS list, targeting specifically Bachelor's-granting schools. This sample yielded 112 responses, 91 of which were complete. For the purposes of this paper, all responses on applicable items have been combined into a single data set and disaggregated by instructional type for all future analysis. In total, 219 respondents were retained: 96 from Bachelor's-granting institutions, 44 from Master's-granting institutions, and 79 from PhD-granting institutions.

**Methods**

The purpose of this study was to describe the range and distribution of instructional practices as reported by abstract algebra instructors interpreted to be implementing a traditional, mixed, or alternative approach; and, furthermore, to investigate similarities/differences in the individual and situational factors influencing those practices. The characterization of instructional type was made using the prompt, *Please indicate the approximate percentage of class time that you are lecturing*, for which we coded the respondents as "Alternative" for responding *Never* or *0-25%*, "Mixed" for responding *25-50%* or *50-75%*, and "Traditional" for responding *75-100%*. This classification resulted in the following distribution of respondents: 17% Alternative (38/219), 57% Mixed (125/219), and 26% Traditional (56/219).

To address the first research question – the range and distribution of instructional practices - three survey items were analyzed. In each instance, the prompt instructed respondents to indicate the prevalence (instances per term / instances per class meeting / percentage of class time) of specific classroom activities/pedagogical practices utilized in their classrooms. To address the second research question – the specific individual factors characteristic of each instructional type – five survey items were analyzed. The first two items gathered demographic information on the teaching experience of the respondents and the latter three items polled respondents as to their beliefs about students, beliefs about teaching, and interest in professional activities as measured by a 4-point Likert scale. To address the third research question – situational factors characteristic of each instructional type – eight survey items, divided into two sets, were analyzed. The first were those that situated abstract algebra within the broader mathematics curriculum and the second subset were those intended to capture respondents' perceptions of departmental support for, and institutional constraints on, innovative teaching.

Group mean scores for each sub-item were computed by instructional type and compared using inference testing procedures such as ANOVA, Chi-square, or the Kruskal-Wallis test, as applicable to the data, with post-hoc testing for pairwise comparisons therein; within each item, the Holm-Bonferroni correction was applied to control for the family-wise error rate affiliated with multiple comparisons when appropriate.

## Results

### Research Question #1 – How can we conceptualize "traditional", "mixed", and "alternative" instruction in upper division mathematics courses?

In order to conceptualize the classroom experience of each instructional approach, the mean reported prevalence of a variety of pedagogical practices was computed and used as a means of comparison. Sorting respondents based on proportion of time spent lecturing highlighted 14 instructional practices that varied significantly between at least two of the three categories, with 8 practices being significantly different (family-wise error rate < .05) on all three pairwise comparisons. Considering reported teaching practices altogether generated these profiles:

- **Alternative** instruction is characterized by spare lecture, with class time split class time (fairly evenly) between showing students how to write proofs, having students work in small groups, having students give presentations, having students work individually, lecturing, holding whole class discussions, and having students explain their thinking. For alternative instruction, when compared to the other instruction profiles, it is less likely for instructors to pause and ask questions, use visual representations, diagrams, and informal explanations. Students in these classes are frequently asked to make presentations to the class and develop their own conjectures and proofs and are sometimes asked to develop their own definitions.
- **Mixed** instruction is characterized by moderate use of lecture lecturing, with significant class time devoted to showing students how to write specific proofs, pausing to ask students questions, and using diagrams, visual representations, and informal explanations to help students with formal ideas. Additionally, there is some class time devoted to students working alone and in small groups, giving presentations, and explaining their thinking. Students in these classes are pretty frequently asked to develop their own proofs, and are sometimes asked to present their work to the class and develop their own conjectures.
- **Traditional** instruction is characterized by heavy use of lecture. During lectures instructors report they are showing students how to write specific proofs and pausing to ask students questions. These lectures often include diagrams to illustrate ideas and informal explanations of formal statements, but are the least likely to discuss why material is useful and/or interesting amongst the three categories. Students in these courses are sometimes asked to develop their own conjectures or proofs.

We do not claim it is surprising that, with less time devoted to lecturing, Mixed and Alternative instructors are spending more time engaging students in mathematical activity (e.g., developing proofs) and in peer-to-peer activity (e.g., working in small groups and giving presentations). Rather, we offer these results to justify using the amount of time spent lecturing as a viable means for differentiating instructors as Traditional, Mixed, and Alternative.

### Research Question #2 - What individual characteristics are associated with instructors that report traditional, mixed, and alternative instruction?

Keeping in mind the goal of elaborating on the "toy model" presented in Figure 1, our investigation of research question #2 has yielded information that draws distinctions on individual characteristics between types of instructors. In particular, at least one significant pairwise comparison existed for 12 of the 17 items under consideration: teaching experience

(2/2), beliefs about teaching and learning (7/10), and interest in various types of scholarly activities (3/5). Traditional instructors (i.e., those reporting lecturing more than 75% of class time) are the most experienced group, hold the strongest beliefs in favor of the appropriateness or necessity of lecture and the most pessimistic views on their students' abilities, and have a stronger interest in mathematical research than educational research. Conversely, alternative instructors (i.e., those reporting lecturing less than 25% of class time) tend to be the least experienced, hold the strongest beliefs in favor of non-lecture activities and the most optimistic views on their students' abilities, and prefer research in teaching and learning to that of abstract algebra. For all items, the Traditional and Alternative groups always occupied the extreme positions on the continuum with the Mixed group in between. This provides further evidence to suggest that separating instructors based on a single characteristic (i.e. proportion of class time lecturing) does result in meaningful categorizations.

**Research Question #3 - What situational characteristics are associated with instructors that report traditional, mixed, and alternative instruction?**

Our investigation of research question #3 failed to provide distinctions on situational characteristics between types of instructors, at least as we have defined them. At least one significant pairwise comparison existed for only 2 of the 8 items under consideration: institution type and time pressure. When considering the distribution of institution type by instructional approach, we found that Traditional instructors are significantly more likely to reside at a PhD-granting institution than either the Mixed ($p < .001$) or Alternative ($p = .001$) groups. Nearly 60% of Traditional instructors are at PhD-granting institutions, whereas only 18% of Alternative instructors are. Furthermore, while we can see that the modal class for all institution types is Mixed instruction, we do see a gradual rise in incidence of Traditional instruction as the terminal degree escalates from Bachelor's (12.5%) to Master's (25%) to Doctorate (41.77%). Collectively, these results indicate a dependency between institution type and instructional approach.

When considering abstract algebra in the broader mathematics curriculum, we found little to suggest that the instructors in the various groups experienced different departmental circumstances. The distribution of responses on both the prerequisite course and follow-up course items revealed some interesting trends, but the lack of statistical significance indicated that these items are likely independent of instructional approach. Alternative instructors tend to be the most likely to work in a department to require a proof-based prerequisite (79%) and the least likely to work in one that offers a subsequent algebra course (58%). Additionally, we observed that the requirement of a proof-writing prerequisite seems to be inversely related to the proportion of time spent lecturing (Alternative > Mixed > Traditional) and that the existence of a follow-up algebra course appears to be directly related (Alternative < Mixed < Traditional); however, there were no significant pairwise comparisons in either case.

The second sub-set of items analyzed included those questions intended to capture respondents' perceptions of departmental support for, and institutional constraints on, innovative teaching. These items focused on departmental expectations and content pressure, the availability of time for teaching and course redesign, travel support for professional development, and freedom to make changes to their abstract algebra course. We found that, not only were there no statistically significant differences for many of these items, the distributions were nearly identical. The lone exception being the question: *Do you feel like your job requirements allow you to spend as much time as you would like on teaching…?* Here we see that about 70% of

Alternative and Traditional instructors responded in the affirmative, whereas slightly less than half of the Mixed instructors felt that way.

## Discussion

The goal of this research study was to investigate the range and distribution of reported instructional practices in abstract algebra instruction and how different individual and situational characteristics are associated with instructors who report different types of instructional practices (see Figure 2). This analysis has allowed us to provide further insight into the “toy model” that Henderson and Dancy (2007) developed. In particular, while prior research suggested the importance of a variety of individual and situational characteristics, our work suggests that a relatively small collection of individual characteristics may be actually be the most important for teaching practices, at least in terms of the broad-strokes characterization of instruction we use here.

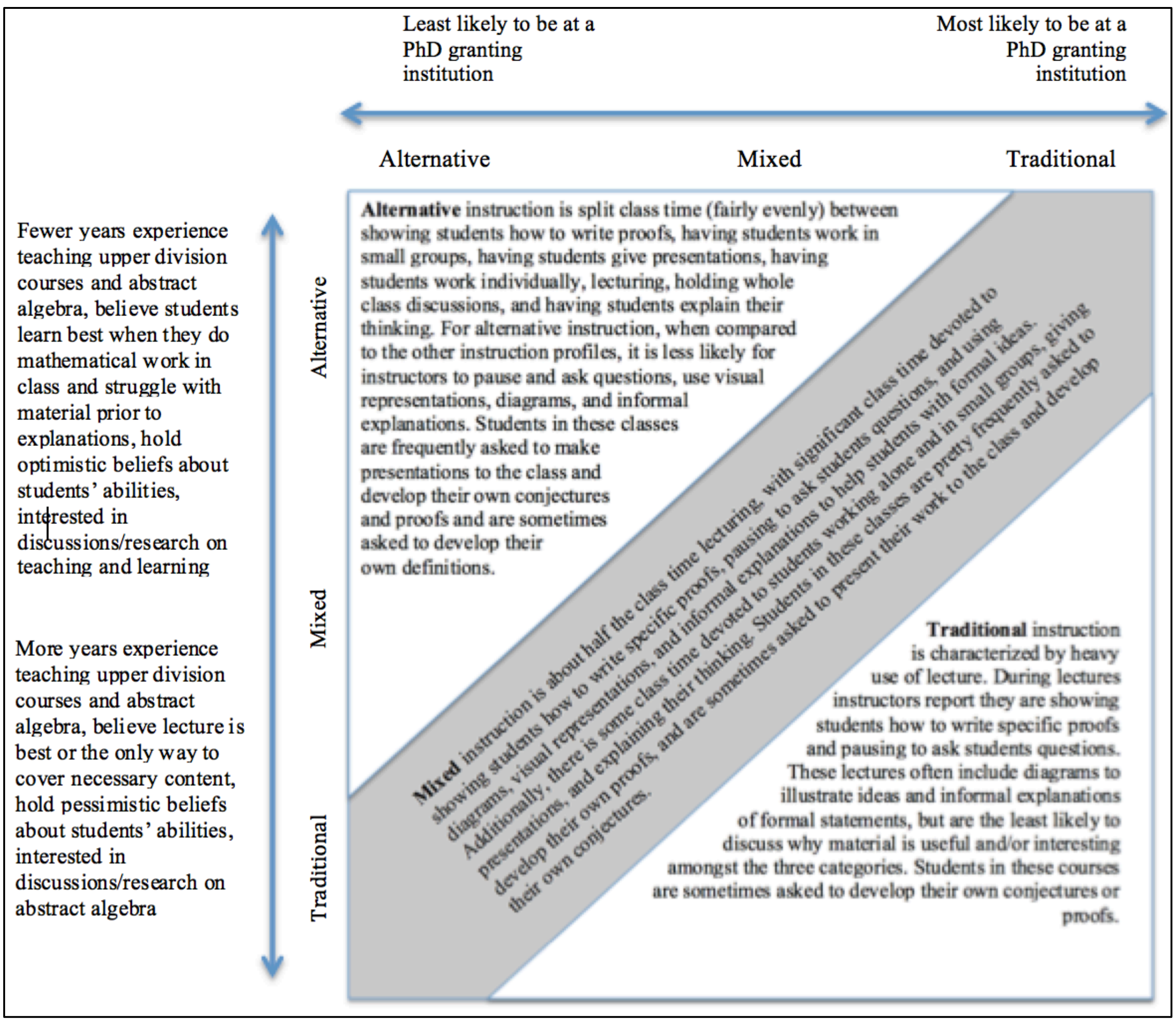


Figure 2. Individual and Situational Factors and Instruction Profiles

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# Graphing as a Tool for Exploring Students' Affective Experience as Mathematics Learners

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*Researching affective issues can be difficult in education; methods like interviews and surveys can place artificial categories on participants' experience and exert biased influence. This lack of tools to study affect calls for better methods. We explore graphing as a potential tool with affordances for studying affect, by reporting results of three separate studies at different timescales where undergraduates graphed affective phenomena like confidence or emotion: two in an introduction to proof course and one in a pre-service teacher content course. By systematically describing each study and looking across the three, we argue that graphing can be a useful technique for representing experience. Its utility lies in aligning research goals with the structure imposed by the temporal axis. More structure along the temporal axis allows researchers access to what a student experiences at predetermined temporal points and less structure allows access to what students themselves find to be salient events.*

*Keywords:* Methodology, Affect, Introduction to Proof, Preservice Teachers

Research is beginning to appreciate the deep importance of affect in the experience of learning mathematics (Ainley, 2006; McLeod, 1992). However, despite this increased recognition of the importance of affect, the field lacks methodological tools to investigate students' non-cognitive, affective, emotional experience during cognitive activity. This methodological paper reports on the approach of affect graphing during learning experiences (building upon the work of McLeod, Craviotto, & Ortega, 1990; Smith & Star, 2007; and Smith, Levin, Bae, Satyam, & Voogt, 2017). We explore the use of affect graphing across three recent studies within undergraduate mathematics education (two studies situated in the context of an introduction to proof course and one study situated in the context of a number and operations content course for pre-service elementary teachers). Particularly notable is how different time scales were engaged in each context: the scale of reflection on work on a single problem, a single class discussion, and finally the scale of reflection was an entire course.

Interviews remain a commonly used method that takes as its object of inquiry the experience of the individual. However, studying affect on the sole basis of verbal protocols is problematic. For example, while interviewers can prompt a subject to report how they are feeling in the moment, one needs to be aware that is an intervention in the experience and may change or shape the perception of the experience. Asking subjects to report their experience in a completely open way can lead to subjects focusing in on very particular moments and not supporting reflection across an entire time interval of interest to the researcher. Lastly, interviews are dependent on interviewees being able to articulate their emotions and feelings in words. Depending on the amount of direction given, participants may need to interpret and respond to categories given to them as opposed to describing their own experience and its ebb and flow in their own terms.

A second potential contrasting approach to studying affect involves surveying participants about their affective experience. Positive implications of such an approach include the ability to generate a larger volume of data with prompts that serve as proxies for experience, beliefs, and participant feelings during problem solving. However, such methods are less responsive to participants' own categories of experience, forcing subjects to again fit their experience into the pre-conceived categories of the researcher. Surveys are also conducted in a way that does not allow for the temporal, moment-by-moment recording of an experience.

While the above discussion of contrast methods is not meant to suggest that adaptations of such methods cannot ameliorate some of the constraints of those methods of data collection, it is meant to point out that other methods (such as the graphical approaches we discuss here) may have advantages in addressing such questions. The field needs better tools for studying affect.

## Framework

This paper differs from the typical empirical report in that we explore the affordances of an innovative methodology. The goal of our paper then is to analyze the ways in which graphing was productive. We therefore present a framework for how we describe each context and compare them to each other. For each context, we provide (1) a *description* of the overall study, (2) the *purpose* of graphing as a tool in this context, (3) what the graph *measured*, (4) *features* of the graph such as timescale, axes, labels, (5) *results* from analyses and suggestion for potential analyses that could be done, and finally (6), a prototypical *example*.

## The Three Contexts

We describe three separate studies in which the approach of graphing was used. Two of the contexts were research projects and one was from a course, as seen in Table 1.

*Table 1. Features of the Three Studies Using Graphing as a Methodological Tool.*

| Course | Population | Axes | Graph Measures |
|---|---|---|---|
| Intro to Proof | Math majors and minors | 1st quadrant | Students' confidence |
| Numbers & Ops | Pre-service elementary teachers | 1st quadrant | Students' confidence |
| Intro to Proof | Math majors and minors | 1st & 4th quadrant | Students' emotion |

The first was a study of undergraduates' confidence reflecting on a completed intro to proof course. The second project examined elementary pre-service teachers' confidence levels in reaction to a class discussion in a numbers and operations course. The third context was also in the intro to proof course but focusing on students' emotions while working on a proof. Each of these studies was conducted by different subsets of the authors of this paper.

We have chosen to order contexts by timescale, from broadest to shortest. We do this to make salient the variations in how the graph was used when the time scale shrinks. In addition, the first and second contexts use graphing to measure the same construct (confidence), while the last context measures emotion; emotion may include confidence but can be multi-dimensional.

### Context 1: Introduction to Proof Undergraduates Graphing Confidence over a Semester

The focus of the first study in which we explore the use of graphing was understanding undergraduates' experience in an introduction to proof course. The population ($N = 14$) consisted of math majors and minors who had just completed the introduction to proof course. In a prior study, Smith et al., (2017) interviewed them about their view of the nature of the course in contrast to past math courses, their sense of success, and how their view of and work on proof tasks may have changed over the course.

The researchers also wanted to tap into the affective dimension of experience – how they felt at different points – so they asked students to graph their confidence in the course across the semester. Interviewers left the room while students drew their graph and when they returned, asked students to talk through their graph. The x-axis measured time, from before the semester started (to account for expectations of the class prior to its start) to right after the final exam. There were otherwise no fixed tick marks along the x-axis, because of the interest in not only how students' confidence changed but *what* influenced shifts in confidence. The y-axis measured confidence. Tick marks on the y-axis were given, corresponding to low, medium, and high confidence. The majority of students drew continuous graphs. The task was kept relatively open to allow students to represent their experience however they chose to, away from our judgment of what may be important milestones in the course.

One analysis is to categorize the *shape* of the graph, to identify common patterns of confidence over the semester. We found five categories of shapes, with two graphs as outliers to categorization. The most common ($n = 4$) shape for confidence was a "W" shape: initial high confidence in the course with a quick drop early in the semester, then an increase over time, followed by a decrease and then a final increase to the end of the course (see Figure 1).

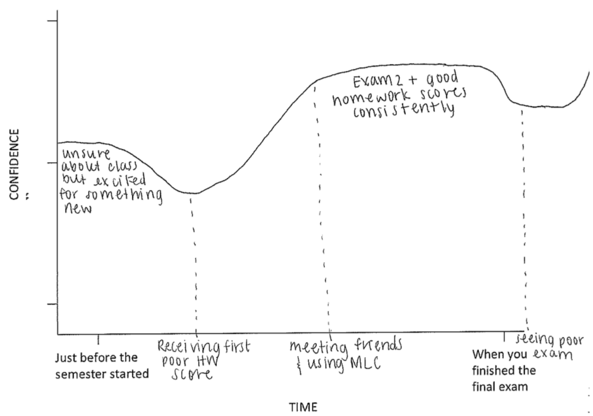


*Figure 1.* Example of the "W" shape, the most common pattern for confidence over time. This figure also shows the kinds of x-axis markers students included as places where their confidence shifted.

The other four shapes in our set of 14 graphs were: (a) continuous increase, (b) concave up parabolic shaped graph, (c) initial increase followed by a sinusoidal wave for the rest of the semester, (d) initial increase followed by decrease, with a final confidence level that was lower than their initial level. The W pattern as the most frequent makes sense given the different nature of a proof-based work and the introduction to advanced mathematics (analysis, linear algebra, number theory) after the half-way point of the course.

Graphing was insightful here in that it gave students a vehicle through which to reflect across an entire semester (half a year). The act of drawing the graph served as a way of recalling and organizing how they felt, in a way that localized interview questions did not capture. In addition, the openness of the x-axis meant students could tell us what events corresponded with rises and falls in their confidence, as opposed to our assumptions that it would revolve around exams for example. With this, we could identify the events that stood out as pivotal moments, relative to the entire experience.

**Context 2: Pre-service Teachers Graphing Confidence over a Class Period**

The second context that used graphing involved a study of pre-service teachers' response to an orchestrated classroom discussion. Two of us have been involved in developing and revising a one-semester course on number and operations for pre-service teachers with an emphasis on justification, specifically on developing our prospective elementary teachers' (PTs') abilities to analyze and critique the work of others. These goals are not easily achieved, as there are issues that arise in orchestrating such class discussions, particularly those that capitalize on incorrect patterns of reasoning that PTs themselves may generate (e.g., Chamberlin, 2005; Silver, Ghousseini, Gosen, Charalambous, Strawhun, 2005). Toward this end we have been developing a strategy for orchestrating discussions for enabling PTs to consider and analyze divergent thinking on mathematical tasks.

The strategy has 4 main stages: (1) Engagement: PTs generate their own ideas for a solution; (2) Interruption, juxtaposition, and re-focusing: The teacher posts two different answers and asks PTs to determine a solution path that would lead to each answer; (3) Articulating the reasoning of another: PTs present ideas for the chain of thinking that led to an answer. This discussion focuses on understanding the reasoning (without bias), along with establishing common ground; (4) Validity: PTs consider how to determine the validity of one approach (and thus why the other approach is not valid). The goal of this orchestration was to destabilize PTs' thinking. Prior analysis of class videotapes indicated this approach's efficacy, but we wanted tools for tracking individual PT thinking at key stages of the orchestration strategy, along with evidence of their understanding at the conclusion of the activity. Graphing was used for this purpose.

Immediately following the activity, PTs were asked to rate their confidence level in their own thinking at each stage on a 5-point scale from low to high, thus creating a "confidence graph" for the activity. At the end of the activity, students were asked to explain the valid strategy in a way that would help someone with the invalid strategy understand why it was invalid. Our data suggest that the orchestration strategy used in this case was successful, both in establishing cognitive dissonance, and also, importantly, in allowing students to come to resolution.

The main patterns in student confidence graphs were the same across two different classrooms with different instructors: the lowest confidence level was at Stage 2b (2.7 out of 5, or 54%) and the highest confidence level was at the end of the activity (4.7 out of 5, or 95%). With respect to the overall shape of PTs confidence graphs, the majority of the students (83%) had at least one point in the activity at which their confidence took a downward turn (indicating some degree of destabilization in their thinking). Analysis of the written student work that accompanied the generation of the confidence graphs, in conjunction with students' professed levels of confidence at the end of the activity, indicated that students came away from the activity with a heightened understanding of the mathematical content of the activity.

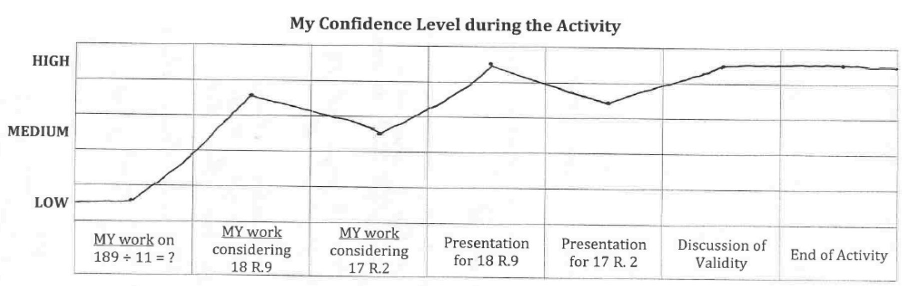


*Figure 2.* Sample confidence graph produced by a student reflecting on their confidence over the course of the class discussion of 189 divided by 11.

Collecting PT's reflections on their affect across the discussion allowed us to capture data that would be difficult to get from other methods like class observation and videotape. Though video reveals an "impression" that at least some of the PTs were deeply engaged in discussions, the confidence graph activity gives researchers (and teachers) a tool for measuring where the entire class is in their understanding and how this shifts over the course of the discussion.

**Context 3: Introduction to Proof Undergrads Graphing Emotion over a Single Problem**

The third study we discuss tracked students' emotions while working on a proof construction task. N = 11 undergraduate students (some math majors, some not) were interviewed four times across the semester, while taking an intro to proof course. In each interview, they were given two proofs to work on for a maximum of 15 minutes each, which were picked intentionally to be challenging, hence problems. Students were encouraged to "think-aloud" while working. After each task, students were asked to describe their process, choose from a given set of emotion words to describe what they felt and then draw a graph of their emotions during that problem.

Graphing as a technique was chosen (a) as a talking aid, to help students articulate their emotions, which in general can be difficult and (b) to succinctly compare patterns of emotion across participants on the same problem and that of a single participant within a problem. The level of intensity in how they felt at various points could be better seen and compared visually.

The graph measured emotion, signed (positive or negative) intensity without specifying the exact emotion. The x-axis was time, from when the student started working on the problem to when they stopped. The y-axis represented emotion, with a tick mark above the x-axis denoting positive emotions (e. g. satisfaction or excitement), a tick mark below the x-axis denoting negative emotions (e. g. frustration or panic), and the axis itself being neutral with no particular emotion, i.e. one's "resting state." All students drew continuous graphs, a line graph across the page. They also marked on the graph reason(s) why emotions shifted.

In sorting the graphs, the analysis revealed 6 general profiles of graphs: (1) overall positive, (2) overall negative, (3) flat, (4) concave down, (5) concave up, and (6) other. Of the 88 graphs, 40% were concave up, 18% were overall positive, 13% were overall negative, 10% were flat, 8% were concave down, and 11% fell into the other category. Concave up graphs suggested that the student overcame struggle(s), whereas overall positive graphs had no issues impacting emotion. Flat graphs showed experiences with little variation in emotion, whether completely flat and above the x-axis because the problem was easy, or right at the x-axis because the student stayed unsure the entire time. Concave down graphs were experiences that started well but where the student got stuck and could not resolve it.

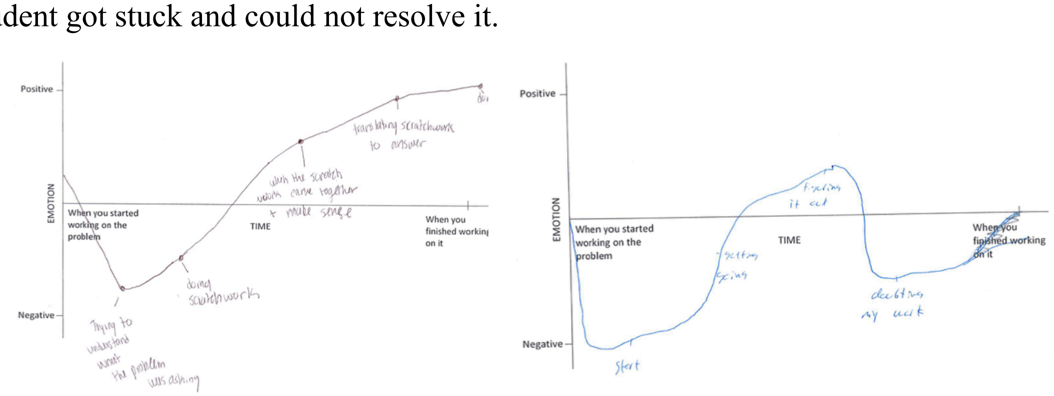


*Figure 3. (Left)* Prototypical example of concave up, the most common graph type. *(Right)* An example of a graph in the Other category, showing rises and falls in emotion.

The other category consisted of graphs with many changes in emotion (such as a “W” shape), large rises and falls, and states of confusion. This other category is a collection of volatile problem solving experiences, due to the size and number of rises and falls in emotion.
The results showed that there were a number of experiences where students successfully worked past a struggle. The graphs were useful in quickly and visually identifying whether students engaged in problem solving behavior - the existence of a struggle.

The choice to collect graphs at multiple timestamps for each student allows for temporal analyses as well. For example, one could look at how students’ emotions while problem solving change over time, as seen in Figure 4.

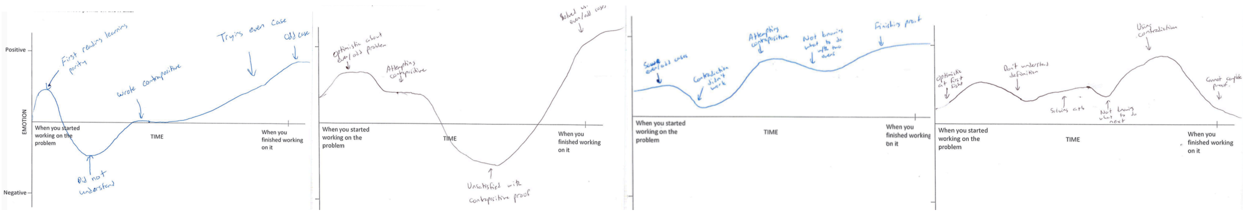


*Figure 4*. Graphs of emotions for one participant over 4 points in time.

Overall, graphing was most useful as a way for students to communicate their problem solving experience in a temporal fashion. Like in the first context, keeping the x-axis unstructured meant students talked about (and annotated on the graph) the events that caused their emotions to shift. The identification of these events and how students interpreted them, and how this changed over multiple points in time especially, was valuable.

## Discussion

We now turn to comparing the use of this new analytic tool across the three contexts and point to directions for future use of the tool. We focus our discussion on the affordances of this tool across the three contexts.

In all three contexts, graphing an aspect of affective experience during learning and problem solving (e.g., confidence, emotion) was used to glean information that would have been challenging to gather using traditional methods such as verbal interview, video of problem solving or class discussion, or survey methods. In all cases, we considered the affective variable over time and assumed that the experience (problem solving, taking part in a class discussion, engagement in a course) influenced the graph that was produced. All three contexts ask participants to reflect on their experience and represent it graphically. As discussed earlier, the produced graphs tap into the utility of this approach for understanding participants’ experience of events of differing time scales: work on a single problem, engagement in a class discussion over an entire period, participation in a course over a semester.

The presentation of the studies and results where this tool was used demonstrates the wide applicability of this methodological tool. While we used the tool with both undergraduate mathematics majors and with pre-service elementary teachers, we can envision that this tool would be possible to use with an even broader range of participants. Mathematics majors were more familiar with graphing and more able to interpret and adapt the tool (e.g., including their own points of salience along the x axis). For the pre-service teacher study, we included more structure along the x axis and grid lines to allow participants to either create a continuous graph or to simply mark whether they had high, medium, or low affect at each of the pre-specified points within the discussion. Many students chose this way of interacting with the given template, creating a bar graph as opposed to a line graph. From a theoretical standpoint, we see that the act of drawing the graph serves as a form of *rendering* for the student, i.e. making sense of an experience. It locates but also provides temporal structuring, allowing students to organize their experience temporally, which helps them communicate their experience to us.

At a general level, there are several affordances to this approach. The method is easy to administer. Participants have a range of agency in terms of what they draw and how (completely open in the case of undergraduate math majors; more structured in case of PSTs). A methodologically attractive feature of the graphs produced is that they capture the students' reflection on their affect over the entirety of the experience as opposed to the interviewee focusing in on one particular part of the experience that was more salient to them. For this reason, giving participants the opportunity to explain their written graphs can allow researchers to elicit data on the relationships between participants' affect at different points in time over the entire experience and also what parts of the experience were most salient. The graphing activity encouraged a negotiation in the representation of particular focal experiences/feelings and a global sense of the experience. Having participants discuss their graph also gave insight into participants' views of the driving forces or reasons behind shifts in confidence or affect. The shared artifact to talk over seemed to help participants craft a narrative not only of how their experience shifted but what was behind those shifts.

While there are numerous affordances of the method, the approach, like any qualitative approach focuses on self-report. However, because our interest is in participants' models of their own experience, it is less critical for us to judge whether or not participants' confidence or emotions actually did increase or decrease in the ways they reported. The important data for us is participants' perception of their own experience, captured very well by the graph. One constraint with capturing data on participants' perception of their experience is that the most vivid data about participants' perceptions comes as close as possible to the experience itself. In the study of problem solving and the study of students' experience of the classroom discussion, the reflections took place immediately after the experience. We felt this was the "best possible" timing so that participants would not be simultaneously reflecting on their experience while also engaging in the focal task. The post-hoc interviews of confidence over the experience of the course were more challenging in this respect because, necessarily, more time had passed between the experience and the participants' reflection on it.

Graphing as a tool has implications for other purposes besides research too. While we focus here on the use of graphing as a research tool, it also works as an in-class tool, as a form of formative assessment. It can function as a support for student reflection or for teachers to check-in with students, as was done in the pre-service teacher context here. We believe graphing is useful for other populations of students also. We focused on the use of this tool with college populations and admittedly, there is reason to believe it is especially useful there because they undergraduates are familiar with graphing as an activity. However, a more structured approach like in the PST context could translate well to less mathematically sophisticated populations.

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Conceptual Blending: The Case of the Sierpinski Triangle Area and Perimeter

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*In this report, we present an analysis of 10 individual interviews with graduate mathematics education students about the area and perimeter of the Sierpinski triangle (ST). We use conceptual blending as a theoretical and methodological tool for analyzing students' reasoning to investigate how students encounter and cope with the ST having zero area and infinite perimeter. Our analysis documents the diverse ways in which the students reasoned about the situation. Results suggest that conceptualizing an infinite perimeter is more accessible to these students than is zero area, that encountering the paradox is dependent on how blends are composed, and that resolution of the paradox comes through completion and elaboration. The analysis furthers the theoretical/methodological framing of conceptual blending as a useful tool for revealing the structure and process of student reasoning.*

*Keywords*: Conceptual blending, Infinite processes, Fractal, Paradox, Student thinking

*It's still hard for me to wrap my mind around the Sierpinski triangle, and that there's infinite perimeter and no area. It makes sense to me individually, but both together at once, I'm still, it's still mind-boggling. – Carmen, graduate mathematics education student*

Straightforward notions of the area and perimeter of geometric shapes are first learned in elementary school, and are revisited and leveraged throughout middle and high school. When dealing with fractals, however, some counter-intuitive situations involving these ideas arise. One such situation, a region with zero area and an infinitely long perimeter, was encountered by a class of mathematics education master's degree students in a chaos and fractals course when investigating the Sierpinski Triangle (ST) shown in Figure 1. As seen in Carmen's introductory quote, this was a non-trivial exercise and caused some students serious consternation.

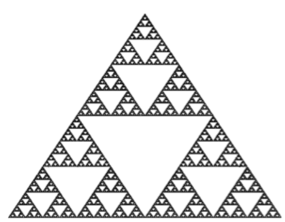

*Figure 1. The sixth step in creating the Sierpinski Triangle.*

To investigate student reasoning about the ST we conducted individual interviews about three weeks after its in-class investigation. Based on the interview data, and using the ideas of conceptual blending, we address the following two related research questions: (1) How do students make sense of (a) area and (b) perimeter of the ST? (2) How do students coordinate the area and perimeter of the ST and cope with the resulting paradoxical situation?

## Theoretical Background

We use conceptual blending theory (Fauconnier & Turner, 2002; Núñez, 2005) as a theoretical and methodological tool for analyzing students' coordination of two infinite processes, one increasing (perimeter) and one decreasing (area). Blending is based on the notion of mental spaces, which are "small conceptual packets constructed as we think and talk, for the purposes of local understanding and action" (p. 40). According to the theory, these mental spaces "organize the processes that take place behind the scenes as we think and talk" (p. 51). Conceptual blending is defined as the conceptual integration of two or more mental spaces to produce a new, *blended,* mental space. An important feature of this new blended space is that it develops an emergent structure that is not explicit in either of the input mental spaces. This emergent structure is generated by three processes: composition, completion, and elaboration.

*Composition* is the selective projection of elements from input spaces into a common space. During composition, distinct elements may be projected on top of each other or *fused,* and common elements may be projected separately. The composition process develops a new space, with the potential for structure not available in either input space. *Completion* is the process of recruiting familiar frames to the blended space, along with their entailments. That is, an individual recognizes certain aspects of a blended space as parts of a familiar frame and brings in additional knowledge, scripts, assumptions, etc., to complete the frame and prescribe structure for the blended space. These frames can serve as tools for *elaboration*, which is sometimes called *running the blend*. Elaboration is the process that leads to the emergence of something new within the blended space, using the tools of the completion process and the elements that compose the blend. These processes, composition, completion, and elaboration, do not necessarily take place sequentially (Fauconnier & Turner, 2002).

Underlying our analyses is our knowledge of previous research related to conceptual blending in other contexts (e.g., Lakoff & Núñez, 2000; Yoon, Thomas, & Dreyfus, 2011; Zandieh, Roh, Knapp, 2011), infinity (e.g., Ely, 2011; Fischbein, Tirosh, & Hess, 1979; Núñez, 2005), paradox (e.g., Dubinsky, Weller, McDonald, & Brown, 2005ab; Sacristán, 2001; Wijeratne & Zazkis, 2015). A review of these works is beyond the scope of this report, but we acknowledge the impact of this prior work for our own and note that our work is some of the first to bring together all these ideas.

## Methods

The study took place in a graduate level mathematics course of 11 mathematics education students (10 of whom agreed to participate in individual interviews). The course was taught by one of the research team members. Students sat in four groups and daily worked on tasks in their small groups and engaged in whole-class discussions of these same tasks. Data was collected as part of a larger study and included video-recordings of each class session, individual task-based interviews conducted at the middle and end of the semester, and copies of all student work.

The focus of the analysis in this paper are students' responses to the following question from the mid-semester interview: *In class, we discussed the Sierpinski Triangle. How do you think about what happens to the perimeter and the area of the ST as the number of iterations tends to infinity?* This question was accompanied by a printout of the ST (as seen in Figure 1), with a follow-up prompt to tell us what they thought about the following claim of a fictitious student, Fred: "The computation shows that the perimeter goes to infinity because the perimeter is given by $3(3/2)^n$ which increases to infinity as n tends to infinity. But, the perimeter can't really be infinitely long, because there is nothing left to draw a perimeter around, since the area goes to zero." This interview task was designed based on the classroom discussion of the ST, which took place two weeks before we began interviewing students. At that time, students seemed to agree that the area went to zero but were unsure of what happened to the perimeter. They publicly considered the possibilities that it went to infinity, converged to some value, or did not exist because there was nothing left for a perimeter to go around. The interview was structured so that we would first gain insight into the students' reasoning about the area and perimeter of the ST, followed by an opportunity for them to respond to Fred's claim.

To identify a student's input space for area (similarly for perimeter), we first marked which of their utterances were about the area. Next, we categorized these utterances into sets of ideas about the area of the ST - including the process by which it is created and the resulting product. In the spirit of grounded theory (Strauss & Corbin, 1998), these ideas were coded and compared iteratively until a coherent set of idea codes emerged. The interviews were divided into two groups and analyzed by different members of the research team. These analyses were then swapped, compared, and vetted.

We investigated students' blending by identifying each of the three processes: composition, elaboration, and completion. To see how a student's blend was composed, we identified which elements of the student's input spaces were brought up as they considered the coordination of area and perimeter (prompted by Fred's paradox). We identified the ways students elaborated their blended spaces by identifying ideas which were not in the input spaces, but emerged as they worked to make sense of the task. Interpretation of completion and elaboration was done first as a group, with all four authors debating each point, then a more detailed pass was made by two members of the team in close comparison with the transcripts, and these analyses were then discussed again among the four authors until agreement was reached.

## Sample Results

During the in-class discussions about the ST there was widespread agreement that the area would go to zero but less agreement that the perimeter would diverge to infinity. We were therefore surprised to find that only six of the ten students concluded that the area of the ST goes to zero but all ten students concluded that the perimeter tends to infinity.

### Area and Perimeter (Research Question 1)

Among students' justifications for their conclusions, we identified seven qualitatively different mental space elements for area and seven qualitatively different mental space elements for perimeter. As a sample we display three of the most prominent different mental space elements side-by-side, with descriptions of the elements and illustrative quotes.

| **Area** | **Perimeter** |
| --- | --- |
| <u>Infinite decreasing process</u><br>Common among all 10 students was the element that area is the result of an infinite, decreasing process. For example:<br><br>Carmen: *So, ok eventually the area gets to zero, but that's if you could do it infinitely many times. And if you actually* | <u>Infinite increasing process</u><br>All 10 students conceived of the perimeter of the ST as the result of an infinite increasing process. Elise's reasoning is typical of this thinking:<br><br>Elise: *You're just like forever adding length to your perimeter, so I feel like your perimeter is forever increasing.* |

| | |
|---|---|
| *conceptualize doing infinitely many times you're never gonna stop.* | |
| Area removed at each step<br>All students except Curtis there was explicit use of the justification that area is removed at each step. Two students computed the first few steps during their interviews.<br><br>Kay: *We're always taking out the middle triangle of each equilateral triangles and we're doing that infinitely so it's like we're taking away area with each iteration.* | Perimeter is added at each step<br>All students except Curtis also pointed to the fact that perimeter is added at each step. Four students accompanied this with computation for the first few iterations.<br><br>Joy: *I think it goes towards infinity because each iteration you're creating more triangles and so you're creating, you're adding to the perimeter.* |
| Change in the rate of change<br>Shani and Kay, who were in the same group, were the only two students who concluded that the area tended to something non-zero. They were also the only two who shared what we refer to as the change in the rate of change for area element, as exemplified in this excerpt.<br><br>Shani: *As we keep taking off little pieces and more become white, it's getting smaller and smaller. Or the amount that it's increasing is getting smaller and smaller and smaller.* | Change in the rate of change<br>Two other students, Elise and Carmen, gave some consideration to the rate at which the perimeter increases and to changes in this rate. For example, Elise argued that<br><br>Elise: *Every time after the first iteration I'm adding more perimeter than I added before. So if I keep adding more then I think it's going to keep going to infinity because I'm just going to keep adding bigger and bigger.* |

**Discussion**. Other reasoning about area and perimeter included reasoning multiplicatively, reasoning about congruent figures, reasoning with geometric series and associated convergence or divergence criteria, and thinking of the ST being composed of leftover or removed pieces. Students primarily made sense of the area and perimeter of the ST as infinite iterative processes. This is not surprising given the construction process students were introduced to in class. What *did* surprise is the fact that, except for Curtis, students used informal additive reasoning to reach their conclusions. The few students who did some computations did so only for the first few iterations and did not generalize the adding of perimeter or removal of area into algebraic expressions from which to take limits. While some students used limit language or referred to convergence criteria, it was not done concretely, despite their mathematics experience.

Given students' informal ways of reasoning, the parallelism between area and perimeter ideas is noteworthy. Each element of reasoning about area had a corresponding element of reasoning about perimeter. While some of these ideas were common (infinite processes, adding area, removing perimeter), others were not. In several cases students' idiosyncratic ways of thinking were consistent within students across area and perimeter. Despite the idiosyncrasies, there was quite a lot of consistency in ways of reasoning across students, both with respect to area and with respect to perimeter.

**Blending Area and Perimeter (Research Question 2)**

One element appears in every student's blended space which did not appear in the area/perimeter section: *infinite creation process*. This element is a result of fusion, wherein two input space elements (here, infinite increasing and infinite decreasing) are projected onto one element. As students were introduced to the ST as something created through an iterative, recursive process affecting both area and perimeter, in a sense the students are re-fusing elements which they originally separated. To organize these ideas, a three-part diagram is used: rectangles represent mental spaces, with the upper rectangles representing the input mental spaces and the lower rectangle representing the blended mental space, and the lines show mappings between the spaces. Due to space constraints, we present only four students and two blending diagrams.

**Joy.** We gained access to Joy's blending process primarily through her response to Fred's argument. Her blended space was composed of the infinite process of creating the Sierpinski Triangle, the area tending to zero, and perimeter tending to infinity. Completion brought into the blended space a metaphor of *perimeter as fence*, along with several entailments. One such entailment is that fences should remain, even if the space they enclose is no longer there. Part of Joy's elaboration based on this frame, as she worked to resolve Fred's paradox, was to say that "we don't count their space, but there is still a perimeter associated with it." Another entailment of the fence framing is that not only do fences have length, but they also take up space. This contributed to another element of Joy's elaboration, that the perimeter will fill in the Sierpinski Triangle, "so eventually in a sense it's all fence." Some parts of Joy's elaboration are grounded in a physical metaphor, and she recognizes this when responding to Fred. She adds to her elaboration that the Sierpinski Triangle is "not a real object," and identifies the juxtaposition of an infinite mathematical process with the physical world as "where the disconnect is."

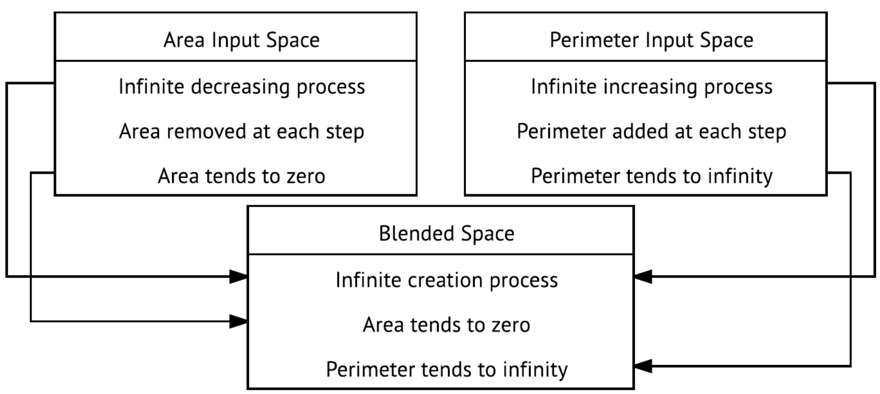


*Figure 2. Blending diagram for Joy's reasoning*

**Elise.** Like Joy, Elise's blended space is composed of the infinite process of creation for the ST, perimeter tending to infinity, and area tending to zero. However, the framing metaphor that completes Elise's space is one of a skeleton, not a fence. She elaborated her blend, saying, "I'm thinking of our perimeter as like, like I guess I think at the end of this I have this skeleton, so I have no area, nothing is left inside" This skeleton metaphor brings with it entailments of bones remaining when flesh has gone, clearly mapping perimeter to bones and area to flesh. In addition, we note that Elise mentions "at the end" in her elaboration, perhaps hinting that she sees the ST as an abstract object at the end of a generating process.

**Curtis.** As with Elise and Joy, Curtis's blended space is composed of an infinite creation process, perimeter tending to infinity, and area tending to zero. Unique to his blended space, however, is his formal, multiplicative formulation of area and perimeter as the limits of infinite sequences. The completion process brings in a zooming frame, saying, "we could say you could zoom in for infinitely, as much as you want, and you could get like these as tiny and tiny as you want, there's still more perimeter to draw" when prompted with Fred's paradox. The second frame we see Curtis leverage is one related to mathematics classes (e.g., Calculus, Analysis) where symbolic manipulations are sufficient. Evidence of this comes from the fact that Curtis did not encounter a paradox when considering an object with zero area and an infinite perimeter on his own, something he elaborated by saying "this isn't like, not physically drawing something like a perimeter, it's kind of just a concept."

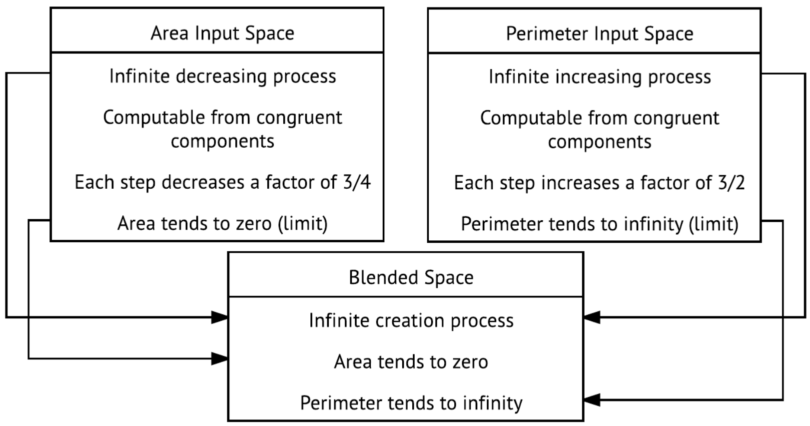


*Figure 3. Blending diagram for Curtis's reasoning*

**Carmen.** Carmen's blended space is, like several others', composed of an infinite process of creation, area tending to zero, and perimeter tending to infinity. The completion of her blend, however, is particularly distinct. She brings in a calculus frame and identifies "analogies to calculus or real analysis," including Riemann sums, that she sees as similar to Fred's paradox. The "calculus arguments" that she references seem to imply, to Carmen, that Fred's paradox is like other paradoxical situations that she has seen in previous mathematics courses. Upon reading Fred's arguments during the interview, Carmen stops to query whether "the perimeter can't really be infinitely long" implies zero perimeter or some non-zero finite length (for Fred). She proceeds to resolve the dilemma by eliminating each, leaving only the possibility that the perimeter is indeed infinite and Fred is wrong. During this episode, two more frames appear. Like Joy, she brings in a fence metaphor for the perimeter and the entailment that fencing should remain, but does not use the idea that fences take up space. Her elaboration using the fence frame, "you have sort of your old triangle fences that you had before [...] we still have this fence around, that big triangle and the center, and we still have those other ones we made before," is how she argues that the perimeter of the ST cannot be zero. Finally, she brings the frame of self-similarity, with the entailment that "we can keep zooming in." The elaboration using this frame is that the perimeter cannot be a finite value, which she explains using a contradiction. Carmen says "I think if we could [stop] then you could say ok it's this number," but the zooming goes on forever, "so that's kind of why it can't be a number."

**Discussion**. Our analysis of students' blending processes, especially as provoked by encountering Fred's argument, revealed how students deal with the paradox of coordinating infinite perimeter and zero area associated with the ST, and how they cope with, or resolve, the cognitive dissonance it provokes. It was sometimes challenging to unpack and distinguish the completion and elaboration processes. We attribute this difficulty in part to the fact that this was the second opportunity in which the students were prompted by Fred's paradox.

All students *composed* a blended space from their area and perimeter input spaces following Fred's prompt, and most of them also *completed* their blended space with additional frames, which then supported *elaboration* of the blend - leading to new implications. In two students' interviews we saw evidence of completion but not elaboration; only for one student we do not have evidence of completion. We saw one commonality across all students' composition processes: the fusion of *infinite (increasing) process* and *infinite (decreasing) process* into a unified *infinite creation process* for the stepwise creation of the ST. This is not to say that there was a shared conception of exactly what happens at each step, only that the process is infinite.

For all but one student, we have evidence of 1-3 distinct frames being used to complete their blended spaces. In all the cases, one of the frames has to do with the nature of mathematics – e.g., the nature of infinite processes. However, four students also used physical frames (fence, skeleton, zooming-in) and their entailments to coordinate area and perimeter and to make sense of that coordination.

## Conclusion

As the 10 students we interviewed were in the same graduate program, part of the same class, had worked together and discussed the Sierpinski Triangle (and, essentially, Fred's argument), we expected to see a certain level of consistency in their responses. However, this was not entirely the case, as seen at every stage of our analysis. To be sure, some ideas about the nature of the infinite iterative process were present in all interviews. But while in class students seemed comfortable with the idea that the area of the ST goes to zero, and concerned about what happens to the perimeter, *all* students' input spaces for perimeter included that it was infinite, and only six of the ten spaces included area going to zero. There were other idiosyncratic elements present in students' input spaces such as Curtis's multiplicative reasoning. There were also idiosyncrasies in terms of the composition of blended spaces. Some students completed their blends with ideas from calculus or analysis, fractal dimension, and metaphors. These frames resulted in varied elaborations. Some related to the nature of the ST, such as "it's not a real object", its non-integer dimension, or that is only the remaining outline; others framed the nature of the paradox itself.

More generally, our analysis methods allow us to point to some of the precise points of departure, from initial ideas to completing frames and final elaborations, one of the methodological implications of our work for future researchers. Along with Zandieh et al. (2014), our articulation of the component process of conceptual blending in a mathematical context allow for nuanced analysis of students' reasoning – though they looked at group blends and types of blends, while we look at more individualistic reasoning. This is particularly relevant for situations where students must bring together multiple ideas. Identifying all three processes - composition, completion, and elaboration - allows us to examine not only the main ideas students mention, but how they are used and enacted, or what leverage they give students in thinking about mathematical objects. This is in contrast to other lenses which make claims about the level of students' understanding, the extent to which their ideas are normative, or the conceptual structures that they might "possess." We are particularly impressed with the analytic power of the completion process, allowing us to articulate the tools by which students elaborate their blends. Thus, our analyses lie fully within the domain of enacting ideas.

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Themes in Undergraduate Students' Conceptions of Central Angle and Inscribed Angle

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*Researchers have investigated students' multifaceted conceptions of angle and their difficulties with connecting angle measure to arcs or circles. In this study, we investigated three undergraduate students' thinking about angles in the context of circle geometry, specifically their conceptions of central and inscribed angle. Conceptual analysis of the data revealed that students involved in the tasks and interviews had various conceptions of these angles that either supported or constrained their ability to complete the tasks. Particularly, conceiving the dynamic transformation of both central and inscribed angles, or identifying their common subtended arc was productive, while considering angle as area or ray pair constrained their thinking.*

*Keywords:* Student Thinking, Angle Conception, Geometry, Preservice Secondary Teachers

The Common Core State Standards for Mathematics (CCSSI, 2010) covers angle content in Grade 2 through High School, starting from identification of angles in planar shapes to angles in trigonometry. These topics highlight the complexity and variety of angle meanings in school mathematics, including angles as "geometric shapes that are formed wherever two rays share a common endpoint," angle measure with reference to a fraction of a circle, angle measure as a turn, and the relationships between central, inscribed, and circumscribed angle (ibid).

Despite the efforts studying students' understandings of angle (Keiser, Klee, & Fitch, 2003; Mitchelmore & White, 1998; Moore, 2013), little attention in mathematics educational research has been given to students' conceptions of angle involved in circle geometry. In the present study, we attempt to gain insights into students' understandings of angle by exploring three undergraduate students' ways of reasoning as they are asked to identify a central angle corresponding to a given inscribed angle in a circle. We illustrate the students' multiple conceptions of angle and how their different ways of thinking affect their ability to identify a corresponding central angle. We conclude by discussing what approaches potentially promote students' understandings of the relationship between central and inscribed angle and instructional implications.

## Background and Motivation

Researchers have discussed how mathematicians in history (Keiser, 2004; Matos, 1990, 1991) and students and teachers (Clements & Battista, 1989; Keiser et al., 2003; Krainer, 1993; Mitchelmore & White, 1998) conceptualized angle concept. There are three viewpoints of angle that occur repeatedly in this literature: (1) angle as ray pair, (2) angle as region, and (3) angle as turn.

Students who conceive angle as ray pair construct an image of angle formed by two rays meeting at a common vertex. Mitchelmore and White (1998) indicated nearly forty-five percent Grade 4 children's responses of their angle definitions reflected they conceptualized angle in a way similar to this. Some third-grade and sixth-grade students' definitions of angle included: "an angle is where two vertices meet and make a point," or "[i]t's when two lines meet each other and they come from two different ways" (Clements & Battista, 1989; Keiser et al., 2003). Students who conceive angle as region consider an angle as a space bounded a ray pair. In this construction, a ray pair will contain two angles (large and small) instead of being one angle itself

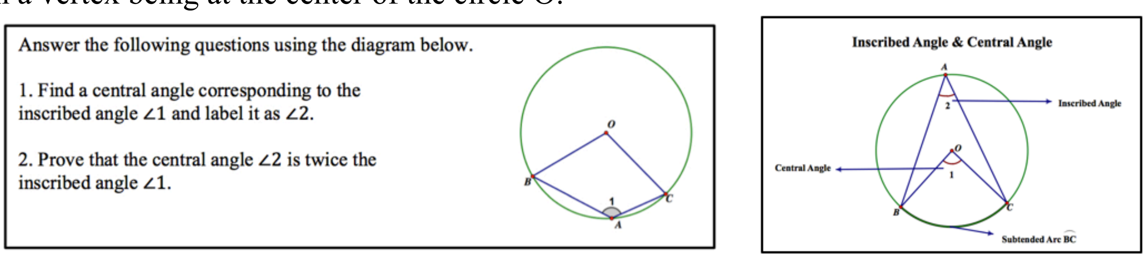
(Krainer, 1993). Fifteen percent of the four-graders' in Mitchelmore and White's (1998) study defined angle as an area. Students who conceptualize angle as turn or opening consider an angle as being formed by a dynamic rotation of one ray from another or angle as describing such rotation. Mitchelmore and White (1998) indicated only 4 out of 36 elementary students interviewed defined angle as turning. Clements and Battista (1989) suggested third graders who had Logo experience were more likely to define angle as a certain amount of rotation. Some children in this study defined angle as "something that turns, different ways to turn," or "when you turn some degrees" (Clements & Battista, 1989).

Noticing that researchers of most of these studies have focused on elementary students' *concept definitions* (i.e., words used to specify a concept) of angle, we consider it necessary to draw attention to their *concept images* (i.e., the total cognitive structure that is associated with a concept, which includes all the developing mental pictures and associated properties (Tall & Vinner, 1981; Vinner, 1991)). The focus of our study is to identify these concept images of angle. Specifically, given the paucity of research on undergraduate and/or teachers' conceptions of angle, our study aims at answering two research questions:

1. What are undergraduate pre-service teachers' concept images of central and inscribed angle in the context of circle geometry?
2. In what ways do these conceptions support and/or constrain their ability to solve circle geometry tasks?

## Methods

We investigated the mathematical thinking of nine undergraduate students majoring in secondary math education from a large public university in the United States. The study consisted of three tasks: a pre-test, a reading task, and a post-test followed by a short interview. Each student completed the series of tasks individually. In the pre-/post-test, students were asked to complete a proof (Figure 1a) with the help of a handout that included a graphical definition of central and inscribed angle (Figure 1b). The normative solution of Question 1 is the reflex angle with a vertex being at the center of the circle O.




(a) (b)

*Figure 1. (a) Pre-/post-test problem, (b) Inscribed and central angle definition handout.*

After finishing the pre-test, they worked on the reading task set up on a computer. We designed two sets of presentations for this task (i.e., Static and Dynamic), and randomly assigned students to them. The Static presentation demonstrated the proof of the Inscribed Angle Theorem in three different cases and the supporting diagram of each case is static (see Figure 2a-c). The Dynamic presentation contained a dynamic diagram with a slider (see Figure 2a-d) The slider allowed students to move the Point C along the circle so that infinite cases could be seen. Meanwhile, the proofs would appear to the right of the figure depending on which static case the current state of the diagram belonged to. The fourth static case: to be tested in the post-test, was omitted from the static presentation and left blank in the dynamic presentation (Figure 2d).

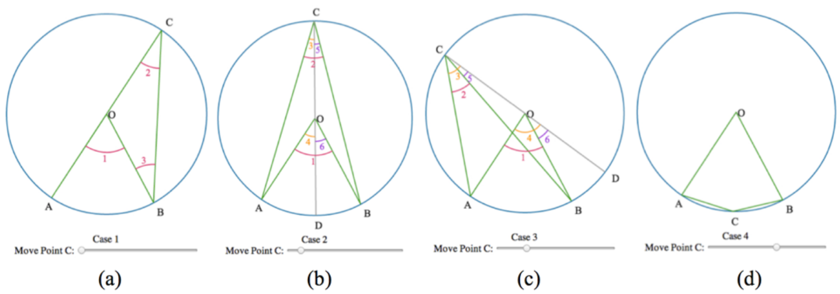


*Figure 2. Snapshots for the four cases in the Dynamic presentation. The Static presentation included Case 1-3 without the slider.*

After the post-test, each student participated in a clinical interview (Clement, 2000) to reflect on their thinking of the three tasks. We audiotaped all interviews and digitized students' written work. The process of how students drew the diagrams and thought aloud was also recorded with a Smart Pen throughout the study. Upon completion of data collection, we transcribed the interviews and incorporated figures and annotations.

In data analysis, we conducted conceptual analysis of an individual (Thompson, 2008) in order to develop models of a student's mathematical thinking. As researchers, although we cannot have access to students' minds, it is possible for us to make inferences about their mathematical thinking in ways that are consistent with our interpretations of their talking and observable actions. Both ongoing and retrospective analysis involved constructing conceptual models of pre-service teachers' meanings of central and inscribed angle. Using the observed conclusions about central and inscribed angle the students reached, we constructed hypothetical mental operations that would viably justify those conclusions that comprised these models.

Due to space constraints, we only report the reasoning of three of the nine students: Joanna (Static group), Hayley (Static group), and Jack (Dynamic group). We choose these three students because their stories establish the existence of highly varied understandings of angle among undergraduate pre-service teachers and the mathematical consequences of those understandings.

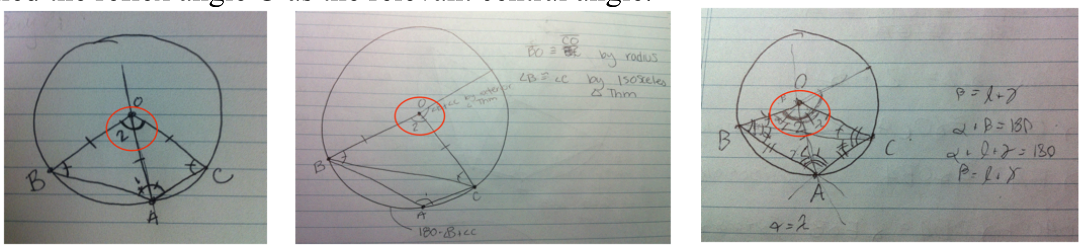
## Results

We organize the three students' conceptions of central and inscribed angle into five themes (Table 1) and describe how these conceptions influence their ability to identify a central angle.

*Table 1. Themes in students' conceptions of central and inscribed angle.*

| Theme | Theme Description | Students |
|---|---|---|
| Outside-Inside | A student considers an inscribed angle as being inside an area bounded by a quadrilateral (within a circle) while a central angle as being outside the same quadrilateral. | Joanna |
| Angle Size as Area | A student considers the openness (as area) of a central angle should be bigger than that of an inscribed angle. | Joanna |
| Angle as Ray Pair | A student considers a central angle as the minor angle constructed by two radii meeting at the center of a circle. | Hayley |
| Shared Arc | A student considers an inscribed angle and a central angle should share a subtended arc. | Hayley |
| Oriented Angle Rotation | A student considers the orientation of an angle as starting from one ray and ending on the other; the orientation of central and inscribed angle should be consistent. | Jack |

In the pre-test, all the three students identified the smaller (obtuse) measure of angle O as the angle measure relevant to Question 1 (see "∠2" in Figure 3a-c). In this section, we will report our analysis of the post-test and interviews, where the three students changed their minds and identified the reflex angle O as the relevant central angle.



(a) (b) (c)

*Figure 3. Diagrams produced by (a) Joanna, (b) Hayley, and (c) Jack during the pre-test.*

**Outside-Inside: Joanna**

When working on the post-test problem, Joanna first realized that the central angle she labeled in the pre-test might be incorrect:

*Joanna*: …But that ["∠2" in Figure 3a] was not the central angle so then I didn't know what to do. So then I got a little bit confused.

*Int*: So you have discovered that this is not the central angle?

*Joanna*: Yeah, I figured it out, because…well because one has to be outside, one has to be inside. And they are both inside, so then I figured out that can't be right…

Joanna claimed that, for the central angle and the inscribed angle, "one has to be outside, one has to be inside." By comparing her own drawing (Figure 3a) and the figure in the handout (Figure 1b), Joanna realized that the central angle she labeled was incorrect, since she thought the area bounded by the correct central angle should not be also "inside" the quadrilateral ABOC. Joanna's "outside-inside" conception can be thought of as relative to the area of the quadrilateral in the circle (see the quadrilateral shaded in orange in Figure 4), with the central angle being outside and inscribed angle inside. We infer that she was conceiving angle as area and the relative positions of the angle-areas of central and inscribed angle. Eventually, Joanna changed her solution and labeled the reflex angle as the central angle.

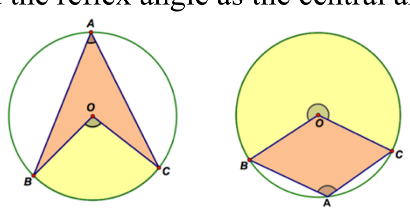



*Figure 4. Interpretation of Joanna's "outside-inside"conception of central and instribed angle.*

**Angle Size as Area: Joanna**

Although Joanna correctly (from our perspective) chose a central angle, she was uncertain about whether the angle was correct, saying "I figured it out but I don't know whether that was right, because I don't think it is." She thought the angle she labeled could not be right because "it was like the way too big to be a central angle, I don't think that's a central angle". Joanna had difficulty with conceiving an angle that is greater than 180°. As Joanna was conceiving angle-areas, here we inferred, Joanna's "too big" (the size of angle) probably referred to the measure of the openness of angles as the areas enclosed by the angles. The reflex nature of the angle might have made it appear to Joanna that the angle was enclosed by the area, rather than the reverse.

Later, she provided an explanation of what confirmed her choice of that central angle. She claimed that the openness of a central angle should "be like…really big" and that it "might be even bigger" than the given obtuse inscribed angle. This idea – a consequence of the theorem she was trying to prove – gave her enough confidence with her selection of central angle to finish the task.

Collectively, Joanna was reasoning with the position (i.e., her "outside-inside" approach) and the size of the angle-areas (i.e. "might be even bigger"). We consider her conception of angle as area in general as the fundamental reasons for her uncertainty about the correctness of her central angle. She continued to use hedge-words in the interview, and despite her success of providing a proof of the Inscribed Angle Theorem, she was not convinced that she had found a correct central angle.

**Angle as Ray Pair: Hayley**

In the interview, we started with asking Hayley about her reasoning in the pre-test:

*Int:* How do you think about this problem? Where did you get stuck?

*Hayley*: Umm…the arc part, like finding the first angle…this was the central angle ["∠2" in Figure 3b], right?

*Int*: What do you think is a central angle?

*Hayley*: The central angle would be in the middle of the circle [moving her hands along the two rays towards the center] cause this is the center, so that's why I put that this "O" is the central angle.

Hayley was describing a central angle as an angle constructed by two radii meeting at the center of a circle. She only conceived of the two radii BO and CO as constructing a single minor angle. Hence, she could only build a correspondence from this singular angle to the subtended minor arc BC and considered angle "O" to have exactly one measure.

Hayley also had a difficulty building a correspondence between central and inscribed angles, instead looking at each angle individually:

*Int*: Do you think going through these three cases would help you identify the central angle?

*Hayley*: I don't know because it is in the same spot. In all those and they are always the same angle in all them ["∠1" in Figure 2a-c]…because they are the same angle, so…yeah, I don't think that will be helpful.

Her awareness of the invariance of the angle location (i.e., "in the same spot") across cases suggested that she was considering the location of a central angle was absolute rather than relative to the inscribed angle. Due to the central angles of the first three cases being less than 180 degrees and thus fitting into her minor angle conception, the presentation was not helpful for Haley to change her previously identified central angle to the reflex angle, and thus she went with the same central angle as a solution for the post-test.

**Shared Arc: Hayley**

Haley's difficulty stemmed from her approach of first identifying a central angle and identifying the arc corresponding to that angle. In the subsequent interview, the interviewer instructed her to instead identify the subtended arc of an inscribed angle first, and then to find the corresponding central angle of that arc. The interviewer and Hayley went through all the three cases using this approach to identify corresponding central angles with given inscribed angles. When Hayley looked at the post-test problem again, she identified the correct central angle for the first time by making use of the shared subtended arc of the central and inscribed angle.

## Oriented Angle Rotation: Jack

In the interview, Jack talked about how he identified the correct central angle by interacting with the dynamic diagrams in the reading task. When looking at the presentation, Jack carefully tracked the angles as the point C moved along the circle and paid particular attention to the transition between Case 3 (Figure 2c) and Case 4 (Figure 2d). He interpreted this process as "take[ing] a limit," by which he meant he was trying to exhaust all the details in between Case 3 and Case 4 to carefully observe how the angles changed and how they were "opened up" differently in this process. He later explained what changed between Case 3 and Case 4 in terms of the angles that made him refined his original choice of the central angle:

> "… you like keep track of the angles as they move because you can see here [Figure 2c], you know these angles stay the same, the same, but they just flipped over [Figure 2d], so you can just sort of generalize it."

Jack was interpreting when Point C went through Point A (from Case 3 to Case 4), the inscribed angle changed its orientation ("flipped over"; Figure 5). Jack may have imagined one ray to be the starting ray (AC), and the other ray to be the ending ray (BC). So as the orientation of the angle flipped (Figure 5; left to right), the direction of rotation also changed (counter clockwise to clockwise). Therefore, the original central angle AOB constructed by AO as the starting ray and BO as the ending ray should also change to the reflex angle AOB constructed by the same starting and ending rays but rotating from a clockwise orientation instead. Another interpretation of Jack's "flipped over" is that before C passes A, the inscribed angle ACB is constructed by AC on the angle's left and BC on the right (facing into angle C from the bottom of circle). After C passes A, the angle is constructed by BC on the angle's left and AC on the right (facing into angle C from the top of circle). So an inscribed angle and its "flipped" angle were orientationally different, and thus the original central angle AOB should also "flip" to its reflex angle with BO on the left and AO on the right (viewing angle C from the top of circle). Both interpretations lead to the same mathematical conclusion, so we consider them equivalent.

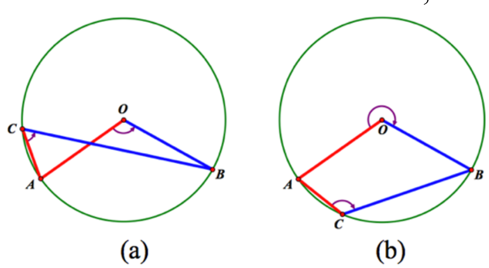



*Figure 5. Interpretation of "Jack"'s "flipped over" as the orientation of an angle changing from (a) counter clockwise to (b) clockwise rotation of one ray from another.*

## Conclusions

The results of our study indicate that, regardless of angle contexts or grade levels, students' understandings of angle as ray pair (i.e., Hayley), angle as rotation (i.e., Jack), and angle as area (i.e., Joanna) persist from elementary students (Clements & Battista, 1989; Foxman & Ruddock, 1984; Keiser et al., 2003; Mitchelmore & White, 1998) to late undergraduates. That these undergraduates' conceptions of angle are similar to the definitions elementary students learn in school should not be surprising. What should be considered significant, however, is the impoverished nature of these images. These advanced undergraduates, many of whom will become mathematics teachers, do not have understandings of angle that have advanced very far beyond ray-pair, rotation, and area. Consequently, all the students struggle to track a changing inscribed angle and thus have difficulties in finding its corresponding central angle.

Although there is a very large body of work on identifying student definitions of angle (e.g., Keiser et al., 2003; Mitchelmore & White, 1998), there has been little work done on identifying students' concept images of angle *and* the mathematical consequences of these images. We have not merely identified Joanna as having an area-meaning, Haley as having a ray-pair-meaning, and Jack as having an orientation-meaning (possibly a rotation-meaning) of angle. We have also shown that these meanings directly cause the students' struggles and successes. Joanna, who conceived angle measure as area had difficulties with conceiving angles greater than 180° since these angles are "too big" to enclose an area. Additionally, her "outside-inside" approach is too specific to the particular situation to be a generalizable understanding of angle. It will easily fail in the situations where no reference objects or shapes (i.e., the closed figure: the quadrilateral) can be identified, or where it is necessary to conceive of a single angle that changes between "inside" and "outside" among cases. The case of Hayley suggested that students who had a ray-pair conception may not inherently or easily conceive of the structure of two segments as having two measures, and therefore constructing two angles. Only perceiving the minor angle constructed by two segments potentially results in students' difficulties with conceiving angles of 0°, 180°, 360°, and larger than 360° (Keiser, 2004). Fortunately, Haley's conception of angle measure as arc provided her a foundation to perceive the shared subtended arc, which was critical in her success. Finally, Jack's image of an angle as having an orientation (or a rotation) supported him to correctly keep track of the inscribed angle in the dynamic situation.

Contributing to the previous work on classification of students' angle conceptions (Clements & Battista, 1989; Keiser et al., 2003; Krainer, 1993), these three students' profiles indicate how the multiple meanings of individual students can interact. Their concept images of angle may not simply be "angles are areas" or "angles are arcs." For instance, Haley's difficulty with perceiving the major arc is generated by the complex combination of angle as a ray pair and angle as arc (i.e., one ray pair only corresponds to one angle measured by one arc).

Lastly, our findings highlight the need for supporting student understandings of angle in the context of circle and arc. The presence of the circle context of the tasks did not inherently lead the students to incorporate circles and arcs into their identification of central angle. Ultimately making use of the circle context was critical to the success of both Hayley (who found a common subtended arc) and Jack (who imagined the angle orientation changing as the vertex moved around the circle). Despite our attempt to assist Joanna to identify the subtended arc shared by an inscribed and central angle, she did not consider this approach as useful since she did not imagine the arc of a circle as having a role in angle measure. We hypothesize that if these students had an image of angle measure as arc (Moore, 2013), they would be more comfortable with relating central and inscribed angles using their shared arcs.

In order for teachers and researchers to be better able to recognize, explain, and respond to student thinking, and identify ways to assist them, we need to further explore their various conceptions of angle and angle measure, attend to the nuances of student thinking and its mathematical consequences, and be sensitive to the students' awareness (or lack of awareness) of the relationships between ray-pair, area, angle measure, circle, and arc.

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# Developing Proof Comprehension and Proof by Contradiction Through Logical Outlines

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*Proof is central to the curriculum for undergraduate mathematics majors. Despite transition-to-proof courses designed to facilitate the shift from computation-based mathematics to proof-based mathematics, students continue to struggle with mathematical proof. In particular, there are few tasks beyond writing proofs that are specifically designed to develop students' understanding of the proofs they read and the proof methods they utilize. The purpose of this paper is to introduce and discuss the merits of two such tasks: constructing and comparing logical outlines of presented proofs. Grounded in APOS Theory, this paper will illustrate a case study that suggests students can improve their understanding of the proofs they read as well as a particular proof method - proof by contradiction – through these two tasks.*

*Key words:* Proof Comprehension, Proof by Contradiction, Transition-to-proof course, APOS Theory

Proof is central to the curriculum for undergraduate mathematics majors. Despite transition-to-proof courses designed to facilitate the shift from computation-based mathematics to proof-based mathematics, students continue to struggle with mathematical proof (Samkoff & Weber, 2015). Instructors of these courses have stressed that students' ability to understand the proofs they read (*proof comprehension*) is of utmost importance and yet, there are few tasks beyond writing a complete or partial proof of some statement that are designed to improve students' proof comprehension. In short, writing proofs have been the primary tasks used to assess students' understanding of the proofs they read. Noting this, Mejía-Ramos et al. (2012) developed a proof comprehension assessment model that split students' understanding of the proofs they read into two categories: local and holistic. *Local* types of assessment focused on one, or a small number, of statements within a proof whereas *holistic* types of assessment focused on students' understanding of a proof as a whole. Utilizing this assessment model, two groups of researchers developed teaching experiments aimed at improving students' proof comprehension. A brief description of their design and results follows.

Samkoff and Weber (2015) developed a teaching experiment to assess whether certain proof-reading strategies, identified in Weber and Samkoff (2011) and aligned with the previously mentioned proof comprehension assessment model, would aid student understanding. They found that: (1) specific prescriptive guidance helped students implement the strategies more effectively, (2) these strategies were beneficial to students, and (3) that there were impediments to proof comprehension that could not be addressed by these strategies (Samkoff & Weber, 2015). These results suggest that while the proof comprehension model by Mejía-Ramos et al. (2012) may assess student understanding of proof, it cannot, alone, be used as a pedagogical tool to develop instruction for a transition-to-proof course.

Hodds, Alcock, and Inglis (2014) developed a booklet containing self-explanation training focused on the logical relationships within a mathematical proof. Through a series of three experiments, they found that: (1) students who received the self-explanation training scored higher on a comprehension test, (2) self-explanation training increased cognitive engagement with a proof, and (3) a short self-explanation training session within a lecture improved students' proof comprehension and that this comprehension persisted over time (Hodds et al., 2014). These results suggest that focusing on the logical relationships within a mathematical proof can improve students' proof comprehension.

To contribute to the paucity of tasks designed to improve proof comprehension, the authors of this study first utilized APOS Theory to model how students may come to understand the proofs they read and, by extension, how they come to understand a particular proof method: proof by contradiction. These models were then used as a guide to address the following research question:

> *Can outlining given proofs and comparing these outlines enhance students' proof comprehension and overall conception of proof by contradiction?*

The following section briefly describes APOS Theory and the preliminary cognitive model we developed for proof by contradiction to address this research question.

## APOS Theory

APOS Theory is a cognitive framework that considers mathematical concepts to be composed of mental Actions, Processes, and Objects that are organized into Schemas. An *Action* is a transformation of Objects by the individual requiring memorized or external, step-by-step instructions on how to perform the operation. As an individual reflects on an Action, he/she can think of these Actions in his/her head without the need to actually perform them based on some memorized facts or external guide; this is referred to as a *Process*. As an individual reflects on a Process, they may think of the Process as a totality and can now perform transformations on the Process; this totality is referred to as an *Object*. Finally, a *Schema* is an individual's collection of Actions, Processes, Objects, and other Schemas that are linked by some general principles to form a coherent framework in the individual's mind (Dubinsky & McDonald, 2001). Utilizing the mental constructs of Actions, Processes, Objects, and Schemas, an outline of the hypothetical constructions students may need to make in order to understand a concept can be developed, referred to as a *genetic decomposition* (Arnon et al., 2014). This genetic decomposition is then used as a foundation to develop instructional materials. A preliminary genetic decomposition for proof by contradiction is provided below.

### Preliminary Genetic Decomposition for Proof by Contradiction

1. Action conception of propositional or predicate logic statements as specific step-by-step instructions to construct proofs by contradiction for the following types of statements: (I) implication, (II) non-existence, and (III) uniqueness.
2. Interiorization of each Action in Step 1 individually as general steps to writing a proof by contradiction for statements of the form (I), (II), and (III).
3. Coordination of the Processes from Step 2 into developing a single Process of a proof by contradiction.
4. Encapsulate the Process in Step 3 as an Object by utilizing the law of excluded middle to show proof by contradiction is a valid proof method. Alternatively, encapsulate the Process in Step 3 as an Object by comparing the contradiction proof method to other proof methods.
5. De-encapsulate the Object in Step 4 into a Process similar to Step 3 that then coordinates with a Process conception of other proof methods to prove statements that require two or more proof methods.

In particular for APOS Theory, there is a focus on repeatable transformations that can be reflected on and subsequently generalized by the individual. For proof by contradiction, the repeatable transformation is logically outlining presented proofs (described in Step 1). That is, as students continue to read and reflect on the logical structure of presented proofs (and thus develop their proof comprehension), they can generalize their understanding of these example proofs to develop an internal conception for proof by contradiction based on the structure of the statement proved (described in Step 2). As students encounter different logical structures of proof by contradiction based on the structure of the statement to be proved, they can compare these specific logical structures to develop an internal, general conception for any type of proof by contradiction (described in Step 3). This report will focus on a single student's experience in dealing with tasks designed to induce the mental constructions described by Steps 1, 2, and 3 in the preliminary genetic decomposition. The following section will give an overview of the study's design and a description of the particular tasks this paper will focus on.

## Methodology

This report is situated in a larger research project on how students develop an understanding of proof by contradiction within a transition-to-proof course, *Bridge to Higher Mathematics*, at a public R1 university in the southeastern United States. To test the validity of the preliminary genetic decomposition, a five-session teaching experiment was developed and implemented in Fall 2016. These sessions were conducted primarily out-of-class and so the number of sessions a student participated in varied. Of the initial 27 participants, only two completed all five sessions.

This report will focus on two particular tasks developed as part of this teaching experiment: Outlining and Comparing. *Outlining* tasks asked students to logically outline a presented proof by contradiction. These tasks were included to prompt students to identify the logical argument within a presented proof by contradiction. *Comparing* tasks asked students to compare two or more logical outlines of presented proofs. These tasks were used as a reflection tool for students to consider the necessary logical lines of a general proof by contradiction and how these lines logically relate.

Data for this report consists of Yara's responses to these two tasks during the teaching experiment. Yara was a senior Mathematics major with a minor in Educational Psychology. Beyond the required prerequisite courses for *Bridge to Higher Mathematics,* she had already taken *Mathematical Statistics*, *Methods of Regression and Analysis of Variance*, *Foundations of Numbers and Operations*, and *Applied Combinatorics*. However, none of these courses required proof writing and thus *Bridge to Higher Mathematics* was her first experience with formal proofs. She completed all five sessions of the teaching episode. This report focuses on Yara as she was the most elaborative in her responses and provided the most data through which to analyze and support how her understanding of the proofs she read as well as her understanding of proof by contradiction evolved throughout the teaching experiment. The following section will describe how we analyzed her responses.

## Data Analysis and Results

All five of Yara's teaching episode sessions were video recorded and then transcribed. Transcripts of these five sessions with Yara were organized and subsequently analyzed using MAXQDA, a qualitative data analysis software. First, sections of the transcripts were grouped by task. Then, Yara's level of understanding proof by contradiction, according to APOS Theory, was analyzed per task. This analysis provided a tool to identify which task or tasks aided her in developing an understanding of the proof method. Due to space constraints, the rest of this section will provide examples from a subset of these sessions on how two tasks, *Outlining* and *Comparing*, aided Yara in developing both a deeper understanding of the proofs she read as well as a more robust understanding of proof by contradiction in general.

**Outlining Task**

As mentioned previously, *Outlining* tasks asked students to logically outline a presented proof by contradiction. For Outlining task 1, students were given propositional representation for the statement and the majority of lines in the proof. For Outlining task 2, students were given predicate representation for the statement only. Finally, for Outlining tasks 3, 4, and 5, students were not given any logical representation. During these tasks, students were encouraged to use either propositional or predicate symbols to outline the logical structure of the proof. Due to space limitations, this report will focus on Outlining task 3.

The presented proof (Figure 1) and Yara's response to Outlining task 3 (Figure 2) are presented below.

**Statement:** The equation $5x - 4 = 1$ has a unique solution.
**Proof:** Assume the equation $5x - 4 = 1$ does not have a unique solution. Then either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. Note $x = 1$ is a solution of $5x - 4 = 1$. Thus there are at least two distinct solutions to the equation $5x - 4 = 1$, call them $y$ and $z$. As both $y$ and $z$ are solutions of the equation $5x - 4 = 1$, $5y - 4 = 1$ and $5z - 4 = 1$. Then $5y - 4 = 5z - 4$ and so $y = z$. Therefore it is not true that there are at least two distinct solutions to the equation $5x - 4 = 1$. This is a contradiction, as we assumed that either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. Therefore it is not true that the equation $5x - 4 = 1$ does not have a unique solution. In other words, the equation $5x - 4 = 1$ does have a unique solution.

*Figure 1: Presented proof for Outlining task 3.*

| **Statement:** $\exists!\, x$ s.t. $P(x)$ |
|---|
| 1. Assume $\sim(\exists!\, x$ s.t. $P(x))$ |
| 2. $R \vee Q$ |
| 3. $\sim R$ |
| 4. $Q$ |
| 5. $5y - 4 = 1 \wedge 5z - 4 = 1$ (Algebra) |
| 6. More algebra ($y = z$) |
| 7. $\sim Q$ |
| 8. $\sim(R \vee Q)$ |
| 9. $\sim(\sim(\exists!\, x$ s.t. $P(x)))$ |
| 10. $\exists!\, x$ s.t. $P(x)$ |

*Figure 2: Yara's logical outline of the presented proof for Outlining task 3.*

Yara provided a desired representation of the statement as $(\exists!\, x)\big(P(x)\big)$ and first line of the outline as $\sim(\exists!\, x)(P(x))$. Then, she switched to propositional logic and initially represented the statement "then either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$" as $P \vee Q$. An excerpt of her thought process behind this representation is provided below.

*Yara*: And then *[long pause]* and then either there is no solution to the equation or there is at least 2 disinked, I mean, 2 distinct solutions to the equation $5x - 4 = 1$. *[pause]* So it would be the or? Like $P$ or $Q$?

*Teacher*: Alright. *[pause for writing]* $P$ or $Q$. So do these have any relation to the original one?

*Yara*: No?

*Teacher*: So if this one doesn't have a relation, then maybe we should call it something else. Like $R$ or $Q$.

*Yara*: Oh! To separate that $P$ from that $P(x)$.

Note that she saw the 'or' in the statement and immediately suggested the representation $P \lor Q$. After reflection, she clarified that this $P$ should be changed to separate it from $P(x)$. This suggests the cue word 'or' prompted the representation $P \lor Q$ as a standard representation for an 'or' statement. After teacher's prompting and suggestion to use a different notation, she realized that $P$ should be separate from the initial statement $P(x)$. This suggests that in her initial thinking, Yara automatically used $P \lor Q$, a standard notation for an 'or' statement, without considering the relationship of that statement to the previous statement. In terms of APOS Theory, this excerpt illustrates a possible Action conception of proof by contradiction in relationship to this particular task. However, analyzing further her proof outline, it appears that she is at a higher level of understanding. We illustrate this below.

Overall, her outline contained the two key steps of a proof by contradiction: assuming the negation of the statement is true (line 1) and arriving at a contradiction (line 8). In addition, she verbally described the logical argument of the proof and how lines in the proof related. For example, when she reached the contradiction line in the presented proof, she stated:

> Then... this is a contradiction as we assumed that there is either no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. So it would be $Q$ and not $Q$? Or would we not have to put that because we have it [$R \lor Q$]... It's already labeled out. [...] Okay, so then not... I was just trying to make sure I had it in my head right like, that [$R \lor Q$] would go into not $R$ and not $Q$.

Her first sentence quoted the line from the presented proof. She then immediately considered the representation $Q \land \sim Q$ - the standard representation of a contradiction. Representing a statement by focusing on cue words (i.e., contradiction means $Q \land \sim Q$) is indicative of an Action conception of proof by contradiction and suggests, as in the previous paragraph, that Yara did not attend to the logical relation between lines in the proof. However, she then recognized that she would not represent this particular contradiction with $Q \land \sim Q$ as she already represented part of this contradiction with $R \lor Q$. Indeed, her final comment "I was just trying to make sure I had it in my head right like, that [$\sim (R \lor Q)$] would go into not $R$ and not $Q$" suggests that she recognized the logical equivalence $\sim (R \lor Q) \equiv \sim R \land \sim Q$ and thus recognized that the contradiction $\sim (R \lor Q) \land (R \lor Q)$ was reached. That is, she recognized and verbally described the logical reasoning behind how a contradiction was reached in this particular proof, which is indicative of a Process conception of proof by contradiction. In addition, she generalized lines 5 and 6 in her outline as "algebra" and thus described the purpose of the algebraic manipulations in the overall argument. In other words, Yara was able to use the logical outline to describe the purpose of specific lines in the proof and thus exhibited local comprehension of the presented proof.

## Comparing Task

As mentioned previously, *Comparing* tasks asked students to compare two or more logical outlines of presented proofs. These logical outlines were provided by the teacher based on the Outlining tasks. For example, Table 1 illustrates the side-by-side logical outlines from Outlining tasks 1, 2, and 3 that were presented to students for Comparing task 2.

*Table 1: Side-by-side logical outlines from Outlining tasks 1, 2, and 3.*

| Outlining Task 1<br>**Statement:** $P \to Q$ | Outlining Task 2<br>**Statement:** $(\nexists x)(P(x))$ | Outlining Task 3<br>**Statement:** $(\exists!\, x)(P(x))$ |
|---|---|---|
| 1. Assume $\sim (P \to Q)$<br>2. $P \wedge \sim Q$<br>3. $\sim Q_k$<br>4. $(\sim Q_k \wedge P) \to Q_k$<br>5. $Q_k$<br>6. $Q_k \wedge \sim Q_k$<br>7. $\sim (\sim (P \to Q))$<br>8. $P \to Q$ | 1. Assume $\sim (\nexists x)(P(x))$<br>2. $(\exists x)(P(x))$<br>3. $P(n)$<br>4. Using $P(n)$, get to a contradiction.<br>5. $\sim (\sim (\nexists x)(P(x)))$<br>6. $(\nexists x)(P(x))$ | 1. Assume $\sim (\exists!\, x)(P(x))$<br>2. $\sim (\exists x)(P(x)) \vee (\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$<br>3. Show $P(n)$ for some $n$.<br>4. $(\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$<br>5. $P(x) \wedge P(y) \to x = y$<br>6. $(\nexists x, y)(P(x) \wedge P(y) \wedge x \neq y)$<br>7. $\to\leftarrow$ (lines 2, 3, and 6)<br>8. $\sim (\sim (\exists!\, x)(P(x)))$<br>9. $(\exists!\, x)(P(x))$ |

When prompted to compare the outlines in Table 1, Yara grouped lines together and described a general purpose for each group of lines (see Table 2).

*Table 2: Yara's general approach for Comparing tasks 1 and 2.*

| Comparing Task 1 | Comparing Task 2 |
|---|---|
| 1. Assume $\sim P$<br>2. Rewrite $\sim P$<br>3. Look at specific value of step 2.<br>4. Work (Algebra)<br>5. Get Contradiction<br>6. $\sim$ Assumption<br>7. $P$ | 1. Assume $\sim P$<br>2. Negate $P$ (Rewrite $\sim P$)<br>3. Use math skills to get to a contradiction.<br>4. $\sim$ Assumption<br>5. $P$ |

Yara's approach for proof by contradiction contained both the key steps of a proof by contradiction and descriptions of how these key steps are logically related (e.g., that lines 3 and 7 logically implied line 8). Comparing her general approach between Comparing tasks 1 and 2, we see the she condensed steps 3, 4, and 5 in task 1 into the single step “Use math skills to get to a contradiction.” Consider the following exchange as Yara compared the logical outlines from Outlining tasks 1 and 2.

*Yara*: So I guess it just, maybe it like, depends on the proof, and what you are trying to prove. Whether you do algebra or... umm...
*Teacher*: So what do we do in that one [*outline during Outlining task 2*]?
*Yara*: In this one, it says to use $P(x)$, get a contradiction. So we did algebra, right?
*Teacher*: Yeah, we did algebra that time as well.
*Yara*: So this one you do... which math skills do you use? Because math skills could mean plenty of things. It could be, like, one of them induction whatever…

From the above excerpt, it is clear that Yara's expression ‘math skills’ stands for ‘mathematical knowledge’ since it includes algebraic skills as well as other proof methods such as induction. Yara stated that the steps in the proof depend “... on the proof, and what you are trying to prove." Sometimes, these steps might mean performing some algebra while in the other situation it may mean using a different proof technique. We interpret this to mean that Yara has generalized the notion of a proof by contradiction and exhibited an Object conception of proof by contradiction. She obviously was able to think of these two proof outlines as two entities that could be compared, de-encapsulated each one of them into the processes they came from, and compared separate lines in each outline to distinguish their similarities and differences.

## Discussion

Both the Outlining and Comparing tasks enhanced Yara’s understanding of the presented proofs in addition to enhancing her understanding of proof by contradiction, as suggested by the preliminary genetic decomposition. These results suggest that the two tasks may be useful in developing transition-to-proof students’ proof comprehension as well as their understanding of particular proof methods as they provide a repeatable transformation (outlining the logical structure) that can be reflected on and subsequently generalized by the individual (through comparing logical outlines). While a robust implementation of tasks to transition-to-proof students at a variety of universities would be necessary to validate these tasks, we find these initial results to be encouraging.

## Implications for Teaching Practices

This report presented two non-traditional tasks that aided students in developing proof comprehension as well as a robust understanding of proof by contradiction. That is, outlining the logical argument of presented proofs by contradiction (Outlining tasks) and comparing these outlines in order to develop general steps for the proof method (Comparing tasks) differ from the traditional proof writing tasks of “definition-theorem-proof” format transition-to-proof courses (Weber, 2004). This is not to say instructors should abandon proof-writing tasks. Rather, we suggest that Outlining and Comparing tasks should be used in conjunction with traditional proof writing tasks to improve and assess a different aspect of proof: comprehension. The tasks introduced in this report join the tasks based on proof reading strategies by Samkoff and Weber (2015) and the self-explanation training tasks by Hodds et al. (2014) as some of the first tasks designed to improve students' proof comprehension.

Moreover, Outlining and Comparing are the first tasks designed to improve students’ comprehension of a particular proof method: proof by contradiction. This is critical as research suggests this method is difficult for students to construct and comprehend (Antonini & Mariotti, 2008; Brown, 2017). These tasks may also provide students a fundamental understanding of the proof method so that other validation tasks, such as critiquing sample proofs and proof editing, may be utilized to further improve on their conception of proof by contradiction.

Finally, these tasks are compatible with other Constructivist frameworks (e.g., Vygotsky’s Social Constructivism) and can be used to develop other proof methods (e.g., mathematical induction). Therefore, these two tasks can be used in any transition-to-proof course to develop students’ proof comprehension as well as their understanding of particular proof methods.

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Development of the Inquiry-Oriented Instructional Measure

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*In this article we discuss the Inquiry Oriented Instructional Measure (IOIM). The development of the IOIM was a multi-phase, iterative process that required analyzing current research literature and videos of classroom instruction and piloting the measure with both experts and novices. The process resulted in identifying multiple instructional practices that support the successful implementation of Inquiry-Oriented Instruction (IOI) at the undergraduate level, and creating a rubric for evaluating the degree to which one's classroom instruction is reflective of these practices. Our goals with this paper are to share the development process and elaborate on the rubric so as to contribute to the knowledge base regarding the implementation of IOI.*

*Keywords:* Inquiry-oriented instruction, Instructional measure, Teaching

Student-centered forms of instruction have been shown to have many positive outcomes for undergraduate mathematics students. Empirical studies demonstrate that Inquiry Based Learning (IBL) is a more equitable form of instruction and leads to greater affective and cognitive gains when compared to non-IBL teaching methods (Laursen, Hassi, Kogan & Weston, 2014; Kogan & Laursen, 2014). These outcomes directly align with the goals of recent calls for improving undergraduate STEM education. Ferreni-Mundy and Gucler (2009) noted all of the calls for reform in STEM education surrounded increasing student understanding of concepts, providing equitable access to students, and transitioning away from traditional teaching approaches to those that are student-centered and involve strategies that encourage active learning. With the push to increase the quality of STEM education, implementing such forms of instruction is important.

Instructional measures are one tool that can be used by various groups within the community to support the successful reform of undergraduate education. Researchers can utilize measures to assess the effects of instructional interventions, and practitioners can use measures to improve their instruction. In addition, measures can provide a vernacular and specific descriptions of instructional practices that promote instructional change. In this paper we begin with a brief discussion of a National Science Foundation funded project, *Teaching Inquiry-oriented Instruction: Establishing Supports* (TIMES). We outline the general design and provide a detailed account of the development of the inquiry-oriented instructional measure (IOIM), a measure for evaluating the degree to which a lesson consists of practices that reflect inquiry-oriented instruction (IOI).

## Background

This work stems from the TIMES project, the goal of which was to scale up inquiry-oriented curricular materials (including developing instructor materials) for Abstract Algebra (Larsen, Johnson, & Weber, 2013), Differential Equations (Rasmussen, 2007), and Linear Algebra (Wawro, Rasmussen, Zandieh & Andrews-Larson, 2015). These curricula are research based and have been continually refined over the past two decades to scaffold student reinvention of mathematical concepts. The grant led to widespread dissemination and implementation of the curricula by recruiting mathematics instructors from across the United States. These instructors participated in activities intended to support them in implementing instruction that aligned with the four underlying instructional principles of IOI (Kuster, Johnson, Keene & Andrews-Larsen, 2017). The IOIM was developed as part of this project to help evaluate the efficacy of the support activities.

Generally, evaluation tools used for research and practice serve specific purposes; purposes that align with the goals of the research being performed. Common observation protocols and instructional measures include the instructional quality assessment (IQA), the mathematics quality of instruction (MQI), and the reformed teaching observation protocol (RTOP). The IQA was developed with a focus on "opportunities for students to engage in cognitively challenging mathematical work and thinking" (Boston, Bostic, Lesseig & Sherman, 2015, p. 160) and revolves around assessment of cognitive demand (Boston, 2014). The MQI was developed to aid in drawing connections between teacher knowledge and classroom instruction (Hill et al., 2008) and focuses on evaluating the quality of the mathematics available to students during instruction. The main goal of the RTOP was to serve as a tool for pedagogical development aimed at improving instruction. The RTOP is designed to measure the degree to which classroom instruction is *reform-oriented* (Sawada et al., 2002). With regard to the TIMES project, we found that these tools and others like them did not fit the specific needs of the project, in that they failed to attend to the nature of IOI to the degree we needed. Our goal was to focus on the instructional practices in which the teacher engaged while in the classroom.

## Overview of the Inquiry-Oriented Instructional Measure

The IOIM is a rubric designed to provide quantitative and descriptive data concerning the enactment of the four main instructional principles of IOI: *generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation* (Kuster et al., 2017). Broadly speaking these principles reflect three characteristics of the role of an IOI teacher: 1) inquiring into student mathematics, both in terms of individual students and in terms of the learning trajectory (Rasmussen & Kwon, 2007; Johnson & Larsen, 2012); 2) being an active participant with the developing mathematics, both in terms of the mathematics of the moment and in terms of the mathematical trajectory intended by the curricular materials (Johnson, 2013; Johnson & Larsen, 2013); and 3) bridging the gap between where the students are and the mathematical goals of the lesson (Wagner, Speer & Rossa, 2007; Speer & Wagner, 2009). The IOIM consists of a set of practices that support the enactment of each of the principles, and a rubric (shown at the end of this report) that measures the degree to which a lesson is inquiry-oriented by examining the quality of the enactment of these practices.

Each of the practices is scored on a 5 point likert-scale from low to high. Generally, within each of the practices, the quality of the mathematical activity promoted by the teacher is what distinguishes a low (1) from a high (5). Take for example Practice 2: teachers elicit student thinking and reasoning. If a teacher evokes solely procedural contributions from students, they score significantly lower (medium-low) than if they routinely have students share their thinking, reasoning and justifications (high).

## Development of the Inquiry-Oriented Instructional Measure

The measure was developed in five phases using data consisting of research literature, videos of classroom instruction from both expert and novice IOI instructors, expert validity checks, and notes from pilot training-sessions. The overall process began with codes and categories that were developed from the data and iteratively refined in the process of creating a descriptive framework of IOI. In addition, new data was sought to specifically address questions and hypotheses as they arose, and subsequently led to the refinement of the framework. Other research methods were incorporated into the phases such as Lesh and Lehrer's (2000) iterative video analysis. In the following sections we outline the work completed in each phase.

### Phase 1: Defining the Task - What is IOI and how do we measure it?

This phase resulted in a general understanding and vocabulary for characterizing IOI and information on how to measure teaching in general. In this phase, we searched through research literature for defining characteristics of IOI and determined that a distinguishing feature was that the teacher, students, and tasks each have a critical and active role in developing the mathematics. After returning to the literature and coding for "teacher", "students", and "tasks", we generated a starter list of instructional practices of IOI and supported these practices with justifications and examples from the literature. After our working definition of IOI was complete, we examined existing instructional measures (e.g., RTOP, IQA, MQI) to determine if they adequately captured our characterization of IOI. Though it was ultimately determined the other measures were not applicable, they did influence the refinement of practices and provided useful descriptions for what some practices looked like when enacted.

### Phase 2: Examining Data - Verifying practices and identifying measure limitations

Although the research literature led to the identification of numerous practices of IOI, the utilization of a measure required being able to *observe* these practices. In this phase, we cycled between analyzing videos and existing literature to verify that the practices identified from the literature were also evident in classroom instruction. In the first pass through the video data, we watched two expert instructors (IO curriculum developers) and three novices. The variation in experience level was purposeful; we intended it to highlight key aspects of instruction. While watching these videos, we documented the classroom events with content logs, coded for critical components, and wrote narratives for each of the practices based on what was observable in the videos. This process resulted in refining the list of practices and their characterizations. Thus, the characterizations of the practices were created in terms of supporting literature and video data.

Once the practices were defined and descriptions of their enactment were created, we delineated across the various levels (i.e., high, medium and low) at which instructors performed each of the critical components. Using the video data, we created a rubric for scoring the quality of the implementation of each component by ranking the various instructors in terms of how well their instructional practices aligned with the tenets of IOI. We then identified themes within the various levels of quality by comparing across the components within each of the scores. The process of ranking the instructors also raised important questions regarding issues such as how these practices connected to each other and how they fit within the four instructional principles.

**Phase 3: Refinement using outside sources**

In this stage, we began seeking resources from beyond IOI research literature and feedback from researchers not directly involved in the development of the measure. First, we asked a researcher not familiar with IOI to code two videos with the drafted rubric. After discussing areas of confusion and working out discrepancies between scores, we began searching through K-12 research literature looking for aspects of K-12 instruction that were commensurate with the practices we identified in IOI. These steps led to refining the practices and led to a better understanding of the principles and the supporting practices. Specifically, the K-12 literature was able to provide descriptions for what we noticed from the IOI video data and language for delineating among the various levels of implementation.

**Phase 4: Sharing to clarify**

In this phase our intent was to pilot the rubric with experts and novices to both clarify connections between the principles and practices and work toward a common interpretation of them. We first asked researchers familiar with IOI but not with the measure to use the rubric to score the same lesson. While this step had multiple benefits, there were two important outcomes. Most importantly, despite no training, the scores across all six researchers (including two rubric developers) were all within one point. Thus, while some reorganization was needed, the descriptions in the rubric were generally meaningful to researchers familiar with IOI.

We then engaged in a pilot training process where we trained three graduate students having no background in IOI on how to use the rubric. During this process we asked the coders to take careful notes of issues that arose for them as they utilized the rubric. We also recorded the meetings when we met to discuss the scores they assigned. From this we concluded that two practices were capturing the same aspects of instruction and removed one of them. We also created resources for coders, including guiding questions, “evidenced by” descriptions, and boldfacing certain words in the rubric.

**Phase 5: Sharing to use**

In this phase, we implemented the full scale training of six graduate students from various mathematics education backgrounds. Training started with having the coders watch video clips exemplifying the different levels of IOI for each of the practices. As training progressed, coders were given more opportunities to watch longer segments of classroom video with a partner or on their own each evening and to justify their own scores using the rubric. In group meetings, coders would then engage in facilitated debates of their scores, which allowed misunderstandings of terms and weaknesses in justifications to be resolved. The coders were also encouraged to articulate in their own words what each practice would look like at high, medium, and low levels as another check of their understanding.

At the end of a week of training, coders were given a test video to determine their readiness to code independently. All coders gave scores within 1 level of the trainer's scores, which allowed them to be released to code videos gathered from TIMES instructors. Five of the six coders then went on to each score eight to twenty-one other videos. (The sixth did not score any videos after training.) In order to insure reliability, the trainer had a meeting with each coder after every fifth video to make sure all scores remained within 1 of the trainer's scores. In seven of the ten meetings, the scores were all within 1 of the trainer. In cases where the coder was off by two levels, they were asked to rewatch and rescore the video in light of the discussion with the trainer before being allowed to continue scoring videos.

## Discussion

In this paper, we outlined our development of a rubric for IOI. Creating a measure for IOI at the undergraduate level presented non-trivial and unique challenges. First, it was necessary for the IOIM to have the flexibility to be utilized across an array of undergraduate mathematics courses. Though, not only does the content differ across introductory courses such as differential equations, linear algebra, and abstract algebra, most notably, the mathematical goals are often vastly different. For instance, an introductory differential equations course is often intended to develop an understanding of solution methods, whereas introductory abstract algebra is often utilized to develop notions of formal mathematical proof. Instead of being overlooked this difference in goals needed to be flexibly built into the measure.

Second, the IOIM needed to incorporate a wide variety of instructional strategies. From a theoretical standpoint, in IOI the teacher navigates along the continuum of pure telling and pure student exploration (Rasmussen & Marrongelle, 2006). From a practical standpoint, flexibility across instructional types was necessary because of the nature of the TIMES project: supporting instructional change. That is, the measure needed to provide information regarding how the participating instructors were incorporating aspects of IOI into their instruction and to what degree they were doing so. These challenges and others greatly influenced the resulting structure of the measure. With this we hope to contribute to a broad community, one consisting of mathematics education researchers as well as practitioners.

# Inquiry-Oriented Instructional Measure

| Practice | Component | Description | 5 | 4 | 3 | 2 | 1 |
|---|---|---|---|---|---|---|---|
| | | | High | Medium High | Medium | Medium Low | Low |
| 1. Teachers facilitate student engagement in meaningful tasks and mathematical activity related to an important mathematical point. | Generate | This practice is characterized by student engagement in mathematical activity related to an important mathematical point. This activity is likely to be supported with meaningful, cognitively demanding tasks. (Jackson et. al, 2013; Hiebert, 1997; Speer & Wagner, 2009). However, we are not evaluating the tasks, rather the quality of the mathematical activity.<br><br>Stein's "doing mathematics" includes conjecturing, argumentation ... all of the things we call engaging in authentic mathematical activity. (Stein, et al., 2008)<br><br>This activity is likely to be supported by providing the students with meaningful, cognitively demanding tasks and supporting their engagement in these tasks through questions that promote the identification and investigation of the mathematical concepts. | Students were engaged in generalizing their thinking to mathematical claims that they then tested/supported with arguments, examples, and counterexamples. The teacher tried to connect the task to the students' prior experiences with guidance that helped facilitate continued engagement in the task. The students were "doing mathematics."<br><br>**"Doing Mathematics"** | While the students were provided opportunities to "do mathematics" the majority of the mathematics during the discussion was performed by the teacher. The teacher used the tasks to engage the students in the conceptual ideas.<br><br>**"Procedures with connections"** | The teacher engaged students in the tasks, focusing on the correct answer or the procedures as opposed to the mathematical ideas underlying those procedures.<br><br>**"Procedures without connections"** | The students engaged in few mathematical practices. The majority of the "learning time" consisted of watching the instructor provide the correct answer. The teacher did not utilize the tasks to support the development of the mathematics. The teacher engaged students in tasks that largely focus on memorization of important mathematical ideas.<br><br>**"Memorization"** | The teacher lectured the material (i.e. there are no mathematical tasks for the students to engage in). The majority of the mathematical activity was carried out by the teacher, with little to no time for students to do anything but watch.<br><br>**No space for student mathematics** |
| 2. Teachers elicit student reasoning and contributions | Generate | Teachers prompt students to explain their reasoning and justify their solution strategies, with the focus on the reasoning the students utilized during the task as opposed to solely focusing on the procedures used. Research on instructional quality indicates that the type of contributions teachers elicit is directly related to the students' opportunities to learn. Thus it is important that teachers elicit thinking and reasoning that "uncover the mathematical thinking behind the answers" (Hufferd-Ackles, Fuson & Sherin, 2004, p.92). | The teacher explicitly asked students to share their approaches to the tasks and their reasoning. The focus was not on what answer they got, but more on how they arrived at that answer, how they were thinking about the problem, the method they took, and a justification for choosing that method. Students may be sharing **thinking/reasoning** and questions without prompting from the teacher.<br><br>Teacher questions and student contributions are about student reasoning, methods, and justifications, with students providing full and meaningful contributions. | The teacher routinely and explicitly asked the students to share contributions pertaining to their thinking and reasoning such as, "Why do you think the examples you generated are working that way?" "How did you do it?" and "What does this mean?" The teacher, however seemed to **interpret and revoice these contributions** for the students without allowing the students to fully explain them or respond to them, which took ownership of the ideas away from the students. Although the student's line of reasoning on the solution path is given, why the student chose that solution path is not. | The teacher typically elicited descriptions about how students solved a task, but not about how they were thinking about and/or approaching the task; **solutions and descriptions of how those solutions were obtained are the focus**. Additionally the instructor often talked about the students' contributions as opposed to allowing them to explain their thinking. The instructor elicited solutions but not reasoning. This includes well-worded questions that are asked in such rapid succession that they lose their ability to elicit students' reasoning. Alternatively, the focus of questions was still more on the correctness of procedures or answers, and not on students' understanding. | The instructor made an effort to elicit student contributions (e.g., reasoning, methods, and justifications) but often these attempts yielded **procedural statements**.<br><br>Questions are segues into lecture. | The teacher asked fill in the blank questions. There were **little to no opportunities for students** to make deep and meaningful contributions. The Instructor does not ask students to share their thinking. |
| | Build | Leatham et al. [22] note that foundational to building on student thinking is the presence of student mathematical thinking that "is likely to advance students' development of important mathematical ideas—whether the student thinking is mathematically significant" (p. 92). Thus, in order to build on student thinking, rich contributions must be elicited. | | | | | |
| 3. Teachers actively inquire into student thinking. | Generate | This practice means that instructors purposely and intently inquire into student thinking for the purposes of determining if and how student generated ideas can be utilized to promote a more sophisticated understanding of the mathematics. The questions asked by teachers not only direct student investigations and provide the teacher with insight into student thinking, they also help students refine and reflect on their own thought process (Borko, 2004; Hiebert & Wearne, 1993; Rasmussen & Kwon, 2007). By asking them questions about how they are thinking - you are supporting them in further engaging in their thinking. | The teacher engaged in conversations with the students about their thinking. These conversations indicate that the teacher was assessing their own understanding of students' thinking. There are instances of probing, rephrasing of student contributions, and representations. (Davis description of interpretive listening and Rasmussen, Kwon, Marrongelle's IO-DM, and Henningsen, Stein 1997))<br><br>It seems as though the teacher is trying to figure out, "How are my students thinking about this?" They are trying to **"build a model of students' understanding."** | The teacher appeared to **actively figure out** the student's mathematical **contributions** (e.g., reasoning, methods, and justifications) - were their conjectures valid, were the students contributions useful in developing the mathematical agenda?<br><br>The teacher appeared to try to answer the question, "Are this student's mathematical **contributions useful?**", as opposed to "What sort of thinking led to this contribution?"<br><br>*This is good pedagogical reasoning on the part of the teacher and very useful for responsiveness and connecting and all sorts of things - but it is not the main point of this practice. | The teacher attempted to figure out how the students solved a problem or task with attention to the concepts underlying that solution method.<br><br>Their focus was on why **something worked**, not how the student was thinking about the mathematics. (E.g., "because of the associative property I can regroup these elements" provides a valid mathematical justification for how to do something but does not provide information about how the student was thinking about the task/mathematics.)<br><br>**Thinking – procedures with connections** | The teacher appeared to be trying to figure out how the students solved a problem with attention to the **validity** of the student's solution **strategies**. That is the focus was on the correctness of the procedures, not on why those procedures worked.<br><br>**Trying to figure out "Can they do it correctly?"** | The instructor's response to student contributions was limited to determining if the contribution (often just "the answer") matched a predetermined and expected answer. In this sense the teacher inquiry served an evaluative role, determining the **correctness of student responses**.<br><br>There is no evidence that the teacher was trying to figure out how the students are thinking, just if their "answer" appears to be correct.<br><br>Trying to figure out "do they have the right answer" |
| | Build | Teacher inquiry serves many functions and roles throughout a lesson (see Hufferd-Ackles, Fuson, & Sherin, 2004; Johnson, 2013; Rasmussen & Kwon, 2007). With regard to building on student contributions, teacher inquiry allows teachers to form models of student thinking and understanding, reconsider important mathematical ideas in light of those models, and formulate questions and tasks which enable the students to build on those ideas (Rasmussen & Kwon, 2007). | | | | | |
| 4. Teachers are responsive to student contributions, using student contributions to inform the lesson | Build | Rasmussen and Marrongelle (2006) state that, "an important part of mathematics teaching is responding to student activity, listening to student activity, notating student activity, learning from student activity, and so on" (p. 414). By doing so, the teacher can generate instructional space where "the nature of student mathematical thinking might compel one to take a particular path because of the opportunity it provides at that moment to build on that thinking to further student mathematical understanding" (Leatham et al., 2015, p. 118). Rasmussen: IO informs questions and tasks that enable students to build important math ideas. | The instructor listened to students' contributions (e.g., reasoning, methods, and justifications) and, when appropriate, used these ideas as a springboard for follow up questions and further exploration by the students. Questions about student contributions were asked to the class. The teacher used intermediate student progress to re-direct and focus small group work towards the important mathematical ideas. The instructor often took advantage of student thinking that they felt promoted the mathematical agenda by shifting focus to a student generated justification.<br><br>Willingness and space to follow/explore unexpected contributions - where the students are doing the exploration.<br><br>**Lots of instructional space: students build on each other's contributions/questions/suggestions** | Student ideas guiding the teacher's exploration - The teacher was willing to follow/explore unexpected contributions (e.g., reasoning, methods, and justifications), and provided space to do so. However the teacher was doing the exploration, not the students.<br><br>**Some instructional space: teacher builds on students' contributions/questions/suggestions** | Student contributions (often answers evaluated for correctness) were used by the teacher as a way to support a predetermined path. Little attention or instructional space is devoted to unexpected contributions (or to eliciting unexpected contributions).<br><br>Additionally, the contributions were often utilized as a way for the instructor to lecture about the topic/idea at hand or lead the students to replying in a predetermined manner.<br><br>**Predetermined contributions to support the predetermined path.**<br><br>*One possible way of getting a score of 3 in this practice, is if a "lecturer" explains a procedure, students practice procedure, and the teacher uses their answers to indicate it is time to teach the next procedure, to do more practice problems, or to do an additional lecture based on how the students did on the procedure. | The teacher rarely considered students' contributions with regard to the mathematical agenda. The use of student contributions was limited to "fill in the blank" responses that indicate is it time to move forward. For example, the teacher lectures for awhile, then the students practice a procedure, and then the instructor explains the correct answer before repeating the process. If necessary, the teacher might lecture further if there were common misconceptions.<br><br>**Limited instructional space** | The teacher was adhering to a pre-determined lesson plan. There was little to no instructional space for exploring student ideas or questions. The teacher's instruction did not make reference to student contributions (apart from answering clarification questions). Essentially the teacher lectured for the whole period.<br><br>**Little to no instructional space** |
| | Shared | When teachers are responsive to student contributions they can create new instructional space (Johnson and Larsen, 2012). In regards to this component, the instructional space is created for the purpose of developing a shared understanding within the classroom community. | | | | | |

| Practice | Component | Description | 5 | 4 | 3 | 2 | 1 |
|---|---|---|---|---|---|---|---|
| | | | High | Medium High | Medium | Medium Low | Low |
| 5. The teacher engages students in one another's thinking (reasoning?) | Shared | Stein and colleagues (2008) provide several examples of how teachers can support students in making mathematical connections between differing student contributions and important mathematical ideas. Some of these examples include asking students to reflect on the contributions of other students, assisting students in drawing connections between the mathematics present in solution strategies and the various representations that may be utilized, and facilitating mathematical discussions about different student approaches for solving a particular problem. Doing so can prompt students to reflect on other students' ideas while evaluating and revising their own (Bransford & Fryklholm, 2000; Engle & Conant, 2002).<br><br>By engaging with one another's thinking, students are able to deepen their thinking, generate new ideas, and make mathematical connections. As discussed by Jackson et al. (2013), "the teacher plays a crucial role in mediating the communication between students to help them understand each other's explanations" (p. 648). | During whole class discussions, the teacher makes instructional space for **students to reflect on the contributions of other students** - e.g., "How are you guys making sense of what they are starting to think about here?" The teacher encouraged students to ask each other questions about alternate approaches and ways of thinking. Additionally the instructor often pointed out ideas other students had while working on the task as a way to support the students' progress. The **teacher and students share responsibility for the mathematical ideas**. Essentially, students interact with each other's ideas without the teacher acting as much of a filter. | During whole class discussion, student ideas were made public and students were asked questions such as "Did anyone think of it a different way?" Most of the interactions left the students with **access to each other's approach but not to each other's thinking/reasoning or justifications**. In general, it did feel like the **student contributions were leading to meaningful class discussion**, as mediated or filtered through the teacher. While the students are responsible for the mathematics, the **teacher is the mathematical authority** in the room.<br><br>Teacher rephrases/revoices student ideas for the purpose of having other students react/interact with it. This is done in a way that does not preserve the student's contribution - or alters/adds/"smooths" in a significant way. The students are interacting with the teacher's version of the student's ideas - as opposed to interacting directly with the other student's ideas. | Students contributed to whole class discussions, during which students were asked questions such as "Did people get the same/different answer?" However, students were not prompted to reflect/compare/assess other students' contributions or ideas or students' responses may not have included much reasoning. **Overall, there was not an explicit attempt to have students make sense of each other's contributions though they did have access to them. The teacher is still the mathematical authority.** | There were opportunities for students to engage in one another's thinking (e.g., small group work).<br><br>However, **during whole class discussion and/or lecture, students did not have access to one another's thinking**. The whole class discussions seemed more like a space to contribute possible correct ideas for feedback from the instructor, as opposed to a space for students to engage in argumentation, explanation, and/or justification.<br><br>For instance, if the only thinking elicited (Practice 2) is a med-low (i.e. procedural statements and answers are shared) then students cannot engage in one another's reasoning. | There is **no student reasoning** shared (in small group or whole group discussion) for the students to engage in. |
| 6. Teachers guide and manage the development of the mathematical agenda | Build | Teachers need to actively guide and manage the mathematical agenda and can do so by: identifying and sequencing student solutions to "ensure that the discussion advances his or her instructional agenda" (Jackson et al., 2013, p. 648); utilizing Pedagogical Content Tools "to connect to student thinking while moving the mathematical agenda forward" (Rasmussen & Marrongelle, 2006, p. 389); or by refocusing the class towards the use of certain student generated ideas, marking important student contributions, and assigning tasks meant to clarify and build on students' ideas/questions. In these ways, teachers can guide and manage the development of the lesson while building on student contributions, developing mathematical ideas in directions commensurate with the mathematical agenda, and maintaining the student ownership of the mathematics. | The teacher monitored small group work, and both recorded and used student contributions during whole class discussions as a way to promote the development of the intended mathematical ideas. The teacher **guided student explorations towards the mathematical goals** of the day. This could include asking students questions to allow them to find a solution in their own way. The teacher's questions and interactions appeared to be directed towards a clear mathematical goal. Additionally the instructor managed both the small group work and the whole class discussions. | There was a **well defined structure within the class**, and the instructor focused the students on the important mathematical ideas. The instructor utilized strong guidance to direct the lesson consisting mostly of student contributions. Specifically the **instructor was responsible for exploring the mathematics, and the students were responsible for responding to that exploration**. This could include asking questions that lead students along the teacher's line of reasoning. | There was certainly a **well defined goal and structure** within the class. The instructor focused the students on the important mathematical ideas. The instructor however did **not** seem to be **utilizing student ideas to progress the mathematical agenda**. | Though there was a goal in mind, the **structure of the class was not well defined**. Additionally, the teacher did not facilitate the progression of the mathematical agenda. | The teacher did **not** appear to have a **clear mathematical goal** in mind. The students are left to explore the mathematics at their leisure. |
| 7. Teachers support formalizing of student ideas/contributions and introduce language and notation after a need/understanding has been established. | Connecting | In inquiry-oriented instruction, as the students reinvent the mathematics, their reinventions build to be commensurate with formal mathematical ideas. The instructor must be able to promote the students' ability to connect their mathematical ideas to more formal mathematics. "The teacher plays a crucial role … in supporting students to link student-generated solution methods to disciplinary methods and important mathematical ideas" (Jackson et al., 2013, p. 648).<br><br>Formal notation is introduced after the students have generated an understanding of what is being notated and a need for it has been established. "In contrast to more traditional teaching in which formal or conventional terminology is often the starting place for students' mathematical work, this teacher [one implementing an inquiry-oriented curriculum] chose to introduce the formal mathematical language only after the underlying idea had essentially been reinvented by the students" (Rasmussen, Zandieh, Wawro, 2009, p. 203). | The students **formalized their own ideas and contributions, doing all of the formalizing except being told that the concept is a standard one with a standard name**. The students were provided with opportunities to make connections between their work and the more formal mathematical notation (e.g., students are asked to translate and/or interpret formal mathematical nomenclature utilizing their own generated ideas). When appropriate, the teacher notated student contributions with formal mathematical language and notation. This was done after the students had a chance to informally develop the mathematical concepts, and when it was likely that the students could make sense of the formal mathematics by connecting it to their previous work.<br><br>There is instructional space for students to formalize their own ideas - where formalizing can mean connecting to standard language and notation. This formalizing draws heavily on the students' work and contributions. There was instructional space for students to translate their contributions and prior reasoning into the newly introduced language and notation. | Teacher does the formalizing of ideas that originated with the students - where formalizing can mean connecting to standard language and notation. This formalizing draws heavily on the students' work and contributions. **The teacher used notation as a way to capture, notate and translate student ideas and contributions. The translation however was done by the teacher, not by the students.**<br><br>The teacher formalized the students' mathematics and made the connections (i.e., there was not an opportunity for students to translate or formalize their own ideas - this was done by the teacher)<br><br>The difference between this and high: with medium-high the teacher records student thinking with notation, but then the teacher does all the translating into the notation. With "high" the teacher introduces notation and then asks the students to do the translation. | Students were given an opportunity to work toward formalizing their own arguments and ideas. However, there is a **significant disconnect between what the teacher introduces and the students' work.** This can happen in a number of different ways. For instance:<br>– The teacher does not draw on the students' ideas/work. Instead, the instructor follows up students' work with a summary that does not directly reference the students' ideas or makes no explicit connections between the students' ideas and the language and notation introduced by the teacher. In effect, the **teacher ignored the formalization students had done.**<br>– The teacher does not attend to the ways in which the students are using formal language and notation, causing **a disconnect in the meaning of words used by the teacher and the students** (e.g., a student uses the term "inverse" or "one-to-one" and the teacher assumes a shared formal meaning).<br>– There are **not enough student ideas** present for formalizing. | **The students were given a brief opportunity to formalize ideas, but they were not successful in formalizing.** This lack of success could be due to insufficient time to complete the task, a poorly framed task, or students not understanding well enough to approach the task. One example of this may be if the students were asked to complete a sequence of tasks, and at the end were asked to come up with a conjecture/generalization about a pattern common to task sequence. Their contributions were ill-formed or not utilized by the instructor.<br><br>Additionally, there were little to no connections between the students' ideas that did arise and the language and notation introduced by the teacher. | **Formal mathematical language and notation were introduced prior to student engagement with a task.** The language and notation were introduced before a need had been established for it. (e.g., Definition/Theorem/Proof)<br><br>**Alternatively, standard mathematical language and notation are not introduced to the students at any point in the lesson.** |
| | Shared | Mathematical language and notation does not only serve as a way to mark important mathematical ideas. In an inquiry-oriented classroom, the introduction of language and notation also serves as a way for teachers to support the use of a common way of thinking about a mathematical idea, which then serves as a launching point for the construction of more formal mathematics. Rasmussen and Marrongelle (2006) indicate that inquiry-oriented instructors often notate student thinking in ways that allow the class to then solve new problems. | | | | | |

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# Professor Goals and Student Experiences in an IBL Real Analysis Course: A Case Study

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*Through an in-depth case study of one real analysis course taught by a very experienced instructor, we gain insight about two goals expressed by advocates of Inquiry Based Learning (IBL) instruction: developing students' persistence in mathematical study and their identity as mathematics learners. The research study was guided by collaborative workshopping research priorities and questions with of a group of experienced IBL instructors. We provide an in-depth characterization of this highly-experienced instructor's conceptualization of his teaching practice in undergraduate Real Analysis; specifically, we identify how his deviation from conventional proof-oriented instruction served to uphold his key goals that students* create proofs *and* overcome challenges*. We then use this characterization of his practice to report on students' experiences learning in the course, especially as related to the professor's two goals.*

*Keywords:* Inquiry-based learning, identity, persistence, instructor goals, motivation

## Motivation for the Study and Research Questions

There currently exist multiple broad movements in undergraduate mathematics education toward various forms of inquiry-oriented instruction (e.g. Dawkins, 2014; Kogan & Laursen, 2014; Kuster, Johnson, Keene, & Andrews-Larson, 2017; Rasmussen & Kwon, 2007). In this context, the term *inquiry* covers a range of particular notions and functions. Kogan and Laursen's (2014) study distinguished Inquiry Based Learning (IBL) courses that spent more than 60% of class time on student-centered activities from non-IBL courses in which instructor speech occupied more than 85% of class time. We rather focus on particular values and goals endorsed by practitioners of inquiry-oriented instruction. This project was initiated by the authors' participation in a collaborative workshop hosted by the American Institute of Mathematics (http://aimath.org/pastworkshops/iblanalysisrep.pdf). The workshop brought together IBL real analysis instructors and mathematics education researchers focused on real analysis to foster professional partnerships and to outline some agendas for research on IBL real analysis instruction. One such research agenda focused on how IBL instruction influenced students' persistence in mathematical study and their identity as mathematics learners. In response, we formulated the current study of the teaching practice of one highly experienced IBL instructor (Professor X) and his students' experiences. We pursued the following questions:

1. What goals for student development does the IBL instructor articulate throughout teaching the course and reflecting on student progress?
2. How does the professor structure the course and his interactions with students to achieve his articulated goals and provide all students with appropriate opportunities to overcome the challenge of creating proof?
3. How do student interact with the course structure and instructor to navigate through the course and how do these trajectories achieve or challenge the instructor's learning goals?

## Theoretical Perspectives on Inquiry in Mathematics Instruction

Rasmussen and Kwon (2007) provide an influential definition of inquiry for undergraduate mathematics instruction. In their view, classroom inquiry includes both 1) student inquiry into mathematical tasks that are meaningful and accessible to them and 2) the instructor's inquiry into students' mathematical reasoning. The first type of inquiry helps students see mathematics as a human activity in which they participate. The latter allows the professor to build instruction on student thinking. As we shall argue later, Professor X's practice was compatible with both criteria, though student inquiry was more prominent and Professor X built on student thinking in more indirect ways. Since it is rooted in Realistic Mathematics Education (Freudenthal, 1973; Gravemeijer, 1994) this tradition of inquiry emphasizes a range of mathematical activities such as defining, conjecturing, theoremizing, and proving. Professor X instead almost exclusively invited students to prove mathematical claims. He provided the vast majority of definitions and statements to be proven in the course script (though students were not always told whether the given statements were true or false). Kuster et al. (2017) describe four principles of this tradition of inquiry oriented instruction: generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation. Of these, Professor X only focused on the first because he wanted to maintain the independence of student contributions in overcoming challenges.

Professor X is more directly aligned with the tradition of inquiry studied by Kogan and Laursen (2014). Professor X does not allow any collaboration among students in his course, which was a key component of the positive experiences Kogan and Laursen reported. Still, those authors go on to explain, "Public sharing and critique of student work may serve as vicarious experiences that enhance self-efficacy and link effort, rather than innate talent, to mathematical success" (p. 197). This explanation of how peer presentations may influence students' mathematical mindset (Good, Rattan, & Dweck, 2012) suggests affective mechanisms that would still be present in Professor X's class, though students worked independently.

## Literature Review

Numerous studies attest to the significant challenges students face in learning the definitions and logic native to real analysis. Conceptual difficulties abound with limits (Oehrtman, Swinyard, & Martin, 2014; Pinto & Tall, 2002), monotonicity (e.g. Alcock & Simpson, 2017; Bardelle & Ferrari, 2011), cardinality (Shipman, 2012), completeness (Durand-Guerrier, 2017), and compactness (Dubinsky & Lewin, 1986). Definitions in analysis frequently include multiple quantifiers that evoke non-normative interpretations among students (Dubinsky & Yiparaki, 2000). Real analysis also includes proof methods students need to learn (Weber, 2001) such as universal generalization (Durand-Guerrier, 2008), absolute value inequalities, and constructing functional relationships between quantified terms. Students' must also reason fruitfully about examples and visual representations to coordinate their concept image with the concept definition (e.g. Alcock & Simpson, 2004; Tall & Vinner, 1981). As mentioned above, Professor X did not present on these definitions, but rather provided students with tasks to prove or disprove and expected students to learn about these concepts relatively independently. As we shall report, Professor X reorganized these concepts to facilitate student learning, though it is beyond the scope of this report to fully explore this conceptual reorganization.

## Methods

The bulk of the research data gathered consisted of Professor X's reflections on the class meetings and student learning as facilitated by the two researchers. Two weeks into the class, after Professor X had time to get to know the students and allow them to settle into the routine of the course, we created an *Initial Summary Document* including summaries of the students based on Professor X's observations and past experience. Using categories created by the expert participants at the AIM workshop, we prompted Professor X to classify students' early proving capabilities, from Novice to Master. We then established two online collaborative documents, an *Interactive Diary* and a *Student Summaries*. In the interactive diary, Professor X recorded interesting and noteworthy episodes from the class as well as general comments about the course notes or classroom culture. Professor X also created notes on the work and progress of individual students in the Student Summaries file, based on student presentations, office hour visits, and homework. The researchers provided comments and questions in each document. We used different colors and date stamps for each entry to make it easy to track the dialogue among speakers and to notice what still required response. We also conducted monthly phone interviews to discuss the interactive diary and student summaries entries, adding to each as possible.

At the end of the semester, the researchers requested interviews from 15 students, five from each of three ranges of success at the end of the semester as perceived by Professor X. Three students from the lower success range (faux initials CR, TJ, and BT), four students from the middle range (LN, KC, SQ, and RU), and three students from the higher range (TC, OI, and OO) agreed to be interviewed. We crafted questions to elicit their perceptions of: 1) the course, 2) the IBL instructional methods, 3) professor and peer interactions in the class, 4) how they progressed in the class, 5) how they benefitted from the class, 6) what they struggled with, 7) what they were most proud of, and 8) what helped them most throughout the course. We conducted a 60-90 interview with each of the 10 participants with online videoconference software so that we could share images of their work and record the session. We centered large portions of the interview on artifacts of the students' work to ground the conversation in the class activity. The researchers then conducted one final interview with Professor X to discuss the work and interview responses of each of the 10 interviewees and his overall assessment of the course.

After all data was collected, the researchers reviewed the student and professor interviews, presentations, homework, and notes files, first to document any insights relative to Research Questions 1 and 2 on Professor X's goals for the course and his enacted strategies to achieve those goals throughout the course. We then made subsequent passes through the data focusing on Research Question 3 for one of the 10 interviewees at a time. We then wrote a narrative detailing each of the 10 student's experience in the course using code words for the 35 emergent categories from the initial analyses when possible. We refined categories and clustered students by similar experience, resulting in 6 subgroups of the 10 students. Rereading the narratives within each cluster, we identified broad characteristics that separated the students into subgroups, resulting in an emergent hypothesis about the effects of student buy-in, goal orientation, and achievement in the course defining their overall experience. We detail the various categorizations with case studies in the results section.

## Results

The following articulates our model of Professor X's conceptualization of his own teaching practice relative to the primary goals of *creating proofs* and *overcoming challenges*. The emergent categories from our model appear in italics. They are organized to portray how they all coherently operated in service of Professor X's two primary goals. We then present three accounts of student experience representing variations within the IBL learning environment.

### Creating Proofs

For Professor X, creating proof is the heart of mathematical practice and students should not complete a course of study in mathematics without learning how to prove independently. To push students to create proof is to engage them in *real mathematics*. Professor X also believes it gives students *deeper understanding and ownership* over what they learn.

**Proof competencies.** To successfully apprentice students in creating proofs, Professor X attended to their growth in terms of three requisite skills for proving: *writing/logic*, *ideas*, and *details.* This view of proof writing competence was embedded in his *assessment structure* on homework assignments in which the grades corresponded closely to the proving competencies he wanted to foster. Students could *rewrite* a proof if they earned below a B. Professor X focused the first four weeks of the course on developing writing and logic by allowing students to turn in proofs that other students presented for homework. After that point students could only submit a problem before it was proven in class.

**Means of learning to prove.** Students were required to *turn in proofs as homework* once per week and *present completed proofs* at intervals throughout the semester. Professor X was readily available for *office hours* and had many students discuss their proofs with him before presenting to the class. In this context he provided *differentiated feedback* targeted at the competence he perceived that the student needed to develop. Students were expected to learn from observing *peer presentations*, which allowed them to see proof approaches that were more accessible to them and help them see proving as an activity in which they could engage.

**Reformulated content**. To help students independently create proofs in the complex context of Real Analysis, Professor X *reformulated some of the mathematical content*. For instance, the definition of supremum was divided into Right Most Point (RMP) and First Point to Right (FPR), which respectively apply when the supremum is in the set or is not. Neighborhoods were described by order relations rather than absolute value inequalities. Because he valued proof creation, Professor X consistently *ignored elegance and efficiency* in student proving. He generally avoided *demonstrating standard methods*, but rather *legitimized student proofs* so long as they were valid and covered all cases

**Differentiated feedback.** Many students praised Professor X's feedback. This feedback consisted of *praising and validating student work* and *providing minimal prompts* to move them forward. Professor X articulated gaps in student proofs as lemmas, an instance of *mathematizing student contributions*. Professor X consistently tried to provide minimal feedback so that students retained *ownership* over their created proofs. Providing counterexamples to student proofs was also a way that Professor X *expanded students' concept image* over time.

**Creating mathematicians.** A final aspect of Professor X's practice related to creating proof was his overarching goal of training new mathematicians. Professor X persistently *invited appropriate students to further study in mathematics*. This invitation had an affective influence, since it represented an expert's high assessment of students' mathematical ability.

**Overcoming Challenges**

The other overarching aspect of Professor X's view of the value of inquiry-based instruction was pushing students to attempt and complete challenging proof tasks. Stronger students benefitted because they often had never faced challenge in mathematics courses before. Weaker students gained *confidence* by independently writing proofs and presenting them to their peers.

**Task difficulty.** One key means by which Professor X encouraged students to attempt and overcome challenges was including tasks in his notes that varied markedly in difficulty. Students could not always assess a task's difficulty, leading them to try hard tasks inadvertently. Stronger students who could identify easier tasks turned in proofs for *simple* and *complex tasks* to maintain good grades while spending more time working on *challenge tasks*. Finally, some of the students that Professor X identified (and who self-identified) as mathematically weaker reported attempting problems later in the notes so they had more time to work on them before others presented them in class. Professor X at times withheld *feedback* or guidance from students he perceived as mathematically strong because he wanted to maintain the challenge of the task and their independence in creating a proof. For most students, Professor X very *intentionally praised what was good* in their proof attempts to encourage them to persist in the challenges. Professor X valued how strong students presenting incorrect proofs *legitimized the struggle of creating proofs* on their own, which for him represented *real mathematics*.

**The Case of BT**

Initially, BT reported being very intimidated by the course, especially because she had to present proofs to her peers. She reported that she started learning more after she could not copy other students' work. Her homework average was a 90, suggesting she earned A's often on her first try. Professor X held a higher view of BT's proving ability than she did. English is not BT's first language, which affected her confidence. Partly due to low confidence, she only presented twice during the semester and thus earned a B presentation grade.

BT reported two significant moments that helped her feel more confident. First, she solved a problem independently and was able to present it to the class, about which she reported:

> For this one, I feel like I figured it out on my own…. So that's why I feel like this was my proud moment proof.… After [this problem] I could definitely see myself growing. That's when I saw that I went from here to here [*raising her hand to indicate levels*]. That's when I started thinking mathematically more…. I felt like I started understanding more, and in a way I started enjoying the class more. I wasn't able to always understand what was put on the board, but I feel like I could grasp the idea, and, in a way, if I ever sat down and worked on it long enough I would be able to prove it.

Later, Professor X personally invited her to take Analysis 2 on the strength of her performance. Regarding Professor X's praise and invitation to take Analysis 2, BT said,

> And I was, "Huh. I might be good in this." I didn't see it, but him telling me this stuff definitely helped me believe it.… Maybe I did get something out of this. Because I don't see it in myself.… He made a big difference…. So in a way I feel like him telling me, "you can do it." It was the push that I needed. Ok, if he sees it then, you've definitely got it. You just need to work on it.

BT was aware of what she did not understand. She looked for problems that made sense to her and often went ahead in the notes to new topics that others had not studied yet. She anticipated this gave her more time and she wanted to make sure the students in the front row who "knew their stuff" would not present it before her. She was also intimidated by indirect proof, even though she felt this led her to write longer case-based proofs. Later in the course, she deliberately attempted proof by contradiction to give herself a challenge. Professor X agreed with BT's assessment of her improvement. She quickly grew to solid B-level work completing basic proofs. She did not achieve "master level" by producing more complex proofs.

**The Case of KC**

KC was a strong student who expressed appreciation for most all of Professor X's goals and values for IBL instruction. In fact, as a preservice teacher he said he wanted to use IBL in his own future high school classroom because he valued the way it helped him learn. Professor X gave very minimal feedback to KC because he thought he could figure out what was wrong and fix it. KC praised the quality of Professor X's feedback, especially how he could tell him how to correct his proof without letting him know whether the statement was true or false.

KC reported working for long periods on the proofs from this class and enjoying learning. He compared the work in this class to mathematicians' proving, except he felt they had much greater guidance and support from Professor X through the definitions, tasks, and feedback. KC enjoyed challenging problems and often got hooked on trying to see if he could solve them. He had been working extensively on P22 and planned to continue his work after the semester ended if he had not proven it yet. While appreciating the challenge, he said it was humbling to find a problem that he could not yet solve. Throughout the semester, KC was willing to attempt hard problems and Professor X commented that he presented his work well because he had thought hard about the problems at length. Professor X said it was clear that KC 100% liked the course and did hard problems and clear that he had it all in his head, with a "complete and firm grasp of everything."

**The Case of SQ**

Only one of the 10 interviewees, SQ, was overtly critical of the IBL nature of the course. Others expressed beliefs that implicitly diverged from Professor X's goals. As a preservice teacher, he thought IBL could have uses, but that instruction should usually be more direct.

SQ was very strong mathematically, but expressed frustration over the challenging nature of the course. Professor X told him he would appreciate challenges when he found something he loved and successfully overcame them, but SQ disagreed. He resented the way Professor X pushed him to do more. He wished Professor X guided him toward more efficient approaches.

SQ reported only working on the course homework for the hour before class started, but he performed well due to strong mathematical ability. He thus sought easier tasks and tried to avoid challenges. The variation in task difficulty both allowed SQ to find tasks he could complete easily before class started and allowed Professor X to implicitly push SQ toward more challenging problems if he attempted them after misjudging their difficulty. Professor X offered a telling interaction between the two:

> He probably wasn't doing any more work than looking for the easier problems, then came by my office one time to ask about a question…. I said that's all very good work and very nice and I'm really looking forward to seeing what you do from here, and he said "well I don't really want to work on it anymore." And I said you wouldn't be taking the class if you didn't want a challenge. You didn't come to college because you didn't want challenges. You came to college precisely because you do want challenge. He says, "No. Challenges make me anxious." And he actually was vibrating and sweating. And I noticed that after that when he would ask a question, he would be very nervous. Like it made him very uncomfortable when I would challenge him with a question. And yet I was doing it because he was clearly talented.

SQ seemed to understand Professor X's goals and intentions, even though he did not buy in.

## Discussion

Our results are structured as an answer to Research Question 1, identifying Professor X's primary instructional goals of *Creating Proofs* and *Overcoming Challenge*. We organized his classroom strategies to align with these goals in a partial answer to Research Question 2. We now summarize the range of student experiences of the resulting classroom environment illustrated by critical distinctions among our three cases, BT, KC, and SQ, thus answering Research Question 3. We will then return to Research Question 2 and discuss how the various strategies employed by Professor X afforded a wide range of students create their own meaningful proofs and overcome relevant challenges.

## Student Buy-in and Goal-Orientation

Our clustering of student experiences and exploration of the range of variation within each group resulted in distinctions along two primary dimensions: students' goal-orientation and their level of buy-in for Professor X's IBL instruction. Dweck & Legget (1988) demonstrated that individuals who viewed intelligence as innate and fixed in an achievement situation typically adopted a goal to demonstrate proficiency, and they persisted only in cases of perceived success while avoiding challenge when they perceive failure. In contrast, individuals who viewed intelligence as malleable and able to grow with use typically adopted a goal to increase their competence, and they persisted seeking challenge regardless of success. BT represents a case of high buy-in to the course goals through a performance orientation. She primarily avoided hard problems and developed pride in being able to complete many of the easier ones for a high grade. This success improved her confidence and she later sought some challenge by branching out to trying proof by contradiction. KC represents high buy-in with a learning orientation. He enjoyed the challenge of the class, even seeking to continue work on difficult problems after the class ended. He appreciated Professor X's IBL approach because it helped him learn and feel like a mathematician, and he wanted to adopt the approach in his own future teaching. SQ represents low buy-in with a performance orientation. Although he and Professor X both assessed that he was more than capable of doing the hardest work in the course notes, he reacted negatively and viscerally to the challenge. He expressed feelings of frustration over both the style of the course and his struggle on problems that he could not immediately solve. We observed no students with learning orientations that did not buy in. We thus generated six categories because we subdivided each of the three categories above between moderate and high achievement in proving.

## Differential Engagement of Professor Goals

Professor X consistently expressed goals of developing his students' ability to construct their own proofs and overcome meaningful mathematical challenges. Engaging a variety of students in the class, Professor X adopted several strategies that allowed students to differentially benefit from the course. Based on his judgment of their ability, he sought to give each "a problem worthy of their intellect." He subsequently offered support and feedback to enable them to be successful yet retain intellectual ownership of that success. He challenged students at different levels by offering differentiated feedback withholding (what he judged to be) just the right amount for each student to succeed. He valued success at multiple levels: 1) writing meaningful mathematics and logic, 2) developing key ideas for proofs, and 3) effectively attending to all details for a rigorous argument. He also enabled all students an appropriate entry point by reworking the content to more conceptually accessible units that afford proof without clever techniques, including problems at a wide range of difficulty throughout the course notes. Allowing students to improve and resubmit homework problems supported his focus on overcoming challenges. Professor X continually fostered his students' confidence, initially by allowing them to turn in presented proofs at the beginning of the semester, and always finding some aspect of their work to genuinely praise. Understanding that students would place different value on developing mathematical reasoning, Professor X took the long view of such difference, saying "It doesn't surprise me that many kids have different perspectives, and that's totally ok with me." He simultaneously valued what they got out of it for their current priorities and maintained hope that many would someday come back for graduate study in mathematics.

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## Partitioning a Proof: An Exploratory Study on Undergraduates Comprehension of Proofs

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*In this paper, we explore eleven undergraduate students' comprehension of two proofs taken from an undergraduate abstract algebra course. Our interpretation of what it means to understand a proof is based on a proof comprehension model developed by Mejia-Ramos, et al. (2012). This study in particular examines the extent to which undergraduate students are able to modularize a proof using the proof's key ideas. Additionally, eleven doctoral students in mathematics, referred in this paper as experts, were asked to provide modular structures for the same proofs that the undergraduate students received. We employed experts' modular structures of the proofs to analyze that of undergraduates'. The main finding of the study is that, contrary to experts' proof modularization, undergraduates partitioned the proofs in a way that failed to highlight how key components of the proofs are logically linked, suggesting an inadequate proof comprehension.*

*Key words:* Proof, Proof Comprehension, Modularization, Abstract Algebra.

Mathematics majors are expected to spend ample time on reading and writing proofs. However, despite its importance in undergraduate mathematics education, research on proof comprehension is limited. In fact, much of the proof literature focuses on students' aptitude to construct or validate proofs and less on their ability to comprehend proofs (Mejia- Ramos et al., 2012; Mejia-Ramos & Inglis, 2009). Mejia-Ramos and his colleagues (2009) systematically investigated a sample of 131 studies on proofs and they found that only three studies focused on proof comprehension. They hypothesize that the scarcity of the literature on proof comprehension is perhaps due to the lack of a model on what it means for an undergraduate student to understand a proof. In this study, we used an assessment model for proof comprehension that was developed by Mejia-Ramos, et al. (2012) to explore undergraduates' comprehension of proofs. In particular, this study seeks to examine the to extent to which undergraduates are able to modularize a proof to enhance their proof comprehension.

### Theory: Assessment Model for Proof Comprehension

Mejia-Ramos, et al. (2012) proposed that one can assess undergraduates' comprehension of a proof along seven facets. These seven facets are organized into two overarching categories: local and holistic. A local understanding of a proof is an understanding that a student can gain "either by studying a specific statement in the proof or how that statement relates to a small number of other statements within the proof" (p.5). Alternatively, undergraduates can develop a *holistic* comprehension of a proof by attending to the main ideas of the proof. According to the model, students' holistic comprehension of a proof can be assessed by asking students to identify a modular structure of the proof. A good modular structure of the proof shows an understanding of how key components or modules of the proof are logically connected to obtain the desired conclusion.

## Review of the Literature

Research looking into students' comprehension of proofs is relatively sparse. In Weber's (2012) study mathematicians reported that they measured their students' understanding of proofs by (1) asking students to construct a proof for a similar theorem to the one that was proven in class, and/or (2) asking them to reproduce a proof; and some said they do not assess their students' understanding of a proof. However, one cannot accurately capture students' comprehension of a proof by having them reproduce it (Conradie & Frith, 2000).

There are fewer studies on what students do when they read proofs for understanding. For example, Inglis and Alcock (2012) conducted a study that compared and contrasted beginning undergraduate students' proof-reading habits to those of research-active mathematicians. By studying their participants' eye movement while reading a proof, they concluded that undergraduate students, compared to the experts in their study, spend more time focusing on the "surface feature" of a mathematical proof. Based on this observation, the researchers suggest that undergraduates spend less time focusing on the logical structure of the argument; this, in turn, seems to explain why students often have difficulty understanding the logical structure of a mathematical argument, as evidenced elsewhere in the literature ( Selden & Selden, 2003).

Recent studies on novice proof readers suggests that undergraduates are not successful in gleaning understanding from the proof they see during lecture ( Lew et al, 2015). For example, students interviewed in Lew et al.'s (2015) study did not comprehend much of the content the instructor desired to convey, including the method used in the proof. Students interviewed in Selden and Selden's (2003) study also failed to understand a proof holistically since they were fixated on verifying each line and put little emphasis in attending to the overarching methods used in the proof. One purpose of this study is to build on the growing body of research on proof comprehension.

## Research Methodology

### Participants and Research Procedures

This study took place in a large public university in the northeastern United States. The content of the proof used in this study come from an introductory abstract algebra course. In the chosen research setting the standard textbook used is *Abstract Algebra: An introduction* by Hungerford (2012). The goal of the course (as stated in the syllabus) is to introduce students to the theory of algebraic structures such as rings, fields, and groups in that order.

Since the main purpose of this study is to explore undergraduates' comprehension of proofs—in particular, proofs that appear in an introductory abstract algebra course—the lead author personally approached undergraduates who had taken or were enrolled in an introductory abstract algebra course. Eleven undergraduates agreed to participate in this study and were assigned pseudonyms S1-S11. At the time of the study, six of the eleven undergraduate participants (S3, S5, S6, S7, S8, and S9) were enrolled in an introductory abstract algebra course. Seven participants—S1, S2, S3, S5, S6, S7, S8, and S9—were pursuing a major in secondary mathematics education and said they intended to be high school mathematics teacher. The remaining four students were mathematics majors.

In addition to undergraduates, we used eleven doctoral students, to conduct a fine-grained analysis of undergraduates' proof comprehension. At various times, we asked the doctoral students to provide, in writing, modular structures of the proofs. To avoid confusion, in the remainder of this paper we will refer to these doctoral student participants as experts.

In this study undergraduates were given two proofs, proofs A and B found in appendix 1 and 2, and were asked to read for understanding. We chose these proofs for various reasons, including their pedagogical value. For instance, proof A was chosen because it illustrates conditions that one can impose on integral domains to make them fields. Undergraduates were asked to read the proof until they felt they understood it and were encouraged to write and/or highlight on the proof paper as well as to think out loud while reading. Once a participant finished reading a proof, we asked her to (1) partition a proof into its modular structure and (2) explain the purpose of some assertions and how they are logically connected to prove the claim.

**Data Analysis**

Recall that eleven doctoral students in mathematics were asked to provide a modular structure for both proofs. All eleven modular structures of proof A that the doctoral students provided were studied carefully and resulted in what will hereafter be referred as the expert's modular structure. The expert's modular structure of proof A reads as follows:

> First, fix an arbitrary non-zero element $a$ (lines 1-2 in the integral domain $R$. Second, using $a \in R$, construct a map from $R$ to $R$, and then show this map is injective (Lines 1-6). Finally, using results from about maps between finite sets, argue that the map is surjective. It follows then that $a$ has a multiplicative inverse.

Similar to proof A, doctoral students' modular structures for proof B were also studied carefully and the following synthesized expert's modular structure emerged:

> First, if $H = \{e\}$ then the claim follows trivially. Second, consider the case where $H \neq \{e\}$. Using the properties of subgroups and the well-ordering axiom, proof B, in lines 2-4, argues for the existence of the smallest positive integer $k$ satisfying $g^k \in H$. Third, it shows that $< g^k > \subset H$. Finally, using the minimality of $k$ and the division algorithm the proof establishes that $H \subset < g^k >$. It follows then, $g^k$ is the generator of $H$.

Undergraduates' modular structures of each proof were then analyzed in relation to the expert's using the rubric described in Table 1.

Table 1

*Rubric to assess undergraduates' identification of modular structure of proofs*

| Rating | Criteria |
|---|---|
| Very poor | • Participant failed to provide a partition of the proof<br>• Participant wrote something completely irrelevant or incorrect.<br>• Participant seems to have copied the claim or a significant part of the proof word for word<br>• Over all, participant described how the proof is structured in way that is *very different* form the expert's modular structure for the proof. This means participant's modular structure of the proof failed to capture the purpose of each module and how they are logically connected. |

| | | |
|---|---|---|
| Poor | | • Participant wrote something relevant to the proof, but he/she failed to discuss how each module is related to one another<br>• Participant may have repeated the claim or some ideas or sentences from the proof word for word<br>• Overall, participant described how the proof was structured in a way that has *little resemblance* to the expert's modular structure for the proof |
| Satisfactory | | • Participant partitioned the proof into modules in way that resembles the expert's partition of the proof, but does not always describe the logical relationship between modules<br>• Participant did not state clearly state the purpose of some module or components of the proof<br>• Overall, participant described how the proof was structured in a way that has *some resemblance* to the *expert's* |
| Good | | • Participant explained the purpose of each module and how the modules together prove the theorem<br>• Overall, participant's description of how the proof was structured is *very similar* to that of the expert's |

## Results

An overwhelming number of undergraduates in this study provided modular structures that suggested that they either *poorly* or *very poorly* understood how the *key ideas* of the proof are logically linked to prove the claim. More specifically, most undergraduates did not identify the purpose of some of the key arguments of the proofs. For example, six students did not correctly address the purpose of showing that the kernel of $f_a$ is trivial (see lines 3-5 of proof A). Table 2 below summarizes our assessment of undergraduates' modular structure of proofs A and B.

Table 2. *Undergraduates' modular structure of proofs A and B*

| Evaluation | Undergraduate students | |
|---|---|---|
| | Proof A | Proof B |
| Very poor | S1, S2, S8, S9, S11 | S1, S2, S6, S7, S8, S9 |
| Poor | S3, S5, S6, S7 | S3, S10, S11 |
| Satisfactory | None | S5 |
| Good | S4, S10 | S4 |

As shown in Table 2, nine out of eleven undergraduate participants provided a structure for proof A that has either little or no resemblance to the expert's. Some undergraduate students, for instance S1 and S2, wrote something that either only amounts to repeating the claim word for word or is a general comment that can be said about any proof, not just proof A. For example, when asked to break the proof into components or modules specifying the logical relationship between each of the modules, S1 wrote: "proof was structured step by step". Another student, S2, said that "[the proof] was structured as a list of consecutive steps…" Note that both S1 and S2 do not provide any thoughts on the modular structure of proof A. Other participants, while correctly describing the goal of the proof A, offered a structure of the proof that is vague. For example, S3 wrote: "The proof started with stating some definitions. Then set some constraints and stated what the goal was. Proved bijection and then the goal which was that every non-zero element has a multiplicative inverse." S3's description of modular structure of proof A's is vague in that many proofs begin with definitions, constraints, and goals. Also, observe that S3 does not mention how the map $f_a$ is defined and the role it plays in proving the claim. S8, on the other hand, attended more to the writing style of the proof and much less about its content. When asked to provide a modular structure of proof A, she wrote:

> The proof had a plan. Step 2 in the proof explains where the proof is going. Step 3 also guides the reader forward letting them know when they are going next. It uses moving language all throughout, words like next, finally, since…

Far from describing key components of the proof and indicating how they are organized to prove the claim, S8 appears to focus on words of the proof rather than the idea of the proof. Also, when asked why in proof A the kernel of the map was shown to be trivial, S8 erroneously stated it was "to support the fact that there exists [sic] no zero divisors." However, the purpose of showing the map was trivial is to show that it is injective. Her response entails that she did not recall one of the properties of integral domain which is the absence of zero divisors. Therefore, it could be the case that her inadequate knowledge of meaning of important terms of the proof such as integral domains might have resulted in insufficient comprehension of the logical structure of the proof.

By contrast, two participants, S4 and S10, offered a structure of proof A that indicated some comprehension of the proof. For instance, S4 structured proof A as follows:

> Lines 1-2 set up that which to be prove
> Lines 3-6 prove that $f_a: R \to R, f_a: x \mapsto ax$ is surjective
> Lines 7-3 prove that $f_a$ is surjective
> Line 9 proves that $\forall a \in R, a \neq 0_R, a$ has a multiplicative inverse

As evidenced above, S4's modular structure of proof A does not have too much detail. Yet, it captures all *key ideas* of the proof that is noted in the expert's modular structure. In particular, both S4 and S10 described the *key* components of proof A; namely, how the map $f_a$ from $R$ to $R$ is constructed and that it is a bijection.

As shown in Table 2, a majority of undergraduates presented a modular structure for proof B that two researchers independently deemed *very different* from the expert's modular structure presented above. For instance, six students, S1, S2, S6-S9, provided a modular structure that is vague and misses *key ideas* of proof B. In describing the modular structure of proof B, S7 wrote: "the proof was divided into components that each proved a 'lemma' that was needed for the next mini proof. All these proofs were needed to prove the claim." Note that S7 does not indicate what the lemmas are and how they were used in the proof. Indeed, what S7 wrote regarding proof B can be said for just about any proof. Moreover, S7 does not correctly identify the purpose of assertions in lines 5-8. She wrote that "…the purpose of these lines [5-8] is to show there does not exist a smaller power of $k$…"

S4 is the only participant who described a modular structure for proof B in a way that was very similar to the expert's. S4 wrote:

> First, the trivial case (lines 1-2). Next, show that there is a smallest positive integer $k$ such that $g^k \in H.$ Finally, prove that $H =< g^k >$ by showing that for all $g^i \in H, i = nk.$ (lines 5-9).

First, observe that S4 correctly noted that proof B proceeds by cases. Also, he included key components of the proof such as establishing the minimality of $k$ and using the division algorithm to ultimately show that the any subgroup of a cyclic group is also cyclic. Finally, when S4 was asked to describe the goal of lines 5-8 in proof B, he correctly indicated that the purpose of arguments or statements in lines 5-8 is to show that $g^k$generates *H.*

To summarize, undergraduates in this study demonstrated limited comprehension of proofs A and B. Indeed, six out of eleven provided a modular structure that related *very poorly* with that of the experts, suggesting limited proof comprehension. One plausible explanation for participants' poor proof modular structures has to do with lack of familiarity with the tasks we asked them to do in this study. Stated differently, undergraduates, including those in this study, are rarely asked to partition a proof and asking them to do so might not necessarily reflect their understanding of the proofs. Furthermore, some undergraduates in this study may have viewed these proofs as not long enough to warrant breaking them apart. We suggest that future studies can improve on this study by first showing participants examples on what it means to modularize and then ask students to identify a proof's modular structure.

## Appendix 1: Proof A

**Direction:** Please feel free to write any of your thoughts while reading the proof below. Also please think-out-loud while reading the proof. Note that the numbers only indicate each line in the proof for follow up questions. Below you will find a proof of the following claim.

**Claim**: Let *R* be a finite integral domain. Then *R* is a field.

*Proof*. 1. Let *R* be a finite integral domain whose multiplicative identity is $1_R$ and whose additive identity is $0_R$.

2. Since *R* is a commutative ring, it suffices to show that every nonzero element in *R* has a multiplicative inverse.

3. Let *a* be a fixed nonzero element of *R* $(a \neq 0_R)$. Consider the map $f_a: R \to R$ defined by $f_a: x \to ax$. We first show that the kernel of $f_a$ is trivial.

4. Note that kernel of $f_a = \{x \in R: f_a(x) = 0_R\} = \{x \in R: ax = 0_R\}$.

5. Since *R* has no proper zero divisors, $ax = 0_R \Longrightarrow a = 0_R$ or $x = 0_R$. But, $a \neq 0_R$ thus $x = 0_R$.

6. Therefore kernel of $f_a = \{0_R\}$ and so $f_a$ is injective.

7. Next, note that $|R| \geq |f_a(R)|$. Since $f_a$ is injective, it follows that $|R| = |f_a(R)|$.

8. Because $f_a(R) \subseteq R$ and $|R| = |f_a(R)|$, we have that $f_a$ is surjective.

9. Finally, since $1_R \in R$, we have that $\exists x \in R$ such that $f_a(x) = ax = 1_R$. So $a$ has a multiplicative inverse. Therefore, $R$ is a field.

## Appendix 2: Proof B

**Direction:** Please feel free to write any of your thoughts while reading the proof below. Also please think-out-loud while reading the proof. Note that the numbers only indicate each line in the proof for follow up questions. Below you will find a proof of the following claim.

**Claim:** Any subgroup of a finite cyclic group is cyclic.

*Proof*. 1. Suppose that $G =< g >$ and $H \leq G$.

2. If $H = \{e\}$, then $H =< e >$. Otherwise, $\exists i \in \mathbb{Z}, i \neq 0$ such that $g^i \in H$.

3. Then, $g^{-i} \in H$. It follows that one of $i$ or $-i$ is a positive integer.

4. The well ordering axiom guarantees that there is a smallest positive integer *k* such that $g^k \in H$. We will show that $H =< g^k >$.

5. Clearly, $< g^k > \subseteq H$ because $H$ is closed under the operation of *G* and $g^k \in H$.

6. Suppose that $h = g^i \in H$.

7. By the division algorithm we know that $i = nk + r$ for some $0 \leq r < k$.

8. Then, $r = i - nk$. We have that

$$g^r = g^{i-nk} = g^i g^{-nk} = g^i (g^k)^{-n} \in H.$$

9. Since $g^i$ and $(g^k)^{-n}$ are both elements of *H* and *H* is a group, it follows that $r = 0$ or we would have a smaller thank *k* positive power of *g* in $H$. Conclude that $H =< g^k >$

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Informal Content and Student Note-Taking in Advanced Mathematics Classes

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*This study investigates four hypotheses about calculus instruction: (i) that lectures include informal content (ways of thinking and reasoning that are not captured by the formal symbolic statements), (ii) that informal content is usually presented orally but not written on the blackboard, and (iii) that students do not record the informal content that is only stated orally but do if it is written on the blackboard, and (iv) that professors often most want students to learn the content they state informally. Via interviews, we also explored why professors chose to write on the board, or not, content. We recorded 5 calculus mathematics lectures and photographed the notes of 78 students. We found that informal content was common, although most informal content was presented in a written form. Typically students recorded formal content while not recording informal content.*

*Keywords:* Lecture, Calculus, Student Learning

Many, if not all STEM majors will be exposed to a calculus course in their study, even if they intend to major in something other than mathematics. In the Fall semester of 2010 alone, over 300,000 students were taking a calculus course at the undergraduate level (Blair, Kirkman, & Maxwell, 2013). It's estimated by the Department of Commerce that STEM jobs will increase by 17% from 2008 to 2018, as opposed to 9.85 for non STEM fields, such an increase in the job market requires an equal increase in new STEM majors (Langdon, et al, 2011). Yet, relatively few students begin their undergraduate careers as STEM majors and very few students transfer in. In particular, Green noted that "not only do the science have the highest defection rates of any undergraduate major, they also have the lowest rates of recruitment from any other major" (1989, p. 478), meaning, there is (nationally) a net loss of students over the undergraduate program (Hilton & Lee, 1998). Often Calculus is both a stepping-stone and barrier into these majors and in order to increase the number of STEM majors, the number of students who succeed in calculus must increase as well. The failure rate and rate at which even successful Calculus 1 students who do not go on to Calculus 2 is high. The lack of success and persistence limits a student's opportunity to pursue a STEM career.

A major recent research project has explored the impact of various characteristics of calculus classes and how they influence student success (Mesa & Burn, 2011). They have described the typical curriculum as including limits and continuity, derivatives, integration, sequences and series. Faculty tended to focus their instruction on procedural fluency, with other aspects being less common. More, they found that lecture was the most common mode of instruction (82% of the time), and that typically presentations involved symbolic manipulations with some graphical representations as well. Yet, research on note-taking (and recall) as well as advanced mathematics classes suggests that further investigation of the actual instruction of calculus could help explain some of student's frustrations and difficulties with the class. Mathematics professors certainly hope that students gain more than procedural fluency from their classes—they want (and believe) that students need to gain understanding. While they may not have the same types of definitions for understanding that mathematics educators do, they certainly include being able to flexibly use procedures and be able to explain why the procedures work. This report seeks to explore the seeming difference between what professors intend students to gain and what students believe that classes focus on.

A pair of previous studies in proof-based mathematics classes motivated this exploration. A qualitative study (Lew, Fukawa-Connelly, Mejia-Ramos, & Weber, 2016) found that students might fail to learn what the professor intended from a lecture, even when the professor repeatedly emphasized the point. In particular, if the professor's main learning objective was about informal ways of thinking, the professor typically spoke it aloud without writing it on the board and the students failed to notice it during the lecture and did not record it in their notes. A subsequent quantitative study (Fukawa-Connelly, Weber, & Mejia-Ramos, 2017) of 11 lectures and 96 students showed that informal mathematics is regularly part of lectures in advanced mathematics, that such content is typically presented orally without the professor recording anything on the board, and that students do not typically record orally presented content. While the study did not test the claim that failure to record content in their notes lead to students being unlikely to learn it, or the converse, that recording lead to learning, there is substantial literature on the efficacy of note-taking as a learning tool. In particular, students normally forget content they do not record in their notes (c.f., Einstein, Morris, & Smith, 1985; Kiewra, 1987).

Following the previous studies of calculus and proof-based mathematics, we investigated three hypotheses and one question:

1. When lecturing about calculus instructors regularly discuss informal aspects of mathematics. Specifically, these lecturers represent mathematical concepts using informal representations, discuss methods that can be useful for completing related mathematical tasks, give informal explanations of concepts and processes, and give heuristics for approaching different types of problems.
2. When lecturers discuss informal aspects of mathematics, they usually make their comments orally and do not record them on the blackboard. The blackboard is reserved for formal mathematics, most prominently, worked examples, as well as definitions, theorems, and proofs.
3. When lectures make these comments orally, students usually do not record these comments in their notes.

The question that we explored is, "what rules and heuristics guide instructor's choices about recording content on the board (or not)?" We wanted to better understand instructor's decision-making in order to understand the conditions of instruction, and, possibly, what might be malleable about their practice.

## Methods

### Participants

We recruited participants by sending e-mails to every instructor teaching a calculus course at an institution granting doctorates in mathematics (TAs leading recitation sections were not considered instructors; for simplicity, we subsequently refer to all participants as professors, even if that is not the person's actual job). Our email asked the instructor if we could record one of their lectures and invite their students to participate via a researcher photographing their notes. A subset of professors was also asked to participate in a short post-class interview about their instruction. There was no selection process for professor interviews. Professors were not told the purpose of the study. The professors were also asked not to let the students know that we would be conducting research on their note-taking during the lecture (the professors did often want to announce that a researcher would be present and studying the class). 5 different professors participated, and the content of their course is summarized in Table 1 below.

*Table 1. Overview table of instructor, class and content*

| Instructor | Overarching Course-content | Description of content in the lesson we recorded |
|---|---|---|
| M1 | Calculus 2 | Ratio and Root tests for series convergence |
| M2 | Calculus 1 | Fundamental Theorem of Calculus |
| M3 | Calculus 1 | Fundamental Theorem of Calculus |
| M4 | Calculus 1 | Fundamental Theorem of Calculus |
| M5 | Calculus 2 | Representation of a function as a power series |

**Data collection**

For each participating, a member of the research team attended a class meeting in which an exam was not given. The researcher audiorecorded the lecture, while transcribing everything that the professor wrote on the blackboard in the researcher's notes. We also attempted a rough-count of the number of students in the class, although because we sat near the front of the room in order to have good-quality audio recordings, it was impossible to ensure that the counts were completely accurate. Each lecture was scheduled to be 70 minutes, although some were shorter than 70 minutes because the professor chose to return exams and then address individual questions. At the end of the lecture, the researcher made an announcement to the class inviting students to share their notes with the researcher, even if their notes were not of high quality or the student did not take notes at all. Collectively, 96 students across the 5 lectures agreed and the researcher photographed the notes that the students took for that lecture (no students indicated that they took no notes). Each class had between 20 and 30 students in attendance. For those professors who were interviewed, the lead author would meet with the professor outside of class and asked the professor to explain what the most important learning goals for the class were and why the professor chose to convey the ideas in the way that he or she did.

**Analyzing the lectures**

Each lecture was transcribed. The authors coded the lecture for every time one of the following were presented: definitions, propositions, proofs, examples, heuristics, pictures or graphs (these required further analysis), rules, charts (e.g., tables), conceptual examples, and contextual (real-world) examples and described the mathematical content of the lecture (often by referring to the title of the section the professor was presenting on). Any disagreements between the two researchers were resolved by discussion. We developed our categories primarily via the literature, using Fukawa-Connelly, et al's (2017) codes related to proof, and, adding codes related to the presentation of procedures given the different focus of the calculus courses. We added codes for procedures (general statement of procedures) and examples of procedures (e.g., illustrating a process or algorithm). We applied each code at the sentence-level, aggregating sets of sentences in order to capture the complete instance of a particular code. Sometimes different codes would be interspersed; for example, the professor might begin an example, give a heuristic, and then complete the example. In such a case, we would aggregate the example sentences into one unit and the heuristic sentences into another. No sentences were double-coded. In the text below, we refer to definitions, propositions, and proofs as formal mathematics because the previous literature (c.f., Davis & Hersh, 1981) has called them such. Moreover, because the calculus class is focused on procedures (Mesa & Burn, 2017), we also include any step-by-step instructions of how to complete a procedure or algorithm as formal mathematics. We refer to the other content that we coded for as informal mathematics. For space reasons, we concatenate our description of coding schemes, noting that the coding scheme for *definitions, propositions,* and *proofs* were taken directly from Fukawa-Connelly, et al (2017) and that for *informal representation* was adapted to be appropriate for calculus. Because we expected two different kinds of examples; those that illustrate definitions (or concepts) and those that illustrate processes, we differentiated between them, adopting Fukawa-Connelly et al's definition and coding of *example* as our *example of a definition*, while adding a coding structure for an *example of a process*. We used the same rule as Fukawa-Connelly, et al to code whether content was written (either on the board or in student notes).

**Analysis of professor's claims**

Prior to the interviews, we identified what we believed to be the primary goals that the professors had for student learning and prepared questions about them. There were two primary types of claims that the professors made; the first, described their intended learning goals for the students. For each of the learning goals that a professor stated and the related descriptions of when and how they attempted to convey that content. We aligned those with our instances of coded content. We indicated what type of content was coded for and the mode of presentation. Because we had correctly identified the professor's learning goals from our attendance of the lecture, we were able to ask what motivated their choice of presentation mode for the content. We used open-coding to develop summary codes for decisions, and then summarized them as to whether the reasoning relied on large-scale beliefs about students, or content or structure of the course, or factors specific to the intended content.

## Results

Table 2 presents the number of instances of each category, the percentage of instances that were written on the blackboard or only printed orally, and the percentage of possible instances that these comments appeared in students' notes (for example, there could have been up to 383 total recorded instances of Oral Heuristics collectively in student notes, only 30 instances were recorded).

*Table 2. Summary of content and recording in notes*

| | | Instances in all lectures | Recorded in students' notes |
|---|---|---|---|
| **Definition** | Total: 2 | | |
| | Oral | 0 (0% of all instances) | 0% (0 out of 0) |
| | Written | 2 (2%) | 82 |
| **Rule** | Total: 18 | | |
| | Oral | 9 (%) | 11% (20 out of 180) |
| | Written | 9 (9%) | 24% (45 out of 185) |
| **Example of process** | Total: 23 | | |
| | Oral | 6 (6%) | 0% (0 out of 69) |

| | | | |
|---|---|---|---|
| | Written | 17 (17%) | 80% (219 out of 276) |
| **Informal Rep.** | Total: 14 | | |
| | Oral | 8 (8%) | 10% (17 out of 168) |
| | Written | 6 (6%) | 14% (13 out of 89) |
| **Proof** | Total: 1 | | |
| | Oral | 0 (5%) | 0% (0 out of 0) |
| | Written | 1 (95%) | 64% (9 out of 14) |
| **Graph** | Total: 2 | | |
| | Oral | 0 (0%) | 0% (0 out of 20) |
| | Written | 2 (2%) | 78% (14 out of 18) |
| **Heuristic** | Total: 33 | | |
| | Oral | 7 (7%) | 8% (30 out of 383) |
| | Written | 26 (26%) | 40% (34 out of 85) |
| **Theorem** | Total: 8 | | |
| | Oral | 0 (0%) | 0% (0 out of 20) |
| | Written | 8 (8%) | 83% (105 out of 126) |

These data largely confirm the first two hypotheses that we test in the paper. First, there were 356 instances of mathematicians presenting mathematical methods, conceptual content, modeling mathematical behaviors, and examples across the 11 lectures, or over 32 instances per lecture. This corroborates the growing body of research that mathematicians do not solely focus on formal mathematics in advanced mathematics. Second, for method, informal representation, and modeled mathematical behaviors, most of these comments were made orally and not written on the blackboard. The presentation of examples was an exception. Examples usually were written on the blackboard; we believe that this is because this allowed the mathematics professors to perform formal calculations and derivations with the examples. Third, when professors presented their comments orally, these comments rarely were recorded in students' notes. However, if they wrote their comments on the blackboard, they usually were recorded in students' notes. When the formal content was not written on the blackboard, the students do not record it. This suggests that what students record in their notes is determined primarily by the mode of presentation, rather than the type of content being presented.

## How the Professor Conveyed Content and Why

We illustrate three aspects of instruction; what the professor hoped to convey to students, how it was conveyed and why the professor conveyed it that way, and whether the students recorded it. The most important idea that the professor wanted to convey was, "that the anti-derivative is the same thing as the definite integral … that these two processes are inverses of each other." We interpreted the second statement, "that these two processes are inverses of each other" as relating differentiation and anti-differentiation. Thus, we took the professor's statement to mean that she wanted students to take away that anti-differentiation and the definite integral are the same thing and that differentiation and anti-differentiation are inverse processes.

To convey that differentiation and anti-differentiation are inverse processes, the professor described them that way orally once, without recording it on the board. She also described anti-differentiation as "going backwards" to the original function four separate times, always orally. The professor also referred to "undoing" a process another 4 times, in describing differentiation. None of the students recorded any of the orally-stated claims. More, she twice drew illustrations of the relationship in general form. One such instance was after presenting the FTC Part 1 (see *Figure 1*). She added a line noting, “that is, integration is identical to anti-differentiation.”

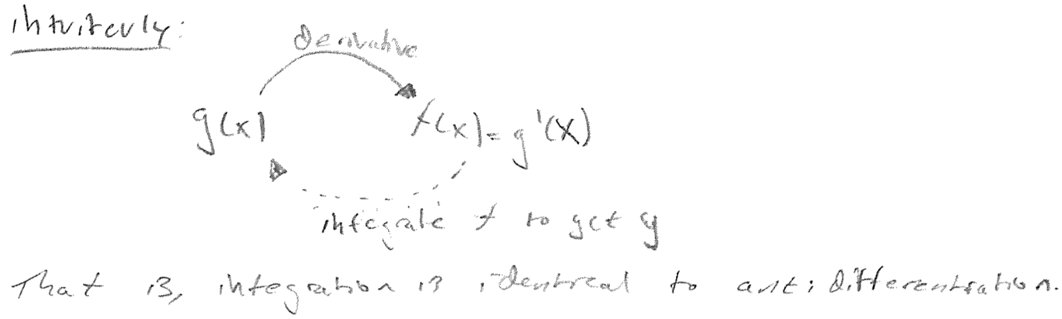


*Figure 1. Diagram after the FTC Part 1*

This diagram carried both of her intended claims and 7 of 9 students recorded the diagram while one student recorded the sentence without the diagram. The professor also drew a third illustration of the relationship for a specific pair of functions within the context of an example of a process. While 6 of the 9 students in the class recorded the example in their notes, none of them drew the diagrammatic representation included in the example. Thus, 7 of 9 students made any recording that captured the second of the two primary ideas that the professor attempted to convey out of 13 possible times that she expressed the idea, and 8 of 9 captured at least one instance of the first primary idea she wanted to convey. This was the only time she stated the first primary idea informally.

The professor gave a general explanation of her thinking related to writing content on the board. She said:

> The students who tend to be at the top of the class, I can take care of their questions by just saying it without writing it down. And then students who tend to get overwhelmed by the details and maybe perform a little bit lesser on different quizzes and things, the more I ... I have to be careful about what I write down because if I write down too much information that tends to overwhelm them and they're not able to separate the forest from the trees. Does that make sense?

We interpreted the professor here as making a collection of claims; first that the best students in her class would acquire the ideas even if they are only ever stated orally. Second, that the weaker students need something different than the strong students as “too much information tends to overwhelm them.” The idea of “separating the forest from the trees” we interpreted as giving priority to certain types of mathematical proficiencies. In particular, because the professor always wrote algebraic examples of procedures on the board as well as formal mathematical statements, she appears to be claiming that these examples and formal statements are the “forest” that the students must apprehend while the informal ideas are the “trees”—which she claims “might overwhelm them.” That is, we interpreted her as claiming that it is not necessary that the weaker students come to understand the more informal ideas.

## Discussion

Our findings offer support for the generality and validity of the following claims:

1. When lecturing about calculus instructors regularly discuss informal aspects of mathematics. Specifically, these lecturers represent mathematical concepts using informal

representations, discuss heuristics that can be useful for completing mathematical tasks, give informal explanations of concepts and processes.

2. When lectures make these comments orally, students usually do not record these comments in their notes.

We did not support the third claim. Instead, instructors recorded 26 out of 33 heuristics on the board. A corrected version of the claim is that:

- Lecturers recorded all definitions, theorems and proofs that were included in the class on the board. They also recorded 74% of examples and 79% of heuristics. Most content recorded on the board (43 of 71 instances, 61%) consisted of examples and heuristics.

Yet even though heuristics were generally recorded on the board, they were not often included in student's notes. We note that examples were recorded in student's notes at an 80% rate. As a result, we suggest two, related, concerns; first, that we have potentially mis-cast examples as informal content. As the calculus class focuses on procedural fluency, examples of processes represent an important category of content that may deserve a different categorization. Another possibility is that the notion of formal and informal is inappropriate for a calculus class; rather, the focus should be on content that describes, illustrates, and justifies procedures. When such content is presented in 'entirely mathematical' text (e.g., text for which a standard mathematics definition/meaning exists) this should be considered the 'formal' corpus of the calculus course. This data suggests that students are differentiating between the types of content that they chose to record in their notes. More investigation is needed to explore their decision to record or not record content in their notes, and, what, if any mathematical or presentation cues they use.

Finally, we have investigated why professors chose to present content in the way that they do. Our investigation reveals that instructors are thoughtful about even this level of detail in their lectures. In particular, we showed an instructor who had considered the range of students in the class and the relative importance, for the students, of the different types of content. The professor we showed here indicated two different main ideas that she wanted the students to take-away from the class, both stated using informal language. She repeatedly stated one them during the class, including drawing three diagrams that illustrated the relationship. She only presented the second 'main' idea one time, but did so in writing. We note that none of her conversation here was specific to the content presented, instead, focusing on the nothing that some students would understand the orally presented content and some needed a written presentation that focused on the "forest," which we understood to be the most important ideas. This appears to be slightly at odds with her claim that these were the most important ideas that she wanted to convey. More investigation is warranted into the decision-making, but, we caution that attempting to force instructors to specify conditions specific to particular pieces of content might lead to post-hoc justifications when the decision was not made consciously, perhaps being habit or culture.

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# Conventions or Constraints? Pre-service and In-service Teachers' Understandings

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*Several researchers have noted that it is important for students and teachers to be able to differentiate between what is mathematically critical to a concept or representation and what is a convention maintained for the purposes of communication. In this report, we describe two studies examining the extent to which pre-service and in-service teachers (PSTs and ISTs) understand graphing conventions either as conventions or as rules that must be unquestionably maintained. We highlight the extent to which conventions are pervasive in both PSTs' and ISTs' meanings for graphs and related ideas (i.e., function and rate of change) and describe why such meanings are problematic.*

Keywords: Conventions, Graphs, Preservice Teacher Education, In-service Teacher Education

Hewitt (1999, 2001) distinguished between arbitrary and necessary information in mathematics curriculum and learning. He described arbitrary information as that which students need to be informed about by an external source (e.g., the name of an object or representational conventions), whereas necessary information students can deduce for themselves. In addressing graphs and coordinate systems, Hewitt (1999) described aspects of coordinate systems that are necessary (e.g., the need for a starting point or origin and orienting vectors or quantities) and noted:

> These are some aspects of where mathematics lies within the topic of co-ordinates, rather than with the practising of conventions. I am not saying that the acceptance and adoption of conventions is not important within mathematics classrooms, but that it needs to be realised that this is not where mathematics lies. So I am left wondering about the amount of classroom time given over to the arbitrary compared with where the mathematics actually lies. (p. 5)

Whereas mathematicians and mathematics educators are likely to agree with Hewitt's distinction, the extent to which students and teachers maintain understandings consistent with his description is an open question. Hence, in this report, we present the findings from two studies, one with pre-service teachers (PSTs) and one with in-service teachers (ISTs), intended to address the question, "In what ways do pre-service and in-service teachers understand graphing conventions?" In this report, we highlight the extent to which conventions are pervasive in both PSTs' and ISTs' meanings for graphs and related ideas (i.e., function and rate of change).

## Theoretical Perspective

Of relevance to this report, when discussing students' use of notation and representational systems with respect to conventions, Thompson (1992) described two ways in which an individual can use a convention: (a) using a convention unthinkingly and possibly unknowingly and (b) using a convention with an awareness that she is conforming to a convention (i.e., convention *qua* convention). Thompson (1992) elaborated, "To understand a convention *qua* convention, one must understand that approaches other than the one adopted could be taken with equal validity. It is this understanding that separates convention from ritual" (pp. 125). We leverage Thompson's distinction in the context of teachers' graphing activity, arguing a teacher's use of graphs entails a *convention qua convention* if the teacher maintains a convention with the *awareness* of maintaining a convention (i.e., understands the convention as one way to represent some idea among other equally valid choices). We claim a teacher's use of graphs entails the *habitual use of "convention"* if the "convention" is a necessary or inherent aspect of a teacher's meanings for graphs and associated topics. In this case, what we as researchers perceive to be a convention is not a convention qua convention with respect to that teacher's meanings; hence, we intentionally use quotations to indicate this difference in perception. As we illustrate in the results section, what an observer understands to be a convention can instead be habitual to a PST's or IST's use of graphs to the extent that the teacher unknowingly assimilates situations in ways that entail the "convention". Alternatively, the teacher might consider using graphs in some different way, but the teacher does not conceive such a way equally valid due to her or his system of meanings necessitating that the "convention" be maintained.

## Relevant Literature

Understanding a convention *qua* convention involves an individual being aware of a variety of equally viable representational choices while understanding that a particular choice is customary to a group of individuals. Speaking on various conventions practiced in U.S. and international school mathematics, Zazkis and Mamolo (Mamolo & Zazkis, 2012; Zazkis, 2008) hypothesized that students are hindered in making such distinctions when they only have experiences in which educators maintain particular conventions. Mamolo and Zazkis argued that a potential outcome of educators unquestionably maintaining conventions is that students are not afforded opportunities to develop understandings suitable for novel (e.g., alternative coordinate systems) and unconventional situations.

International and U.S. education researchers (Akkoc & Tall, 2005; Breidenbach et al., 1992; Even, 1993; Montiel, Vidakovic, & Kabael, 2008; Oehrtman, Carlson, & Thompson, 2008) have documented that students often associate function in graphical situations with little more than a ritual application of the vertical line test, a common procedure taught in U.S. school mathematics. As an example, Montiel et al. (2008) identified that students were inclined to apply the vertical line test when investigating relationships in the polar coordinate system. Because some students' meanings entailed carrying out an action tied to the Cartesian coordinate system and a particular axes orientation, those students claimed that relationships such as $r = 2$ do not define a function. In this and other examples (e.g., Breidenbach et al., 1992), the researchers posed graphs that they understood to be representative of functions, yet the students' meanings for functions and their graphs did not afford such understandings.

Our purpose is not to rehash the well-documented claim that students often understand function in unsophisticated ways (see Leinhardt, Zaslavsky, and Stein (1990), Oehrtman et al. (2008), and Thompson and Carlson (2017) for extensive reviews). Rather, our purpose is to draw attention to a particular feature of students' meanings that is more deeply-rooted and problematic than researchers have previously reported. Namely, we infer that the students in these studies drew upon meanings in which what we perceive to be conventions of a particular coordinate system had become features inherent or intrinsic to those students' meanings. For instance, what we perceive to be the convention of representing a function's input along the Cartesian horizontal axis was something the students used habitually (i.e., "convention").

## Methods

In order to explore and better understand PSTs' and ISTs' understandings of conventions, we conducted two studies that used similar tasks (see Task Design). In the first study, we designed and conducted 90-120 minute semi-structured clinical interviews (Ginsburg, 1997) with 31 PSTs enrolled at a large state university in the U.S. The PSTs were entering their first semester in a four-semester preparation program for secondary mathematics teachers. Each PST began the program during her or his junior year (in credits), and each PST had completed at least two mathematics courses past Calculus II. We chose participants from the volunteer pool whose schedules aligned with the researchers' schedules.

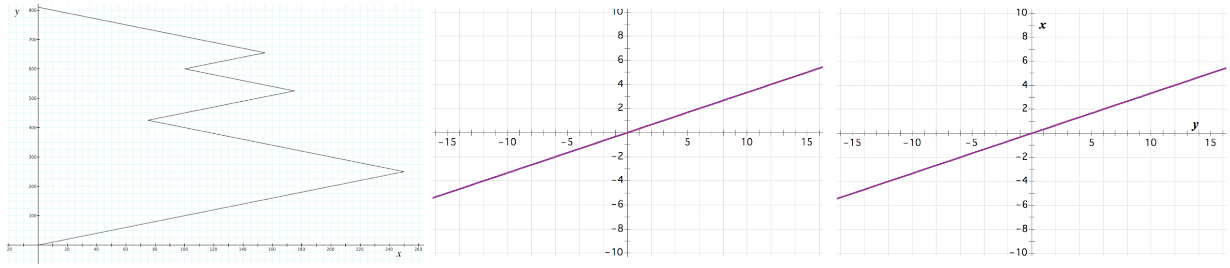
We videotaped the clinical interviews and digitized all written work. We analyzed the data using selective open and axial analysis approaches (Strauss & Corbin, 1998) and conceptual analysis (Thompson, 2008). We identified instances of PST's activity that offered insights into his or her meanings. We used these instances to develop hypothesized models of the student's meanings and we compared a PST's activity across instances and tasks in order to test and improve our interpretations of her or his activity, including identifying themes across instances and tasks. Lastly, we compared across students in order to identify compatible and contrasting meanings. The research team met throughout the data analysis phase in order to refine models of students' meanings and clarify themes in the students' meanings and uses of graphs.

In the follow-up study, we adapted our original tasks for an on-line survey completed by 45 ISTs. The ISTs were geographically distributed across the U.S. and were enrolled in a fully online graduate mathematics course designed specifically for ISTs. We coded the ISTs' responses using open and axial approaches (Strauss & Corbin, 1998) and thematic analysis (Braun & Clarke, 2006). Members of the research team analyzed a subset of the ISTs' responses and we met to discuss our observations, identify commonalities across responses, and adapt or create new codes to capture more ISTs' responses. We iterated this process four times as we refined our codes to capture all ISTs' responses; after obtaining final codes, a second researcher recoded approximately 65% of the data to check for inter-rater reliability. We obtained Cohen Kappa values of 0.78 and 0.85 for the two tasks described, indicating a high level of agreement.

## Task design

We designed each task to include what we perceive to be an unconventional feature with respect to the use of graphs in U.S. school mathematics. Because we did not expect the PSTs or ISTs to spontaneously interpret the displayed graphs as entailing unconventional aspects, we designed tasks to include specific claims with respect to features that we intended to be unconventional, often through hypothetical student responses. By including hypothetical responses focused on aspects we considered unconventional, we were able to infer the extent that something was an inherent or habitual aspect of the PSTs' and ISTs' uses of graphs.

To illustrate, we provided the graph in Figure 1a and posed a variant of, "What about a student who claims that this graph represents $x$ is a function of $y$?" With respect to ***Figure 1***b, we presented the graph as the work of a hypothetical student who graphed the relationship $y = 3x$. We asked the participants to describe how the hypothetical student might have been thinking when creating the graph. The follow-up prompt included a graph with the axes labeled (***Figure 1***c), and we explained that a hypothetical student clarified his graph of $y = 3x$ by labeling the axes as given in the second graph (i.e. $x$ on the vertical axis and $y$ on the horizontal axis). Both tasks illustrate our intent on designing graphs that can be conceived as mathematically viable (albeit unconventional) as presented with respect to the given prompts and claims.



(a) (b) (c)

*Figure 1.* (a) Is $x$ a function of $y$? (b) and (c) A hypothetical student's work to graphing $y = 3x$.

## Results

We structure the results section by first presenting the PSTs' responses to each task. We then synthesize the ISTs' responses to both tasks. We conclude by comparing the PSTs' and ISTs' responses.

### PSTs Responses

Table 1 summarizes the PSTs' responses to the claim, "$x$ is a function of $y$." Ten of the 25 ('Not true' and 'Unsure') PSTs maintained that the graph does not represent a function due to the graph not passing the vertical line test, because there exist $x$-values for which there is not a unique associated $y$-value, or a combination of both. For these 10 PSTs, "function" immediately drew to mind an action that entailed treating (implicitly or explicitly) $x$ or the quantity represented along the horizontal axis as the input quantity (i.e., "convention" as habit). For the seven PSTs who maintained that the statement is true on the condition that the graph is rotated 90-degrees counterclockwise, they understood the phrase "$x$ is a function of $y$" to necessitate a particular axes orientation—an orientation in which the defined input values are represented horizontally—they required (i.e., habitual use of "convention") that the graph be rotated before considering the validity of the claim with respect to properties of the $x$-$y$ pairing.

Finally, we interpreted seven of the 25 PSTs' actions to suggest they did not require $x$ or the horizontal axis to represent input values. Yet, five of the seven students hesitated with the claim "$x$ is a function of $y$" and described that they had a tendency to imagine the graph oriented so that the values defined as the function's input were represented along the horizontal axis. Some students first rotated the graph to determine that the statement is true and then paused when we asked if the statement was also true when considering the graph as given. Ultimately, each of the seven students understood the graph *as given* to be representative of $x$ as a function of $y$.

*Table 1. Codes, counts, and sample PSTs' responses to the statement, "x is a function of y."*

| Code (value) | # | Sample Responses |
|---|---|---|
| True (1) | 7 | Yeah I guess if you do it this way [*writes* '$x(y)$' *on paper*]…for every $y$ there is exactly one $x$. And for every $y$ [*puts marker on vertical axis on graph and moves it horizontally to a point where it hits the curve*] yeah, there's exactly one $x$…I've never thought about it that way but yeah, he's right…awesome way of thinking about that. |
| True, if graph is rotated counter-clockwise 90-degrees (2) | 8 | So she said $x$ is a function of $y$. That'd be, that'd be looking at it this way [*turning the paper 90-degrees counterclockwise*] and saying look there's no [*motioning hand over the graph as if doing the vertical line test*], there's no crossing…So, I mean that's true, but you'd have to flip the whole graph…[*redraws graph in rotated orientation, labeling the horizontal axis as y and the vertical axis as x*] That'd be $y$ and that'd be $x$. So $x$ is a function of $y$. And that's a function…[*Interviewer returns PST's attention to the graph as given*] No, because $x$ isn't a |

| | | |
|---|---|---|
| | | function of $y$. This [*motioning to her sketch*] is the graph of $y$ as a function of $x$. |
| Not true or unsure (3) | 10 | Okay. Um [*pause*] $x$ is a function of $y$. [*long pause*]…Well you know something's not a function if [*placing her marker in a vertical line over the graph*]… if two different inputs can give you the same output… Which you have here obviously that, you know, these one two three four five six $x$-values give you different $y$-values [*using her marker to mark points on the graph in a vertical line*]. I mean these, the same $x$-value can give you six different $y$-values. |

Table 2 presents a summary of the PSTs' responses to the hypothetical student who graphed $y = 3x$ as shown in Figure 1c. We interpreted each of the 20 PSTs (Table 2, the last two categories) who deemed Figure 1c as incorrect or who expressed uncertainty about the hypothetical solution to hold meanings which entailed the habitual use of "convention." These "conventions" included assigning $x$-values to the horizontal axis, maintaining particular axes directions for positive and negative values (which arose after rotating the graph), using the horizontal axis as an input quantity (and inferring from the given equation that $x$ represents input values), or a combination of these. Only 11 PSTs maintained that the graph as given in Figure 1c unquestionably represents $y = 3x$. These PSTs identified the graph's departure from convention, and specifically its departure from a customary axes orientation. They also claimed that the departure does not influence the correctness of the represented relationship between $x$ and $y$.

*Table 2. Codes, counts, and sample PSTs' responses to the prompt and graph associated with Figure 1c.*

| Code (value) | # | Sample Responses |
|---|---|---|
| Hypothetical student unquestionably constructed a correct graph (1) | 11 | He graphed it completely right. That's $y$ equals three $x$…he's not wrong. He just has a different perspective than the traditional $x$-$y$…that's just counter to tradition and normal classroom settings. But I think it's smart of him to understand that it's [the convention] not glued. |
| Hypothetical student constructed a graph that is both correct and incorrect (2) | 11 | It's wrong with like how we normally write graphs…So he should lose points because he wrote the graph in like really incorrectly to what, how the graph should be written. Like the horizontal axis should always be $x$ and the vertical axis should always be $y$. But if you're looking at it based on did he understand that, when $y$ equals three, $x$ equals one, like he understood that, um, relationship between $x$ and $y$. |
| Hypothetical student did not construct a correct graph or uncertain if the hypothetical student constructed a correct graph (3) | 9 | They messed up the placement of $x$ and $y$…They are looking at it like this right now [*rotating graph 90-degress counterclockwise*]…If you are looking at it this way, it's a negative slope [*tracing graph downward left-to-right*] and it should be a positive slope [*tracing imagined graph upward left-to-right*]…slope is wrong. |

## ISTs' Responses

For brevity's sake, we do not present the ISTs' responses to each task, as they are compatible with the PSTs' responses. Instead, Table 3 provides the codes we created to capture the ISTs' responses to the hypothetical student work for both tasks, sample responses to the task associated with the graph in Figure 1a, and counts of the number of IST responses coded within each category for each task. We note that only 12 and 25 IST responses for the tasks associated with Figure 1a and Figure 1c respectively indicate they understood conventions qua conventions for these particular tasks. The remaining 33 and 20 ISTs respectively maintained understandings that entailed a habitual use of "convention".

*Table 3. Codes description, counts, and sample responses of ISTs pre survey.*

| Code (value) | Sample Responses to the task in Figure 1a | Fig 1a | Fig 1c |
|---|---|---|---|
| The student's mathematical statement is correct despite breaking from conventions. (1) | That's great! I am so glad you were able to apply the "vertical line test" in a horizontal orientation and realize that you would have a function. You are correct in saying that $x$ is a function of $y$. | 12 | 25 |
| The student's mathematical statement is true but the student is incorrect because he/she broke from conventions. (2) | I think the student is understanding that $x$ can be a function of $y$ but they are not displaying it correctly through the graph. | 9 | 7 |
| The student's mathematical statement is incorrect or the IST did not address the student's mathematical statement. (3) | It was not a good explanation and $x$ is not a function of $y$, $y$ is a function of $x$. The value of $y$ depends on $x$. They also did not describe what would make it a function. | 24 | 13 |

## Comparing PSTs' and ISTs' Responses

In order to compare the PSTs' and ISTs' responses, we assigned numerical values to each of the three categories within each coding scheme (shown in parentheses in each table). Table 4 provides the average scores for PSTs and ISTs across both tasks. Although these values appear similar, we used a two-tailed Mann-Whitney U-test to examine if there was evidence that the PSTs' and ISTs' responses indicated that they were from different populations. There was not a statistically significant difference between the populations for either task.

*Table 4. Average scores of PSTs and ISTs and p-values obtained from a Mann-Whitney U-test.*

| | Figure 1a | Figure 1c |
|---|---|---|
| PSTs | 2.12 | 1.94 |
| ISTs | 2.27 | 1.73 |
| *p*-value | 0.4777 | 0.2937 |

## Discussion and Concluding Remarks

At the most fundamental level, our findings are significant in that PSTs and ISTs who have completed advanced mathematics courses have developed mathematical understandings that, at best, limit their ability to engage effectively with these topics in situations that we designed to be unconventional. The fact that the ISTs' and PSTs' responses were similar indicates that teaching experience may not have an influence on creating shifts in teachers' meanings with regards to graphing conventions. This finding underscores the importance of giving both populations opportunities to develop more sophisticated meanings for various ideas that are not constrained by, what to us as researchers, are conventional choices. Even more significant is that so many of these teachers (or soon-to-be teachers) held meanings that led to claims and actions that, although often internally viable to them, were contradictory from our perspective and suggested their habitual use of "convention." We do not have the data to comment on the effects of an interaction in which a teacher makes these comments to an actual student who claimed $x$ to be a function of $y$ or who produced an unconventional graph of $y = 3x$, but it is not hard to imagine that the student would be left wondering what he or she did wrong and possibly conclude that axes-variable label pairs and orientations are critical features of a mathematical idea or established rules that must be followed rather than arbitrary conventions (i.e. Hewitt 1999, 2001).

We hypothesize that the PSTs' and ISTs' meanings 'worked' for them throughout their schooling. Due to the pervasive role of conventions in school curricula and instruction, they were likely able to repeatedly assimilate their experiences to these meanings with little or no perturbation. Slope or rate of change associations based on a direction of a line are likely to 'work' in situations that maintain the "conventions" upon which those associations were constructed; function meanings that inherently or tacitly entail the vertical and horizontal axes as representing a function's output and input, respectively, 'work' in situations that maintain those "conventions." Second, due to the PSTs and ISTs repeatedly having the opportunity to construct and re-construct meanings that 'work' without perturbation, they developed a system of meanings that are internally rational and consistent. Such meanings are compatible with what Thompson and Harel (Thompson, Carlson, Byerley, & Hatfield, 2014) called *ways of thinking* – meanings that become so routine or habitual that a person (consciously or subconsciously) anticipates situations involving the associated concept to entail that meaning.

Our intention is not to discredit conventions, nor to convey that conventions are unimportant. Nor do we intend to imply that curricula and educators can realistically be expected to address every convention in mathematics. Instead, we agree with researchers (i.e., Hewitt, 1999, 2001; Thompson, 1992; Zazkis, 2008), who have argued for ensuring that students and teachers become aware of conventions as choices that do not impact the underlying mathematics. One reason for educators to support individuals in becoming explicitly aware of conventions specific to a particular group or field is that conventions vary within and among fields (i.e., it is standard in economics to represent the independent variable on the vertical axis and the dependent variable on the horizontal axis). Hence, collaborating successfully across discipline boundaries requires that an individual become operative with the conventions and practices common to each field, or at least that an individual hold meanings that enable her or him to accommodate conventions of other fields. Another important and more fundamental reason that educators should support individuals in becoming aware of conventions is the restrictions in individuals' ways of thinking that result from constructing a system of mathematical meanings dependent on "conventions". If the goal of mathematics education is to prepare students to provide correct answers in canonical settings, then such restrictions are not an issue. However, if our goal as educators is to support students in constructing a generative mathematics that helps them organize their experiences among and within fields, as well as take on advanced and abstract mathematical ideas that require students to differentiate between what is essential to an idea and what is not, then it is important that they construct mathematical knowledge that has assimilatory capacity in canonical and unconventional settings.

In closing, we argue that our results call into question the entrenched place of conventions in school curricula and instruction. Addressing this issue requires that those designing curricula and instruction take more seriously the negotiation of conventions among students and their teachers. In short, if students and teachers are to understand a convention *qua* convention, then they need opportunities to come to understand mathematical ideas in ways that enable a subsequent negotiation of conventions within the context of those ideas. That is, a productive negotiation of conventions should occur in conversations where a mathematical idea—which is understood as remaining invariant in canonical and unconventional contexts—remains the focus, as opposed to conversations that obscure conventions.

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Development of the Elementary Algebra Concept Inventory for the College Context

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*This study describes the creation and validation of the first concept inventory for elementary algebra at the tertiary level. A 22-item multiple choice/multiple answer instrument was created through a combination of literature review, syllabus review, and collaboration with instructors. The instrument was then revised and tested for content, construct and concurrent validity as well as composite reliability, using a circular process that combined feedback from experts (mathematicians, instructors, and mathematics education researchers), cognitive interviews with students, and field tests using both classical test theory and item response theory. Results suggest that the inventory is a valid and reliable instrument for assessing student conceptual understanding in elementary algebra, as conceptualized in this study.*

*Keywords:* elementary algebra, conceptual understanding, concept inventory

Elementary algebra and other developmental courses have consistently been identified as barriers to student degree progress and completion. Only as few as one fifth of students who are placed into developmental mathematics ever successfully complete a credit-bearing math course in college (see e.g. Bailey, Jeong, & Cho, 2010). At the same time, elementary algebra has higher enrollments than any other mathematics course at US community colleges (Blair, Kirkman, & Maxwell, 2010). Moreover, groups traditionally underrepresented in higher education and in STEM fields are significantly more likely to be placed into elementary algebra. For example, the National Center for Education Statistics reported that from 2003 to 2009, 51.6% of African Americans and 49.5% of Hispanics were enrolled in developmental mathematics courses in college, compared to only 39.4% of whites (2012).

There is evidence that students struggle in these courses because they do not understand fundamental algebraic concepts (see e.g. Givvin, Stigler, & Thompson, 2011; Stigler, Givvin, & Thompson, 2010). Conceptual understanding has been identified as one of the critical components of mathematical proficiency (see e.g. (National Council of Teachers of Mathematics (NCTM), 2000; National Research Council, 2001), and many research studies have documented the negative consequences of learning algebraic procedures without any connection to the underlying concepts (see e.g. J. C. Hiebert & Grouws, 2007). However, developmental mathematics classes at community colleges currently focus heavily on recall and procedural skills without integrating reasoning and sense-making (Goldrick-Rab, 2007; Hammerman & Goldberg, 2003), often because there is pressure for students to pass standardized exit exams that can be exclusively procedural in nature. This focus on procedural skills, divorced from conceptual reasoning, can create a vicious cycle in which the developmental students most in need of explicit instruction in conceptual understanding do not receive it.

At the same time, no validated assessments currently exist to assess conceptual understanding for elementary algebra in the postsecondary context. As a result, instructors cannot systematically detect which incorrect or underdeveloped algebraic conceptions are impeding student progress, and thus they cannot target instruction to address these conceptions explicitly. For these reasons, our team developed an elementary algebra concept inventory (EACI), which we tested for validity and reliability. Research questions included:

1. To what extent does the concept inventory have content, construct, and face validity?
2. How strong is the composite reliability of the instrument?
3. Does the instrument show concurrent validity in being able to distinguish between students with low versus high levels of conceptual understanding in elementary algebra?

## Conceptual understanding

The definition of conceptual understanding (and its relationship with other dimensions of mathematical knowledge, particularly procedural fluency) has been much debated and discussed (e.g. Baroody, Feil, & Johnson, 2007; Star, 2005), with as yet no clear consensus. Conceptual understanding and procedural fluency (as well as other mathematical skills) are strongly interrelated (e.g. J. Hiebert & Lefevre, 1986; National Research Council, 2001). However, it can be important to focus on conceptual understanding explicitly, since without explicit instruction in concepts, students may interpret mathematics as a sequence of algorithms, arbitrarily applied, without understanding (e.g. J. C. Hiebert & Grouws, 2007) and may be unable to correctly apply procedures (e.g. Givvin et al., 2011; Stigler et al., 2010).

This study recognizes the interrelatedness of conceptual understanding with other mathematical skills, and defines it this way: An item tests ***conceptual understanding*** if *logical reasoning* grounded in mathematical definitions is necessary to answer correctly, and it is *not* possible to arrive at a correct response *solely* by carrying out a procedure or restating memorized facts. We define a ***procedure*** as a sequence of algebraic actions and/or criteria for implementing those actions that could be memorized and correctly applied with or without deeper understanding of the mathematical justification. For example, consider the following questions:

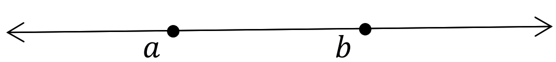
***Sample procedural question***

If $a < b$, which of the following expressions must ***also*** be true? There may be **more than one** correct answer—select **ALL** that are true.

a. $-a < -b$
b. $2a < 2b$
c. $\frac{a}{2} < \frac{b}{2}$
d. $a - 1 < b - 1$
e. $a + 1 < b + 1$

***Sample conceptual question (similar to one from the inventory)***

Consider the numbers $a$ and $b$ on the number line below. Which of the following must be true? There may be **more than one** correct answer—select **ALL** that are true.




a. $-a$ is to the left of $-b$ on the number line
b. $2a$ is to the left of $2b$ on the number line
c. $\frac{a}{2}$ is to the left of $\frac{b}{2}$ on the number line
d. $a - 1$ is to the left of $b - 1$ on the number line
e. $a + 2$ is to the left of $b + 2$ on the number line

In the procedural question, it is possible to answer correctly with no understanding of the mathematical reasoning behind the properties of inequality; technically a student could correctly answer this question by using a standard memorized algorithm for manipulating inequalities, even if they do not understand the concepts behind it. For the conceptual question, it is not possible to answer it completely correctly using only memorized procedures without understanding. For example, there is no obvious expression or equation on which to apply procedures. Students could translate the information given by the number line into the inequality $a < b$ and then apply procedures, but this would require conceptual understanding of how to translate between graphical representations of the number line and inequalities.

**Conceptualizations of algebra domains in the research literature**

There is no one clear consensus about what the core concepts of algebra are. Attempts have been made to categorize algebra by the types of structures or types of actions that are involved (see e.g. Aké, Godino, Gonzalo, & Wilhelmi, 2013; Bell, 1996; Gascón, 1994-1995; Godino et al., 2015; Kaput, 1995; Kieran, 1996; Lee, 1997; Lins, 2001; Mason, Graham, & Johnston-Wilder, 2005; Pinkernell, Düsi, & Vogel, 2017; Rojano, 2004; Smith, 2003; Star, 2005; Sutherland, 2004; Usiskin, 1988), and national standards for algebra have been developed (Common Core State Standards Initiative, 2017; Mathematical Association of America, 2011; National Council of Teachers of Mathematics (NCTM), 2000), although those elementary algebra standards are aimed at K-12 rather than adult learners, which may be problematic since adult learners have been shown in some cases to use different mathematical reasoning from K-12 students (Masingila, Davidenko, & Prus-Wisniowska, 1996; Scribner, 1984).

In order to develop a list of concepts fundamental to elementary algebra at the tertiary level, we conducted a literature review to compile a list concept domains that are both common in research in algebraic thinking and relevant to elementary algebra curricula in the college context (Wladis, Offenholley, Licwino, Dawes, & Lee, n.d.). We began by consulting reviews on the topic (e.g. (Kieran, 2006; Kieran, 2007; Wagner & Kieran, 1989) as well as papers that were cited by, or that cited these reviews. Then, since the last of these larger systematic reviews was published in 2007 (Kieran, 2007), we searched nine mathematics education research journals, nine general education research journals, and five sets of mathematics education conference proceedings, to find any research focused on algebraic thinking; references listed in these papers were also explored. This systematic review (limiting to topics relevant to elementary algebra in the college context) led to a classification of the existing research into four common domains:

- (C1) **Variables and symbolic representation** (see e.g. Bardini, Radford, & Sabena, 2005; Bloedy-Vinner, 2001; Dubinsky, 1991; Furinghetti & Paola, 1994; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Kuchemann, 1978; Malisani & Spagnolo, 2009; Philipp, 1992; Sfard, 1991; Stacey & MacGregor, 1999; Ursini & Trigueros, 2004; Usiskin, 1988)
- (C2) **Equality/Equivalence** (see e.g. Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Kieran, Boileau, Tanguay, & Drijvers, 2013; Knuth et al., 2005; Knuth, Stephens, McNeil, & Alibali, 2006; Mevarech & Yitschak, 1983; Rittle-Johnson & Alibali, 1999; Steinberg, Sleeman, & Ktorza, 1990; Zwetzschler & Prediger, 2013)
- (C3) **Algebraic structure sense** (see e.g. Christou & Vosniadou, 2008; Christou & Vosniadou, 2012; Hoch & Dreyfus, 2004; Hoch & Dreyfus, 2005; Hoch & Dreyfus, 2010; Hoch, 2003; Hoch & Dreyfus, 2006; Hoch, 2007; Linchevski & Livneh, 1999; Menghini, 1994; Musgrave, Hatfield, & Thompson, 2015; Novotná & Hoch, 2008; Tall & Thomas, 1991; Thompson & Thompson, 1987)
- (C4) **Functions, proportional reasoning, and covariation** (see e.g. Blanton & Kaput,

2005; Breit-Goodwin, 2015; Carlson, Oehrtman, & Thompson, 2005; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Oehrtman, & Engelke, 2010; Goldenberg, Lewis, & O'Keefe, 1992; Heid, 1996; Heid & Blume, 2008; Oehrtman, Carlson, & Thompson, 2008; Sitomer, 2014; Thompson, 1994)

In addition, elementary algebra syllabi were also consulted (30 community colleges randomly selected from IPEDS) and participatory action research was conducted with five experienced elementary algebra instructors to outline domains of elementary algebra in the college context and to generate appropriate questions for the inventory (Wladis, Offenholley, Lee, Dawes, & Licwinko, 2017; Wladis, Offenholley, Licwinko, Dawes, & Lee, 2017).

## Results and Discussion

### Initial Pilot Testing

For an initial pilot test of the instrument, the inventory (V1) was given to a sample of 23 college algebra students at the Borough of Manhattan Community College, City University of New York (BMCC/CUNY) in order to identify problematic questions, issues with wording, and to assess how long it would take students to complete particular problems. Based on these results, the wording of questions was simplified, and some questions were broken into simpler parts. The revised inventory (V2) was given to a group of 160 students who had recently taken elementary algebra; they were asked not only to complete the inventory, but to give feedback on the difficulty and clarity with which problems were posed, as well as to give written explanations of their experiences with the individual questions, and any other feedback that they wanted to share. The time that students spent on each problem was also tracked. Problems that were identified as particularly difficult or unclear or were were missed by a significant number of students, or on which students spent a particularly long time were revisited and revised jointly by the whole group of instructors. Roughly half of the inventory questions was revised at this stage.

### Content and Face Validity

In order to assess content and face validity, the revised inventory (V3) was given to 52 instructors who had recently taught elementary algebra at the community college or university level. Instructors came from four different states, many different ethnic/racial and immigrant backgrounds, included faculty who taught at both two- and four-year colleges, and included faculty with a variety of different degree backgrounds. In addition to taking the test, instructors were also asked to give feedback on each question, assessing the clarity and difficulty and giving suggestions for corrections or improvements. They were also asked to assess the topics and concepts covered by the questions and were asked to suggest concepts that were overrepresented, underrepresented, or missing from the current version of the concept inventory. Based on the feedback of faculty, the instrument was revised further.

For example, the following question was included on V3, but was then replaced with other questions in V4 because it was pointed out that it could be answered correctly by applying procedures without understanding:

Which of the following equations are true? There may be **more than one** correct answer—select **ALL** that apply.

a. $2x + 3x = 5x$
b. $2x + 3x = 5x^2$
c. $x^2 + x^3 = x^5$

d. $x^2 \cdot x^3 = x^5$
e. $x^2 \cdot x^3 = x^6$

This question was subsequently replaced with two different questions aimed at testing the conceptual understanding of the algebraic reasoning behind combining like terms and properties of exponents, one of which is similar to the question below:

A student has written: $2x^2(3y-4) + 5x^2(3y-4) = 6x^2(3y-4)$

Which of the following statements is true?

a. The students' work is incorrect because the correct answer is $6x^4(3y-4)$.
b. The student's work is incorrect because the two terms cannot be combined.
c. The student's work is correct because 2 and 5 can be seen as coefficients for the common expression $x^2(3y-4)$.
d. The student can only simplify this expression if they distribute the $2x^2$ and the $5x^2$ first, and then combine like terms.

**Reliability, Structure, and Convergent Validity**

The resulting instrument was administered to elementary algebra students across four semesters; we report results from the first two semesters (V4 and V5) here. V4 was administered to 28 sections of elementary algebra taught by 23 different instructors, once during the first week of classes (pretest) and once during the last week of classes (posttest). This resulted in 484 completed pretests and 315 completed posttests. Item response theory was used on V4 to assess the extent to which individual items provided good discrimination and were at an appropriate level of difficulty. Items that seemed to be very difficult or very easy, as well as those with negative or low discrimination were revised (V5). V5 was administered to 33 sections taught by 21 instructors, resulting in 431 completed pretests and 192 completed posttests.

**Construct Validity**

Twenty students who were currently enrolled in elementary algebra were invited to participate in cognitive interviews. Each student was asked to complete the items on their own, and then were asked to participate in a "retrospective think-aloud" interview (Sudman, Bradburn, & Schwarz, 1996), in which they explained what possible approaches to answer the question they considered while completing the questions. Research suggests that concurrent and retrospective think-aloud protocols reveal comparable information, and that retrospective think-alouds may be preferable to concurrent think-aloud protocols because they are less likely to have a negative effect on task performance, especially for complex or challenging tasks with a high cognitive load (see e.g. (Van Den Haak, De Jong, & Jan Schellens, 2003).

Students were then asked to explain what they were thinking about as they considered each question, and then to explain for each answer choice why they did or did not select it. The cognitive interviews focused in particular on several main aims: whether student answers were consistent (i.e. (in)correct thinking yielded (in)correct answers; whether the activities and thinking elicited by the questions was conceptual and in the algebra domain intended by the question; and whether students used test-taking strategies. Cognitive interviews revealed strong consistency of student responses, with almost all students choosing (in)correct answers only when they exhibited evidence of (in)correct reasoning. The vast majority of student responses (including incorrect responses) exhibited some degree of conceptual thinking—for example, even when students were using incorrect reasoning, they were often attending to structural aspects of the expressions or equations on the exam, rather than attempting to apply procedural algorithms to them. There was a low-incidence of use of test-taking strategies.

## Reliability and Internal Consistency

Both structural equation modeling (SEM) and item response theory (IRT) were used to assess the reliability of the instrument.

**Classical test theory: Using structural equation models for confirmatory factor analysis.** Confirmatory factor analysis using structural equation modeling (SEM) was used to model items as predictors of a single latent construct. Average variance extracted (AVE) ranged from 0.76-0.99, indicating very good convergent validity; composite reliability (CR) ranged from 0.99 to 0.9998, indicating excellent reliability (Hair, Anderson, Tatham, & Black, 1998), consistent with requirements for high-stakes testing (Nunnaly, 1978); the standardized root mean square residual (SRMR) ranged from 0.06 to 0.08, suggesting that the model fit was good and supporting the operationalization of the inventory as having a single latent construct (Hu & Bentler, 1999). RMSEA was acceptable, ranging from 0.061-0.064, further supporting the goodness of model fit (MacCallum, Browne, & Sugawara, 1996).

**Item response theory.** First hybrid three-parameter logit models were explored, with items grouped into three separate groups (based on the underlying probability of randomly guessing that item correctly), each with its own pseudoguessing parameter (Birnbaum, 1968). However, since pseudoguessing parameters were not significantly different from zero, a two-parameter logic model was used. Reliability was assessed using the test information function (tif), where $reliability\ = 1 - \frac{1}{tif(\theta)}$. In IRT, reliability is dependent upon the value of theta, with $\theta = 0$ representing a mean score on the instrument, and other values ta representing the number of standard deviations (SDs) that a score is above or below the mean (e.g. $\theta = -1$ is one SD below the mean). Peak reliability of the test ranged from $\theta = -1.11$ to $0.56$, suggesting that the instrument is most reliable for students who are around the mean in algebraic conceptual understanding as measured by this instrument. Both post-tests obtained excellent reliability ($\geq 0.9$) (Nunnaly, 1978) around the peak, from about 1.5 SDs below the mean to about 0.5 above the mean (see Table 1). Farther away from the peak the reliability remained strong, with acceptable reliability within a minimum of two SDs of the mean on either side on all test administrations (see Table 1). This suggests that the reliability of the test is quite strong for a broader range of students across the ability spectrum. We note that the peak reliability is higher, and overall reliability somewhat stronger, for the posttests than the pretests; this is not surprising, since students have been exposed to algebraic instruction prior to the posttest, but not necessarily prior to the pretest. For example, this may help students to be more familiar with terminology or symbolic representation used on the exam, and that may lead to more reliable results.

*Table 1. Reliability values for various values of $\theta$*

| **$\theta$ values** | **V4 pretest** | **V4 posttest** | **V5 pretest** | **V5 posttest** |
|---|---|---|---|---|
| Peak reliability | $-1.11$ | $-0.52$ | $0.56$ | $-0.44$ |
| Acceptable[a] reliability ($\geq 0.7$) | $[-3.4, 2.1]$ | $[-3.7, 2.4]$ | $[-2.9, 3.6]$ | $[-3.7, 2.5]$ |
| Good[a] reliability ($\geq 0.8$) | $[-2.4, 0.6]$ | $[-2.7, 1.6]$ | $[-1.7, 2.6]$ | $[-2.9, 1.7]$ |
| Excellent[a] reliability ($\geq 0.9$) | *NA* | $[-1.5, 0.5]$ | *NA* | $[-1.4, 0.5]$ |

[a]Based on the criteria set forth by Nunnaly (1978)

## Pre-Test Versus Post-Test Scores

Mean and median scores on the post-tests each semester were not significantly different from the pre-test scores, suggesting that on average, students are likely not gaining any conceptual understanding after one semester of instruction in a traditional algebra class. This is in line with the findings of concept inventories in other subjects, where it was found that some types of instruction could improve aggregate student gain stores, but that on average traditional instruction did not improve outcomes (see e.g. Epstein, 2013; Hake, 1998).

*Table 2. Test scores, calculated by percentage of questions answered correctly, for each test administration*

| | **pretest** | **95% CI** | **posttest** | **95% CI** |
|---|---|---|---|---|
| V4 | 54.2% | [53.3%, 55.1%] | 52.9% | [51.6%, 54.2%] |
| V5 | 56.4% | [55.5%, 57.3%] | 54.2% | [52.3%, 56.1%] |

**Limitations**

We note that this concept inventory is based on a very specific conceptualization of elementary algebra and of conceptual understanding for this subject as well. Because of this, student scores on this inventory reflect only these constructs, and do not necessarily reflect other ways in which conceptual understanding in elementary algebra may be conceptualized.

One major limitation with many previously-developed tests and instruments historically has been that they were developed among a predominately white, middle class population and then were applied to wider populations, including many ethnic minorities and lower-SES students for whom they were not necessarily valid. The City University of New York, which will be used for this research, is highly diverse with an undergraduate population that is roughly 40% first-generation American, more than 75% students of color, roughly 20% first-generation college students, and more than 50% eligible for Pell grants. This diversity makes CUNY an excellent source of information about the validity of the instrument among this population; however, more work with various populations, including rural and suburban populations, is necessary.

**Implications**

This research demonstrates that it is possible to create valid instruments that can reliably measure some aspects of conceptual understanding in algebra. Some future avenues of research would be to explore further validation of this inventory, e.g., to determine to what extent scores on the EACI differ from scores on validated exams that test procedural fluency in algebra. In addition, future studies that consider which factors correspond to higher versus lower gain scores for whole classes in elementary algebra could help to shed light on which teaching approaches and curricula, etc. can increase student conceptual understanding in elementary algebra.

For practitioners, this study illustrates that it is possible to create questions that target conceptual understanding in elementary algebra specifically. Instructors wishing to create their own questions that would assess this skill could follow some of the process described here and in (Wladis et al., 2017; Wladis et al., 2017). In addition, this study raises important questions about curricula and assessment. Students in this study showed no gains on average in conceptual understanding over the course of one semester of algebra instruction. At the college in which this instrument was tested, all of the learning outcomes on the syllabi are entirely procedural, and as a result, it is likely that most instructors (even those who employ more active learning techniques, of which there were many in this study) do not directly address concepts in their teaching. While this study presents no conclusive evidence of the relationship between these two things, the patterns observed by looking at the pre- and post-test results of the EACI for students in elementary algebra in this study suggest that teaching algebra procedures alone likely does not in and of itself lead on average to gains in conceptual understanding.

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## An Undergraduate Mathematics Student's Counterexample Generation Process

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*This paper illustrates the processes and struggles involved in a student's generation of a counterexample. The data involves one student's at-home proving while working on homework for his introduction-to-proof course. In this paper, we present an episode where a student engaged in substantive efforts in order to generate a proof by counterexample. We compare and contrast this episode against results from the literature on example generation to provide insights regarding the similarities and differences between example and counterexample generation as they relate to proof.*

*Key words:* counterexample, example space, disproof, undergraduate, at-home proving.

### Introduction

Examples and counterexamples are in many ways inextricably linked. Goldenberg and Mason (2008) emphasized this association when they wrote, "In a mathematical context there is little difference between an example and a counterexample: it all depends where your attention is anchored, and what you are attending to" (p. 184). To illustrate this idea, consider $1/\pi$. It is simultaneously an example of an irrational number and a counterexample to the claim that for all real numbers $x$, $x^2>x$. Given this link between examples and counterexamples, it might be reasonable to extrapolate that the process of generating examples and the process of generating counterexamples are similar. However, since one counterexample can disprove a statement and an infinite collection of examples are insufficient to prove a statement, the underlying reasons for counterexample and example generation in a proof context are fundamentally different. This disparity in purpose may affect the process meaning that in the context of proving, example and counterexample generation processes maybe very different. These two conflicting arguments for why counter-example generation and example generation might be different or similar to each other provide the impetus for the propose study.

### Related Literature and Theoretical Perspective

The mathematics education literature has explored the role example generation plays in proving but currently has less to say about the process of counterexample generation and how it relates to proving and example generation. Moreover, the literature provides no definitive answer to how the processes of example generation and counterexample generation relate to each other.

Research on learners' example generation can be broadly partitioned into two categories: studies involving tasks which prompt learners to generate examples and studies involving problem solving tasks which do not explicitly prompt for example generation. Studies with tasks that prompt for example generation often use "and then another" tasks, where students are asked to generate increasingly more examples of a particular mathematical concept (e.g., Watson & Mason, 2002, 2005; Zaslavsky, & Peled, 1996; Zazkis & Leikin, 2007). Typically, these tasks follow a predictable trajectory where first a learner generates immediately accessible examples of the concept (e.g. Goldenberg and Mason, 2008). The learner then works to generate new examples by combining examples and/or varying parameters. The second type of study involves students working on tasks which do not specifically prompt for example generation and highlights instances where example generation occurs as a problem solving strategy. These studies illuminate the utility of example generation as a strategy rather than the process of example generation itself.

### Counterexamples

Much of the mathematics education research relevant to counterexamples has focused on their pedagogical uses (e.g., Zazkis & Chernoff, 2008, Zazkis, 1995). Such work, however, does not discuss how students might evaluate a claim to determine that it may be false or subsequently generate counterexamples toward the end of proving a claim to be false. Overall, research on counterexample generation is relatively sparse. Meanwhile, several researchers have worked with graduate students who were prompted to evaluate the truth of mathematical claims (e.g., Alcock & Inglis, 2008, Weber, 2009). Such evaluation typically involves either the generation of a proof that establishes the claim as true or generation of a counterexample showing the claim to be false. However, these studies had either students produce counterexamples in such a quick manner that little can be inferred about the counterexample generation process (Weber, 2009) or focused on documenting the differences in the number of examples/counterexamples generated rather than the process by which they were generated (Alcock and Inglis, 2008). We begin to address this gap in the literature by discussing an episode where a student engaged in substantive efforts in order to generate a proof by counterexample.

## Methods

The research study was conducted at a large university in the southwest United States. Data were collected from two introduction-to-proof courses taught by the second author of this paper over two semesters. In this study, we discuss work related to the prompt:

*True or False and why: If $a$ and $b$ are both irrational then $a^b$ is irrational.*

This task's prompt was intentionally chosen because it does not indicate whether the statement is true or false. As such, the data related to this task includes both students' work to determine whether the claim is true/false and, in the cases where a successful proof was produced, an exploration which led to an appropriate counterexample which forms the basis of a proof. Thus, this task affords the opportunity to not only investigate the process of constructing proof via counterexample, but also investigate the process of how a student may realize the necessity for a counterexample.

Each student in the course was provided with a Livescribe® smart pen and notebook and was instructed to include all work done while working on the assignments, from their initial thoughts on the problems to the final solutions to be submitted for grading. We received 56 assignments with student work relevant to the prompt. The data presented here comes from one of these students.

## Results

Given the limited space, this paper's analysis focuses on a single student, Alex, and his associated work on the task. This work involves multiple shifts in notation. It also involves shifts between (1) attempts to formally prove the statement, (2) attempts to disprove the statement via counterexample (counter example generation), and (3) his work on proving related results. The core observation we wish to convey is that these three activities inform each other, with insights and notation from one affecting work in the others. This means that the counterexample generation process described here is more nuanced (and draws from more sources) than the processes of example generation described in the literature. Given the multiple shifts in activities and notation we include Figure 1 below to provide a top level view of these shifts. This also aides the reader in keeping track of how segments of the proving process relate temporally to the process as a whole and when segments of the process are omitted due to space limitations.

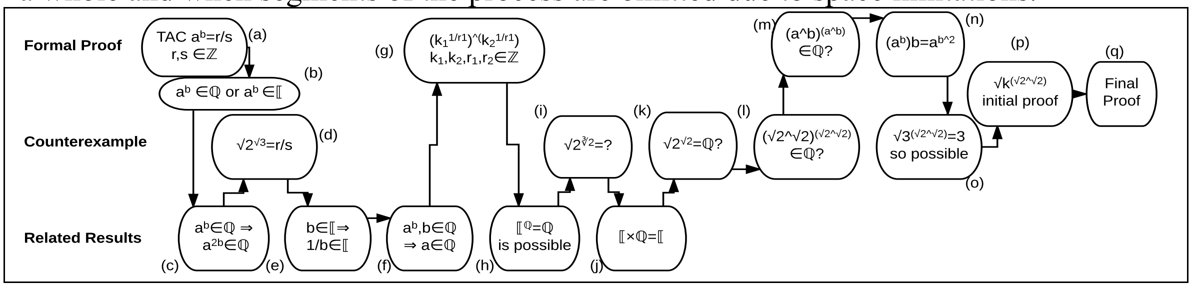



**Figure 1.** Alex's work on the counterexample task.

***Segments a-b: Alex's initial approach to the task***

In Alex's initial approach to the problem, he wrote "$a, b \in \mathbb{I} \Rightarrow a^b \in \mathbb{I}$", repeating the statement being considered in the task but using the symbol $\mathbb{I}$ to represent the set of irrational numbers, and "[toward a contradiction] assume $a^b = \frac{r}{s}$ $r, s \in \mathbb{Q}$". From here, Alex attempted to syntactically manipulate this equation and arrived at the statement, "$ar^b = s^b$". However, this did not show the contradiction he sought and adjusted his approach to consider both possibilities of the rationality of $a^b$ by explicitly indicating "either $a^b \in \mathbb{I}$ or $a^b \in \mathbb{Q}$". This is the first time where we see Alex considering these two possibilities of rationality. Since he is simultaneously considering both approaches, in figure 1, segment (b) sits between the counterexample and formal proof trajectories.

***Segments c-f: Irrational numbers as square roots***

After making a note of the set of rational numbers is closed under multiplication, Alex then wrote "Counterexample $a = \sqrt{2}$ [,] $b = \sqrt{3}$[.] Assume $\sqrt{2}^{\sqrt{3}} = \frac{r}{s}$". Here Alex considers the use of specific examples of irrational numbers $\sqrt{2}$ and $\sqrt{3}$, in an attempt to find a counterexample. We note that Alex's choice of irrational numbers is consistent with the typical first examples of irrational numbers in example generation literature (e.g., Goldenberg & Mason, 2008). However, after noting that $\sqrt{2}^{\sqrt{3}}$ is not rational, Alex quickly abandoned this specific counterexample strategy in favor of returning to working with abstract representations to seek a contradiction. We offer Alex's speedy dismissal of his examples $\sqrt{2}$ and $\sqrt{3}$ as evidence that Alex may prefer general counterexamples.

Next, Alex further evoked his knowledge of irrational numbers and exponents, proving the lemma "if $b$ irrational, then $\frac{1}{b}$ irrational". With this lemma established, he attempted to utilize it to create the desired contradiction. In particular, under the assumption that $a^b$ is rational, he wrote, "then $(a^b)^{1/b} = a^1 = a$". This establishes that it is possible to take $a^b$ to an irrational power to yield $a$. For example, setting $a = 2^{\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}}$, yields a counterexample to the claim. While Alex did not use this approach, we will see that this was a productive stepping-stone toward his eventual solution.

***Segments g-j: Irrational numbers as n-th roots***

In the next portion of Alex's proof progression, Alex explicitly restricted his consideration of irrational numbers to only roots of natural numbers. This is consistent with his solely considering irrational numbers in his earlier counterexample generation attempts. More specifically, he wrote "if $b$ irrational then $b$ is an $r$-th root of some number say $b = {k_0}^{1/r_0} \in \mathbb{I}$" and similarly defined the variable $a = {k_1}^{1/r_1} \in \mathbb{I}$ (two lines below, he wrote that $b^{r_0} = k_0 \in \mathbb{N}$.) We note that this example space of irrational numbers is again consistent with the example generation literature (e.g., Goldenberg & Mason, 2008), which identifies n-th roots as the second most accessible class of examples after square roots of non-square integers. Based on this restriction, we cannot say if Alex's example space is restricted to roots or if this represents a restricted evoked example space for the purpose of this task.

Regardless of his understanding of various forms of irrational numbers, Alex uses this new (restricted) representation of irrational numbers to continue his formal, syntactic exploration of $a^b$ as a rational number, where $a$ and $b$ are irrational. In particular, Alex represented $a^b$ as $({k_1}^{1/r_1})^{({k_0}^{1/r_0})}$ and considered the expression $(a^b)^{({k_0}^{1/r_0})}$. He proceeded to again attempt to prove the statement is true via contradiction using this new notation, resulting in a dead end. Beside this work, he wrote several observations related to the original statement. In particular, he acknowledged that a rational number raised to a rational power yields a rational number and that it is possible for an irrational number raised to a rational power to yield a rational number. He justified this latter observation via the specific example $(\sqrt{2})^2 = 2 \in \mathbb{Q}$.

Next, Alex wrote "Shows possible $a, b \in \mathbb{I}$ and $a^b \in \mathbb{Q}$ counterexample?" followed by the use of specific numbers ($a = \sqrt{2}$ and $b = \sqrt[3]{2}$) in attempt to generate a counterexample. After noting this did not yield the desired result, Alex returned to his formal exploration using the $({k_1}^{1/r_1})^{({k_0}^{1/r_0})}$ notation. Once again, these attempts were not fruitful and Alex considered the properties of the products of rational and irrational numbers as shown in Figure 2. In this figure, we see Alex once again moving between general and specific attempts to generate a counterexample.

***Segments l-m: Numbers of the form $(a^b)^{a^b}$***

In the subsequent portion of Alex's proof progression, he expanded his consideration of irrational numbers beyond $a$'s and $b$'s which are roots of integers; he introduced the use of irrational numbers that are composed of a base and an exponent both of which are irrational numbers. This can be viewed as the counterexample generation process being filtered through the task (e.g., Zazkis & Leikin, 2007) where the previously generated pairs of irrational numbers, $a$ and $b$, are filtered through the task to place $a^b$ under consideration. When $a^b$ is not rational, and thus does not serve as a counterexample to the statement, it can instead serve as a new example of a single irrational number.

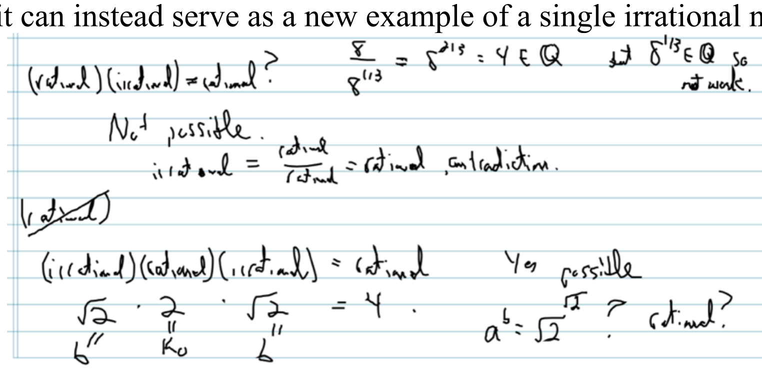


**Figure 2.** Alex's consideration of the products of irrational and rational numbers.

This behavior first emerged as Alex wrote $(a^b)^{a^b} = a^{b \cdot a^b} = a^r$ indicating the leftmost expression as an irrational base to an irrational exponent and the rightmost expression as a rational number. Alex further considered the possibilities of rationality of the products and powers of irrational and rational numbers. Here $(a^b)^{a^b}$ structure of in his proof attempt is informed by his work with $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}^{\sqrt{2}}}$ when attempting to generate a counter example.

Further, Alex noted that $\sqrt{2}\sqrt{2} = 2$, which led to a reiteration that it is possible to create a rational number by taking an irrational number to a rational power. In this reiteration, he wrote "$a^r = \mathbb{I}^{\mathbb{Q}} \in \mathbb{Q}$ if $a = \sqrt{2}$, $r = 2$". Given his infrequent use of specific examples and the immediacy of Alex's use of this example following the statement indicating an irrational number raised to a rational power can yield a rational number, we interpret this as Alex reverse engineering a rational number from irrationals and to apply this to the task at hand.

Despite this use of specific examples, Alex next attempted to construct a proof as shown in Figure 3. We see in Figure 3 that Alex's progress was limited when trying to formalize his counterexample. In subsequent attempts to formalize his counterexample, Alex returns to his syntactic representations of irrational and rational numbers rather than attempting to generate a specific counterexample. We offer this as further evidence that Alex may have a preference toward using general counterexamples.

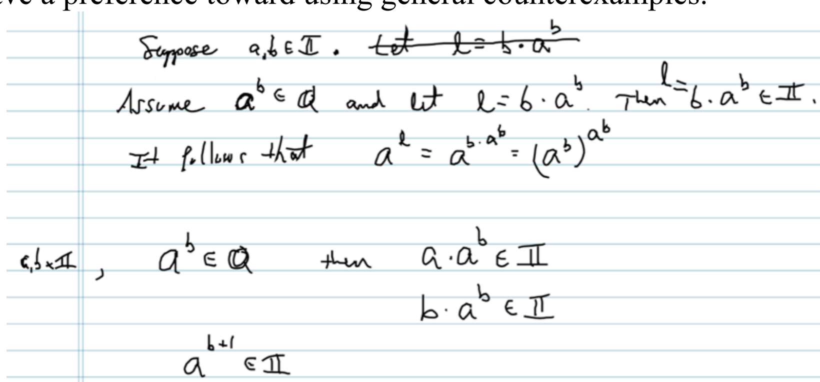



**Figure 3.** Alex's attempt to formalize a counterexample.

***Segments n-q: Numbers of the form $(a^{\sqrt{2}})^{\sqrt{2}}$***

Following his unsuccessful attempt to formalize his counterexample in a proof, Alex builds off of his previous approach by modifying the variables and applying specific numbers. Rather than using strictly $a$'s and $b$'s, Alex applied numbers to his previous expression "$a^b = (a^b)^b = a^{b^2} = \sqrt{3}^{\sqrt{2}^2} = \sqrt{3}^2 = 3 \in \mathbb{Q}$" and noted "so possible." This building off of his previous approach is consistent with the literature on example generation (e.g., Goldenberg and Mason, 2008; Zazkis & Leikin, 2007) and is an instance where his attempt to create a proof using abstract $a, b$ notation informed his later generation of a counter example. Further, based on Alex's note of "so possible" next to this expression, we believe Alex has knowingly identified a counterexample that shows the possibility of an irrational base to an irrational exponent to be equal to a rational number. However, it is unclear whether Alex realized that this single counterexample was sufficient to form a basis of a proof that the claim is false. In fact, rather than using the counterexample he generated, Alex continued working towards a general counterexample. One explanation for Alex's preference for general counterexamples may be that his behaviors are an instantiation of the belief that abstract mathematical objects must be the focus of proving activity.

In the proof sketch shown in Figure 4, Alex used $a = \sqrt{k}^{\sqrt{2}}$ and $b = \sqrt{2}$ to show a contradiction that it is not true that if $a, b \in \mathbb{I}$, then $a^b \in \mathbb{I}$. In figure 1, both this and the final proof appear between the formal proof and counterexample axes because they are proofs that utilize counterexamples. It is noteworthy that after having generated a suitable, specific counterexample above ($a = \sqrt{3}^{\sqrt{2}}$ and $b = \sqrt{2}$), Alex generated an additional counterexample ($a = \sqrt{k}^{\sqrt{2}}$ and $b = \sqrt{2}$). Thus, we highlight that the counterexample generation process does not necessarily stop when a counterexample is found. Specifically, in Alex's case, we see that the motivation for this additional counterexample may be focused on supplying a general counterexample. Based on the evidence available, we cannot be sure whether this proof using a general counterexample is a result of 1) a belief that proving activity must usually center around work with abstract mathematical objects, or 2) a lack of understanding of the role a single counterexample plays in relation to a mathematical claim.

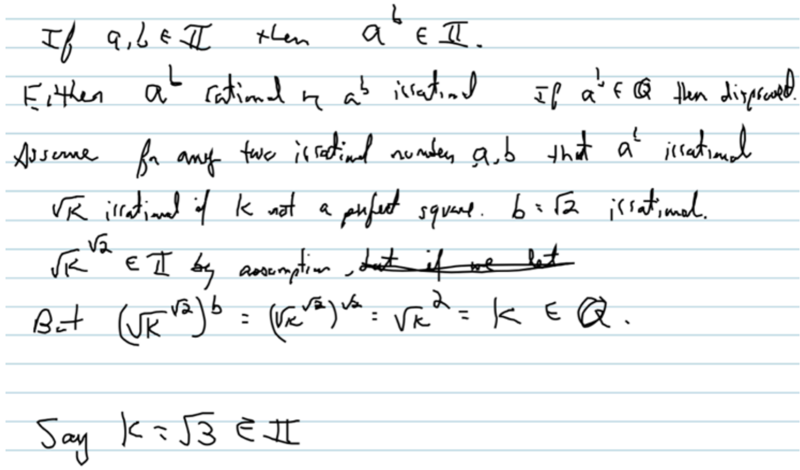
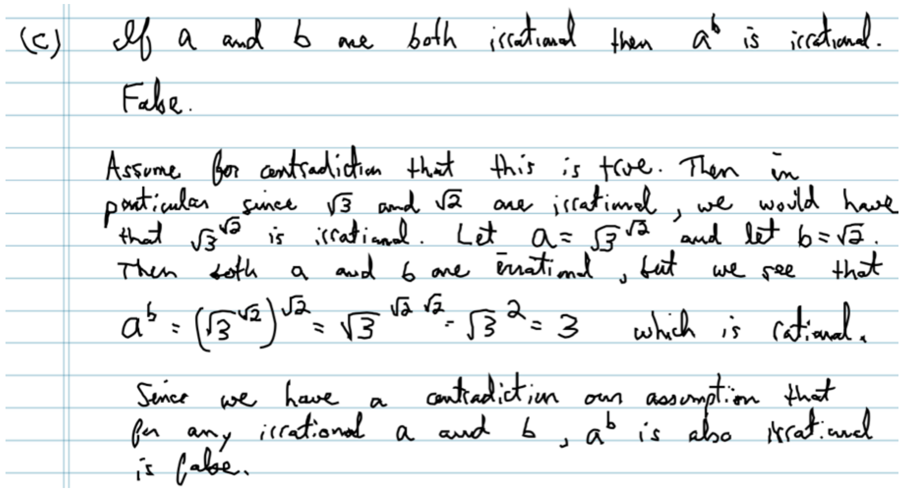



**Figure 4.** Alex's proof sketch.

Meanwhile, in Alex's final proof, we see that he used the specific counterexample he first presented where $a = \sqrt{3}^{\sqrt{2}}$ and $b = \sqrt{2}$ to show a contradiction that it is not true that if $a, b \in \mathbb{I}$, then $a^b \in \mathbb{I}$. It is unclear what influenced Alex's change from using k=2 to k=3. Moreover, we do not have evidence to indicate why Alex shifted from his more formal approach shown in his proof sketch to using the specific counterexample where $a = \sqrt{3}^{\sqrt{2}}$ and $b = \sqrt{2}$. One possible interpretation of this shift in Alex's approach is the above proof sketch's reliance on the assertion that "$\sqrt{k}$ irrational if $k$ not a perfect square" – a non-trivial claim that warrants a justification.



(c) If $a$ and $b$ are both irrational then $a^b$ is irrational.

False.

Assume for contradiction that this is true. Then in particular since $\sqrt{3}$ and $\sqrt{2}$ are irrational, we would have that $\sqrt{3}^{\sqrt{2}}$ is irrational. Let $a = \sqrt{3}^{\sqrt{2}}$ and let $b = \sqrt{2}$. Then both $a$ and $b$ are irrational, but we see that $a^b = \left(\sqrt{3}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{3}^{\sqrt{2}\sqrt{2}} = \sqrt{3}^{2} = 3$ which is rational.

Since we have a contradiction our assumption that for any irrational $a$ and $b$, $a^b$ is also irrational is false.

**Figure 5.** Alex's final homework submission.

## Conclusion

There are several important points to be made about Alex's work on this problem. When he generated examples of pairs of irrational numbers $a, b$ and considered the rationality of $a^b$, this process was consistent with the example generation literature. This is true not only in terms of his movement from directly accessible examples to less accessible examples, but also the filtering of the example generation through the task itself. However, generating a counterexample was not Alex's primary goal. Rather he was attempting to assess the veracity of the claim and generate a proof that justified that veracity. As a result, the proving process was more complex than generating pairs of irrational numbers $a, b$ in the hope that one had the property that $a^b$ is rational.

The process also included several failed attempts to prove the claim was true via contradiction and the incorporation of several representations of $a$ and $b$. Moving between these attempts to generate a counterexample (as seen in Figure 1) influenced the generation process in fundamental ways. For example, his use of the pair $a = \sqrt{2}$ and $b = \sqrt[3]{2}$, when attempting to generate a counterexample was influenced by his attempts to use other roots ($b = {k_0}^{1/r_0} \in \mathbb{I}$). Alex's proof attempt encouraged a particular type of example and thus made that notation readily accessible to his counterexample generation process.

## Discussion

This proposal offers evidence that proving activities that are not focused specifically on generating examples can influence counterexample generation. That is, proof attempts can influence a students' counterexample generation. The mutual influence of counterexample generation and proof attempts points to the complexity of the process of evaluating the truth of a mathematical claim currently absent in the processes discussed in the example generation literature. This observation regarding the complexity of evaluating the truth of a mathematical claim points to the limitations of the example generation literature for informing this related process. Possible avenues for future research include investigations of learners' potential preference for abstract or general counterexamples to specific ones and more targeted and systematic investigations into the processes by which students generate counterexamples. A natural next step may be to conduct interviews with students working on similar counterexample generation tasks.

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Mathematics Graduate Teaching Assistants' Growth as Teachers: An Unexamined Practice

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*In recent years, providing teaching professional development for graduate teaching assistants has become more common in mathematics departments in the US. Following this trend, mathematics education researchers have begun to conduct studies on professional development programs and on graduate students as novice teachers. The purpose of this literature review is to examine the current status of research in this field and make recommendations for future research on graduate teaching assistants and professional development. In examining the literature, we utilize an existing framework for collegiate teaching practices and focus on studies that attended to growth. As a result of this literature review, we recommend that researchers begin developing models or theories for how and why graduate students grow as teachers.*

*Keywords:* Post-secondary, professional development, graduate teaching assistants, teaching practices, growth

## Introduction

In recent years there have been many efforts made to improve the quality of instruction in first-year undergraduate mathematics courses, which often have low pass rates. For many students, these are the only undergraduate math courses they experience. Also, graduate teaching assistants (GTAs) are often the primary instructors for these courses. In an effort to improve the quality of instruction in these courses, it has become increasingly common for math departments to offer teaching professional development (PD) for their GTAs. In addition, math departments have begun evaluating the impact of these PD efforts, but the results have been mixed.

At the K-12 level, researchers have found that PD interventions aimed at improving teaching tend to be more effective when they are theory-based and use an explicit model accounting for features of the PD, school contexts in which teachers work, and how the PD is intended to shape the practices of teachers participating in it. In a similar vein, it is reasonable to expect that attempts to improve teaching among GTAs would benefit if informed by models of how these novice college mathematics instructors learn to teach.

With this in mind, we reviewed the research on how mathematics GTAs learn to teach. Our intended goal was to identify and characterize the models and theories that have informed studies of GTAs' growth as teachers. While there are many aspects of GTA teaching in which growth might occur, we chose to focus explicitly on teaching practices in this paper because they directly relate to the professional work that GTAs engage in. We sought to take stock of what is known about improving GTAs' teaching, what gaps there may be, and how to move forward. To do this, we asked:

1. *What GTA teaching practices do researchers attend to?*
2. *How do researchers attend to GTAs' growth as teachers over time?*
3. *What models or theories of growth do researchers use?*
4. *What stances do researchers take regarding GTAs' growth as teachers?*

In our literature review, we searched three major research databases from 2005 to 2016: Education Resources Information Center (ERIC), PsycINFO, and Web Of Science, the RUME proceedings from 2010 to 2016, and the AMS Notices from 2005 to 2016. We chose these as the foundational sources because undergraduate-focused math education research is often published in these sources.

We found there is little empirical or theoretical research that explicitly or implicitly describes GTAs' growth. Here we define growth as the process of changing along an identifiable trajectory. For something to be considered growth, it must be true *that* something has changed and that exactly what has changed can be *identified.* We take our definition of teaching practices from our theoretical framework, which will be taken up again later. In later sections, we will review the results of our search, summarize the few results we did find, and discuss how researchers attended to GTAs' growth as teachers. After providing some background and the results of the literature search, we propose a refinement of Speer, Smith, and Horvath's (2010) framework for collegiate teaching practice. The refinement emerged from our analysis in order to address more recent research on GTAs and their teaching. Finally, the central claim that we make is that *GTAs' growth as teachers is a largely unexamined practice*.

**Background**

In many departments, GTAs are assigned to teach first-year undergraduate courses, including remedial math, college algebra, pre-calculus, calculus, and mathematics for pre-service K-12 teachers. Since students are stakeholders in instruction, we first highlight ideas from research published about student experiences in lower division undergraduate courses. In calculus, student experiences vary greatly (Bressoud, Mesa, Rasmussen, 2015; Burn, Mesa, & White, 2015). Students traditionally under-served by *status quo* K-12 education continue to be at a disadvantage in post-secondary settings (Bahr, 2010; Cuellar, 2012; Kena et al., 2016; Nuñez, Hurtado, & Calderón Galdeano, 2015). Difficulty passing initial college mathematics courses has a negative impact on persistence of STEM-intending students (Thompson, Castillo-Chavez, et al., 2007). While these phenomena are influenced by a variety of factors, instruction is a key element in student success. For that reason, research on how professional development can help improve the quality of instruction is an important facet of research on undergraduate mathematics education.

While providing teaching professional development for GTAs has only recently become more common, there is a wealth of research on professional development for pre- and in-service K-12 teachers (Chen & McCray, 2012; Desimone, 2009). However, measuring effective PD can be difficult and results are mixed and influenced by many external factors (Guskey & Yoon, 2009). Moreover, these studies focus primarily on professional development for teachers who have earned at least a bachelor's degree in education, stressing the importance of discipline-specific scaffolds for teacher learning. In contrast, GTAs often have a great deal of experience in doing mathematics but little to no formal training in teaching and instruction. Consequently, they are a different audience for professional development than K-12 teachers.

## Frameworks

To guide our analysis of teaching practices, we drew on Speer et al.'s (2010) framework for examining teaching practices at the collegiate level. Speer et al. define *teaching practices* to be the instructional judgments, decisions, and actions employed by instructors in and outside of the classroom. Note that this is distinct from *instructional activities*, which are activities used to organize student learning and stimulate student engagement with classroom resources (e.g., using group work). Teaching practices and instructional activities are interwoven and the distinction between them is often not made clear, or even mentioned, in research publications. Both are important teaching elements examined by researchers and were present in the articles we analyzed. See Table 1 for a full description of the teaching practices identified in Speer et al.'s (2010) framework.

*Table 1. Framework for collegiate teaching practice of Speer et al. (2010)*

| Teaching Practice | Description |
|---|---|
| Allocating time within lessons | Deciding how much time to allocate among topics and within individual class periods |
| Selecting and sequencing content within lessons | Choosing and sequencing the mathematical content presented in an individual class period; for example selecting problems and deciding which theorems to present |
| Motivating specific content | Introducing, motivating, and providing a rationale for sequencing topics, specifically to promote student engagement |
| Posing questions, using wait time, and reacting to student responses | Deciding what questions to ask, how long to wait for a response, and how to respond to students' answers |
| Representing mathematical concepts and relationships | Deciding which mathematical ideas to present in the classroom and how to present them |
| Evaluating and preparing for the next lesson | Reflecting and evaluating on individual class periods, and using (or not using) this information to inform the next lesson |
| Designing assessment problems and evaluating student work | Developing assessment problems by considering content coverage, expected difficulty level, sequencing of problems, and relevance to particular elements of content |

## Methodology

All articles considered for inclusion in the review were peer-reviewed and contained at least one search term from each of four categories: teaching, domain, level, and participants (see Table 2 for exact search terms). This yielded 1,889 articles. We read each abstract to determine if an article could reasonably address our research questions. We double-coded until we reached consensus on the criteria for inclusion, with an inter-rater reliability of 97%. Our intent was to focus on mathematics GTAs, but we also decided to code for STEM fields in general that way we could keep track of articles that might be relevant if we decided to extend our literature review. After discussion, we agreed to seven articles that were relevant. To capture other relevant research on this topic, we then read the abstracts of the RUME proceedings for the years 2010 through 2016 (we restricted our time frame due to infrequency of relevant articles), again coding for inclusion, and found 17 relevant articles. Finally, we searched the AMS Notices using an advanced Google Scholar search for the years 2005 through 2016 (using the same search terms in Table 2, excluding the "domain" category). This yielded an additional two articles, which gave us a total of 26 articles relevant to our research questions.

Each article was then open coded for teaching practices, attention to growth over time, use of an explicit or implicit model or theory of growth, and stances on growth. Six articles were double-coded, at which point the team discussed preliminary findings and how to adjust the coding procedure. After consensus was reached, the rest of the articles were coded. Using the Speer et al. (2010) framework, we conducted a second cycle of coding that categorized our open codes to fit into the given framework.

*Table 2. Search Terms*

| Category | Terms |
|---|---|
| Teaching | teach*, instruct*, "professional development", PD, training, TD |
| Domain | STEM, math*, science, physics, chemistry, biology, statistics, engineering |
| Level | undergrad*, collegiate, tertiary, college |
| Participants | "graduate student", GST, GSI, GI, novice*, "future faculty", beginning, GTA, TA |

## Findings

While all 26 articles focused on GTA teaching, the participants involved in the studies still varied. The majority included graduate student participants who were currently teaching or who would be teaching in upcoming semesters. In addition, one mixed-methods study included "novice college math instructors," who were defined as instructors with less than seven years of teaching experience and included non-graduate students. A few studies explicitly stated that the graduate students were in their first or second year as instructors, but many studies did not specify how long the GTAs had been teaching. In addition, the researchers gathered data in a variety of ways, including interviews, classroom observations, observations of GTA PD classes, and surveys. Details of our findings are given below and summarized in Tables 3 and 4.

### Teaching Practices and GTA Growth

Of the 26 articles, we were able to utilize Speer et al.'s (2010) framework to categorize the teaching practices studied in 14 of the articles. Each of the specified teaching practices was addressed in at least one article, suggesting that their framework is consistent with the current research efforts surrounding GTA PD. However, we suggest two refinements to the framework based on our review. First, we suggest adding "anticipating student thinking" to the framework. We found eight articles out of the 26 that examined this teaching practice. This teaching practice is implicitly part of "selecting and sequencing content within lessons" and "motivating specific content," but we suggest that it be explicitly stated. Also, anticipating student thinking plays a central role in facilitating productive mathematical discussions (Stein, Engle, Smith, & Hughes, 2008) and is something that novice teachers often struggle with. Second, we suggest that the practice of "representing mathematical concepts and relationships" be refined to include verbal descriptions. One of the articles we coded with this teaching practice specifically focused on "speaking with meaning" (Musgrave & Carlson, 2016). Although Speer et al.'s (2010) description does not explicitly exclude verbal representations, it emphasizes *what* is shared rather than *how* it is shared. This refinement also reflects recent work on teachers' coherent mathematical meanings (Thompson, Carlson, & Silverman, 2007).

*Table 3. Number of articles addressing the teaching practices in adapted Speer et al. (2010) framework*

| Teaching Practice (* Adapted) | Number of Articles |
|---|---|
| Anticipating student thinking* | 8 |
| Allocating time within lessons | 2 |
| Selecting and sequencing content within lessons | 4 |
| Motivating specific content | 1 |
| Posing questions, using wait time, and reacting to student responses | 5 |
| Representing mathematical concepts and relationships including how concepts are described in words* | 3 |
| Evaluating and preparing for the next lesson | 4 |
| Designing assessment problems and evaluating student work | 3 |
| Does not examine any specific teaching practice | 12 |

After coding for teaching practices, we then coded to identify articles that focused on how GTAs grow over time, provided models or theories of growth, and took stances regarding growth. These are discussed in the next few subsections.

*Table 4. Number of articles that attended to growth*

| | Number of Articles |
|---|---|
| Focused on growth over time | 11 |
| Used models or theories of growth | 3 |
| Took a stance regarding growth | 13 |
| Did not attend to growth | 9 |

**Growth over time.** We found 11 articles out of the 26 that addressed growth in GTAs' teaching practice over time. For example, Raychaudhuri and Hsu (2012) conducted a longitudinal study to explore how beliefs and pedagogical approaches evolve over the span of a year. Based on their preliminary analysis, Raychaudhuri and Hsu present stages of GTA beliefs regarding students moving from teacher-centered knowledge to student-centered knowledge. Musgrave and Carlson (2016) studied GTAs' descriptions of average rate of change before and after one semester of PD. They found that graduate students who participated in the PD described average rate of change more conceptually than their counterparts, but still struggled to verbalize the meaning of average rate of change. In another study, Duncan (2016) used a teaching experiment methodology to examine how one GTA's mathematical meanings and instructional planning decisions changed while creating a hypothetical learning trajectory (HLT) on angles, angle measure, and the radius as a unit of measurement. Duncan's results suggest that having GTAs work through tasks in a researcher generated HLT can cause changes in what GTAs identify as being important to teach.

**Models or theories of growth.** We found three articles out of the 26 that employed a specific model or theory of growth. Beisiegel (2011) utilized Lave and Wenger's (1991) theory of legitimate peripheral participation to describe the process by which GTAs gain knowledge and understanding about the practice of teaching post-secondary mathematics. In particular, Beisiegel studied how "the attention to legitimate peripheral participation in a mathematics department [might] prevent graduate students from adopting alternate modes of teaching" (p. 20). In a study examining the teaching philosophies of GTAs, Nepal (2014, 2015) used a cognitive apprenticeship model, which stems from situated cognition and Vygotsky's (1978) sociocultural theory. Nepal applied this model to explain how and why GTAs' teaching philosophies change as they "accumulate knowledge about teaching and learning gradually through the interaction with other people and their own teaching experiences" (2015, p. 5). Some papers mentioned a model or theory of growth but did not incorporate it as a key aspect of the study. For example, Yee, Rogers, and Sharghi (2016) claimed that reflecting, revising, and collaborating help GTAs "actively engage with teaching theories" (p. 1458) and "develop a community of practice" (p. 1459). However, an explicit model or theory of growth was not referenced or used.

**Stances regarding growth.** Thirteen articles took stances on teaching quality that referenced knowledge for teaching, cognitive demand, pedagogies, and student achievement. Firouzian (2014), Speer and Firouzian (2014), and Firouzian and Speer (2015) cited the large body of work on mathematical knowledge for teaching (MKT) as evidence for why it is important for GTAs to develop MKT. Roach, Noblet, Roberson, Tsay, and Hauk (2010) cited Smith and Stein's (1998) work on cognitive demand to describe the (limited) variety in cognitive demand in the questions TAs asked. Finally, Yee et al. (2016) drew upon *Principles to Actions* (National Council of Teachers of Mathematics, 2014) as evidence for why specific teaching practices are associated with effective teaching.

## Discussion

Based on our findings, we argue that GTAs' growth as teachers is a largely unexamined practice. We assume that the purpose of most, if not all, GTA PD programs is to foster growth as teachers, but were surprised to find that only a small percentage of the research on GTA PD addresses growth. Of the 26 articles we reviewed, nine of them did not focus on growth at all. Only three of them provided explicit models or theories of growth, but none of these focused on teaching practices. Current research gives a general sense of what GTAs may be doing in the classroom, but how their teaching practices change as they gain experience and participate in PD is understudied. It should also be noted that the four studies that address growth demonstrate that growth is possible, so further research is warranted. As a result, we suggest that the field would be greatly enhanced by additional longitudinal studies exploring how GTAs grow as teachers.

In particular, it would be beneficial to begin developing an accepted definition of GTA growth. We argue that part of this process is being clearer on our stances as a research field on teaching quality and how these relate to models or theories of growth in teaching. Moreover, most studies assume a common understanding of the term "teaching practice" rather than attempting to define it. This leads to a lack of clarity about which aspects of teaching are analyzed in research as well as the researchers' stances on teaching quality. It is striking that there were no articles that were explicit both about their stance on teaching quality *and* a model or theory. We call for future research to take an explicit stance on teaching quality and how teaching quality is related to models and theories of GTA growth.

Finally, our findings suggest a need for explicit models or theories of growth in teaching linked to stances on teaching quality. There has been some progress on this (e.g., development of MKT by Thompson, Carlson, & Silverman, 2007), but more development is needed. We call for the research community to begin developing the models of growth that will allow research on GTA PD to grow into a rich body of literature such as exists in the research literature on K-12 teacher PD.

## Future Directions

Although we did not find many articles on mathematics GTAs' growth as teachers to include in our literature review, there was a larger body of work on STEM GTAs in general. Since there are many similarities between mathematics and other STEM disciplines, it would be fruitful to see how researchers have attended to STEM GTAs' growth as teachers. In particular, if studies on STEM GTAs attend to teaching practices or define GTA growth or use explicit models or theories of growth, then we could build upon this in the RUME community. However, it is also important that we attend to the ways in which the teaching and learning of mathematics differs from other STEM disciplines.

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# Our Mathematical Ideas are Part of Our Identity

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*This paper explores the notion that our mathematical ideas are part of our identity. This notion, which was a significant result of a qualitative dissertation study, will be explored in depth through an examination of data in connection with the educational research related to identity. The story of Binary, a first generation college student completing a transition-to-proof course in his final semester of college, provides the context in which we explore this complex notion.*

*Keywords:* Identity, Nature of Mathematics, Mathematician's Practice, Transition-to-Proof

One day in Foundations of Higher Mathematics, a student called Binary[1] says, "I think no one really understands my idea. I think people are just so focused on $2k + 1$ that they are not looking at the bigger picture here." This comment came on the second day of classroom discussion, the bulk of which consisted of students debating the merits of Binary's suggestions regarding a proof of the claim that "the sum of any two odd numbers is even." Notice the language Binary uses: "I think no one really understands *my* idea." Binary identifies the idea as his own. In this paper, we explore the notion that the idea is part of his identity.

## Theoretical Frameworks On Identity

Identity is a powerful construct used to examine and understand the interactions between learners and mathematical content and practices (e.g. Boaler & Greeno, 2000; Langer-Osuna 2015). Gee (2000) defined identity as "being recognized as a certain 'kind of person' in a given context" (p. 99). Identity is related to how an individual views her or himself, and how others view an individual. Drawing on work done by Boaler and Greeno (2000) and Martin (2000), Cobb and Hodge (2010) distinguished between core identity, normative identity, and personal identity. Observing student behavior, the authors noted that core identity refers to how students "viewed themselves and who they wanted to become" (p. 187). Normative identity does not refer to how students view themselves or one another as individuals, but rather, the focus is on how students become a "mathematical person" or "doer of mathematics" (p. 187). An important component or normative identity is mathematical competence (Cobb, Gresalfi, and Hodge 2009). What students perceive as necessary to become a doer of mathematics may be in conflict with who they are and want to become (their core identity). Personal identity is developed as students "participate in (or resist) the activities of particular groups and communities, including those of the mathematics classroom" (Cobb & Hodge, 2010, p. 187).

To understand the overlapping but separate notions of personal identity and normative identity, we can use an additional framework called figured worlds. A figured world is "a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (Holland, Lachicotte, Skinner, & Caine, 1998, p. 52). In mathematics education, this

[1] Students in the course completed an assignment in which they chose a number type that best represented themselves. For this assignment, the student we call Binary, a computer science major, wrote this description of why he chose Binary as his number type: "Very simple once you get to know me. But can be very confusing if you don't."

framework has been used to understand how students navigate one figured world of the mathematics classroom and the figured worlds of their other identities, such as being an athlete, under-represented minority, etc. Boaler and Greeno (2000) described an inquiry-based mathematics classroom as a different figured world than a traditional classroom, the latter of which may be much more misaligned with promoting the development of students' normative identities. Nasir, Hand, and Taylor (2008) examined the lack of intersection between students' figured worlds of the mathematics classroom and the basketball court. Langer-Osuna (2015) extended this notion to examine the challenges of under-represented minority students in mathematics, as the figured world of a mathematics classroom overlaps much more with that of dominant cultures versus that of non-dominant cultures. In all of these cases, the figured world of mathematics is misaligned with the various figured worlds of students outside of the mathematics classroom, thus leading to unsuccessful learning.

## Methodology and Analysis

The theoretical frameworks of identity will be used to shed light on a finding from a qualitative dissertation study. The importance of identity emerged from the data, and the construct was not used to frame the original study. The original purpose of the study was to create a humanistic framework for the nature of pure mathematics. Two questions guided the study: What is the nature of pure mathematics? And what should undergraduate students in a transition-to-proof course understand about the nature of pure mathematics? In seeking to answer these questions, the researcher incorporated the methodological framework of heuristic inquiry (Moustakas, 1990). This form of inquiry has roots in humanistic psychology, and it leverages the researcher as instrument. Douglass and Moustakas (1985) wrote, "It is the focus on the human person in experience and that person's reflective search, awareness, and discovery that constitutes the essential core of heuristic investigation" (p. 42).

To understand the nature of mathematics and consider what students should understand about the nature of mathematics, the researcher (first author) collaborated with a graph theorist and co-taught an undergraduate introduction-to-proof course. In regards to this discussion on identity, to the most relevant data collected come from the transition-to-proof course and the researcher's reflective journal. The data gathered from the course included audio recordings of discussions the researcher had with a co-instructor, audio of whole-class discussions, student homework, classwork, exit tickets, and all other class materials. Twenty-three students from the course agreed to participate in the study.

The researcher employed the processes of heuristic inquiry and other qualitative analysis techniques to arrive at several possible characteristics of the nature of mathematics that may be valuable for students to know and understand. The end of heuristic inquiry is what Moustakas (1990) called the creative synthesis:

> Finally, the heuristic researcher develops a creative synthesis, an original integration of the material that reflects the researcher's intuition, imagination, and personal knowledge of meanings and essences of the experience. The creative synthesis may take the form of a lyric poem, a song, a narrative description, a story, or a metaphoric tale. In this way the experience as a whole is presented, and, unlike most research studies, the individual persons remain intact. (p. 51)

The creative synthesis for this study includes the IDEA Framework for the Nature of Pure Mathematics and ten stories that illustrate key characteristics of the nature of mathematics.

## Results

A main result of this study is the IDEA Framework for the Nature of Pure Mathematics and ten corresponding stories that illuminate the characteristics of the framework. The IDEA framework consists of four foundational characteristics: Our mathematical ideas and practices are part of our ***I***dentity; mathematical ideas and knowledge are ***D***ynamic and forever refined; mathematical inquiry is an emotional ***E***xploration of ideas; and mathematical ideas and knowledge are socially vetted through ***A***rgumentation. In this paper we present a story, *If Nobody Agrees With You*, highlighting the notion that our mathematical ideas are part of our identity.

### If Nobody Agrees With You

One day in Foundations of Higher Mathematics some small groups are working to create group proofs for different theorems and presenting their proofs of those theorems to the class. The Yellow Team create and present the poster shown in Figure 1.

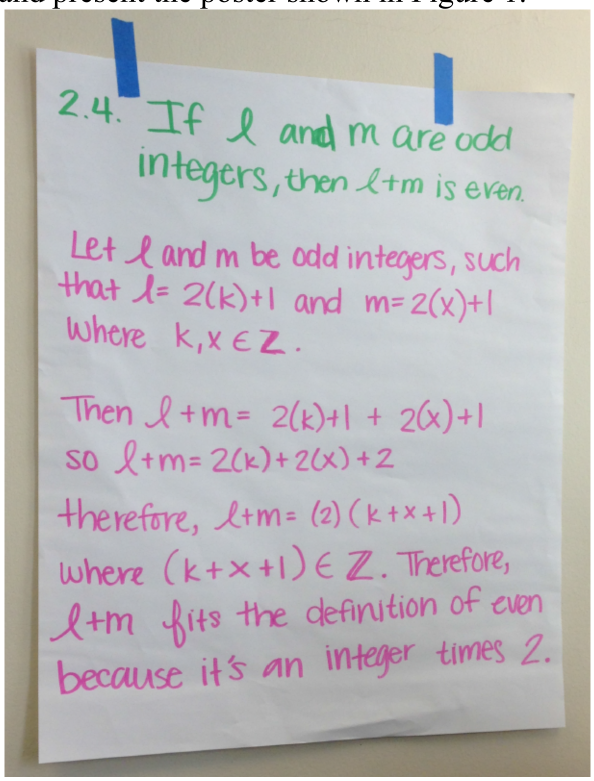


*Figure 1. Yellow Team's Poster*

In general, the class as a whole likes the group's argument and does not have many questions. Infinitely Repeating Decimal's comments are representative of the class: "It is pretty straightforward, it doesn't get more concise than that." However, Binary does have an important question. He asks, "If instead of having the *x*, if you did put *k*, would it make the argument less strong?" We see Binary is referring to the original delineation of the odd numbers *l* and *m* as $l = 2k + 1$ and $m = 2x + 1$ where *k* and *x* are integers. Would the argument be less strong if we simply defined $l = 2k + 1$ and $m = 2k + 1$? One of the presenters, Odd, responds "Oh, so like *k* here and *k* here?" (pointing to the *k* and *x* in the original delineations). "Yeah," Binary replies. Odd explains, "Yeah, because basically what you did was, you didn't say that it's any two odd integers, you said the same integers. So you basically just said $l + l$." Binary tries to explain that he thinks $k$ is sufficient, by appealing to the definition of odd number; but he is interrupted as another student, Whole, interjects: "If you want to stick with $k$ you could be like $k$ subscript 1 and $k$ subscript 2." The dialogue continues:

Binary: I just feel like if they both were $k$ it would still make a strong argument.

Odd: If $l$ equals $2k+1$ and $m$ equals $2k+1$, as $k$ increases the other $k$ is increasing so $l$ and $m$ would always be equal.

Binary: But it's still going to be odd. That's what I am saying.

Odd: It's an odd number but it's the same odd number. So it doesn't cover any combination of two odd numbers. So basically you would only be able to say like 3+3 or 6 or 7+7. But with this, because these are different, we could say let $l$ be equal to 3 and $m$ be equal to 7.

Infinitely Repeating Decimal: If you really wanted them to be k's you could use subscripts like $k_1$ and $k_2$.

Odd: Yeah.

Infinitely Repeating Decimal: You just need to show that they are not the same integer.

Binary: Yeah. I am just trying to figure out if I just left that as $k$, and he had that. Would I get less points?

At this point several members of the class laugh and some chime in that indeed the argument would receive less points. Dr. Amicable and Surreal[2] (the co-instructors) bring the class's attention back to the mathematics. Dr. Amicable asks for a show of hands to see how many students understand Odd's explanation, and asks Infinity to explain in her own words.

Infinity: Yeah. If you made both the variables, the $k$ and the $x$ the same. If you did make them both $k$ then they would be the same number. So you would get the same outcome of $l$ and $m$. And so your numbers wouldn't vary. So, like he was talking about how the answer would be consistently the same throughout. … If you make both of those $k$'s in the equation where x is. Then you are going to get the same exact number.

Dr. Amicable: Hmm. So is it right, if I were to summarize what you are saying Infinity; would it be right to say that essentially we are changing this condition to if $l$ and $m$ are the same odd integers? Then $l + m$…

Infinity: Yea. Because you would be plugging in…. You would be using the same variables.

Dr. Amicable: Okay. Binary what do you think?

At this point the instructors, Dr. Amicable and Surreal, expected Binary to jump on board with the class consensus. But he stands firm.

Binary: I am just saying. In general if you were to use examples, then yeah. But just in general I feel like no matter what you put in, by the definition of an odd number, it's going to come back to the exact same thing.

Whole (interrupting): You have to have some kind of variance in it when you are adding two.

Binary: But not in the way he did, maybe in another proof, yes. But…

Whole: It's the same principle though.

Composite: Okay. But if you do use $k$ in both $l$ and $m$... If you say $2k + 1$ is equal to $l$ and $2k + 1$ is equal to $m$. You are going to go to the next step where it adds, and you will have $2k + 1$ plus $2k + 1$. And then instead of being $2k + 2x$ it's going to be $2k + 2k$ which equals $4k$. And you are going to have $4k + 2$ which is not going to be something that looks like the definition of an even number.

Binary: [emotional] Yes you will. You pull out 2.

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[2] Surreal is the researcher and first author.

Odd: It would still be even.

Binary: It would still be even.

Binary stated "It would still be even" with conviction. Then several people in the class begin to discuss. Of course, the result $4k + 2 = 2(2k + 1)$ is in the form of an even number and several members of the class agree. But they also stress that by using $l = 2k + 1$ and $m = 2k + 1$, the result is "more narrow" because "you've got to cover the spectrum." Subsequently the students and instructors begin to consider examples that may serve to change Binary's mind. Binary still does not. He says, "I understand what ya'll are saying. It makes perfect sense to me what ya'll are saying. Don't get me wrong. What ya'll are saying is 100% correct. But I'm saying that this way still satisfies everything to me for an odd number." Time runs out for the class and the students are required to write one big idea for the day and one question they have for their exit tickets. Binary's big idea was "If no one agrees with you, you're wrong."

After class, Surreal and Dr. Amicable were very happy about the discussion students had regarding Binary's question. As students left the room, Even asked if she could stop by Dr. Amicable's office. Later on, Dr. Amicable and Surreal had this e-mail exchange:

Dr. Amicable: … We will need to chat about Even's concern. In brief, she felt that Binary was under attack today in class and it made her feel very uncomfortable. She recognizes that Binary may not have felt under attack, but she felt that for him. I appreciated her coming to share her feelings. Perhaps we can address the best ways to critique and also remain professional in our classroom setting (at the beginning of next class), but I'd like to chat with you about your thoughts.

Surreal: Okay we can chat. I did not think Binary was under attack. Only his idea was under attack! But I also recognize that students have never experienced mathematical argumentation and so it may be hard for some of the students to deal with it. Although I do not have immediate thoughts about what we would tell students, I think engaging in a dialogue with students may be productive.

Dr. Amicable: I agree. I did not see it as an attack on Binary either, but it wouldn't hurt to talk with the students about critiquing an idea rather than a person.

Notice that Surreal's initial thought was that Binary was not under attack. "Only his idea was under attack!" But are our ideas not also our selves? When we criticize another person's ideas, are we not criticizing the person as well?

The next day in class Surreal began by asking students to talk about their big ideas and questions from the previous class. Many of the students said they had conversations outside of class about Binary's idea. Others said they were trying to think of new ways to convince Binary of their point of view. Infinitely Repeating Decimal's big idea was "I'm really struggling to figure out a different way to represent that $l = 2k + 1$ and $m = 2k + 1$ only satisfies $l = m$. There has got to be a way though!" He expanded upon this idea during class.

> Infinitely Repeating Decimal: You know, to me, the discussion we had on Tuesday, it was very clear that set of restrictions only satisfies $l = m$, but to someone else if it is not clear—like they think that can be interpreted differently. I think you have an obligation to make sure that everyone is on the same page. Whether one person or another changes their position, I think it is very important that everybody agrees on a given definition or theorem, etcetera. But I couldn't figure out any other way to represent that, to possibly represent it in another way that might make it more clear.

The classroom dialogue continues:

Real: Yeah I think clearly we spent a lot of time in class on it the other day, and it's an important point. Generality, or proving that something is universally true, is more valuable than obviously proving specific cases. I think it is important that we help Binary get to that point. But I just wasn't sure exactly how to persuade him that we needed to differentiate the k's in that specific example to ensure that we have a general case that our proof covers all the bases.

Surreal: So it seems like Infinitely Repeating Decimal and Real are thinking, "We've got this idea and we want to convince Binary of it." And Binary felt, I think; how did you feel Binary? [Recall Binary's big idea: "If no one agrees with you, you're wrong."]

Binary: The big idea is like, yeah I agree with ya'll 100%. But ya'll are not listening to me when I say that. With Infinitely Repeating Decimal, what he's trying to say, I completely agree. But I'm not looking at it as just "$k$." When I see that definition of odd—For me, I feel like two times any number in the world plus one would be odd. So I'm feeling like, when I see a definition I am taking that definition plussing that definition to get this new definition. So when I see that I just take $k$ and I make it like $z$or like $2z + 1$ equals odd, and that's how I'm seeing it. [3] So even though the $k$ only satisfies $l = m$. To me, I feel like just the definition alone, no matter the variable, is enough to prove the theorem.

The dialogue continues. Dr. Amicable and Surreal discuss the nature of mathematical argumentation for mathematicians, and the importance of criticizing ideas rather than people.

Dr. Amicable: I know when I have been to math conferences, and mathematicians are presenting their work, sometimes it gets fairly heated in the room, right? … I have seen things that were similar to what we saw in class on Tuesday where it's back and forth like "I'm not sure I understand why you can say that because I see it this way." … Actually there is a lot of emotion involved in mathematics. … we need to keep thinking about one another's ideas, and really try to understand the other idea. The more we can understand someone else's idea, the deeper our own understanding will become. Alright?

Binary: I think no one really understands my idea. I think people are just so focused on 2k+1 that they are not looking at the bigger picture here.

Whole (interrupting): Well earlier before class today, Complex brought it up and we were talking about this ... And he brought it up that what you were saying break it down by the definition of variable—what the actual definition of variable is …

Over the course of two days, this was at least the fourth time that Whole interrupted Binary. Complex shares an idea, and Surreal, wanting to move on to other course content, asks Binary to explain his idea one last time.

> Binary: For me when I see $k$, I pretty much in my head, I put an odd number times $2$, and I put the $z$ which means any real integer, plus 1. So I know if I see $2z + 1$ that represents any possible odd number. So I put $(2z + 1) + (2z + 1)$ equals an even number. That's what's in my head. So when I see $z$ I know I can put in any number imaginable and get an even number. And for this example I feel like that was enough proof. You didn't need an example. You didn't need any other variables, and that is the idea that I had.

A couple weeks later one of the instructors, Surreal, in his research journal, wrote:

> Emotions today. I remember Binary hasn't talked the last two class periods. … We said we were not criticizing Binary, just his ideas. But Even took it as criticisms of him. Our ideas are our selves. Mathematics involves criticism of people's ideas and argumentation.

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[3] It is unclear if Binary is referring to z as an integer or ℤ as the set of integers.

> Students are not ready for a class in which their ideas (and hence their selves) are criticized against the "objective" standard. … I guess what typically happens is that students are told the "right" ideas. Take away the creative act. … I have a vision of pre-service teachers afraid to speak in class. Mistakes are okay! Push our thinking forward as a community. Courage and humility.

## Discussion

For many students, the mathematics classroom and the learning of mathematics represent different and non-overlapping figured worlds with their other identities (Boaler 2000, Langer-Osuna 2015). In many of these cases, students do not see themselves as mathematical persons and struggle to understand the relevance of the mathematical ideas to other parts of their lives (Nasir 2008). Such failure in mathematical learning stems from the students' divergent normative and personal identities. Contrary to the examples in existing literature, we identify and examine in this paper a unique case study, where the student's (Binary) figured worlds of mathematics and his other identities seem inseparable (at least in the eyes of the student Even). The data revealed that Binary did not speak in whole class discussions for the two class periods following those described here. The convergence of Binary's normative and personal identities, instead of leading to successful mathematical learning, is also the key in hindering Binary's progress in his progression toward being a mathematician in the transition-to-proof course.

Binary emphasized that "no one really understands *my* idea" and "if no one agrees with you, *you're* wrong." When his ideas are challenged, a classmate Even also points out that Binary himself is being attacked. In these students' minds, the notion of mathematical idea, and thus an individual's normative identity, are one and the same as the individual's personal identity. There is no separation. However, for a professional mathematician (or mathematics educator as in the case of Dr. Amicable and Surreal), the separation is clear. Surreal points out that "[he] did not think Binary was under attack" and that "only his idea was under attack," and Dr. Amicable agrees. When discussing this separation with students, Dr. Amicable brings up her experiences at professional conferences where "mathematicians are presenting their work, [and] sometimes it gets fairly heated in the room." Dr. Amicable also explains to students that "there is a lot of emotion involved in mathematics" and that "[students] need to keep thinking about one another's ideas, and really try to understand the other idea." The focus of Dr. Amicable's point is clearly on the ideas rather than on individuals.

## Reflections and Questions

What can undergraduate instructors do so that students come to see mathematical ideas as part of their own identity in a productive manner? In terms of identity, we believe we can learn from a study of school mathematics done by Magdelene Lampert 27 years ago. Drawing from the work of Póyla and Lakatos, Lampert (1990) argued that courage and humility are mathematical virtues that students must develop if they are to participate in authentic mathematical discourse. Students must have courage to put forth their own ideas for examination by the classroom community. They must also have humility, and understand that their ideas may need to be revised in light of this public examination. More research is needed to understand how these virtues can be cultivated at the undergraduate level.

Boaler (2015) claimed that, "children are wrongly led to believe that all of the ideas already have been had and their job is simply to receive them" (p. 172). Do we teach undergraduates the same? Or do we support students in being creators of mathematical ideas? If the instructor tells students what the right ideas are, then it is up for the students to conform. Student identities may be in conflict when becoming mathematically competent means submitting to authority in a particular classroom (Cobb & Hodge, 2010). If students are to see mathematical ideas as part of their own identity, then the mathematics classroom needs to be a site of idea-generation rather than a site of indoctrination into what is "right." However, what is normative, and what it means to be mathematically competent, in inquiry-based classrooms is significantly different than what students experience in traditional courses. Students need more opportunities for idea generation in lower-level undergraduate mathematics classrooms so that by the time they reach their upper-level courses, they have more confidence in their own ideas, and hence their selves.

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# Connecting the Study of Advanced Mathematics to the Teaching of Secondary Mathematics: Relating Arcsine to the Study of Continuity, Injectivity, Invertability, and Monotonicity

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*Prospective mathematics teachers are usually required to complete courses in advanced mathematics to be certified to teach secondary mathematics. However, most teachers do not find these advanced mathematics courses as relevant to their teaching. In this paper, we describe a novel way to teach real analysis to future teachers that connects the content of real analysis to the activity of teaching secondary mathematics. We illustrate this method by describing a module that links the study of the relationship of continuity, injectivity, and strict monotonicity in real analysis to the teaching about the arcsine function and solving trigonometric equations in secondary mathematics. We describe a teaching experiment in which this module was implemented and present evidence of the efficacy of this instruction.*

*Keywords:* Inverse; Real analysis; Teacher preparation; Trigonometry

In the United States and elsewhere, prospective secondary mathematics teachers are required to complete extensive coursework in undergraduate mathematics to become certified to teach secondary mathematics. This coursework usually includes advanced upper-level coursework for mathematics majors (e.g., CBMS, 2001), with many institutions currently requiring that future mathematics teachers complete the equivalent of an undergraduate degree in mathematics (Ferrini-Mundy & Findell, 2001). However, many teachers find the advanced mathematics courses that they complete as irrelevant to their teaching (e.g., Wasserman et al., 2015; Goulding, Hatch, & Rodd, 2010; Rhoads, 2014; Zazkis & Leikin, 2010). In this paper, we focus on how we can design advanced mathematics courses to better meet the needs of prospective teachers.

## Relevant literature

### The influence of advanced mathematics on subsequent teaching

Although prospective secondary mathematics teachers are usually required to complete many courses in advanced mathematics, several scholars have noted there is little research on whether or how these courses influence prospective teachers' future pedagogical practice (e.g., Deng, 2008; Moriera & David, 2007; Ticknor, 2012). Here, we discuss two findings that suggest that completing such courses have only a modest effect on prospective teachers' pedagogical behavior. First, large-scale studies have found a weak relationship between the number of advanced mathematics courses that a teacher has completed and the achievement of that teacher's students (Darling-Hammond, 2000; Monk, 1994).

Second, when practicing secondary mathematics teachers have been asked how their experiences in advanced mathematics courses have influenced their teaching, many teachers claimed that their advanced coursework did not contribute to their development as teachers (e.g. Goulding, Hatch, & Rodd, 2000; Rhoads, 2014 Ticknor, 2012; Zazkis & Leikin, 2010). For instance, Zazkis and Leikin (2010) surveyed or interviewed 52 practicing secondary mathematics teachers about how their understanding of advanced mathematics influenced their teaching. The majority of the participants in Zazkis and Leikin's study claimed that they rarely used their knowledge of advanced mathematics in their teaching and few could cite any specific instances of their knowledge of advanced mathematics actually informing their teaching. Wasserman et al. (2015) found that this occurred even when the teachers demonstrated an understanding of the advanced mathematics that they were taught.

**Reasons why advanced mathematics may not benefit prospective mathematics teachers**

Researchers have proposed two reasons for why advanced mathematics courses might not benefit prospective mathematics teachers, even if the prospective teachers understood the content that they were studying. The first reason relates to what Klein (1932) has referred to as a "double discontinuity" between K-12 mathematics and advanced mathematics: the K-12 mathematics that students learn bears little resemblance to the advanced mathematics that is taught at universities and the advanced mathematics that prospective K-12 mathematics teachers learn in university is irrelevant to their future pedagogical practice. In the last decade, researchers have explored this double discontinuity in more detail.

A primary reason that advanced mathematics can inform the teaching of secondary mathematics is because there is an overlap between the content covered in advanced mathematics and the content and disciplinary practices covered in secondary mathematics (Wasserman & Weber, in press). For instance, a first real analysis course deals with concepts such as the real numbers, functions, continuity, and inverse functions, all of which are important concepts in high school algebra, trigonometry, pre-calculus, and calculus. Even though the same concepts and disciplinary practices are covered in advanced mathematics courses and secondary mathematics courses, the way these concepts and practices are treated differs significantly. For instance, Moriera and David (2007) presented a theoretical analysis of how advanced mathematics courses framed concepts from the secondary curriculum. Moriera and David noted that in advanced mathematics courses, concepts usually were introduced using a single canonical formal representation. For example, the familiar concept of fractions was defined as an equivalence class of ordered pairs in $\mathbf{Z} \times \mathbf{Z}\backslash\{0\}$ where $(a, b)$ and $(c, d)$ were equivalent if $ad = bc$. However, Moreira and David argued that effective teaching of secondary mathematics often required the use of multiple representations, many of which were visual but not necessarily formal. For example, fractions might be represented both numerically and pictorially as pie charts, which students will not usually witness in an advanced mathematics course. Similarly, continuity is defined in advanced mathematics formally in terms of epsilon-delta definitions. This treatment bears little resemblance to the informal graphical manner in which continuity is treated in secondary mathematics (e.g., Tall, 2012; Winslow, 2013). Consequently, teachers who study concepts such as fractions and continuity in advanced mathematics may see few implications for teaching secondary mathematics because the advanced treatment of these concepts will not meet the needs of their students (Deng, 2008).

A second disconnect between the activities that in which university students engage in advanced mathematics and in which instructors engage while teaching secondary mathematics (e.g., Ticknor, 2012). For instance, students in advanced mathematics spend a substantial amount of time studying and producing proofs. However these proofs would be usually be inappropriate to use in secondary mathematics classrooms because they employ technical vocabulary, abstraction, and methods of reasoning beyond what secondary students are capable of following (Wasserman et al., 2015). It is not obvious how studying and writing proofs should inform pedagogical activities such as designing activities, grading students' work, and providing informal explanations that a secondary student can understand. Further, prospective mathematics teachers often think these links are non-existent (Wasserman et al., 2015).

## Theoretical perspective

### Why prospective mathematics teachers must take advanced mathematics: A trickle down model

From our perspective, the anticipated benefits of having prospective teachers complete a course in advanced mathematics can be modeled by the "trickle down" model presented in Figure 1 that considers the relationships between i) advanced mathematics; ii) secondary mathematics; and iii) *teaching* secondary mathematics. This model highlights that most of the material covered in an advanced mathematics course consists of advanced mathematics, where little attention is paid to secondary mathematics. However, the hope is that the advanced mathematics provides an opportunity for the prospective teacher to better understand certain aspects of the content of secondary mathematics. For instance, by learning the zero divisor property about rings in abstract algebra, the prospective teacher may develop a deeper understanding for why you can solve polynomial equations by factoring polynomials (e.g., Murray & Star, 2013). Or by engaging in disciplinary practices such as proving, the prospective teacher may develop a better appreciation about the nature of those disciplinary practices (e.g., Even, 2011). Some instructors of advanced mathematics may be explicit about the connections between advanced mathematics and the content of secondary mathematics, but in often prospective teachers are asked to make the connections themselves. Next, the expectation is that prospective teacher's better understanding of the secondary mathematics content will inform their future *teaching* of mathematics. In our experience, exactly how prospective teachers should teach differently is rarely discussed in advanced mathematics courses. Prospective teachers are expected to use their understanding of advanced and secondary mathematics to improve their teaching on their own or the connections between advanced mathematics and teaching secondary mathematics will be provided in a subsequent education course (Murray & Star, 2013).

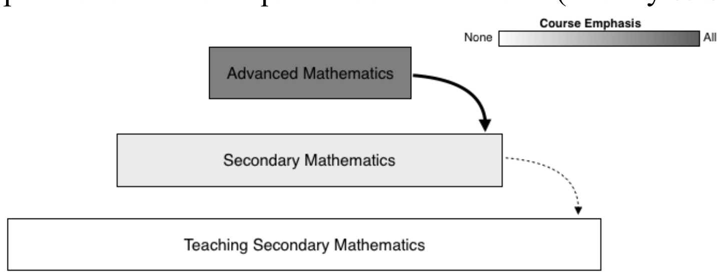


*Figure 1.* Implicit model for real analysis courses designed for teachers

### Our alternative model for teaching advanced mathematics to prospective teachers

We present an alternative instructional model for how advanced mathematics can be taught to prospective teachers in Figure 2. We begin by presenting a realistic pedagogical situation from secondary mathematics. From there, we discuss the secondary mathematical concepts that are in play and problematize the mathematical challenges inherent in the situation that we provide, highlighting fundamental issues that lie beneath the surface that are handled in real analysis. Next we discuss the issue in terms of real analysis. We cover the associated concepts with a formal treatment and make explicit what connections this has for high school mathematics. Finally, and importantly, we describe how this knowledge can inform our response to the initial pedagogical situation that we posed in the beginning of the lesson.

We designed our pedagogical situations to satisfy three criteria. First, there should be a relationship between the real analysis being taught and a topic from the Core Curriculum State Standards in Mathematics (CCSSM, 2012). We do this so that the topic is present in the secondary mathematics curricula and we are not merely preparing students to engage in enrichment activities. Second, the pedagogical situation invites or requires students to engage in what Deborah Ball and her colleagues refer to as a "High Leverage Practice" (HLP), where an HLP "is an action or task central to teaching" (TeachingWorks, 2013). HLPs include providing explanations or models to explain a concept and analyzing and critiquing instruction for the purposes of improving it. We used HLPs so the teachers were engaging in activities that are central to their practice. Finally, we strove to create situations that PSTs would perceive as authentic.

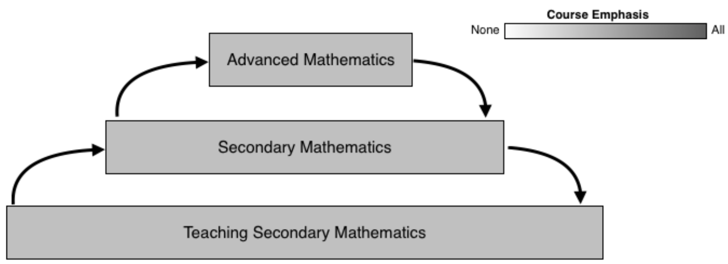


*Figure 2.* Our model for real analysis courses designed for teachers

## Research methods

### Broad research context

The data reported in this paper are part of a larger study supported by the National Science Foundation. Our analysis focuses on the 7th of our 12 modules. The real analysis covered in this module include definitions and theorems concerning the relationship between continuity, injectivity, and strict monotonicity. Particularly important is the theorem that a continuous function is invertible on an interval if and only if the function is strictly monotonic on that interval. The secondary mathematics that we cover involves introducing the arcsine function to a trigonometry class and grading and providing feedback on a student's incorrect solution to a trigonometric equation. Hereafter we refer to this model as the Trigonometry Module.

In this paper, we report on the third iteration of a teaching experiment in which the Trigonometry Module was implemented. The Trigonometry Module was initially informed by a study in which we probed 14 prospective and in-service teachers understanding of inverse and the arcsine function as well as the relevance of real analysis for understanding these topics (Wasserman et al., 2105). We developed and implemented the Trigonometry Module, first in an unpublished constructivist teaching experiment (Steffe & Thompson, 2000) with three PSTs and then in the context of a university real analysis course with 32 prospective and in-service teachers (Wasserman, Weber, & McGuffey, 2017). Following the principles of design research (e.g,. Cobb et al., 2003), we used our analysis of the first two iterations of our Trigonometry Module to refine our models of PST's thinking, how we anticipated the PSTs would engage in our activities, and the activities themselves. For the sake of brevity, we do not describe these refinements here, but will do so during our talk.

**Data collected from this iteration**

The third iteration of the Trigonometry Module occurred at a large state university in the northeast United States. At this university, mathematics education undergraduates were required to complete a mathematics major and a real analysis course was a requirement for this major. In spring 2017, the mathematics department offered x sections of real analysis, one section of which was advertised as a special section that was taught by our research team; the third author of the paper was the lead instructor of the course. This section was advertised as a special section of the course for prospective teachers, although the course was open for all mathematics majors. In total, 17 students enrolled in the course, 13 of these students were in the mathematics education program, two expressed an interest in teaching, and two did not express an interest in teaching.

The class met three times a week. Roughly one out of the three weekly class meetings was devoted to implementing a real analysis module, one of the three weekly class meetings consisted of a traditional lecture covering real analysis content that we did not find relevant to teaching secondary mathematics (e.g., compactness and uniform continuity), and one of the three weekly class meetings was a workshop in which students were given practice and assistance solving problems and writing proofs. When we implemented our modules, we had the students sit in four groups of three to five students. For three of the groups, a member of the research team observed and facilitated the group's discussion. We collected the following data: We videorecorded all of the instructor's actions, we audiotaped each group as they worked on the activities in the module, we had electronic copies of the students' reflective journal entries, performance on a pre-test and post-test, and their homework, and we archived the instructor and researchers' field notes.

**Analysis**

Following the principles of design research (Cobb et al., 2003), prior to conducting our instruction, we had anticipated models for how the PSTs would understand the central concepts of our module, such as inverse function and the arcsine function, desired understandings that we wanted students to develop by the end of the module, and a hypothetical learning trajectory (Simon, 1995) for how students' engagement with the module's activities would foster these desired understanding. For each activity in the module, we had anticipated behaviors for how the PSTs would engage with each activity. These theoretical models were informed by experiences teaching trigonometry (Weber, 2005), prior laboratory studies in which we interviewed PSTs about their understanding of inverse functions and the arcsine function (Wasserman et al., 2015), and significantly by the first two iterations in which we implemented the Trigonometry Module. In the retrospective analysis (Cobb et al., 2003), we analyzed the extent that PSTs' actual behavior aligned with our anticipated behaviors and developed theories to account for any revisions.

In our analysis of the pre-test and post-test data, we first coded PSTs' responses for mathematical correctness. One item asked students how they would introduce the arcsine function to their class. We noted whether PSTs said in their explanations that the arcsine function was the inverse of the sine function with the domain of $[-\pi/2, \pi/2]$ (a correct response) or that the arcsine function was the inverse function of sine with no domain restrictions (an incorrect response). Another item presented PSTs with the a situation in which a student presented the following solution to a trigonometric equation:

$$\begin{aligned} \sin(2x) &= .7 \\ \arcsin(\sin(2x) &= \arcsin(.7) \\ 2x &= 0.7754 \end{aligned}$$

x = 0.3877 + 2πk

This solution contains two mistakes. The correct solution is x = 0.3877 + πk and 1.1831 + πk. So the student work contained two errors. When taking the arcsine of both sides, the student neglected one of the solutions in the [-π,π] interval (π – arcsin(.7) = 2.3662) and the student added the 2pk at the last step, rather than the third, thus missing solutions of the form 0.3877 + πk when k is an odd integer. In evaluating the PSTs response to the student's work, we documented which mistakes, if any, the PST identified. After coding for the mathematical accuracy of the PSTs' responses, we analyzed the ways in which the PSTs would respond to students qualitatively and interpretatively using thematic analysis (Braun & Clark, 2006).

## Results

### Lesson and behaviors

Given space constraints, we give only a quick synopsis of what transpired in our implementation of the Trigonometry Module, but a more extensive analysis will be presented in our talk. As we anticipated from our prior research (Wasserman et al., 2015), PSTs did poorly on the pre-test. Most thought that sin $x$ and arcsin $x$ were inverse functions and few spotted either of the mistakes in the student solution that they were asked to evaluate.

To step up to secondary mathematics, the instructor provided the students with the definitions of injective functions and (strictly) monotonic functions. Then PSTs were asked to explore the relationship between continuity, strict monotonicity, and invertability by debating about whether four statements were always true, sometimes true, or never true. For instance, one statement was that if a function was strictly monotonic on an interval, then it had an inverse on that interval. (This is always true). The purpose of these activities was to engage PSTs in "productive struggle" as they wrestled with these ideas so that the subsequent real analysis would be motivated. Next, we stepped up to real analysis by having the lecturer present two theorems. The first theorem was that strictly monotonic functions were always invertible. The second theorem was "Let $I \subseteq \mathbb{R}$ such that I is an interval. Suppose f(x) is a continuous function from A to $\mathbb{R}$. Then f(x) has an inverse if and only if f(x) is strictly monotonic."

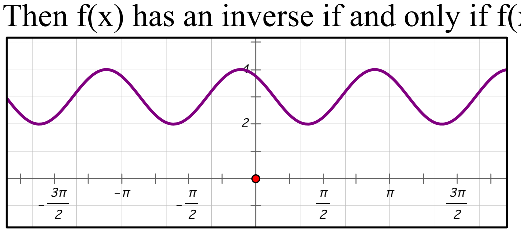


*Figure 3.* Graph of *f(x)* that students were asked to consider

We had the PSTs step back down to secondary mathematics. PSTs were given a worksheet specifying that "If we want an inverse for a continuous real-valued function *f(x)* but *f(x)* is not one-to-one, **by convention,** we seek to find the largest interval *A* on which *f(x)* is monotonic such that *A* contains 0 and at least one positive number". They were then asked to revisit the relationship between sin x and arcsin x. They were also shown a graph of *f(x)* in Figure 3 and asked to identify the conventional domain restriction in which *f(x)* would have an inverse, if *f(x)* would have an inverse on domains such as [π/4, π/2], and how they could justify their answers using the theorems that were previously discussed.

The lecturer presented a proof of the following theorem: Suppose *f(x)* is a continuous real-valued function *f(x)* that is one-to-one on an interval *I*. Suppose $a \in f(I)$. Then, $x = f^{1}(a)$ is unique solution to the equation $f(x)=a$ in the interval $I$. The lecturer discussed how in secondary mathematics, periodicity and symmetries were used to find the solutions to equations of the form $f(x) = a$ outside of the domain in which $f(x)$ was conventionally restricted. Finally, in stepping back down to practice, in the post-test, participants were again asked how they would introduce sine to the students and then they were asked to provide feedback to students who solved the trigonometric equation cos(3x) = .5, making errors similar to the students on the pre-test.

**Pre-test and post-test comparison**

On the pre-test, when asked how they would introduce arcsine to students, only one of the 10 PSTs mentioned domain restrictions. Eight PSTs said they would begin their presentation explaining the nature of inverse and that arcsin x was the inverse of sin x with no mention of domain restrictions. On the post-test, nine of the ten PSTs explicitly mentioned domain restrictions (with the remaining PSTE vaguely saying that he would explain that "arcsin x undoes the sine function to a certain extent"). Many PSTs used creative student-centered activities to illustrate the points (e.g., presenting students with the graph of sin x on the blackboard and inviting students to erase parts of the graph until the function had an inverse).

On the pre-test, only one PST found both errors in the student-generated solution, one PST found one error, one PST gave an ambiguous response, and the other seven PSTs found no errors, focusing instead on issues such as the student "rounding too early". On the post-test, seven PSTs found both errors, two PSTs found one error, and one PST gave the same ambiguous response that he did on the pre-test. The PSTs generally provided feedback to the student by pointing out how arcsin x was only the inverse of the sine function on its restricted domain.

## Summary and Discussion

In this proposal, we presented evidence that by studying real analysis while participating in our Trigonometry Module, PSTs were better able to engage in high leverage practices about teaching inverse trigonometric functions. This includes providing an explanation or a model for explaining what the arcsine function is as well as evaluating and providing feedback to a students' argument. We have provided evidence that PSTs lacked the mathematical knowledge to engage in these HLPs effectively before our module; they gave mathematically incorrect explanations about the meaning of arcsin x and they did not recognize mistakes that a student made in his solution to a trigonometric equation. After the class, most PSTs did not make these errors. In the talk, we will also document how they provided pedagogical responses that were not only mathematically correct, but thoughtful and appropriate.

More broadly, we have described a pressing issue—PSTs are required to take advanced mathematics courses but are not benefitting from doing so. We have described an innovative method for addressing this problem by linking the content of real analysis to the high leverage practices that PSTs must engage in. Finally, we have provided an illustration of a module built in accordance with our theory and refined from several iterations of design research, along with evidence that we have achieved our desired learning goals when we implemented this module. Consequently, what we are presenting is a theoretically driven existence proof that our innovative model has the potential to make advanced mathematics relevant for practicing teachers.

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# An APOS Study on Undergraduates' Understanding of Direct Variation: Mental Constructions and the Influence of Computer Programming

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*This study explores undergraduates' understanding of direct variation before and after instruction using computer programming to teach generalization over the concept. Based on an initial genetic decomposition for direct variation, the four math/CS researchers developed a research design that included lessons featuring computer programming and mathematical proof writing activities. This study shares results from an application of the instructional research design to N=33 undergraduates interested in teaching. Lessons were from a secondary education math methods course. Follow up interviews were conducted with seven participants. The analysis, using APOS as a framework, categorized mathematical behaviors at the Action, Process or Object level. The study identified obstacles that may have prevented progression through deeper levels of understanding such as deficient prerequisite skills and an affinity for routine algebraic manipulation rather than considering underlying relationships. The student data demonstrated how computer programming activities influenced undergraduates' mental images.*

*Key words:* Direct Variation, Generalization, Computer Programming, Pre-Service Teachers

## Introduction

The ability to generalize is considered an essential skill for reasoning about and deeply understanding mathematical concepts by mathematics education researchers. Many researchers have investigated how to explicitly induce students to develop generalizations in the context of mathematical explorations (Tall et al., 1991). Many mathematics education researchers believe that using computer programming activities designed to parallel the construction of an underlying mathematical process may stimulate or accelerate the development of the associated mathematical construction (Dubinsky and Tall, 2002; Authors, 2012). In prior work, we developed an explicit method for motivating students to generalize into mathematical constructions using computer programming exercises and proof writing based on the theoretical perspective of APOS theory. In this study, we extend our previously published preliminary report examining student's understanding of direct variation (Authors et al., 2016). The research questions we investigate are: (1) Does our genetic decomposition of direct variation adequately describe the observed students' constructions; and (2) Do our instructional treatment's computer programming activities influence students' mental constructions as described in the genetic decomposition?

## Proportional Reasoning and Theoretical Framework

Mathematics education researchers have dedicated considerable energy to proportional reasoning with elementary and middle school students, high school students, undergraduates and graduates. Collectively these researchers have shown that students and adults have difficulty with problems involving proportional reasoning (Noelting 1980; Vergnaud, 1983; Hart, 1988; Lesh et al., 1983; Kaput and West, 1994). We found that explanations for the difficulties undergraduates experience with the concept of direct variation are sparse in existing literature. Hence this study will contribute to the literature on how undergraduates understand direct variation by examining students' mental constructions and exploring how computer programming activities support the development of mathematical constructs.

The theory of reflective abstraction was described by Piaget (1985) as a two-step process, beginning with reflection on one's existing knowledge, followed by a projection of one's existing knowledge onto a higher plane of thought. Further, Piaget (1985) and Dubinsky et al. (2005a, 2005b) wrote that during the process of cognitive development, reflective abstraction could lead to the construction of logico-mathematical structures by the learner. The conviction that reflective abstraction could serve as a powerful tool to describe the mental structures of a mathematical concept led Dubinsky to develop APOS theory.

In APOS theory the mental structures are Action, Process, Object, and Schema. A mathematical concept develops as one acts to transform existing physical or mental objects. Actions are interiorized as Processes and Processes are encapsulated to mental Objects. It is tempting to view the progression as linear, but APOS practitioners hold that learners move back and forth between levels and hold positions between and partially on levels. In other words, the progression is not linear. This nonlinear behavior and the resulting mental structures may explain the different ways learners respond to a mathematical situation (Arnon et al., 2014).

## Genetic Decomposition for Direct Variation

The genetic decomposition was developed as a conjecture of the mental constructions, Actions, Processes, and Objects, that may describe the construction of mental schema for the concept of direct variation as it develops in the mind of the learner. The genetic decomposition served as a model for the design of this research study as well as the analysis of the results. It was also the basis for the computer activities in the lessons that were developed for the students. The pervasive impact of the genetic decomposition is consistent with an APOS theoretical framework (Asiala, et al., 1996).

The prerequisite concepts to start the construction of direct variation are an Object conception of multiples of a number, a Process conception of variable and an Object conception of constant. The notion of equality (=) needs to be used as a relation between corresponding elements of two sets. The learner must have a Process level conception of one-to-one correspondence between two sets X and Y, and be able to recognize and compare corresponding members.

### Action

The Actions needed are simple algebraic manipulations involving division and/or multiplication of numbers. The learner will apply the Actions to substitute in known values and solve for an unknown value in the equation. For example, she or he might find a constant ($k$) by dividing the first value ($x$) by the second value ($y$), and then multiply a subsequent number by $k$ to find the answer. Each activity is viewed by the learner as a single instance, isolated from subsequent similar instances. At this level, $k$ is viewed as a specific value, not as an arbitrary constant. The learner may or may not see the relationship between $x$ and $y$, they may work several examples without seeing a general pattern.

The same Actions described above can take place in different settings with different representations of the relation, such as a table, mapping, graph, and an analytical example.

### Process

These Actions are interiorized into Processes as the learner repeats the Action with different values of $k$ or different values of $x$ or $y$. They might iterate through values of $x$, but instead of checking specific numbers, the student can determine in general and in his or her imagination, for example, that as values of $x$ increase, corresponding values of $y$ will increase. The learner recognizes a general behavior that $x$ and $y$ vary.

As the learner iterates over $x$, this Process with $x$, $y$, and $k$ is coordinated into a new Process where the learner can view a sequence of numbers X and can determine if elements $x$ in a set X vary with corresponding values $y$ in a set Y without multiplying each value of $x$ by $k$ but by imagining each value of $x$ as a multiple of its corresponding value of $y$. While they imagine multiplying by $k$ or dividing by $x$ and $y$ to get $k$, they may not see that $y$ is locked into a value by $x$ and $k$, into a pattern that is carried out no matter what value is given. They may or may not see the rate of variation as a constant rate.

**Object**

The Process of checking if elements of a sequence of numbers X are equal to a constant multiple $k$ of corresponding values of Y, (or quotient of $x$ and $y$ is constant) encapsulates into an Object when the individual is able to apply Actions or Processes to it. The Actions that can be carried out on the Process conception of direct variation include comparing and contrasting it with other generalized properties of multiples such as doubling or halving, and to interpret the role of varies directly in the possibility that the two sets X and Y have a constant $k$ when any corresponding elements are divided. For example, they may understand that the ratio between corresponding elements of X and Y is a constant $k$. They may double the values in X and observe that values in Y are doubled. Then they may halve the values in X and observe that values in Y are halved, and so on. The learner may generalize the process that the subsequent values are determined by $k$, the constant of proportionality, locked in a pattern that is carried out no matter what value of $x$ you select. Another Action on the process may be reversing the process to determine X when $k$ and Y are known.

## Methodology

We applied our instructional treatment to the concept of direct variation for this study. Our investigation was carried out with 33 upper level undergraduates who were interested in teaching mathematics. Each subject participated in a complete lesson including the pre-test, response sheets, and post-test. The format of the lesson was as follows. A brief introduction to the programming environment was given along with the code template shown in Figure 1. A cursory review of the relationship distance is rate times time ($d=rt$) was also presented. Using the code template with an increasing rate and fixed time, participants were asked to complete the program to output the associated distance. Learners were encouraged to experiment with their computer programs and make observations about any relationships. Once this initial table was constructed, the participants were ushered through a series of program modifications and written responses. For example, they were asked to add columns to their programs to depict the doubling or halving of the rate with time fixed and the resulting distance. Programs were modified to show the results of doubling, tripling, and halving the rate with time fixed. Written responses to questions and reflections on their observations were recorded by the participants on their response sheets including generalizations of behavior. Observations on variation and direct variation were solicited as general expressions and participants were taught how to denote the general expressions in mathematical language. For example, participants might observe that if rate doubles and time is fixed, then distance doubles. The instructional treatment was designed so that repetition with various program modifications would stimulate the desire to generalize the observed behavior and make conjectures about the mathematical construct. The final stage of the lesson involved making conjectures and convincing arguments. Participants were shown how to use general expressions to support, or refute, a conjecture using mathematical language. They were then asked to attempt their own convincing arguments with the general expressions they recorded during their inquiry. All of the participant's responses were collected on written response sheets during the lesson. Additional data was collected in the form of interviews. We recorded interviews with seven of the participants which were then transcribed and analyzed. All of the collected data was reviewed and scored using APOS theory. We devised a ranked set of scores to denote pre-action, action, process, and object levels for the direct variation concept based on our genetic decomposition and recorded scores for each subject's pre-test, response sheets, post-test, and where applicable interview data. In the event that authors disagreed, a discussion and further analysis of the data was used to reach consensus.

```
print("r", "t", "d", sep="\t")
t = 5
r = 1
while r < 11:
  print(r, t, "?", sep="\t")
  r = r + 1
```

*Figure 1. Computer programming template for lesson*

## Results

In the discussion that follows, R denotes the researchers and U0001 to U0033 identify undergraduates. Results are presented that show how student mathematical behavior correlated to the genetic decomposition. Results also illustrate the influence of computer programming on students' ability to generalize over the concept of direct variation.

### Action

Student responses to questions were scored at Action level based on the description in the genetic decomposition. Action level responses were analyzed by the authors for common mathematical behaviors. Student responses during the lesson and in interviews following the lesson fell into three categories of mathematical behavior:

- Category 1. Using specific values or thinking about specific instance
- Category 2. Balancing the equation
- Category 3. Substituting a value in the equation

**Action Category 1. Using Specific Values or thinking about specific instance.** In the follow-up interview, the researcher asked the student to explain their thinking on a response.

R: What were your thoughts on this? (pointing to post-test response)

U0001: I like having values just cause[sic] it helps distinguish what we're already going over like variables are fine but when I actually have a number to place with the variable it makes it easier to keep up with where I'm going and what I'm doing. So I would place a random value somewhere just so I know how to get from point A to point B.

**Algebraic manipulations of a general expression.** It is not unexpected that students at the Action level for a concept would use specific values to direct their problem solving. Surprisingly, this study found that ten of the eleven Action level students did not rely on specific values but performed algebraic manipulations on a general formula. What looked like a general argument, which might imply an Object conception, was instead an explicit, step-by-step procedure to balance the equation. This is similar to Frith, et al. (2016) who found students could work proportion problems applying “mechanical knowledge or algorithmic procedures” without actually reasoning about the relationship. Mechanics of algebra included either trying to balance the equation (9 instances) or to substitute general expressions into the equation (8 instances). Students at this level did not meet the prerequisite skills, as defined in the genetic decomposition, two students were at the pre-action level for the concept of multiples, eight did not meet the prerequisite process level for the concept of variable, two did not meet the prerequisite for constant, and one did not meet the prerequisite for the concept of one-to-one correspondence.

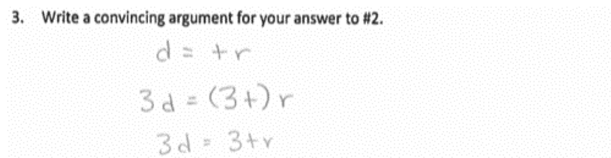


*Figure 2. Action Category 2 – Balancing the equation*

The snip of student U0005 in Figure 2 shows a typical response in Action Category (2). The student carried out the step by step procedure, multiplying both sides of the equation by a constant, e.g., if *d=tr* then *3d=(3t)r*. This student wrote in their response of a “need to balance”, as they multiplied both sides by 3.

The snip of student U0029 in Figure 3 shows a typical response in Action Category (3). The student carried out the step-by-step procedure, substituting *3r* for *r* in the equation *d=rt*. This work demonstrates a lack of the prerequisite requirement for a process understanding of variable, as *d* takes on the role of the first distance and the second distance.

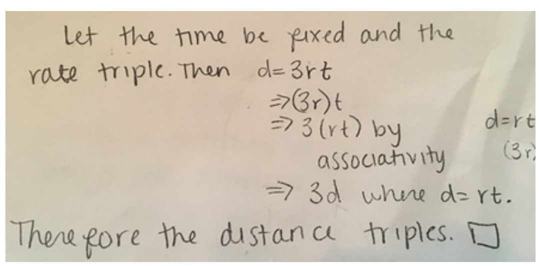


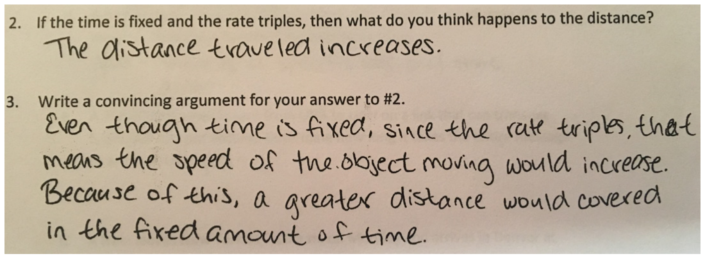
*Figure 3. Action Category 3 – Substitution*

**Process**

Of the 14 students at the process level, 10 demonstrated the notion of varies without demonstrating a notion of varies directly. Student concept of varies fell in two categories: (1) *x* increased (or decreased) then *y* increased (or decreased) or (2) *x* increased (or decreased) by some multiple, then *y* increased (or decreased). In either case, whether or not they repeated the given information about *x*, for their part in the solution they did not mention the multiple. They did not indicate an awareness of the “locked relationship” between *x* and *y* that is determined by the constant of proportionality *k*.

A typical response for varies in Process Category (1) was demonstrated by student U0003 who described a dependence between rate and distance where the rate increased then the corresponding distance “will increase as well”. The parenthetical statement by the student “The same time frame in a quick pace" indicated they were imagining a process in their mind, where rate and distance varied in a coordinated way.

The snip of work from U0007 in Figure 4 shows a typical response for varies in Process Category (2). The student described a dependence between rate and distance where the rate tripled then the corresponding distance traveled increased. They are imagining a process where an object is moving at a faster speed so "a greater distance would be covered in a fixed amount of time". There was a Process in their mind where rate and distance varied in a coordinated way.


2. If the time is fixed and the rate triples, then what do you think happens to the distance?
The distance traveled increases.

3. Write a convincing argument for your answer to #2.
Even though time is fixed, since the rate triples, that means the speed of the object moving would increase. Because of this, a greater distance would covered in the fixed amount of time.

*Figure 4. Process Category (2) - Varies*

In neither case did the students in Process Category (2) demonstrate a knowledge of the "locked in" relationship between $x$ and $y$ that is a part of direct variation and is fully determined by the constant of proportionality.

**Object**

The responses that indicated Object level understanding of direct variation, according to our genetic decomposition, fell in two categories: (1) Relationship between X and Y locked in place by $k$, (2) Elements of X were dependent on values in Y and the dependency determined by $k$. Although 32 of the 33 students correctly identified two general expressions relating $d$, $r$, and $t$, only two students demonstrated an Object level knowledge of the fixed relationship between X and Y, determined by $k$, before the instructional treatment.

### Influence of Computer Programming on Generalization

The influence of writing computer programs to explore the concept of direct variation was demonstrated by 16 of the 33 students. These students referenced their programming activities in their responses, in multiple instances, even though neither the question (nor the instructor) suggested responding with program code. Students naturally and intuitively adopted language from their programs. Twelve of the sixteen, who referenced programming in their responses concerning general expressions, improved at least one level during the instructional treatment, while two stayed the same and two went down a level. The students who referenced their programs when asked to give a general expression fell into two categories: (1) Computer Input: Print Statements and (2) Computer Output. In both cases illustrated below by typical responses, the students imagined generating code in their mind, and copied their imagined code onto their response sheet.

The response from Student U0007 in Figure 5 shows a typical response for Computer Category (1). The student imagined writing a computer program with the displayed print statement as an input statement. The response below was after the first computer programming activity. The print statement was stuck in between the answers for Response #3 and Response #4. It appears as a transition between the English statement "the distance is also doubled" and the general expression *(2r)t*. The transitory work is seen as the student wrote “*d = r2*t*” above the print statement “*(r*2)*5”*.

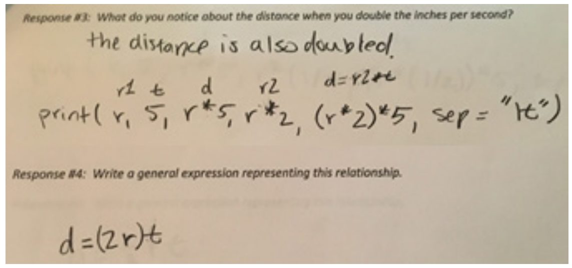



*Figure 5. Category (1) – Computer program*

Two students demonstrated evidence that running a computer program in their mind and reflecting on the output in table form was a transition from English language to mathematical language. The response from Student U0002 demonstrated the typical response for Computer Category (2) by constructing a table for Response #5 in the left margin after concluding, “halving the rate also halves the distance”. The same student then responded in Response #6 with “d2 = $\frac{rt}{2}$.”.

In the following interview snip, U0004 described how they developed a “mindset of generalizing” during the instruction. They did not demonstrate a general notion using variables in their expression until after the first programming activity.

R: Just describe when you were writing the last couple of proofs or either one of the proofs
U004: I was thinking more of just the letters and generalizing it after we had done those together and the ones on the other response sheets because I think I was in a mindset of generalizing it…
R: Right
S: So the way I wrote it out, I put more notation the second time on the post-test.

## Conclusion

In this study, students explored direct variation through an explicit method for teaching generalization that uses computer programming and convincing arguments. The researchers found that scoring and assessing undergraduates' conception of direct variation was complex due to the task-dependent and context-dependent nature of conception. The genetic decomposition adequately described the students’ constructions observed in the data. We noticed students at the Action level tended to manipulate algebraic expressions without understanding the underlying structure. We found many students in our study have a notion of vary but not directly varies. We observed some students who needed to construct the property vary, at the Process level, before constructing the property varies directly, at the Object level and conjectured that a notion of vary is a prerequisite to directly vary. Therefore, we have modified our genetic decomposition to account for this in future studies. We found that prerequisite deficiencies corresponded with the inability to progress through levels of understanding as measured by APOS. We found that students naturally turned to their computer programs to help find general expressions for the concept. Some students considered the inputs to their programs and others reflected on the outputs of their programs when asked to write general expressions for observed relationships. The programming activities influenced students and served as a catalyst to move from purely English descriptions of their conceptions to using mathematical symbols and a “mindset of generalizing”.

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Building Models of Students' Use of Sigma Notation

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*Summation notation is a widely-used standard that can represent all kinds of sums. Despite its utility, the literature on this topic points to the notation being difficult for students. Our research project gives insight into how students think about summation notation and why it is so challenging. This report builds off of the first phase in our project, which proved the existence of students' uncertainties with elements of the notation. Survey data from 285 undergraduates suggested that uncertainties are common amongst students. We also found that the act of encoding a sum in sigma notation is more cognitively demanding than interpreting a summation notation expression. In this paper we present models of students' ways of thinking about summation notation.*

*Keywords:* Summation notation, calculus, cognitive models

Sigma Summation notation is a widely-used standard for expressing all kinds of sums, from infinite series to probabilities to approximate area under a curve. However, there is little in the research literature regarding this notation. There is consensus that the notation presents difficulties for students. For example, we were able to find papers discussing issues that contribute to student struggles in statistics (Ramsey, 1999) and engineering (Armstrong & Croft, 1999) that mention summation notation. Little and Jones (2010) provide empirical evidence that the notation is challenging for students and they observed that, "The use of algebraic language such as sigma notation and iteration formulae added substantially to the difficulty of questions" (p. 141). However, Little and Jones' study provides no information regarding why students might struggle with this notation or how they might come to make better sense of it.

Summation notation appears in some research articles on infinite series (e.g., Martínez-Planell, Gonzalez, DiCristina, & Acevedo, 2012) and integration (e.g., Sealey, 2014) but is not the focus of these studies. Martinez-Planell et al., focus on the distinction between seeing an infinite series as an infinite sum and seeing it as the limit of a sequence of partial sums. They do not investigate possible challenges involving the sigma notation used to represent partial sums and infinite series. Sealey presents a framework for understanding Riemann sums that includes a "summation layer" but acknowledges that her study, "did not require students to use the notation of $\sum_{i=1}^{n} f(x_i)\Delta x$ to represent the Riemann sum, nor did it explore students' understanding of this notation" (p. 243). Additionally, Brijlall and Bansilal (2010) proposed a genetic decomposition of the Riemann sum that explicitly attended to summation notation. Specifically, they included two abilities related to summation notation in their model proposing that students who understand Riemann sums 1) "can represent finite or infinite sums as expanded sums, when given its compact form using sigma notation" and 2) "can represent finite or infinite sums in compact form using sigma notation when given its expanded representation." (p. 133-134). While this study provides no information about whether (and if so, why) students struggle with these two activities, this pair of activities provides a structured way to think about what it may mean for students to be able to make sense of and work fluently with this notation.

What little information exists suggests that students struggle with summation notation and that this likely contributes to difficulties with concepts that rely on this notation such as probability, infinite series, and integration. However, the research literature contains very little insight into *how* students think about this notation and *why* it is challenging. Our research project endeavored to investigate how students think about summation notation and was completed in two phases.

In the first phase, summation notation was not the focus. We conducted an exploratory design experiment with two students, Betty and Kathy, where we actively investigated post-calculus students' understanding of integral. During the experiment, summation notation came up and the students displayed some interesting thinking about how the notation functioned. When we investigated their thinking we found that Kathy and Betty were aware that sigma notation provides a shorthand way to write a sum. They seemed keenly aware of the kinds of information needed to expand a sum (or that should be captured in the notation when encoding a sum). However, they were unsure exactly how this information was supposed to be recorded in the notation. For example, they were uncertain whether the value on top of the sigma was supposed to represent the terminal input value or the number of terms (Strand, Zazkis, & Redmond, 2012).

After working with Betty and Kathy we wondered if their uncertainties are common amongst other undergraduate students. For this reason we designed a second phase specifically targeting student thinking about summation notation. The results from the second phase of our project will be the focus of this paper. We aim to address the following research questions: (1) If students do struggle with sigma notation, what kinds of difficulties do they have and how might such difficulties be explained? (2) Is there a difference in difficulty between expanding (interpreting) sums expressed in summation notation and compressing (encoding) expanded summations using summation notation?

## Methods

### Survey Instrument

Our survey instrument consisted of three tasks. The first task involved encoding a Riemann sum, which required the student to express the input variable of a function as a function of the index variable. The second task was a basic encoding task in that the index variable could serve as the input variable. The third task was an interpretation task. We conjectured the first task was the most difficult and the third task to be the least difficult. This study focuses on student responses of the second and third tasks. Here we will describe the two tasks (in reverse order) and the rationale for their design.

The Interpreting Task (Task 3) instructed students to write out the expanded ("longhand") sum represented by a given sigma notation expression (Figure 1). We expected a student adhering to the standard convention of summation notation to write, "$(2+1)^2+(3+1)^2+(4+1)^2+(5+1)^2+(6+1)^2+(7+1)^2$." Specifically, the "2" below the sigma represents the starting value of the index, the "7" above the sigma represents the terminal value of the index, and each step in the index is incremented by 1. We chose to start the index at two so that we would be able to tell if the seven above the sigma sign was interpreted as the number of terms or the terminal value of the index. Betty and Kathy vacillated between these two interpretations of the index and so we wished to know how common this particular difficulty was. In general we wished to see what kinds of errors students might make in interpreting the different elements of summation notation (the index, the sigma, the summand, the numbers above and below the sigma).

**Task 3:** Write out the given summation longhand:

$$\sum_{k=2}^{7}(k+1)^2$$

*Figure 1. The Interpreting Task.*

The Encoding Task (Task 2) asked students to encode the sum of the first ten odd integers using summation notation (Figure 2). There are many possible expressions that would follow the standard convention of summation notation but an example would be "$\sum_{k=1}^{10} 2k-1$ " or possibly "$\sum_{k=0}^{9} 2k+1$". With this task we were interested in what challenges encoding with summation notation would present to the students. We were also interested in exploring the relative difficulty of encoding and interpretation tasks.

**Task 2:** Using $\sum$-notation, write an expression for the sum of the first ten odd integers.

*Figure 2. The Encoding Task.*

The Interpreting and Encoding Tasks were exactly the same as tasks given to Betty and Kathy during the design experiment. We chose to give these tasks to Betty and Kathy so that we could investigate what kinds of errors students might make with summation notation in less complex contexts than the first task. The consistency across the two phases of the projects allowed us to leverage Betty and Kathy's reasoning when we analyzed the survey data.

**Participants**

Two hundred eighty five undergraduate students participated in the second phase of our study. These students were enrolled in a course in the calculus sequence, differential equations, linear algebra, or a course in an undergraduate advanced calculus sequence and received extra credit for their participation. We invited students enrolled in this set of courses in hopes of receiving responses from students with various summation notation experiences. In total, 567 students were enrolled in at least one of 15 sections of the 8 courses. 50.3 percent (285/567) of the students completed at least part of the survey, 42.3 percent (242/567) of the students completed both Task 2 and Task 3.

**Analysis**

There are four stages of analysis for the Interpreting and Encoding Tasks. During the first stage we went through each survey and counted how many had errors of any kind; these were sorted by task. In the first pass we looked to see if each solution was perfect; if it was then the response was marked 'correct' and if not it was marked as 'incorrect'. We did not attempt to analyze or describe the errors at that stage.

In the second stage of analysis we recorded error types made on the Interpreting Task. To do so we first read each survey's responses and discussed common errors. We came up with 6 categories of error types, including: gave a sum with seven summands, substituted only even values for $k$, not enough summands, used sigma with the expanded sum, used only one value as the input, and other.

In the third stage we turned to the Encoding Task. While coding error types for the Interpreting Task involved low inference, we felt this would not be the case for the Encoding Task. For instance, one student produced the following expression: "$\sum_{i=0}^{10} i + 1$" (Figure 4). We could conjecture that the student incorrectly substituted index values starting at $i = 0$ and ending at $i = 10$; however, this would rely heavily on our interpretation. Later, we will argue that this is probably not how this student would explain their encoded sum. In order to match the level of inference used to code the Interpreting Task, we recorded each piece of the given summation expression. For instance, we recorded what (if anything) was written below and above the sigma. We also recorded the type of expression within the sigma into the following categories: outputs odd values when incrementing the inputs by 1 (e.g., $2x - 1$), outputs odd values when not incrementing the inputs 1 (e.g., $x + 1$), and seemingly unproductive expression (e.g., $\frac{(-1)^x}{x}$). In this stage we also sorted responses that we tagged as 'correct' into three categories: correct and summed $2i + 1$ (or equivalent) from $i = 0$ to $i = 9$, correct and summed $2i - 1$ (or equivalent) from $i = 1$ to $i = 10$, or correct with other encoding.

The last stage of analysis involved comparing each student's Interpreting Task response to the Encoding Task response. This process included separating surveys based on error type and then looking at the frequency of the Encoding Task tags. For example, we found that 16 of the 27 students that gave an expression with seven summands in the Interpreting Task also wrote a "10" above the sigma in the Encoding Task.

## Sample Results

### A Model for Interpreting Summation Notation

We found that most students were able to answer the Interpreting Task correctly; 73.2 percent (186/254) of the students that attempted the task correctly answered the question. We will first attempt to explain the thinking of the 69 students that incorrectly answered the Interpreting Task. We found the most common error type (27 responses) involved summing seven terms. There were 4 different subcategories within this error type, which are listed with their frequency in Table 1.

*Table 1. Categories for summing seven terms.*

| Category | Frequency (n=27) |
|---|---|
| Summed $(k+1)^2$ from $k = 2$ to $k = 8$ | 16 |
| Substituted 2 seven times | 6 |
| Did not increment by 1 | 3 |
| Did not substitute values for $k$ | 2 |

Summing seven terms suggests that these students took the number atop the sigma as the number of summands. Kathy and Betty also considered this during the first phase of the project. While this is not the standard convention (i.e., the value above the sigma represents the terminal index value), it *is* a viable convention and it is analogous to a "count loop" in computer science.

However, the non-standard usage is still problematic because a user who respected the standard usage would be unable to recreate the expanded sum the students meant to encode.

We can test whether or not the 27 students truly take the number above the sigma to be the number of summands by considering their responses to the Encoding Task. The 27 students should write "10" above the sigma when they are prompted to encode the sum of the first *ten* odd integers if they are using the nonstandard convention.

The following section will investigate how students who 'summed $(k+1)^2$ from $k=2$ to $k=8$' responded to the Encoding Task.

**Subcategory 1: Summed $(k+1)^2$ from $k=2$ to $k=8$.** Figure 3 gives a typical response of the 16 students under the "summed $(k+1)^2$ from $k=2$ to $k=8$" subcategory. Responses in this subcategory include 7 summands, each of which correspond to an output of the function $f(k)=(k+1)^2$ with $\{2,3,\ldots,8\}$ as its domain.

$$(2+1)^2+(3+1)^2+(4+1)^2+(5+1)^2+(6+1)^2+(7+1)^2+(8+1)^2$$

*Figure 3. Typical Interpreting Task response with 'summed $(k+1)^2$ from $k=2$ to $k=8$' tag.*

Now consider an Encoding Task response from one of the students that made a subcategory 1 error in the Interpreting Task (Figure 4). Notice, this student did indeed place a "10" above the sigma. In total, 10 of the 16 responses did so as well.

$$\sum_{i=0}^{10} i+1$$

*Figure 4. Student response to Encoding Task.*

Two of the 6 students that did not place a "10" above the sigma left the Encoding Task (seemingly) incomplete; both responses did not include an expression to sum. Additionally, it appears as though another student misread the Encoding Task to say 'write an expression for the sum of odd numbers less than ten.' (See figure 5.) This is evidenced by the list "{1, 3, 5, 7, 9}". Under this assumption, the number above the sigma, "6", corresponds to the number of odd values less than 10.

$$\sum_{i=1}^{N}(x_i)4x, \{1,3,5,7,9\}$$

$$\sum_{i=0}^{6}(N+1) \quad 0,1,,3,,5,7,9$$

*Figure 5. Possible misinterpretation of Encoding Task prompt.*

It is also interesting to note that it is possible for the students to assign the number above the sigma to mean the total number of summands *and* give a 'correct' answer to the encoding task. In this situation, a student could interpret "$\sum_{k=1}^{10} 2k-1$" to mean: substitute natural numbers for

$k$ until there are 10 summands. We found 3 of the 10 student responses that placed a "10" above the sigma was tagged as 'correct'.

However, this convention is problematic whenever the starting variable is not equal to 1. In this situation, the number of summands does not equal the number of values to substitute using the standard convention. There were 40 Encoding Task responses that were tagged 'correct' because they gave the expression "$\sum_{k=0}^{9} 2k + 1$". This response suggests that these students do *not* take the number above the sigma to mean the number of summands since the prompt asked students to sum the first *ten* odd numbers. We found that only 1 of the 40 students that answered "$\sum_{k=0}^{9} 2k + 1$ " to the Encoding Task also wrote seven summands for the Interpreting Task.

**Task Hierarchy**

While most students were able to correctly answer the Interpreting Task, only 38.4 percent (96/250) of the students that attempted the Encoding Task wrote a correct response. We conjectured that encoding is more cognitively demanding than interpreting a summation-notation expression. This is because if a student were able to encode a sum, then the student would be familiar with the structure and workings of the elements of the summation notation; that is, the student would have the ability to accurately interpret a given summation-notation expression. Additionally, in order to correctly verify their encoding, a student would necessarily be able to interpret their own summation notation accurately.

With respect to the Tasks we expected this to play out in a contrapositive manner. That is, we expected that students who could not successfully complete the Interpreting Task would not be able to successfully complete the Encoding Task. Table 2 shows the number of students that correctly answered both tasks correctly, neither of the tasks correct, Interpreting Task correctly but Encoding Task incorrectly, and Encoding Task correctly but Interpreting Task incorrectly.

*Table 2. Combination of correct responses to Interpreting and Encoding Task.*

| Tasks Correct | Frequency (n=242) |
|---|---|
| Both Interpreting and Encoding Tasks | 86 |
| Neither Interpreting nor Encoding Tasks | 54 |
| Only Interpreting Task | 93 |
| Only Encoding Task | 9 |

Notice, of the 64 participants that were not able to answer the Interpreting task correctly, 9 participants were able to answer the Encoding task correctly. This seemed to be a high percentage until we looked further into how the 9 students responded. In particular, 4 of these students attempted to answer the Interpreting Task by expanding $(k + 1)^2$. One of these students then made an algebraic error (unrelated to the notation). The 3 other students distributed the sum across the three terms, evaluated the first two summations correctly, and then incorrectly reduced "$\sum_{k=2}^{7} 1$ " to "$1 \cdot 7$" (Figure 6). It seems as though evaluating the sum of a constant function over an index set might be more difficult for students since the students were able to correctly evaluate the first two terms after they distributed the sum.

$$\sum_{k=2}^{7}(k+1)^2$$

$$= \sum_{K=2}^{7} (K^2+2K+1)$$

$$= \sum_{K=n}^{7} K^2 + 2\sum_{K2}^{7} K + \sum_{K=2}^{7} 1$$

$$= 2^2+3^2+4^2+5^2+6^2+7^2+2(2+3+4+5+6+7)+(7\times1)$$

$$= 4+3+16+25+36+49+2(27)+7$$

$$= 200 \text{ Ans.}$$

*Figure 6. Student 'incorrect' response to Interpreting Task.*

An additional 2 students answered the interpreting task incorrectly by summing seven terms. These students then answered the Encoding task by writing "$\sum_{k=1}^{10} 2k - 1$". This is a situation that we described before in the previous section. That is, it is likely that these students interpret the number above the sigma to mean the number of summands, not the terminal index value. When students assign the initial value of the index to be 1 (and increment by 1) both ways of thinking will produce the same expression.

We also hypothesized that the ability to interpret a summation notation expression would not be sufficient for successfully encoding with the notation. This is because encoding sums with summation notation requires an understanding of functions and their domains above and beyond what is required to accurately interpret a given summation-notation expression. Specifically one must *construct* a function; this entails coordinating the construction of a rule with the construction of an appropriate domain (i.e. indexing set, in this case). For example in the Encoding Task, a student must first construct a function that will output odd values when the student inputs natural numbers (e.g., $f(x) = 2x - 1$). Then the student must restrict the natural numbers in such a way that the indexing set will output the first ten odd numbers (e.g., $\{1, 2, 3, \dots, 10\}$). In contrast, to interpret a student does not have to coordinate the construction of a rule with the construction of an appropriate domain. Instead, the student only coordinates the inputs from the given index set with the outputs of a given function. For this reason we expected a number of students to successfully complete the Interpreting Task who would not successfully complete the Encoding Task. Table 1 shows that this was indeed the case: 93 participants were able to successfully complete the Interpreting Task but were not able to complete the Encoding Task.

## Conclusion

We found that undergraduates struggle with summation notation. Like Betty and Kathy, many students were unsure about the structure and workings of the elements of summation notation. In particular, our data suggest that some students are unaware of (or at least do not follow) the standard summation notation conventions. However, often students do follow a non-standard convention (e.g., assign the number above the sigma to mean the number of summands, increment by a value other than 1, etc.). The conventions we witnessed are viable options; however, they are problematic because they are not standard. In particular, issues may arise when communicating with others that adhere to the standard convention.

We also found that encoding is more cognitively demanding than interpreting a summation-notation expression. We presented evidence supporting that being able to expand sums expressed in summation notation is necessary to being able to encode an expanded sum using summation notation. However, the ability to interpret an expression in sigma notation is not enough to encode a sum in sigma notation.

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# Developing Understanding of the Partial Derivative with a Physical Manipulative

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*Multivariable calculus education is an area of growing investigation, and in this study we specifically target the topic of partial derivatives. Data was collected on students learning in an innovative curriculum using physical manipulatives. We trace the complex path as students developed both their mathematical knowledge and their use of the artifacts at their disposal, and analyze the interaction between them. Implications for the classroom and for research are noted.*

*Keywords*: multivariable calculus, partial derivative, instrumental genesis, utilization scheme

## Introduction & Literature Review

Multivariable calculus education has seen a surge of interest in recent years. A presidential panel emphasized it as a key course for the introduction of ideas that are complex and essential for STEM students (PCAST, 2012). Some research has been done on student conceptions in multivariable calculus of function (e.g. Martinez-Planell & Trigueros-Gaisman, 2012), derivative (Martinez-Planell, Trigueros-Gaisman & McGee, 2015), and integral (Jones & Dorko, 2015). Significant research has been done on student conceptions for single variable calculus, much of it in the last several decades, and some research has addressed how students construct multivariable calculus knowledge on those foundations (Dorko, 2016).

One central idea is that of rate, which underlies the derivative. During the typical curricular progression, students must develop an increasingly sophisticated conception of rate. They encounter a single constant rate with first linear single-variable functions, then a single nonconstant rate with nonlinear single-variable functions, then multiple nonconstant rates with multivariable functions. The first conception, including coordinating change in the input and output variables into a quotient, has been shown to be essential to developing the second conception (Pustejovsky, 1999; Zandieh, 1997).

The results reported here are from the implementation of an innovative curriculum designed to introduce important topics from multivariable calculus through student exploration with physical manipulatives. Students use, as representations of two-variable functions, surfaces which are molded from clear plastic and have a dry erase surface. Accompanying tools include an inclinometer (used to measure the slope at a point on the surface in a given direction), and domain mats (dry erase sheets with coordinate lines or contour lines). Students in small groups complete activity sheets in-class, which emphasize collaborative learning, student inquiry, and measurement with quantitative reasoning (for further details see Wangberg & Johnson, 2013). A previous investigation examined one aspect of how students learn about tangent plane and linear approximation using these tools (Fisher, Samuels & Wangberg, 2017).

## Research Question

Even in light of the recent burst of research in multivariable calculus, little research has been done into how students develop conceptions of the partial derivative for functions of two variables. Further, for students who use physical tools to complete tasks and answer questions in their activities on multivariable calculus, little is understood about how they enhance their understanding of calculus, or what role the tools play in that process. At the nexus of these issues lies the following research question:

How do students develop conceptions of the partial derivative during exploration with a physical manipulative?

## Theoretical framework

Verillon & Rabardel (1995) presented Rabardel's theory of instrumental genesis to explain the complex process by which a person engaged in achieving a goal adopts the use of some assisting object. The material object when first introduced is an artifact. For it to be a productive tool, the user must attach to the artifact a role in completing the present task. Actions and behaviors cognitively organized by the user for a class of situations comprise a utilization scheme. Schemes can be constructed personally by the user as derived schemes, or received in a social context as adopted schemes. The process of instrumental genesis produces an instrument, an artifact endowed with a set of utilization schemes for tasks, which is therefore a combination of material object and cognitive structures. During instrumental genesis, the artifact shapes the user through interactions which enhance the user's understanding of the subject matter, a process known as instrumentation. Additionally, the user shapes the artifact by developing utilization schemes for interacting with the artifact, a process known as instrumentalization. Thus, as user and instrument develop their partnership, each one causes a transformation in the other. Subsequent to the development of the theory, instrumental genesis was applied in mathematics education to understand student use of graphing calculators, computer algebra systems (Artigue, 2002), and dynamic geometry software (Leung & Chan & Lopez-Real, 2006).

## Methodology

The data for this report were obtained from four students who worked as one group on an activity sheet designed to introduce the concept of the partial derivative. The first author, present as the instructor, asked questions to help make student thinking explicit and to encourage discussion and resolution of any disagreements within the group. The session was video recorded, the recording was transcribed, and the data were coded for instances of instrumental genesis by each author. Any differences of opinion were discussed until agreement was reached.

In the activity the students were tasked with measuring the partial derivative at a point on the surface using the inclinometer. The inclinometer used by the students had two rods, one round and one square with a bubble level attached, connected at the ends by a joint (see Figure 1).

Students could successfully complete the activity sheet with a utilization scheme for finding the partial derivative consisting of the following utilization schemes: one for the direction of the derivative, aligning the parts of the inclinometer in the proper vertical plane; one for the tangent line, positioning the round rod tangent to the surface at the selected point; one for representing the "run" ($\Delta x$ or $\Delta y$ in this context), positioning the square rod horizontal using the level; one for representing the "rise" ($\Delta z$), indicating a vertical displacement between the rods; and one for measuring change between two values for a variable, for which two possibilities are using a ruler or laying the inclinometer on the domain mat grid and counting boxes. As a result, they could calculate the quotient and find the partial derivative.

*Figure 1. Student using the surface, inclinometer, and ruler over a domain mat.*

The recorded session was split into episodes. Each episode consisted of discussion on approximately the same topic or in the same vein. Each episode was then coded with respect to utilization schemes. The students introduced two other utilization schemes in addition to the five schemes which comprised the partial derivative scheme described above: a scheme representing the normal line and a scheme measuring the interior angle of the inclinometer. Each scheme, when mentioned, was coded as attempted and completed (C), attempted with partial progress (P), or attempted with no progress (N), with an indication if it occurred specifically in the two-dimensional *y=f(x)* context (2).

## Results

Here we present some results on the student work to find the partial derivative for a two-variable function at a given point, where the function is represented by a 3-dimensional plastic model. In the activity given to the students, the input variables *x* & *y* represented position, and the function value *T(x,y)* represented temperature. The following episode is presented with its coding and short description.

*Interviewer*: Okay, so, how would you use the same structure, the same orientation [as in the *y=f(x)* context], if you used it on the surface?
*Student A*: (put inclinometer on surface, round leg tangent, see Fig 2)
*Student B*: Like this?
*Interviewer*: Okay, so now describe to me what you are doing there.
*Student A*: Well I'm basically taking the point (lifting inclinometer and indicating the point), and I'm putting this on there, like, tangent.
*Interviewer*: Okay, so go ahead, do that.
*Student A*: (placing inclinometer tangent again) So this has to be...
*Interviewer*: Okay, so you got that one tangent.

*Figure 2. Student A places the inclinometer tangent to the surface.*

In the excerpt, the group had just transitioned from the two-dimensional, *y=f(x)*, context to the three dimensions, *z=f(x,y)*, context. The group, for the first time, demonstrated a complete utilization scheme for tangency, as well as directionality. However, with the square leg not horizontal, they did not implement the complete utilization scheme for $\Delta y$ (which they had done before). They did not attempt to represent $\Delta z$, or to measure any displacements.

The coders identified 13 episodes in the recorded student activity, and the above description provides an example of how the coding was executed. The results are summarized in Table 1.

## Discussion

Analyzing the students' actions through the lens of instrumental genesis gave valuable insight into describing both their struggles and achievements.

### The Role of Instrumentation

In instrumentation, the artifact shapes the user through interactions which enhance the user's understanding of the subject matter. One example of successful instrumentation occurred in the episode below, in which students used the level to make the square rod parallel to the *xy*-plane to help represent displacements in the domain.

*Student B*: What is this for (indicating the level)?
*Interviewer*: Yeah, so what is that for?

*Table 1. Utilization schemes and how they arose in each episode of the activity.*

| | | **EPISODE NUMBER** | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **UTILIZATION SCHEME** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** |
| **Partial derivative schemes** | **represent run** | | | C | | C2 | | C2 | N | C | N | | C | C |
| | **represent rise** | | | C | | C2 | | | | | | | C | C |
| | **measure change** | | | C | | C2 | C | | | | | | C | C |
| | **directionality** | C | C | C | | | C | | C | N | N | | C | C |
| | **tangent line** | | | | P | C2 | | C2 | C | C | C | C | C | C |
| **Other schemes** | **normal line** | C | | C | | C2 | C | | | | | | | |
| | **measure angle** | P | P | | | | | | | | P | | | |

C: attempted and completed. P: attempted with partial progress. N: attempted with no progress. 2: in 2-dimensional *y=f(x)* context

*Student C*: To make it parallel.
*Interviewer*: That's to make sure that it's parallel. So why do you want it to be parallel?
*Student B*: *xy* parallel, right?

In this excerpt, the students observed a physical aspect of the artifact, the bubble level. Then, they proceeded to assign to it a functionality connected with mathematical content, being parallel to the *xy*-plane.

One early example of failed instrumentation occurred when the inclinometer shaped the thinking of one student in unproductive ways.

*Interviewer*: So tell me, tell me what you need to do, and then tell me how you're going to use the tool to do it.
*Student B*: First of all, the, perpendicular, the point, and we can find the right point, the bubble in the circle, and we can get the theta. So it means, this surface, this plane (indicating the *xy*-plane) and this plane (indicating the square leg) is parallel, so it means we can get this theta (indicating the angle, see Figure 3), and this theta is same. So-
*Student C*: -we can get-
*Student B*: -if we know theta...I don't know.

*Figure 3. Student B indicates the angle on the inclinometer.*

In this excerpt, the students focused on the angle presented by the tool. Using the angle was the very first idea suggested. One can hypothesize the reason, based on the appearance of the artifact. The primary physical characteristics of the artifact are two legs connected at their ends, and this mimics the standard instantiation of an angle. Further, the mobile joint allows for manipulation to any angle, which mimics the standard method to compare different angles. It is indeed possible to calculate the slope knowing the angle, however the students decided it was too complicated and did not pursue it. Thus, they did not devise a way to measure either the angle or the rate with this strategy, so this did not lead to new mathematical knowledge.

**The Role of Instrumentalization**

In instrumentalization, the user shapes the artifact by developing schemes for interacting with the artifact based in existing knowledge. One example of successful instrumentalization included using the inclinometer to represent the tangent line (as detailed in the first excerpt in the results section).

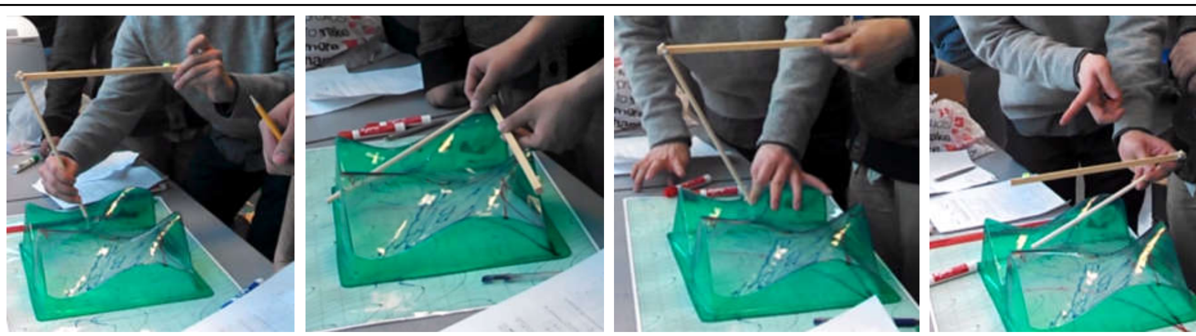
One example of failed instrumentalization included an attempt to measure $\Delta z$. The students previously demonstrated awareness of 3-dimensional rectangular coordinates, thus possessed the requisite mathematical knowledge, however the ruler was positioned between the ends of rods and not vertically (see Figure 4).

*Figure 4. Student C measures the length between the rods' ends.*

**Extending Utilization Schemes from the 2-Dimensional *y=f(x)* to the 3-Dimensional *z=f(x,y)***

One recurrent unproductive move during attempts to find the slope was placing the inclinometer's round rod perpendicular to the surface for $z=f(x,y)$. One possible explanation is that the normal line is uniquely determined, while the tangent line is not. The students did not offer a justification for placing the inclinometer perpendicular to the surface, despite multiple prompts to do so. Yet when subsequently presented with the two-dimensional $y=f(x)$, they quickly placed the inclinometer tangent and found the derivative at a point.

Previous research has documented how the transition from single to multi-variable calculus presents significant challenges (Dorko, 2016; Jones & Dorko, 2015). The analysis with instrumental genesis revealed how students were grappling with an issue more pervasive for two variable functions, that of directionality. Early on in the activity, the student group lined up the entire inclinometer (both legs) in the appropriate direction (see Figure 5a). However, when forced to grapple with other considerations, particularly a utilization scheme for the tangent line by making the round leg tangent (while making the square leg parallel simultaneously), the complete directionality scheme was lost (see Figure 5b). Subsequently, the correct directionality scheme returned but the correct tangent line scheme again disappeared (see Figure 5c). Only then was the group able to merge all the correct schemes simultaneously to form the utilization scheme for finding the partial derivative (see Figure 5d).



*(5a) (5b) (5c) (5d)*

*Fig 5. Students losing and regaining the utilization schemes for directionality and tangent line.*

**Equivalence between Utilization Schemes**

One interesting deduction made by the students was the equivalence between certain utilization schemes. When calculating a derivative for $y=f(x)$, the students' utilization scheme to measure both the rise and the run consisted of laying the inclinometer flat on grid paper and counting boxes. When calculating a derivative for *z=f(x,y)*, their utilization scheme for measurement at one moment consisted of counting grid boxes. Later in the activity, it consisted of using a ruler. They used the results in equivalent fashion, referring to them in both cases as *dx* (or *dy, dz, dT*), and subsequently dividing the two numbers to calculate the rate. Members of the group were satisfied with both methods, and no one insisted on switching for either scenario.

The reasons for this difference in practice may come down to previous experience. For *y=f(x)*, by common instructional practice, students would have prior experience determining the slope of a straight line by counting grid boxes, and the inclinometer already lay on the grid paper when they reached this step. For *z=f(x,y)*, students in this class had prior experience determining the z-value (i.e. height) of a point using a ruler. Although the students were drawn to different utilization schemes in different contexts, their actions reflect that they used different schemes to get the same outcome.

**Adaptation of a Previous Utilization Scheme**

When students found the slope for *y=f(x)* using the inclinometer, they used what for them was a well-known process in a well-known scenario, but with an artifact they had only recently encountered. This was an example of adaptation of a previous utilization scheme. For the utilization scheme for tangent line on the surface, students initially used a pen as their tool to represent a tangent line on the surface. Subsequently, they used the (round leg of the) inclinometer to manifest the tangent line.

Another type of adaptation of a previous scheme occurred when the students moved from discussing derivative in the *y=f(x)* context to discussing it in the *z=f(x,y)* context. This progression actually occurred twice, first with the actual tools and second in discussion only. For *y=f(x)*, the students quickly and effectively created a utilization scheme to find the derivative. (As discussed previously, modifying the scheme for *z=f(x,y)* did not happen quickly or without struggle, either the first or second time.) In this case, the artifact was the same but the scenario had changed.

**Utilization Schemes Disappear and Reappear**

Students seemed to "forget" what they already knew, only to "remember" it subsequently. Students produced a utilization scheme for directionality almost immediately, and it stayed present in their manipulations and discussion for some time. However, in the process of grappling with certain obstacles that seemed to give them great difficulty, directional fidelity disappeared. The utilization scheme for tangent line took the longest to appear for the first time. Perhaps the group's greatest challenge was changing the relationship of the inclinometer round rod to the surface from normal to tangent. During the discussion before it happened, the group laid the inclinometer on the surface in new ways which ignored their previous directionality scheme.

One possible explanation is that the more challenging obstacles produced so much cognitive load (Sweller, 1988) that the students could not simultaneously consider or maintain directionality. Only after it was resolved, reducing cognitive load, could students return to considering their already-determined utilization scheme for directionality.

**Instrumentalization during Development of a Utilization Scheme**

Student mathematical knowledge and artifacts can interact in dynamic ways during instrumental genesis. At one point, Student A said "dee-T. Hold on, something over 12, I think. Dee-T should be 12. No, dee-T should be this one (measuring in the z-direction). Yeah, four." He incorrectly coordinated the numerator and denominator in a slope calculation, before correcting himself. The apparent reason was that the triangle is upside down from typical usage, with the horizontal side higher in space than the vertical side (as in Figure 1). During the development of the scheme, the student confronted the signal from the inclinometer regarding the spatial relationship between $\Delta T$ and $\Delta x$ (referred to by the student as dee-T and dee-x, respectively), initially accepting it before ultimately, and correctly, rejecting it. During this act of instrumentalization, it was necessary for the student to determine which information from the artifact to utilize (the lengths), and which to ignore (the relative positions of $\Delta T$ and $\Delta x$).

### Linking Utilization Schemes

Students linked utilization schemes, to form what one might call a utilization super-scheme. They linked five schemes to form a scheme for finding the partial derivative. The formation of this linkage was clear when working on subsequent questions, and students found partial derivatives quickly, using the utilization scheme formed in the present activity.

## Conclusion

Multivariable calculus is an essential course with numerous crucial ideas for students pursuing STEM. Innovation has been encouraged to improve learning and retention; concomitant with that is a need to analyze and understand student learning in these innovative contexts.

### Implications for the Future

The students studied here had a tremendously difficult time generating a tangent line for a two-variable function. It was the last idea proposed and utilization scheme generated, and arose only after the interviewer introduced the *y=f(x)* context and scaffolded from there. Instructors might consider emphasizing tangent lines and planes early in multivariable derivative instruction to overcome this obstacle.

The transition from single variable calculus to multivariable calculus is one that students will continue to have to make, and one which presents considerable challenges. Previous studies considered the transition for the equation of a variable equal to a constant (Dorko, 2016) and integration (Jones & Dorko, 2015). In the current study, students needed to transition ideas such as tangent line and derivative. Further study of all aspects of this key transition is essential.

### Summary

In this report we engaged in a study of four students in a group learning about the partial derivative through the use of a physical model while completing an activity sheet. We continued from previous work (Fisher, Samuels & Wangberg, 2017) our original approach to extend use of the instrumental genesis framework to contexts involving physical manipulatives. The use of the physical manipulatives illuminated the gaps in student knowledge, and also provided a path to fill them in. It is important to note that the data analyzed here covers the work of only four students, at a particular place and time, and we make no claims regarding generalizability to other students in other contexts. The students described here encountered numerous challenges as they extended their robust knowledge in the 2-dimensional *y=f(x)* context to the 3-dimensional *z=f(x,y)* context. They struggled to find a rate for a two-variable function, plumbing various parts of their mathematical knowledge while manipulating the artifacts at their disposal.

Reflected in their work were the role of instrumentation and instrumentalization as the students engaged in the mental constructions which turn the artifacts into tools. On this complex journey, the students devised utilization schemes, during which the student learning developed and manifested in noteworthy ways. This culminated in the development of the scheme for finding the partial derivative, which required coordinating the developed schemes. Thus, analyzing student actions through the lens of instrumental genesis proved effective and insightful to describe student learning activity in this context.

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Epistemological Beliefs About Mathematics and Curriculum Goals in the Cognitive Domain: a Case Study of Preservice Secondary Mathematics Teachers

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*Beliefs have long been recognized as a "hidden variable" in mathematics education. Epistemological beliefs are an inherent, although often implicit, component of curriculum goals in the cognitive domain. Connections between acquiring and accessing higher order cognitive strategies and epistemological beliefs are gradually becoming better understood. Israeli guidelines for mathematics teacher preparation emphasize of view of mathematics as a complex body of knowledge and knowing mathematics as a dynamic process. We present a case study of Israeli preservice secondary mathematics teachers' epistemological beliefs about mathematics, assessed via concept maps at the beginning and end of undergraduate studies. A mixed-methods approach was used to analyze maps. Results suggest that students' beliefs shifted to align with Israeli goals. Implications for STEM curriculum design are discussed.*

*Keywords:* Preservice Secondary Mathematics Teachers, Epistemological Beliefs, Concept Maps, Curriculum Design

## Introduction:

Complex interactions between cognition, metacognition and epistemological beliefs impact selection and application of cognitive strategies (Hofer & Sinatra, 2010). Epistemological beliefs, therefore, play an inherent role in education policy; they are implicit, often hidden (Leder, Pehkonen, & Torner, 2002), components of curriculum objectives in the cognitive domain.

We present a case study assessing alignment of students' beliefs with goals for the preparation of secondary mathematics teachers in Israel. During the three-year course of studies, beliefs that mathematical knowledge is simple and disconnected became less central; beliefs that mathematics is absolute knowledge of procedures evolved to a nuanced system of beliefs about a mathematics of procedures, concepts and processes.

## Theoretical Background:

Learning objectives in undergraduate STEM education include retention of knowledge over time, the application of knowledge to solve unfamiliar problems, and commitment to lifelong learning (Fairweather, 2008) which are associated to higher order cognitive processes (Fink, 2003). Acquisition of higher order thinking skills is a necessary, although not sufficient, component of solving complex problems and responding innovatively to changing circumstances (Binkley et al., 2012). Students must also internalize a deep approach to learning (Bromme, Pieschl, & Stahl, 2010) as an active, self-driven process of deep thought and construction of knowledge and understanding (Marton & Saljo, 1976).

Students' epistemological beliefs about the nature of knowledge, knowing and learning (Hofer & Pintrich, 1997; M. Schommer, 1990) are connected to their adoption of surface or deep learning approaches (Vermunt, Van Rossum, & Hamer, 2010) Beliefs that knowledge is absolute--facts with or without accompanying understanding--correspond to a surface, outcome-oriented approach to learning. Beliefs that knowledge is dynamically constructed and context-dependent aligns with a deep-learning, process-oriented approach (Vermunt et al., 2010).

Epistemological beliefs and metacognition are connected (Schraw & Moshman, 1995); the ability to access the cognitive tools needed to engage in critical thinking and problem solving is mediated through beliefs about knowledge and knowing (Hofer & Sinatra, 2010; Kuhn, 1991). Holding non-availing beliefs impedes acquisition of higher cognitive skills (Schraw, Crippen, & Hartley, 2006) and limits the range of cognitive strategies that are accessed (Hofer, 2004; Louca, Elby, Hammer, & Kagey, 2004). Since beliefs impact the acquisition and application of higher order cognitive skills (Hofer & Sinatra, 2010; Schraw et al., 2006), they are an inherent, although often hidden (Leder, Pehkonen, & Torner, 2002) component of curriculum objectives.

In mathematics, higher order cognitive skills include discovery, making connections and building understanding, which characterize (deep) *conceptual knowledge* (Hiebert & Lefevre, 1986; Star, 2005). Mathematics-related beliefs are strongly related to the cognitive processes of mathematics (McLeod & McLeod, 2002). For example students who believe that knowledge is simple, isolated facts (M. Schommer, 1990) are less likely to use higher level cognitive strategies and have lower levels of achievement (Cano, 2005); a belief that knowledge is simple affects self-regulation in learning (Muis, 2007) and has been shown to negatively impact student achievement in remedial mathematics courses in university (Briley, Thompson, & Iran-Nejad, 2009). Fostering dynamic beliefs about mathematics as process of discovery (Ernest, 1991; Grigutsch & Törner, 1998) is an inherent component of cognitive curriculum goals; neglecting epistemological beliefs as a component of mathematics education goals is a potential source of inequity of access to career and educational opportunities for which mathematics is a gateway (Leder, Pehkonen, & Törner, 2002).

Epistemological beliefs of mathematics education students have been widely studied (e.g., Ball, 1990; Schmidt et al., 2008) because mathematics teachers' beliefs impact their teaching practice (Beswick, 2005; Blömeke & Delaney, 2012; Thompson, 1992), student learning (Staub & Stern, 2002) and student achievement (Tatto et al., 2008). While some countries, for example the United States, explicitly include belief-related goals in guidelines for mathematics teacher preparation (CBMS, 2012), the guidelines of other countries, *e.g.,* Israel do not contain explicit belief-goals (Gutfreund & Rosenberg, 2012).

Despite the absence of explicit belief-related goals, the importance of addressing belief-development in programs for mathematics teacher preparation has long been recognized (Brownlee, Purdie, & Boulton-Lewis, 2001; Wilkins, 2008). For example, a cross-country study of 23,000 pre-service mathematics teachers from six countries examined the structure of mathematics teacher training programs and the mathematics-related beliefs of the programs' students at the end of their studies. Following Grigutsch (1998) beliefs were characterized as static/absolute and/or dynamic. End-of-program beliefs varied by country and by program across all six countries graduates held dynamic beliefs, but the extent to which graduates also held absolute beliefs varied by country (Schmidt et al., 2008). In addition, preliminary research indicates that the number and type of mathematics and mathematics education courses in mathematics teacher preparation programs impact beliefs (Blömeke, Buchholtz, Suhl, & Kaiser, 2014) with more opportunities to learn mathematics education courses leading to more dynamic beliefs. The connection between programs' explicit or implicit belief goals and graduates' beliefs was not examined, raising the question of the role implicit or explicit expectations play in belief development.

Israeli guidelines for preparation of secondary mathematics teachers (Gutfreund & Rosenberg, 2012), referred to as "Gutfreund guidelines" in what follows, do not specifically address beliefs, however they include content goals characterizing mathematics as a process of knowledge development, which are aligned to dynamic beliefs about mathematics (Grigutsch & Törner, 1998). Pedagogical goals address higher order cognitive skills; also aligned with dynamic beliefs, and, additionally, a belief that mathematical knowledge is complex, rather than simple (M. Schommer, 1990). Developing epistemological beliefs of mathematics as a complex body of knowledge and a dynamic process of inquiry is, therefore, an implicit goal of the Gutfreund guidelines.

## Research Goal

The previously unexplored connection between belief expectations and students' end-of-program beliefs provided a rationale for studying the epistemological beliefs about mathematics held by Israeli mathematics education students at the beginning and end of the program of studies and analyzing these beliefs in terms of their alignment to the Gutfreund guidelines (2012).

## Methodology

This case-study was conducted within the framework of a regulated B.Ed. program mathematics education at an Israeli college of education; graduates are certified to teach Israeli secondary mathematics.

### Sample

Twenty-five students began the mathematics education program in the 2014-15 academic year. Data was collected before students began the course of studies. A second set of data was collected 3.75 years later at conclusion of the program. Twenty-two of the 25 students completed the program. All students who completed the program agreed to participate in the study. The initial data of the students who did not complete the program was excluded.

### Data Collection

Various methods have been employed to collect data on beliefs of pre-service mathematics teachers, including interviews and classroom observation (e.g., Ball, 1990) and Likert-type surveys assessing level of agreement with statements reflecting a pre-determined set of mathematics-related beliefs (e.g., Tatto et al., 2008). Interviews and observations provide an in-depth picture of beliefs, but they are time intensive in terms of both data collection and data analysis. Likert surveys are an important tool for gathering and analyzing large data sets, but they cannot access beliefs that are not included in the survey (Grigutsch & Törner, 1998) and may yield "false-positive" agreement with some beliefs (Philipp, 2006).

We employed concept maps (Novak & Gowin, 1984), which visually represent abstract knowledge and understanding, to collect students' beliefs about knowing mathematics. Primarily used to assess knowledge and understanding of content, concept maps have also been used to capture meta-cognitive views about thinking (Ritchhart, Turner, & Hadar, 2008). We adopted Ritchhart's methodology to collect students' epistemological beliefs: they were asked to reflect on what it means to know mathematics; to generate a list of words and phrases that came to mind; to arrange their ideas in a hierarchy of importance/centrality to the notion "knowing mathematics"; to connect related ideas with lines and to briefly describe the connections. Students created concept maps before beginning the course of studies and again 3.75 years later at the completion of the program course-work.

## Data Analysis

We used a two-step process to analyze the items (words and phrases) on the two sets of concept maps. First-stage qualitative analysis used a constant comparative paradigm; students' responses were read and reread to discover commonalities and recurring themes (Strauss & Corbin, 1990). This inductive process uncovered a structured system of categories describing different ways of knowing mathematics. Student maps included many items. When a single item seemed to relate to more than one way of knowing, it was categorized under each of the appropriate categories. Each item on a map was assigned a rank from 1 to *n* indicating its distance from the center of the map, with items closest to the center assigned a rank of one. The items were coded by category and rank.

The coded maps were used to define "beginning of program" and "end of program" matrices. Each category was assigned to a variable and each map was assigned to two rows in the appropriate matrix; each item on a map corresponded to two matrix entries-- the first denoting its category and the second, its rank. The ranking was used to assign a weight to each item; On a map with *n* ranks, rank *n* items (those items furthest from the center) were assigned a weight of $\frac{1}{n}$, rank *n*-1 items were assigned a rank of $\frac{2}{n}$, items closest to the center (rank one) received a weight of one.

For each map, the sum of the weights of the items in the category was computed, labeled as the [*category name*]-belief score. A [*category-name*] average weight was computed for each map by dividing the belief score by the number of map items in the category. Ratios of each belief score to the sum of all belief scores were computed both for the complete set of categories and for various subsets. Subset ratios will be described in the findings. The belief scores, average weights and belief ratios are dependent variables of the maps. For each student, differences between beginning-of-program and end-of-program values were computed, defining dependent variables of the students in the program.

## Findings

Students related to knowing mathematics in complex and varied ways. Two main ways of knowing mathematics, *i.e.,* epistemological beliefs, emerged from the categorical analysis of the maps: knowledge of *mathematical content* comprised of topics (such as algebra) or skills (such as addition); and *attitudes* toward mathematics. Students' attitudes were expressed in terms of cognitive, behavioral and affective components, *e.g.,* perseverance (behavioral), satisfaction (affective), and success (cognitive). The categorical structure is shown in Figure 1.

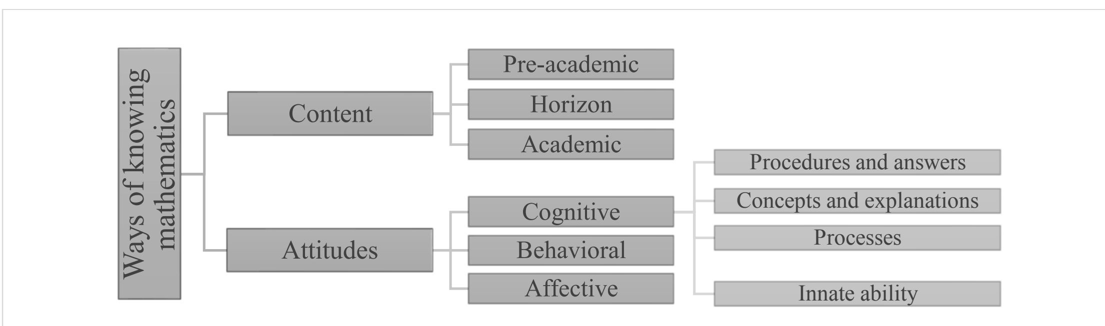


*Figure 1. Categorical structure of beliefs*

In the initial stage of quantitative analysis, four belief scores were computed corresponding to the content category (*content*) and the three components of the attitude category (*affective, behavioral, cognitive*.) The ratio of each score to the sum of the four scores was then computed. Mean ratios at the beginning and end of the program are shown in Table 1.

*Table 1. Differences in belief ratios*

| | Belief Ratios | | | |
|---|---|---|---|---|
| | Content | Affective | Behavioral | Cognitive |

| | | | | |
|---|---|---|---|---|
| Beginning of Studies | $\mu = 0.355$<br>$\sigma = 0.245$ | $\mu = 0.033$<br>$\sigma = 0.060$ | $\mu = 0.118$<br>$\sigma = 0.171$ | $\mu = 0.494$<br>$\sigma = 0.217$ |
| End of Studies | $\mu = 0.149$<br>$\sigma = 0.189$ | $\mu = 0.044$<br>$\sigma = 0.096$ | $\mu = 0.099$<br>$\sigma = 0.140$ | $\mu = 0.708$<br>$\sigma = 0.195$ |

Students at both the beginning and end of the program related to knowing mathematics as attitudes towards the subject more strongly than knowing specific mathematical skills and topics. Due to the small sample size and wide variation between students, effect size and statistical significance do not provide meaningful data. None-the-less, the results suggest that trend was stronger at the end of the program; the content ratio at the beginning of the program was 0.355 ($\sigma = 0.245$) and the content ratio at the end of the program was 0.149 ($\sigma = 0.189$). Same-student comparisons confirm this finding; the content ratio of 77% (*n*=17) of the students decreased over the course of studies.

**Mathematical Content**

Mathematical content was included on most maps: 86% (*n*=19) of the beginning-of-program maps and 64% (*n*=14) of the end-of-program maps. There were three categories of mathematical content: *pre-academic content* such as geometry, solving equations and order of operations; *horizon content*, which in Israel bridges secondary and post-secondary mathematics, such as three-dimensional geometry and vectors; and *academic content* such as cyclic groups and infinity. (Note: In Israel, high school encounters with infinity, such as horizontal asymptotes, are algorithmic and are not associated to the symbol or concept of infinity.) Four students included horizon content and four students included academic content. One of the four included both horizon and academic content. For only one student did the number of academic items exceed the number of pre-academic items. This finding suggests that program graduates do not relate to academic mathematics, *i.e.,* the mathematics of mathematicians (Beswick, 2011).

Students did not connect different areas of mathematics on their maps, suggesting that at both the beginning and end of the program students' viewed mathematics content as a disjoint set of skills and topics rather than as a connected system (e.g., Beswick, 2005), *i.e.,* they had a simple knowledge belief about the structure of mathematics (M. Schommer, 1990).

The mean weight of the content items changed over the course of studies. The mean weight was 0.68 ($\sigma = 0.32$) on the initial maps and 0.32 ($\sigma = 0.26$) on the final maps. Items closest to the center of a map have a weight of one, therefore the results suggest that content became less central to students' views about knowing mathematics over the course of studies. Same-student comparisons confirm this finding; for 82% (*n*=18) of the students, the mean content weight of the second map was less than the mean content weight of the first. These findings suggest that the strength of students' belief in simple mathematical knowledge decreased over the course of studies.

**Attitudes toward mathematics**

Students' attitudes toward mathematics included cognitive, affective and behavioral components (Figure 1). Statistical analysis indicated that and behavioral components of attitudes became less central to students' epistemological beliefs about mathematics over the course of studies; the centrality of cognitive beliefs was stable.

**Cognitive Beliefs**

Four distinct sub-types of cognitive beliefs emerged from the data: three categories of cognitive processes associated to knowing mathematics and one relating to who knows mathematics. The three sub-types of cognitive processes are listed below:

- *Procedures and answers:* Mathematical knowledge consists of procedures; it is absolute. Knowledge is demonstrated by implementing mathematical procedures and achieving correct outcomes. “Correct” presentation is (sometimes) included.
- *Concepts and explanations*: Mathematical knowledge consists of understanding the concepts underlying procedures. It is absolute. Knowledge is demonstrated by an ability to explain or understand explanations of procedures and concepts.
- *Processes*: Mathematical knowledge is based on concepts which can be intuited, and discovered (or rediscovered). New knowledge can be constructed from existing knowledge. Knowledge is demonstrated by connecting concepts, relating mathematics to real life, asking and answering questions about mathematics, and creating (or recreating) new ways to solve problems.

The fourth sub-type cognitive belief, labeled *innate-ability*, describes a belief that knowing mathematics requires a certain type of intelligence (C. Dweck, 2006; C. S. Dweck, Chiu, & Hong, 1995). Thirty-six percent of the students (*n*=8) began the program holding a belief that knowing mathematics requires “intelligence” or a “mathematical head.” No students expressed this belief at the end of the program, indicating that these future teachers had, indeed, internalized the idea that intrinsic ability is not a pre-requisite for learning mathematics.

Fostering dynamic beliefs is implicit in the guidelines for Israeli mathematics teacher preparation (Gutfreund & Rosenberg, 2012). We therefore separately analyzed map items in the three categories of cognitive processes that emerged from our data, computing procedures-and-answers, concept-and-explanations and processes beliefs scores as well as the ratio of each of these belief scores to the sum of the three scores. Differences between beginning- and end-of-studies mean ratios are shown in Figure 2.

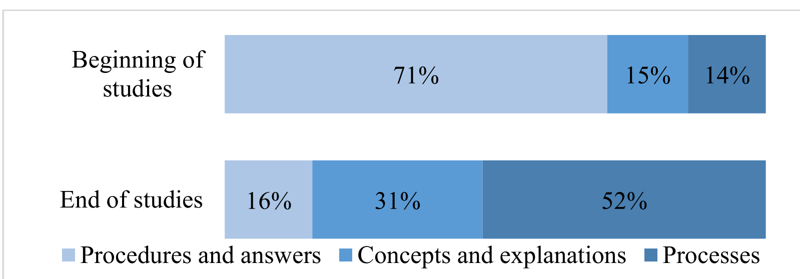


*Figure 2. Cognitive process ratios*

Cognitive beliefs expressed by beginning students were overwhelmingly focused on procedures and answers. By the end of the program, students’ cognitive beliefs had shifted to process; 52% of their beliefs expressed mathematics-as-processes. The mean weights for items in each of the three categories of cognitive processes support this finding. On the initial maps, the mean weight of procedures-and-answers items was 0.775 ( $\mu = 0.166$ ); on the final maps the mean weight was 0.480 ( $\mu = 0.316$ ). Same-student comparison showed that the mean weight of procedures-and-answers beliefs decreased for 77% (*n*=17) of the students. In contrast, the mean weight across all students of process beliefs increased from 0.338 ( $\mu = 0.388$ ) to 0.829 ( $\mu = 0.152$ ) and the mean weight of process beliefs increased for 77% (*n*=17) of the students. These findings indicate that over the course of studies students’ views shifted from mathematics as procedures to mathematics as processes.

## Discussion

This study analyzed pre-service secondary mathematics teachers' beliefs about knowing and learning mathematics at the beginning and end of their studies, as expressed through concept maps. Our findings document how their epistemological beliefs about mathematics changed over the course of studies, evolving to align with beliefs that support the cognitive objectives detailed in the Israeli guidelines for the preparation of secondary mathematics teachers (Gutfreund & Rosenberg, 2012). Prior research indicates that education impacts beliefs about the structure and stability of knowledge (Marlene Schommer, 1998), therefore the overall picture that emerged supports the idea that the program of studies impacted student beliefs about the complexity of mathematical knowledge and the cognitive components of mathematics. Other factors, such as age, may also have impacted beliefs.

Our results indicate that a view of knowing mathematics as knowing isolated content (simple knowledge) decreased over the course of studies. This has positive implications for these students' abilities to access and apply higher order cognitive strategies (Cano, 2005) and self-regulate their learning (Muis, 2007). The Gutfreund guidelines (2012) highlight these abilities as main components of mathematical literacy, deemed an essential component of teacher preparation. Our findings indicate that end-of-program beliefs align with the goals for mathematical literacy expressed in the guidelines.

Gutfreund guidelines (2012) for content knowledge of mathematics include understanding mathematics as a creative process rather than a finished, polished product; guidelines for pedagogical knowledge guidelines include supporting learning of both low and high order cognitive processes and stress equity of opportunities to learn mathematics vis a vis gender and differing abilities. These goals are aligned with dynamic, process-oriented beliefs about mathematics (e.g., Blömeke & Delaney, 2012; Briley et al., 2009). Our findings document that students' beliefs changed dramatically from an almost exclusive focus on mathematics as procedures and outcomes to a nuanced set of beliefs where mathematics includes procedures, concepts and dynamic processes.

The connection between the structure of mathematics teacher preparation programs and graduates' beliefs have shown that differences in opportunities to learn mathematics education courses impact graduates' beliefs. Our findings indicating post-program alignment with (implicit) belief goals present another avenue of exploration: the connection between program goals and epistemological beliefs. As a first step belief-alignment to the Gutfreund guidelines (2012) of other Israeli mathematics teacher preparation programs should be assessed. Belief-alignment in other countries should also be evaluated, including comparing alignment when belief goals are explicitly stated and alignment when they are implied.

## Conclusion

This case study showed that the epistemological beliefs of students completing a three-year undergraduate program in mathematics education are consistent with goals in the guidelines for the preparation of Israeli secondary mathematics teachers (Gutfreund & Rosenberg, 2012) and are aligned with belief expectations implicit in the goals.

The attention paid to the epistemological beliefs of graduates of mathematics education programs (e.g., Blömeke & Delaney, 2012) should be expanded to other undergraduate STEM programs. Connections between acquisition and application of higher cognitive strategies and epistemological beliefs (Hofer & Sinatra, 2010) suggest a role for beliefs in setting and meeting STEM goals of complex problem solving and lifelong learning (Fairweather, 2008).

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# Exploring the secondary teaching of functions in relation to the learning of abstract algebra

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*Secondary mathematics teachers regularly take advanced mathematics courses, but many regard them as unrelated to their work as teachers. In accord with a novel instructional approach (Wasserman et al., 2017), we designed materials for an abstract algebra course that connect to the teaching of functions in secondary schools. In this paper, we describe findings from a small-scale teaching experiment employing design research, which provides evidence that particular tasks were productive for accomplishing some of the mathematical and pedagogical aims.*

*Keywords:* Functions, abstract algebra, secondary teacher education

Prospective secondary mathematics teachers (at least in the United States) are frequently required to take a large number of mathematics courses, including advanced courses such as abstract algebra and real analysis, to obtain certification to teach secondary mathematics. This is ostensibly with good reason – much of the content in secondary school is connected to and can be informed by ideas studied in these advanced mathematics courses (e.g., CBMS, 2012). Yet secondary teachers regularly report that completing such courses provides little professional value and does not influence their subsequent instruction (e.g., Zazkis & Leikin, 2010). This raises the challenging problem of designing tasks and modules in advanced mathematics courses that make meaningful connections to secondary mathematics teachers' future professional work.

In this paper, we explore the findings from a small-scale teaching experiment with two students in a secondary mathematics teacher education program. These students engaged with materials designed in accord with a novel instructional model for teaching advanced mathematics courses to secondary teachers (Wasserman et al., 2017). In particular, this paper looks specifically at their engagement with and reflections on abstract algebra content in relation to the teaching of functions in secondary school.

## Literature and Theoretical Perspective

### Advanced Mathematics Courses in relation to Secondary Teaching

Given the strong connection between ideas studied in advanced mathematics courses and the content of school mathematics, one would expect such courses to have an influence on secondary teaching. Yet findings from various studies appear to indicate the opposite. Monk (1994) examined the relationship between the number of university mathematics courses that a teacher completed and the learning outcomes of their students. The key finding was that courses beyond a fifth course – i.e., an advanced mathematics course – had little to no effect on the learning outcomes of that teacher's students. Zazkis and Leikin (2010) found that, according to practicing secondary teachers' self-reports, knowledge of advanced mathematics was rarely used and had little direct influence on their classroom practices. Other studies have reported similar results (e.g. Goulding et al., 2003; Rhoads, 2014; Wasserman et al., 2015).

This disconnect brings up the challenge of how to leverage the content of advanced mathematics in ways that are relevant to secondary teachers. Distinguishing between connections to the content of secondary mathematics and connections to the teaching of secondary mathematics, Wasserman (2016) analyzed school mathematics standards (CCSS-M, 2010) to identify four areas – arithmetic properties, inverses, structure of sets, and solving equations – where knowledge of abstract algebra might influence school mathematics' instruction. In general, the connections explored by Wasserman (2016) were specific to abstract algebra. That is, they were regarding *abstract-algebra-specific-content*, such as a group. We highlight this as a means to distinguish such content from other content that would also be related to the study of abstract algebra, but not necessarily unique to the study of abstract algebra, such as a function. We consider this to be *non-abstract-algebra-specific-content*. These sorts of connections have been given less attention in the literature.

**A Novel Instructional Model**

From Wasserman et al.'s (2017) point of view, the belief that completing a course in advanced mathematics will improve prospective or practicing teachers' (PPTs) ability to teach secondary mathematics has been based on a traditional view of transfer from the cognitive psychology literature (e.g., Perkins & Salomon, 2002). More specifically, there is an assumption that as a byproduct of learning advanced mathematical content, PPTs will better understand secondary mathematics content and will consequently respond differently to instructional situations in the future – a tenuously presumed "trickle down" effect (Figure 1a). Given the notorious difficulties in achieving this type of transfer, it is less surprising that PPTs' experiences in abstract algebra (or other advanced mathematics) often does not influence their teaching.

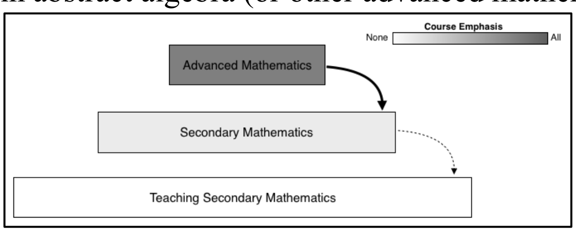


*Figure 1a. Implicit instructional model for advanced mathematics courses designed for teachers*

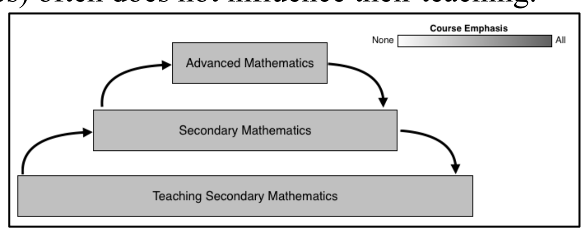


*Figure 1b. Our instructional model for advanced mathematics courses designed for teachers*

In Figure 1b, we present Wasserman et al.'s (2017) alternative instructional model for teaching advanced mathematics (including abstract algebra) in ways that can inform a PPT's pedagogical practice. This model is based on the premise that knowledge that PPTs learn should be inherently practice-based and applicable to the actual activity of teaching (e.g., Ball, Thames, & Phelps, 2008). Our model is composed of two parts: building up from and stepping down to practice. To *build up from (teaching) practice,* the abstract algebra content is preceded by a practical school-teaching situation. The building-up portion provides a context that sets the stage for the study of abstract algebra in ways that are both relevant to teachers' practices as well as well-suited to being learned in abstract algebra, which also aims to ease the challenges associated with transfer (e.g., Barnett & Ceci, 2002). The second part, *stepping down to (teaching) practice*, then uses the mathematical ideas from abstract algebra as a means to reconsider the secondary mathematics and relevant pedagogical situations. Stepping down to practice explicitly clarifies the intended mathematical and pedagogical aims of the abstract algebra content. In between building up from and stepping down to practice, the abstract algebra topics are covered by the instructor in ways true to its advanced nature with formal and rigorous treatment.

## Methodology

In accord with Wasserman et al.'s (2017) instructional model, our research team designed five modules that were intended to connect content typically covered in an abstract algebra course – including binary operations, groups, isomorphisms, subgroups, and rings and fields – to various teaching situations. The modules included some of the *abstract-algebra-specific-content* connections identified by Wasserman (2016). For the purposes of this paper, however, we elaborate only on one module, the Functions Module, which leveraged the abstract algebra content of binary operations and isomorphisms as a means to converse, broadly, about functions and to reflect on the secondary teaching of functions. That is, the connection in this module was not about binary operations and isomorphisms per se, but instead used them as instantiations of and an opportunity to discuss functions – an example of a *non-abstract-algebra-specific-content* connection. Figure 2 gives an overview of this module.

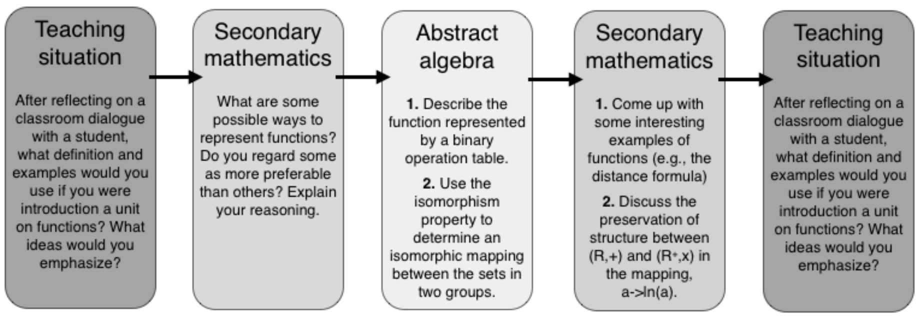


*Figure 2.* Overview of the Functions Module

Using design research (e.g., Cobb, et al., 2003) within a teaching experiment, the study engaged participants with some specific mathematical ideas and secondary mathematics teaching situations. Researcher-hypotheses were tested against participants' ways of thinking during the sessions. Two students (PPTs) enrolled in a program in secondary mathematics teacher education agreed to participate. One was a pre-service teacher, the other an in-service teacher with five years of experience (but not currently teaching). We collected and analyzed two sources of data: (i) a (transcribed) video-recording of PPTs engaging in the materials; and (ii) an (transcribed) audio-recording of a post-teaching-experiment semi-structured interview.

In our analysis, we compared what actually transpired during the teaching experiment to our hypothesized responses. First, we considered responses to the teaching situation. We characterized important aspects of PPTs' initial responses to the teaching situation, and their responses at the end of the module, and identified differences. Second, we considered whether these differences were in accord with our hypotheses, and, if not, whether they were meaningful instructional changes. Third, for each of these instructional differences, we then analyzed PPTs engagement with all facets of the module and their post-interviews to identify instances where their thinking appeared to shift in relation to the difference identified, and then to consider why this may have been the case. In this paper, we report on one instructional difference identified in the Functions Module that was in accord with our hypotheses, and discuss the aspects of the module that appeared to be most-closely associated with why PPTs responded differently.

## Results

We organize the presentation of results from our analysis in terms of their support for two particular claims: 1) PPTs indicated their future teaching of functions would include novel mathematical examples and more non-mathematical examples, not just numerical ones; and 2) PPTs' struggle to view a binary operation table through a functional lens progressed through four stages and was productive for acquiring a deeper understanding of function, and was influential on their reported approaches to teaching the function concept to secondary students.

**Claim 1**

The first claim is that PPTs indicated their future secondary mathematics instruction would include novel mathematical examples and more non-mathematical examples of functions, not just numerical ones. As mentioned, this instructional change indicated by PPTs was essentially in accord with researcher hypotheses. We consider three sources of data in support of this claim: i) their initial reaction to the teaching situation; ii) their reflection back on the teaching situation at the end of the module; and iii) their interview responses after the module.

During PPTs initial discussions about a teaching situation (which is omitted here for the sake of space), they responded to the question: "If you were introducing a unit on functions, what definition and examples would you use? What ideas would you emphasize? Explain your reasoning." The definition they had mentioned already was that every input has a unique output. Their initial examples were pictorial mappings that demonstrated the idea of uniqueness with an example, $\{(1,1),(2,2),(3,3)\}$, and non-example, $\{(1,1),(2,2),(3,3),(3,2)\}$. Further examples included tables and graphs, and used a step function to reinforce uniqueness – that it was a function, but if you had two closed circles (at the same *x*-value) it would not be. They also included various other types of functions (linear, quadratic), using their equations to talk about inputs having unique outputs. The key point is that their initial examples of functions were numerical (i.e., in $R \times R$) – which was in accord with our hypotheses – and they used different representations of these kinds of functions to exemplify the issue of uniqueness.

After engaging with the material in the module, the PPTs reflected back on their responses. In contrast to functions with numerical inputs and outputs, their discussion focused almost exclusively on incorporating more abstract examples, especially real-world examples (e.g., people to birthdays, piano keys to notes). These examples emphasized the "mapping between two sets of objects" part of function in addition to the "uniqueness" part.

*Interviewer*: Uh, talk about, maybe some of the things you might do, uh, definitions and examples you might use or see with students.

*A01*: So, the birthday?

*B03*: Yeah, and I liked, I liked the piano, or anything that's, you know, not so mathy, I guess."

*Interviewer*: The birthday, piano, real world, so why that?

*A01*: I just think they help them connect, like, what the idea of a function is.

*B03*: Yeah, and sometimes I feel that in math, you have to do…

*A01*: Only numbers.

*B03*: Yeah, like add, subtract, multiply, and divide, yeah, numbers, and…where was I going with this? I don't know. There's this idea a function is outside of just add, subtract, multiply, and divide… it helps identify the idea that the function is just some mapping we describe by however we want.

During the interview after the module, we probed further into some of their thinking. Here, they mentioned part of the rationale for doing so was: "Just to give [students] other examples of things that are functions besides what we traditionally talk about in an algebra classroom." They also indicated, "I have other examples of things that are functions now that I didn't have before…And maybe some of these are too complicated to show them, but it would cause me to maybe stop and think about…maybe there's another mathematical thing that I could show them outside of the traditional $y = x + 3$… that is a function that's not normally something we would talk about as a function." One such example they considered including was the quadratic formula, i.e., the function, $(a,b,c) \to \left(\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}\right)$, which was one they had come up with previously during the module when asked to identify interesting examples of functions in secondary mathematics.

**Claim 2**

The Functions Module was designed to leverage two aspects of abstract algebra as a means the motivate discussions about function. The first was viewing a binary operation table through a functional lens; the second was leveraging isomorphisms to discuss an abstract example of a function mapping. As it turned out, the first was especially important for PPTs' reflections on secondary teaching – the second, less so.

During the post-interview, the PPTs singularly identified the binary operation table task – which is discussed in more detail below – as being particularly influential. Also, however, we briefly trace two other facets of the module that made their way into PPTs' responses: i) their mention of the "piano" example was connected to the function that was included in the module as a precursor to the isomorphism activity (but was not the isomorphic mapping itself); and ii) their "quadratic formula" example was one they identified during the module. Notably, for (ii), their initial reaction was that the quadratic function would *not* be a function because each $(a, b, c)$ does not map to a unique output – there are two; later, they acknowledged it would be if the output set were pairs. This reiterates the idea that the objects being mapped to or from – and not just the idea of uniqueness – is important in determining whether something is a function, an aspect PPTs emphasized more readily in their teaching responses at the end of the module.

The primary activity in the module they identified as productive was the binary operation table task – where they were given the additive (mod 12) binary operation table and asked to "Describe the function (i.e., mapping) that this binary operation table represents." It was their (unexpected) struggle on this activity that appeared to have been especially productive for developing a deeper sense of function.

*Interviewer:* …So what were the main ideas that you got going through the abstract algebra content? …

*B03:* A deeper understanding of the function being something besides what I traditionally always thought about a mathematical function to be…

*A01:* I think that was the one that I had the hardest time—like the binary operation…

*B03:* And that one was really hard to think about cause it took us forever…it took us forever for us to figure out what the domain was.

*A01:* …it was a good place for us to get stuck.

*B03:* That's where I feel we, at least for me, I turned the corner about thinking about a function outside of just some linear situation… The fact that your domain can actually be an ordered pair…

We view their discussion here as indicative of a relationship between the binary operations task, which forced them to wrestle with and broaden their conception of function, and the real world and novel mathematical examples they mentioned including at the end of the module.

Since PPTs engagement in the binary operation task was profound, we looked further into reasons for why this may have been the case. In doing so, we identified four conceptual shifts that the PPTs went through as they came to view the additive (mod 12) binary operation as a function. During what we refer to as Stage 1, the PPTs had an *equation-view* of function. Their initial reactions to the task were:

*A01:* So, you're just, like, saying that $0 + 0 = 0$?

*Teacher-researcher:* Mmhm.

*A01:* Ok. …

*B03:* So, we just say, like, it's taking all the integers $0$ to $11$, and then…this is what we're mapping to?

*A01:* I don't know, I'm so confused… I'm so confused. I don't if this is the input, or…

*B03:* Are both of these inputs?

*A01:* … wouldn't these be outputs?... Unless it's like $x + y = z$. I don't know.

We point out that their initial, admittedly confused, attempts to view this as a function were by defining equations: $0 + 0 = 0$ and $x + y = z$. Now, these equations describe individual facts as well as more general truths about the binary operation table at hand. However, this equation-view was, ultimately, unproductive. For the next several minutes, the participants struggled to determine the domain – they cycled back and forth between thinking it was and was not "0 to 11." Their difficulties with the domain made their efforts to list actual elements in the mapping nearly impossible. The shift to Stage 2, a *mapping-view* of function, was facilitated by prompts to describe the mapping informally and to determine specific elements in the domain and range.

*B03:* It takes two of them… So it takes, it takes…if we do $A \times A$[1], we get all the ordered pairs, and then the added pairs get added together…like the two pieces of the pairs get added together to get that, but I don't know how we would write that.

*A01:* Ohhh.

*Teacher-researcher:* So… You don't have to be technical at this point. Just show me… Not just describe it, but show me things that map to things…

*A01:* $0 + 0$ maps to $0$, $1 + 0$ maps to 1. It…

*B03:* Go all the way up to, like, $11 + 1$, and $11 + 1$ maps to $0$.

*Teacher-researcher:* So what're you mapping? So what's the domain and what's the range?

*A01:* This [e.g., $0 + 0$] is our domain right? Cause this is being mapped to this [e.g., $0$].

Although this may seem a trivial difference, we argue that viewing the binary operation table as $(0 + 0) \to 0$ and *not* $0 + 0 = 0$ was an important conceptual shift: it fostered their ability to identify, or at least get closer to identifying, elements in the domain and the range. The next shift, to Stage 3, a *multivariable-view* of function, was facilitated by the teacher stating that "the '+' is actually fairly irrelevant…" Their response was, "So we can just list the ordered pairs?... So now our domain is all these ordered pairs… And our range is over here." In other words, this shift allowed them to recognize the mapping as $(0, 0) \to 0$, which is more clearly indicative of the multivariable domain input and which removes the "+" from the domain. Last, the shift to Stage 4, a *dependent-view* of function, was facilitated by a student-teacher interaction.

*Teacher-researcher:* Ok. So the general set, the domain is what?...

*B03:* So $A \times A$ is the domain, and $A$ is the range.

*Teacher-researcher:* … And so probably the easiest way to describe this is as a function is to say our function is taking things of the form here, it's taking two inputs, and it's mapping it to what? So if I have these inputs $A$ and $B$, it's mapping it to…?

*B03:* $A + B$.

This last shift, guided by the instructor, was important, in that it allowed writing the mapping not as $(a, b) \to c$, but rather as $(a, b) \to (a + b)(mod\ 12)$. In other words, it established the element in the range set as being dependent on the input variables ("+" as part of the output, not input), which led easily to the participants recognizing the equation form of the function as: $f(a, b) = (a + b)(mod\ 12)$. These four stages appear to be conceptual shifts in PPTs' thinking on the binary operations task that facilitated their coming to a deeper understanding of function.

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[1] We note their use of the Cartesian product was likely prompted by the definition given, which was: A function $\phi$ mapping set $A$ to set $B$ is a relation between $A$ and $B$ (i.e., $\phi \subseteq A \times B$) such that each $x \in A$ is related to exactly one element in $B$.

## Discussion and Conclusion

This study explored how PPTs engaged in and reflected on materials designed to connect the secondary teaching of functions to content in abstract algebra (binary operations and isomorphisms). The purpose was to explore materials with some non-abstract-algebra-specific-content connections that might be used in an abstract algebra course with (or for) secondary teachers. We discuss three points with regard to the primary claims from the findings.

First, the binary operation table task (much more so than the elaboration on isomorphisms) was productive for deepening PPTs' notions of function. It was during this activity that PPTs struggled, productively, to view a familiar operation through a functional lens. The four-stage process they went through lends some insights into the conceptual challenges they faced. Notably, these mirror, or perhaps elucidate, some shifts that secondary students also go through in their transition to understanding even more basic functions. Functional relationships are regularly introduced through an *equation-view* with two variables, e.g., $y = x + 3$. These then need to be understood as a *mapping*, $x \rightarrow y$, in particular between pairs of *numbers* (the multivariable stage was, essentially, about identifying objects in the domain and range), for which the *dependent* relationship between the two variables provides a more useful characterization of the mapping, $x \rightarrow x + 3$. This mapping, then, can be given the more formal and typical equation notation of a function: $f(x) = x + 3$. According to their own reports, engaging in this process with a more abstract example helped the PPTs recognize the broader ubiquity of functions, such as the quadratic formula mapping from 3-space to 2-space, the derivative relationship as a mapping between functions, etc.

Second, we present similarities and differences between the types of functions the PPTs indicated they might use as examples after the module – more abstract real-world examples (e.g., piano keys to notes) and novel secondary mathematics examples (e.g., quadratic formula). Both of these are more abstract, by which we mean that the sets being mapped to or from are typically not just a set of numbers, but rather a set of objects, letters, coordinate pairs, etc. However, there are some differences between real-world examples and novel secondary mathematics examples. First, real-world examples are already regularly introduced in secondary classrooms – but oftentimes only to be quickly discarded and forgotten. Given that PPTs' discussions valued looking at different representations of functions, including visual ones, this makes sense: real-world examples are harder to represent in multiple ways via tables, equations, graphs, etc. Now, "graphing" such real-world functions might in fact be an interesting exercise. However, in contrast, novel secondary mathematics examples such as the quadratic formula provide a similar sense of abstractness, but also may have the advantage of having other easily-identifiable representations to discuss (e.g., explicit formulas, graphs, etc.).

Third, we make a dual point about the PPTs' reflections on their teaching. On the one hand, the kinds of examples they ultimately discussed incorporating into their own teaching were in accord with the aims of the module. Indeed, one of the goals was that teachers should select examples that exemplify more nuances with the idea of function, which such abstract examples helped accomplish. On the other hand, many of the ideas came directly from the module materials or from their engagement with the module materials. Indeed, they even mentioned potentially having secondary students look at the (mod 12) binary operation task. We have seen this tendency before, of PPTs "transporting" materials from a teacher education setting, in the exact form they experienced them, to their teaching (Wasserman et al., under review). It exposes a tension in teacher education, and suggests that teacher educators may need to be more explicit about how general ideas (not just materials) might be adapted for and applied to teaching.

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Convergent and Divergent Student Experiences in a Problem-Based Developmental Mathematics Class

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*In recent years low success rates in traditionally taught pre-college mathematics classes has led to new courses that use group work and problem solving to teach the required content. Early results examining student outcomes are promising, but say little about students' classroom experiences. This study uses interviews from six students and one instructor in a single class to explore differences between student experiences and the intentions of the instructor. Although several students expressed positive perceptions of the class, tensions arose between students who wanted to learn efficiently versus the classroom expectation that students stay together in their groups. Practices such as copying and dictation arose, at least partially, as coping mechanisms for students caught between these conflicting values. Future work should examine alternative grouping methods and ways of using early indicators of need to provide additional support.*

*Keywords:* Community college, developmental mathematics, group work, problem solving

Community colleges, although initially conceptualized as a place to prepare students for advanced study, now serve an incredible range of missions and students (Dougherty & Townsend, 2006). Students enter these schools at dramatically different stages of life, ranging from recent high school graduates with plans to earn a PhD, to adults returning to school after many years in the workforce or at home raising families (Cohen, Brawer, & Kisker, 2013). As a result, the mathematics background of these students is wide: spanning from seeing the material the first time, to having taken advanced coursework. Nowhere within community colleges is this truer than in pre-college, or *developmental*, mathematics classes, where instructors must meet the challenge of addressing the needs of this unique and complex population in a single classroom.

For many community college students, achieving their educational plans requires completing a developmental mathematics class, which are intended to provide the knowledge and skills necessary for success in credit-bearing college-level classes. However, low success rates (Attewell, Lavin, Domina, & Levey, 2006; Bailey, 2009) mean that developmental courses often play a gate-keeping function, a fact that is particularly concerning given that African Americans (Attewell et al., 2006) and individuals from lower-socio economic backgrounds (Hagedorn, Siadat, Fogel, Nora, & Pascarella, 1999) disproportionately enroll in developmental classes.

In recent years, developmental mathematics educators have moved to address the high failure rates by implementing mathematics curricula that use real-world problems and group work to help make the curriculum more accessible, echoing the reform efforts from the 1980s and 1990s in K-12 mathematics (National Council of Teachers of Mathematics [NCTM], 1989, 2000). In addition, these classes, often called *Mathematical Literacy*, are intended to support students in learning how to see mathematics in their daily lives. Studies of K-12 classrooms have shown that in problem solving and group work contexts, some students may resist the instructional norms (Lubienski, 2000). In addition, group work, although promoting opportunities for learning, can also lead to power struggles within groups (Esmonde & Langer-Osuna, 2013) and lack of opportunity to learn for some students (Baxter, Woodward, & Olson, 2001).

The most famous of the *Mathematical Literacy* classes are the Carnegie Pathways (Carnegie, n.d.a, n.d.b), with research from early implementations of these Pathways yielding tentatively positive results (e.g., Sowers & Yamada, 2015; Yamada, Bohannon, & Grunow, 2016; Yamada & Bryk, 2016; Norman, 2017). However, this work focuses on student success rather than on students' experiences with the curriculum. Given the large instructional shift, the uniqueness of the developmental population, and the diverse mathematics backgrounds of the students, it is important to understand how students experience these classes and what the individual instructor intended for students, paying particular attention to how these perceptions differed between students with different reactions to the class. This study sets out to do exactly this, asking:

1. How do perspectives on specific aspects of the course differ among students who have a positive, neutral, or negative reaction to the course?
2. How do students' experiences with the course compare with the intentions of the course instructor/developer?

## Methods

All data were collected from students in a single *Mathematical Literacy* classroom at Fields Community College (FCC; all names are pseudonyms), taught by an instructor who had participated in the development of the course. The course was not a Carnegie Pathway. Data draw from interviews with the instructor and students conducted outside of class and classroom audio recordings of the interviewed students' groups during the Spring 2015 semester.

### Sample

This study focuses on six of 22 students from a single *Mathematical Literacy* classroom who consented to participate in a single interview outside of class and contained more than 8 (of 24 possible) hours of audio of them in their groups. All students in the observed classroom were invited to take part in data collection. Everyone who indicated interest was interviewed. Table 1 provides basic demographics and the mathematics backgrounds of the interviewed students.

*Table 1. Interviewee demographics, mathematics backgrounds, and class outcomes*

| Name | Demographics | Mathematics background | Expected grade |
|---|---|---|---|
| Carley | White female<br>19 years old | Started developmental at lowest level; Trigonometry in high school | |
| Carrie | Asian female<br>20 years old | First developmental class<br>AP statistics in high school | |
| Craig | White male<br>25 years old | First developmental class<br>Trigonometry in high school | |
| Dave | White male<br>20 years old | First developmental class<br>Statistics in high school | |
| Emilia | Black female<br>19 years old | First developmental class<br>Trigonometry in high school | |
| Tyrone | Black male<br>48 years old | Started developmental at lowest level; GED | |

### Data Collection

Most student interviews occurred during the eighth or ninth week of the semester. Audio data were collected throughout the semester, but I focus on the data collected in week seven.

Focusing on this subset of classroom data provides alignment between the observation data and the experiences students shared during their interviews. In addition, during week seven all 22 students were still actively attending class, meaning the class contained its full range of diversity, both mathematically and demographically. The instructor interview occurred the last week of instruction.

**Analysis: Student Interviews**

To examine the similarities and differences in students' perceptions of *Mathematical Literacy*, depending on the type of experience a student had in the class, three main stages of analysis took place. I discuss each stage in more detail in the following sections. The majority of the coding was done by two researchers trained in mathematics education.

**Stage 1: Classifying student experiences.** Students' experiences were classified as positive, neutral, or negative using their response to the interview question "would you recommend *Mathematical Literacy* to others who were considering taking the class?"

**Stage 2: Coding for emerging themes.** Interviews were coded for emerging themes related to their perceptions of the class and classroom phenomena using multiple rounds of open coding (Creswell, 2014; Emerson, Fretz, & Shaw, 2011). This study focuses on data related to three main, mutually exclusive codes: *Group Work*, *Problem Solving*, and the *Instructor*. *Group Work* referred to students' discussions of working in groups, relationships with group members, or reflections on working in groups. *Problem Solving* related to students' discussions about the problem-rich curriculum and experiences engaging in mathematics. The *Instructor* code related to students' reflections on the instructor, their relationship with her, and their experiences working with her individually or with their group. Within each of these three main codes, mutually exclusive sub-codes were developed.

Elements of the classroom are inherently closely related, which occasionally made mutually exclusive coding difficult. For example, sometimes students spoke about their group interactions with the instructor. Broadly, this discussion fell into both the *Group Work* and *Instructor* codes. In instances like this, the default code was always *Instructor*.

**Stage 3: Contrasting student experiences.** For each student, the final list of codes from Stage 2 were tabulated for each individual. Using these tabulations, I identified patterns using the mixed-methods-analysis technique of matrices (Miles & Huberman, 2013), which organizes data along two or more dimensions, one of which is ordinal, to identify patterns between cases. Each matrix cell contains project data and the entire matrix can be used to draw inferences and detect patterns. For this project, I apply matrices with a convergent-divergent purpose in mind, using students' experience type (i.e., positive, neutral, or negative) as the ordinal dimension and final codes along the other, looking for patterns in how students spoke within codes.

As part of this analysis, I draw on interview segments, combined with examples from the classroom audio, to explore similarities and differences in how students with positive, neutral, or negative perspectives spoke about the classroom. I include these classroom examples not to causally link the perceptions students shared to a particular classroom event, but rather to illustrate examples of the classroom phenomena students identified.

**Analysis: Instructor Interview**

The initial round of coding of the instructor interview relied on the same three main codes as students (i.e., *Mathematics Curriculum*, *Group Work*, and *Instructor*), with appropriate adjustments made for the fact that the subject of discussion had switched from perceptions of the classroom to intentions for the classroom. Only the components of the interview that related to the instructor's experiences of these three things were considered.

I wrote these results to represent the intended curriculum with respect to each of the three codes. After analyses of the student interviews were complete, I returned to the instructor's interview, rereading it with a lens toward the student interview sub-codes. The analysis of the instructor's intentions was then refined to reflect the student sub-codes, noting places where the instructor's responses did not have comparable student codes.

## Results

The research questions of this study examine how students with different experiences in *Mathematical Literacy* vary in their perspectives of the class and the classroom phenomena, contrasting these with instructor intentions. I start with the classifications of student experiences and then present the results from the instructor, followed by the students. In this brief report, I focus on the results from the main code of *Group Work* (this code had the most material). Results for *Problem Solving* and *Instructor* will be included in the full report and presentation.

### Student Recommendations

Student recommendations fell into one of three categories: positive, neutral, or negative. Students who recommended the class tended to provide an overwhelming positive response. For example, Craig started answering the question with "I would now….especially if they were like me." Those coded as negative tended to qualify their answers, saying that the class might be appropriate for some students, but not for them personally. For example, Tyrone recommended the class for students "if they're up for a challenge," but would not recommend it for "people like me." Dave did not indicate his personal feelings, thus, his response was coded as neutral. Table 2 presents the recommendations of the students crossed with their anticipated grades.

Table 2. *Students' recommendations for the class and their expected grades*.

| | Recommendation | | |
|---|---|---|---|
| Expected grade | Yes | Neutral | No |
| A | Craig, Carrie | Dave | |
| B | | | |
| C | Carley | | |
| Did not complete class | | | Emilia, Tyrone |

### Convergent and Divergent Perspectives on Mathematical Literacy

To explore the instructor's intentions and how students with different experiences in *Mathematical Literacy* vary in their perspectives of the class and the classroom phenomena, I organize the remaining results with (a) the instructor's perspective and (b) results related to the patterns within the student codes.

**Group work: Instructor's perspective.** Group work was an important part of the course design, for Ms. Ann, the instructor, who, together with her colleagues at FCC, decided that in order to get students to do the mathematics the way they desired, "lecture classes aren't just going to be able to work. They need to be having these conversations [about math] in class." Thus, the choice to implement group work was driven by the curriculum objectives.

During her interview, Ms. Ann explicitly discussed how she created groups, explaining that she liked to spread her top- and low-performing students evenly between groups, but within this also considered “personalities and attendance” to create groups that provided a productive environment for all her students. She tried to include at least one “strong” member in each group. Thus, the instructor explicitly considered the range of abilities within the groups so as to provide as many students as possible with access to others who were fairly comfortable with the material.

An underlying assumption of much of Ms. Ann’s discussions related to group work was creating conditions where students worked together and discussed many instances of reaching out to students to help manage group relationships and keep students working together.

Ms. Ann acknowledged that group work allowed some students to minimize the amount of work they contributed, but explained that the class grading structure meant that most of these students would not pass the class without some degree of personal understanding of the content. She also noted that the group project rubrics allowed students to grade each other, but she observed that “the students are not always willing to throw each other under the bus,” which she found frustrating because it limited her ability to hold students individually accountable.

***Group Work*: Students’ perspectives.** *Group Work* sub-codes fell into six categories: (a) *Group Dynamics*, (b) *Togetherness*, (c) *Checking In*, (d) *Copying*, and (e) *Accountability*. I discuss the main findings for each of these sub-codes below.

***a. Group Dynamics*.** Many of the students with more positive experiences explicitly noted that groups usually contained students with diverse mathematics levels. Dave commented that sometimes groups have “someone who knows a lot about something with someone who doesn’t know anything about it” and Carrie observed, “everyone is at a different levels [*sic*] and they all kind of contribute their own things.” These remarks suggest that students, although perhaps not explicitly aware of the mathematical backgrounds of their group mates, recognized that a range of background knowledge existed within their groups.

***b. Togetherness*.** Although a few students talked about the benefits of togetherness, the majority of the students’ talk related to *Togetherness* related to divisions within groups.

Emilia and Tyrone, students who would not recommend *Mathematical Literacy*, both noted they usually found themselves behind. Tyrone commented, “sometimes I might be behind. I’m always behind. And then I look, ‘hey where you at?’ I’m just like man, ‘you all just go ahead—I’ll catch up.’” For both, a lack of togetherness resulted in being left behind. For example, a diagrams of Emilia’s group for a day near the time of her interview (Figure 1) shows she lagged behind that day and rarely spoke. When she did speak, she was usually talking to the instructor about problems her group mates had already discussed. Although not shown here, similar lag patterns were observed for many of the students who did not complete the class. This suggests that an early lack of togetherness in groups might signal the need for additional intervention.

The four other interviewed students did not mention feeling left behind. However, Carley and Craig gave examples of the ways they strove to bring groups together, while Carrie and Dave distanced themselves from this responsibility. For example, Dave noted that “it’s really difficult to get things done when you’re in a bad or…not a good group…I mean once you’ve got a good motion going then there’s no reason really you should have to stop.” It should be noted that Carrie and Dave were in groups with Emilia and Tyrone respectively around the times of their interviews. As noted earlier, Emilia rarely spoke with her group, despite Carrie saying in her interview that usually everyone had someone to work with. Audio recording of Dave and Tyrone’s group demonstrates that Tyrone was helped in his group, but not usually by Dave.

The lack of togetherness the students note and that is illustrated in Figure 1 show that togetherness was an issue. Dave and Carrie touched on reasons why this might be the case, noting that slowing down could be disruptive or distract from completing assignments quickly.

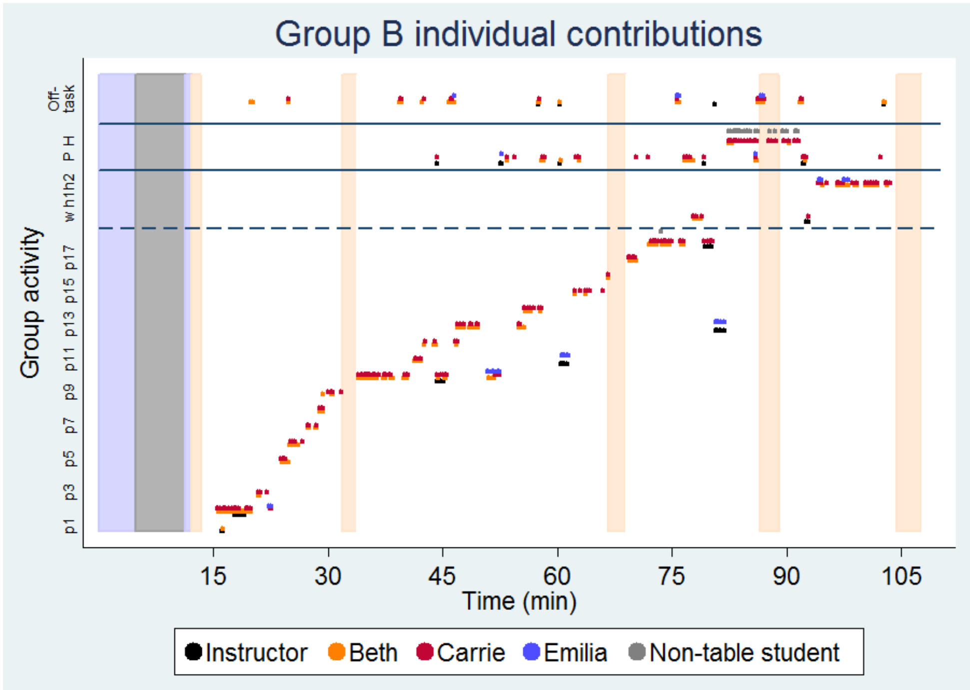


*Figure 1. Individual contributions in group B. Each dot indicates when an individual started a new speaking turn. Recording of individuals started when the instructor opened the classroom up for group work (here, around minute 15). The light blue regions indicate when the instructor was returning assignments and checking in with students. The grey regions are when the students were engaged in a quiz or an activity related to the study. The light orange regions are when the instructor was lecturing. White regions are when the class working in groups. The Group activity labels mean, in ascending order: problems 1 through 17 (skipping even numbers on the labels), written reflection task (w), group homework assignments 1 and 2 (h1, h2), group planning discussions (P), helping other groups (H), and off-task talk.*

***c. Checking In.*** All six students talked about asking others for help or being asked by group members if they needed assistance and described the help they received from their group mates as useful. However, Emilia and Tyrone both mentioned times when they had needed help but encountered barriers to receiving aid. Emilia expressed feelings of stress when others checked on her, noting that in one group "they would like stop occasionally to see if I need help but it just, it makes me feel like 'Oh my god! I need to step it up.'" Tyrone discussed a group member who "really knows her shit, but I don't even speak to her." When pressed about the relationship, he indicated that he thought "she just don't like me." In both cases, the students who would not recommend the class did not trust that their group would provide the help they needed or wanted.

***d. Copying.*** All the students acknowledged copying occurred within the groups, but clear divisions existed. Both Tyrone and Emilia, who negatively recommended the class, said they copied. Emilia said her group at the time "just tell me to copy down the answer," suggesting group-sanctioned copying to quickly address Emilia's questions or catch her up to the rest of the group occurred. Tyrone admitted to initiating the copying "so I can go back and look at it and do it like that….I always, like, go back and look at it so I can understand it." For Tyrone, copying was a strategy for learning, allowing him access to the content he could not cover in class. For Tyrone and Emilia copying was a coping strategy for the lack of togetherness in their groups.

The four other students admitted copying occurred but did not admit to themselves doing so. Three of these students said that they had let others copy, but none mentioned encouraging the practice. Instead, they distanced themselves. For example, Carley, in talking about a woman who often copied, said "I'll let her copy, but it's…just going to hurt you in the long run." Thus, while the students universally acknowledged copying, the roles they played in the practice varied.

***e. Accountability.*** Distribution of the workload and a lack of control were themes common among the students with more positive recommendations, identifying that the workload on group assignments was not always even and they lacked control over group assignments. Most of these students described conflicting feelings about trying to regulate or report their peers. In contrast, Tyrone and Emilia, the students who negatively recommended the class, said little about their experiences with group-graded assignments. Emilia did not mention group-graded assignments at all. Tyrone, rather than talking about the fact that the quality of the work was sometimes out of his hands, noted that group grades could hide the fact that not everyone in the group understood.

The contrast in experiences might be at least partially understood by a classroom instance during which Carrie and Emilia's group negotiated a graded group assignment. Emilia was responsible, by a class policy, for writing up the group answers. During the group conversation about the problem, Emilia functioned primarily as a scribe, with her group members effectively dictating answers. Thus, Emilia's group members managed the work to produce an acceptable product *efficiently*, meaning that Emilia lost the opportunity to reflect with and learn from her group, even when positioned by classroom rules to act as a critical person in the discussion.

Note that the scribe work Emilia did in this example differs from the copying discussed earlier. Here, Emilia was completing an assignment where each person in the group received the same grade, regardless of who did the assignment. In contrast, when copying, the students were doing so for work graded for completeness, so only the student who copied stood to lose.

## Discussion and Conclusions

The results presented here demonstrate some of the consequences of forming groups with diverse mathematics backgrounds and demonstrates how these conditions mean classroom goals can come into conflict. During her interview, the instructor suggested that she *relied* on a diversity of mathematics knowledge within groups to provide the best opportunity for students to learn effectively from the curriculum. Although students recognized that groups often contained a large range of knowledge levels, not all students felt they received the support they needed, while those in a position to help did not always believe supporting others was a productive use of time. Through this lens, the decision to not always support their group mates can be viewed as a rational choice, even if this is not particularly kind or fair. Many of the classroom practices that students discussed were consequences of, or coping mechanisms for, addressing the range of needs within the groups. An uneven workload on assignments was a consequence of having high-knowledge students not trusting their slower moving group mates to do the work. Copying was, for at least some students, a coping mechanism to help them quickly acquire access to the course materials when they could not participate fully in the discussion during class.

The resulting lack of togetherness in some groups did not meet the instructor's intentions and could be an indicator, if it occurs early in the semester, that a student needs additional support. Although the instructor did notice and work to address the lack of togetherness within groups, these measures were not always enough. Future iterations of *Mathematical Literacy* should experiment with group structures that prioritize knowledge levels differently. In addition, an early lack of togetherness in mathematically diverse groups might be an early and actionable indicator a student requires additional support. While this study shows that not all students had positive experiences in *Mathematical Literacy*, some did. Refinements and reflections on ways to better meet the needs of students could do a lot for future *Mathematical Literacy* students.

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Reasoning About One Population Hypothesis Testing: The Case of Steve

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*Hypothesis testing is a key concept included in many introductory statistics courses. Yet, due to common misunderstandings of both scientists and students, the use of hypothesis testing to interpret experimental data has received criticism. With statistics education on the rise, as well as an increasing number of students enrolling in introductory statistics courses each year, there is a need for research that investigates students' understanding of hypothesis testing. This paper describes results obtained from a larger study designed to investigate introductory statistics students' understanding of one population hypothesis testing. In particular, we present on one student's understanding of the concepts involved in hypothesis testing, Steve, who provided us the best spectrum of different levels of knowledge according to APOS Theory, our guiding theoretical framework. Based on this data, we suggest implications for teaching.*

*Keywords:* Hypothesis Testing, Introductory Statistics, APOS Theory

## Introduction

The use of statistics is crucial for numerous fields, such as business, medicine, education, and psychology. Due to its importance, according to the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) College Report*, more students are studying statistics, and at an increasingly younger age (GAISE College Report ASA Revision Committee, 2016). As a result, the *GAISE College Report* calls for nine goals for students in introductory statistics courses. One of these nine goals is that "Students should demonstrate an understanding of, and ability to use, basic ideas of statistical inference, both hypothesis tests and interval estimation, in a variety of settings" (p. 8).

Hypothesis testing is conducted in order to analyze a claim about a population parameter, based on sample statistics. It involves formulating opposing statements—the null hypothesis and alternative hypothesis—about the population parameter of interest. The goal of hypothesis testing is to determine whether or not to support the original claim, based on whether we reject the null hypothesis. To do so, a sample statistic is measured or observed and converted to a standardized value called the test statistic. The test statistic is then used to calculate the probability, called the *p*-value, of obtaining a test statistic at least as extreme, under the assumption that the null hypothesis is true. If the *p*-value is too low, then we reject the null hypothesis. Once a decision is made, a conclusion can be formed about the claim.

With statistics education reform on the rise, as well as an increasing number of students enrolling in introductory statistics courses each year, there is a need for research that investigates students' understanding of hypothesis testing, a concept taught in almost every introductory statistics course (GAISE College Report ASA Revision Committee, 2016; Krishnan & Idris, 2015). While previous research in this area has focused on students' misconceptions pertaining to hypothesis testing, our study sought to turn attention to what students understand and how they come to understand it. We focus our attention on the following research question:

*How do students reason about the concepts involved in one population hypothesis testing while working two problems involving real-world situations?*

In this paper, we focus on answering this question for one particular student, Steve, who elaborated the most in his interview, and thus, provided us with the richest data.

## Literature Review

Research has revealed that although students are able to perform the procedures surrounding hypothesis testing, they lack an understanding of the concepts and their use (Smith, 2008). Providing a survey of research on students' understanding of statistical concepts, Batanero et al. (1994) stated that hypothesis testing "is probably the most misunderstood, confused and abused of all statistical topics" (p. 541). Students appear to experience a "symbol shock" (Schuyten, 1990), which provides an obstacle for students interpreting particular questions (Dolor & Noll, 2015; Liu & Thompson, 2005; Vallecillos, 2000). Vallecillos (2000) found that students have trouble with not only the symbols, but also with the formal language and meaning behind the concepts involved in hypothesis testing, including words such as "null" and "alternative" when referring to the hypotheses. Students interviewed were not able to accurately describe what these terms mean and how they impact the decision to either fail to reject or reject the null hypothesis (Vallecillos, 2000). Williams (1997) made a similar observation. She found that, due to the tedious process behind hypothesis testing, students were not able to connect the statistical concepts back to the context of the problem. She further stated that, "the biggest hurdle is reaching a statistical conclusion, and the real meaning of the original question may be forgotten in the process" (p. 591).

Students' difficulty with understanding hypothesis testing can oftentimes be attributed to how it is taught. Textbooks and instructors frequently give a specific step-by-step script to follow when performing hypothesis testing, which does not provide students the opportunity to see the process as a whole. Link (2002) described this as a six-part procedure, which leads many students to look for keywords and phrases as guides when solving hypothesis testing problems. He found evidence that students were able to correctly substitute values into a formula selected from a formula sheet, but they did not have an understanding of the logic behind the overall procedure of hypothesis testing.

## Method

The focus of our larger study is on university students who are enrolled in an introductory statistics course at a large public institution in the southeastern United States. For this particular institution, students were required to spend three academic hours per week in a computer lab, completing assignments through Pearson's *MyStatLab.* Data collection took place during Fall 2014 and Spring 2015. All students enrolled in six sections of an introductory statistics course (approximately 240 students) were invited to participate in a problem solving session and semi-structured interview pertaining to hypothesis testing. Twelve students volunteered to participate. During the problem solving session, each participant worked alone on two hypothesis test questions, similar to problems they had already seen. They were encouraged to use Excel when needed, since the use of it was required as part of the class. The first question asked the student to conduct and interpret a hypothesis test for a single population proportion. The second question asked the student to conduct and interpret a hypothesis test for a single population mean. The questions were as follows:

1. In a recent poll of 750 randomly selected adults, 588 said that it is morally wrong to not report all income on tax returns. Use a 0.05 significance level to test the claim that 70% of adults say that it is morally wrong to not report all income on tax returns. Use the *P*-value method. Use the normal distribution as an approximation of the binomial distribution.
2. Assume that a simple random sample has been selected from a normally distributed population and test the given claim. In a manual on how to have a number one song, it is stated that a song must be no longer than 210 seconds. A simple random sample of 40 current hit songs results in a mean length of 231.8 seconds and a standard deviation of 53.5 seconds. Use a 0.05 significance level to test the claim that the sample is from a population of songs with a mean greater than 210 seconds.

Immediately following the problem solving session, the students participated in a semi-structured interview that was video-recorded. There were ten interviews, eight with one participant each and two with two participants each. During the interviews, participants were asked to elaborate on their solutions and thought processes. Conducting the interviews was divided among five members of the research team, who all followed the same protocol. The data (interview transcriptions, written work, and Excel files) were analyzed and coded according to the levels of conceptions in APOS Theory (described below). The research team deliberated until an agreement was made regarding the codes.

### APOS Theory

Action–Process–Object–Schema (APOS) Theory is a constructivist framework for describing how an individual might develop his or her understanding of a mathematical concept (Arnon et al., 2014). It emphasizes the construction of cognitive structures called Actions, Processes, and Objects, which make up a Schema. These structures are constructed through reflective abstraction, particularly through the mental mechanisms of interiorization, reversal, coordination, encapsulation, and generalization. The construction of these structures signify levels in the learning of a mathematical concept. An Action is a transformation of Objects in response to external cues. The primary characterization of an Action is the external cue, which could be keywords or a memorized procedure. Reflection on a repeated Action can lead to its interiorization to a Process. While an Action is an external transformation of Objects, a Process is an internal transformation of Objects that enables an individual to think about the transformation without actually performing it. Once a Process is conceived as a totality and the individual can perform transformations on it, the Process is said to have been encapsulated into an Object. While a component of APOS Theory is the development of a genetic decomposition, i.e., description of how an individual might develop an understanding of a mathematical concept, our genetic decomposition is omitted in this paper due to space limitations.

## Results

While performing a hypothesis test, it is necessary for an individual to formulate the hypotheses about a population parameter, evaluate the test statistic, find the *p*-value, compare the *p*-value to the significance level, form a decision about the null hypothesis, and form a conclusion about the claim. Through these objectives, students construct mental structures called **hypotheses**, **test statistic**, **p-value**, **decision**, and **conclusion**, each of which can be conceived as an Action, Process, or Object. In this section, we provide examples of how the mental structures of **hypotheses**, **test statistic**, **p-value**, and **decision** emerged in the reasoning of one particular student, Steve. As we will show, these constructions emerged as Processes or Objects in Steve's reasoning. We use bold font when referring to the primary mental structures that make up our genetic decomposition, to distinguish them from other uses of these terms. For simplicity, we do not use a different font to distinguish between the different levels corresponding to a concept. Note that we are not seeking to classify Steve in terms of his understanding, but instead, present evidence we found of his reasoning. Due to space limitations, we omit discussing **conclusion**.

**Hypotheses**

The mental structure, **hypotheses**, can be conceived as a transformation—an Action or Process—that acts on the claim of the hypothesis test and returns the null and alternative hypotheses. As an Object, additional transformations can be performed on **hypotheses**. Steve exhibited both a Process conception and Object conception of hypotheses.

To illustrate Steve's reasoning of **hypotheses** as a Process, the following excerpt is considered from Question 1 of the instrument.

> Um, well, when you're doing null and alternative you always focus on the claim they give you. Um, so 70%, and just to make things easier, uh we do the null is equal to .7, and then the alternative would be whatever you're asking, in this case you're asking, is it 70%. So you use not equal to 70%.

Steve acknowledged, in general terms, that the claim is used to formulate the hypotheses. We consider this to be evidence of a Process conception of hypotheses.

To illustrate Steve's reasoning of **hypotheses** as an Object, the following excerpt is considered from Question 2 of the instrument.

> OK. I just did the same thing I did with proportion, and I said the null is equal to um 210, in this case, and uh the alternative is greater than 210. But the only reason I said that is because um in this bottom line of the question says, test the claim that the sample is from a population um with a mean greater than 210.

Steve used the phrase, "in this case," to indicate that in his mind he distinguished his procedure for Question 2 from his procedure for Question 1. Despite the fact that the questions on the instrument pertained to two different contexts, Steve said, "I just did the same thing I did with proportion." In order to be able to describe his procedures as the same, while also distinguishing between them in the different situations in which they arose, he had to have compared them, which is evidence of an Object conception of hypotheses.

**Test Statistic**

The mental structure, **test statistic**, can be conceived as a transformation—an Action or Process—that acts on various population parameters and sample statistics and returns a standardized value, namely the test statistic, which is the number of standard deviations a sample statistic is away from the distribution's center, or expected value. As an Object, additional transformations can be performed on **test statistic**. Steve exhibited both a Process conception and Object conception of test statistic.

To illustrate Steve's reasoning of **test statistic** as both a Process and an Object, the following excerpt is taken from Steve's discussion of Question 1, in which he described what accounted for an extreme value of the test statistic.

> But going back on it, it makes sense, you know, if you've got a *p*-hat that, that's very very different from your, from your *p*, you know, 78 is a whole 8% off of uh the 70%. And also your test statistic is very large. I'm not totally sure what a test stat is, but it reminds me of *z*-scores, and I remember when you have a *z*-score that gets above 3, it starts to get pretty, pretty crazy. So 5 is huge, which is also the reason that you're getting a bunch of zeros or very close to 1.

Steve appeared to have encapsulated into an Object the Process of calculating a *z*-score for proportions, in order to consider how it resulted in an extreme value of the test statistic. He explained that a large value of the test statistic resulted from having a value of the sample statistic that is very different from the value of the population parameter in the null hypothesis. APOS Theory acknowledges, in general, that it is necessary to de-encapsulate an Object back into a Process, which appears to be the case with Steve. That is, he de-encapsulated his test statistic Object back into a Process to consider the difference between $\hat{p}$ and $p$. We should note that based on Steve's statement, "I'm not totally sure what a test stat is, but it reminds me of *z*-scores," he appeared to have constructed isolated Processes for each test statistic, which he needed to further coordinate in order to construct a single test statistic Process.

***P*-value**

The mental structure, **p-value**, can be conceived as a transformation—an Action or Process—that acts on the test statistic and returns a probability—a number between 0 and 1. As an Object, additional transformations can be performed on **p-value**. Steve exhibited both a Process conception and Object conception of p-value.

To illustrate Steve's reasoning of **p-value** as both a Process and an Object, we consider the following excerpt from Steve's discussion of the *p*-value for Question 1, in which he explained various procedures for calculating the *p*-value, depending on the situation.

> *Steve:* Well, whenever you're finding a *p*-value you're doing a .DIST function, and when you're doing proportions, it's NORM, and when you're doing means, it's T. So in this case we used NORM.S.DIST cause I think the other formula is silly. But uh since it's a two-tailed test I couldn't just stop there. I had to 1 minus that and then double it.
> *Interviewer:* OK, OK. And you did the 1 minus, why?
> *Steve:* Um because if you don't do 1 minus, it ends up being something very very close to 1. So a bunch of .9999…, and you can't double that. Whenever I got stumbled, I was like, oh wait, do I, uh, do I double the 1 minus or it by itself. Well, you can't go over 1. It has to be between 0 and 1.

Steve explained, in general, that an Excel .DIST function is used to calculate a *p*-value, and he said the result "has to be between 0 and 1." Steve's description in general terms of the transformation on the test statistic that resulted in the *p*-value and recognition of the p-value as a probability is evidence of a Process conception of *p*-value. Furthermore, Steve described situations in which you would use NORM.S.DIST versus T.DIST. Although Steve was not completely correct in stating that you always use T.DIST in the context of means, he clearly compared different procedures for calculating the *p*-value and considered situations in which these procedures would arise. Thus, we consider this to be evidence of an Object conception of *p*-value.

**Decision**

In hypothesis testing, we make a decision about whether or not to reject the null hypothesis by comparing the *p*-value to the significance level, which, in this course, was a predefined upper bound for the *p*-value. In particular, if the *p*-value is less than or equal to the significance level, we reject the null hypothesis. The mental structure, **decision**, can be conceived as a transformation—an Action or Process—that compares the *p*-value to the significance level and returns the decision about whether to reject the null hypothesis. In particular, **decision** compares the *p*-value and significance level as areas or probabilities. As an Object, additional transformations can be performed on **decision**. Steve exhibited a Process conception of decision.

To illustrate Steve's reasoning of **decision** Process, we first consider the following excerpt from Steve where he demonstrated that he compared the *p*-value and significance level as areas.

> Oh wait! Wasn't the *p*-value supposed to be from the edge? So wasn't the *p*-value supposed to be like this … [*draws on paper*] … the stuff on the outside? I remember now. It was um . . . I don't see how that relates to those, but I know it relates to the significance level 'cause your .05 is going to be outside of that.

Steve explained how he was able to graphically represent the *p*-value (see Figure 1). Finding that the *p*-value is less than the significance level, he drew the region whose area is the *p*-value inside the region whose area is the significance level, evidence that he compared the *p*-value and significance level as areas. To clarify, when Steve said, ".05 is going to be outside of that," we interpret it to mean that the rejection region is not strictly contained in the region whose area is the *p*-value. In addition to considering this to be evidence of a component of a **decision** Process, we also consider this as further evidence of a **p-value** Object.

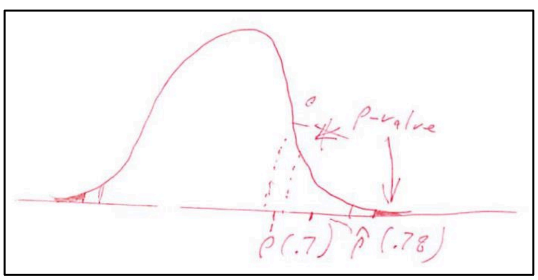


*Figure 1: Steve's graph of the p-value for Question 1.*

The previous excerpt established that Steve was able to compare the *p*-value and significance levels as areas, which we consider to be a necessary characterization of a **decision** Process. To further illustrate Steve's reasoning, we consider the following excerpt about whether or not to reject the null hypothesis for Question 1. Note that part of this excerpt was discussed previously in the section on test statistic.

> *Interviewer:* OK, so, and how did you arrive at your conclusion? What did you arrive at?
> *Steve:* I just remembered anytime the *p*-value is less than the, uh, significance level you reject the null, uh, I think [*laughs*]. But going back on it, it makes sense, you know, if

> you've got a $p$-hat that, that's very very different from your, from your $p$, you know, 78 is a whole 8% off of uh the 70%. And also your test statistic is very large. I'm not totally sure what a test stat is, but it reminds me of $z$-scores, and I remember when you have a $z$-score that gets above 3, it starts to get pretty, pretty crazy. So 5 is huge, which is also the reason that you're getting a bunch of zeros or very close to 1 […] So it's interesting, we always go all the way out to the $p$-value, but you can pretty much tell from your test statistic if it's correct or not.

Initially, Steve rejected the null hypothesis based on a memorized rule, suggestive of a **decision** Action. However, he reflected on this Action and related an extreme test statistic to a small $p$-value. As a result, Steve explained that depending on the magnitude of the test statistic, you could potentially form a decision about the null hypothesis without comparing the $p$-value to the significance level. The ability to describe the result of a transformation without needing to perform all of its steps is evidence of a Process conception.

## Discussion and Concluding Remarks

Since the number of students enrolling in introductory statistics courses each year is continually increasing, it is important to explore students' reasoning of hypothesis testing (GAISE College Report ASA Revision Committee, 2016; Krishnan & Idris, 2015). This report, part of a larger study, focused on examples of how the mental structures of **hypotheses**, **test statistic**, **p-value**, and **decision** emerged in the reasoning of one particular student, Steve. Steve's constructions of the mental structures emerged as Processes or Objects in his reasoning. Steve exhibited a Process conception of hypotheses by acknowledging that, in general, the claim is used to formulate the hypotheses. In another situation, Steve exhibited an Object conception of hypotheses by being able to compare procedures for formulating hypotheses between two different problems. Steve illustrated test statistic as both a Process and an Object by describing what accounts for an extreme value of the test statistic in a situation. Steve exhibited an Object conception of $p$-value by being able to explain and compare various procedures for calculating the $p$-value, depending on the situation. Lastly, we found evidence that Steve illustrated a Process conception of decision by being able to describe the results of his decision without going through the steps of comparing the $p$-value to the significance level. In this case, he related a large test statistic to a small $p$-value.

Our results suggest that concepts involved in hypothesis testing are related through the construction of higher order transformations, operating on Processes that have been encapsulated into an Object. It has been widely recognized in APOS Theory literature that encapsulation of a Process into an Object is difficult to achieve, a possible explanation for why hypothesis testing is such a challenging topic for students. However, we found evidence of these constructions of higher order transformations in Steve's rich descriptions of the concepts.

With textbooks and instructors frequently introducing the topic by giving a step-by-step script to follow, what Link (2002) describes as a six-part procedure, construction of higher order transformations becomes even more difficult as this instruction leads students to look for keywords and phrases as guides when solving hypothesis testing problems. Based on the results, it is important when teaching to develop questions for students that motivate them to think and explain beyond a procedural approach. Creating activities with guiding questions will encourage students to think such as Steve, and to develop deeper knowledge of hypothesis testing.

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Framework for Students' Understanding of Mathematical Norms and Normalization

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*Mathematical norms and normalization of vectors are important concepts used throughout the mathematical and physical sciences; however, very little research has been done on students' understanding of these concepts. To remedy this lacuna, this report presents a framework that can be used to model, explain, and predict the ways students reason about and solve problems involving norms and normalization.*

*Keywords:* norms, normalization, student understanding, vectors, linear algebra

Normalization of particular vectors from various vector spaces (e.g., $\mathbb{R}^n$, $\mathbb{C}^n$, function spaces) is mathematically important in various contexts. Some examples include directional derivatives in multivariable calculus, states of quantum mechanical systems in Physics, and the development of orthonormal bases through the Gram-Schmidt process in Linear Algebra and Numerical Analysis. Despite the wide applicability of normalization within mathematics and science, students' understanding of norms and normalization seems not to have been studied. Research that has examined students' understanding of absolute value have come close to the topics (e.g., Almog & Ilany, 2012; Sierpinska, Bobos, & Pruncut, 2011), but have not directly addressed them. This lack of research into students' understanding of norms and normalization must be remedied.

In this report, I present a framework for students' understanding about mathematical norms and normalization. This framework aims to address the following research questions: (a) What are the various components involved in understanding mathematical norms and normalization, and how are they interconnected? and (b) How does a students' understanding of those components impact their thinking and solution strategies when working on problems involving norms and normalization?

I first explain the theoretical lens of the Emergent Perspective that I adopt within this report, and why the development of models of student thinking and understanding are important. Next, I describe the methods for the study and framework development. An explanation of the framework is given afterwards, with a focus on how the various components fit together and interact with one another. I then use data from two students to illustrate how the framework can be used to model, explain, and predict students' reasoning about normalization problems. Lastly, I discuss how the results illustrate elements of understanding norms and normalization that were particularly powerful for students in their reasoning about normalization problems, and the implications these have for teaching and future research.

## Theoretical Lens

The Emergent Perspective (Cobb & Yackel, 1996) coordinates psychological constructivism (von Glasersfeld, 1984, 1995) and social interactionism (Bauersfeld, Krummheuer, & Voigt, 1988) into a version of social constructivism that views mathematical learning as both individual construction and enculturation into the mathematical community. As an elaboration on the Emergent Perspective, Rasmussen, Wawro, and Zandieh (2015) added the importance of understanding the conceptions individual students bring to bear in their mathematical work. The main goal of this research is to gain a better understanding of students' conceptions about norms and normalization, and create a framework for modeling, explaining, and predicting students' reasoning about these concepts.

## Methods and Framework Development

The framework developed and used herein was inspired and influenced by Zandieh's (2000) framework for student understanding of derivatives and Lockwood's (2013) model of students' combinatorial thinking. Similar to the work of Lockwood (2013), I used a *conceptual analysis* (von Glasersfeld, 1995) or "a detailed description of what is involved in knowing a particular (mathematical) concept" (Lockwood, 2013, p. 252) to create this framework of students' understanding of norms and normalization. This conceptual analysis involved an iterative process of moving among my own theoretical thinking about the constructs involved in understanding norms and normalization, relevant literature, and the student interview data.

Although research examining students' understanding of norms and normalization is scarce, research on students' understanding of absolute value is relevant, as the absolute value is an example of a norm. Important findings include: the power in understanding multiple ways to define the absolute value (e.g., delete the negative sign, $|x| = \sqrt{x^2}$) in solving different problems (Wilhelmi, Godino, & Lacasta, 2007); and the power of understanding absolute value as the magnitude of a number or its distance from zero (Almog & Ilany, 2012; Sierpinska, Bobos, & Pruncut, 2011). These ideas impacted the development of the framework, and may be important for students' understanding of norms and normalization, as I illustrate later.

The data used in the development of the framework consists of hour-long, video-recorded, semi-structured interviews with individual students at two different collection sites: nine junior-level quantum mechanics students from a university in the northwestern United States; and two junior-level linear algebra students and two sophomore-level multivariable calculus students from a university in the southeastern United States. Students at the first site were asked questions about several linear algebra concepts including normalization, while students at the second site were only asked questions about norms and normalization. Although interviews from both sites informed the framework development, the data used within this report to illustrate the utility of the framework come from the second collection site.

In analyzing the student data, I first watched the sections of the interview in which students explained their understanding of normalization and normalized vectors from $\mathbb{R}^2$ and $\mathbb{C}^2$, writing a summary of each student's thoughts afterwards. Next, the transcript or video of the interview was coded (Maxwell, 2013) for each student, with some codes influenced by the state of the framework at the time of coding. Lastly, I examined the framework to see how well it could be used to model and make sense of each student's thinking and reasoning about norms and normalization, making modifications as necessary. The framework, as it stands in the next section, has gone through several revisions and refinements based on this analysis, as well as feedback received through poster presentations at two conferences (Watson, 2017a, 2017b).

## Framework for Students' Understanding of Normalization

Figure 1 presents a visual representation of the framework. I contend that understanding normalization essentially involves three major components, namely the norm of a vector, procedures for normalizing a vector, and what a normalized vector is (as conveyed by the three large ellipses in Figure 1). I expand on the contents of these ellipses in the following subsections. The lack of directional arrows in the figure is deliberate, as any component could inform how a student thinks about any of the other components, although when normalizing a vector, students

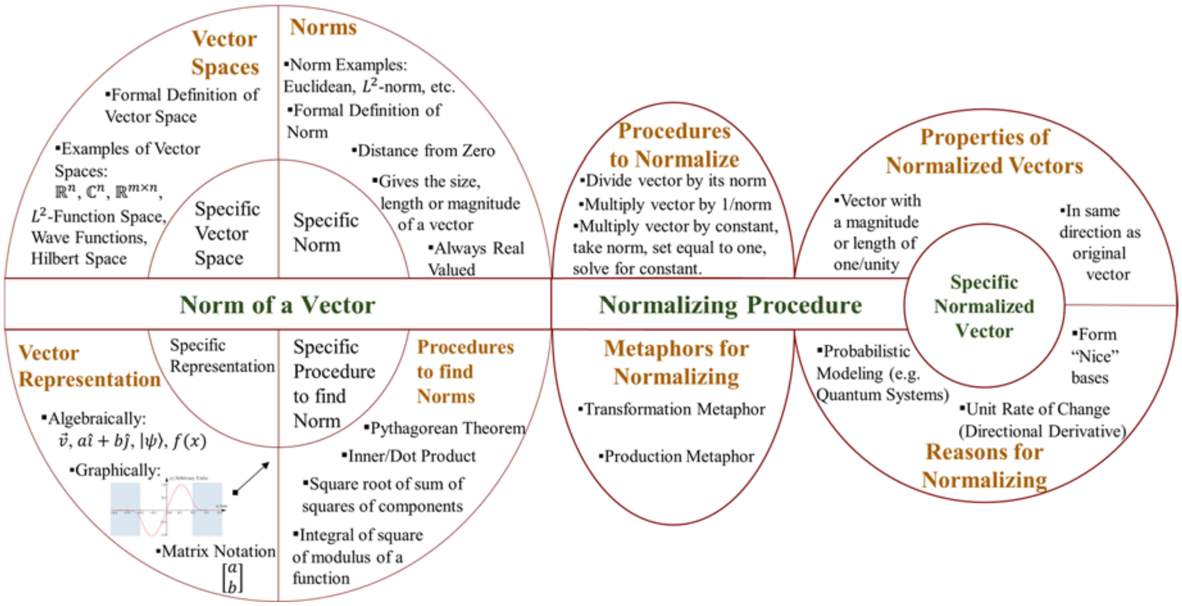


*Figure 1: Framework for students' understanding of norms and normalization*

generally find the norm, perform a normalizing procedure, and end with a normalized vector (i.e., left to right in the figure). Finally, students' understandings of norms and normalization do not necessarily include all of these components and connections; as such, when using the framework to model a student's understanding, components and connections presented in Figure 1 could be scarcer or even missing for a particular student's model.

**Norm of a Vector**

I have found four elements that can influence or determine how a student finds the norm of a vector, namely the vector space the vector is an element of, the representation chosen for the vector, the particular norm function to be used, and the procedure chosen for finding that norm (these four aspects are represented by the inner ellipse on the left of Figure 1). Additionally, a student's broader or more general understanding of vector spaces, representations of vectors, norms, and different procedures for finding norms, discussed in further detail below, can also inform and influence how the student finds the norm of a specific vector

A student's understanding of vector spaces could include examples of several vector spaces, such as $\mathbb{R}^n$, $\mathbb{C}^n$, or $L^2$-function space, although many students only have experience with vectors in $\mathbb{R}^n$. Mathematically sophisticated students may also be able to draw on their understanding of the formal definition of vector space. Altogether these can influence how a student thinks about a specific vector, which can inform their normalization of it.

A student's understanding of vector representations could include examples of algebraic notations (e.g., letter with special marking, functions, Dirac Notation), graphical notations (e.g., graphs of functions, points on a Cartesian coordinate system, directional arrows), and matrix notation (i.e., column or row vectors), with each representation choice affecting how a student thinks about finding the norm of a vector. Furthermore, a student's understanding of why we use representations, and their ability to select the best or most useful representation for a given task—which is part of Meta-Representational Competence (diSessa, Hammer, Sherin, & Kolpakowski, 1991; diSessa, 2004; Wawro, Watson, & Christensen, 2017)—could also impact a student's thinking about norms and normalization.

For many undergraduate students, the only norm they are explicitly aware of is the Euclidean Norm on $\mathbb{R}^n$, as most have only heard the term "norm" in conjunction with real vectors. However, mathematically sophisticated students may also know examples of other norms, the formal definition of norm, and important properties of norms (e.g., always real valued). Any of these ideas about norms can shape how a student finds the norm of or normalizes a vector.

There is also great variety in how students approach finding the norm of a vector for a given norm. For instance, a few ways students can find the Euclidean norm of a vector in $\mathbb{R}^2$ are to take the square root of the sum of the squares of the components, take the square root of the dot product of the vector with itself, or graph the vector and use the Pythagorean Theorem to find the length. While all of these are correct, each procedure can influence how students think about the norm, and their understanding of multiple procedures and connections between them could be drawn upon at any time.

**Normalizing Procedure**

There are several different ways a student can normalize a vector, such as dividing the vector by its norm, multiplying the vector by the reciprocal of its norm, or multiplying the vector by an unknown constant *before* finding the norm, setting it equal to one, and solving for this normalization constant. Moreover, there seem to be essentially two metaphorical expressions (Zandieh, Ellis, Rasmussen, 2017) students call upon when normalizing a vector which influence how they think about, and even notate, the normalized vector. The *transformation/morphing metaphor* views normalizing as a procedure that transforms or morphs the original vector into the normalized one, as when a student talks about "shrinking" the original vector down to a length of one. The *production metaphor*, on the other hand, views normalizing as a procedure that produces a vector that is in the "same direction" as the original vector, but has a length of one.

**Normalized Vector**

A student's understanding of normalized vectors includes ideas about properties of normalized vectors and reasons why normalization is important. The properties could include normalized vectors having a norm, length, or magnitude of one, and being in the same direction as the original vector. Reasons for normalization students have in their understanding could include probabilistic modeling (such as in quantum mechanics), looking at unit rates of change (such as with directional derivatives in multivariable calculus), or even simply a rule or procedure that must be carried out as a part of some algorithm.

## Using the Framework to Model Students' Understanding of Norms and Normalization

I now demonstrate how the framework can be used to model students' understanding using data from two students, Luke and Spencer. Luke was a physics/mathematics double major who came from an advanced linear algebra course, and Spencer was a computer engineering major who came from an introductory multivariable calculus course at the time they were interviewed. The interview consisted of having them solve three problems related to norms and normalization (an absolute value problem $|x - 7| = 3$; normalizing a vector in $\mathbb{R}^2$; normalizing a vector in $\mathbb{C}^2$ with components $3$ and $3i$), and explain their own understanding of absolute value, norms, and normalization in general.

Although Spencer did briefly mention that the absolute value can tell you how far away a number is from zero, he struggled to make use of this fact in solving problems, and did not see how absolute value could be related to mathematical norms. In fact, Spencer always described a process for normalizing vectors when asked about norms, and essentially seeing “norm” and “normalize” as the same idea:

*I*nterviewer: Do you see a difference between norm and normalization, or are those just, like, so related that...?

*Spencer*: I mean, I see 'em pretty related. Um, I don't really see a difference between them, honestly.

Furthermore, what Spencer understood by normalization seemed particularly narrow, as evidenced in his work on normalizing the vector $\boldsymbol{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Before even starting the problem, he changed the vector $\boldsymbol{v}$ to the form $2\hat{\imath} + 5\hat{\jmath}$, and proceeded by finding the square root of the sum of the components, $\sqrt{2^2 + 5^2}$, to get $\sqrt{29}$. He then explained that he would need to divide by $\sqrt{29}$ to get one, and proceeded to divide each component by $\sqrt{29}$ to arrive at his solution of $\frac{2}{\sqrt{29}}\hat{\imath} + \frac{5}{\sqrt{29}}\hat{\jmath} = 1$. I then asked him what normalization means to him, and why he chose that particular procedure:

*Spencer*: Normalize to me is basically making, like, getting the, getting this one, basically. … Like, so, when you use the distance formula, obviously, like, this is the distance formula [pointing at the square root of the sum of the squares]. But, um, you want, like, the distance to be [pause]. You want it to be one. … Um, like, but other than that, I never, I've never actually asked why?

*Interviewer*: Why did you choose that procedure to normalize?

*Spencer*: It's really the only one that I know. And it's the most recent one in my head. We just went over it, I think, last week. So, that's, that's the most recent one. And, I think that's the only one that I know as of now. I think that's it.

Spencer did seem to realize that the distance formula is an integral part of normalization, but the discussion above continues to affirm that Spencer did not have a strong understanding of norm, and rather understood norm and normalization as being one in the same, namely a procedure he was taught to carry out. In fact, when he tried to make sense of normalization graphically, he plotted the original vector and normalized vector as points on a Cartesian plane, but was never confidently able to describe the relationship between the two points, or what it meant for the normalized vector to have a magnitude of one.

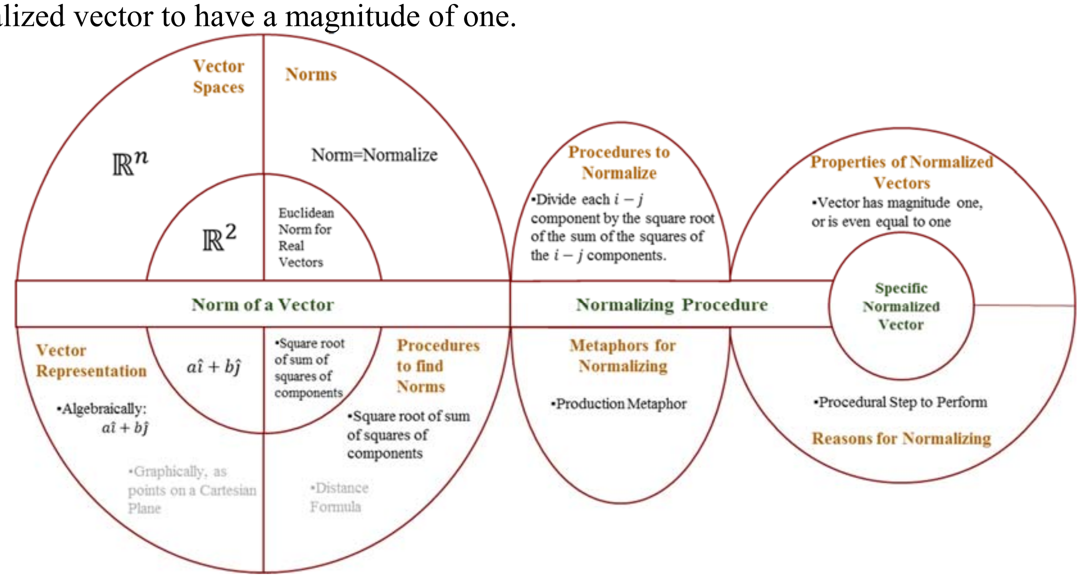


*Figure 2: Model of Spencer’s understanding of normalization*

Based on the above discussions with Spencer, I present a model of his understanding of normalization in Figure 2, which uses the framework as an organizational tool. This model can help us make sense of Spencer's understanding, but also gives us power to predict how he might approach normalizing vectors from unfamiliar vector spaces. More specifically, we would predict that Spencer would rely on the only procedure he knows for normalization, and probably encounter moments of uncertainty and confusion in the process.

As predicted by the model, when Spencer was asked to normalize the vector $\begin{bmatrix} 3 \\ 3i \end{bmatrix}$, he again attempted to write the vector as $3\hat{\imath} + 3i\hat{\jmath}$, which alone bothered Spencer, having the imaginary $i$ and the normalized basis vector $\hat{\imath}$ in such close proximity. He then used his method of finding the square root of the sum of the squares of the components, and arrived at a value of zero.

*Spencer*: So, um, [pause]. I don't think it can be normalized? 'Cause I got zero.

*Interviewer*: OK, what do you mean by...?

*Spencer*: … you can't really divide anything by zero. … I just don't think it can be normalized at that point.

Spencer's approach to solve this problem could be modeled by replacing the $\mathbb{R}^2$ in the model above with $\mathbb{C}^2$, with all other components essentially the same. As soon as he got zero, however, his procedure broke down, leading him to declare the vector as something that cannot be normalized. Spencer was not able to draw on any generalized understandings of norms or normalized vectors to help him rethink his procedure for normalization. Still, by the end of the interview Spencer realized his understanding of normalization was limited, and he even mentioned the possible existence of other types of normalization.

To see a stronger understanding of norms and normalization, I present my model for Luke's understanding in Figure 3, and highlight important aspects of it. First, Luke had a strong understanding of absolute value as representing a number's distance from zero, and

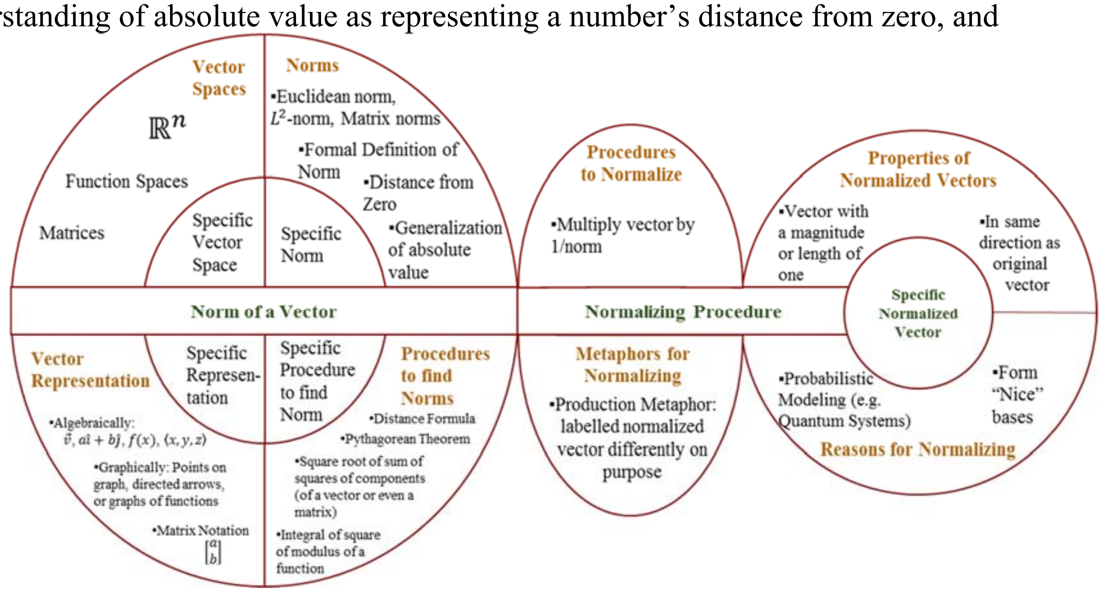


*Figure 3: Model of Luke's understanding of normalization*

understood norms as a generalization of this idea, namely giving the distance a vector is from the zero vector. Second, this distance from zero conceptualization was not limited to $\mathbb{R}^n$, as Luke was able to describe what the norm would be for several vector spaces, and how this could be thought of as giving a distance from zero. As an example, consider his description of a possible norm for the vector space of matrices:

*Luke*: So, if you have a matrix ... that would be, like, you know, you could define the norm where it's, like, just the square of the sum of all of 'em [the entries of the matrix]. Or, whatever you wanted to do. And that would be the distance from the zero matrix in the same sort of way, because, the zero matrix would be all zeros.

Third, Luke was confidently able to use and move among multiple vector representations and procedures for finding norms. And fourth, Luke understood the result of normalization, and why it is important for the creation of orthonormal bases, as well as probabilistic modeling in quantum mechanics.

Based on Luke's model, we would predict Luke to successfully draw upon his strong understanding to make sense of normalizing vectors from vector spaces that he was not familiar with. This was evidenced in his work to normalize the vector $\begin{bmatrix} 3 \\ 3i \end{bmatrix}$. Luke was somewhat unsure of how to proceed with normalizing this vector, and his first attempt was similar to Spencer's above, arriving at a norm of zero. However, unlike Spencer, Luke was able to draw on his understanding of norms, and immediately recognized this could not be correct:

*Luke*: So, I did something wrong, 'cause that's not the zero vector.

*Interviewer*: So, why did you think you did something wrong?

*Luke*: Well, the way a norm is defined, says that like, the norm of a vector can only be zero if that vector is zero. And, these entries are not zero. [I: OK]. So, there's something wrong with how I've been doing this. And, I bet, if I just took the modulus of each one first, then it would work.

Even more striking is the fact that Luke was able to propose a modification (taking the modulus of both components first) that was viable and mathematically sound, and he went on to correctly normalize this complex vector.

## Discussion and Conclusion

Luke's understanding of norm representing a distance from zero was particularly powerful for making sense of norms in multiple vector spaces, even vector spaces that were unfamiliar to him. This coincides with Sierpinska et al. (2011) who explained the power in understanding absolute value as a distance from zero:

> Definitions based on the notion of distance are important in applications and in mathematical theory, in particular in generalizations of absolute value to norms in higher dimensions and general vector spaces, and in generalizations of limits and continuity in topology. (p. 280)

This also relates to my own findings on the importance of understanding norms for a strong understanding of normalization, including its importance for multiple contexts within science and mathematics (Watson, 2017a, 2017b).

In this report I have presented a framework for students' understanding of mathematical norms and normalization. This framework has identified the components that go into understanding mathematical norms, normalization, and normalized vectors. Furthermore, I have shown how students' solution methods for mathematical problems involving these concepts can be thought about, made sense of, explained, and even predicted by using the framework to model students' understanding. It is hoped that this framework will be helpful for future research into students' understanding of norms and normalization, as well as their understanding of related concepts such as metrics and metric spaces within real analysis and topology. Furthermore, this framework could be used by instructors to think about ways they might best help their students develop robust understandings of norms and normalization within a variety of courses where these concepts are used, such as helping students conceptualize norm as a distance from zero.

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# Mind the 's' in Individual-With-Contexts: Two Undergraduate Women Boosting Self-Efficacy in Mathematics

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*Investigations of students' identities are shedding light on the processes that generate differential learning. In this paper, we expand the construct of dispositions to account for what individuals carry through their bodies and beliefs across contexts. While bringing back the attention to individualized dispositions, we avoid the trap of innateness. After we create a three-layered framework of dispositions to explain why and how two undergraduate women, Bettie and Melissa, develop differential confidence in mathematics through a semester-long number theory class. We learn that individualized dispositions can change throughout time, and desensitize or make individuals vulnerable to gendered, racialized, and sexualized stigma.*

*Keywords:* Self-efficacy, Equity, IBL, Situated Learning, Dispositions.

The scholarship on identity, specifically in Mathematics education, has demonstrated the mutual construction of individual identities, preferably called positionings, and the contexts within which these identities emerge (Cobb & Bowers, 1999; Greeno, 1998; Hand & Gresalfi, 2015; Lave & Wenger, 1991; Martin, 2000; Nasir, 2002). The revised concept of *dispositions* is particularly illuminating. Following a situated approach, Cobb and Gresalfi (Gresalfi, 2009; Gresalfi & Cobb, 2006) define dispositions as ways of engaging with an activity organized by sets of ideas, perspectives and values. In an IBL activity, dispositions can include: "working together or individually," "providing developed or succinct explanations to a groupmate," "offering critical comments to others," "using the textbook (when and how)," "preferring to study with others or alone" and others. Contexts and individuals determine which dispositions emerge and become patterns of engagement. For instance, a student may actively participate in a group and be a passive listener in another type of group or class (Hand & Gresalfi, 2015). In this paper, we expand the construct of dispositions and create a framework to explain why and how two undergraduate women in a number theory class, Bettie and Melissa, developed differential confidence in mathematics.

Bettie and Melissa self-reported a low self-efficacy at the start of their number theory class. During exit interviews at the end of the semester, Bettie reported a robust increase in self-efficacy while Melissa reported only a relative confidence in her mathematical ability. Their number theory course used predominantly small-group work; after the third week students stayed in the same groups. Bettie and Melissa were the only women in their five-member groups.

As we conducted a situated approach analysis of Bettie's and Melissa's group work, we observed how group dispositions explained the differential roles they took in groups and classroom dispositions. Particularly, a classroom norm to not look at textbook solutions, shaped the construction of their differential self-efficacies. In addition to classroom and group dispositions, Bettie and Melissa activated dispositions that were produced in historic contexts or outside the studied classroom. Melissa enacted a collaborative disposition and Bettie a solitary reflective disposition. The collaborative disposition rendered Melissa vulnerable to groupmates' unresponsiveness, while the latter left Bettie immunized.

The reported analysis shows three interlaced layers of dispositions: classroom, group and individualized dispositions. These offered different learning opportunities to Bettie and Melissa, and fostered different mathematical identities (see Figure 1).

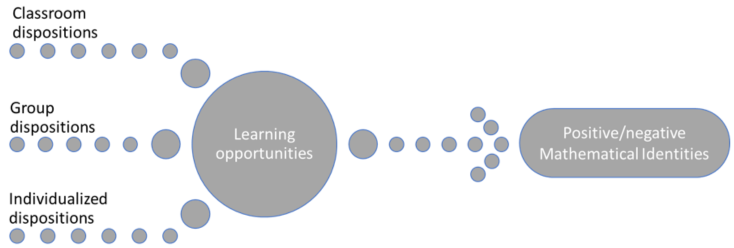


Figure 1: Three layers of dispositions construct learning opportunities that shape mathematical identities.

*Classroom dispositions* are constructed through interactions between an authority, either institution or instructors, and students. Social and socio-mathematical norms (Yackel, Rasmussen, & King, 2000), as enacted in a classroom, are the best examples of classroom dispositions. When students work in small-groups, *group dispositions* are constructed. For example, if group members assign a high mathematical authority and social influence to a group member, group dynamics may easily turn into a tutoring disposition. Esmonde (2009) showed that mechanisms of group dispositions are independent from, but may connect to, classroom dispositions. The new construct to a situated approach introduced herein is *individualized dispositions*, by which we conceive the interconnectedness of individual-with-context**s**.

When we think of individuals as embedded within a context, we must account for the ways individuals navigate through and connect multiple contexts simultaneously or consecutively. We propose two mechanisms by which dispositions transfer from one individual-with-context to the same individual in a different context. First, dispositions can transfer across contexts through the individual's body. Performativity theorists (Butler, 2010; Kramsch, 2010) have shown that repetitive enactment of dispositions leave embodied marks in individuals. To use an external resource, Erickson (2004) observed that linguistic and gestural behaviors common to an environment, e.g. home or church, may become reflexes that individuals carry to the educational environment. Second, dispositions may translate into beliefs or narratives that individuals can carry into other contexts. Boaler and Greeno (2000) documented students, who enacted classroom dispositions aligned with traditional teaching. They believed that mathematics lack creativity, which influenced their decisions of major choice in college. Sfard and Prusak (2005) documented narratives produced during family meetings about the mathematical ability of East-European immigrants to Israel. The narrative identities transferred to classroom behaviors: students from these communities were renown of persevering at solving challenging math problems. We shall call the transferred dispositions *individualized dispositions,* since transfers occur through the individual's body or beliefs.

The selection of relevant concomitant and historic individual-with-contexts is challenging. Relevant transferred dispositions must be triggered by current individual-with-context. Thus, we find mediated and unmediated interviews to be an optimal data collection method to investigate individualized dispositions. Further discussion is in the methods section.

## Methods

### Activity and Participants

Data reported was collected from a semester-long number theory class at a Northern California University. The class met for one hour ten minutes twice a week. Professor Hoffmann, the instructor, used group work as the predominant form of teaching in class for the first time in his career. He gave students worksheets of theorems and problems to solve in class with their groups. Students could find the proofs of theorems in the assigned textbooks, but Hoffmann encouraged them to "use [their] brains not the textbooks" to solve the problems. A weekly homework, consisted of writing the solutions to selected problems from the worksheets, was assigned to be submitted individually and graded. Students took a midterm and final exams.

The class included students from diverse ethnic backgrounds and mixed gender, 10 women and 13 men, the majority of whom were majoring in Mathematics for teaching. The African-American ethnicity was not represented in this class. Students composed their groups at their will and stayed in the same group by the third week of class. There were total of 5 groups of 4 or 5 members each. One group did not participate in the research.

Melissa and Bettie were selected for focal study, because both worked with four men in their groups and reported low self-efficacy at the start of the class. Melissa was majoring in Mathematics for teaching and Bettie in Mathematics for liberal arts. Melissa's group (GM) and Bettie's group (GB) had three vocal members and one predominantly silent member. GM's members were Robert (Math for teaching major, vocal), Tom (Math for advanced studies, vocal), Emil (Math for liberal arts, vocal), Tito (Math for teaching, silent). GB's members were Ted (Math for teaching, vocal), Jeremy (Math for advanced studies, vocal), John (Applied Math, vocal) and Boutros (Math for teaching, silent).

### Data Sources

Starting from the second week of class, four group's group work, including GM and GB, were *videotaped*. Students took *early and exit surveys* and submitted *individual memos* after every group session. They participated in an *early individual interview*, where they were asked about their motivations for majoring in mathematics, feelings about the mathematics discipline, history with mathematics classes in high school and college, experience with group work in classes, and first impressions about current group members. They also participated in *individual interviews by the end of the semester*, where they were asked about their understanding of number theory, confidence in the material, the changes throughout the semester in their learning methods and behavior in group work, and the roles their groupmates tended to play.

Students participated in SCNI interviews (Stimulated Construction of Narratives about Interactions; see El Chidiac, 2017) every other week. In the SCNI interviews, participants watched a video of their recent group session and commented on their social interactions. The SCNI interviews were conducted individually and within twenty four hours of the end of class. The SCNI sessions of GM and GB took place within five hours of the end of the class. GM participated in one and GB two out of the four SCNI interviews. Emil (GM) and John (GB) did not participate in any interview.

## Results

### Developing Two Types of Self-Efficacy

At the beginning and throughout the semester Melissa showed low levels of confidence in relation to her peers. She stated in the early interview on 9/17 that “every single person I've met has understood more than I have which is really you know, makes me feel very insecure about my decision to be a math major when everybody's you know - smarter than you.” During the exit interview Melissa showed a shift in her perception.

> Melissa: So, then I started contributing more […] cause at first I felt like I was not the smart one of the group, and I'm not. But, I also felt, feel now that I'm at least at a level playing fieldish, more at a level playing field of like, brain capacity [laughs] I guess, and um knowledge in general. Um, because I don't know sometimes I shock myself when you know, I get something and then like Robert doesn't get it or Tom doesn't get it and I'm just like why don't why don't you understand, it's this, come on now. You know. Um. But that makes, that makes me feel more confident in myself and it makes me want to participate more cause then you know, I'm not actually like helping them, but I'm helping them in my not understanding. (Interview 12/02)

Melissa developed a relative type of self-efficacy. She constructed a confidence not on her own mathematical ability, but based on noticing her groupmates were not as smart as she previously believed. Even at the end of the semester, Melissa felt she did not fully understand the material (“I'm helping them in my not understanding”). She started the class with an individualized disposition from her prior proof course, in which she “struggled” (Interview 9/17). She reiterated this perceived inability later in the semester, “I'm just really bad at writing proofs” (SCNI 11/05). We note that in both interviews, Melissa revealed an individualized disposition about her mathematical ability, namely *evaluating one's own mathematical ability in contrast to one's perception of others' abilities*.

Like Melissa, Bettie started the class feeling she was not smart enough. In the early interview on 9/22, she stated, “I feel like I'm not super um I don't know I guess smart so it takes me a while to understand things I have to see it done a couple times and like I have to do it a couple times to like completely fully understand it.” Unlike Melissa, Bettie's perception of her non-smartness depended on her slow learning processes and not her comparison to other people's abilities.

Like Melissa, Bettie expressed a struggle with proofs in previous classes. But unlike Melissa, she was satisfied of her learning in this number theory class. In the exit interview on 12/02, she stated, “for me I feel like theory in general is just like learning proofs and . I don't know . it's just been really difficult for me. But . uhh out of all the proof classes I've taken this is probably the most that I've . like . learned . I guess you can say. Cause a lot of the time I kinda just got by. and I feel like this one I'm actually understanding like . why.”

When asked about their learning methods, both Melissa and Bettie stated at the start of the class that they learn by memorizing. On 9/22 Bettie said she used to learn by “keep doing it, repeating it, just memorizing.” When asked how she prepared for the midterm, Melissa said she “wrote up cheat sheets,” looked through her notes and homework and “tried to memorize it all.” By the end of the semester, Melissa reported no change of learning methods (“I pretty much learn the same,” Interview 12/02). However, Bettie emphatically affirmed a move away from learning by memorizing towards learning by understanding. On the exit interview (12/02), answering whether she endured a change in her ways of learning, she responded as follows.

Bettie: "yeah definitely. I kinda just . not really change it but it made me realize what . my style of learning is . kinda thing. I would just study and I didn't really know what was beneficial and what wasn't. now I realize I need to read, obviously . I have to read through the book. I have to . like do the homework . like slo::owly at my own pace and like do it myself. and . um . that's like the only way I'm gonna retain anything or like know what I'm doing. Because before I would just like do the homework. but I wouldn't really . like know what I was […] I was just like copying and pasting . finding answer online . and writing it out and like hoping it was the right answer. but now that . I'm actually like reading the book, working with friends, like doing the homework, actually doing the homework myself, I just feel like this is just what I need to start doing."

Bettie and Melissa differentially boosted their mathematical self-efficacy: Bettie improved her learning methods, while Melissa only repaired her perception of groupmates' mathematical abilities. This differential improvement of self-efficacy was reflected in their exam achievements as well (Table 1). Given the self-reported significance of group learning and use of textbooks, we shall investigate them next. [For lack of space, we leave the study of textbook usage to the extended paper]

Table 1: Melissa's and Bettie's grades on the midterm and final exams

| | *Melissa's grade* | *Bettie's grade* |
|---|---|---|
| *Midterm exam* | 45% | 53% |
| *Final Exam* | 20% | 61% |

### Learning During Group Work

We watched the videos of GM and GB group sessions and coded the instances when Bettie and Melissa offered explanations (Figure 2), contributed mathematical ideas (Figure 3) and sought explanations (Figure 4) by group sessions over the semester. Melissa increased her participation in the group work after 10/15. This evidence corroborates her comment (above), her increase of confidence made her "want to participate more" (see quote above). Bettie's pattern of participation in group work remained low throughout the semester. Overall, Melissa participated more in group work than Bettie. However, the latter surprisingly showed more learning gain than the former. Why was Melissa's participation in group work not conducive to learning?

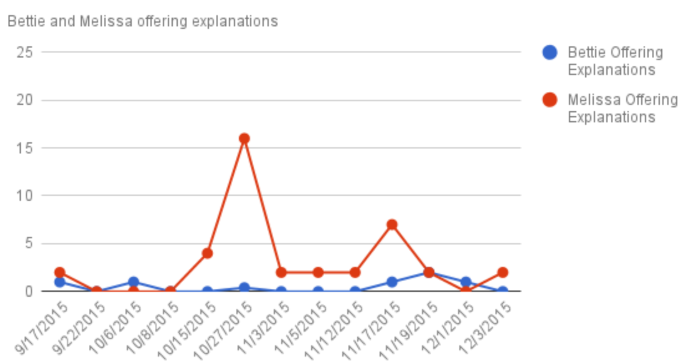


Figure 2: Instances of offering explanations for Melissa and Bettie over the semester.

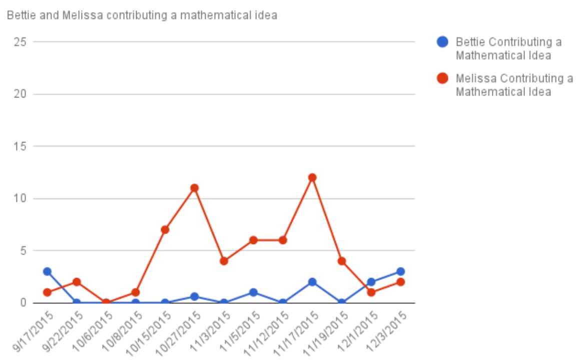


Figure 3: Instances of contributing mathematical ideas, for Melissa and Bettie over the semester

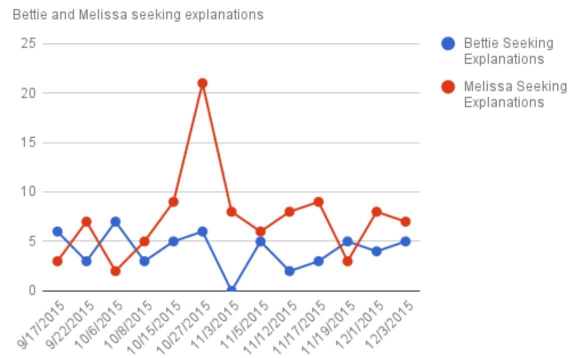


Figure 4: Instances of seeking explanations, for Melissa and Bettie over the semester

To answer this question, we looked at the individual interviews. During the early interview (9/17), Melissa was asked how she felt about the current group. She responded, "yeah I feel very focused um not necessarily like I know what's going on but I feel like I'm not thinking about anything else but what's in front of me on the worksheet and like trying to help […] more more than trying to figure out what's going on I'm trying to help my friends who know better like give them my ideas of what I might be thinking to help them put put it together because once they put it together then they can explain to me what's going on." Later in the same interview, as she was describing the role of her groupmates, she commented, "And so they [Tom, Robert and Emil] are kind of like the three main brains I feel like um and they all just bounce ideas off each other and I try to like this if they stop talking I just kind of like ask a question to get their brains going again because I mean they know way more than I do. I feel like. and then they're also in modern right now. Emil's taking modern and Robert's in modern and I'm pretty sure Tom is taking modern, Tito's taking modern right now. I'm just like over here like I'm still in linear. so they know a little bit more about what's going on." During the SCNI on 9/24, Melissa reiterates, "I don't know where to go from here I hope someone [small laugh] gets something and like, I normally would ask a question to someone, you know to clarify something and usually that gets their brains going, but […] I tried that, and nobody really responded to me, besides Emil."

At the beginning of the semester, Melissa activated a group disposition of helping her groupmates find the solutions and deactivated the disposition of understanding the materials. She took up the role of stimulating the brains of her groupmates because she thought they are more resourceful than she. Nonetheless, the helping group disposition echoed a similar individualized disposition. During the early interview on 9/17, Melissa confessed, "I feel like it's kind of hard for me to relate to [the courses in college] since I'm not going to be using it while I'm teaching I mean I'll have it in the back of my head you know for that one kid that's going to ask me 'well why?'" For Melissa, learning is conditioned by how knowledge can help others.

As Melissa spent more time in group work and realized she could be resourceful, she faced a new challenge: unresponsiveness from her groupmates. During the SCNI interview on 11/05, Melissa paused the video and commented, "oh number two I understood and I'm kind of pissed because, I shout out my idea and no one really listened to me because in the end that's kind of almost exactly what we ended up proving. which kind of pisses me off because I wish someone would have realized, I wish I could have said it better so that they would have understood what I meant." She reiterates this struggle at the end of the same SCNI and the exit interview.

> Melissa: I would see that they weren't listening to me […] So I had an idea so I would a- like, act dumb, and kind of you know, ask them about, you know well what about this, what if like we do this or this or this kind of thing. Well like, what about that, to kind of lead them in the direction that I'm already on. […] they can't be told […] how to think. […] So, I just like, ask a question that maybe has them change their perspective to understand what I'm trying to say. (Interview 12/02)

Bettie also faced lack of recognition from groupmates. In the exit interview on 12/02, she stated in a colloquy style, "if it has to do with arithmetic I feel like I'm just . I feel like . I can do it. like I. Maybe they don't take me as like serious so when I have the answer they're like "whatever like it's probably wrong." but I usually do get the right answer and I feel like "hah" like "told you.""

Unlike Melissa, Bettie did not attend to her groupmates. In fact, during the early interview, she thought her group had only four, instead of five, members, of whom she knew only Jeremy by name. During group work, she mostly worked alone, reading from the textbook and writing on her notebook. During the SCNI interview on 10/15, she confessed, "when I hear people talking and I don't understand I just zone them out because it confuses me more so I just like keep do . I just keep looking on my own." Later on in the same interview, she reiterated and generalized her group disposition, "that's basically all I do [in group work]. like when I'm in class. I just listen to what [my groupmates] are saying and look at the book . cause if I don't understand it then . when they're like talking . I don't know. so I just zone people out . until I look at it myself because . otherwise it just confuses me more." During group work, Bettie took up a group disposition geared towards building her own understanding of the mathematics. In effect, her dominant participation role was "seeking explanations" (compare graphs in Figure 2, Figure 3 and Figure 4). In the interviews, Bettie reported an individualized disposition of seeking to understand mathematical concepts. When she was stuck on homework, Bettie affirmed during the early interview on 9/22, "I go online […] yeah I work alone […] most of the time […] I just like to read over because I like to understand things cause like it's really frustrating when I'm just like copying work I have to really just like understand what I'm doing and why I'm doing it so I kind of just like to work alone because it takes me a pretty long time to figure out a problem."

## Conclusion

Both women, Melissa and Bettie, were not recognized as reliable resources by their groupmates. However, this stigma was more detrimental to Melissa than Bettie. Due to Melissa's group and individualized helping dispositions, the developmental path of her mathematical identity led her to take on the role of helping others to find answers to problems. This path was blocked twice throughout the semester. Melissa could overcome the first barrier, i.e. her perception that her groupmates are more resourceful than she, but struggled with grabbing her groupmates' hearing. In contrast, the developmental path of Bettie's mathematical identity went through coping with her slow learning processes in group. She was oriented towards understanding the mathematics, the disposition of which made her immune to others' recognition of her abilities.

Throughout the analysis, we showed that individualized dispositions undergirded all observed group dispositions. Dispositions that students carry from other contexts into the classroom are more significant for identity development than it is commonly being thought of.

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# First Results From a Validation Study of TAMI: Toolkit for Assessing Mathematics Instruction

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*Many researchers consider observation to be a 'gold standard' for measuring classroom practices since self-report surveys may be prone to bias. In this paper, we explore how the design of survey instruments and observation protocols affects the trustworthiness of the data collected. We describe our process of developing well-aligned observation and survey instruments in order to reduce sources of measurement error. We present results from a large-scale test of these instruments in 176 observations of 17 different math courses. Our results indicate that when survey instruments are designed to describe what happens in a course, rather than evaluate the quality of the instruction, and when those survey results are compared to observation protocols measuring teaching in the same way, self-report surveys are largely trustworthy.*

*Keywords:* Measurement, Instruction, Observation, Surveys

Describing and assessing instructional practices in science, technology, engineering, and mathematics (STEM) courses is a difficult undertaking. Various methods exist, each with their own advantages and disadvantages (American Association for the Advancement of Science, 2012). Two commonly used methods in RUME studies are classroom observation and instructor surveys. Observational data is collected by a neutral third party so it is sometimes considered more objective or 'accurate' than self-reported survey data, which may be prone to subjective bias (Ebert-May, Derting, Hodder, Momsen, Long, & Jardeleza, 2011).

However, observations have their own sources of measurement error. Observers often need significant amounts of training to ensure sufficient inter-rater reliability (IRR) – the degree to which different raters agree on their ratings of what they observe (Smith, Jones, Gilbert, & Wieman, 2013). Though IRR is usually used as a standard for judging quality of observation data through adherence to a particular protocol, IRR only captures whether observers apply the protocol consistently, and does not address other inherent sources of variability in observation data. Multiple observations of the same instructor are needed to make confident inferences about their teaching as a whole (Pianta & Hamre, 2009), since two or three class sessions may not represent the entire class (Hill, Charalambous, & Kraft, 2012). But if more observations are needed, this adds significantly to the time and cost of collecting observation data. This drives the need for validity comparisons (Hill, Charalambous, & Kraft, 2012): if researchers can confidently substitute a survey for observations, then they can increase the number of participants involved while decreasing the costs and time invested.

A number of studies have attempted to answer the question of whether survey data can be as trustworthy as observations, and they have come to different conclusions. Ebert-May et al. (2011) found that while most instructors said in surveys that they changed their courses to become more active and learner-centered, most were observed using traditional lectures and teacher-centered instruction. A similar study conducted in a K-12 context (Fung & Chow, 2002) found a mismatch between the teachers' conceptions of their teaching style and observed practices with teachers again overestimating the interactivity and student-centered characteristics of their teaching. However, college faculty members studied by Smith et. al (2014) were "generally self-aware" of how often they used methods related to lecturing and presentation.

If observations are considered more objective, mismatches are interpreted as 'inaccuracy' or 'bias' in self-report. However, it may be that mismatches are at least partially caused by a misalignment between survey and observation instruments. In this paper, we present results of our own validation study comparing an instructor survey and observation protocol. We first describe the framework and process we used to create two well-aligned instruments, and then present results from a large-scale study using them. We explore the questions:

1. What design choices affect alignment between surveys and observation protocols?
2. When using well-aligned instruments, in what ways does instructor self-report of instructional practices agree or disagree with observation data?

## Conceptual Framework: Types of Observation Protocols

Observation protocols are characterized along multiple dimensions (Hora & Ferrare, 2012). The two main dimensions are descriptive vs. evaluative and segmented vs. holistic. Descriptive protocols aim to simply capture or describe what is happening in a class, whereas evaluative protocols rate the quality of a class. Segmented protocols divide a class into short time segments, usually 2 or 5 minutes long, with coders recording features of the class during each period. Thus, the whole class is characterized by the sum and sequence of short segments. In contrast, holistic protocols aim to characterize the class as a whole. The observer may take notes during class then use the notes as evidence to rate the class across multiple criteria. Additionally, protocols may focus on the instructor or students, may or may not take subject matter into account, may require high or low inference by the observer, and may vary in the degree of structure.

These dimensions can be combined in many different ways, but the two main dimensions capture the largest differences between protocols. A descriptive protocol may measure how frequently a practice such as group work occurs using a segmented approach (e.g. group work occurred during 17 of 25 two-minute intervals) or a holistic approach (e.g. "about ¾ of class time"). Evaluative protocols may measure the quality of group work by asking how students engaged in group work or whether the group task was structured effectively. Again, that can be done in a segmented way (e.g., marking student engagement during each interval) or holistically (e.g., rating on a 0-4 scale whether "students were productive and engaged in group activities").

| | **Segmented** | **Holistic** |
|---|---|---|
| **Descriptive** | **TDOP** (Hora & Ferrare, 2013)<br>**COPUS** (Smith, Jones, Gilbert, & Wieman, 2013)<br>**RIOT** (West, Paul, Webb, & Potter, 2013) | **SPROUT** (Reimer, Schenke, Nguyen, O'Dowd, Domina, & Warschauer, 2016) |
| **Evaluative** | **PORTAAL** (Converse, Eddy, & Wenderoth, 2014) | **RTOP** (Sawada, et al., 2002)<br>**MCOP2** (Gleason, Livers, & Zelkowski, 2017)<br>**M-Scan** (Walkowiak, Berry, Meyer, Rimm-Kaufman, & Ottmar, 2014) |

*Figure 1. Framework for classifying observation protocols.*

In Figure 1, we characterize some common observation protocols used in STEM courses based on their main design features; individual items may fall into the other quadrants. For example, some items on the MCOP2 are more descriptive than evaluative. Segmented protocols tend to be very detailed and granular, whereas holistic protocols zoom out to capture a broader view. The choice of a protocol should match the goals for its use. For example, if the goal is to offer formative assessment for instructor growth, a descriptive protocol such as RIOT, which focuses on instructor/student interactions, offers a lower-stakes measure conducive to constructive discussion. In contrast, RTOP is more evaluative and may be viewed as judgmental rather than constructive.

Surveys items about instructional practices can also be classified as descriptive or evaluative. Surveys may ask instructors to describe and quantify their behaviors over a period of time (e.g. "How often did you lecture this semester?"), or ask them to reflect on and evaluate their own teaching based on criteria such as the quality of their interactions with students, the types of activities used in the course, or their perceived ability to explain difficult concepts.

## Methods

Now, we describe how we used this framework to design a descriptive instructor survey along with a well-aligned segmented, descriptive observation protocol. We then describe how we collected and compared data from both instruments in college math courses. The instruments are part of the Toolkit for Assessing Mathematics Instruction (TAMI) that we are developing.

### Development of the Survey Instrument

Our survey came out of prior work evaluating professional development workshops (Hayward, Kogan, & Laursen, 2016). Our goal was to assess initial changes in the use of particular instructional practices, not to evaluate how well instructors were using these practices, since these skills may take years to develop. So our questions asked instructor to report the frequency of use of different practices. We designed our survey to be administered shortly after the conclusion of a course and to use descriptive rather than evaluative items, asking "*what did you do*?" versus "*how well did you do it?*" For this project, we conducted think-aloud interviews to adjust the items and answer choices to better align with instructors' conceptualizations of their own practices.

On our final survey, instructors first report how frequently they use 11 different classroom practices commonly seen in college math courses including group work, whole class discussions, formal lecture, interactive lecture, and student presentations. Frequencies are measured with a 7-point scale with concrete descriptors from 'never,' to 'about once a month,' to 'every class.' Then, instructors report the duration of use for each of the practices they used: 'a few minutes,' '1/4 class,' '1/2 class,' '3/4 class,' or 'entire class.' Open-ended items ask instructors to describe patterns in practices or rare events (e.g. computer lab for the last 3 classes.) Finally, instructors supply text descriptions of 'lecture,' 'presentations,' and 'group work' in the courses.

### Development of Observation Protocol

**Background.** A 2011 paper by Ebert-May et al. is often used to argue that self-report is not 'accurate.' They compare biology instructors' self-reported practices to observations coded with the Reformed Teaching Observation Protocol (RTOP) (Sawada, et al., 2002). Many instructors reported using active learning strategies, but RTOP scores were more in line with lecture-based teaching. The authors concluded, as their title suggests, that 'what we say is not what we do.'

However, our own analysis of the RTOP instrument reveals that this protocol is not well aligned with the data collected through self-report. The RTOP is evaluative and holistic, whereas the self-reported survey items are more descriptive and behavior-oriented. We were curious how much of the discrepancy between observation and survey methods was due to a difference in what is being measured, rather than an 'inaccuracy' in self-report. In other words, were the answers really different, or were they comparing answers to two different questions?

**Design of our observation protocol**. Our existing survey functioned well for evaluation work, and internal consistency provided evidence that it was trustworthy (Hayward & Laursen, 2014). It is a descriptive, frequency-based behavioral survey, so we wanted a descriptive, segmented, behavior-oriented observation protocol to match. Measures in the TDOP (Hora & Ferrare, 2013) offered more detail than we needed, but the COPUS (Smith, Jones, Gilbert, & Wieman, 2013), a simpler modification of the TDOP intended for STEM courses, did not quite align with practices we saw in college mathematics courses, especially with its focus on clicker use.

We modified the COPUS by changing a few codes and adding others to better align it with practices we saw in mathematics classes and with survey items we previously developed through interviews with mathematics instructors. We incorporated the ICAP framework (Chi & Wylie, 2014) as a research-based measure of the nature of student engagement. We included some end-of-class holistic items that are evaluative and descriptive. Thus, while most of our protocol is segmented and descriptive, it also includes items from each of the other quadrants. The main portion aligns well with our survey items, but adding items from other quadrants allows us to analyze how misalignment may affect the comparison between survey and observational data.

### Sample

Our sample included 176 in-person class observations from 17 courses. Our average of 10.4 sessions per course is many more than is typical (1-3 observations per course), but was necessary to ensure that we obtained a truly representative sample (Weston, Hayward, & Laursen, 2017). The data included 4789 two-minute observations, or nearly 160 hours of observations carried out over two terms at three public universities in courses on algebra, calculus, geometry, statistics, and mathematical modeling. Class meetings were 50 minutes long, meeting three or four times per week, or 75 minutes long, meeting twice per week. All courses were on semester schedules.

### Reliability of Observations

In piloting our observational protocol we assessed inter-rater reliability (IRR). Overall IRR was high, with raters agreeing on 93% of their observations over each two-minute period and varying only modestly by the type of item. Modest variations were also found for how well raters agreed when rating activities for different teachers, from 91% to 96% depending on which teacher was observed. These results are on par with those of other published protocols.

### Analysis Methods

Comparing survey to observational results was complicated by differences in the frequency of measurement, with surveys given once at the end of each term, and observations taking place multiple times throughout the term. To make a fair comparison, we aggregated observations at the course level by taking averages across all classes observed. We also aggregated within similar types or formats of classes such as classes primarily devoted to lecture, group work or a mix between these formats. Aggregating this way makes a fair comparison to survey items that ask instructors to estimate the proportion of time spent in classes "*when you used this activity.*"

We made two types of comparisons. First was the comparison of the instructors' report of average amount of time within each class devoted to specific activities (such as lecture) compared with the average observed time devoted to this activity. Observational averages took values between 0 and 100%; survey values asked teachers to estimate rougher proportions of class time spent doing the activity (e.g., "entire class," "¾ of class"). For this analysis we used a fairly liberal criterion and considered bivariate points a match if the observational value fell between the two nearest boundaries for survey proportions – these 'match' ranges are represented as vertical green bars in Figures 2-4. For instance, if an instructor estimated he lectured ½ of the class, any observed value between the next nearest survey responses of 25% and 75% was considered a match. Lower and higher boundaries were set at 10% and 90% of class time. Analysis with the full data set will use correlation coefficients and other tests of congruence such as the Kappa coefficient.

Second, we created an interactivity index based on the number of questions teachers and students asked during lecture. To create a three-point scale aligned with our three categories, "formal," "some interaction," and "interactive," we counted frequencies for six question/answer types – 2 for students and 4 for instructors. Each item was scored as its tertile (1,2, or 3) of the frequency distributions. We averaged these scores, and again split into tertiles for the final index.

**Results**

We compared observation and survey data from all participants who were observed in-person, totaling 176 observations of 17 courses, although most comparisons used 13 or 14 courses depending upon which activities were reported. Results from this analysis are presented below. We are currently integrating data from 141 additional observations from 16 courses observed via video camera and results from the full analysis will be available by February.

We first examined the match between survey responses and observational data for lectures. The survey question about lecture asked: "*On average, when you used this method, did you use it: Entire class, ¾ class, ½ class, ¼ class, a few minutes.*" We asked about three types of lecture: formal (little or no question or answer), some interaction, and frequent interaction. We compared the dominant mode of lecture reported in surveys to the observed averages of time spent lecturing in classes (Figure 2). For the most part, instructor estimates were aligned with what we observed. Four out of 13 cases were considered misclassifications, for a "true" classification rate of 69%. However, two of these errors were very near the classification boundaries. One instructor drastically underestimated the amount of lecture used, relative to what was observed.

We also compared survey instructor ratings of their own lecture interactivity to observed interactivity in classes (Figure 3). The tertiles for the interactivity index were 1–2, 2–2.5, and 2.5-above. Again we saw moderately high congruence between how instructors rated the interactivity of their courses and observed interactivity. Three out of 14 teachers highly overestimated the interactivity in their classes, and one teacher slightly underestimated the interactivity of his/her teaching. Overall accuracy was 69%.

We also compared averages for instructors' reported use of group work (Figure 4). We found that most instructors were at the extremes – either they mixed a fairly small amount of group work with lecture or other activities, or devoted the whole class to working with groups, usually on one day of the week devoted to recitation. Two instructors slightly overestimated the amount of time their students spent working in groups, one slightly underestimated, and one made a large overestimation. Overall accuracy was 71%.

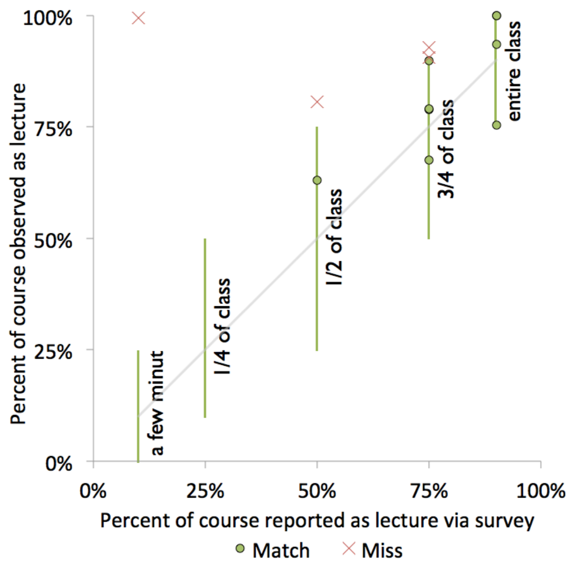


*Figure 2. Comparison between survey and observations of lecture.*

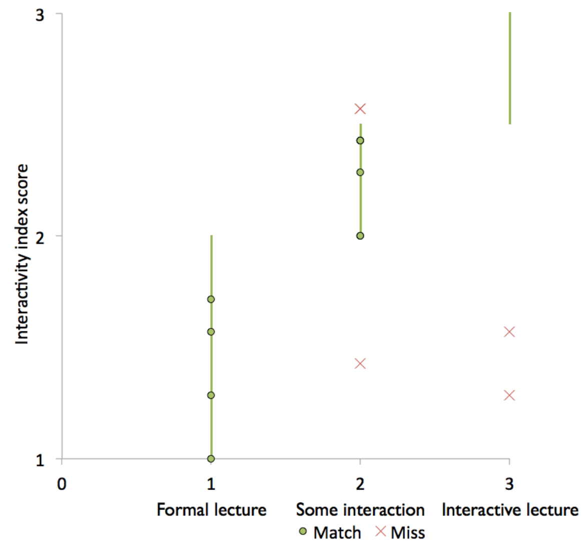


*Figure 3. Dominant lecture mode defined by survey and interactivity index from observations.*

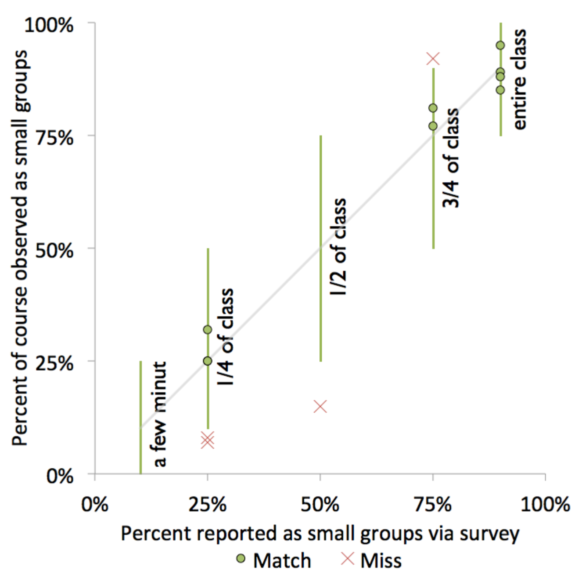


*Figure 4. Comparisons between survey and observations of group work.*

## Discussion

We made three validity comparisons with our initial data. Overall, it seems that when survey and observational measures are aligned, instructors' self-reported practices are aligned with observation data. For the two comparisons of proportions of time spent lecturing and in structured group work (Figures 2 and 4), most instructors seem to have an accurate idea of the proportion of class time spent on each activity. These descriptive comparisons are fairly straightforward, and instructors likely remember their basic lesson structure over the course of a term. Not surprisingly, some underestimate the amount of class time they spend lecturing, and overestimate the time students work in groups. However, most of these differences are relatively small and are related to extreme values; at the upper ends, instructors tended to underestimate and at the lower ends, they tended to overestimate. Only one participant had a large discrepancy, which may be due to differences in how our definition of “lecture” differed from the instructor’s.

Estimates of teacher-student interactivity (Figure 3) were also fairly accurate in terms of the number of matches. However, the discrepancies are large; those who reported the most interaction were some of the lowest-rated in observations. These results are interesting when interpreted through the design framework. Our interaction index is a norm-referenced measure. This means ratings are based on comparisons to other courses in the dataset rather than to an outside, objective standard. When we collapsed the three types of lecture on the survey (formal, some interaction, and interactive), self-reports aligned well with observations. However, when we used a more evaluative approach by comparing the type of lecture with our interaction index, discrepancies were large. So while instructor self-report was quite accurate with strictly descriptive measures (i.e. duration of lecture or group work), there were greater discrepancies with this more subjective, evaluative index. Past studies claiming that instructors were not ‘accurate’ in self-report relied heavily on evaluative measures, and our results suggest that using descriptive instruments instead of evaluative may help reduce these discrepancies.

Although observations are commonly thought to be objective, it is impossible to remove all forms of bias. We found that using segmented, descriptive items helps to reduce bias. However, protocol designers still must decide how to define items. Their perspectives bias what ‘counts’ for items. For example, when coding question and answers, we designed our protocol to only count when the instructor provided a real opportunity for students to respond. Many times, instructors asked rhetorical questions, or answered their own questions so quickly that students had no opportunity to respond. So we did not count these as questions. It is entirely possible that instructors frequently used these types of questions, but our coding would reflect very little interaction. The ‘error’ for those who drastically overestimated the interactivity of their lectures relative to our observations may originate in our definition of what counts as a question.

Despite multiple claims that self-report is not ‘accurate,’ the issue of trustworthiness is much more nuanced than how well it compares to observation data. Survey data may be prone to self-report bias, but there are also many sources of variation or error in observation data. These include observation protocol and survey design alignment, coding definitions of what ‘counts,’ and variability in day-to-day activities and representativeness of the observation sample compared to the whole course. Our results suggest that when survey and observation instruments are well designed and properly aligned, surveys may be a trustworthy, efficient, and less costly method of measuring teaching practices.

It remains an open question whether it is possible to use a survey to measure changes in teaching practice following professional development interventions. Issues of bias increase when instructors are expected to change their practices, and some may consciously or unconsciously overestimate the time spent on inquiry-based activities, and underestimate their time in teacher-centered lecture. Survey items must also be sufficiently sensitive to capture differences before and after the intervention. Future work should test the surveys in such intervention conditions.

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**A Department-Level Protocol for Assessing Students' Developing Competence with Proof Construction and Validation**

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***Abstract:*** *This methodological paper describes a protocol for assessing the development of students' competence with proof, created by the assessment committee within the Department of Mathematics at Western Michigan University. The assessment protocol we describe evolved over a period of 20 years and aims to collect information that is meaningful and actionable for improving mathematics instruction within the department. While there are several unique features of Western Michigan University that have created a context in which such work can be undertaken at the level of the department, we believe that this case will be of interest to mathematics departments seeking to find ways to measure their students' developing competence with proof.*

**Keywords:** Proof construction, Proof validation, Assessment

## Introduction

The role of mathematical reasoning and proof in the undergraduate major's mathematical training is crucial. Becoming skilled at constructing mathematical proofs requires the mastery and coordination of a number of different skills and the creative ability and mental dispositions to bring all those skills and the appropriate content knowledge to bear on a particular statement. These skills and abilities can be roughly divided into two categories – comprehension of an argument and construction of an argument, with validation of a purported proof involving skills from both categories.

Early work on proof construction and comprehension established different classification schemes students used to understand mathematical proofs and typologies to classify student generated proofs (Balacheff, 1988; Harel and Sowder, 1998). More recently, Mejia-Ramos et al. (2012) developed a multidimensional model of that revealed the complexity of proof comprehension. They present seven aspects of understanding a proof and then generated possible items that could enable teachers/researchers to assess students' understanding of these facets of proof comprehension. The Mejia-Ramos et al. model incorporated many of the aspects identified by the Selden and Selden (1995, 2003) in their work on student validation of purported proofs.

Successful construction of a proof requires the coordination of skills related to logic, proof structure and types of argumentation, content knowledge, and creativity. Atwood (2001) identified seven obstacles in writing proof – three related to beginning the process and four to completing the process. Thus far, much work on proof construction has focused on course level interventions to improve student abilities to construct proofs (e.g., Selden, Benkhalti, & Selden, 2014). In terms of work on assessing student proof attempts, Andrew (2009) describes a "Proof Error Evaluation Tool (PEET)" that would help instructors provide consistent feedback to students regarding typical errors. Andrew identified two main categories of difficulties students had in writing proofs – eight types of errors related to proof structure and six related to conceptual understanding. He suggests that the tool be used as a rubric for the instructor and as a way of making the assignment of writing a proof more transparent to the student by providing them with the PEET as a reference when writing proofs and when examining the feedback provided by the instructor. Some of the "errors" identified relate less to the validity of the student's argument and more to the superficial issues that influence the readability of the argument (e.g. legibility of the writing, the lack of a diagram to guide the reader, lack of succinctness in argument).

Despite the growing literature on the difficulties students face in constructing and validating arguments, a challenge in the literature remains creating a way to define and track the development of proof competence over longer periods of time and experience, such as over the course of one's undergraduate program of study (Stylianides, Stylianides & Weber, to appear). It is this challenge that the protocol we present in this paper aims to address.

**Development of a Protocol to Assess the Longitudinal Development of Proof Competence: The Case of Western Michigan University**

Western Michigan University Mathematics Department is a medium-sized department within a public university with Carnegie classification of High Research Activity. Assessment has long been a priority for the university and the department. Several years ago, in response to calls at the university level, all departments were asked to identify one student learning objective that they would focus on collecting data about and improving with respect to. The mathematics department coalesced around the learning objective

*"Students will have the ability to detect invalid arguments and construct different types of valid mathematical arguments at an appropriate level of sophistication."*

In constructing an assessment protocol around students' development of proof competence, guided by the literature on proof, the assessment committee chose to approach the task of tracking students' developing proof competence by focusing on these two, distinct, but related competences of proof construction and proof validation.

*Task Criteria*

The first step in putting our department-wide assessment protocol in place was to find candidate tasks. We had several criteria for the sort of tasks we needed to create. The tasks needed to be suitable for students at many levels (i.e. the content of the task should engage high school math and not content students would not have experience with until later in their programs). At the same time, the task needed to allow us to see growth in sophistication of both proof approach and growth in sophistication of ability to communicate mathematically. We were particularly interested in tasks that would submit to several different approaches. Finding tasks that worked well for our purposes took several iterations and pilots. In constructing items to assess our dual objectives of proof construction and validation, we used the data from our proof construction task to generate sample arguments that became the basis of our proof validation task.

*Implementation Cycle*

We collect data from all students in our program every semester. In the fall semesters, we administer a proof construction task and in the spring semester students work on a proof validation task. Our current protocol involves a two-year cycle of tasks. We knew that we did not want to give the *same* exact task every year because students would begin to know the task and be less generative in their methods. Our data collection efforts are aimed at students in our program (e.g., math majors). However, in many cases, a student may not declare a mathematics major until later in their undergraduate career. Thus, in order to collect data on math majors' proof development, we also sample our lower division courses broadly and collect data from all students enrolled in our courses. The timing of the assessments is also deliberate. We aim to conduct the fall assessments early in their program before having the influence of particular coursework. Thus, we aim to administer the fall assessment within the first weeks of the fall semester. We made a conscious decision to administer the spring assessment as late as possible in the spring semester to allow the maximal time for growth over the course of a single year.

*Assessing Performance: The Development of Proof Construction Rubric*

Generating the data on the proof construction and validation tasks led to the need to have a systematic way to score and interpret the data. The validation tasks are multiple choice (students evaluate several arguments that are given to them) and are thus easy to score and interpret. The proof construction task, however, presents many more issues with respect to scoring and interpretation. Discussions at the department level about what was valued in assigning codes to student work led to discussions about how to handle/interpret issues such as how students communicated their arguments (symbolic versus with words) or whether written work revealed that students had the kernel of the idea of a proof, but were not yet able to complete it. Engaging the faculty in looking at students' proof work was extremely generative in that it revealed interesting and significant differences among faculty in their expectations and interpretations of student work. Thus, the effort to articulate what was valued by faculty in students' proof work and to develop a common language for making consistent decisions about student-generated proof led directly into the generation of the proof rubric. After many iterations and sessions that involved testing the emerging rubric against student data collected in past iterations, as of Fall 2015 we had a working prototype of the protocol.

The rubric that the department is currently using can be found in the appendix. As an overview, we point out that the rubric can be thought of in three main sections. The first section of questions concern whether or not the proof was valid and whether or not it was communicated at a very high level. We consider an answer of yes to these two questions to be the "high bar" that should be reserved only for the papers that meet our top expectations. If the paper receives a yes to these two questions, the following questions on the rubric do not need to be answered. The next section of the rubric contains questions that pertain to establishing whether markers of minimal growth towards proof writing competence exist in the student work sample being scored. In contrast to the first section, we see this section as a kind of "low bar" that lets us see whether the student's proof attempt contained initial chains of argument or evidence of mathematical reasoning. The final section of the rubric pertains to specific challenges or hurdles to writing a valid proof that are common reasons why the proof sample did not rise to a "yes, yes" profile. These include questions related to whether the proof sample contained an algebra error, or whether students made unwarranted assumptions, or did not provide sufficient evidence for the claims they were making in the argument they were presenting.

*Faculty Involvement in Scoring*

Collecting longitudinal data on mathematics majors' developing proof competence is a large endeavor. Our aim was to create a data collection and analysis protocol that would be sustainable over the long term in the department and that would inform conversations about how to improve curriculum and pedagogy. Thus, a high amount of faculty involvement across the entire department is necessary for the ongoing success of the protocol. As we geared up to administer the assessment to our focal classes in Fall 2015, the assessment committee held faculty development sessions to train faculty on the interpretation and use of the common rubric. The faculty development sessions involved the use of sample student work generated in the pilot phase and scoring it using the rubric (then coming together and discussing discrepancies in interpretation). The assessment committee created open times for faculty to meet and discuss scoring with committee members. Such opportunities were important as qualitatively analyzing and coding student work is a practice that is not familiar to many mathematics faculty members (who may be more familiar with grading student work solely for correctness and assigning points).

*Assignment Protocol and Resolving Discrepancies in Scoring*

Early on in our assessment work, the task of scoring student work samples fell on the department assessment committee. As mentioned above, in order to move towards a more sustainable model, we have sought ways to fairly distribute the work of scoring. In our most recent iteration of the two-year cycle, we used the following assignment procedure for scoring the tasks. Lower level courses were assigned to be scored by one person (not the instructor). This is because we were aware that we were casting a wide net in collecting this data and that much of the student work generated would not be by students who ended up pursuing mathematics majors. For any course aimed specifically at mathematics majors, we decided to have two independent scorers (not the instructor). Our general practice was to have a member of the assessment committee be one of the two independent scorers for papers from upper level courses. Once assignments had been made of which faculty would be scoring (roughly an equal number distributed to each faculty member, with the committee having more responsibility), we sent faculty members a link to the scoring rubric in google forms. This allowed for a master spreadsheet of all proof scores to be generated automatically.

Having multiple people score the upper level student papers opened up the possibility that individuals will diverge in their scoring of the student work. For each person marking the paper, the assessment committee generated a spreadsheet (directly from google forms) that included the student's identifier, the person's name who marked the paper, and then entries for every element of the rubric (displayed in 1's for yes and 0's for no). Then, for each statement in the rubric, the assessment committee reviewed differences among the scores. Electronic copies of all exam papers by class were saved to a common workspace and the committee went directly to the pdfs of the student work in order to discuss the meaning of discrepant scores. When two committee members were involved in marking the paper, it was possible to have an immediate discussion that resulted in resolution of the discrepancy. If a scoring discrepancy involved a faculty member not on the committee and did not appear to be an obvious error, the committee assigned .5 to each score given instead of assigning a 1 or a 0.

*Reporting the Findings*

Currently, analyses presented to the faculty of student results on the assessment (usually presented the semester after data is collected) have included a cross sectional analysis of students' performance on particular competence within the rubric (e.g., "X% of students in Course Y or below were able to produce valid arguments on the assessment item.") Over time, we are interested in reporting patterns in longitudinal for students in our program.

*Discussion*

In the above sections, we have described in detail the protocol that we use to assess the development of students' competence in generating and validating proofs. There are a number of issues that are under ongoing discussion. While the development of proof protocol that we have put in place and are implementing is powerful, there are some limitations of the choices that we have made in developing our protocol. For example, one thing that we would like to learn more about is how students work to prove statements that are more closely related to the content that they are learning in their upper division coursework. By design, the statements that students engage with in our longitudinal assessment needed to be independent of the coursework that students take in our program. We are exploring the possibility of augmenting our protocol with course-embedded assessments that would be asked of students as part of their normal coursework but collected as part of the departmental profile of their developing proof competence. Other future directions include conducting follow-up interviews to establish alignment between scores on students' written work and what we assess as their understanding in an interview setting in which we can probe their thinking. With respect to faculty development, interview data may be useful for engaging faculty in analyzing student work. The topic of math faculty development around the pedagogical implications of what can be learned from student work (including the student work generated by our proof assessment protocol) is a topic for ongoing investigation.

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## Appendix: Proof Construction Rubric

***Rubric Section 1***

**Question 1 -- Is the proof valid?** A valid proof should be free of algebraic errors, unjustified claims, missing cases, and the imposition of additional hypotheses. Students who use the fact that a^2-a+1 has no real roots must justify this claim.

**Question 2 -- Does the student sufficiently communicate their ideas (whether correct or incorrect)?** Is it easy to follow their line of thinking or interpret what they have done? For example, if a student wrote "I don't remember what a reciprocal is, but if I did, I would assume that there is a real number such that the sum of it and its reciprocal equals one," it would be coded "yes." If the student is only reiterating or clarifying the problem, then code as "No."

***Rubric Section 2***

This section should be marked only if the student did NOT write a valid proof ("No" on Question 1).

**Question 3 --- The student appears to understand that the problem is about a + 1/a compared to 1** The rest of their work, if any, can be at any level. A student who writes something like a * 1/a = 1 did not correctly interpret the problem.

**Question 4 -- The student engaged with the problem, doing some work (possibly incorrect) beyond just interpreting or rephrasing the statement** There is some evidence that the student engaged with the problem and persisted in their attempt. For example the student tried out specific numbers, sketched a graph, or did some algebraic fiddling (beyond rote steps like adding or simplifying fractions without a comparison to 1).

**Question 5 -- Some mathematical reasoning exists (possibly built from incorrect assumptions or definitions)**

**Question 6 -- The student used words to express at least one complete idea in support of their argument** If the student is only reiterating or clarifying the problem, then code as "No."

***Rubric Section 3***

This section should be marked only if the student correctly interpreted the problem ("Yes" on Question 3).

**Question 7 -- The student made an algebraic error, or their use of language or notation interferes with progress** These errors could range from a simple sign error to conceptual errors such as extending the zero product property to another integer.

**Question 8 -- The student ignored cases or imposed additional hypotheses (explicitly or implicitly)** For example, a student may successfully complete the proof under the assumption that a is strictly positive. Or a student may cover the case $a < 0$ and after this implicitly assume that a is nonnegative, for example stating that if $a < 1$ then $1/a > 1$. Also code proof by example(s) as "yes."

**Question 9 -- The student made claims that they did not attempt to justify** Examples of "yes": a student wrote "the equation $a^2 - a + 1 = 0$ clearly has no real solutions and so there cannot be any such real number" or arrived at the equation $a^2 - a + 1 = 0$ and then simply stated "therefore there cannot be any real number such that the sum of it and its reciprocal is 1." Attempting to justify a claim, but doing so incorrectly or incompletely, would be coded "no."

**Question 10 -- The student introduced a framework that *could* be used to write a successful proof** Manipulating the expression $a + 1/a$ (vs. the equation $a + 1/a = 1$) is exploratory work and not a framework. Introducing a multivariable scheme is unlikely to lead to a successful proof, so also rate this as "No." For a "yes", unresolved issues or gaps in the proof (including missing cases) could be resolved using skills that the student has either demonstrated already, or which are easily accessible to a calculus student (e.g., testing trivial cases). Simply testing the values $a = 1, 2, 3$, etc. would be coded "No."

Proof Norms in Introduction to Proof Textbooks

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*We present a textual analysis of three of the most common introduction to proof (ITP) texts in an effort to explore proof norms as undergraduates are indoctrinated in mathematical practices. We focus on three areas that are emphasized in proof literature: warranting, proof frameworks, and informal instantiations. Each of these constructs have been connected to students' ability to construct, comprehend, or validate proofs. We carefully coded all the proofs and supplemental material across common sections in the textbooks. We found that the treatment of proof frameworks was inconsistent. We further found that textbook proofs rarely used explicit warranting and informal instantiations. We conclude by reflecting on the impact of inconsistent proof norms and unsubstantial focus on supportive proof components for students in ITP courses.*

*Keywords:* Proof, Proof Norms, Textbook Analysis, Warranting, Proof Frameworks

Proof is an essential aspect of the mathematical discipline (de Villiers, 1990; Hanna, 2000; Hersh, 2009; Rav, 1999), and as such proficiency in all areas of proof is important for students in undergraduate mathematics programs to gain. In order to meet this goal, university mathematics departments offer introduction to proof (ITP) courses to help students learn about the argumentative process of proof specific to mathematics. One of the main objectives of the ITP course is to improve the undergraduate student's ability to construct formal proofs. Despite this objective, numerous studies have documented the difficulties that students have in making the transition to advanced mathematics and in their ability to construct formal proofs (e.g. Moore, 1994; Selden & Selden, 2003; Weber & Alcock, 2004). In spite of this research and the hypothetical function of the ITP course, until recently little focus has been put on the nature of these classes.

In this study, we focus on the intended curriculum of ITP courses as reflected in textbooks. Curricular materials provide a significant factor in the learning and development of reasoning and proof (Konior, 1993; Stylianides, 2014). As such, the aim of this study is to understand both the implicit and explicit messages that ITP textbooks send to the reader about the nature of mathematical proof. Specifically, we analyzed three research-based aspects of proof: proof frameworks (Selden & Selden, 1995, 2003; Weber, 2009), explicit warranting (Alcock & Weber, 2005; Inglis & Mejiá-Ramos, 2008; Pedmonte, 2007; Toulmin, 2003), and diagrams as well as other informal reasoning (Samkoff, Lai & Weber, 2012; Weber & Alcock, 2004). We found that ITP proofs often lacked consistency in terms of frameworks and warranting, and generally overlooked diagrammatic and other informal reasoning.

## Theoretical Framing and Background

Underlying our work is the assumption that curricular materials reflect and impact the nature of a mathematics course. In general, we argue in alignment with Zhu and Fan (2006):

> ...textbooks are a key component of the intended curriculum, they also, to a certain degree, reflect the educational philosophy and pedagogical values of the textbook developers and the decision makers of textbook selection, and have substantial influence on teachers' teaching

and students' learning (p. 610).

While not a perfect substitute for the classroom, textbooks provide a substantial set of example proofs that students experience. Furthermore, writers of these texts are implicitly endorsing a set of norms for proofs in this setting. In this way, textbooks provide an artifact for exploring the norms for ITP-level proofs and a substantial resource impacting students' ultimate learning.

We also frame our work in terms of proof as a social construct. Research has established that what constitutes a valid proof varies based on context and even from individual to individual (e.g. Moore, 2016; Weber, 2008). In this way, we may expect that textbooks may reflect similar variation in proof norms. We focus on three areas of norms: proof frameworks, warranting, and diagrams and other informal reasoning.

**Proof Frameworks**

Selden and Selden (1995) introduced the construct of *proof framework* to capture the "top-level" structure of proof that comes directly from unpacking a statement. For example, if the mathematical statement to be proven is, "For all integers $x$, if $x$ is odd, then $x + 1$ is even integer" the *complete* proof framework for a direct proof of this this statement might look like the following:

*Proof:* Let $x$ be an integer. Suppose $x$ is odd…
…then $x + 1$ is an even integer. ■

Notice this is a direct proof. A contrapositive proof framework would unpack the contrapositive statement.

The literature reflects that ability to produce an accurate proof framework has a relationship to other activities such as constructing, validating, and comprehending proof (e.g. Mejiá-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; Selden & Selden, 1995, 2003; Weber, 2009). Both Selden and Selden (1995) and Weber (2009) found that many students do not typically check proof frameworks and may lack awareness as the essential role of an appropriate proof framework in a proofs' validity.

**Warranting**

One of the most fundamental acts in mathematics, especially in proof and proving is that of warranting: justifying assertions (see Hanna 1991, 1995; Healy & Hoyles 2000). According to Toulmin (2003) arguments, and by extension proofs (see Alcock & Weber, 2005; Inglis & Mejiá-Ramos, 2008; Pedmonte, 2007), have a formal structure which is defined by the interplay of at least three fundamental constructs; that of *data*, *warrants*, and *claims*. In the tradition of Toulmin (2003), a claim is a statement or assertion of what is true, the data is the grounds by which the assertion of truth is made, and the warrant justifies the connection between the data and the claim by, for example, invoking a definition or rule. For instance, we can use Toulmin's scheme to analyze the following statement: "Since x is odd, then by the definition of odd, $x = 2n + 1$ for some $n \in \mathbb{Z}$." The claim being made in this instance is that "$x = 2n + 1$ for some $n \in \mathbb{Z}$," the data on which rests the truth of this assertion is "Since $x$ is odd," and the warrant that connects the data and claims is "by the definition of odd."

Warranting, explicitly connecting data and claim, is an important ability for students to learn. Alcock and Weber (2005) asserted that, "Failure to consider the warrants used in a proof will not only cause students to be unable to validate proofs reliably, but… can also prevent them from gaining conviction and understanding from proofs presented in their classrooms" (p. 133). Furthermore, Alcock and Weber claimed that instructors for proof-oriented course do not commonly discuss warrants and that textbooks are also infrequent in explicit language on the subject. Within proofs, warrants are often left implicit. The ability to infer these implicit warrants is an essential skill for understanding proofs in advanced mathematics and should be part of students' enculturation into proof based mathematics (Weber & Alcock, 2005).

### Diagrams and Other Informal Reasoning

Informal reasoning plays an important role in the learning and construction of proofs (Fischbein, 1983; Hanna, 1991), moreover, it is the multifarious interplay of these intuitions with the rigorous and abstract aspects of mathematical ideas that are the cornerstone of advanced mathematics (see Mariott, 2006). Informal reasoning may take many forms including that of exploring examples or diagrams.

Diagrams and other example based instantiations can aid students in understanding a statement, and gaining a level of conviction in a theorem and its proof (see Alcock & Weber, 2008, 2010; Samkoff, Lai, & Weber, 2012, Weber & Mejiá-Ramos, 2015). As moving from informal to formal reasoning is an important factor in the creation of mathematical ideas (Raman 2003; Weber & Alcock, 2004), we explored the degree to which textbooks leveraged informal instantiations including: using numbers to explore computational cases (e.g., substituting values as test cases), building example sets to explore set interactions (e.g. unions, intersection, Cartesian products), or testing the behavior of specific set members under a particular mapping.

## Methods

### Textbook Sample

In this study we analyzed three textbooks (see Table 1) which are among the most used textbooks for the standard ITP course in the United States. According to David and Zazkis (2017), these textbooks represent roughly 27%[1] of the market share for textbooks used by departments and instructors for the ITP course. All other standard ITP texts had less than a 4.2% market share.

*Table 1. Introduction to Proof Textbooks*

| Title | Publisher | Authors | Market Share | Year |
|---|---|---|---|---|
| Mathematical Proofs: A Transition to Advanced Mathematics (**Book A**) | Pearson Education | Chartrand, Polimeni, & Zhang | 11.3% | 2013 |
| A Transition to Advanced Mathematics (**Book B**) | Brooks/Cole | Smith, Eggen, & St. Andre | 10.6% | 2011 |
| Book of Proof (**Book C**) | Richard Hammack | Hammack | 4.9% | 2013 |

We selected the sections and chapters that aligned with content found in a typical Standard ITP course: formal logic, number and set theory, relations, functions, and cardinality of sets (David & Zazkis, 2017). We grouped the sections into *introductory material* which consisted of all sections prior to the three *content specific* sections of functions, relations, and cardinality of sets. For each of the pertinent sections and or chapters of the textbooks, we read the vast majority of proofs, as well as any explicit commentary that each book provided about the construction of said proofs. All told, we analyzed arguments for 345 mathematical statements, of which we identified 280 were proofs or at very least the outline of a proof. See table 2 for a breakdown of number of proofs by section in each text.

[1] David and Zazkis (2017) shared information on 154 universities use of textbooks, 12 of which used lecture notes only, meaning that 38 of the remaining 142 classes used one of these three texts.

*Table 2. Mathematical Statements and Proofs*

| | Book A | | Book B | | Book C | | Total | |
|---|---|---|---|---|---|---|---|---|
| Section | Statement | Proof | Statement | Proof | Statement | Proof | Statement | Proof |
| Intro. Material | 56 | 50 | 85 | 64 | 46 | 46 | 187 | 160 |
| Relations | 9 | 9 | 17 | 15 | 3 | 3 | 29 | 27 |
| Functions | 10 | 10 | 34 | 23 | 6 | 4 | 50 | 37 |
| Cardinality of Sets | 21 | 17 | 44 | 30 | 14 | 9 | 79 | 56 |
| Overall | 96 | 86 | 180 | 132 | 69 | 62 | **345** | **280** |

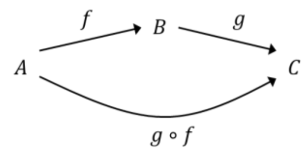
Each of the three textbooks were analyzed using thematic analysis (Braun & Clarke, 2006). The analysis began with open coding proofs from Book A within the categories of proof frameworks, warranting, and diagrams/informal reasoning. This set of codes of was condensed and categorized. The robustness was tested in the next text: Book B. These coding scheme was further expanded when new codes emerged from this text. For the purpose of consistency and in being faithful with the method of constant comparison, prior to coding Book C, we recoded Book A using the full set of codes. Once the second coding of Book A was completed and codes were refined and condensed, coding of Book C began, and it was at this point that saturation occurred and the coding cycle ceased as no new codes arose from coding Book C.

**Analytic Framework**

In this section we share the most relevant sections of our coding framework. Proof frameworks were coded as either complete, incomplete, or non-existent. All explicit warrants were identified based on their type: definition, theorem, or algebra. We identified two types of informal categories: diagrams (visual representations) and informal reasoning. See Table 3 for elaborations of the codes,

*Table 3*. Analytic Framework for Coding Proofs

| Category | Code: Description | Example |
|---|---|---|
| Proof Frameworks | **Complete:** A proof has both the antecedent and consequent of the original statement represented in accordance with the proof method being employed<br><br>**Incomplete:** A proof which only has one or the other of the antecedent or consequent represented in accordance with the proof method being employed.<br><br>**Non-Existent:** A proof which has neither the antecedent nor consequent represented. | For all integers x, if x is odd, then x + 1 is even integer.<br><br>*Proof:* Let $x = 2n + 1$ for some $n \in \mathbb{Z}$…<br>…then $x + 1$ is an even integer. ■<br><br>Code: Incomplete (the antecedent is not explicitly unpacked to: "Let *x* be an integer." |
| Warrants | **Definition**: The authors use a definition, property, axiom, or other fact *accepted* in the text as a warrant to connect some data and claim within a proof. | "By the distributive property we have that $3 \cdot (x + y) = (3 \cdot x) + (3 \cdot y)$…" |

| | | |
|---|---|---|
| | **Theorem**: The authors use a theorem, corollary, lemma, or other fact *proven* in the text as a warrant to connect some data and claim within a proof. | Code: Algebra (reference to the field property distributive) |
| | **Algebra:** The authors use an algebraic field axiom as a warrant to connect some data and claim within a proof. | |
| Informal | **Strategies**: Any instance where the authors present syntactic strategies for proof production. |  |
| | **Semantic**: Any instance where a proof or its supplementary material references a diagram or present other semantic explorations whether to simply clarify an idea or as a means to further the proof. | Code: Diagram (this figure was directly reference in a proof). |

## Results

### Proof Frameworks

We found that the three books varied in terms of how often they presented a complete proof framework (CFP). Book A and Book B provided CFP roughly 1/3 of the time while Book C provided CFP 68% of time (see table 4). Roughly a quarter of proofs from Book A and Book B did not include either the proof framework antecedent or conclusion. A student reading Book C is exposed to significantly more complete proof frameworks than a student reading Book A or Book B.

*Table 4. Complete Proof Frameworks and Non-Existent Proof Frameworks*

| Text | Complete | % Complete | Non-Existent | % Non-Existent |
|---|---|---|---|---|
| Book A | 32 | 37% | 22 | 26% |
| Book B | 50 | 38% | 27 | 20% |
| Book C | 42 | 68% | 2 | 3% |
| Overall | 124 | 44% | 51 | 18% |

Additionally, we found that complete frameworks were present more consistently in the introductory materials, as proof methods were being explicated, and then used less and less as we continued through each book (see Table 5). Conversely, non-existent frameworks were less frequent in the introductory material, but become more common as the texts progress. Finally, proof frameworks in the supplementary material were treated in a manner roughly parallel to how they were treated in the body of the proofs themselves.

*Table 5. Complete Proof Frameworks (Comp.) by Topic*

| | Book A | | Book B | | Book C | |
|---|---|---|---|---|---|---|
| Sections | Comp. | % Comp. | Comp. | % Comp. | Comp. | % Comp. |
| Intro. Material | 25 | 50% | 34 | 53% | 33 | 72% |
| Relations | 2 | 22% | 2 | 13% | 3 | 100% |

| | | | | | | |
|---|---|---|---|---|---|---|
| Functions | 3 | 33% | 2 | 9% | 3 | 75% |
| Cardinality | 2 | 12% | 12 | 40% | 3 | 33% |

**Warranting**

We found explicit warranting to be an uncommon occurrence in all three textbooks. In Table 6, we present the use of explicit warrants within the categories of definitions (DEF), theorems (THM), algebraic field axioms (ALG). We also provide the number of proofs and number of statements. Overall only 6% of statements included explicit warranting throughout all three texts, and a little more than a third of all proofs had *any* explicit warranting of any kind in them. Thus the reader of any of these texts is unlikely to regularly be exposed to explicit warrants. In all three texts, field axioms were implicitly warranted in all cases. Book C provided the most warrants often explicitly warranting with definitions.

*Table 6. Proofs with Warrants, Statements, and Total Warrants in ITP Texts*

| | Proofs with Warrants | Statements in Proofs | Warrants | | | | Total by Statement (%) |
|---|---|---|---|---|---|---|---|
| | | | DEF | THM | ALG | Total | |
| Book A | 18 | 808 | 5 | 19 | 0 | 24 | 3% |
| Book B | 47 | 885 | 21 | 38 | 0 | 59 | 7% |
| Book C | 37 | 633 | 51 | 20 | 0 | 71 | 11% |
| Total | 102 | 2326 | 77 | 77 | 0 | 154 | 6% |

When we expanded our analysis to the supplemental material, we found that Book B and Book C often included warrants in parenthetical comments. An additional 46[2] explicit warrants can be found in parenthetical comments. This reflects that warranting is (a) not meant to be part of the proof product; (b) but warranting is part of the proving process.

**Informal Reasoning and Diagrams**

We found diagrams and other semantic explorations to be the most sparsely represented construct of all coded entities. Conversely, the authors regularly offer insight akin to Weber's (2001) strategic knowledge as strategies were especially prevalent.  In Table 7 we present the use of strategies and semantic explorations as part of the argumentative process. The authors show a bias toward presenting strategies for how to produce a proof rather than the exploring instantiations to better understand the underlying premise being proven.

*Table 7. Use of Informal Reasoning and Diagrams*

| | Proofs | Strategies | Strategies (%) | Semantic | Semantic (%) |
|---|---|---|---|---|---|
| Book A | 86 | 40 | 47% | 6 | 5% |
| Book B | 132 | 72 | 55% | 8 | 6% |
| Book C | 62 | 10 | 16% | 3 | 5% |

[2] Three of these warrants were related to algebraic field axioms.

| | | | | | |
|---|---|---|---|---|---|
| Total | 280 | 122 | 44% | 15 | 5% |

## Discussion

Through our textbook analysis, we found proof frameworks, warranting, and informal reasoning occurred inconsistently and often infrequently in typical ITP textbooks. Proof frameworks were by far the most treated entity of the three as each text explicitly touched on the idea that there is an overarching logical shell implied by the mathematical statement to be proved and the chosen proof method. This was a point that was touched on early by each of the three texts, but was not used consistently throughout the texts. In general, the texts convey a message that a proof framework need not be explicit part of a proof. As we know students struggle to produce, identify, and understand the role of proof frameworks (Mejía-Ramos et al., 2012; Selden & Selden, 1995, 2003), leaving such a framework implicit may be further increasing the difficulty in seeing the importance of these structures.

Our analysis bore out Alcock and Weber's (2005) conjecture that textbooks do not treat warranting and its importance explicitly. None of the three texts explicitly addressed the role of warranting. Further, explicit warrants were a rarity. The textbooks reflected a norm for the ITP setting that field axiom claims never need warrants. Claims relying on definitions or theorems sometimes need explicit warrants. There is no reliable message to be found concerning the use of warranting within these texts. For the reader this means that coming to a deeper understanding of explicit warrants will be difficult, let alone coming to have an understanding about the importance of being able to infer warrants. This means that the impetus is on the instructors of ITP courses to introduce what a warrant is and the role that it plays in the argumentative process, but also to explore how they are used implicitly and the role the reader of a proof has in inferring the implied warrant (see Alcock & Weber, 2005, Weber, 2004).

Finally, informal reasoning, much like that of warranting, had no explicit conversation surrounding the subject in any of the three texts. None of the three texts present a clear picture of how informal reasoning may guide the production of formal proof and the place that informal ideas and representations such as diagrams play in a formal proof. The exception to this occurs in the cardinality section where maps are written informally. This more informal treatment is leaves us to wonder the lasting affect that seeing a large body of formal proofs, followed by some very informal proof on a particular subject will have on students' ability to understand and gain conviction in ideas surrounding cardinality in their future classes.

Our study implications are limited in the degree to which commonly used textbooks reflect the implemented curriculum. We see this exploration as highlighting that there is a clear under treatment of warrants and information exploration in common textbooks. We do not wish to claim this means there is an under treatment of these topics within classes. However, our textbook analysis paired with research on the typical nature of proof-based courses (e.g. Alcock & Weber, 2010; Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016), builds a strong argument for this being the case.

Furthermore, the textbook analysis unearthed inconsistent proof norms particularly around proof frameworks and warranting. If a set of typical textbooks designed to enculturate students in proof production contain fundamental inconsistencies, how can we expect our students to understand the importance of these constructs? As instructors and researchers, we must be aware of the messages our curricular materials may send.

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# Transforming students' definitions of function using a vending machine applet


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*The purpose of this study is to examine the understandings of functions that students developed and tested while engaging with a Vending Machine applet. The applet was designed to purposefully problematize common misconceptions associated with the algebraic nature of typical function machines. Findings indicate that the applet disrupts students' algebraic view of function and supports their transformation of meaning schemes for the function concept.*



*Keywords:* Functions, Calculus, Teaching with Technology


The concept of function is central to the study of undergraduate mathematics, science, and engineering (e.g., Cooney, Beckmann, & Lloyd, 2010; Dubinsky & Harel, 1992; Leinhardt, Zaslavsky, & Stein, 1990). However, research has revealed persistent and common misconceptions among undergraduate students with respect to the definition of function (Vinner & Dreyfus, 1989), use of function notation (e.g., Oehrtman, Carlson, & Thompson, 2008), and connections between function representations (e.g., Brenner et al., 1997; Clement, 2001; Dreher & Kuntze, 2015; Stylianou, 2011). Hence, there is a need for the development and study of interventions to help address misconceptions such as these among undergraduate students so that they are set up for success in their future studies.

To this end, we designed and studied the implementation of an applet-based learning intervention focused on disrupting undergraduate students' understanding of the function concept. The purpose of this study is to examine the understandings of functions that students developed while engaging with a Vending Machine applet designed for students to test and develop their own definitions for function.

## Background Literature

Much of the research on student understanding of function has occurred in the context of college algebra, precalculus, or calculus classes. Through these studies there has been a careful identification of common understandings that students develop related to the concept of function. One common student understanding is that functions are defined by an algebraic formula (Breidenbach et al., 1992; Carlson, 1998; Clement, 2001; Sierpinska, 1992). This is not surprising since functions are typically introduced as specific function types, such as linear and quadratic functions, in the middle school and high school curriculum (Cooney et al., 2010). Thompson (1994b) found that not only do students view functions as algebraic formulas, they often view functions as two expressions separated by an equal sign. While an equation view of function is not inherently wrong, it is narrow and can lead to difficulties for students as they work with functions in different contexts and with different representations (Cooney et al., 2010).

Along with an algebraic view of functions as representations of particular objects (e.g., graphs, expressions) rather than a relationship between inputs and outputs, research has also shown that students often rely on the graph of an equation and the vertical line test to differentiate a function from a non-function (Breidenbach et al., 1992; Fernandez, 2005). This can lead to conceptual difficulties in determining functions from non-functions, including the tendency to apply procedures to determining functions from non-functions (Breidenbach et al., 1992; Fernandez, 2005). Students whose view of function is algebraic and who use procedural techniques to identify functions and non-functions struggle to comprehend a general mapping of input values to a set of output values (Carlson, 1998; Thompson,1994a). The consistency of problematic understandings of function found across studies of students speaks to the need for pedagogical practices to specifically disrupt and correct these ideas. This is especially important given that function is a unifying concept among many undergraduate mathematics courses. Students need to understand both what a function is (i.e., the definition of function) and how to identify one across contexts and representations (e.g., Carlson, 1998; Thompson, 1994a; Breidenbach et al.,1992). Yet, particular attention to students' understanding of the definition itself has not been widely researched.

## Theoretical Framework

As we consider undergraduate students' learning related to function, we adopt a theoretical lens of transformation theory (Mezirow, 2009). Transformation theory is consistent with constructivist assumptions, specifically that meaning resides within each person and is constructed through experiences (Confrey, 1990). Mezirow (2009) describes four forms of learning that lie at the heart of the theory: elaborating existing meaning schemes, learning new meaning schemes, transforming meaning schemes, and transforming meaning perspectives (p. 22). Meaning schemes are the specific expectations, knowledge, beliefs, attitudes or feelings that are used to interpret experiences (Cranton, 2006; Peters, 2014). In the context of this study, an undergraduate student might transform his/her meaning scheme for function by rejecting her prior conception of function as a graph that passes the vertical line test and adopt a broader view of function that includes numerical and algebraic representations.

Learning by transforming meaning schemes often begins with a *disorienting dilemma*. This stimulus requires one to question current understandings that have been formed from previous experiences (Mezirow, 2009). It is this type of learning experience that we are particularly interested in - both designing stimuli for it, and the ways that meaning schemes are transformed as a result. Given the evidence that undergraduates often have a view of function that is limited to algebraic expressions and their associated graphs (e.g., Carlson 1998; Even, 1990) and that such understandings typically result in a "vertical line test" related definition of function (e.g., Carlson, 1998), we designed an experience that would problematize these understandings, thereby creating a stimulus for transformation.

One strategy that has been suggested for resolving common misunderstandings related to function is the use of a function machine as a cognitive root. The idea of a *cognitive root* was introduced by Tall and colleagues as an "anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built" (Tall et al., 2000, p.497). As an example of a cognitive root for function concepts, Tall et al. suggest the use of a *function machine* (sometimes referred to as a function box). The machine metaphor Tall and colleagues describe is typically a "guess my rule" activity where the inputs and associated outputs are provided and students are challenged to determine what happened in the function machine (i.e., determine the function rule). While students are presented with a machine to embody the function concept, the rules used by the machine are algebraic in nature. In their studies using such machines proved quite promising as a cognitive root for function, yet some students still struggled with connecting representations and determining what is and is not a function (McGowan et al., 2000). Given the promise of a machine metaphor as a cognitive root for function coupled with our desire to present a disorienting dilemma for undergraduates, we set out to design an applet as a learning experience.

## Design of the Applet

The Vending Machine applet (version 2.0) was designed to trigger a disorienting dilemma in students' understanding of function. The applet contains no numerical or algebraic expressions, but instead was built on the metaphor of a vending machine. Our Vending Machine applet (https://ggbm.at/qxQQQ7GP) is a GeoGebra book consisting of four pages. The first two pages contain two soda vending machines each with buttons for: Red Cola, Diet Blue, Silver Mist, and Green Dew. When the user presses a button (input), one or more cans appear in the bottom of the machine (output). To remove the can(s) from the bottom of the machine, the user clicks the "take can" button. On each of the first two pages, one machine is labeled as a function and the other is labeled as not a function. The non-function machines each have at least one button that produces a random can when pressed (i.e., the resulting can is not predictable based upon the button that is pressed). The directions ask the user to explore Machines 1-4 on Pages 1 and 2 and make a conjecture about why Machines 1 and 3 are functions and Machines 2 and 4 are non-functions.

Pages 3 and 4 of the applet allow the user to test and adapt their conjecture through interacting with 10 additional vending machines, Machines A through J. The functionality of each machine was designed to address misconceptions from the literature on distinguishing functions and non-functions. Examining Table 1, you will notice that the understandings we are trying to disrupt are the notion of what represents an element in the range (Machines B, I, & J), students occasional use of the term "unique" when thinking about outputs (Machines B & I), and the notion that functions should be "predictable" (Machines A, C, I, & J) - meaning that if one knows the function rule and is given an output, it is possible to determine what input resulted in that output.

*Table 1. Machine output for each button pressed*

| | Button Pressed | | | |
|---|---|---|---|---|
| | Red Cola | Diet Blue | Silver Mist | Green Dew |
| Machine A | red can | blue can | silver can | random can |
| Machine B | two silver cans | green can | red can | blue can |
| Machine C | random can | random can | random can | random can |
| Machine D | silver can | green can | red can | blue can |
| Machine E | red can | silver can | silver can | green can |
| Machine F | blue can | silver can | green can | red can |
| Machine G | green can | green can | green can | green can |
| Machine H | red can | red can | silver can | silver can |
| Machine I | random pair | blue can | silver can | green can |
| Machine J | red can | blue & random can | silver can | green can |

## Method

The purpose of this study was to examine the understandings of functions that undergraduate students developed while engaging with the Vending Machine applet. We specifically address the following research questions: 1) How do students define function? and 2) How do students change their definition of function as a result of engaging with the Vending Machine applet?

### Data Collection

A total of 123 students at six post-secondary institutions, that ranged in size, location, and focus, participated in the study. These students were undergraduates who had completed Calculus I. Prior to their use of the applet, students were asked to write a definition of function in their own words based on their current understanding, i.e., a pre-definition. This was done toward the beginning of the course or before functions were discussed, and no explicit instruction or discussion of functions had yet occurred. This data was collected by the instructor and students subsequently engaged with the Vending Machine applet outside of class, recording their interaction via a screencast. During the following class session students were asked to define function once again, a post-definition. The data used for this particular paper are the students' pre-and post- definitions.

### Data Coding and Analysis

Given that our goal was to make sense of students meaning schemes within their written responses (i.e., text data) the use of content analysis (Creswell, 2007) as an analysis technique is appropriate. Specifically, we used *directed content analysis*. Directed content analysis uses existing theory or prior research to identify key concepts as initial coding categories for recognizing patterns in text responses (Hsieh & Shannon, 2005). For example, one set of initial codes were defined based on prior research that has identified student conceptions of functions as objects (e.g., expressions, graphs, tables of values) and relationships (e.g., mappings) (e.g., Dubinsky & Harel, 1992; Breidenbach et al., 1992). Through open coding, additional codes were defined as they emerged from the data. For example, students often included examples within their definitions so we created a set of codes to capture the nature of these examples (e.g., graph example, expression example).

Our completed codebook included 16 codes. To establish reliability in our coding a subset of 25 randomly selected definitions were coded independently by all six members of the research team and the number of agreements were divided by the number of assigned codes. The team had 93.1% agreement, so the codebook was considered reliable (Miles & Huberman, 1994). Once reliability was established, definitions were coded independently by six coders, with all definitions double-coded by pairs of coders. Pairs then compared codes and discussed and resolved differences (DeCuir-Gunby, Marshall, & McCulloch, 2011).

Prior to attending to aspects of the text individually, the pre- and post-definition were taken as a whole and coded in terms of their *accuracy* (i.e., correct, incorrect, or close to correct). Key elements of a correct definition were 1) the definition was not limited to a specific type of function (e.g. linear or quadratic), or to a particular representation (e.g., equation), and 2) the definition addressed the idea that functions map each input to one and only one output. Definitions coded as close to correct included those that indicated each input has one and only one output, but were not classified as correct because they were not general enough (e.g., the definition limited a function to a particular representation, such as an equation).

Next, each definition was coded regarding whether the definition indicated a function was a relationship (e.g., mapping), an object (e.g., equation, graph), or neither (see Table 2). We referred to this set of codes as *focus*, as they indicated how the students “saw” function. This coding was intended to be mutually exclusive, although some exceptions were found. Finally, definitions were coded according to whether or not they attended to output, as this was another aspect of the definition that we expected to be problematic based on the literature (Carlson, 1998; Even, 1993). After coding was completed, results for each code were summarized and analyzed for patterns and themes that provided insight to transformations of students’ meaning schemes related to the definition of function.

## Results

### Pre-definitions

Our results show that the vast majority of students initially incorrectly defined function and their definition focused on a function as an object (Table 2). Only 24 students (19.4%) defined function correctly or close to correct. Note that with respect to focus there is a fourth category, *both object and relationship.* This is because, although the categories of relationship, object, and neither were meant to be mutually exclusive, there were definitions that referred to a function as both a relationship and an object. For example, one student defined a function as “an expression that is representative of a relationship between two or more variables.” Referring to a function as an expression would generally be categorized as an object, but in this case the student also refers to a function as a relationship. Finally, there were initially 47 students (38.2%) who paid attention to output in their definitions. For example, “for each value of x, there can only be one and only one y” and “function is when there is a specific output given an input” were both coded as “attention to output”. Definitions that made mention of the vertical line test were also coded as paying attention to output.

*Table 2. Code occurrences n pre- and post- definitions*

| Code | Example | Pre (%) | Post (%) |
|---|---|---|---|
| | Accuracy | | |
| Correct | A function is a relation in which for every input there exists exactly one output. | 6 (4.8) | 8 (6.5) |
| Close to Correct | A function is a mathematical equation in which a single input only yields one result. | 18 (14.6) | 46 (37.4) |
| Incorrect | An expression that is representative of a relationship between 2 or more variables | 99 (80.5) | 69 (56.1) |
| | Focus | | |
| Relationship | A mapping from a domain to a codomain (or range) | 14 (11.4) | 18 (14.6) |
| Object | An equation with an x-input that gives a y output. | 86 (70.0) | 64 (52.0) |
| Both object & relationship | An expression that is representative of a relationship between two or more variables | 7 (5.7) | 10 (8.1) |
| Neither | f(x) → y; i) unique y value for every x; ii) one to one | 16 (13.0) | 31 (25.2) |

**Changes in definition**

In this section, we report results in terms of how students' definitions changed after interacting with the applet. In terms of accuracy, the majority of students (52.8%) persisted in an incorrect definition of function. However, considering students who moved from incorrect definitions to definitions that were either correct or close to correct, over one-fourth (27.6%) of students improved their definition of a function. Furthermore, only 4% of students regressed in their understanding of function, i.e., from correct or close to correct to incorrect, or from correct to close to correct.

To better understand the aspects of function that are still problematic in definitions we look at focus and attention to output. In terms of students' focus in their definitions of function, a total of 90 students (73.2%) did not change (Table 3). Furthermore, the majority of students (69.1%) started with a definition of function that was classified as an object, and most (52%) persisted in that view. In terms of students' attention to output, 45 of the 47 students who initially made special reference to the output of a function maintained a focus on output in their definitions after engaging with the applet. On the other hand, of the 76 students who did not make any special reference to output in their initial definitions, 60 (78.95%) of them did so after engaging with the applet. Overall, 85.3% of all students attended to output in their revised definition.

*Table 3. Occurrences of pre and post definition characteristics*

| | Post-Definition Characteristics | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Pre-Definition Characteristics | Incorrect | Close to correct | Correct | Object | Relationship | Both O & R | Neither | Attend to output | No attention to output |
| Incorrect | 65 (52.8) | 31 (25.2) | 3 (2.4) | | | | | | |
| Close to Correct | 3 (2.4) | 14 (11.4) | 1 (0.8) | | | | | | |
| Correct | 1 (0.8) | 1 (0.8) | 4 (3.3) | | | | | | |
| Object | | | | 64 (52.0) | 8 (6.5) | 1 (0.8) | 12 (9.8) | | |
| Relationship | | | | 1 (0.8) | 10 (8.1) | 0 (0.0) | 6 (4.9) | | |
| Both O & R | | | | 0 (0.0) | 1 (0.8) | 5 (4.1) | 1 (0.8) | | |
| Neither | | | | 3 (2.4) | 0 (0.8) | 0 (0.0) | 11 (7.9) | | |
| Attend to Output | | | | | | | | 45 (36.6) | 2 (1.6) |
| No attention to output | | | | | | | | 60 (48.7) | 16 (13.0) |

## Discussion

With the essential role function plays in college mathematics, it is imperative that students have an understanding of function beyond an algebraic understanding. To address this, we created an applet, building from Tall and colleagues (2000) suggestion of a function machine as a cognitive root. Our Vending Machine applet (version 2.0) was designed to provoke disorientating dilemmas related to students' understanding of function which promote reflection and ideally shift students' meaning schemes related to definition of function away from an algebraic view. Since little research has been conducted on undergraduates' definition of function since Vinner & Dreyfus (1989), one of the goals of the study was to examine the current definitions of a function from a large sample of undergraduate students from six universities. From examination of students' definitions before engaging with the applet, our results showed that only approximately 5% of students could correctly define a function. The majority of students in this study did not include in their definitions the two key elements that define a function: 1) it is applicable across different representations; and 2) functions map each input to one and only one output.

From examining changes in students' definitions, the Vending Machine applet seemed to support students in moving toward a correct definition of function, and promoted greater awareness of the importance of the output in relation to the definition of function. However, since 55.2% of the students' post-definitions were limited to a specific representation (i.e., graph, equation) the applet did not seem cause a dilemma for students to move away from their algebraic view of functions in their definitions. An examination of students' screencasts (see Martin, Soled, Lovett, & Dick., under review) showed that students did not experience the dilemmas we had designed for as they worked through the pages of the applet. The first two pages of the applet, that told students which machine was a function and not a function, seemed to cause students to only used what they learned from these machines when determining if the other ten machines were a function or non-function.

Even though the Vending Machine applet (version 2.0) seemed to help students move towards a correct definition of function, for many students the experience did not provoke a dilemma regarding function as an object (e.g., equation, graph). It is possible that the ways in which the students interacted with the applet might have prevented the dilemma from occurring. Thus, we felt the applet could be improved, and as such the applet has been revised. Version 3.0 now consists of four pages in a slightly different format and includes two new machines. The first three pages each contain two vending machines (similar to version 2.0) except the directions say "Which one is a function?" Our intent is that this version will allow students to deepen their understanding of function by applying their knowledge of the function concept to the vending machines instead of using their experiences with the first four machines in version 2.0 to determine whether the other machines are functions or non-functions.

## Conclusion

The results of this study suggest that the Vending Machine applet has the potential to be a powerful tool (cognitive root) for disrupting students' limited view of function and supporting their transformation of meaning schemes related to the concept of function. However, to determine whether or not these findings are generalizable, the use of the newest version of the applet (3.0) needs to be studied on a larger scale. Our plan is that through further study and revision, we will produce a transformative cognitive root that disorients students' algebraic view of function, remedies existing misconceptions, and on which conceptual understanding of function concept can be built.

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# The Effect of Self-Efficacy on Student Performance in Calculus

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*Self-efficacy is an important variable that has been used to study students' performance at all educational levels and in many content areas. In the report, we discuss the results of a quantitative study considering self-efficacy in college Calculus and its correlation to other variables available in a large scale study. Ultimately, our findings contradict existing findings regarding the effect of self-efficacy on class performance. We add to these results an interesting finding regarding the effect of self-efficacy on student's study habits: while time spent on course homework does not mediate the effect of self-efficacy, more time spent on course preparation by students with high self-efficacy tends to decrease their expected final course grade. Results contribute to math instructors' understanding of their teaching and may help with the construction of more effective instruction.*

*Key words:* Calculus, Self- Efficacy, Class Performance, Homework Hours Spent, Classroom Research.

## Introduction

Self-efficacy describes a person's perception of her own potential to master a specific task, and has been shown to have a powerful effect on achievement (Bandura, 1986; Bandura, 1997). Self-efficacy affects behavior by influencing one's choices and actions (Pajares, 1996). Bandura (1997) proposes four sources of self-efficacy: personal experience (referring to prior outcomes), vicarious experience (what one observes), social persuasions (feedback received from peers and others), and psychological state (ones mood), but personal experience and previous performance levels influence self-efficacy above all else (Chen & Zimmerman, 2007). In this research report, Bandura's theory of self-efficacy grounds and supports the work.

Within the context of mathematics, self-efficacy has been shown to be a better predictor of performance than measures of math anxiety or prior experience with math (Pajares & Miller, 1994; Pajares & Miller, 1995), and appears to be tantamount in importance to even intellectual ability (Pajares & Kranzler, 1995). High self-efficacy correlates positively with greater aspirations, greater commitments, and a greater ability to recover from setbacks; high math self-efficacy correlates with greater persistence on long and difficult problems, and greater accuracy of computation (Collins, 1982; Hoffman & Schraw, 2009). Hackett (1985) found a positive relationship between self-efficacy and ACT scores, which agrees with numerous studies showing a powerful link between high self-efficacy and high performance (Fast et al., 2010; Pajares & Miller, 1994; Pajares & Miller, 1995; Pajares & Kranzler, 1995); Peters, 2013). Meece, Wigfield, & Eccles (1990) found that high math self-efficacy corresponds to students valuing math more highly, and expecting to succeed. The study also found that students' performance expectations predict math anxiety, but that math anxiety only indirectly relates to subsequent performance. Performance expectations do, however, predict actual performance. Students who rate math performance as relatively important tend to have lower math anxiety than students who rate math performance as unimportant (Meece, Wigfield, & Eccles, 1990).

A positive relationship has been demonstrated between math self-efficacy and gender (men tend to have higher math self-efficacy) (Hackett, 1985). However, while Hackett (1985) found a positive relationship between gender and math achievement, other studies have found no such relationship (despite confirming that men tend to have higher math self-efficacy than women) (Meece, Wigfield, & Eccles, 1990; Peters, 2013). However, self-efficacy mediates the influence of gender on math performance (Pajares & Miller, 1994). Math self-efficacy has even been shown to mediate the effects of prior math experience on performance, itself the foremost source of self-efficacy (Pajares & Miller, 1994).

Our analysis utilizes the Mathematical Association of America's (MAA's) study of *Characteristics of Successful Programs in College Calculus* (NSF, DRL 0910240) data to revisit these issues by addressing the following two questions:

**Research Question 1**: Is the effect of student math ability on expected final grade moderated or mediated by student's self-efficacy, such that confidence in ability can overcome lower actual ability?

**Research Question 2**: Does a student's self-efficacy effect the amount of work he or she does to prepare for class, and does self-efficacy moderate the effect of class preparation on a student's expected final grade?

## Methods and Data

As mentioned before, we utilized the *Characteristics of Successful Programs in College Calculus* (CSPCC) data set and conducted factor analysis to construct a measure of self-efficacy by combining students' answers to several questions into a single variable. We then created mixed effects ordered logistic regression models with random intercepts using this measure and other data from the survey. The CSPCC data was collected and made available for reserachers by MAA based upon work supported by the National Science Foundation under grant DRL REESE #0910240 between 2009 and 2015.

The survey data concerns students' experience and performance in Calculus I. The entire survey consists of two parts, a pre-survey administered at the beginning of Calculus I, and a post-survey administered at the end of the Calculus I course. From this data set, we used the expected course grade from the post-survey as the dependent variable. From the pre-survey we use SAT and ACT math scores along with three separate questions that assess self-efficacy as independent variables. One additional independent variable was taken from the post survey: the amount of time spent on homework. As control variables, we used age, gender, mother's education, and father's education from the pre-survey.

The survey data included responses from 13,965 high school students who completed at least one of the surveys. Of these students, 10,506 are eliminated, because they only completed one of the surveys, and the questions we are using to complete the analysis come from both surveys. After using this list wise deletion criteria on just the dependent variable, the final sample of college student respondents is 3,459 from 14 universities. Of those 3,459, 10 did not report a final grade, reducing the sample to 3,449 (in the full sample 6,144 students did not report a final grade). Additionally, the two measures of student math ability, SAT and ACT math score, are also missing across some observations. We converted ACT math scores to SAT scores using the conversion method recommended by The College Board, so as to maximize the sample size of our model. After converting the ACT score, and eliminating observations where respondents did not report either score, the final sample is 2,973. The other independent variables were also missing in some responses. When included in a model together, the final overlapping sample size(*n)* is 2787. However, the exact *n* of each of our models changed depending on the variables included. Table 1 presents the summary statistics and missing observation counts across all of the variables used in our models.

Table 1: Summary and Missing Data Statistics

| | Mean | SD | Min | Max | Missing |
|---|---|---|---|---|---|
| Gender | .4531755 | .497892 | 0 | 1 | 24 |
| Mother's Ed | 3.66272 | 1.057391 | 1 | 5 | 12 |
| Father's Ed | 3.759239 | 1.132856 | 1 | 5 | 10 |
| Calc I Grade | 3.159311 | .837746 | 0 | 4 | 10 |
| Last HS Math Grade | 3.704399 | .6118699 | 0 | 4.33 | 194 |
| Self Efficacy | .0253085 | .9761658 | -4.654925 | 1.17197 | 23 |
| Age | 18.81306 | 1.794104 | 14 | 38 | 5 |
| SAT Math Score | 650.3742 | 78.44545 | 18 | 800 | 477 |
| AP Math Class | 1.556512 | .4968852 | 1 | 2 | 60 |
| *N* | 2787 | | | | |

### Linear Regression Results for Research Question 1

As mentioned above, previous research found self-efficacy to be a better predictor of mathematics achievement (Pajares & Kranzler, 1995). We reexamined this question and asked whether self-efficacy moderates or mediates the effect of ability on student performance in their Calculus I course. To assess the potential moderating effect, we first constructed a measure of self-efficacy. We used principal component factor analysis to construct a variable that combines three survey questions that evaluate a student's self-efficacy. Factor analysis is a statistical technique for data reduction. It reduces the data by generating linear combinations of "factors" that reconstruct a group of related variables (Hamilton 2013, Ch.11). More specifically, principal component analysis (PCA) conducts an eigen decomposition of the correlation matrix between the selected variables. The eigenvectors represent uncorrelated linear combinations of the variables that capture most of the variance across the variables. In other words, principal component analysis can reduce many variables down to a single variable that captures the most amount of covariance between the variables. We use the first factor loadings to construct a measure of self-efficacy.

The three survey questions that generated the data we use in the PCA are as follows and responses to these questions can range from 0 (strongly disagree) - 5 (strongly agree):

**Question 29 Know**: I believe I have the knowledge and abilities to succeed in this course.
**Question 29 Understand**: I understand the mathematics that I have studied.
**Question 29 Confident**: I am confident in my mathematics abilities.

A histogram of the constructed measure is presented in Figure 1


Figure 1: Histogram of Constructed Self Efficacy Variable

With this measure constructed, we interact self-efficacy with SAT math score and students' grade in the last high school math class they completed. Table 2 presents the results of the mixed effects ordered logistic regression models with random intercepts for respondent's university department. The interaction terms are not statistically significant, implying that there is not a strong consistent moderating effect of self-efficacy on course ability.

Table 2: Moderation Analysis for Question 1

| | (1)<br>Grade | (2)<br>Grade |
|---|---|---|
| Female | 0.0975<br>(0.0737) | -0.0750<br>(0.0699) |
| Mother's Ed | 0.0283<br>(0.0410) | 0.0572<br>(0.0388) |
| Father's Ed | 0.000235<br>(0.0388) | 0.0727*<br>(0.0356) |
| HW Hours | -0.00850<br>(0.00763) | -0.0308*<br>(0.00686) |
| Age | 0.0198<br>(0.0198) | 0.00324<br>(0.0130) |
| Self Efficacy | 0.365<br>(0.264) | 0.440*<br>(0.184) |
| SAT Math Score | 0.00720*<br>(0.000533) | |
| Last HS Math Grade | | 0.631*<br>(0.0576) |
| Interaction | 0.000424<br>(0.000412) | 0.0530<br>(0.0503) |
| $N$ | 2905 | 3172 |

Standard errors in parentheses
$*p < .05$

To analyze whether self-efficacy mediated the effect of their actual ability on their grade in Calculus I, such that actual ability causes self-efficacy, which in turn determines a student's grade, we start by examining whether all variables are statistically significant predictors of expected final grade when included on their own. The results of models (1), (2) and (4) shown in Table 3 show that the coefficients on all variables are positive and statistically significant. Secondly, we tested whether the coefficients on SAT math score or high school math grade lose significance when self-efficacy is included in the model, which would indicate a mediating effect of self-efficacy The results, presented in Table 3 are not consistent with the presence of a mediating effect. Both SAT math score and high school math grade remain statistically significant and positively signed when efficacy is included in the model (models (3) and (5) in Table 3).

Table 3: Regression Model for Mediation Analysis in Research Question 1

| | (1) Grade | (2) Grade | (3) Grade | (4) Grade | (5) Grade |
|---|---|---|---|---|---|
| Female | -0.0466 | 0.00704 | 0.0961 | -0.245* | -0.0729 |
| | (0.0717) | (0.0673) | (0.0737) | (0.0678) | (0.0699) |
| Mother's Education | 0.0221 | 0.0488 | 0.0285 | 0.0556 | 0.0563 |
| | (0.0403) | (0.0373) | (0.0410) | (0.0381) | (0.0388) |
| Father's Education | -0.00626 | 0.0818* | -0.000884 | 0.0891* | 0.0725* |
| | (0.0379) | (0.0343) | (0.0387) | (0.0349) | (0.0356) |
| HW Hours | -0.0149* | -0.0284* | -0.00880 | -0.0397* | -0.0309* |
| | (0.00741) | (0.00666) | (0.00762) | (0.00668) | (0.00686) |
| Age | 0.0127 | -0.0142 | 0.0205 | 0.00206 | 0.00306 |
| | (0.0200) | (0.0115) | (0.0197) | (0.0128) | (0.0130) |
| SAT Math Score | 0.00836* | | 0.00711* | | |
| | (0.000519) | | (0.000528) | | |
| Self Efficacy | | 0.678* | 0.635* | | 0.630* |
| | | (0.0360) | (0.0402) | | (0.0378) |
| Last HS Math Grade | | | | 0.772* | 0.623* |
| $N$ | 2922 | 3357 | 2905 | 3191 | 3172 |

Standard errors in parentheses
* $p < .05$

## Linear Regression Results for Research Question 2

The work ethic of a student is an important determinant of her performance. However, the willingness of a student to invest her time and energy in preparing for the course (as a proxy for this, we used the Homework variable) is certainly impacted by self-efficacy. We included this important Homework variable in our analysis of student performance, and evaluated whether self-efficacy moderates the effect of homework hours invested on student performance, or whether homework hours mediates the effect of self-efficacy. Self-efficacy may determine the amount of time and energy a student decides to invest in her course. To assess the presence of a mediating effect of homework hours on self-efficacy, we first examined whether self-efficacy is correlated with the amount of homework hours students expend. We then evaluated whether homework hours and self perception of ability are statistically significant predictors of expected final grade when included on their own. Finally, we tested whether the effect of efficacy loses significance when homework hours is included in the same model. The results of these models, presented in Table 4, are not consistent with the proposition that homework hours mediate the effect of self-efficacy on student's achievement in Calculus I. In fact, the results imply the opposite direction of causality, but given that self-efficacy is evaluated prior to students completing the course, the causal relationship implied by the model is not plausible.

To evaluate the moderating effect of self-efficacy on homework hours, we create an interaction term between the two variables and again include it in a mixed effects multinomial ordered logit models with random intercepts for university. The results of the model are presented in Table 5.

Table 4: Regression Model for Mediation Analysis in Research Question 2

| | (1) HW Hours | (2) Grade | (3) Grade | (4) Grade |
|---|---|---|---|---|
| main | | | | |
| Efficacy | -0.495* | 0.635* | | 0.635* |
| | (0.0866) | (0.0402) | | (0.0402) |
| Gender | | 0.0961 | -0.0466 | 0.0961 |
| | | (0.0737) | (0.0717) | (0.0737) |
| Mother's Ed | | 0.0285 | 0.0221 | 0.0285 |
| | | (0.0410) | (0.0403) | (0.0410) |
| Father's Ed | | -0.000884 | -0.00626 | -0.000884 |
| | | (0.0387) | (0.0379) | (0.0387) |
| HW Hours | | -0.00880 | -0.0149* | -0.00880 |
| | | (0.00762) | (0.00741) | (0.00762) |
| Age | | 0.0205 | 0.0127 | 0.0205 |
| | | (0.0197) | (0.0200) | (0.0197) |
| SAT Math | | 0.00711* | 0.00836* | 0.00711* |
| | | (0.000528) | (0.000519) | (0.000528) |
| $N$ | 3408 | 2905 | 2922 | 2905 |

Standard errors in parentheses
$*p < .05$

Table 5: Moderation Analysis for Question 2

| | (1) Grade |
|---|---|
| Female | 0.104 |
| | (0.0738) |
| Mother's Education | 0.0290 |
| | (0.0411) |
| Father's Education | 0.00288 |
| | (0.0388) |
| HW Hours | -0.0131 |
| | (0.00765) |
| Age | 0.0232 |
| | (0.0198) |
| SAT Math Score | 0.00709* |
| | (0.000529) |
| Self Efficacy | 0.799* |
| | (0.0604) |
| Interaction | -0.0254* |
| | (0.00677) |
| $N$ | 2905 |

Standard errors in parentheses
$* p < .05$

The interaction term is negative and statistically significant. Examining the marginal effect of homework hours at different levels of self-efficacy produces some surprising results. When we examined the marginal effect of a five hour increase in homework hours at various levels of self-efficacy on a student's expected grade, in Figure 2, we observe that a one hour increase in the amount of time a student with low self-efficacy spends on his or her homework decreases the probability that she receives a D or an F. However, every additional hour of study time increases the probability that she earns a B by much more than it increases the probability that she earns an A. The effect of increased homework hours has a similar effect for students with average self-efficacy, in that the larger amount of study time increases slightly the probability that the student earns an A, and decreases the probability that the student earns a D or an F.

Interestingly, for students with high self-efficacy, every hour of study time decreases the probability that the student earns an A by about .008, but increases the probability that the student earns a B (by .005) or a C (by .003). Increased homework hours has no statistically significant effect on the probability that a student with high self-efficacy receives a C or lower. Implied in this relationship is that self-efficacy has the largest positive impact on grade when a student studies the least. Figure 1 presents the change in the predicted probability of receiving each grade for a one unit increase in self-efficacy at different amounts of homework hours. This figure demonstrates that high self-efficacy increases the probability that a student receives an A in the course, having completed on average zero hours of homework per week, by .162, increases the probability that the student receives a B by .148, and decreases the probability of an F by .004. However, the effect of self efficacy on the likelihood that a student earns an A decreases to .072 when the students spends 18 hours per week on homework. Therefore, high self-efficacy proves more impactful on a student's grade when he or she does not spend a substantial of time studying. For those students with high self-efficacy, putting in more prep time is an indication that they are more like to earn a B than A in the course.


(a) Marginal Effect of Homework Hours on Grade

(b) Marginal Effect of Self-Efficacy on Grade

Figure 2: Marginal Effects from Moderation Analysis for Research Question 2

**Conclusion**

In this study we were unable to confirm findings of past studies that suggest that self-efficacy is more important than actual ability. Mathematics ability and experience correlate with self-efficacy, but self-efficacy does not overtake the effect of experience and ability. Additionally, we inquired as to whether student's self perception mediates or moderates the impact of their study habits on their course grade. Ultimately, when considered in aggregate, the regression analyses presented in this paper seem to suggests the following conclusions. Students have a fairly accurate perception of their math ability. Those who are are not confident in their ability because their ability is not very strong put in more effort preparing for class, and benefit from increased time spend studying and completing homework. However, this increased effort most greatly increases the odds that they earn a B in the class. Those with very high ability, and who are confident in that ability, do not expect to earn a higher grade as a result of increased time spent on course preparation, but do expect to earn high grades. In fact, students with high math ability and high self-efficacy are most likely to expect to earn an A. For those students with average confidence, applying themselves and putting more time in on homework and class prep tends to increase the likelihood that they earn an above average grade. Finally, self-efficacy has the largest effect on a student's grade when the student did not spend a substantial amount of time preparing for class.

For mathematics instructors, the take home lessons from this study are simple and commonsensical. First, confidence is not a replacement for hardwork, experience, and preparation in mathematics achievement. Apart from especially gifted mathematics students, homework and study time boost exam performance for most students. Over time, sucessful experiences in mathematics courses compound and create a virtuous cycle whereby students become more confident and this confidence, in conjunction with a strong work ethic, in turn contributes to further good performance.

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# Replacing Exam with Self-Assessment: Reflection-Centred Learning Environment as a Tool to Promote Deep Learning

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**Abstract**

*A new blended learning environment encompassing a wide variety of formative assessment was developed for a large undergraduate mathematics course to promote deep learning approach. In order to enhance reflection, the final exam was replaced by students' self-assessment. At the end of the course, a cluster analysis found four student clusters differing in their deep and surface learning approaches. Analysis of open feedback questions suggests that the contextual factor most commonly associated with deep learning approach was innovative assessment. Our findings lead us towards understanding how to foster deep learning approach in different kinds of learners.*

**Keywords**
Learning Approaches, Deep Learning, Blended Learning Environments, Self-Assessment

## Introduction

Student learning in higher education can be described by means of students' approaches to learning. These approaches are often divided to deep and surface learning approaches (Biggs, 2012). The adoption of these approaches is known to be connected with student characteristics but also with teaching methods and assessment (Baeten et al., 2010). In order to support deep approach to learning in a large mathematics course, we created a blended learning environment designed to offer a wide variety of feedback to the students and to promote reflection throughout the course. Based on our earlier findings on self-assessment in higher education (Tuohilampi, Rämö, Häsä, & Pekkarinen, 2017), we left out the course exam and replaced it with formative self-assessment. Moreover, the assessment methods were not separate components, but instead they were woven into the students' everyday activities in the course. This was achieved by innovative use of digital course components.

## Background of the Study

In our study, we seek to understand learning environments as diverse and complex structures that have the potential to alter students' learning strategies through reflection. Below, we briefly introduce the theory of learning approaches and the previous attempts to promote deep learning within different learning environments. We connect this background with the theory of blended learning environments, since in our learning environment the adoption of deep learning approach is promoted using both physical and digital elements.

**Approaches to Learning.** Approaches to learning consist of the combination of motivation and learning strategies of the student – very often these approaches are divided into deep and surface approaches (Biggs, 1987; Biggs, 2012; Entwistle 1991). A deep approach refers to a true intention to understand the content to be learned (Diseth 2003). It is linked with the idea of intrinsic motivation, "interest in ideas", and is allied to deeper pedagogical approaches that foster personal understanding (Entwistle 2000; Diseth 2003). Surface learning approach, on the other hand, is linked with using the least amount of effort to reach the minimal required outcomes (Biggs 1987; Garrison & Cleveland-Innes 2005). It can be said that the focus for surface approach learners is the completion of the task, not the growth with learning.

Deep learning approach has been valued more in the context of higher education (Garrison & Cleveland-Innes 2005). Sadler-Smith (1997) found a significant positive correlation between performance and deep learning in the context of higher education. Diseth (2003) observed the same connection; furthermore, he found that surface approach correlated negatively with performance in his study. However, deep and surface learning approaches are not to be seen as fundamental traits of students, as students have been observed to change their learning strategies with situational demands (Marton & Säljö 1976).

**Creating Learning Environments that Support Deep Learning Approach.** A large number of studies have investigated attempts to cultivate the deep learning approach within student-centred learning environments in higher education. However, the results of those studies are not consistent (Baeten et al. 2010). In their meta-analysis, Baeten and colleagues determined the factors needed to encourage deep learning approach; these were 1) contextual factors (teaching, assessment), 2) perceived contextual factors (how students perceive teaching and assessment) and 3) student factors (such as age and gender). The most successful strategies were determined within each of these factors. Those were found to be 1) innovative assessment and student-centred teaching, 2) satisfaction with the overall quality of the course and 3) intrinsic motivation. All these factors (contextual, perceptual and personal) need to be considered when designing a deep learning focused learning environment.

Innovative assessment methods prove to have a significant role in enhancing the adoption of deep learning approach within a learning environment. In their meta-analysis, Sluijsmans and colleagues found that self- and peer-assessment discouraged passive learning that was connected with surface approach (Sluijsmans et al., 1998); however, the use of these assessment methods was observed to require a lot of training. Increasing the variety of assessment in an active learning environment can increase the students' feeling of responsibility for their own learning and therefore enhance deep learning approach (Wilson & Fowler 2005). However, innovative assessment methods can also lead to adoption of more surface approaches when the terms of assessment do not require deep learning approaches (Gijbels & Dochy, 2006; Struyven et al., 2006). Learning environments can especially foster the adoption of deep learning approach if they encourage students to reflect on their own learning (Sobral, 2001; Waters & Johnston, 2004).

The research on the processes of deep learning within learning environments has focused on finding patterns inside the whole sample, which has been considered to be a weakness (Wilson & Fowler 2005; Baeten et al. 2010). In this study, we try to identify those subgroups of the student sample that are not able to benefit of the various feedback forms that we offer to foster reflection through our blended learning environment.

**Blended Learning Environments.** At its simplest, blended learning refers to providing digital learning materials in a physical classroom environment (Garrison & Kanuka 2004, 96–97; Singh 2003, 52–53). In this study we use a more complicated model to capture the diversity of support mechanisms of the reflection-centred learning environment. According to Manninen and colleagues (2007), blended learning environments can be seen to be composed of five dimensions that are separate yet overlapping: 1) physical environment (the space and buildings around learning situation), 2) social environment (the social interaction of learning situation), 3) digital environment (learning technology, ICT), 4) local environment (learning in the "real world", learning where the skills are needed), 5) didactic environment (learning as a centre for the learning environment). The learning environment that we developed reflects all the introduces elements (Figure 1).

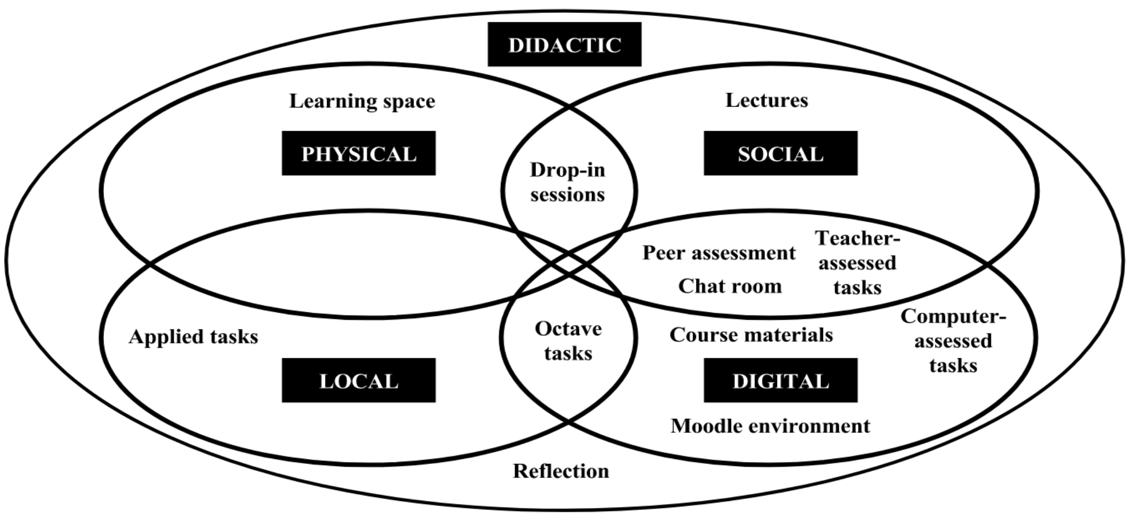
In our study, we seek to examine the connections between our learning environment and adoption of deep learning approach. However, unlike many course settings described in the literature (Baeten et al., 2010), ours is based exclusively on reflection. In our course, the whole culture of assessment is flipped, as we replace the traditional course exam with a variety of formative, digital assessment methods, such as self-assessment, that require a high amount of reflection. This led us to call our setting a "reflective-centred, blended learning environment". In the next section, we provide more details about the course setting.

**The Pilot Course with a Blended Reflective-centred Learning Environment**

The blended, reflective-centred learning environment investigated in this study was created for the course Linear algebra and matrices I. The course was taught in the Open University of the University of Helsinki during six weeks in May–June 2017. The course had 164 students, most of whom were degree students from the University of Helsinki, majoring in mathematics or a related discipline. Teaching in the learning environment was based on the Extreme Apprenticeship Model (Vihavainen, Paksula, & Luukkainen, 2011; Rämö, Oinonen and Vihavainen, 2016). It is a teaching model in which students take part in activities resembling those of experts. A central feature in the model is formative assessment.

All the digital features of the course were implemented in the digital learning platform Moodle. All course materials, such as lecture notes and problem sets were offered electronically, and all submissions were also handled electronically. Figure 1 shows the components of the learning environment divided into the five different dimensions indicated by Manninen and colleagues (2007).

Figure 1
*The reflection-centred learning environment divided into dimensions (Manninen et al. 2007).*




Each week, students were given a set of problems to solve. Some of the problems were digital tasks created with a system for automatic assessment called Stack (Sangwin 2013). For these tasks, instant automatic feedback was offered. Others were manual tasks completed with pen and paper and then scanned for submission. For a subset of the manual coursework, the students received written comments from the teachers or peers. The problems included real life applications of the topics discussed. Some tasks involved the use of a typical mathematics software Octave. For solving the problems, students were offered guidance in drop-in sessions that took place in a specially designed learning space. The students could also ask for help anonymously in an online chat room.

There was no final exam. Instead, grades were determined by formal self-assessment.

Students based their self-assessment on a learning objectives matrix which contained the objectives of the course, concerning both mathematical content and transferable skills. The latter included skills such as reading mathematical text and giving and receiving feedback. The students assessed their mastering of each topic and awarded themselves a grade for the course, with written reflection justifying their choice. The self-assessment was compared automatically to the coursework completed by the student, and if the two agreed, the grade given by the student to her/himself was confirmed. In case of discrepancies, the student was asked to justify their opinion or suggest themselves another grade.

## Goals of the Study

The aim of this study is to analyse the blended reflection-centred learning environment described above from two perspectives: 1) What were the levels of deep and surface learning experienced by the students, and what kind of subgroups of students were there in terms of their learning approaches? 2) Which contextual factors (Baeten et al. 2010) did the students connect with deep and surface learning approaches? These questions need to be asked in order to develop the reflective-centred learning environment in the direction that would foster deep learning approach. We need to find the student groups that require help in developing their learning strategies, and separately, we need to know which components of the course are best suited to promoting deep approach.

## Data Collection and Analysis

Learning approaches were tested with a ETLQ-questionnaire validated in Finland (Parpala, Lindblom-Ylänne, Komulainen, & Entwistle, 2013). The deep approach and surface approach subscales both consisted of four items ($\alpha = .62$ and $\alpha = .75$, respectively). The statistical measurements were conducted with IBM SPSS Version 24.

The qualitative data concerning the second goal of the study was collected after the pilot course with the same questionnaire as the learning approaches. The descriptive, one-shot questionnaire was designed with the guidelines of Lodico and colleagues (2010, p. 159–171). The questions were used both in research and in the development of the course, and the questions concerned, for example, the experiences about self-assessment ("*How did you experience the fact that there was no exam in this course?*") and support ("*How have you been able to benefit from the feedback during the course?*"). The questions about supportive elements in the reflective-centred learning environment were based on the interview questions by Mumm and colleagues (2015). The data was collected in Moodle.

The qualitative analysis was based on content analysis by Miles and Huberman (1994). The pool of all the open answers of the course feedback was used as a source for data. First, the content analysis was started as conventional (Hsieh & Shannon, 2005). We searched the data for all the expressions of deep and surface approach; this step was influenced by the previous knowledge about the theory of those approaches. Since the length of the answers varied, the analysis unit was chosen to be a coherent idea present in the text (Schreier, 2012, p. 131–134). There were 74 units where the students described their learning approach as deep or surface; these units were then reduced, deep and surface approaches separately. This way we looked for the contextual factors of the learning environment that influenced the learning approaches. The reduced expressions were grouped into categories, which were then grouped again as category classes and subclasses. Finally, the found category classes were connected with the theory of the elements of a blended learning environment (Manninen et al., 2007).

## Findings

**Levels of deep and surface approach.** Overall, deep learning approach (M = 3.89, SD = .66) was reported to be higher than surface learning approach (M = 2.07, SD = .70) after the course. A cluster analysis was conducted to determine whether there were any subgroups of students in terms of learning approaches. Deep and surface approach factors were considered to be the cluster variables, since that choice is aligned with the research question and the chosen variables encode the maximum amount of information about the students' learning approaches (Theodoridis & Koutroumbas 2006). Since there was no preconceived idea about the correct number of clusters, a hierarchical cluster merging was used for exploratory design (Antonenko et al. 2012). Ward's algorithm (Ward 1963) was chosen for clustering algorithm to decrease the differences among the clusters. The scores of the variables were standardized to Z-points before the analysis.

The data was first analyzed in the form of a dendrogram. Observing large gaps between the cluster sets (Olson & Biolsi 1991) identified three or four separate clusters. The number of clusters was ensured by performing a discriminant function analysis on the data (Romensburg 1984). The solution with three clusters predicted cluster membership by 92 %, whereas the solution with four clusters predicted that by 95,5 %. Since the four-cluster solution divided the students who reported a lot of surface approach into their own yet small group, it was selected as the most appropriate one for this study. The differences of the means of the main variables between the clusters are shown in Table 1.

Table 1
*Deep and surface approach, task points and course grades of the four clusters.*

| | | Surface approach | Deep approach | Tasks with | | | Grade |
|---|---|---|---|---|---|---|---|
| | | | | automatic feedback (max. 64) | no feedback (max. 40) | teacher feedback (max. 9) | (scale 1–5) |
| **Total** | M | 2.07 | 3.89 | 52.71 | 31.94 | 7.93 | 4.10 |
| n = 113 | SD | .70 | .66 | 10.45 | 8.76 | 1.70 | 1.45 |
| **Cluster 1** | M | 1.90 | 3.14 | 51.16 | 30.10 | 7.57 | 3.75 |
| n = 30 | SD | .41 | .51 | 10.68 | 8.19 | 1.96 | 1.74 |
| **Cluster 2** | M | 1.45 | 4.38 | 56.17 | 35.86 | 8.25 | 4.58 |
| n = 36 | SD | .28 | .41 | 7.65 | 7.34 | 1.76 | 1.20 |
| **Cluster 3** | M | 2.40 | 4.06 | 51.76 | 30.92 | 7.94 | 3.97 |
| n = 36 | SD | .28 | .41 | 10.75 | 8.56 | 1.57 | 1.45 |
| **Cluster 4** | M | 3.60 | 3.73 | 48.30 | 27.00 | 7.80 | 3.63 |
| n = 10 | SD | .39 | .48 | 14.92 | 11.45 | 1.03 | .92 |

The four clusters were compared using ANOVA. The assumption regarding the homogeneity of variance was met for all the other variables (Levene test, $p$ = .06– .73). The clusters 1 (n = 30), 2 (n = 36) and 3 (n = 36) are considered to be normally distributed by their variables. Cluster 4 (n = 10) was tested; all the variables were normally distributed (Kolmogorov–Smirnov test, $p$ = .61–.20). ANOVA was then used to identify differences between the clusters regarding the variables shown in Table 1.

Unsurprisingly, significant differences were found regarding surface approach (df = 3, F = 128.9, $p < 0.001$) and deep approach (df = 3, F = 45.19, $p < 0.001$). Apart from these differences the only significant difference between the groups was found regarding the points from non-assessed tasks (df = 3, F = 312.27, $p < 0.05$). Small yet insignificant differences were found between points from automatically assessed tasks (df = 3, F = 243.68, $p$ = .081) and course grades (df = 3, F = 4.73, $p$ = .078).

Finally, Bonferroni Correction Post Hoc Test was conducted to find out the exact clusters that had the most significant differences between them. All the clusters differed significantly in terms of surface and deep approach ($p < 0.05$) except clusters 3 and 4 that only differed in terms of surface learning. The students in clusters 1 and 4 were shown to have completed significantly less non-assessed exercises than the students in cluster 2 ($p < 0.05$). No other differences were found.

**Contextual factors connected with learning approaches.** The contextual factors connected with deep and surface learning approaches were investigated with a qualitative content analysis of the open answers of the students. The categories of contextual factors that promoted deep learning approach are shown in Table 2. The number of expressions found in each category is reported in brackets.

It was found that innovative assessment was the main contextual factor to enhance deep learning in the reflection-centred learning environment. Students also reported that student-centred course materials supported deep learning since they provided information about the exact learning goals and the relations between them. Interestingly, all the expressions found were connected with the digital learning environment, as seen in Table 2.

Table 2
*The contextual factors (Baeten et al., 2010) reported to promote deep learning approach.*

<table>
<tr><th>Categories</th><th>Category subclasses</th><th>Category classes</th><th>The dimension of the learning environment (Manninen et al.2007)</th></tr>
<tr><td>No exam (22)</td><td rowspan="2">Replacing exam with self-assessment</td><td rowspan="7">Innovative assessment</td><td rowspan="2">Digital & didactic</td></tr>
<tr><td>Self-assessment tasks (13)</td></tr>
<tr><td>Variety of feedback (4)</td><td rowspan="5">Various forms of feedback</td><td rowspan="2">Digital, social & physical</td></tr>
<tr><td>Formative feedback (3)</td></tr>
<tr><td>Feedback from peers (4)</td><td rowspan="2">Digital & social</td></tr>
<tr><td>Teacher-assessed tasks (1)</td></tr>
<tr><td>Automatic feedback (2)</td><td rowspan="3">Digital</td></tr>
<tr><td>Learning objectives matrix (6)</td><td rowspan="2">Course materials</td><td rowspan="2">Student-centred course materials</td></tr>
<tr><td>Lecture notes (1)</td></tr>
</table>

Students linked deep learning strategies directly with the learning environment in their open answers, but that was not the case with surface learning approach. Instead, surface approach was, for example, connected with traditional exams; students described that in other courses they might attempt to memorise the lecture notes a couple of days before the exam. Also, some students linked the non-assessed tasks with surface learning by mentioning that they completed them with lesser effort than the rest of the tasks, the reason being that the non-assessed tasks were unmotivating since no feedback was provided.

## Discussion

In our study, we found that the reflective-centred, blended learning environment, based on various formative assessment methods such as self-assessment, was a promising course experiment in terms of promoting deep learning approach. Quantitative and qualitative analyses were conducted to explore the levels of learning approaches within the student population and to find the course components that were connected with the processes of deep and surface learning.

Four student clusters differed from each other in terms of deep and surface learning approaches. Two of those student groups might be considered to be "at risk" as they reported either a high level of surface or a low level of deep approach. According to Wilson and Fowler (2005), there is a need to foster "deep shift" within the students that are typically surface-oriented learners. Our cluster analysis found only 10 students reporting high levels of surface approach, which might mean that our learning environment was able to foster deep learning even among those students that would typically be surface-oriented.

A qualitative content analysis showed that the students formed fewer connections between surface learning strategies and the course components than between deep learning and course components. This, together with the fact that the size of the cluster of students reporting high surface approach was small, indicates that in the future there might be a bigger pressure to promote deep learning approach than to prevent surface learning approaches in this kind of course context. These arguments should, however, be tested in a similar setting with pre-tests and deeper qualitative data.

The course grades did not differ significantly between the different clusters. We argue that this is because of the carefully built support system that allowed the students with less productive learning approaches to complete a large number of tasks, which then enabled them to assign themselves high grades. However, it was found that the students in "at risk" clusters completed less non-assessed tasks than their peers. Non-assessed tasks were also one of the only contextual factors that were linked with surface approach in the course feedback data. It might be that the lack of feedback for these tasks discouraged the students with less deep learning strategies from trying them, as these students were not prepared to reflect on their progress when there was no formal external assessment and no deep learning was therefore required. A similar effect has been observed in previous research, namely that there is a connection between tasks that do not require deep learning and the emergence of surface approach (Gijbels & Dochy, 2006; Struyven et al., 2006). In the future, there is a need to explore which elements of our non-assessed tasks promote surface approach.

In students' open answers, the reflection-centred learning environment was largely connected with expressions related to deep approach especially in terms of innovative assessment methods, which is in line with previous research (Baeten et al. 2010). Interestingly, all course components that students connected with deep learning approach were part of a digital learning environment (Manninen et al. 2007). Based on this finding and the large amount of deep learning approach reported by the students, we suggest that blended learning environments are viable surroundings for promoting deep approach. This finding has value especially in the context of large-enrolment courses, where digital tools can be used to alleviate the demand for resources needed to foster student-centred teaching.

There is a need to develop the digital self-assessment component of the course since it was connected with deep learning in our data. Studying the "at risk" clusters might help us in understanding how to engage different kinds of learners to reflect on their learning, and furthermore, to take responsibility of their own learning processes.

**Limitations of the Study.** The open questionnaire answers did not allow us to analyse further nor validate the four clusters of students in terms of learning approaches. Also, we focused on the contextual factors of the learning environment; a substantial analysis of the perceived contextual factors would shed more light on the roles of the different course components in supporting deep learning. In order to address these issues, broader qualitative data will be collected in the next implementation of the course, especially in the form of student interviews.

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## Pedagogical Considerations in the Selection of Examples for Definitions in Real Analysis

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*This study investigates mathematicians' pedagogical practices and associated beliefs about the use of examples to instantiate definitions in a real analysis textbook. We used task-based interviews, asking participants to revise the introductory presentation of a concept, including definitions and examples, to be of higher pedagogical quality. All mathematicians believed that examples and counter-examples are important in learning about a concept. In this report, we concentrate on how mathematicians take the collection of examples and student thinking into account when deciding on which examples to use and the types of criteria they use to determine an appropriate collection of examples for a definition.*

*Keywords:* Real Analysis, Examples, Instruction

Leinhardt, Zazlavsky, and Stein wrote that "A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases" (1990, p. 6). Similarly, researchers have asserted that "exemplification is a critical feature in all kinds of teaching, with all kinds of mathematical knowledge as an aim" (Bills & Watson, 2008, p. 77). This study explores the use of examples as a part of pedagogical practice in proof-based courses, and, in particular, real analysis. In these courses, one important way of presenting mathematical subject matter is via examples. In particular, a recent study of 11 proof-based undergraduate mathematics lectures (each between 60 and 75 minutes) included, among the findings, that 65 examples were presented across the lectures, with every professor discussing at least one example, and the median professor discussed 5 examples during a single lecture. That is, the presentation of examples appears to be a common part of the pedagogical practice of mathematicians while giving instruction about proof-based mathematics.

### The Pedagogical Importance of Examples

Authors have claimed that examples are important in developing conceptual understanding (Mason & Watson, 2008; Vinner, 1991) and knowledge and use of examples is a mark of expertise in mathematics (Michener, 1978). Examples have been claimed to help students develop understanding of mathematical definitions (Antonini, 2006; Leinhardt, Zazlavsky, & Stein, 1990), and, examples can help students interpret, create, and prove mathematical theorems (c.f., Cuoco, Goldenberg and Mark, 1996; Lakatos, 1976). As part of the theorem generalization process, examples have been described as essential for generalization and abstraction (Antonini, et al, 2011). The perceived pedagogical power of examples (Antonini, et al, 2011; Bills & Watson, 2008; Mason & Watson, 2008) has led to, among others, the exploration of graduate students' use of examples to determine the truth of conjectures (Alcock & Inglis, 2008) and of the principles K-12 teachers use in selecting examples to use with their students (Rowland, 2008; Zodik & Zaslavsky 2008). For example, teachers use examples to motivate basic intuitions or claims about new material (Michener, 1978). Similarly, there is evidence that asking students to generate boundary examples can help clarify the need for criteria in a definition or hypotheses in a proof (Mason & Watson, 2001). Interviews with mathematicians suggest that they also attribute some of these pedagogical values to examples (cf. Alcock, 2010; Michener, 1978;

Weber & Mejia-Ramos, 2011; Weber, 2012). More, they report using examples as part of their presentation of proofs (Alcock, 2010; Weber, 2012), and to instantiate claims or definitions (Alcock, 2010). Observational studies provide evidence that these claims are representative of their pedagogical practice (c.f., Fukawa-Connelly & Newton, 2014; Mills, 2014).

Mills (2014) observed four mathematicians teaching advanced mathematics courses and found that they used examples to motivate the statement of a theorem, instantiate a concept, or illustrate results. She did not describe or classify which examples the instructors used or their associated rationale for these choices. Fukawa-Connelly and Newton (2014) provided some insight in this regard by investigating one professor's use of examples of the concept of group in an abstract algebra class, drawing on the notion of the enacted example-space. They found that relatively few examples were made part of the content of the class, but that each of them was used repeatedly. Finally, Cook and Fukawa-Connelly (2015) surveyed and interviewed algebraists about the examples of groups and rings that the believed to be most important for students to know at the end of an introductory group theory course. They found that algebraists typically named classes of groups (e.g., the cyclic groups) rather than concrete examples and that there was relatively little consensus about the set of examples that students should know. The rationales that the mathematicians provided for their choices focused on the familiarity and ease of instruction about examples (using words like *simple* and *nice*), the historical foundations of the subject, the ability to demonstrate different ideas and concepts (including helping students avoid inappropriate generalizations), those that allow helpful visual representations, and experience the prevalence and variety of groups and rings. Finally, we note that Peled and Zazlavsky (1997) differentiated between different types of counter-examples frequently used by mathematics teachers. The described specific (e.g., the integers) that can show a statement is not true, and general (e.g., where one length in a figure might be a variable) that can help explain why a statement is not true.

The current study build on the literature in two important ways. First, we note that only two of these studies identify specific examples that mathematicians use in their teaching, and only one study directly reports on the corresponding rationale for the mathematician's choice of examples. Yet, that study explored examples at the most general level; asking about the entire collection of groups and rings that students should know *at the end of a course*, rather than exploring the examples used to explain what a group or ring is to a student. While Fukawa-Connelly and Newton (2014) examined the examples used to instruct students on the concept of a group, they did not interview the professor and so were unable to provide any of the instructor's rationale for his pedagogical decisions. We found evidence that mathematicians attend to several different categories of information when considering the examples in their text: individual examples and their properties, the connection of examples and its properties, what should be explicit in the text and how/where is should appear, and aspects of student (or reader) thinking and their relation to the examples and concepts. In this paper we advance two claims:

1. Mathematicians attend to properties of the collection of examples: the size, diversity, and ordering of this collection as well as whether it has duplicates.
2. Mathematicians attend to the aspects of student/reader thinking while working with the text: interaction with prior knowledge, intuition/informality and expected lack of intuition, and teaching general cognitive skills.

## Methodology

### Rationale

In this study, mathematicians were given four (researcher-created) introductions to concepts from real analysis textbooks, each of which included a definition statement (or more than one) and may have included examples and discussion of the concept. The mathematicians were instructed to revise these introductions to improve their pedagogical quality. We assert that the mathematicians' pedagogical thinking and values can be observed through the additions to and deletions from the text, their evaluations of the elements present in the text, and the rationales they give for their revisions.

This approach can show strong evidence that the mathematicians do care about an aspect of an introduction, but their silence on an aspect is not evidence that it is unimportant to them. Because of this asymmetry, we designed the introductions to be diverse with respect to all aspects that we identified in our literature review (including the types of revisions that Lai, Weber, and Mejia-Ramos (2012) identified), pilot interviews, and personal teaching experience, including the presence and number of examples, formality and abstraction, motivation, precision, and presence of normative notation.

We used participants' revisions, evaluations, and rationales for the revisions to form hypotheses about what they believed a good pedagogical introduction to a concept should include, with a focus on what the appropriate exemplification of a definition would be. We note that the exemplification appropriate for a text may not be appropriate for a lecture and we do not intend our work to make any claims about exemplification in lecture.

## Method

**Participants.** The first author invited mathematicians to participate in the study. He solicited the participation of 10 mathematicians, and we do not have any a prior reason to believe these participants more interested in or capable at mathematics teaching than other mathematicians. The research expertise of the participants included analysis, applied math, functional analysis, analytic number theory, and geometric topology and their teaching experience ranged from two years as a graduate teaching assistant to over twenty years as a teaching-focused institution. All of them had taught or were preparing to teach real analysis. We assigned all mathematicians a single-initial designator (that is not either of their initials) and refer to them with gender-neutral pronouns in order to protect their anonymity.

**Materials and Procedures.** Each participant met individually with the first author for a task-based interview. Participants were presented with four revision tasks sequentially. For each, they were given a 1-page, complete textbook introduction to a concept that we believed to be mathematically correct, told that the target audience for the text was a student in a junior-level real analysis course, and asked to revise it to improve its pedagogical quality. After they finished revising each introduction, they were asked to describe each change and their rationale for it, about broad categories of changes if they did not make them (including changing the collection of examples), about the aspects of the introduction they left unchanged, whether anything in the introduction was atypical, about their goals, and whether any of their comments would have been different in some medium other than a textbook.

Participants were told to "think aloud" as they were making their revisions. Because this task asked participants to improve the pedagogical presentation of the definition and because we explained that it would be for a textbook for a specific undergraduate course, we assumed the participants would treat these as pedagogical presentations. At times, we did need to clarify that the presented materials constituted the whole of the presentation of the definition and that the next items would be propositions and proofs. Each interview was audio-recorded and subsequently transcribed, and participants were asked to make any needed written changes (although they often specified multiple changes aloud while writing relatively few in comparison); any written productions were subsequently scanned.

**Analysis.** To analyze the types of revisions performed, we used an open coding scheme in the style of Strauss and Corbin (1990). While we were sensitive to the prior literature, because none of it related to textbook authorship and none focused on real analysis, we believed that an open-coding scheme would be more appropriate. For each revision (including purely evaluative statements) that a participant made on the revision tasks, we made a general description of the edit and what aspect of the presentation it was describing and used these aspects as category names. We then went through the transcript and noted the reason given for the edit (if any). For the purposes of this study, we then collected all revisions related to examples, and ignored those about other aspects of the introduction. We again engaged in another round of open coding, developing categories of codes that described any revisions and related codes for the rationales that participants provided. As appropriate, we coded new instances using categories that we had already developed or created new categories as needed. As we coded, we continually refined our coding manual, including revising names and definitions of categories, and, noting which sets of categories were orthogonal and which were overlapping, such that a particular instance should not carry codes from both categories. Once the categories were formed, we recoded all the data and resolved any remaining issues through discussion.

## Data and Results

All of the mathematicians indicated that they believe that examples should be part of the pedagogical presentation of a definition. For example, on Definition 4, we presented the definition without any examples and all of them mathematicians indicated that their revision would include examples. Some suggested that 5 was the appropriate number of examples to include with the pedagogical presentation of a definition. Their descriptions of their goals for examples suggested that they were thoughtful about which examples would be included, the collection and sequencing of the examples, the relationship between the examples and the text, and, the range of examples presented. We chose to highlight 2 primary findings from these interviews that illustrate the types of thinking that mathematicians exhibited with respect to examples, first quantitatively then with specific quotes.

The interviews produced 184 distinct comments about examples to be coded, with a total of 626 codes assigned. Of these 116 codes were given for comments about individual examples, 62 about the collection of examples, 74 about the text and its explicit elements, and 45 about student thinking. These data support the claim that the mathematicians attend to all four of these aspects of the examples in a textbook introduction to a concept.

We have asserted more specifically that mathematicians attend to properties of the collection of examples: the size, diversity, and ordering of this collection as well as whether it has duplicates. The code Collection and its subcodes were present in the comments for all four definition tasks (7,17,12,26) for a total of 62 coded items. Similarly, Collection and its subcodes were present in the comments from all ten participant interviews (7,6,5,6,8,7,5,4,5,9). We have also asserted that mathematicians attend to the aspects of student/reader thinking while working with the text: interaction with prior knowledge, intuition/informality and expected lack of intuition, and teaching general cognitive skills. Thinking and its subcodes were present in all four definition tasks (14,9,9,13) for a total of 45 coded items. Similarly, Thinking and its subcodes were present in the comments from all ten participant interviews (3,5,4,4,2,2,3,9,9,4).

**Collection of Examples**

All of the mathematicians attended to both individual examples and the collection of examples. The most common subcode of Collection was Diversity, which captures the participants' comments that the collection has or should have individual examples with different properties. For example, they valued having both examples and non-examples, "extremes of behavior", simple and complex examples, and various representations.

In this subsection, we illustrate how the professors claimed that examples should help students make sense of what the concept is. Dana, in discussing Definition 1, was very explicit in describing the types of thinking that examples needed to support:

> I'm trying to head off any confusion about exactly what the definition is of the increasing function. I'm trying to expose students to kind of broaden their universe in their head of what mathematical functions are. They're not just the functions that you differentiate in your calculus class. They can include functions that are not smooth and aren't defined everywhere. … to familiarize them with different types of examples.

We interpreted Dana as claiming that examples serve a number of roles in helping students come to understand a concept. First, that students might be confused about a concept and that examples can mitigate that. Second, that examples can force students to consider unfamiliar instances of known concepts, to "broaden their universe," when considering special classes of the previously known concept. In doing so, there are particular types of variation that professors might attend to, for functions that might be "not smooth" or "not defined everywhere." While not every professor was as explicit as Dana, they all indicated the importance of examples in helping students broaden their collection of examples to include more 'exotic' (which they often described as unfamiliar or complex) cases of common concepts.

We further illustrate how professors attended to how examples (including counter-examples) might support students in interpreting concepts. The first important way that professors believed examples can support student understanding of the definition of a concept is by helping to interpret the concept and distinguish it from other, similar, concepts (all of the professors made comments indicating that this was a consideration). For example, many of the professors made explicit statements about the value of counter-examples. For example, Kai claimed, "having counter-examples is just as important as having examples, knowing what something is and what isn't." Similarly, Cody claimed, "I often say to my students, "For every new definition we learn, we want to think of "an example but we also want to think of a counter-example." Cody then provided a specific instance, related to Definition 1, for which a counter-example would be helpful, "Yeah. Also find a function which is not increasing or strictly increasing." We interpreted Cody's statement as claiming that a function that is not increasing or strictly increasing would help students to understand how Definition 1, strictly increasing, is distinct from increasing or not-increasing, that is, the counter example could help illuminate the meaning of the inequalities in the statement of Definition 1. Brett explained why counter-examples are helpful, "If you just have a positive example of something, that doesn't help you really compare. Unless you have both a positive and a negative example it's hard to use that." We interpreted Brett as claiming that comparing examples with different features, especially one that has all of the needed features of a concept with one that does not, is one way that a learner might come to understand a concept, and, without a counter-example among the collection of examples, students would be unable to make such comparisons.

Similarly, the professors claimed that examples allow students to distinguish a concept from other, similar concepts, such as between a lower bound and a greatest lower bound. The idea of using examples to help students distinguishing between similar concepts was mentioned by all the professors at least once and at least once during the discussion of each definition. We illustrate this with two representative quotes of professor's the claims about using examples to differentiate concepts. For example, in discussing Definition 2 (lower bound and greatest lower bound) Kai claimed:

> I would add one more example, or I would modify an example. There is an example of the infimum, there are examples of lower bounds or examples of not lower bounds but in example 1, I would add that it is a lower bound but it is not the greatest lower bound because that thing gives counter-example to what an infimum is. It's a lower bound but not the greatest lower bound because –pi/2 is a greater lower bound; so I would add that.

In this case, the professor is asking to add or modify an example so that one of the examples will be a lower bound but not a greatest lower bound. Brett asked for a similar revision, "You can probably say it's something like -pi is a lower bound but not an infimum, or you can say that …" Both of these professors were using examples to illustrate the difference between two, similar, concepts.

Second, the professors were very explicit that they attended to whether individual examples had properties that were not required by the definition, and, when all of the examples in the collection had the same extra properties. That is, the professors attended to whether examples had unnecessary properties. In discussing the definition of a convergent sequence, Morgan noted, "Note that the limit doesn't have to be attained though it can be and I will probably give an example of that." This comment specifically notes a property that the definition of convergent does not require, that the limit be attained, and that the professor thinks that a good pedagogical presentation would include examples with both cases. Similarly, Harper noted that, "I don't particularly see it in the numerical example that they give, but the pictorial example includes it, meaning that the sequence doesn't have to be on one side of the limit. I believe the numerical examples that I see right now, all of them stay on one side of the limit, but the diagram does not, so that is fine." Our interpretation of Harper's comment is that the definition of a sequence does not require that the sequence "stay on one side of the limit," meaning that this is a property that is not required. Second, Harper attended to whether each of the examples had this property, evaluating each of the numerical examples and the diagram, and noting that in the collection, at least one example does not have the unneeded property. Similarly, Dana gave a positive evaluation of an example illustrating a property of infima, "I like that example, the fifth one, because it shows that the infimum may be a member of the set or it might not. Either one is possible. I like this example, the set's an interval." Here, we interpreted Dana's evaluation as first stating the unnecessary property, whether the infimum is a member of the set, and then giving a positive evaluation of the fact that the collection of examples includes at least one example that does not have that property. In the presentation, we will further support these claims and illustrate additional ways that the professors believed that examples could support student understanding of a concept.

### Student Thinking

All of the mathematicians attended to student thinking in relation to the examples. The majority of the codes in this category were for Thinking rather than a subcode, indicating either that there were diverse aspects of thinking under consideration that did not come together into subcodes or that the participants most often talked about student thinking in the context of the particular concept rather than general principles about thinking. The most common subcodes captured participants' comments that examples would or should be familiar or unfamiliar to students, that working with examples in the context of a particular example taught a general skill, and that students would or would not have intuition about the examples. While the participants valued building intuition for the concepts, they also valued non-obvious examples about which students would not have intuitive conjectures or about which their intuition would be wrong because these both generate pedagogical situations that are useful for teaching students to work carefully with definitions and proofs.

For real analysis, the most common source of salient prior knowledge was calculus. Jesse observes that "analysis is trying to formalize calculus. An increasing function is something that they've seen in calculus a lot. They've never really talked too much about lower bounds and greatest lower bounds in calculus." Moving beyond description of familiarity for students, Morgan suggests revising a scatterplot to the graph of a familiar function:

> I guess I'm thinking of taking a standard graph of 1/x and thinking of what the infimum of the values of the function are. Which is sort of related to what this example was going to do but it seems to me that it's again this is a graph that they have seen before however I think that it is ... Would have presented it in a context or something that they had not thought before. Actually, they have thought about asymptotes so maybe it would be a way to relate this new concept with something that they have seen before."

Conversely, Kai suggests that "Examples that they have not seen before or wouldn't necessarily come up with on their own, I think, would be really good for them to see."

The goal, according to Harper is "to make students understand the definition as deep as they possibly can, hence the types of examples that I provide. That's the nature of how I studied things back when I was a student. It's like looking for things in the definition which are weird, which are non-intuitive yet are included in this definition." And for Kai, "learning how to be very intentional in applying the language and notation of the definition, I think, is a good skill." Examples "help students see the pathology of things that can happen" (Jesse).

## Discussion

This study has made two main contributions. First, it provides a fine-grained analysis of the factors that instructors consider when selecting examples to instantiate a particular definition as part of a textbook. The instructors all claimed that examples (and non-examples) are important in helping students understand a definition, and claimed that their goals in example selection are aimed at exactly this; helping students understand the particular definition. Their descriptions of their goals for examples suggested that they were thoughtful about which examples would be included, the collection and sequencing of the examples, the relationship between the examples and the text, and, the range of examples presented. In doing so, it provided further evidence that mathematicians are thoughtful about their instruction. More, one of the criteria that we described show that mathematicians take student thinking into consideration in their choices of examples; attempting to avoid common errors, 'head off' inappropriate overgeneralizations such as by ensuring that the collection of examples does not all share a particular unneeded property, and support the construction of a rich example space. Due to limited space, we have only been able to discuss two of our categories of codes and those two without much detail. We will develop these and other themes in more detail and show more data in a presentation in at the conference.

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Figurative Thought and a Student's Reasoning About "Amounts" of Change

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*This paper discusses a student coordinating changes in covarying quantities. We adapt Piaget's constructs of figurative and operative thought to describe her partitioning activity in terms of the extent that it is constrained to carrying out particular sensorimotor actions on perceptually available material, and we relate such descriptions to her thinking about quantitative amounts of change. We conclude the paper by discussing how characterizing these nuances of student thinking in terms of figurative and operative thought contributes to current literature on covariational reasoning and conceptualization of concept construction.*

*Keywords:* Cognition, Piaget, Covariational Reasoning, Amount of Change

Researchers have shown that students' *quantitative* and *covariational reasoning*—the mental actions involved in conceiving measurable attributes changing in tandem (Carlson, Jacobs, Coe, & Hsu, 2002; Thompson, 2011)—are critical for their learning of function and rate of change (Ellis, 2011; Johnson, 2015; Thompson & Carlson, 2017). Stemming from the complexities of students' thinking, these researchers have called for investigations that identify nuances in students' covariational reasoning. We answer these researchers' calls by drawing on Carlson et al. (2002) and Saldanha and Thompson's (1998) notions of covariation to characterize a student's reasoning about amounts of change in various contexts. We extend extant literature using Piaget's (2001) notions of *figurative* and *operative* thought to explain the extent a student's reasoning was constrained to sensorimotor actions and perceptual results from those actions.

**Quantitative Reasoning, Covariational Reasoning, and Partitioning Activity**

Thompson (2011) described that the mental construction of a *quantity* involves "conceptualizing an object and an attribute of it so that the attribute has a unit of measure" (p. 37). Despite Thompson's use of "measure," he emphasized that reasoning about a specified quantity's value is unnecessary when reasoning quantitatively; sophisticated conceptions of quantity entail reasoning about a quantity's magnitude (i.e., amount-ness) while anticipating that it has an infinite number of measure-unit pairs (Thompson, Carlson, Byerley, & Hatfield, 2014). Such distinction between a quantity's magnitude and its measures enables us to account for reasoning about covarying quantities that is not constrained to the availability of values; focusing on a quantity's magnitude affords characterizing mental activity in terms of perceptual material associated with a quantity's amount-ness (e.g., a segment that represents a quantity of distance).

An individual imagining variations in a quantity's magnitude (and hence value) is positioned to reason covariationally. When reasoning covariationlly, "a person holds in mind a sustained image of two quantities' values (or magnitudes) simultaneously…one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (Saldanha & Thompson, 1998, p. 299). Building on Saldanha and Thompson's (1998) covariation, Carlson et al. (2002) specified mental actions involved in coordinating quantities, among which students' coordination of amounts of change of one quantity with respect to changes in another (Mental Action 3 in their framework) is central to our work here. An individual coordinating amounts of change imagines quantities' magnitudes accumulating in successive states (and possibly anticipates continuous covariation between these states; see Thompson and Carlson (2017)). To illustrate, a student reasoning about covarying quantities B and K can envision the magnitude ||B|| accumulating in equal accruals, construct the accumulation of ||K|| in terms of corresponding accruals, and coordinate those accruals in ||K|| to conceive ||K|| increasing by decreasing amounts with respect to ||B|| (see Figure 1a-c for an illustration with respect to the Taking a Ride task in Figure 3a). Because coordinating amounts of change involves the activity of constructing a magnitude's accumulation in terms of accruals, we use *partitioning activity* to refer to students' mental and sensorimotor actions associated with their producing and reasoning about these increments that may represent amounts of change.

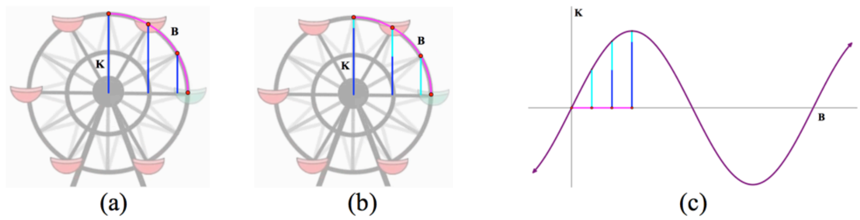



*Figure 1. As quantity B increases by equal amounts (denoted in pink), quantity K increases (denoted in dark blue, (a)) by decreasing amounts (denoted in light blue, (b)), which can be represented in a Cartesian system (c).*

## Figurative and Operative Thought

*Table 1. Figurative and Operative Partitioning Activity*

| Partitioning Activity | Foregrounded Actions of Partitioning Activity |
|---|---|
| *Figurative Partitioning Activity* | Repeating sensorimotor actions of partitioning tied to particular perceptual material and results; Potentially constrained to *available* perceptual material; Conceived invariance is with respect to sensorimotor actions and their perceptual results |
| *Operative Partitioning Activity* | Sensorimotor actions subordinate to mental actions (e.g., quantitative and covariational reasoning); Can anticipate partitioning activity on available or hypothetical perceptual material; Conceived invariance is with respect to coordinated mental actions and their transformations |

We have found the theoretical distinction between *figurative* and *operative* thought (Piaget, 1976, 2001; Steffe, 1991; Thompson, 1985) useful in developing models of students' partitioning activity. Piaget (1976, 2001) characterized figurative thought as based in and constrained to sensorimotor actions and perception, and he described operative thought as the coordination of mental operations so that these coordinations dominate figurative material (i.e. sensorimotor actions and perceptual material). We emphasize that characterizing a student's thinking as operative does not imply her thinking does not entail fragments of figurative material. Likewise, characterizing a student's thinking as figurative does not imply that her thinking does not entail operative schemes. A researcher's sensitivity to these distinctions is an issue of "figure to ground" (Thompson, 1985, p. 195). When a student's thinking foregrounds carrying out repeatable (mental or sensorimotor) actions and the results of those actions, it is figurative; when a student's thinking foregrounds the coordination of actions and transformations of those actions and their results, it is operative. The issue of foregrounding is important for describing students' partitioning activity because such activity necessarily entails figurative material (e.g., drawing and producing graphs and partitions) and likely entails operative schemes (e.g., understanding a coordinate systems in terms of directed distances). In characterizing students' partitioning activity in terms of figurative or operative thought, we thus make the distinctions in Table 1.

To illustrate these distinctions, consider a student determining if the graphs in Figure 2a and Figure 2b represent the linear relationship $y = 3x$. With respect to Figure 2a, a student who engages in figurative partitioning activity could imagine the graph in terms of the successive movements of one axes mark to the right (denoted in blue) and then three axes marks up (denoted in red), and associate such movements with a positive slope (Paoletti, Stevens, & Moore, 2017). With respect to Figure 2b, the student could conceive movements to the right (denoted in blue) along the graph as corresponding to movements down the graph (denoted in red), and associate such movements with a negative slope. In each case, the student's thought is dominated by carrying out or repeating particular sensorimotor actions to the extent that associations (e.g., a line falling left-to-right necessarily has a negative slope) are tied to that activity and its results. Hence, the student concludes that the two graphs are different.

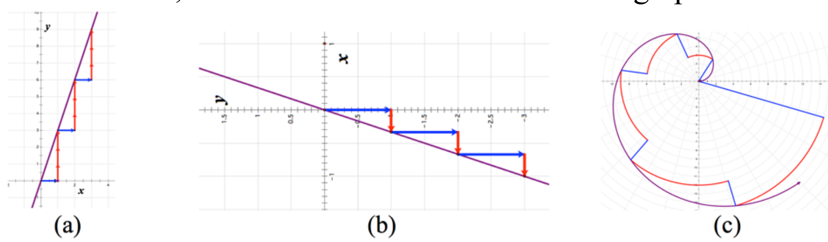



*Figure 2. (a) A graph that represents the relationship of y=3x in a Cartesian coordinate system, (b) a rotated graph of (a), and (c) a graph that represents the relationship of r=3θ in a polar coordinate system.*

In comparison, a student who engages in operative partitioning activity could conceive that both graphs are such that any directed change in $x$ corresponds to a directed change in $y$ three times as large as that in $x$. The student's partitioning activity is operative because she can coordinate and transform activity specific to each graph to conceive an underlying invariance that dominates figurative differences in activity. The student might anticipate re-presenting invariant partitioning activity in other contexts or coordinate systems (e.g., polar coordinates, Figure 2c). The anticipation of re-presenting partitioning activity aligns with Moore and Silverman's (2015) *abstracted quantitative structure*: a structure of related quantities a student has internalized as if it is independent of specific figurative material (i.e., representation free).

## Methods

This paper reports results of a semester-long teaching experiment (Steffe & Thompson, 2000) with prospective secondary mathematics teachers (PSTs; Lydia, Emma, and Brian). They were in their first semester of a four-semester secondary mathematics education program at a large university in the southeast United States. We conducted 10-11 teaching sessions (1 to 2 hours each) with each PST. The project principal investigator (the second author) served as the teacher-researcher (TR) at every teaching session. At least one other research team member was present as the observer(s). Each session was videotaped and digitized for analysis. In both ongoing and retrospective analyses efforts, we conducted conceptual analysis (Thompson, 2008) to develop models of PSTs' mathematics. Specifically, our iterative analyses efforts involved constructing hypothetical mental actions that viably explained the PSTs' observable and audible behaviors. We continually searched the data for instances that the models could not account for, and we modified our models or we attempted to explain developmental shifts in a PST's meanings. In this paper, we focus on the case of Lydia because of particular aspects of her partitioning activity that were consistent throughout the teaching experiment. We consider it important to characterize her ways of thinking in order to add nuances to our prior conceptualizations of students' quantitative and covariational reasoning.

## Task Design

We describe Lydia's activity on three related tasks: (1) Taking a Ride, (2) Which One, and (3) Circle. Taking a Ride included an animation of a Ferris wheel (Desmos, 2016) (see Figure 3a) and focused the students on constructing the covariational relationship between the *height* of the green rider above the horizontal diameter of the wheel and its *arc length* traveled (the sine relationship; Moore (2014)). Which One (Figure 3b) was presented after students' first encounter of Taking a Ride. It included a simplified version of a Ferris wheel (left) with the position of a rider indicated by a dynamic point. The topmost line segment (shown in blue, right) represented the arc length the rider had traveled counterclockwise from the three o'clock position. Students could vary the segment length by dragging its endpoint with the dynamic point on the circle moving correspondingly. We asked the student to determine which of the six red segments, if any, could accurately represent the rider's height above the horizontal diameter as the rider's arc length varied. Segment 1 is a normative solution and segments 2-6 vary with either different directions or rates. In students' initial attempt on these two tasks, we did not prompt them to graph because we wanted to gain insights into their reasoning with displayed magnitudes in contexts that minimized the influence of their previously constructed graphing meanings. For Circle (Figure 3c), we asked students to graph the relationship between the *horizontal distance* and the *arc length* associated with a dynamic point (i.e., the cosine relationship). Collectively, we designed the series of tasks to provide different figurative material to tease apart the extent that a student's reasoning was dominated by figurative or operative thought.

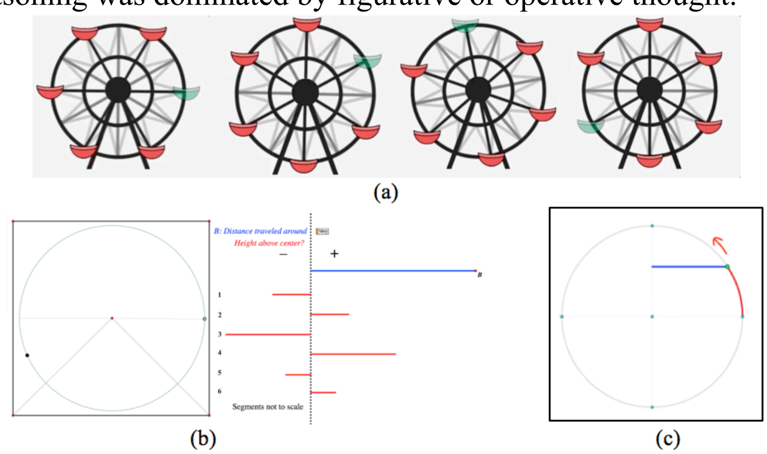



*Figure 3. (a) Snapshots of Taking a Ride, (b) Which One (with segment numbers labeled), and (c) Circle.*

## Results

In this section, we illustrate Lydia's partitioning activity with a focus on the figurative material constituting her partitioning activity as she considered a variety of representations.

### Re-presenting Partitioning Activity

In the first teaching session, we worked with Lydia on Taking a Ride (Figure 3a). With much effort, Lydia constructed what we perceive to be successive amounts of change of height for successive, equal changes in arc length (see her construction in Figure 4a-c). Noticing that the blue segments (in Figure 4c) decreased in magnitude, Lydia concluded that, "[A]s the arc length is increasing... [the] vertical distance from the center is increasing ... but the value that we're increasing by is decreasing." Suggesting she was excited that she had identified this relationship, she explained with enthusiasm, "I just discovered this by myself." This revealed that her activity of drawing partitions and identifying amounts of change was novel to her at the time.

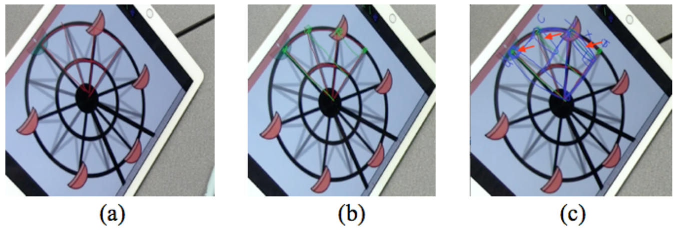



*Figure 4. Lydia (a) partitioned the distance traveled in equal increments, (b) identified height of the green rider in each successive state, and (c) identified amounts of change of height.*

Immediately following this task, we presented the Which One task (Figure 3b). After some explorations, Lydia claimed that she would like to choose a red segment that is moving at a constant rate. She eliminated four of the six segments and had hard time deciding which of the other two segments was moving constantly (Figure 5a). She then decided to orient one of them (a normatively correct solution) vertically, and put it inside the circle (Figure 5b). She then confirmed that the length of that segment matched the height of the dynamic point for different states (Figure 5c). When asked if the segment entailed the amounts of change relationship constructed in the initial Taking a Ride task, she responded:

*Lydia*: Not really…Um, I don't know. [*laughs*] Because that was just like something that I had seen for the first time, so I don't know if that will like show in every other case…Well, for a theory to hold true, it like – it needs to be true in other occasions, um, unless defined to one occasion.

*TR*: So is what we're looking at right now different than what we were looking at with the Ferris wheel?

*Lydia*: No. It's – No…Because I saw what I saw, and I saw that difference in the Ferris wheel, but I don't see it here, and so –

*TR*: And by you don't see it here, you mean you don't see it in that red segment?

*Lydia*: Yes.

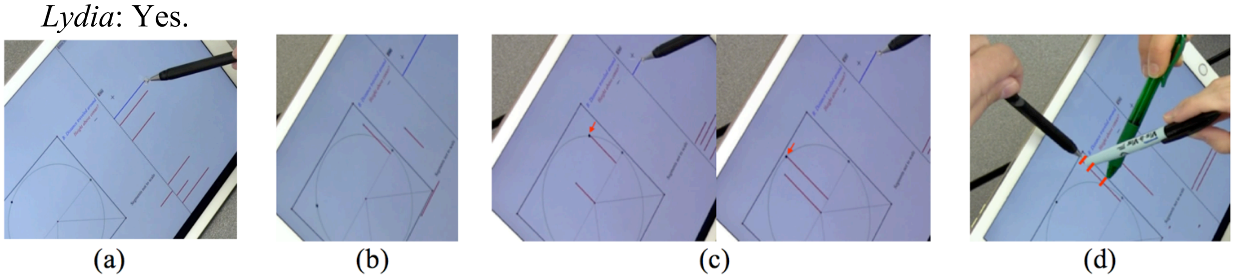



*Figure 5. (a) Lydia was working on the Which One task, (b) checking the red segment point-wisely, and (c) we were assisting Lydia to identify amounts of change in height.*

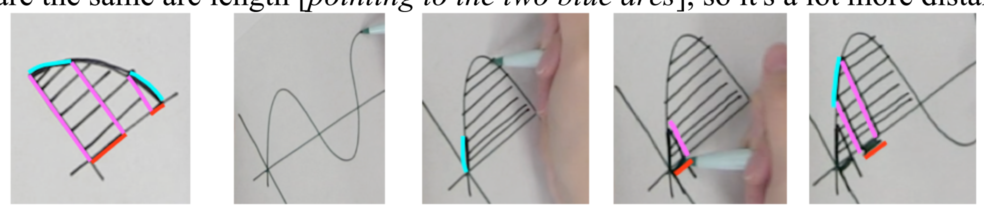
Lydia described height increasing by decreasing amounts as a "theory" that needed to be tested in this new situation. Her knowing that the red segment worked for each state did not imply by necessity that the red and blue segments existed in a covariational relationship consistent with that between height and arc in Taking a Ride. Following this exchange, and after the researchers created perceptually available material by using pens to denote amounts of change of the red segment (Figure 5d), Lydia responded in surprise that her "theory" held true.

We characterize Lydia's partitioning activity as figurative due to her difficulty re-presenting such activity from one context to another. She identified successive height accruals on the Ferris wheel (Figure 4c), but her understandings of amounts of change were rooted in carrying out activity and creating perceptually available increments in that context. When moved to a context with magnitudes changing continuously, she did not anticipate or re-present her partitioning activity. That is, as she considered successive red segment states in Which One, she was unable to hold in mind the red segment associated with a prior state to compare it to a current state.

**Situations, Graphs, and Figurative Material**

The TR began the fourth session by asking Lydia what she recalled from the previous sessions, in which she worked on Taking a Ride (Figure 3a) and Circle tasks (Figure 3c). She started with drawing the first quarter of a circle (see Figure 6a for work):

> **"**So we kind of said as the arc length is increasing in the first quadrant that our X distance is decreasing [*drawing the horizontal segments within the circle from bottom to top in Figure 6a*], and then…distance will decrease more in the same amount of space. So like from here to here [*highlighting the bottom blue arc*], then we'll say these are the same arc length [*highlighting the top blue arc*]…so we're going to take this point here [*marking a point at the top of the far-right pink segment*] and then drag it down [*drawing the far-right pink segment*], we've only lost this much [*highlighting the shorter red segment*]. And then from here [*drawing the middle pink segment*] to here [*tracing the far-left pink segment*] we lost this distance [*highlighting the longer red segment*], but we're saying those are the same arc length [*pointing to the two blue arcs*], so it's a lot more distance."



(a) (b) (c) (d) (e)

*Figure 6. (a) Lydia's drawing of the circle situation, (b) Lydia's graph, and (c)-(e) her construction process.*

Lydia's partitioning activity appeared compatible with that from previous sessions, and thus the TR asked Lydia how such activity related to graphing the relevant relationship. Lydia drew a graph (Figure 6b) and explained how the graph related to her partitioning activity in Figure 6a:

> "As we go up in arc length [*highlighting the blue curve in Figure 6c*]…that distance is decreasing [*drawing the horizontal segments from bottom to top in Figure 6c*], and so we see that here [*drawing the pink segment in Figure 6d*] is like this [*highlighting the red segment in Figure 6d*], and then [*highlighting the blue curve and drawing the pink segments in Figure 6e*] ... here is this [*drawing the red segment in Figure 6e*]. So that's the same conclusion we had gotten from the circle, so then we can say that this circle relates to this graph."

Lydia's partitioning activity across the situation and graph included: (a) drawing horizontal segments emanating from the circle and curve (see Figure 4a and 4c), (b) tracing arcs from lower end points to higher end points on the circle (denoted in blue, see Figure 6a) and tracing a curve on graph in the same manner (denoted in blue, see Figure 6c and 6e), (c) drawing vertical segments from the end points produced by the arcs and curve to a horizontal segment or line (denoted in pink, see Figure 6a, 6d and 6e), and (d) drawing horizontal segments between two pink segments and comparing their lengths (denoted in red, see Figure 6a, 6d, and 6e). We characterize Lydia's partitioning activity as figurative due to it foregrounding repeated sensorimotor actions that produce similar perceptual results (e.g., partitioning along something curved, drawing vertical segments, and drawing and comparing horizontal segments).

Providing additional evidence that Lydia's partitioning activity was figurative, later in the teaching session, Lydia drew a similar graph (Figure 7a) in order to discuss the relationship between "height" and "arc length". Her activity included tracing from left to right two equal horizontal segments (denoted in red, Figure 7a), drawing vertical segments from end points of the vertical segments to the curve of her graph (denoted in pink, Figure 7a), and tracing two corresponding curves on her graph (denoted in blue, Figure 7a). She compared the lengths of these curves and concluded that the increases in height became smaller. Similarly, on a circle, she traced two horizontal segments (denoted in red, Figure 7b), drew vertical segments (denoted in pink, Figure 7b), and traced and compared two arcs on the circle (denoted with blue, Figure 7b). Again, Lydia's figurative partitioning activity involved her carrying out same sequence of sensorimotor actions on her graph and circle, the elements of which entailed similar perceptual results (e.g., the sequence of drawing horizontal and vertical segments, and curves).

(a) (b)

*Figure 7. (a) Lydia's new with drawn partitions, and (b) Lydia's circle with drawn partitions.*

**Discussion**

Characterizing a student's thinking of amounts of change in terms of figurative or operative partitioning activity is significant in that it allows us to describe nuances in Carlson et al. (2002)'s covariation framework and, more generally, mental actions involved in quantitative reasoning (Thompson, 2011). A student's amounts of change understandings can differ in the extent that her partitioning activity is restricted to particular sensorimotor actions and the perceptual results of these actions. In this paper, we illustrated that a particular student's partitioning activity was figurative because it involved her seeking to repeat sensorimotor actions in a particular order across various situations. Furthermore, her partitioning activity was constrained to having perceptually available material. Consequently, when confronted with a novel situation in which these figurative elements were absent or carrying out the sensorimotor actions failed (e.g., Which One), she had difficulty re-presenting partitioning activity.

von Glaserseld (1982) defined *concept* as "any structure that has been abstracted from the process of experiential construction as recurrently usable…must be stable enough to be re-presented in the absence of perceptual "input" (p. 194). Characterizing partitioning activity as we have enables us to extend and apply this definition in the context of students' reasoning about relationships between covarying quantities. When a student abstracts her partitioning activity so that it is not tied to particular figurative material, thus mentally anticipating transformations of such (e.g., changing orientations or representations), she has constructed a *concept* related to this relationship (e.g., the concept of sine or rate of change). As Lydia's activity indicates, it is important for researchers to consider students' activities among a variety of contexts before making claims about their covariational reasoning and meanings. Moving forward, we call for continued explorations into how students reflect upon their partitioning activity and abstract quantitative relationships and structures (e.g., rate of change).

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Mathematical Knowledge for Teaching Examples in Precalculus: A Collective Case Study

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*The purpose of this collective case study is to examine mathematical knowledge for teaching examples in precalculus. The instructors involved in the study were experienced graduate teaching assistants who were teaching their course for the third time and were identified as good teachers. Utilizing a social constructivist and cognitive theory approach, I analyzed video recordings of enacted examples. The central question that guided this analysis was: What is the mathematical knowledge for teaching examples in precalculus? The goal of this study is to examine undergraduate mathematical knowledge for teaching from the perspective of practice, instead of relying on existing frameworks. As a result of this study, the author developed a model of mathematical knowledge for teaching examples in precalculus that includes knowledge of representations, students, instruction, specialized content, and connections when enacting high cognitive demand examples.*

*Keywords:* Mathematical knowledge for teaching, undergraduate, precalculus, cognitive demand, examples

## Introduction

Mathematical knowledge for teaching (MKT) has been defined as the "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball, Thames, & Phelps, 2008, p. 395). While MKT has been studied extensively at the elementary level (Ball et al., 2008; Carpenter & Fennema, 1991; Heather Hill, Sleep, Lewis, & Ball, 2007; Ma, 2010) and at the secondary level (Krauss, Baumert, & Blum, 2008; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012; Rowland, Huckstep, & Thwaites, 2005), research on MKT at the undergraduate level is still a growing field (Speer, Smith, & Horvath, 2010). The goal of this study is to contribute to that field by building upon the link between MKT and cognitive demand (Charalambous, 2010) in order to study mathematical knowledge for teaching examples in precalculus from the perspective of practice.

### Problem

Often, it is assumed that earning a degree in mathematics is what initially qualifies ones to teach at the undergraduate level. Historically, undergraduate instructors learned to teach by following the role model of mentors. However, Bass (1997) points out that there is much that cannot be learned through observations alone. To address lack of teaching preparation, many doctoral programs today offer teaching professional development (PD) for graduate teaching assistants, who will make up the future workforce of undergraduate instructors (Bressoud, Mesa, & Rasmussen, 2015; Ellis, 2014). While offering some teaching PD is better than none, the content of what is being taught is an important aspect to consider.

Of course, pedagogical knowledge is a component of teaching and should be included in GTA PD. However, studies have shown that despite their formal mathematical education, GTAs still lack mathematical knowledge that is needed for effective teaching (Kung & Speer, 2009; Speer & Hald, 2008). In these studies, the authors rely on existing frameworks for MKT that where developed at the K-12 level. While it is reasonable to assume that K-12 and undergraduate MKT are similar, Speer points out that there are important differences between K-12 and undergraduate teachers that need to be attended to (Speer, King, & Howell, 2014). Therefore, the goal of this study is to examine MKT at the undergraduate level from the perspective of practice, instead of relying on existing frameworks.

## Significance

As previously stated, there is little research on MKT at the undergraduate level. But why is it important to study MKT to start with? First, studies have found that *pure content knowledge* is not a predictor of teaching quality and student achievement (Begle, 1972; Greenwald, Hedges, & Laine, 1996; Hanushek, 1981, 1996). However, studies at the K-12 level have shown that *MKT* is a predictor of teaching quality and student achievement (Hill et al., 2008; Hill et al., 2007; Krauss et al., 2008). This knowledge is not usually taught in content courses, hence why many GTAs seem to be lacking MKT. While no measures of MKT at the undergraduate level exist, it is reasonable to assume that this positive relationship still exists at the undergraduate level. Therefore, if we can identify what MKT at the undergraduate level looks like and integrate it into GTA PD programs, we can have a positive impact on undergraduate education.

The other question that is reasonable to ask is why focus on precalculus? As the number of students needing to take introductory math courses for their degree increases, the teaching burden of math departments increases (Ellis, 2014). Approximately 1,000,000 college students take introductory level math courses each year (Gordon, 2008). Of these, approximately 85-90% are non-STEM intending (Rasmussen, Ellis, Lindmeier, & Heinze, 2013) and success rates are typically around 50% (Gordon, 2008). Even for STEM-intending students, studies have found that difficulty passing introductory-level courses is contributing to the "leaking pipeline" of students leaving STEM (Thompson et al., 2007). Therefore the instructional quality of precalculus has a large impact on undergraduate students.

## Background

While research on MKT at the undergraduate level is sparse, there does exist a large body of research on K-12 MKT. While my goal is to examine MKT at the undergraduate level from the perspective of practice instead of using existing frameworks of MKT that were developed at the K-12 level, the two are bound to be closely related. In an effort to situate my study within the existing field of research on MKT and avoid the assumption that I am attempting to study MKT at the undergraduate level in an epistemological vacuum, I will first present a broad overview of existing research on MKT. Also, I chose to study MKT by building upon its relationship with the cognitive demand of tasks. This decision was motivated by Charalambous' (2010) exploratory study, which found that MKT and the cognitive demand of enacted tasks are positively related.

**Mathematical knowledge for teaching.** Following the studies that showed that subject matter knowledge was not a predictor of teaching quality and student outcomes, Lee Shulman (1986; 1987) proposed that researchers begin studying pedagogical content knowledge. Shulman defined pedagogical content as going "beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (1986, p. 9). Shulman situated pedagogical content knowledge in contrast to subject matter knowledge, which is "the knowledge, understanding, skill, and disposition" of a subject matter (1987, p. 8). Since then, math education researchers have begun looking into professional knowledge for teaching mathematics. Hill, Rowan, and Ball (2005) found that elementary teacher's MKT was a significant predictor of student gains. Similarly, Baumert et al. (2010) showed that secondary teachers' MKT was a predictor of student outcomes. In both of these examples, the mathematical knowledge that is specific to the work of teaching is not *usually* taught in general undergraduate mathematics courses. Therefore, using the number of math courses taken beyond calculus is not the same as measuring content knowledge for teaching.

Speer, Smith, and Horvath (2010) conducted a literature review to search for empirical research on the practices of collegiate teachers of mathematics. As a result, the authors identified only five articles, indicating that "collegiate teaching *practice* remains a largely unexamined topic in mathematics education" (p. 100). Since then, more studies have been published specifically on MKT at the undergraduate level (Bargiband, Bell, & Berezovski, 2016; Callingham et al., 2012; Castro Superfine & Li, 2014; Firouzian & Speer, 2015; Hauk, Toney, Jackson, Nair, & Tsay, 2013; Jaworski, Mali, & Petropoulou, 2017; Musgrave & Carlson, 2017; Rogers & Steele, 2016; Rogers, 2012; Speer & Wagner, 2009; Vincent & Sealey, 2015). However, some of these studies utilize existing frameworks for MKT that were developed at the K-12 level, which can be problematic (Speer et al., 2014). Therefore, the purpose of this study is to contribute to this growing body of research by examining MKT at the undergraduate level from the perspective of practice.

**Cognitive demand and task unfolding.** Smith and Stein (1998) define *lower-level demand tasks* as "tasks that ask students to perform a memorized procedure in a routine manner" and *higher-level demand tasks* as "tasks that require students to think conceptually and that stimulate students to make connections" (p. 269). Stein, Remillard, and Smith (2007) also created a framework to describe the temporal process of task unfolding and factors that contribute to this transformation. In this process, teachers utilize a written task to formulate their intended task, which in turn influences the enacted task. Each phase in this process is motivated by the goal of producing student learning and is influenced by factors, such as teacher's beliefs and knowledge. In 2010, Charalambous found that there was a connection between elementary teachers' MKT and their ability to enact tasks at a high level of cognitive demand. It is this relationship between MKT and cognitive demand that I plan to build upon in this study.

### Purpose and Research Question

The purpose of this collective case study is to examine mathematical knowledge for teaching examples in precalculus. I will do this by first examining cognitive demand in order to identify examples that were enacted at a high level of cognitive demand. Building upon Charalambous' (2010) results, I believe that these examples will provide me with fertile ground for examining MKT. While I believe that MKT influences every stage in the process of task unfolding, this report will focus on the final stage of task unfolding. The central question that guides this study is: *What is the mathematical knowledge for teaching examples in precalculus?* To narrow the focus of this study, I will primarily attend to answering the following two subquestions:

1. What mathematical knowledge enables instructors to enact examples at a high level of cognitive demand?
2. How can we characterize this knowledge?

## Methodology

### Theoretical Framework

In order to study teacher knowledge, I will utilize a social constructivist lens as well as cognitive theory of the teaching process. A social constructivist lens assumes that "multiple realities are constructed through our lived experiences and interactions with others" (Creswell, 2013, p. 36). Social constructivist researchers believe that reality is shaped by individual experiences, utilize an inductive method of emergent coding, and often collect observational data. Schoenfeld's (1998, 1999) cognitive theory of the teaching process attends to teacher knowledge (as well as goals and beliefs) and how it influences decision-making. The reason why I chose this framework is because it attends to the reasons why a teacher makes certain instructional decisions and what knowledge enables them to do this. Also, it complements Stein et al.'s (2007) task unfolding framework in many ways.

**Setting and Participants**

For the purposes of this study, precalculus courses are defined to include the College Algebra, Trigonometry, and combined College Algebra + Trigonometry courses. The participants from this study were all instructors at the same large public university in the Midwest. At the university involved in the study, second-year graduate students make up the majority of the instructors for precalculus. Since second-year graduate students are teaching their own class for the first time, I chose to exclude them from my data set and instead only recruited participants who were teaching a precalculus course for at least the third time. The participants in this study included one Trigonometry instructor (Greg) and three College Algebra + Trigonometry instructors (Alex, Emma, and Kelly). All of them were graduate students in their third, fourth, or fifth year who had already earned their M.S. and were working towards their Ph.D. in mathematics. While they all were teaching their prospective course for the third time, they had 2.75 years of collegiate teaching experience on average. Also, all of the participants in this data set were recruited because their department had identified them as good teachers.

**Design and Procedures**

In order to answer my research questions, I am utilizing a collective case study design (Stake, 1995). In order to examine MKT more generally, I included multiple instructors and collected data on multiple examples. Since I have included a limited number of participants, there is little is known about mathematical knowledge for teaching precalculus, and I seek to propose new theoretical insight into MKT, I chose to utilize an exploratory case study (Yin, 2014). The unit of analysis I am focusing on is the examples enacted by precalculus instructors. Studying teaching from the perspective of practice can be difficult, so I utilized the frameworks of cognitive demand and task unfolding to help make the knowledge the teachers were using more visible. Building upon Charalambous' (2010) finding that MKT and cognitive demand are positively related, I utilized cognitive demand as a way to identify examples that would provide me with rich opportunities to examine MKT. Second, studying teaching through the task unfolding framework (Stein et al., 2007) allowed me to see the instructors' decision-making and examine how their mathematical knowledge enabled them to enacting examples.

Coding proceeded in two stages that concentrated on cognitive demand and then knowledge. In the first stage, I utilize the Task Analysis Guide (Smith & Stein, 1998) to code the cognitive demand of enacted example. Examples that were coded as enacted at a high level of cognitive demand were then analyzed in the second stage, which has two cycles. In the first cycle, I utilized inductive descriptive coding (Miles, Huberman, & Saldaña, 2014) to identify mathematical knowledge that enabled the instructors to enact the example at a high level of cognitive demand. This round of coding would help me to answer my first research question. To answer my second research question, I conducted a second cycle of pattern coding in order to identify emergent themes and relationships between the codes that resulted from the first cycle. A detailed description of this methodology can be found in Author (2017).

## Results

### Task Unfolding by Cognitive Demand

I will report the results from the first stage of analysis in brief, since the second stage of analysis primarily answers the research questions. In total, there were 39 examples included in the full data set. Of those, 13 examples were either included in the written lesson guide but not used by the instructor or included in their lesson plan but not enacted during class time. While these examples still involved the teacher utilizing their mathematical knowledge to make instructional decisions, this paper focuses on enacted examples, so they will not be discussed. Of the remaining 26 examples, 14 of them were enacted at a high level of cognitive demand. It is also important to note that all 14 of these examples were coded as procedures with connections tasks (Smith & Stein, 1998).

### Mathematical Knowledge for Teaching

In the second stage of coding, four main domains of knowledge emerged: representations, students, instruction, and specialized content. In addition, knowledge of connections between and within these domains was also a prominent domain of knowledge that emerged. For each of these domains, I will describe some of the related sub-codes and give examples of the mathematical knowledge that the instructors used in relation to these categories.

**Representations**. Since procedures with connections tasks are "usually represented in multiple ways" (Smith & Stein, 1998, p. 348), it is not surprising that representations emerged as a main domain of knowledge. Several instructors depended on *knowledge of representations that reflected student thinking*. For example, Alex introduced exponentials by having students compare simple and compound interest. After letting her students work on the problem for a while, she noticed that many students were working calculating compound interest recursively, so she drew a table that organized their calculations by year. Emma, on the other hand, recognized that her students were struggling to connect verbal descriptions of function transformations to their final graphical representations, so she drew the associated graph for each individual transformation. In teaching her students about the long-term behavior of polynomials, Kelly utilized *knowledge of accessible representations that still capture complexities* (e.g., $y = x, x^2, x^3$) in order to strip away unnecessary distractions and help her students focus on the important features.

**Students.** Instructors relied upon their knowledge of students in varying ways. Greg used *knowledge of common student struggles* and removed the goal statement from the written lesson guide in order to force his students to make connections between the problem and the content they had previously learned. Both Alex and Kelly applied their *knowledge of students' abilities* and designed their examples around tasks that students would struggle with, but were within reach. This also required the instructors to have *knowledge of student understanding.* Instructors also utilized *knowledge of appropriate questions to ask*, *knowledge of how to probe student thinking, knowledge of how to interpret student thinking,* and *knowledge of how to respond to student thinking* as they collaboratively worked through examples with the input of students. Emma also had to interpret and respond to student thinking, although she did so in the context of reviewing student quizzes and selecting an example that addressed a common mistake many students made. Another general sub-code that was categorized as knowledge of students was *providing explanations to students*.

**Instruction.** The two most common sub-codes that fell under the domain of knowledge instruction were *knowledge of instructional sequences* and *knowledge of problem scaffoldings.*

To help her students construct an exponential equation, Alex sequenced instruction so that students worked informally with concepts before they were formally defined, utilized familiar problems to reintroduce ideas, and provided motivation for topics. She also scaffolded their inquiry by introducing a table. Emma scaffolded problems by building connections between algebraic and graphical representations and sequenced instruction by first utilizing familiar, but inefficient, methods before introducing new, but more efficient, methods. Also, Greg utilized *knowledge of how to guide instruction towards the mathematical point* by choosing to not pursue a student suggested idea that might detract from the main goal of the example.

**Specialized Content.** While knowledge of course content influences all of the domains, some sub-codes related primarily to specialized content knowledge that goes beyond the content covered in the course. For example, instructors had to rely on their specialized *knowledge of reasonable and appropriate examples*. While some of this was planned, other times it was something that instructors had to do on the spot. For example, Alex initially introduced function compositions generally. However, she decided to make the example more concrete and constructed functions that were reasonable and appropriate. In order to come up with accessible representations that still captured complexities, Kelly drew upon her *knowledge of critical and non-critical features* of functions and their long-term behavior. In explaining why a certain answer was incorrect, Emma utilized *knowledge of how errors impact the final solution*. While these may all be examples of content knowledge that the instructors would like their students to develop, they were not part of the intended learning outcomes for the course and therefore make up specialized content knowledge that the instructors drew upon when teaching.

**Connections.** Given that all of the examples were coded as procedures with connections tasks, connections emerged as another main domain of knowledge. However, this domain is different from the others in that it is not independent, but rather captures knowledge of relationships between and within the other four domains. Instructors relied upon their knowledge of connections in a variety of ways. For example, Kelly drew upon her *knowledge of related topics* in order to illustrate how the multiplicity of zeros relates to the behavior of a polynomial function at its zeros. In order to help students understand the purpose of an example or a single step, Alex and Emma relied on their *knowledge of connections between mathematical computations and problem-solving goals.* In many cases, instructors combined their knowledge of connections and their pedagogical skills in order to build *knowledge of how to help students make connections*.

## Discussion

In analyzing the data, I found that knowledge of representations, students, instruction, specialized content, and connections enable instructors to enact examples at a high level of cognitive demand. Since knowledge of connections is really knowledge of how the other domains are connected, I represented this model as a pyramid (Figure 1) with specialized content as the base and connections as the edges. In addition to making connections to different domains, knowledge of connections can also be used within a single domain. Finally, knowledge of students, instruction, representations, and connections are all situated within and build upon knowledge of course content, but I chose to not focus on this type of knowledge in my model.

## Conclusions

Given that examples are an important part of teaching, this model can be used in designing teaching PD opportunities for GTAs. In particular, PD should be designed to help GTAs develop knowledge of representations, students, instruction, specialized content, and connections. This model benefits the community of math education by providing a decomposition of the knowledge used by instructors when teaching examples in precalculus. While it is similar to other models of MKT, it is also different in several important ways. First, the domains of knowledge are inherently connected. Second, while knowledge of representations and connections are implicit in many of the other models, they are not explicitly emphasized.

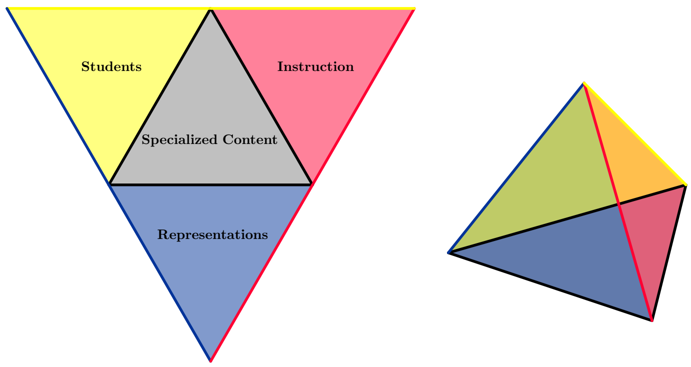


*Figure 1 Proposed model for mathematical knowledge for teaching examples in precalculus.*

## Limitations

First, as noted previously, the five domains of knowledge are not assumed to be independent. From a quantitative standpoint, this is a limitation of the model, but I believe it accurately reflects the interconnected nature of teaching. Second, since all of the high cognitive demand examples were coded as procedures with connections tasks, this model may overemphasize knowledge of connections and representations. However, "doing mathematics" may not be well suited for examples and it may be reasonable to assume that most high cognitive demand examples are procedures with connections tasks. Also, since this study was a collective case study and all of the instructors were graduate students, it may not be generalizable.

## Future Research

There is still much work that needs to be done to understand MKT at the undergraduate level, but this study provides a starting point for future investigations. In particular, it would be interesting to extend this study in several different directions. First, expanding the sample size and including instructors with a variety of backgrounds and teaching experience would test whether or not the model could be generalizable. Second, observing enacted examples that are "doing mathematics" tasks (Smith & Stein, 1998) would help further refine the model and test whether or not "procedures with connections" tasks had a large influence on the knowledge domains that emerged. Third, in order to understand post to better understand MKT at the undergraduate level at large, it would be beneficial to collect classroom data that focuses on more than just examples. Finally, my intention is to dig into the entire process of task unfolding and see what knowledge instructors use in the planning stage and utilize pre- and post-observation interview data to dig further into the knowledge used by instructors when teaching precalculus.

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# Using Quantitative Diagrams to Explore Interactions in a Group Work and Problem-Centered Developmental Mathematics Class


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*Despite low success rates and an academically vulnerable population, classes taught at the pre-college (or developmental) level have rarely been examined by mathematics education researchers. Mathematical Literacy, a recent developmental curriculum innovation, aims to better meet the unique needs of developmental students through group work and problem-centered materials. Using a novel quantitative representation of the classroom and descriptive statistics, this study examines the productivity of developmental mathematics students, their engagement with each other and the instructor, and their access to the curriculum in a Mathematical Literacy implementation. We find that students' engagement is regular, productive, and frequently involves the instructor. However, some students have less access to group discussions. Future implementations should focus on early identification of such students.*



*Keywords:* Developmental mathematics, community college, quantitative methodology


Enrolling over one million students (Blair, Kirkman, & Maxwell, 2013), mathematics classes offered at the pre-college level (often called *developmental*) serve those enrolled in academic tracks requiring advanced coursework, but not deemed knowledgeable enough to take college-level, credit-bearing mathematics classes. Most developmental mathematics is taught at community colleges (Blair et al., 2013), which serve students from drastically different stages of life, ranging from fresh out of high school to adults returning to school after many years (Cohen, Brawer, & Kisker, 2013). Problematically, many of the students who start developmental classes never finish (Bailey, 2009; Bailey, Jeong, & Cho, 2010), meaning that developmental coursework serves a gatekeeping function. Thus, examination of the issues surrounding the curriculum and instruction of these classes is critical.

Classroom-level research on developmental mathematics curriculum and instruction is limited, although calls for more high-quality investigations of community college mathematics are common (e.g., Condelli et al., 2006; Mesa, in press; Mesa, Wladis & Watkins, 2014; Speer, Smith, & Horvath, 2010). Most existing work focuses the instructional methods teachers use, rather than on the interplay between the curriculum, students, and instruction (Mesa, in press). Existing work shows that developmental teachers tend to use methods emphasizing skill acquisition (Grubb et al., 1999). Given that many students have previous taken and failed to learn the material (Hoyt & Sorensen, 2001), developmental students may benefit from teaching that uses different approaches.

## Mathematical Literacy at Fields Community College

Fields Community College (FCC; all names are pseudonyms), a large community college in a small Midwestern city, has recently become involved in the *Mathematical Literacy* movement (Statway and Quantway are the most well-known and widely implemented of these classes [e.g., Hoang, Huang, Sulcer, & Yesilyurt, 2017]). In addition to supporting student content learning, the designers of *Mathematical Literacy* aimed to (a) make the content relevant to the academic needs of the developmental students, and (b) highlight how mathematics informs students' lives.

To meet these goals, the *Mathematical Literacy* curriculum centers around real-world problem solving facilitated through group work, echoing the calls of the National Council of Teachers of Mathematics' (NCTM; 1989, 2000) push for more problem-centered instruction in the K-12 curriculum. Although similar in intent, *Mathematical Literacy* is tailored to meet the needs of developmental students, who, given their diversity in terms of demographics, life stage, and career objectives (Cohen et al., 2013), create a unique classroom of self-selecting, but often skeptical students. The differences between the developmental and K-12 populations, combined with the focus of *Mathematical Literacy* on group work, raise the question of whether and how these students will engage with the curriculum and their groups.

This study examines a *Mathematical Literacy* classroom, through a theoretical perspective focused on student enactment of the task as implemented (Stein, Grover, & Henningsen, 1996) and their patterns of interaction with the instructor. In particular, we ask:

1. Within their groups, how productive is student engagement with the curriculum materials?
2. Does everyone have equal access to group discussions within their groups?
3. What are students' patterns of interaction with each other and the instructor as they work?

To investigate these questions, we introduce a new method for examining classroom participation and productivity, demonstrating the utility of the method by showing results related to student talk within groups and instructor movement throughout the classroom.

## Methods

All the data for this study come from FCC, which implemented a *Mathematical Literacy* curriculum over the 2014-2015 school year. Data were collected in a single classroom taught by an instructor who had participated in the course development.

### Sample

The classroom started with 24 students, which adjusted to 22 students (6 men and 16 women) within the first week of the semester. Fourteen were White, six were Black, two were from Asian backgrounds, and one was Hispanic. Nineteen students agreed to be audio recorded in their groups. Because groups shifted throughout the semester and everyone in the group needed to assent to recording, at most 14 students were in audio-recorded groups at any given time.

### Data Sources

Data come from field notes and classroom audio, collected during weeks one, seven, 13, and 15 of a 16-week semester. This report focuses on the data from week seven. The full paper will include data from other weeks. Class met three times a week for 110 minutes and the first author was present for the entire class period. During observations, the instructor was audio recorded for the entire period. Student groups were audio recorded using table microphones. Recording started when the class transitioned to small-group work and stopped after the groups were gone. Field notes were taken.

### Analysis

The first two research questions examine how students engaged with the curriculum during group work and the equality of their participation. Using descriptive statistics and quantitative representations of the groups, we examine observed patterns of behavior within groups. For the third question, we perform a similar analysis, but look at patterns of engagement at the classroom level, focusing on instructor location and talk. All our results rely on (a) descriptive statistics on participation patterns, and (b) diagrams created from coded and timestamped transcripts of the classroom. Examples of these diagrams occur in Figures 1 and 2 in the results section. The methods for extracting descriptive statistics and generating the diagrams are described in the next few sections.

**Preparation of transcripts.** All group audio was transcribed in full. Turns were timestamped and began when a new person started talking or there was an extended gap in students' conversation. Transcripts were coded for:

1. Whether the instructor was present for each speaker turn.
2. The activity of the students in their groups.

To code the activity of students in their groups, we included isolated, non-related comments of individuals that were not remarking about something that others at the table were talking about. This decision reflects the fact that we were interested in group, not individual, behavior. At times, separate conversations occurred simultaneously. In such instances, turns were coded separately for the activity, and an indicator identifying that two conversations were occurring was added to the transcript. Group activities were coded as one of four main categories, which are elaborated on in Table 1:

1. Problems from the workbook, coded at the problem level.
2. Homework assignments, coded at the assignment level.
3. Class-related activities not directly related to a graded assignment.
4. Off-task talk.

*Table 1. Group activity sub-codes.*

| Code | Description |
|---|---|
| Problem From Workbook | Students work towards a solution on a problem from the workbook. These were coded at the individual problem level. |
| Homework Assignment | Students work on one of the two group homework assignments. These are coded at the assignment level (review or project). |
| Helping Other Groups | When students from other groups visited the recorded table and engaged in discussion with one or more group members about a problem the table had already solved. |
| Planning and Other Class-Related Talk | Negotiations about which assignment they should work on, group roles, times they could meet outside of class, check-ins to see if anyone needed help, organization talk as they transition to a new assignment, or discussions about mathematics not directly related to their work for the day. Classroom related discussions not related to completing the workbook problems or group homework assignments. |
| Off-task | Students talking about non-mathematical issues that did not directly contribute to their classroom assignments or the nature of mathematics. |

**Computation of times.** We computed the total time spent on each activity within groups by adding turn times together. This method of time calculation assumes that the group activity during periods of silence remained the same until the next utterance. Although this may not always be accurate, there is no reason to expect bias in favor of any activity.

**Creating group- and classroom-level diagrams.** Diagrams were created from timestamped and coded transcripts. All diagrams plot time against classroom or group activities and use colored regions to indicate different classroom-level activities (e.g., group work time, lecture). Group-level diagrams have two additional layers of information:

1. The *y*-axis lists the curriculum-level activities described in Table 1, with on-task activities lower and off-task group activities higher in the diagram.
2. Markers in the diagrams are linked to individuals to keep track of people through time as they move between group activities. Each color represents a unique speaker.

For all diagrams, markers indicate the start time of an individual's speaking turn, corresponding to the curriculum activity of the group at that time.

To visualize the whole class, we created diagrams that included multiple groups on a single plot (Figure 2 is an example). To minimize the overall complexity, three main modifications were made from the group-level diagrams:

1. Individuals within the same group are assigned the same color, rather than separate colors.
2. The instructor is tracked by her presence at the table rather than by her individual contributions using a larger, black marker.
3. Problem numbers are not explicitly labeled. To separate on- and off-task utterances, a dashed line cuts through the region for each group.

After creation, the diagrams were examined for participation patterns and deviations from the patterns. The diagrams are in some ways similar to the Chronologically Ordered Representations of Discourse and Features Used (CORDFU) diagrams that Luckin (2003) proposed. Our diagrams improve on CORDFU diagrams by using color strategically to separate individuals from each other, organizing the *y*-axis strategically, and clumping individuals around group activities to allow quick comparison of individuals.

## Results

### Group Engagement with Curriculum Materials

The average number of turns per individual suggests active, sustained conversations over the class, although individual participation varied greatly (Table 2). Emilia, in Group B, spoke an average of 38 times each day (about 3.4 minutes), while Craig, in Group D, spoke an average of 424 times a day (about 45 minutes). Groups spent an average of about 15 minutes off-task (Table 3), which includes a take a 10-minute break they were encouraged to take each class. Note that these averages reflect only table-level talk. Classroom observations documented students often texting or examining their phones during class. While these results only speak to off-task *talk*, when students were talking within their groups they generally discussed class-related issues. Thus, groups were productive in completing the materials.

### Access to Group-level Discussions about Material

The descriptive statistics suggest that student participation was not even within groups, but do not provide information on whether individuals had the opportunity to participate in group discussions about the material. To examine this question, we examined group-level diagrams. Several groups, in their diagrams, showed a closely working unit, with all members of the group either working on problems or going off-task together. However, this was not always the case. Figure 1 shows a group that clearly split into two overlapping sub-groups to cover the material. Closer inspection shows that in that group, although a student named Tyrone talked frequently (Table 2), it appears that he was supported through the material primarily by Sarah, rather than working through the material with his entire group. For example, Tyrone did not participate in completing the written task with the group on that day. This assignment, which will be elaborated on in the full paper, was one of the few opportunities for students to reflect on the mathematical content of the lesson.

*Table 2. Average number of turns and length of group contributions by individual per day.*

| Table | Individual | Days present | Average number turns (*SD*) | Total time (min) speaking (*SD*) |
|---|---|---|---|---|
| | Dave | 3 | 199(41) | 12.2(2.4) |
| | Felicia | 3 | 140(29) | 9.3(2.9) |
| | Sarah | 3 | 231(36) | 21.1(7.1) |
| | Tyrone | 3 | 371(71) | 30.4(8.0) |
| | Beth | 3 | 260(38) | 40.5(13.8) |
| | Carrie | 3 | 240(18) | 30.5(11.3) |
| | Emilia | 3 | 38(22) | 3.4(2.3) |
| | Henry | 3 | 189(21) | 17.4(3.3) |
| | Gabby | 3 | 116(22) | 9.3(3.2) |
| | Jen | 3 | 253(11) | 44.3(7.0) |
| | Carley | 3 | 231(82) | 15.4(15.4) |
| | Craig | 3 | 424(33) | 45.3(2.6) |
| | Fiona | 2 | 223(35) | 15.1(3.5) |
| | Helen | 2 | 206(131) | 14.3(9.1) |
| Total[a] | 14 | | 223(96) | 22(13.8) |

[a] This row reports the average contribution for each student, weighting each student equally.

*Table 3. Average time spent within groups on different group activities.*

| | Average[a] time (*SD*) | | |
|---|---|---|---|
| Activity | Total | Without instructor | With instructor |
| Workbook | 49.5 (5.9) | 41.8 (4.5) | 7.7 (1.5) |
| Reflections | 2.2 (1.9) | 2.9 (1.5) | 0.6 (0.6) |
| Group homework | 15.5 (13.7) | 11.9 (7.8) | 0.4 (0.3) |
| Planning and other class-related talk | 7.9 (2.4) | 6.0 (1.9) | 1.7 (0.6) |
| Helping | 1.9 (3.4) | 2.5 (3.0) | 0.0 (0.1) |
| Off-task | 15.8 (7.6) | 13.9 (6.8) | 1.9 (1.7) |

[a] Time was measured in minutes and averages were calculated by first computing the average time spent on each activity by group over the three days for which we have detailed records. We then averaged over the four groups.

## Students' Patterns of Interaction

Inspection of the classroom-level diagrams allow for easier comparisons of groups within a single day and more clearly show the instructor's patterns as she moved between groups. Figure 2 presents the class-level diagram for the same day shown in Figure 1. The diagram shows the instructor stopping at each group several times over the course of the period, although she spends markedly more time with Group A than the other groups. Interestingly, much of her time with Group A on the day shown in Figure 2 was off-task, rather than related to the curriculum materials. The graph also shows the instructor seems to arrive at groups while they are still actively solving a problem, rather than at the end of the problem (to help them finish the problem) or when they are off-task. There was not a time for whole-class discussion.

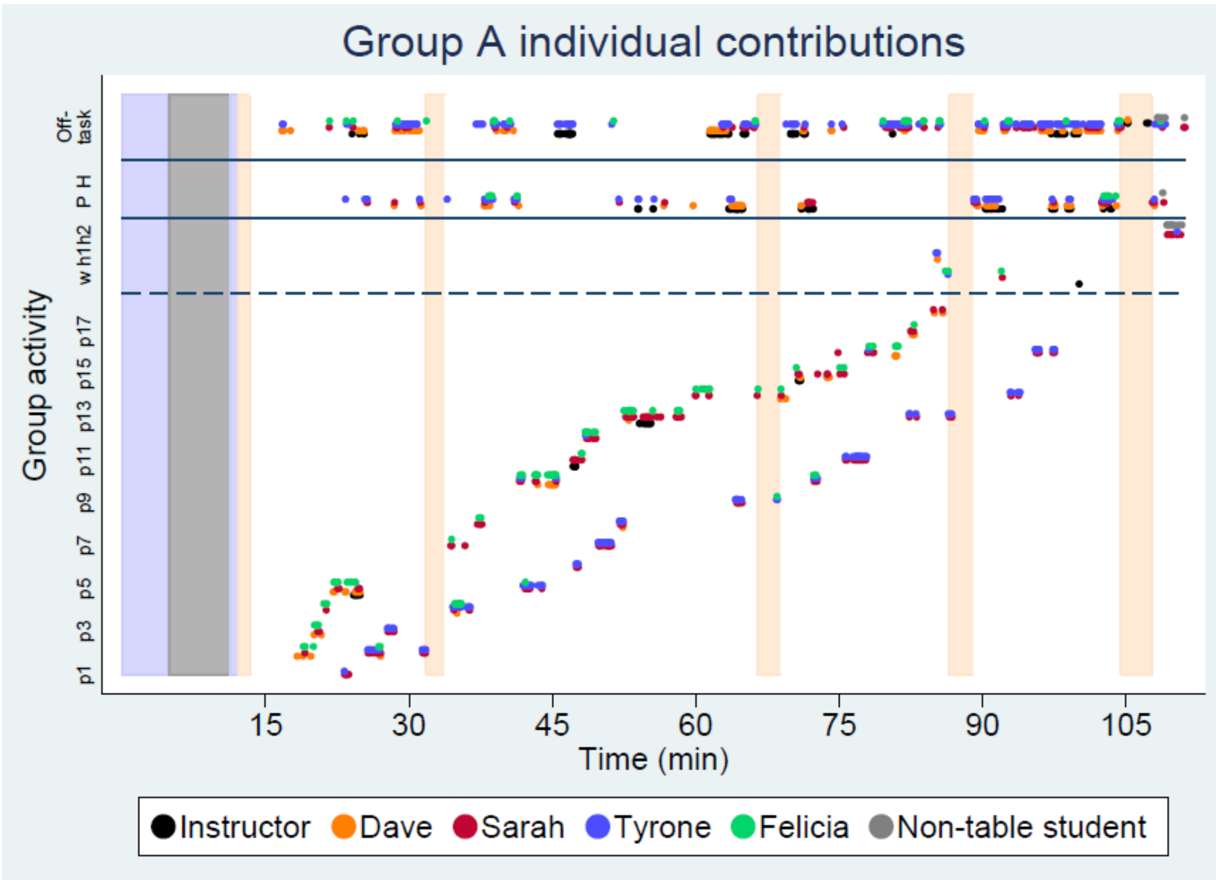


*Figure 1. Individual contributions to group activities for Group D. Each dot indicates when an individual started a new speaking turn. Recording of individuals started when the instructor opened the classroom up for group work (here, at minute 18). Light blue regions indicate when the instructor was returning assignments and checking in with students. Grey regions are when the students were engaged in a quiz or an activity related to the study. Light orange regions are when the instructor was lecturing to students and white regions are when the class was expected to be working in groups. The Group activity codes mean, in ascending order: problems 1 through 17 (skipping even problems) in the workbook, written reflection task (w), group homework assignments (h1 and h2), group planning and class-related talk (P), helping other groups (H), and off-task discussion.*

## Discussion & Conclusions

The *Mathematical Literacy* curriculum used at FCC was structured to provide students with multiple opportunities to engage in real-world problem solving. Student enactment of this curriculum showed that (a) students were regularly on-task and engaged with the curriculum materials, and (b) that student engagement with the curriculum was primarily within their groups. Whole-class discussions of the material were rare to non-existent but the instructor provided ample time for students to engage with the problems while the she monitored and supported their progress through regular visits. The instructor’s dramatic move away from lecture, the instructional method that dominates many developmental classrooms (Grubb et al., 2009), demonstrates that such an instructional shift is possible within developmental mathematics.

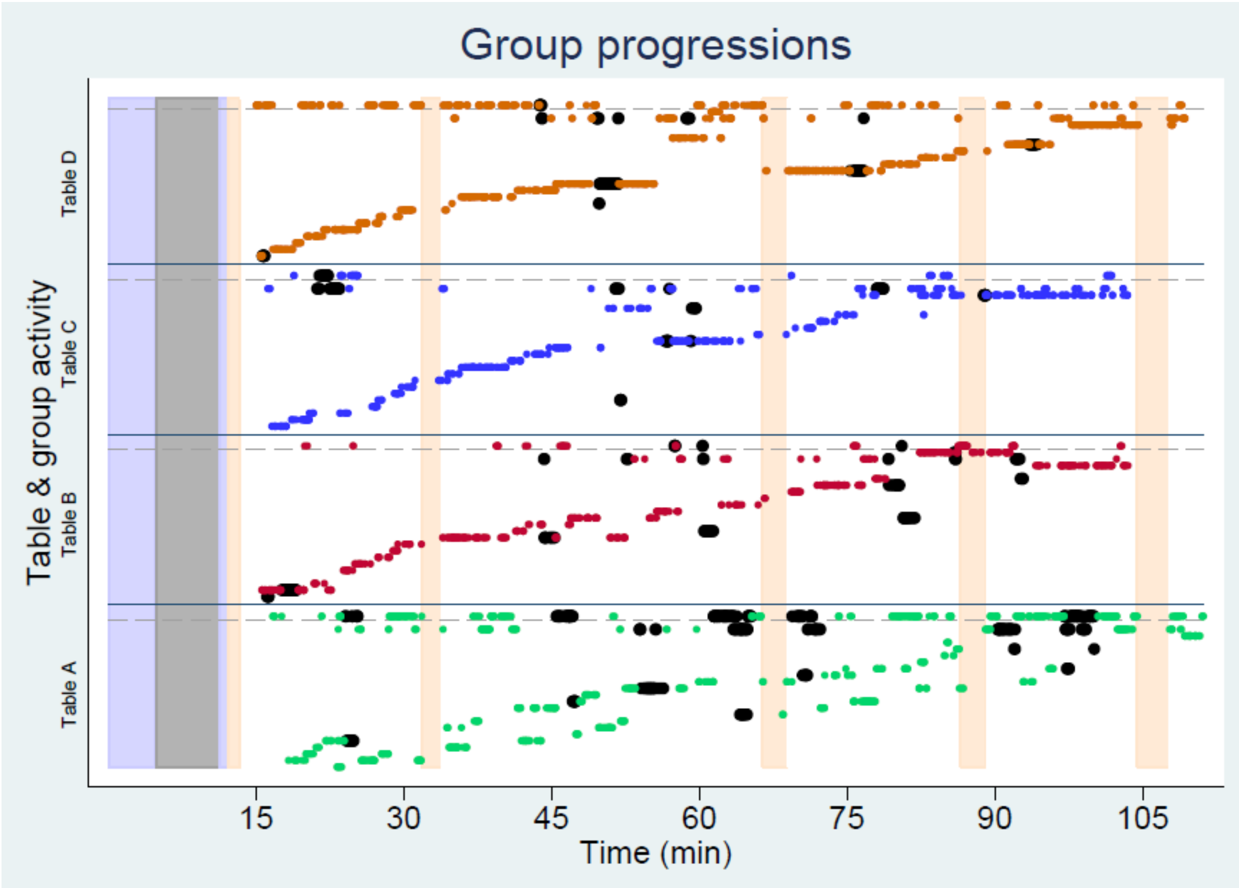


*Figure 2. Progression through group and classroom activities. See note for Figure 1 for an explanation of the shaded regions. Markers indicate that an individual started speaking at that time with respect to a particular classroom activity. Differently colored markers indicate different groups. The larger, black markers always indicate the presence of the instructor at the table, regardless of who spoke. Classroom activities are separated as they are in Figure 1; the dashed lines indicate the location of the off-task line for each group.*

Moreover, the students, although often having had many years of negative experiences in mathematics and only a short time frame to acclimate to group work and problem solving, can and did engage for sustained periods of time with mathematics problems in their assigned groups, a result that runs contrary to similar findings in K-12 mathematics: Wood and Kalinec (2012) offer one of the few looks at a group's time spent on a mathematics problem, and found that approximately 50% of the fourth graders' time was spent in off-task conversation. Adult students have been shown to have highly productive group work sessions in non-mathematical contexts (Barkaoui, So, & Suzuki, 2008). Our results suggest that developmental students in this context seem to exhibit a similar ability to self-regulate, perhaps because they have high achievement goal orientations (Mesa, 2012) and value efficiency in their education (Cox, 2009).

Although the regular engagement of the students and small amount of off-task talk is encouraging, the case of Tyrone and students like him who do not seem to have access to group level conversations about the material means that future implementations of *Mathematical Literacy* should consider ways to better meet such students' needs. Lastly, the use of quantitative diagrams, in addition to providing one method for helping to identify such students, can also be adopted as a research methodology that allows for the blending of qualitative coding with quantitative representations of the classroom.

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Evaluation of Impact of Calculus Center on Student Achievement

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*Many universities are spending resources to establish math tutoring centers. Sharing information about the effectiveness of such centers is crucial to determine how to allocate resources. We illustrate methods of evaluating tutoring centers. We investigate the question, "what is the association between students' attendance at the Colorado State University Calculus Center and their grade in Calculus II?" We found a statistically significant positive correlation between students' tutoring center participation and their grades.*

*Key words:* Resource Center, Calculus, Student Support, Tutoring, Evaluation

The Characteristics of Successful Programs in College Calculus study (CSPCC) recommended that universities have "proactive student support services" and found that tutoring centers foster "student academic and social integration" (Bressoud, Mesa & Rasmussen, 2015, p.viii) Ninety-seven percent of the 118 US institutions that responded to the CSPCC survey question about tutoring centers had a tutoring center (Bressoud, Mesa & Rasmussen, 2015, p. 70). Tutoring centers in both the UK and the USA are asked to evaluate their success to secure and maintain funding (Personal Communication, Mills, 2017; Matthews et. al. 2012). In addition to evaluation for funding, tutoring centers should be evaluated to determine the "optimal strategies for delivery of support" (Kyle, 2010, p. 104). Tutor training, education level of tutors, format of tutoring, use of technology, location of center and more varies from center to center (Bressoud, Mesa & Rasmussen, 2015; Perin, 2004, p. 563-564).

## Conceptual Framework: What counts as success?

It is difficult to measure if tutoring centers achieve their goals using data that is commonly collected. One goal of tutoring centers is that students learn "mathematics worth knowing" (Thompson, 2008, p. 46). We want them to understand calculus as a sensible tool to understand the rate of change and accumulation of real-world quantities (Thompson, Byerley & Hatfield, 2013). However, good scores on calculus tests do not imply students are learning mathematics worth knowing. The CSPCC study collected Calculus 1 final exams from 253 US universities. They found "the exams generally require low levels of cognitive demand, seldom contain problems stated in a real-world context, rarely elicit explanation, and do not require students to demonstrate or apply their understanding of the course's central ideas" (Tallman, Carlson, Bressoud & Pearson, 2016, p. 105).

Another goal of tutoring centers is to help students complete STEM degrees. Centers help students become socially and academically integrated into the university, which helps retain first year students (Bressoud, Mesa & Rasmussen, p. 82; Solomon, Croft & Lawson, 2010; Tinto, 1997). We recognize women are more likely to switch out of a STEM degree even if they are equally qualified as men (Ellis, Fosdick & Rasmussen, 2016).

These goals are important, yet hard to directly measure given data commonly collected. Many centers report the difficulties of both running a tutoring center and gathering and analyzing quality data (Matthews, et. al, 2012). Despite the acknowledged limitations, we define success as a positive correlation between a student's attendance at the Calculus Center (CC) and the student's score in Calculus II after controlling for other variables impacting success. Future studies could consider students' scores on validated assessments on calculus concepts and students' persistence to graduation with a STEM degree.

## Literature Review: Evaluating Tutoring Centers

Matthews, et. al ( 2012) wrote the most complete literature review of evaluation of tutoring centers located in the UK, Ireland, and Australia. We discuss a subset of the studies reviewed, plus additional studies from the US.

Some studies found positive statistical relationships between student success in courses and tutoring center attendance (Dowling & Nolan, 2006; Cuthbert & MacGillivray, 2007; Mac an Bhaird, Morgan & O'Shea, 2009). All of these studies suffer from the difficult to avoid self-selection bias. Students who are more likely to use tutoring center are more likely to share other characteristics that impact grades such as motivation. Some studies used qualitative data to evaluate the effectiveness of centers. For example, Carroll and Gill (2012) qualitatively evaluated a tutoring center using student evaluations.

Not all studies found a positive relationship between tutoring center attendance and grades. For example, Walker and Dancy (2007) found that students who attended a physics tutoring center had 20 percent lower mean exam scores than those who never attended (p. 138). They hypothesized that students who struggled self-selected to use the tutoring center.

Cooper (2010) found that a drop-in multi-subject tutoring center helped increase students' GPAs and persistence in college. Students who came to the tutoring center at least 10 times had on average 0.2 higher GPA and were 10% more likely to persist in college. However, Cooper (2010) did not find a relationship between students' tutoring center attendance and performance in particular courses.

### Evaluating the Impact of Tutoring

There have been hundreds of articles about the impact of one-on-one tutoring. Topping (1996) reviewed the literature about peer-tutoring for undergraduate students. Although many studies "suffered from problems of self-selection to groups" (p. 335), Topping found evidence that having advanced undergraduates tutor newer undergraduates improved tutee's grades and was cost efficient (p. 338). Leung (2015) conducted a meta-analysis of studies on peer tutoring in all subjects at the K-16 level. Leung computed a weighted mean effect size, for studies on tutoring, finding a significant weighted mean effect size, $d$=.43, $p$<.001, for undergraduate tutoring. This was found to be a larger effect than tutoring at kindergarten and elementary levels but smaller than that for secondary education. Leung found significant effect sizes at all academic ability levels and all school levels, but the meta-analysis does not address the differences in going to a tutoring center versus having a one-on-one tutor.

Colver and Fry (2015) noted "a vast majority of research that is available relies exclusively on correlational, qualitative, or other similarly limiting methodologies that make it difficult to glean insight into the causal impact that tutoring might have on student success" (p. 16). Annis (1983) randomly assigned students to read articles under control, tutor, or tutee conditions. Students who tutored others had significantly greater learning gains than those who were tutored. Lidren and Meier (1990) randomly assigned psychology students to receive frequent, minimal, or no tutoring. They found a statistically significant positive relationship between tutoring and success on class exams. Arco-Tirado, Fernández-Martín, and Fernández-Balboa (2011) randomly assigned undergraduates to receive tutoring on study skills and found that there was no statistically significant relationship between tutoring and success.

## Causation versus Correlation

Administrators want to know if tutoring centers or some other intervention is a better use of funds. Tutoring centers would like to show the center *caused* student success. Demonstrating causal relationships requires random assignment to the treatment condition and students can not be randomly assigned to use or not use a tutoring center. A correlation between tutoring center attendance and course grades does not imply a causal relationship because students self-selected to use the tutoring center. It is possible that the weaker students are more likely to self-select to tutoring and that we could expect tutored students to have lower grades (Munley, Garvey & McConnell, 2010;Walker & Dancy, 2012). On the other hand, we could argue that more motivated students are more likely to use tutoring and are also more likely to engage in many other behaviors that will increase their grades. We are not the first to note that many studies of tutoring should include control variables "to rule out the possibility that students with better skills (higher GPA) are more inclined to seek help than those with poorer skills (lower GPA (Perin, 2004, p. 580).

Most of the studies Matthews, et. al. reviewed did not provide evidence that students' improvements in grades were caused by the tutoring center because the studies did not control for self-selection bias. For example, Pell and Croft (2008) used tables to compare the percentage of students who earn various grades and tutor center attendance. They did not control for other variables. MacGillivray and Croft (2011) advocated for tutoring centers to use more rigorous methods to evaluate tutoring center success. They wrote "the essential concept is to compare performance relative to a base measure for those who used [the tutoring center] with the same relative performance for those who did not" (p. 15). They suggested use of students' prior GPA, results on a first assessment, and diagnostic test data as possible baseline measures. They noted two studies that used diagnostic testing as a baseline measure (Dowling & Nolan, 2006; Bamforth, Robinson, Croft & Crawford, 2007). Mac an Bhaird, Morgan, & O'Shea (2009) used students' performance in past school-level examinations as a baseline. Although MacGillivray and Croft (2011) noted that general linear models are useful for analyzing the relationship between many variables and student performance, they only noted one study of tutoring centers (MacGillivray & Croft, 2003) that used general linear models.

Munley, Garvey and McConnell (2010) used the student's high school rank, SAT math score, current college GPA, number of credits the student is enrolled for, freshman or sophomore status, gender, race, participation in Greek life, student attendance of recitation session led by graduate teaching assistants, and course instructor as control variables. They found that students who were tutored did not have statistically significantly different grades. MacGillivray and Croft (2011) also suggested similar control variables and also suggested using a diagnostic test.

General linear models are considered useful in evaluating the impact of education interventions in general. As detailed by Theobald and Freeman (2013), the most commonly used methods to analyze learning gains pre-post test data in undergraduate STEM education -- raw change scores, normalized gain scores, normalized change scores and effect sizes -- fail to control for observable student characteristics; hence, researchers should instead use linear regression to control for observable factors.

## Statistical Methods

Colorado State University established a Calculus Center (CC) in August 2016. The tutoring is provided by faculty who teach calculus, graduate teaching assistants, and undergraduate learning assistants who also attend the course that they tutor. The data was collected from four large sections of Calculus II taught by three different instructors.

We will model the average relationship between performance in Calculus 2 and the number of visits to the CC by estimating a generalized linear model (GLM) of the binomial family with a logistic link function. The dependent variable is each student's total score minus attendance, midterm 1 and graph extra credit scores and is used as a measurement of performance. Performance will be modeled as a binary grouped variable: the sum of 636 independent homogeneous Bernoulli trails, implying that Performance has a binomial distribution with parameter 636 (Gujarati & Porter, 2009 p. 557; McCullagh & Nelder, 1989 p. 102). This estimation technique models the sigmodal, non-linear relationship implicit to bounded endogenous variables that is neglected by ordinary least squares estimation.

The parameter of primary interest to this paper is the number of visits to the CC. The initial regression will have six required variables that control for student motivation and mathematical ability and 13 additional test parameters. The required variables include: three diagnostic math questions, midterm 1, attendance, high school GPA, and an indicator of low previous performance (denoted LLP) taking value of one if student reports a C or lower in previous calculus class. The test variables are number of visits to CC, honors section indicator, the number of times the student took Calculus 1, the number of times the student took Calculus 2, the student's total credit hours, honors status indicator, first generation indicator, minority indicator, masters or second bachelors indicator, international student indicator, age, and male indicator. The final models are obtained by running all possible subsets of the test parameters and selecting the model with the least exogenous variables within 2 of the minimum corrected Akaike information criterion (AICc): a change in the AICc that is less than two is negligible (Burnham & Anderson, 2002; Cavanaugh, 2009). AIC is a goodness of fit measure which eliminates the subjective judgment in hypothesis testing (Akaike, 1974). AICc is AIC with a larger penalty for additional parameters. Using the minimum AICc in lieu of p-values is done for predictive accuracy as it minimizes the distance between the true model and candidate model (Burnham & Anderson, 2002).

In estimating student performance, ordinary least squares regression (OLS) is commonly used. However, performance scores are bounded and OLS estimates are not. In general, using OLS with a proportional dependent variable that is bounded between 0 and 1 is only valid if most observations are within 0.3 to 0.7. Approximately 20% of our observations satisfy this criterion. Hence, OLS estimates may entail non-normal and heteroscedastic residuals, unbounded predictions and a reduction in explained variability in the dependent variable (Gujarati & Porter, 2009). Therefore, we use logistic regression with a non-binary response variable. While logistic regression is most commonly used with a binary response variable, it can also be used with a bounded proportion that falls between 0 and 1 as suggested in Papke and Wooldridge (1996). One disadvantage to logistic regression is the large sample requirement relative to OLS. As a general rule, logistic regression requires at least 30 observations per predictor variable. However; some argue that there should be at least 50 observations per predictor variable (Burns & Burns, 2009).

To help address the potential problem of heteroscedasticity (Gujarati & Porter, 2009), we will report two models. The first will use all observations and the second will omit all observations for which $Cooks\ D_i > 3 * Mean(Cook's\ D)$ (Cook, 1977). Cook's Distance (Cook's D) is a measure of each observation's leverage and residual values. It is used to identify influential outliers in a predictor set. By removing influential observations with the Cook's D criteria, we investigate if the estimates are robust to outliers that may be caused by our inability to properly control for previous mathematical ability and motivation.

Students with high Cook's D values correspond primarily to three groups: low performance students, students who checked into CC more than 60 times, and high preforming students without intervention. Points with high Cook's D values should be examined for validity (Stevens, 1984). Some students came to the CC every week between their classes to work on homework for other classes. Some high-performing students never attended the CC because they did not need tutoring. Finally, some students did not use center and showed no signs of effort to pass the class. We are most interested in students who were attempting to pass the class, using the CC to study calculus, and were not already so strong mathematically that they did not need tutoring. These observations justify dropping the observations with high Cook's D values from the model.

The empirical model is as follows:

(1) $$L_i = \ln\left(\frac{P_i}{1-P_i}\right) = \beta_0 + \beta_1 CC_i + \beta_j \text{Control}_{\text{ji}} + \epsilon_i$$

where the model includes j = 1, …, k control variables and i = 1, …, n observations with

(2) $$P_i = \frac{\text{Performance}_\text{i}}{636}$$

Equation 1 is weighted by the variance function $V(P_i) = 636\ P_i\ (1 - P_i)$ to achieve a homoscedastic error term. Performance is being modeled as the sum of 636 independent homogeneous Bernoulli trails, $\text{Performance}_\text{i} \sim \text{Binomial}(636, P_i)$, implying

(3) $$E[\text{Performance}_\text{i}] = \frac{\text{e}^{L_i}}{1+e^{L_i}} * 636$$

To measure the accuracy of the model, we will construct the variable Pass Prediction as

(4) $$\text{Pass Prediction}_\text{i} = \begin{cases} 1, & for\ E[\text{Performance}_\text{i}] > 445 \\ & 0,\ \text{otherwise} \end{cases}$$

and the variable

(5) $$\text{Pass}_\text{i} = \begin{cases} 1, & \text{Student}_\text{i} \text{ recieved C or better} \\ & 0,\ \text{otherwise} \end{cases}$$

to estimate the percent correct pass prediction for sampled students as follows:

(6) $$\text{Correct Pass (\%)} = \left(1 - \frac{\sum|\text{Pass Prediction} - \text{Pass}|}{N}\right) * 100$$

We also predict the number of students who, assuming they received the average benefit from visits to the CC and assuming causality, could have passed the class if they had gone to the CC two times per week for 15 weeks and the number of students who passed because of the visits they made to the CC by summing the variables from equations (7) and (8). Let $\widehat{P_i}$ denote predicted performance for student `i'.

(7) $$\text{Pass if used optimal CC}_\text{i} = \begin{cases} 1, \widehat{P_i} - \beta_1 CC_i + \beta_1(30) > 445 \\ \quad 0,\ \text{otherwise} \end{cases}$$

(8) $$\text{Pass from CC}_\text{i} = \begin{cases} 1, \widehat{P_i} - \beta_1 CC_i < 446 \\ \quad 0,\ \text{otherwise} \end{cases}$$

## Results

Models of all possible subsets of the test variables along with the required variables were analyzed to determine the final model, where the AICc of the final model is within 2 of the minimum AICc with the fewest independent variables. The final model includes all required variables and all test variables except Total Credits, Age, and Male.

Table 1

*Minimum AICc Logistic Regression Results*

| | All Observations | | Cook's D Omission | |
|---|---|---|---|---|
| | B | Robust SE | B | Robust SE |

| | | | | |
|---|---|---|---|---|
| High School GPA | 0.1825*** | 0.051 | 0.2034*** | 0.0436 |
| Midterm 1 | 0.0132*** | 0.0026 | 0.0159*** | 0.0015 |
| Attendance | 0.0274*** | 0.0033 | 0.0254*** | 0.0020 |
| Pretest item: chain rule | 0.2363*** | 0.0512 | 0.2546*** | 0.0459 |
| Pretest item: unit conversion | 0.1019* | 0.0522 | 0.0940** | 0.0431 |
| Pretest item: rate of change | -0.0114 | 0.0435 | -0.0139 | 0.0363 |
| Low Previous Performance | -0.1112** | 0.0492 | -0.1581*** | 0.0409 |
| Visits to CC | 0.0063** | 0.0026 | 0.0081*** | 0.0021 |
| Honors Section | 0.0926 | 0.1077 | 0.0655 | 0.0805 |
| # of times taking Calc 1 | -0.1434*** | 0.0366 | -0.1605*** | 0.0303 |
| # of times taking Calc 2 | -0.2521*** | 0.0709 | -0.2816*** | 0.0448 |
| Honors | 0.125 | 0.0857 | 0.1283* | 0.0736 |
| First Generation | -0.06 | 0.0624 | -0.0529 | 0.0498 |
| Minority | -0.0898 | 0.0603 | -0.0632 | 0.0469 |
| Masters/Second BA | 0.3291** | 0.1548 | 0.3407* | 0.2021 |
| International | 0.4576*** | 0.0953 | 0.5320*** | 0.0785 |
| Constant | -1.6393*** | 0.3449 | -1.7322*** | 0.2566 |
| N | 683 | | 636 | |
| Deviance/DF | 30.0656 | | 20.3085 | |
| Correct Pass | 88.1406% | | 91.3522% | |
| Passed if Used CC(Total) | 68 | | 54 | |
| Passed if Used CC (LLP) | 37 | | 34 | |
| Passed from CC Use | 37 | | 30 | |

Note: *$p$<.1. **$p$<.05. ***$p$<.01 and p-values were not used to select model.

In the more common use of logistic regression with a binary dependent variable, the interpretation of the beta values is generally done by examining the $e^{\beta}$, which is the multiplicative change in the odds of an observation being included in the group of interest (commonly labeled as 1) for a one unit increase in the independent variable. With a proportion as the dependent variable, the interpretation is not as straightforward as when inclusion in the group of interest is not the objective. However, the sign of the beta still carries the same general meaning. Negative beta values have an $e^{\beta}$ less than 1 which is a decrease in odds, while positive beta values have an $e^{\beta}$ greater than 1 which is an increase in odds. In the context of this example, negative beta values are associated with variables that are believed to decrease the number of course points earned while positive beta values are associated with variables that are believed to increase the number of course points earned.

Using equation (4), a student's expected performance can be calculated using their given characteristics. Using the all observations model, the expected performance of the "average student" in the course, using average values for numerical variables and modal values for categorical variables, is 78.7% with five visits to the CC over the semester (79.3% using Cook's D omission model). The performance of an otherwise similar student who visits the CC twice per week is 81.2% (82.4% using Cook's D omission model). This has a smaller effect on performance than attendance in the course however. The performance of an otherwise average student who attends class half of a standard deviation above average student is 80.5% (81.0% using Cook's D omission model) while the expected performance of an otherwise average student who had half of a standard deviation more visits to the CC than the average student is 79.1% (79.9% using Cook's D omission model).

Knowledge of some pre-requisite material made a marked difference in students' expected performance. Overall, 71.6% of students did not answer the chain rule pretest question correctly. The students that answered the chain rule item correctly have an expected performance score 3.7 percentage points higher for an otherwise average student (for both models). Similarly, the 29.0% of students who correctly answered the gallons to liters unit conversion question have an expected performance score 1.6 percentage points higher (1.5 using Cook's D omission model). A correct answer on the rate of change question was associated with a lower expected performance score, though only .2 percentage points (for both models) and the association is insignificant.

While the endogenous variable Performance ignores points from midterm 1, attendance and extra credit, the model is still able to accurately predict if the sampled students actually passed the class with approximately 88% accuracy for the all observations model and 91% accuracy after omitting outliers. Out of the 683 students included in the first model, we estimate that 37 passing scores may be attributable to the visits these students made to the CC. We also estimate that 68 students could have passed the class if they had gone to the CC two times per week. Thirty-seven of these 68 students reported a C or worse in prior calculus classes, indicating potential success for the CC as an intervention if students with low prior grades were properly targeted. These three estimates should be taken with caution because they unrealistically assume a causal relationship between performance and visits to the CC and assume each student receives the average gain from their visit to the CC. The estimates, when taken as a percent of observations included in model, do not substantially change when omitting outliers, but do slightly decrease.

## Conclusion

The results of these analyses suggest that increased visits to the CC is associated with a higher likelihood of passing Calculus 2. In addition to controlling for prior student achievement, as other studies have, this study also includes variables to control for same-semester achievement and motivation by controlling for an early test grade and attendance in the course respectively. In addition, the included independent variables can be used to identify which students are at risk of failing and may be able to pass the course with additional assistance from the CC. This information could be used by teachers to target borderline students and encourage them to seek assistance.

As is common with similar studies, the issue of self-selection is a non-trivial one. Due to this, it is not possible to prove that increased scores are a direct cause of receiving assistance from the CC rather than being caused by other lurking variables such as student motivation, uncontrolled for ability, etc. In a separate survey of the same sample we asked students to respond to the following statement: "I believe that I earned a better grade in the course because of the help at the Calculus Center." Twenty-seven percent of the 151 students who responded strongly agreed with the statement and 29% agreed. Only 10% disagreed or strongly disagreed. Of course, students cannot know for certain the cause of their success in class. However, if most students had said that the CC did not impact their grade, it would be evidence that the correlation we found was primarily due to lurking variables.

Despite our inability to demonstrate causality, we still think the results are significant because of the other studies that found no correlation between visits to a tutoring center and course performance (Cooper, 2010; Walker & Dancy, 2007). Anecdotal evidence suggests that having our tutors attend the course they tutor for and meet weekly with the instructor to discuss the math coming up is one of the factors leading to the CC success.

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# Developing Preservice Teachers' Mathematical Knowledge for Teaching in Content Courses


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*In this paper we present evidence that a) providing opportunities for PSMTs to engage with simulations of practice and b) making connections between advanced perspectives on geometry and 7-12 mathematics allows PSMTs to develop MKT in university mathematics content courses.*



*Keywords:* Mathematical Knowledge for Teaching, Preservice Secondary Mathematics Teachers, College Geometry


Mathematics teachers draw on understandings of and connections between various knowledge bases while doing the work of teaching (Hill, Ball, & Schilling, 2008; NRC, 2010). These knowledge bases include but are not limited to typical problems for mathematics content, how that content is situated in the larger mathematical landscape, different ways students might come to know the content, and pedagogical strategies and principles that are specific to that content (Hill, Ball, et al., 2008). These types of understandings, often referred to as mathematical knowledge for teaching (MKT), should be explicitly developed in those people who seek to teach mathematics to others (Morris, Hiebert, & Spitzer, 2009; Silverman & Thompson, 2008).

Preservice secondary mathematics teachers (PSMTs) in the United States face great challenges in developing their MKT due to more demanding state mathematics standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and a lack of opportunity to learn mathematics in ways that apply that learning to teaching situations. In response, the Mathematics Teacher Education Partnership (http://mte-partnership.org) has engaged in systematic research, development, and implementation efforts to improve secondary mathematics teacher preparation. As part of this effort, the Mathematics of Doing, Understanding, Learning and Educating for Secondary Schools (MODULE($S^2$)) Project has developed, piloted, and studied the effectiveness of curricular materials in a College Geometry course. The materials, in the form of three modules, interweave aspects of MKT into a rigorous content course, can stand alone or be used to form a coherent and complete College Geometry course, and develop understanding of advanced content. In addition, the modules provide opportunities for PSMTs to develop their MKT and better understand the nature of the field of mathematics and its practice. The modules are designed to that the activities can be completed by non-education mathematics majors and education majors alike, because the work of and thinking in K-12 classrooms is a valid setting for applied problems in a mathematics course. In addition, because PSMTs will be expected to teach according to standards that view mathematics as a social construction rather than something that students receive (White-Fredette, 2010), it is critical that they learn college mathematics in an environment that both embraces this view and provides opportunity for PSMTs to develop MKT in secondary classroom contexts that also embrace this view. In this paper, we present our efforts to understand how the implementation of the MODULE($S^2$) curricular materials might help PSMTs develop MKT.

## Perspectives on MKT

One way researchers conceptualize MKT is to think of it as several intertwined types of subject matter knowledge and pedagogical knowledge. Hill et al. (2008) define six kinds of this knowledge in their MKT framework:

- Common Content Knowledge: knowledge to solve typical problems in a content area;
- Horizon Content Knowledge: knowledge of the larger mathematical context in which the mathematics one is teaching is situated;
- Specialized Content Knowledge: knowledge of content that is unique to or motivated by teaching situations;
- Knowledge of Content and Students: knowledge of how students think about, understand, or come to know particular mathematics content;
- Knowledge of Content and Teaching: knowledge of pedagogical strategies and principles specific to the mathematics content one is teaching; and
- Knowledge of Content and Curriculum: knowledge of available curricular resources and how to sequence instruction using those resources.

University faculty who teach mathematics content courses to PSMTs operate on two different levels with regard to MKT. First, we draw on our own MKT as we teach. Second, we can attend to how we facilitate developing PSMTs' MKT, not only in teacher preparation courses but also in mathematics content courses (Eli, Mohr-Schroeder, & Lee, 2013). This study investigates this second level. One way researchers might gain insight into *how* PSMTs' MKT develops is to analyze their responses to simulations of teaching practice assignments over a period of time. Simulations of teaching practice provide PSMTs with an opportunity to engage in enacting teaching practices by describing a realistic classroom scenario where student work or student thinking on a mathematics task is shared and PSMTs respond in some way (e.g., plan a class discussion or provide an explanation). Simulations of practice focus attention on key aspects of teaching that may be difficult for novices but are almost second nature for skilled teachers. They engage PSMTs in responding to student thinking – a valuable act in which PSMTs rarely engage (Grossman, Hammerness, & McDonald, 2009). Completing simulations of practice can help develop specific aspects of MKT, such as the ability to analyze student work to better understand the mathematical connections students make (Eli, Mohr-Schroeder, & Lee, 2013). Further, we know that teachers with stronger MKT foster student learning with greater mathematical richness and appropriateness than teachers with weaker MKT (Hill, Blunk, et al., 2008).

Silverman and Thompson (2008) provide a framework that we utilized for analyzing simulations of practice in order to determine how PSMTs' MKT developed over time. Within this framework, instruction is conceived of as the teacher creating space for students to reflect on mathematical ideas and formulate powerful understandings together in "similar and consistent" ways (Silverman & Thompson, 2008, p. 507). In this instructional setting, a teacher's MKT for teaching a particular idea can be measured by the extent to which the teacher has:

- an advanced understanding of the idea "that [carries] through an instructional sequence, that [is] foundational for learning other ideas, and that [plays] into a network of ideas that does significant work in students' reasoning" (Thompson, 2008, p. 32) – known as a *key developmental understanding* (KDU) of the idea;
- developed models of the many ways that students might come to understand the idea – known as *decentering*;
- an understanding of how others might think of the mathematical idea in a similar way;
- an understanding of the types of activities and discussions that might occur during those activities that would support others developing similar understandings of the idea;
- an understanding of how students who have come to understand the idea in this particular way are empowered to learn other related mathematical ideas (Silverman & Thompson, 2008, p. 508).

## MODULE(S$^2$) Geometry Materials

Three geometry modules were each implemented individually as a unit within a College Geometry course during this study. In all three modules, learners were expected to be generators of knowledge while exploring geometry questions and problems. We now describe the modules.

The first module, *Axiomatic Systems*, challenges PSMTs' understandings of axiomatic systems with an opening examination of the concept of straightness in Euclidean and Spherical systems. Further explorations include discussions of other non-Euclidean axiomatic systems (e.g., projective, neutral, and hyperbolic geometries) that require PSMTs to consider how propositions and concepts defined in one axiomatic system transfer to another. Examples of activities that focus on building MKT include analyses of a classroom vignette (involving angles formed by parallel lines and a transversal) in which the teacher suggests that the class could choose a different angle relationship axiom as their starting point. This activity points to ideas about the structure of axiomatic systems and challenges PSMTs to draw on that knowledge as they consider alternative lesson structures. Another MKT development activity engages PSMTs in exploring the midpoint quadrilateral theorem and its related corollaries in Euclidean geometry. In both activities, PSMTs are required to draw on their understandings of deep, underlying concepts that form the foundation of topics taught in high school mathematics.

The second module, *Transformations*, begins with an exploration of bijective functions which map elements from the real plane to the real plane. During this investigation, learners generate definitions of transformations and isometric transformations, and it challenges their ideas of how one might explore transformations of the plane (e.g., point-by-point analysis, algebraic methods, or graphical methods). The series of activities that follow serve to deepen the PSMTs' MKT for transformations as they delve into horizon knowledge (e.g. how isometries of the plane form a cyclic group under function composition) and other related areas of pedagogical content knowledge. Other activities that seek to develop MKT in this module ask PSMTs to examine sample high school student work on constructing reflections and rotations in order to consider what the high school students may have been thinking. By the end of the module, PSMTs define congruent shapes from a transformational perspective, and they see the structure of axiomatic systems at work when they come to understand that they must introduce a reflection axiom. The module culminates with proofs of the triangle congruence theorems from a transformational perspective. Teachers with MKT rooted in understandings of geometry from a transformational perspective can help students engage meaningfully with geometric thinking instead of relying on algebraic or arithmetic methods of solving a problem (Seago et al., 2013).

The final module, *Similarity*, builds on PSMTs' understandings of transformations by adding a dilation to produce similarity transformations. Once PSMTs construct a dilation and describe it with clear mathematical language, they discover the need for having a way to measure. This propels learners into explorations of the Pythagorean Theorem and measuring area within an axiomatic system. In this module, PSMTs explore whether all parabolas are similar, and they prove the Triangle Similarity Theorems. Throughout all three modules, the materials provide opportunities for PSMTs to connect advanced perspectives in geometry to the content of K-12 geometry standards in the CCSSM. We contend that the specific efforts to include an examination of realistic classroom scenarios, sample student work, prevalent misconceptions, and connections between geometric ideas work together to develop PSMTs' MKT.

## Methods

In this investigation, the second author taught a College Geometry course with 16 students that was required of PSMTs in a secondary mathematics certification program but was open to all mathematics majors. All 16 students agreed to participate in this case study designed to answer the question: How do the modules help PSMTs develop MKT? In order to measure whether or not an increase in MKT occurred during the semester, we utilized a nationally validated Geometry Assessment for Secondary Teachers (GAST) (Mohr-Schroeder, Ronau, Peters, Lee, & Bush, 2017) measure. Each PSMT took the GAST at the beginning and end of the course, and the research team scored responses after being trained by GAST staff. In order to gain insight into *how* the PSMTs' MKT changed, we analyzed pre- and post- simulations of teaching practice assignments according to the Silverman and Thompson MKT framework.

We focus on two simulations of practice in this report. In the first simulation of practice, PSMTs viewed the classroom scenario in Figure 1 and responded using the following prompts at the beginning (Pre) and end (Post) of the unit. The **Pre-assessment Prompt** was: *A class has been working on properties of quadrilaterals, specifically proving that the pair of base angles of an isosceles trapezoid are congruent. During the discussion, a student makes the statement shown in the clip. What is the student thinking? How should you respond?* The Post-assessment Prompt included two parts: ***Part 1*** *– Write the words the teacher should say in responding to this student, and* ***Part 2*** *– What do you think the student in the previous depiction meant, and what you say about the Van Hiele level of this this student's understanding?*

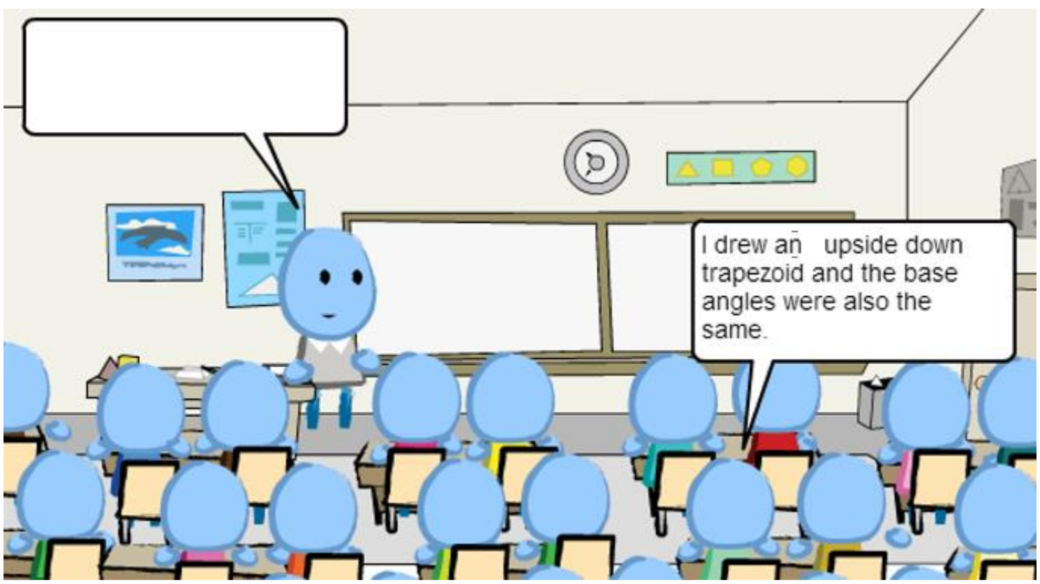



*Figure 1. A teaching scenario for simulation of teaching practice assignment #1.*

The second simulation of teaching practice utilized the student work in Figure 2 and the same prompt for both the pre- and post- assignment. The **Prompt** was: *Your students are working on reflection problems (reflecting segment* ***a*** *over line of reflection* ***r****). While circulating the room and observing the students' work, you encounter the two responses shown in the figures below. Explain how the students may have obtained their solutions and evaluate the result of their work. What feedback would you give the students?*

A research team of four coders analyzed PSMTs' responses to the pre- and post-assignments of both simulations of teaching practice. Here, we describe the five codes we matched to this framework. With regard to KDUs, it was difficult to find evidence to determine a) the level of advanced understanding for a PSMT's expression of a mathematical idea in the text of their response and b) whether or not the understanding was part of a network of ideas that carried through an instructional sequence. Therefore, if PSMTs made statements that were mathematically sound and we could envision the statement as being a critical piece of such a network of ideas, we coded the statement as ***KDU*** to indicate PSMTs may have at least a piece of a KDU. Because decentering means PSMTs understand models of ways students understand an idea that also differentiates their own point of view from another's point of view, we only coded for ***decentering*** when PSMTs provided more than one way of reasoning mathematically about the idea. If students showed evidence of understanding the way in which the student was thinking about the idea, we coded for ***understanding student thinking***. If the PSMTs' response suggested an activity that could be completed to advance student thinking, we coded for ***activity***. Finally, if a response showed evidence that the PSMT understands how understanding a mathematical idea in a particular way empowers the learning of another idea, we coded for ***connections***. As the team began coding, we found it important to also code for each ***incorrect mathematical statement*** as well as ***general discourse moves*** where the PSMTs sought to respond to students, promoting discourse to advance the lesson, but where the response did not specifically draw on particular mathematical ideas that would advance student thinking.

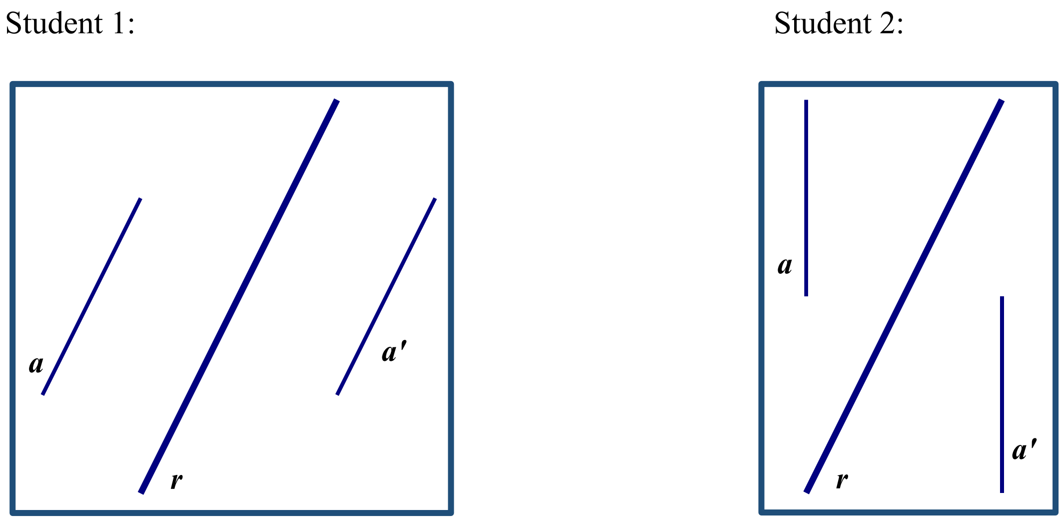



*Figure 2. Sample student work used with simulation of teaching practice assignment #2.*

Two coders analyzed half of the responses and a second pair of coders analyzed the other half. Then, each pair independently analyzed how the other pair coded their data, noting areas of disagreement. In this way, all four coders analyzed all of the simulation of practice data. All four coders then reassembled to negotiate disagreements in coding in order to arrive at a final coding of the PSMTs' responses. Once this step was complete, the team looked for patterns that emerged when comparing the case of each PSMT – focusing particularly on their responses to the simulations of practice pre-assignments and post-assignments.

## Results

A comparison of pre- and post-GAST scores revealed that PSMTs' MKT did indeed increase. The mean score on the pre-GAST was 8.7 out of a possible 16 and the post-GAST mean score was 10.7. A paired $t$-test showed that this difference is significant with a $p$-value of 0.002. An analysis of the simulations of teaching practice showed that PSMTs provided more evidence of the presence of MKT categories on the simulation post-assignment responses when compared to the pre-assignment, particularly with regard to ***KDU***s and ***understanding student***

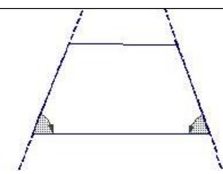
***thinking***. In addition, the number of instances of ***general discourse moves*** decreased dramatically from pre- to post, indicating that PSMTs progressed in their ability to respond to particular mathematical reasoning of students in their responses by the end of the course. In light of this, it is not surprising (though not necessarily encouraging) that there were slightly more ***incorrect mathematical statements*** in the post-assessments, because more PSMTs were making a greater number of mathematics-specific statements in their responses.

Here, we present responses from PSMT11, a case in the course that exhibited typical MKT development. In PSMT11's pre-response to assignment 1 (see Figure 3), we see a ***general discourse move*** asking students to state what they know or what inferences they could make about a particular idea. It is not clear how the mathematical ideas central in the question posed by PSMT11 were foundational to a connected network of ideas that could be used to advance students' understandings of base angle congruence in an isosceles trapezoid.

In contrast, part 1 of PSMT11's post-assignment response indicates ***KDU***s of rigid motions and variations in angle measure that informed the questions posed. Mathematics-specific questions that draw on understandings of rotations (upside down) and changing angle measures are clearly meant to cause cognitive conflict for the student and provide an opportunity for the student to reorganize their thinking about trapezoids. In addition, the question "Would all of our postulates and theorems still hold?" indicates an ***activity*** or class discussion that PSMT11 believes would hold potential to help the student progress in their understanding of the idea under discussion (and we agree). In both paragraphs of the part 2 response, PSMT11 provides evidence of a ***KDU*** of congruence as well as the ability to ***understand student thinking***, even if imperfectly – the student showed evidence of understanding congruency, but PSMT11 did not acknowledge it in the last sentence of the response.

**Pre:**

What do we already know or what inferences can we make about congruent angles in shapes?



**Post:**

**Part 1:** Would it matter if your trapezoid is 'upside-down'? Could our base angles be obtuse instead of acute? How would this affect our conclusion? Would all of our postulates and theorems still hold? Does a shape change its properties based on orientation? Why or why not?

**Part 2:** The student literally turned their trapezoid upside down and concluded that the 'new' base angles were also congruent. Rather than understanding that these 'new' base angles actually represented the summit angles from the right side up trapezoid, the student assumed that since the orientation of the trapezoid changed, then the base angles might also. After finding that this was not the case, the student came to the conclusion that the 'new' base angles were also congruent without deducing that the summit angles of the original trapezoid were congruent along with the base angles.

The Van Hiele level of understanding this student might possess would be considered a Level 0, which refers to Visualization. The student lacks understanding of the parts of the trapezoid functioning together, and rather views the trapezoid as a 'total entity.' This student is able to identify the trapezoid, use geometric language to describe it, and can reproduce it, but they are unaware of the special properties that it may possess, such as parallel lines and congruencies.

*Figure 3. PSMT11's responses to pre- and post- simulation of practice assignment 1.*

In the pre-assignment for simulation 2, (see Figure 4) PSMT11suggested an ***activity*** by posing questions about folding paper to visualize reflections and further student thinking. We also see ***activities*** in PSMT11's post-assignment, but she further elaborated by understanding student thinking and exhibiting multiple KDUs. The statement, "The student 'reflected' the original image but the student failed to notice the orientation of the line of reflection" indicates PSMT11's attempt to ***understand student thinking.*** In contrast to PSMT11's pre-assignment, ***KDU***s were prominent in the post-assignment. For example, she recognized that when an image is reflected over a line, distance is preserved and segments in the image are congruent. Additionally, PSMT 11 used her KDU of reflections to develop activities when she explained that "we must use perpendiculars to measure our distances" (***KDU***) and then posed questions that she could ask to advance student thinking (***activities***). Though we see more KDU in the post-assignment we consider this may be due to the nature of the geometry course.

| |
|---|
| **Pre:** If you folded your paper across the dotted line, would your reflection process still hold? Why or why not? What can you do to your conclusion so that it does hold? What does it mean for something to be reflected? Think about your answer in terms of mirroring. |
| **Post:**<br>***The Student's Solution***: This student noticed that the original image is parallel to the line of reflection, yet they unsuccessfully performed a proper reflection. The student 'reflected' the original image but the student failed to notice the orientation of the line of reflection. It is true that the new image is the same distance away from the line of reflection as the original and the new image is congruent to the original, but if you metaphorically 'folded your paper in half,' the image would not lie on top of itself. The new image's endpoints would not fall on the same perpendicular as the original image's endpoints, which is the main factor contributing to this misconception. The endpoints of each image must fall on the same perpendicular. The midpoint of the perpendicular will fall on the line of reflection.<br>***My Feedback***: What would happen if you folded your paper across the line of reflection? How does this reflection differ from your answer? Compare your answer to the actual reflection. What do you think went wrong? I know you know that the distance from the line of reflection to each image must be congruent and the lines themselves must be congruent. But think about if you were that line, and you looked directly into the line of reflection (like it was a mirror)... would that reflection be skewed to the left? Where would the reflection be? Why? This is why we must use perpendiculars to measure our distances from the line of reflection to our image. Your new image's endpoints will lie on the same perpendicular as the endpoints of your original image. |

*Figure 4. PSMT11's responses to pre- and post- simulation of practice assignment 2.*

## Discussion

In this report, we present evidence of PSMTs' development of MKT. In particular, after learning in a College Geometry course with MODULE($S^2$) curricular materials, we observed the development of PSMTs' KDUs of mathematical ideas and the ability to more fully understand student thinking. In addition, PSMTs significantly decreased their use of general discourse moves. We attribute this advancement of MKT to the PSMTs completing activities that are grounded in the work of teaching. Baumert and colleagues (2010) make a similar argument, that solely focusing on common content knowledge develops "only a limited mathematical understanding of the content covered at specific levels" in the school curriculum (p.167). Providing opportunities for PSMTs to engage with simulations of practice and activities that make connections between advanced perspectives on geometry and 7-12 mathematics allows them to begin to bridge the gap between college coursework and classroom teaching and meet the challenge of developing MKT in their university mathematics content courses.

## Acknowledgement

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Shape Thinking and the Transfer of Graphical Calculus Images

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*Shape thinking has previously detailed how students may view function graphs. Students using static shape thinking view a function graph as if it were a wire where learned rules, formulas and quantities appear as a consequence of the perceived shape. This study presents a case study that demonstrates how static shape thinking can be extended to other graphs seen within calculus. Results demonstrate how one first-semester calculus student perceived a "triangular" shape within a function graph. Quantities appeared as a consequence of this perceived shape and his reasoning on multiple related tasks was influenced by his transfer of this perceived shape onto subsequent graphs. Even the the student's reasoning led to inaccurate responses to interview tasks, his reasoning was accurate and consistent within his perception.*

*Keywords:* Calculus, Shape Thinking, Quantitative Reasoning, Transfer, Derivative

Oftentimes students identify solution procedures by the type of question found in particular locations in the textbook, and thereby reduce calculus to a set of disjoint procedures that are conceptually unavailable when removed from the exact setting in which the procedure was presented (e.g., Bezuidenhout, 2001; White & Mitchelmore, 1996). Students' often view their procedural knowledge as irrelevant in "quantitatively complex situations" and have difficulty with reasoning quantitatively when relevant to even seemingly straightforward applications of mathematical concepts (Lobato & Siebert, 2002, p. 88). The quantitative complexity of the concept of derivative at a point is outlined in Zandieh's (2000) work where student understanding is framed within three "layers" of process-objects pairs. Carlson, Jacobs, Coe, Larsen, and Hsu (2002) asserted that students can struggle with derivatives because of an impoverished understanding of function that lacks a coordination of quantities foundational for reasoning about dynamic relationships captured by functions' rules. Carlson et al. (2002) outlined the mental actions needed for productive reasoning about the derivative of a function. In particular, emphasis was placed on Zandieh's ratio layer and the mental actions of coordinating the amount of change in one quantity with changes in the other quantity.

We speculate that interactive images can support the development of mental actions coordinating amounts of change of quantities. Furthermore, research suggests that engaging students in multiple problems from which to generalize can promote a richer understanding of a concept than might otherwise be achieved (Oehrtman, 2008).

This report is part of a larger study investigating the effects of contextual and graphical images of derivatives from multiple contextual problems using virtual manipulatives (VMs). For this paper, focus is placed on an interesting case and we ask the following question:

> What might students transfer while interacting with images graphically modeling similar quantitative attributes of different situations related to the concept of derivative?

## A Research-Based Approach to Interactive Image Design

This section provides a review of relevant literature as it relates to the design of the images contained within the VMs. VMs were adopted because they can show continuous change in real time (Castillo-Garsow, 2012) and can be used to aid students in making sense of calculus concepts by highlighting connections between multiple representations, developing quantitative reasoning, and supporting exploration of formal limit definitions (Cory & Garofalo, 2011; Thompson, Byerley & Hatfield, 2013; Thomas & Martin, 2017).

All problems presented in this study (Figure 1) were adapted from Oehrtman's (2008) approximation framework. Oehrtman (2009) found that students' spontaneous reasoning using an approximation and error analysis cognitive model for limit closely resembled the formal structure of limits while simultaneously supporting students in productively engaging quantitatively complex situations. Repeated structured reasoning using quantities and relationships between quantities associated with approximations and error analysis can encourage student generalization to shared structures across similar contextual situations.

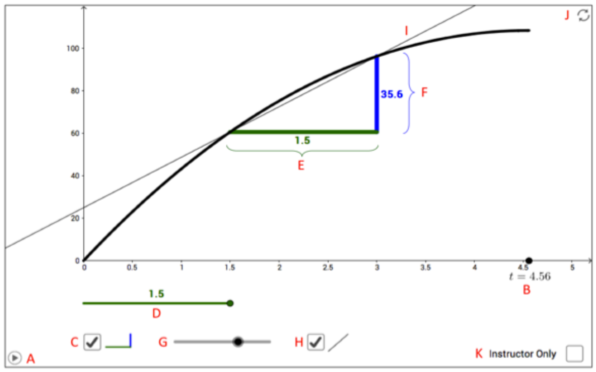
| Bolt | Sphere | Asteroid | Iodine |
|---|---|---|---|
| A bolt (arrow) is fired from a crossbow straight up into the air with an initial velocity of 49 m/s. Approximate the speed of the bolt at 2 seconds. | Approximate the instantaneous rate of change of the volume of a sphere with respect to its radius when the radius is 5 cm. | NASA has determined that asteroid 1999 RQ36 has a 1 in 1000 chance of colliding with Earth on September 24, 2182. […] Approximate the instantaneous rate of change of the gravitational force between the Earth and 1999 RQ36 with respect to distance when the two objects are 10,000,000 m apart. | The half-life of Iodine-123, used in medical radiation treatments, is about 13.2 hours. Approximate the instantaneous rate at which the Iodine-123 is decaying 5 hours after a dose of 6.4 g is injected into the bloodstream. |

*Figure 1. Four approximation problems used in each interview.*

VMs were created by the second author using GeoGebra 5 (Hohenwarter & Fuchs, 2004). Figure 2 presents an overview of key attributes of a graphical VM indicating the interactive capabilities of the VM within the context of the bolt problem.

The mental actions of conceiving of, creating, and making inferences with covarying quantities are a "foundation from which the student can reflect upon to develop mathematical understandings and reasoning" (Moore, Carlson, & Oehrtman, 2009, p. 3). Covariational reasoning is defined as "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). Carlson et al. (2002) went on to describe developmental levels of student's images (Thompson, 1994) of covariation based on the accrual of mental actions progressively supporting more sophisticated covariational reasoning in the context of derivative. In particular, Levels 1 and 2 involved coordinating changes in one quantity with changes in the other quantity. Levels 3 and 4 included the same mental actions as from Levels 1 and 2 and additionally involved coordinating amounts of change of one quantity with changes in the other quantity.

Students typically read the problem prior to interacting with a VM. When they first interacted with a graphical VM, the axes and the play/pause button (Figure 2 A) were the only visible attributes of the VM. We anticipated that the inclusion of the play button might help support developmental Levels 1 and 2. After the animation played, students could reveal (Figure 2 C) depictions of amounts of change. For the graphical VMs, these depictions of amounts of change initially appeared as attributes of multiple "triangles" along the curve (not depicted in Figure 2). Once amounts of change were depicted, the student could adjust (Figure 2 D) the amount of change of the independent quantity and highlight (Figure 2 G) particular intervals. After an interval had been highlighted, the student could reveal a secant line for the highlighted interval (Figure 2 H). The VMs' abilities to reveal, adjust, and highlight depictions of amounts of change were included to support Levels 3 and 4. For the purpose of our research, it was up to the student to conceive of and coordinate measurable attributes of the VM that coincided with




A Play/Pause Button
B Slider for independent variable
C Checkbox toggles amounts of change on/off
D Slider for change in independent variable
E Horizontal line segment
F Vertical line segment
G Slider highlights single interval
H Checkbox toggles secant line show/hide
I Secant line
J Refresh
K Checkbox toggles axes labels on/off

*Figure 2. Interaction points for a graphical VM where particular amounts of change have been highlighted. *Note: The letters A-K and corresponding descriptions on the right did NOT appear on the original VM.*

approximating the requested instantaneous rate. In particular, when interacting with a graphical VM, students needed to conceive of the *lengths* of the horizontal and vertical attributes of "triangles" as representing the amounts of change.

## Theoretical Background: Shape Thinking and Transfer

For this study we focus on one of two modes of thinking based upon the extent to which an individual engages in quantitative and covariational reasoning while reasoning about graphs.

### Static Shape Thinking

Static shape thinking entails a view of a function's graph as "an object in and of itself, essentially treating a graph as a piece of wire (graph-as-wire)" (Moore & Thompson, 2015, p. 784). The mental actions and operations that students engage during static shape thinking are rooted in Piaget's (2001) figurative thought "based in and constrained to sensorimotor experience (including perception)" (Moore, 2016, p. 324). Thus, a student exhibiting static shape thinking relies on the most salient perceptual cues of shape. Equations, function names, rules and properties of the function appear as consequences of shape. This view of the graph of a function may serve the student well in particular situations, such as function translations; however, static shape thinking obfuscates student's ability to view functions as emergent through covarying quantities. Static shape thinking does not mean students lack quantitative reasoning, but that when quantities appear, they appear as a consequence of the shape.

To better describe a student's meanings and ways of thinking while engaging in shape thinking, Moore and Thompson (2015) drew upon Thompson, Carlson, Byerley and Hatfield's (2014) definitions of *understanding*, *meaning*, and *ways of thinking*.

> *Understanding* is an in-the-moment state of equilibrium, which may occur from assimilation to a scheme or from a functional accommodation specific to that moment in time. A *Meaning* is the space of implications that the moment of understanding brings forth—actions that the current understanding implies. *Ways of thinking* are "when a person has developed a pattern for utilizing specific meanings…in reasoning about particular ideas" (Thompson et al., 2014, p. 12). (Moore & Thompson, 2015, p. 784)

### Actor-Oriented Transfer

"Actor-oriented transfer is defined as the personal construction of relations of similarity between activities" (Lobato & Siebert, 2002, p. 89). In this study we adopted transfer as actor-oriented “to understand the interpretative nature of the connections that people construct between learning and transfer situations” (Lobato, 2012, p. 239). In particular, we desired to better understand the idiosyncratic interpretative nature of the connections that students were constructing as they progressed through the four problems (Figure 1) using the provided VMs (Figure 2). Due to the quantitative complexity of the problem situations, we anticipated that the nature of their interpretive engagement would be supported by quantitative and covariational reasoning. Quantitative and covariational reasoning have been shown to support students in conceiving of relevant mathematical structures within contexts and generalizing to a common mathematical structure shared by multiple contexts (Lobato & Siebert, 2002; Ellis, 2007; Thompson, 2011). In particular, students' in-the-moment understanding of a VM is influenced through an interaction of prior learning experiences, their interpretation of the problem, and quantities and relationships conceived concerning the problem's situation and conceived concerning the attributes of the VM. The space of implications that a moment of understanding brings forth can evidence transfer through the quantities and relationships between quantities conceived as similar to quantities and relationships from prior situations and prior VMs.

## Methods and Analysis

Five students from a first-semester calculus course at a medium-sized university voluntarily participated in this study. The participants were A-B range calculus students majoring in a STEM field. Interviews occurred after the concept of the derivative and during definite integral instruction using Briggs, Cochran, and Gillett (2015). Each student was assigned to one type of representation (contextual or graphical) and level of interactivity (VM or static) throughout all interviews. This paper focuses on one student, pseudonym Jeremy, who viewed graphical VMs.

Jeremy participated in four 20 to 45 minute individual interviews where one problem from Figure 1 was presented per interview. He began with the bolt problem since velocity is the most commonly used physical example of derivative within calculus (Zandieh, 2000). In the second interview, he was introduced to the sphere problem and so on. All problems presented different situations but shared a common derivative structure and method for obtaining approximations to support transfer. Using a laptop computer, Jeremy interacted with four VMs, each depicting a different graphical image corresponding to the current problem, and each image similar to the image depicted in Figure 2. He was also provided with a calculator and smartpen.

While interacting with each VM, Jeremy was asked to identify on the image what depicted what he was approximating, approximations, and errors when applicable. Furthermore, he was asked how he could improve any approximation he might have produced. This task was intended to support student use of the VM to explore average rate of change over smaller intervals of the independent quantity. To evaluate his developmental level (Carlson et al., 2002) in relation to the current image, he was asked to complete the statement, “For any fixed amount of the change in (time/radius/distance/time), the amount of change in (height/volume/gravitational force/mass) is (increasing/decreasing/neither).” He was also told to compare rates at different instances, such as comparing the speed of the bolt at two and four seconds. Jeremy was repeatedly asked to indicate attributes of the image that supported his responses to interview questions.

Interview data consisted of written notes from his smartpen and audio and video records, including screen capture and one camera capturing gestures toward the screen. Data were analyzed with the intent to reconstruct in-the-moment understandings and meanings based upon the quantities and relationships between quantities conceived upon the situation and VM image. Records of Jeremy's responses to interview questions were coded for instances of the appearance of quantities and relationships with particular attention paid to amounts of change and his interpretation of how these amounts were represented within the current image. Noting the appearance of quantities together with the current interview question and Jeremy's indicators of attributes of the image corresponding to such quantities, provide evidence for the origin of his reasoning. The analysis of Jeremy's reasoning concerning the bolt problem served as a baseline with which to compare his reasoning on subsequent tasks. To document transfer, special attention was paid to any moment in which Jeremy explicitly applied attributes from previous interview problems even though these attributes might be irrelevant for the new situation.

## Results: A Case Study from a Graphical Virtual Manipulative

This section describes Jeremy's reasoning as he progressed through the four problems.

### Jeremy's Reasoning During the First Interview

In the first interview, Jeremy described getting an approximation by taking the derivative of the function because "the derivative of position is velocity." As the animation played, he said, "It's pretty much showing […] the position with respect to time." He then stated that the vertical axis was showing meters. He noted that he could use the graph to make an approximation, and after he was asked how he would do that, he gave the formula "position divided by time."

**The appearance of quantities based on perceived shape.** After he was directed toward the "clickable features," Jeremy selected the checkbox that toggled the horizontal and vertical line segments denoting amounts of change. He said, "This makes me think of an area function." He then began talking about integrals and Riemann sums he was studying in class, "I think these shapes represent rectangles, it looks weird because it's in a way that I've never seen it." He then drew his own version of the Riemann sum showing an "overestimate" (Figure 3a), and concluded that the area under the function depicted the distance traveled.

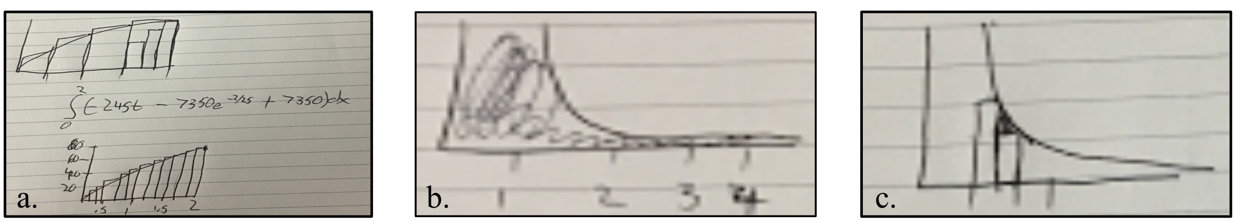



*Figure 3. Jeremy's graphical depictions involving areas and rectangles.*

After he displayed the secant line, he called it the "tangential." In this moment, he said that the "tangential" indicated the rate of change and moved both sliders to increase the number of "triangles" and to get the secant line to "land on" the point corresponding to $t = 2$. He stated that this method would give him the rate of change at two and repeatedly mentioned using the formula "position divided by time" to obtain an approximation. Even though Jeremy had been talking about "tangential" lines and a rate formula, he went on to say that to get a better approximation he would "take the integral from zero to two." Jeremy was viewing an image with many "triangles" depicted as he made this comment. Jeremy eventually reminded himself that since the image was the graph of position he needed to take the derivative to obtain the velocity. He was asked if there was a way to approximate the rate of change without taking the derivative. He said it was just "geometry" and went back to using the rectangles under the curve.

The previous paragraphs have detailed how Jeremy cued off of his perception of the image in front of him and his remembrances of learned rules and facts. For Jeremy, one of the most salient features of the images were the "triangles." When he cued off of the "triangles" or the "geometry" of the image, Jeremy "saw" rectangles and that reminded him of Riemann sums. When he cued off of the "tangential" lines or the physical situation, he was reminded of derivative and rate of change but neglected related quantitative meanings depicted on the graph.

**The appearance of amounts of change.** Once given the fixed amounts of change question, he looked at the image and said, "From 0 to 4.5 the height increases." The question was restated with emphasis on the "*amount of change,*" but he continued to neglect amounts of change. When asked to compare the rate of change at one second to the rate at four seconds, Jeremy finally spoke of amounts of change in reference to rates as he gestured over the graph of the function.

> The rate of change is definitely slower. As you can see, the height from zero to one [gesturing over the $x$-axis from $x = 0$ to 1] is in a difference of forty [upward gesture to the function at $x = 1$], but the height from three to four [gesturing over the $x$-axis from $x = 3$ to 4] is barely twenty [making the same upward gesture to the function but at $x = 4$].

In this moment, the fixed amounts of change and the comparing rates questions appeared to help focus Jeremy on quantities relevant to rate that were depicted on the image.

**Transfer Enabled by Static Shape Thinking During Subsequent Interviews**

When presented with the sphere VM, Jeremy stated, "Oh this is just another approximation problem." He then concluded he could produce an approximation by "doing the same thing as last time" and inquired if the image in front of him was depicting, "straight up triangles?" Clearly, Jeremy had engaged in attempting to transfer his way of reasoning about the first problem to the next problem and that this reasoning was influenced by his "triangles."

**Implications of triangle reasoning.** Throughout all four interviews, Jeremy continued to cue off of the "triangles" and imbue "rectangles" upon the image even though no rectangles were ever depicted. When looking at the triangles in the VM for the sphere he said, "I want to use a rectangle to estimate the area under the curve, and that will tell me the rate of change." He then reproduced the graph from the bolt problem and illustrated a rectangle under it.

As a consequence of Jeremy's "rectangles," he spoke frequently of the area under the curve (Figure 3b) and using the rectangles to approximate the area under the curve (Figure 3a). In addition, his graphical structure based on his perceived rectangles evolved to include quantities related to overestimates, underestimates and error. For example, he described the triangles as depicting error as "area that's not captured" by the area of the rectangles corresponding to an underestimate (Figure 3c). He observed, "the more rectangles I put in, the more accurate the estimation gets," and noted that the VM "program doesn't take the shapes under the curve to infinity." He described the integral as a limit of rectangles with no error, "An integral is pretty much this, but making your shapes [go to] infinity to where there is no error." When asked for detail to explain why area approximated instantaneous rate, he expressed uncertainty but remained steadfast that area under the curve depicted the appropriate rates of change.

**"Tangential line" reasoning.** Throughout the interviews, Jeremy continued to talk about the displayed secant lines as if they were "tangent lines." For him, his "tangential lines" represented the instantaneous rate of change at a point while simultaneously evolving to "fitting" the "hypotenuse" of his "triangle." Even though he had related rate of change to amounts of change during the first interview, in subsequent interviews he had difficulty describing amounts of change in relation to any rate. For example, in the second interview he drew a picture of the graph with several lines illustrated and compared rates where greater rates corresponded to steeper slopes. When asked to describe what the lines showed, he said, “Oh, the rate of change at that point.” When asked to identify the attribute of the lines that was the rate of change, he faltered. By the third interview, his notion of area was influencing how he compared rates, claiming that a “significantly higher” area corresponded to greater rate. Jeremy detailed how the area under the curve in Figure 3b from zero to one corresponded to a greater rate than from three to four because there was significantly more area under the curve from zero to one.

**Learned rules and lower level mental actions.** In addition to his graphical attributes that could produce instantaneous rates, Jeremy repeatedly stated his learned rule that derivatives would give instantaneous rates. Even so, when cuing off of the image, he continued to claim that he could take the integral over an interval to produce an instantaneous rate. Furthermore, in all four interviews, Jeremy repeatedly indicated no more than Level 2 reasoning while responding to the fixed amount of change questions. Neglecting amounts of change, he repeatedly described how the one quantity would change with respect to the other quantity.

## Discussion

The nature of static shape thinking rooted in figurative thought based upon the most salient features of graphs suggests that Jeremy’s ways of thinking about these graphs likely represents a large population of students. Indeed, every student in our larger study spontaneously mentioned seeing “triangles” when viewing graphs.

Jeremy may seem contradictory at moments, yet he did not come into any observable state of cognitive conflict that ultimately led to any abandonment of approximating instantaneous rates using Riemann sums. Why not? Jeremy’s reasoning becomes very structured and consistent throughout the interviews. Consider Jeremy’s reasoning *as if* the problems in Figure 1 could be solved using definite integrals. Jeremy identifies 1) what he is approximating as area under the curve, 2) approximations as areas of rectangles, and 3) error as the area of “triangles.” In addition, 4) he can make approximations more accurate by including more rectangles and 5) has imagined the definite integral as taking the number of “[rectangle] shapes under the curve to infinity” where there is “no error.” Jeremy’s development and repeated use of this reasoning demonstrates this reasoning as a way of thinking concerning these tasks. This way of thinking was likely reinforced by his ability to answer approximation questions with corresponding graphical attributes.

Jeremy’s way of thinking developed through his repeated transfer of “triangular” shape and the space of implications that such transfer brought forth. It makes sense that static shape thinking would likely enable transfer due to the low cognitive demand of figurative thought. Thus, static shape thinking served Jeremy well by supporting transfer and supporting his development of his way of thinking about these tasks. Keep in mind, Jeremy “saw” rectangles.

The interview protocol focused on students’ perceptions of the images and did not include interventions intended to illuminate the irrelevance of area quantities to instantaneous rate. Thus, this study provides insight into how calculus students might perceive these types of graphs when no interventions are provided. Data from our larger study suggests that interventions designed to bring a student’s attention to the context may support students in moving away from static shape thinking. For example, we have observed notable differences between student reasoning when viewing graphical VMs compared to students viewing VMs depicting contextual images.

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# The Relationship Between Students' Covariational Reasoning When Constructing and When Interpreting Graphs

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*Abstract: Graphing tasks require students to engage in at least one of two activities: construct a graph and/or interpret a graph. Ideally, the meanings a student re-presents when constructing a graph are consistent with the meanings the student constructs from his/her sketched graph. However, this coherence is nontrivial. In this paper I present results from clinical interviews with university precalculus students to illustrate how students' graphing actions can be governed by different images of covarying quantities. More specifically, I present two students' mathematical activity to illustrate how these students' imagined quantities to covary in different ways depending on whether they were reasoning about a situation, constructing a graph, or reasoning about that sketched graph. I conclude by hypothesizing that the way a student coordinates two quantities' measures (e.g., asynchronous coordination of varying quantities or static coordination of measures) can inhibit him/her from imagining the same covariational relationship when constructing and interpreting graphs.*

Keywords: Graphing, Covariational Reasoning, Cognition

Researchers have found that students who imagine quantities to covary in a situation are not necessarily able to re-present that imagery graphically (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, Paoletti, Stevens, & Hobson, 2016). Moore et al. (2016) suggested that students' meanings for graphs (such as graphs starting on the vertical axis, being read or drawn left-to-right, and passing the vertical line test) inhibit students from re-presenting images of the phenomenon that include covariational relationships. When students held these meanings for graphs they re-presented imagery that was distinct from how they initially imagined the quantities to covary in the situation. In this paper I extend Moore et al.'s (2016) work by exploring how the images a student constructs of the phenomenon influence both the graph the student constructs as well as the meanings the student constructs from that sketched graph. More specifically, I characterize two university precalculus students' graphing schemes to study the relationship between how the student initially understands the quantities to covary, the understandings he/she re-presents when constructing the graph, and the understandings he/she constructs from his/her completed graph.

## Background

Covariational reasoning is "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). Thompson and Carlson (2017) leveraged past research on variational and covariational reasoning to propose six major levels of covariational reasoning (see Figure 1) that are not constrained to reasoning about specific function types or methods of representation. Thompson and Carlson explain that the level of a students' covariational reasoning depends on three constructions: (1) the quantities the student is conceptualizing, (2) how the student imagines those quantities to vary, and (3) how the student coordinates and unites two changing quantities both in thought and representation. I elaborate on these three constructions in the following paragraph.

| *Major Levels of Covariational Reasoning* | |
|---|---|
| Level | Description |
| Smooth Continuous Covariation | The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and they envision both variables varying smoothly and continuously. |
| Chunky Continuous Covariation | The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation. |
| Coordination of Values | The person coordinates the values of one variable ($x$) with values of another variable ($y$) with the anticipation of creating a discrete collection of pairs ($x$, $y$). |
| Gross Coordination of Values | The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases". The person does not envision that individual values of quantities go together. Instead the person envisions a loose, non-multiplicative link between the overall changes in two quantities' values. |
| Pre-coordination of Values | The person envisions two variables' values varying, but asynchronously, one variable changes, then the second variable changes, then the first, etc. The person does not anticipate creating pairs of values as multiplicative objects. |
| No Coordination | The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values. |

*Figure 1:* Thompson and Carlson's Major Levels of Covariational Reasoning, highest to lowest (Thompson and Carlson, 2017, p. 23)

Thompson (1990, 2011) explained that a quantity is a mental construction of a quality of an object that one can imagine measuring. Students construct quantities by conceptualizing an attribute to be measured and the way in which they would measure it. How the student imagines each quantity to vary constitutes her variational reasoning[1]. A student's conception of time is closely related to her variational reasoning since imagining a quantity's measure to change necessarily involves imagining time elapsing. Thompson (2011) described two ways students conceptualize time: experiential time "the experience of time passing" and conceptual time "an image of measured duration" (p. 27). Both experiential time and conceptual time are essential to covariational reasoning. For example, to construct what Castillo-Garsow (2012) calls smooth images of change one must imagine change in progress so that she imagines a quantity changing in her experiential time. Conceptual time, on the other hand, is essential to coordinate two quantities' measures at distinct moments in time (Thompson, 2011). The final construction Thompson and Carlson (2017) describe is the construction of a multiplicative object. As Saldanha and Thompson (1998) explained, a multiplicative object is a cognitive construction that enables one to hold two quantities in mind *simultaneously*. If one has coordinated two varying quantities through a multiplicative object then she anticipates that as one quantity changes the other quantity is changing as well. As a result, the student is able to hold both quantities in mind as they change together.

## Theoretical Perspective

[1] See Thompson and Carlson (2017) for description of six major levels of variational reasoning.

According to Piaget (1967, 1985), actions are the source of all knowledge. Individuals organize their actions into schemes that include when to apply the action, an anticipation of the result of acting, how these actions work together, and eventually how these actions can chain together. As one engages in mathematical thinking he activates different scheme(s) in order to make sense of the task.

In mathematics, students are often asked to re-present their mathematical activity in the form of diagrams, graphs, formulas, tables, etc. If the student understands the graph (or formula, table, etc.) to be a depiction of his thinking then the student has an image of the mathematical activity he re-presented and the graph is a *representation* of that image. I emphasize the distinction between re-presenting and representing to be able to study student's graphing activity in the case the student does not anticipate he is representing, or creating a picture of, his mathematical thinking to then reason about.

## Methodology

The subjects in this study were three university students: Ali, Bryan, and Sue. At the time of the study these students had recently completed precalculus but had not yet taken calculus. These students were selected to participate in the study because they collectively demonstrated different ways of engaging in covariational reasoning in a recruitment interview (see Frank, 2017 for more details on recruitment and selection). After being selected, each student participated in a two-hour one-on-one task based clinical interview (Clement, 2000). The purpose of the clinical interview was to characterize each student's meanings for graphs.

After completing the interview process I engaged in retrospective analysis by identifying instances that provided insights into the relationship between the understandings the student re-presented when constructing a graph and how the student understood his/her sketched graph. I used these instances to generate tentative models of each student's schemes for graphing. I tested these models by searching for instances that confirmed or contradicted my model and repeatedly refined my model until it accounted for the student's mathematical activity.

## Results

Of the three students who participated in the study two students (Ali and Bryan) conceptualized graphs in terms of varying quantities. Sue, on the other hand, conceptualized graphs as pictures of an object's motion (consistent with Monk's (1992) notion of *iconic* translations). In this section I describe how Ali and Bryan imagined quantities to covary when reasoning about a situation, when constructing a graph, and when reasoning about their sketched graphs.

### *Pre-Coordination of Values: The Story of Ali*

When Ali created a graph from a contextual description of a situation she engaged in two distinct activities. First, Ali generated a shape by tracking one quantity's variation as she imagined that variation in her experiential time. Then, Ali used the properties of the shape she created to reason asynchronously about the variation of the two quantities labeled on the graph's axes. If the shape she created did not match her anticipation of how each quantity varied, then she guessed shapes from her memory of past graphing activities until she picked a shape that matched how she imagined each quantity to vary. This suggests that Ali used distinct and uncoordinated systems of actions when generating graphs (drawing shapes) and understanding her sketched graphs (reasoning about two quantities' asynchronous variation). I will illustrate Ali's graphing scheme with her engagement in the skateboard task (see Figure 2).

A skateboarder skates on a half-pipe like the one shown. The skateboarder goes across the half-pipe and then returns to the starting position.

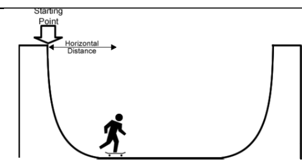
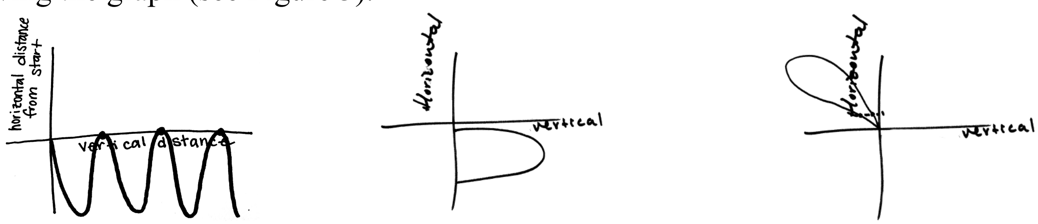



*Figure 2:* Description of skateboard task.

I asked Ali to graph the skateboarder's horizontal distance to the right of the starting position relative to the skateboarder's vertical distance above the ground. Ali made three attempts drawing the graph (see Figure 3).




Ali's first attempt | Ali's second attempt | Ali's third attempt

*Figure 3:* Ali's three attempts to graph skateboarder's horizontal distance from start relative to his vertical distance above the ground.

On Ali's first attempt she drew an oscillating curve in the fourth quadrant (Figure 3). Since Ali imagined the half-pipe below ground, it seems Ali made this graph by tracking how she imagined the skateboarder's vertical distance changing as she imagined the that variation in her experiential time. After drawing the curve, and without prompting, Ali determined her graph was incorrect because "the graph I drew is showing that the vertical distance is increasing the whole time." She went on to draw two more shapes (Figure 3) and each time appropriately reasoned why her sketched graph was incorrect. For example, Ali rejected her second attempt (a side-ways U-shape in the fourth quadrant) since it showed the vertical distance was positive when she wanted to show the vertical distance was negative. After Ali rejected her third graph I asked her to explain her approach to graphing (Excerpt 1).

*Excerpt 1:* Ali's explanation of making graphs by guessing and checking shapes

Int: What are you doing when you are trying to figure out what graph it could be?
Ali: Um. Well I think of like. I either focus. I go back and forth with like okay vertical distance and horizontal distance. So I think of potential like, I guess shapes, that can be drawn and then I'm like does this fit the characteristic of the horizontal distance. If it doesn't then it is out and I think of another one. And so. That's how I usually go about with graphing graphs until I eventually - I'm like this one fits both criteria.

In Excerpt 1, Ali describes her three-step approach to graphing: (1) draw a shape by "think[ing] of potential … shapes that can be drawn", (2) consider what the shape conveys about the variation of each quantity *separately*, and (3) adjust the shape until it matches how she imagined each quantity to vary. This final step is significant because it implies Ali constructed two distinct images of the quantities' covariation; Ali constructed an image of each quantity's variation from the graph that she compared to her image of each quantity's variation from her understanding of the situation. This suggests Ali had an image of the quantities' variation that she could have re-presented when making her graph.

I hypothesize Ali did not make her graph by re-presenting the images of varying she constructed from the phenomenon because she attended to each quantity's variation separately, what Thompson and Carlson (2017) called a *pre-coordination of values*. For example, Ali attended only to the skateboarder's vertical distance when determining the validity of her first and second graphs and attended only to the skateboarder's horizontal distance when determining the validity of her third graph. Additionally, in Excerpt 1 Ali explained that she "go[es] back and forth with like vertical distance and horizontal distance…like does this one fit the characteristic of horizontal distance". I take this as evidence that Ali understood the shape of a graph to show how each quantity varied separately.

By imagining each quantity's variation separately I claim that Ali did not have a single image from having coordinated two quantities' variation that she could attend to when making her graph. In other words, Ali did not have a way to think about making one shape that would convey how the skateboarder's horizontal distance changed *and* how the skateboarder's vertical distance changed. Instead, Ali was constrained to making a graph by re-presenting only one of her images of variation (first attempt in Figure 3) or guessing and checking shapes (second and third attempt in Figure 3). In summary, it seems Ali's asynchronous coordination of the two quantities' variation inhibited her from attending to both quantities' variation when making her graph.

*Coordination of Values: The Story of Bryan*

Like Ali, Bryan demonstrated different conceptualizations of the varying quantities when constructing his graph and when reasoning about his graph. More specifically, Bryan constructed graphs by re-presenting his experience imagining a continuously varying quantity but he did not reason about his graph in terms of a quantity's continuous variation. Instead, he reasoned about his graph by coordinating static states in each quantity's variation. I will illustrate Bryan's graphing scheme with his engagement in the bottle evaporating task[2].

In the bottle evaporating task I asked Bryan to imagine a spherical bottle filled with water that was left outside to evaporate. Then I asked him to graph the height of water in the bottle relative to the volume of water in the bottle as the water evaporated. Before Bryan constructed a graph he reasoned, "When volume is maximum the height should be maximum and when volume is zero height should be zero." This suggests Bryan coordinated two quantities' measures at two moments in time. He proceeded to draw a straight line from the top middle of the plane that fell from left to right (see Figure 4, red line).

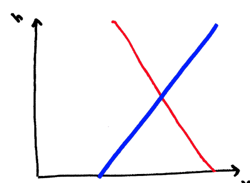


*Figure 4:* Bryan's initial (red) and revised (blue) graph for the evaporating water problem (task adapted from Paoletti & Moore, 2016)

[2] Bottle evaporating task from Paoletti and Moore (2016).

From my perspective, the line Bryan drew was not a re-presentation of the pairs of measures he imagined in the situation. Instead, it seems Bryan made his initial point with the anticipation of showing the simultaneous state of maximum height and maximum value. Then he drew a line by imagining the height of the water decreasing as he imagined the water in the bottle evaporating. This suggests that Bryan constructed his graph by imagining the gross variation of the height of the water in the moment he imagined that variation in his experiential time.

After Bryan drew the line he reconstructed his initial image of two pairs of quantities' measures to reason that his graph should show maximum height and maximum volume. He determined that his graph did not show this relationships saying, "It doesn't make sense. Because over here (*points to start of line in top middle of plane*) it says height is maximum but volume is not maximum (*points to intersection of line with horizontal axis*)." Bryan drew a new graph that was a vertical reflection of his original graph about its midpoint; his graph now decreased from right to left (see Figure 4 blue line). Bryan explained that now he understood his graph to show the height is maximum when the volume is maximum and also show the height is minimum when the volume is minimum.

In summary, Bryan engaged in three distinct activities when completing the bottle evaporation task. First he imagined each quantity's (discrete) variation and coordinated the two varying quantities by constructing pairs of measures, a point's coordinates. Then he drew a line by re-presenting his experience attending to one quantity's gross variation as he imagined it changing in his experiential time. Finally, he reconstructed his initial image of pairs of quantities' sizes to determine if the behavior of the sketched graph matched his anticipation of the relationship between the quantities' measures.

I hypothesize that Bryan did not make his graph by re-presenting his initial image of pairs of measures because he could not anticipate creating these pairs of measures as he imagined a quantity to continuously vary in his experiential time. In other words, it seems that Bryan needed to imagine a static state in the quantities' variation in order to coordinate two quantities' measures. As soon as he imagined one quantity's measure to change he could no longer coordinate two quantities' measures. This implies that the way Bryan coordinated two varying quantities inhibited him from re-presenting his understanding of how two quantities change together when making his graph.

## Discussion

Ali and Bryan both demonstrated different images of covarying quantities when making a graph and when reasoning about that sketched graph. While this highlights the meanings students learn to impose on the products of their graphing actions, the findings from this study suggest that the meanings students construct from their sketched graph are consistent with how they imagined the quantities to covary in the situation. In the examples above, Ali reasoned separately about two quantities' smooth variation both when reasoning about the situation and when reasoning about her graph. Similarly, Bryan reasoned about pairs of measures both when reasoning about the situation and when reasoning about his graph. This suggests that while a student might have distinct experiences making a graph and reasoning about that graph these experiences are actually governed by the same scheme. More specifically, the student's activity making a graph is the result of an accommodation to their scheme for covariational reasoning in order to have actions available to them that persist under variation. For both Ali and Bryan this accommodation involved attending to one quantity as she/he imagined it changing in her/his experiential time. This is significant because it implies that students engage in different levels of covariational reasoning throughout their graphing activity because they are unable to re-present how they initially imagined the quantities to change together.

This study provides evidence that the nature of the student's coordination can inhibit him/her from re-presenting his/her understanding of how the quantities covary in the situation. For example, Ali coordinated two quantities' variation by imagining each quantities' variation separately. As a result, she did not have a single coordinated image to attend to when making her graph. Ali anticipated that she could use whatever shape she made to see the variation of each quantity, but she did not have a way to think about how to make that shape. Instead, she made her graph by guessing shapes until she picked one that appropriately matched how she imagined each quantity to vary.

Bryan coordinated two quantities' variation by coordinating static states in each quantity's variation and constructing the coordinates of a point in the Cartesian plane. However, as soon as he imagined one of the quantities to vary he no longer had an image of a static state in which he could coordinate two measures. As a result, when he attempted to construct his graph he did not continuously coordinate quantities' measures. Instead, Bryan made his graph by imagining one quantity changing in his experiential time. After making his graph, however, Bryan imagined coordinating measures to reason about what his sketched graph represented; he appeared to reason about an infinite collection of points on his graph. In summary, since Bryan's image of plotting points did not persist under variation, Bryan could not re-present his coordination of the quantities' variation as he imagined a continuously changing phenomenon.

Researchers repeatedly emphasize the importance of holding two quantities in mind when constructing a graph (e.g., Moore et al., 2016; Whitmire, 2014). This study provides further evidence that this is a nontrivial construction. More specifically, students need to construct ways to organize their images of varying quantities so that they can hold two quantities in mind as they imagine both quantities to change. I hypothesize that if students hold both quantities in mind then they have something new to represent in a graph – namely the coordination of two quantities. Teaching experiments with Ali and Bryan (see Frank, 2017) suggest that conceptualizing a point as a correspondence point, imagining a graph being made of Tinker Bell's pixie dust, and imagining the phenomenon happening in little chunks (e.g., taking baby steps) might support students in coordinating their images of varying quantities[3] and re-presenting this coordination in a graph .

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[3] Correspondence point didactic object from Thompson, Hatfield, Yoon, Joshua, and Byerley (in press) and Tinker Bell didactic object from Thompson (2013)

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Future Middle Grades Teachers' Coordination of Knowledge Within the Multiplicative Conceptual Field

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*We report theoretical and empirical results generated through studying several cycles of a number and operations content course offered to future middle grades mathematics teachers. A main feature of the course is using an explicit, quantitative definition for multiplication to connect a range of topics in the multiplicative conceptual field (Vergnaud, 1983, 1988). Course topics include multiplication and division with both whole numbers and fractions, proportional relationships, and linear functions. The theoretical results include a mathematical analysis of multiplication as coordinated measurement and a (still emerging) psychological framework that emphasizes coordinating diverse cognitive resources. Empirical data come from clinical interviews conducted with 6 future teachers enrolled in the content course in Fall 2016. One empirical result is the importance of connecting partitioning quantities, dividing measurements by whole numbers, and multiplying measurements by unit fractions when expressing relationships between quantities through multiplication expressions and equations.*

*Keywords:* Quantitative Reasoning, Multiplication, Equations

Improving instruction in topics related to multiplication remains a central challenge for mathematics education. For purposes of the present report, we consider multiplication and division with whole numbers and fractions, proportional relationships, and linear functions of the form $y = \mathrm{m}x$. The importance of these topics has been emphasized by curriculum standards (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2000) and national reports (e.g., Center for Research in Mathematics & Science Education, 2010; National Mathematics Advisory Panel, 2008). Nevertheless, despite several decades of research, the topics listed above pose perennial challenges for both students and teachers, and difficulties with these topics can be a primary obstacle to college readiness (e.g., National Center on Education and the Economy, 2013). The present report comes from an on-going NSF-funded study in which we are investigating ways to help future mathematics teachers develop integrated and coherent understandings of topics related to multiplication.

## Background

We draw on Vergnaud's (1983, 1988) construct of the *multiplicative conceptual field* (MCF) that consists of "all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide" (Vergnaud, 1988, p. 141). Vergnaud included whole-number multiplication and division, fractions, ratios and proportions, linear functions, and further topics in the MCF.

Most research on teachers' understandings of the MCF has concentrated on deficits with respect to particular topics. Although many teachers can use algorithms to determine the product of two fractions or decimals, a host of studies (e.g., Behr, Khoury, Harel, Post, & Lesh, 1997; Eisenhart et al., 1993; Graeber & Tirosh, 1988; Graeber, Tirosh, & Glover, 1989; Harel & Behr, 1995; Sowder, Philipp, Armstrong, & Schappelle, 1998; Tirosh & Graeber, 1990) have reported constraints on in-service and preservice teachers' performance when explaining products of fractions or decimals embedded in problem situations. Similarly, although many U.S. teachers can compute the quotient of two fractions or decimals using algorithms, they often experience difficulties explaining division when it is embedded in problem situations (e.g., Ball, 1990; Borko et al., 1992; Graeber & Tirosh, 1988; Jansen & Hohensee, 2016; Lo & Lou, 2012; Ma, 1999; Simon 1993; Tirosh, 2000).

The small handful of studies on teachers' capacities to reason about proportional relationships report that middle grades teachers perform poorly on test items that, ideally, their students should be able to solve (Post, Harel, Behr, & Lesh, 1991). Teachers can have difficulty distinguishing missing-value problems that ask about proportional relationships from ones that do not (e.g., Cramer, Post, & Currier, 1993; Fisher, 1988; Lim, 2009), can have trouble coordinating two quantities in a proportional relationship (e.g., Orrill & Brown, 2012), can make additive comparisons inappropriately (e.g., Canada, Gilbert, & Adolphson, 2010; Lim, 2009; Son, 2010), and can have trouble conceiving of a ratio as a measure of a physical attribute, such as steepness or speed (Simon & Blume, 1994; Thompson & Thompson, 1994). With respect to problem-solving strategies, teachers can rely heavily on cross multiplication or other formal methods (e.g., Fisher, 1988; Harel & Behr, 1995; Orrill & Brown, 2012), guess at arithmetic operations (Harel & Behr, 1995), and search for key words (Harel & Behr, 1995).

## Theoretical Frame

In contrast to the numerous studies that have emphasized deficits in teachers' understandings of particular topics included in the MCF, in our project we concentrate on emerging competence characterized as developing a coherent perspective that connects various topics related to multiplication. Our conjecture is that teachers might better understand individual topics within the MCF by developing a single lens that ties them together. The framework we present for such an integrated understanding combines mathematical and psychological perspectives.

Figure 1 shows the quantitative definition of multiplication upon which we have converged. It applies to situations in which there is a quantity (the product amount) that is simultaneously measured with two different measurement units (a "base unit" and a "group"). The most important aspects of this definition are (a) writing the multiplicand and multiplier in a consistent order to support a coherent view of multiplication, division, and proportional relationships (e.g., Beckmann & Izsák, 2015) and (b) interpreting $N$, $M$, and $P$ in Figure 1 as numbers that result from measuring quantities in terms of some designated unit.[1] $N$ and $P$ refer to measuring with base units, and $M$ refers to measuring with groups.

| $N$ | $\bullet$ | $M$ | $=$ | $P$ |
|---|---|---|---|---|
| How many base units make one group exactly? | | How many groups make the product amount exactly? | | How many base units make the product amount exactly? |

*Figure 1. A quantitative definition for multiplication based in measurement.*

The definition in Figure 1 can be used to coordinate an important swathe of the MCF—for instance, by viewing division as multiplication with an unknown factor and proportional relationships as instances where values for two of $N$, $M$, and $P$ co-vary while the value for the

[1] Our emphasis on numbers arising from measuring quantities in terms of designated units is consistent with aspects of Thompson's (2010) discussion of quantitative reasoning.

third remains fixed (Beckmann & Izsák, 2015). The definition in Figure 1 is also consistent with the definition for fractions found in the Common Core State Standards for Mathematics (CCSS; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) that presented the fraction $A/B$ and $A$ copies of the unit fraction $1/B$. Figures like that shown in Figure 2a can support the measurement perspective on unit fractions if one asks how many of the long strip make the short strip exactly (1/3). We have found that future teachers have little problem answering such questions and can extend this measurement perspective from unit fractions to non-unit fractions (Figure 2b). This appears to be a reliable foothold for future teachers when extending the measurement definition of multiplication shown in Figure 1 from whole numbers to fractions.


*Figure 2. (a) Interpreting 1/3 from a measurement perspective: 1/3 of the long strip makes the short strip exactly. (b) Interpreting 2/3 from a measurement perspective: 2 (1/3 of the long strip) make the short strip.*

Our psychological perspective is informed by diSessa's (1993, 2006) knowledge-in-pieces epistemology. Knowledge-in-pieces is a constructivist perspective in which learners come to know by using and refining knowledge as they construct interpretations of their interactions with the physical and social environment. The perspective characterizes the evolution from novice to expert knowledge as piecemeal construction, refinement, and reorganization of diverse fine-grained knowledge resources that are connected to varying degrees and whose use is often sensitive to context. Examples of cognitive mechanisms include refining the contexts in which resources are applied, forming new connections among resources, and loosening connections among others. In the present study, we examined the ecology of resources that future middle grades teachers used as they coordinated the definition of multiplication shown in Figure 1 with diverse problem situations that are contained in Vergnaud's (1983, 1988) MCF. Past research has used the knowledge-in-pieces perspective to demonstrate that coming to see diverse problem situations through a common lens can be a significant accomplishment (e.g., Wagner, 2006).

## Methods

In Fall 2016, we recruited six future middle grades teachers who were enrolled in a 2-semester sequence of mathematics content courses. The second author taught both courses. Both courses made extensive use of the definition of multiplication shown in Figure 1. The first course (Number and Operations) covered multiplication and division with whole numbers, the CCSS definition for fractions, the meaning of the equal sign, reasoning from definitions, multiplication with fractions, partitive and measurement division with whole numbers, and connecting division to fractions. The second course (Algebra) focused on proportional relationships, linear equations, and further topics. Teachers in the course were invited to participate in interviews, and the six were selected based on performance on a fractions survey administered the first week of the Number and Operations course.

This report focuses on the first three interviews we conducted during the Number and Operations course. The interviews were spaced a few weeks apart and were coordinated with whole-class instruction, most often so that the interviews provided information about the future teachers' reasoning before specific topics were introduced in the course. The first interview examined how future teachers thought about multiplication as a model of problem situations and how they formed equations of the form $y = \mathrm{m}x$ before the definition of multiplication shown in Figure 1 was introduced in the course. The CCSS definition of fractions had already been introduced and the interview tasks included fractional multipliers and multiplicands. The second interview took place after the definition in Figure 1 was introduced (only with whole numbers) and was designed to access future teachers' facility with the mental operations of splitting and units coordination that have emerged as important in research on children's fractional knowledge (e.g., Steffe, 2003) and also how they formed equations of the form $y = \mathrm{m}x$ at this point in the course. The third interview took place after instruction in fraction multiplication, division as multiplication with unknown factor, the distinction between partitive and measurement division in the context of whole numbers, and the connection between division and fractions. The third interview was designed to see how future teachers reasoned about division in the context of proportional relationships and linear equations, topics that would be covered in the subsequent algebra course. Many of the interview tasks asked future teachers to solve problems using a math drawing. Examples of such drawing include number lines and tape or strip diagrams.

We recorded all of the interviews with two cameras—one focused on the interviewer and research participant and one focused on written work—and collected all of the written work generated during the interviews. A third party transcribed the interviews verbatim. The present report is based on analysis of talk, gesture, and inscription as captured in the videos, transcripts, and written worked generated during the interviews. (In addition to the interview data, we also collected the participants' homeworks, quizzes, and tests assigned in the course.)

We analyzed talk, gesture, and inscription line-by-line for evidence of the knowledge resources that the future teachers appeared to employ. We wrote analytic notes to capture our interpretations of how future teachers were reasoning moment-to-moment. The notes included observations about similarities and differences both within a given teacher across different tasks and across different teachers on the same task. In some cases, we took future teachers' statements as direct and reliable reports of their thinking. In other cases, we made inferences about aspects of future teachers' reasoning that they would not likely be able to report directly.

## Results

Future teachers in the present study employed a complex ecology of cognitive resources when working on tasks across the interviews. To illustrate our results, we make three comments about that ecology that span all six participants and then provide more detailed description of one participant.

First, during the interviews, the future teachers demonstrated facility with whole-number factor-product combinations, algorithms for multiplying fractions, and cross multiplication for solving proportions. We assumed that when future teachers employed these resources, they drew on what they remembered from their K-12 mathematics education. Such resources can be viewed in a negative light when they interfere with reasoning about quantities directly. Although we did observe cases where future teachers determined numerical answers through computations, and thereby circumvented reasoning with quantities, we also observed cases in which future teachers used calculation constructively when solving and explaining problems in terms of math drawings. These data suggest that resources for numerical calculation are not necessarily in opposition to resources for reasoning with quantities but rather could be part of a larger ecology in which numerical calculation and reasoning with quantities support one another.

Second, the future teachers expressed a variety of meanings for multiplication and the equal sign. Meanings for multiplication included widely known ones such as multiplication is about repeated groups and that, in the case of fractions, "of means multiply." For these future teachers, repeated groups and "of" oftentimes appeared to be two disconnected understandings of multiplication rather than different expressions of a single, unified conception of the operation (such as the one shown in Figure 1). Meanings for the equal sign also included several well-known ones, such as an indication to complete a computation, an indication that two numbers co-occur (leading to equations that actually express ratios and look like the classic student-professor error), and the number on the left-hand side is the same as the number on the right-hand side. In one interview, we saw one participant encounter difficulties when he used all three of these meanings for the equal sign when working on a single problem. More generally, multiple meanings for multiplication and the equal sign often led to piecemeal reasoning across tasks which used different combinations of whole numbers and fractions for the multiplier (*M* in Figure 1) and multiplicand (*N* in Figure 1). These data suggested that sometimes future teachers experienced challenges during the interviews not so much because they lacked a particular cognitive resource but rather because they had trouble recognizing when some of those resources might be more useful than others. Knowledge-in-pieces' emphasis on knowledge refinement is well-suited to handle such phenomena.

Third, future teachers employed to varying degrees two mental operations on quantities highlighted in past research on children's fractional knowledge. These are splitting (e.g., Steffe, 2003) and different levels of units coordination (e.g., Hackenberg, 2010). Steffe's splitting operation is a fusion of partitioning and iterating. We asked versions of splitting tasks that made explicit connections to the measurement sense of unit fractions illustrated in Figure 2. None of the participants had difficulty with our splitting tasks. In particular, when presented with a strip like that shown in Figure 2a and told that the strip was a whole number of times (in actual interviews we numbers like 8) longer than a second strip, future teachers had no trouble constructing the second strip or explaining how many of the long strip made the short strip exactly. To illustrate, in case of a diagram like that shown in Figure 2a, all future teachers could explain that 1/3 of the long strip made the short strip exactly. Although, with an appropriate prompt, all the future teachers could express a measurement perspective on unit fractions, we saw differences in performance on tasks designed to elicit units coordination.

For the rest of our results section, we present examples of reasoning from Hanna around whole-number and fractional multipliers (*M* in Figure 1). We will sketch evidence that during the first interview Hanna was more proficient reasoning with and explaining whole-number multipliers than fractional multipliers, even though she also demonstrated a measurement perspective on fractions, and that during the second interview she had begun to coordinate a measurement sense of fractions with multipliers. The examples illustrate how mechanisms like coordination and refinement of fine-grained knowledge resources that are emphasized in the knowledge-in-pieces perspective are a good fit for reasoning we observed. In the full paper, we will present data from others of the six future teachers that also illustrate knowledge coordination and refinement and that provide perspectives on the multiplier and multiplicand that contrast with Hanna's.

We began the first interview with a set of word problems that described (to us) multiplication situations with different combinations of whole-number and fractional multipliers. This interview took place before meaning of multiplication shown in Figure 1 was introduced in the content course. During her first interview, Hanna solved a variety of problems with whole-number multipliers without difficulty, explaining that collecting objects into whole groups cued multiplication for her. For instance, she had no difficulty writing an equation that fit the tennis ball situation: *Jacinda has 4 cans of tennis balls. If there are 3 balls in a can, how many tennis balls does she have in all?* Hanna wrote 4 x 3 = 12 and justified her work as follows: "When you're dealing with different kinds of objects, it's easier to see with multiplication. Like if you had…instead of bags, if you had like 5 soccer balls and then another 5 soccer balls and they didn't even talk about bags, then I'd probably do 5 plus 5." A few moments later during the same interview, we asked her to write an expression or equation that fit the Chili situation: *Nick uses 1/5 of can* [sic] *of tomato paste in his chili recipe. The can contains 4 ounces of tomato paste. How many ounces of tomato paste does he use in his chili?* This time Hanna wrote "1(4) = 1/5 (X)" and explained that she was attempting to use ratios. She then stated that the problem was confusing because "1/5 is talking about of the can, 4 is talking about ounces. So those are two different things." Notice that Hanna attended to a similar feature of both situations—two different things—with very different results. These data provided initial evidence that she did not have a single conception of multiplication that she could apply across problems with whole-number and fractional multipliers.

In subsequent work, also from the first interview, Hanna demonstrated that she did not fully coordinate partitioning a quantity into equal-sized pieces, dividing the value of that quantity by a whole number, and multiplying the value of that quantity by a unit fraction. As she continued to work on the Chili task, she proposed dividing 4 ÷ 1/5 and explained:

> Because you're trying to find a part of the whole and like how much a part of the whole equals. So if you know…you know that the can is 4 ounces and you need…you want to find 1/5…how many ounces 1/5 of that can is then you need to divide 4 ounces by 1/5 to find out how many ounces there are in 1/5.

Later on during the first interview, Hanna worked a more complicated task: *One serving of oatmeal is 1/3 of a cup. For one meal, Chelsea are 2/5 of a serving. How many cups of oatmeal did Chelsea eat?* From the data we could not tell exactly why, but Hanna refined her connections between partitioning and numerical calculation. In particular, she offered a series of explanations, each of which included refinements to previous explanations. Her final explanation coordinated partitioning one serving into 5 parts with dividing 1/3 by 5. She wrote 2 (1/3 ÷ 5) and explained: "So first I divided my 1/3 cup into 5 pieces, because I know that she ate 2/5. So, if I divide it into 5 pieces and then multiply by 2, I can get my 2/5." A few moments later, Hanna acknowledged that she did not know how to compute 1/3 ÷ 5, which we took as evidence that she was not using results of this computation when refining her connections among partitioning, dividing, and actions that could support understanding fractions as multipliers.

Further evidence from the first interview that Hanna did not connect partitioning with multiplying by a unit fraction came on the Pebble task shown in Figure 3. Hanna had no trouble drawing the short path, defining $P$ to be the "amount of pebbles in the long path," and expressing the number of pebbles in the short path with the expression $P \div 8$. At the same time, she explicitly rejected multiplication as a viable option:

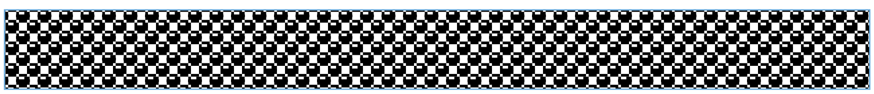
> I don't know how you would use multiplication, because you're trying to find out how much pebbles there are in 1/8 of this long path. So in order to get that, you have to divide the amount of pebbles by 8 to get, right? Yeah, to get that one amount of pebbles. To get…I don't know how you would you use multiplication honestly.

The second interview took place about 3 weeks after the first and after the definition of multiplication shown in Figure 1 had been introduced in the content course. At the time, the definition had only been used with whole numbers. During the second interview, Hanna demonstrated facility with measurement that went beyond her work during the first interview. When working on the task shown in Figure 4, she defined $b$ to be the "beads in short strip" and $B$ to be "beads in long strip." She then generated and explained the equation $B = b$ (LS/SS):

> So first I'm going to see how many short strips can go into my long strip. So I divide my long strip by my short strip, and then I'm going to multiply whatever I get. So how many short strips can go into my long strip times the amount of beads that are in my short strip, and then…then you get how many beads are in your big strip.

Notice first that Hanna appeared to apply a measurement sense to the fraction notation, LS/SS, when discussing "how many short strips can go into my long strip" and second that she connected this measurement sense to the multiplier. Further evidence from the second interview made clear that, although Hanna had made important steps in the right direction, she still had further to go when thinking about fractional multipliers.

The drawing below shows a 1 kilometer long garden path. It is 8 times as long as another garden path. Please draw the other path.



Imagine that the same two paths are covered with pebbles that are of uniform size and are spread evenly on both paths. The dots shown in the long strip indicate the pebbles.

If you knew the amount of pebbles covering the long path, how could you express the amount of pebbles covering the short path?

*Figure 3. The Pebbles task from Interview 1.*

Some strips of fabric have tiny beads sewn onto them. The beads are spread uniformly across each strip.

If you knew the amount of beads in the short strip, how would you find the amount of beads in the long strip?

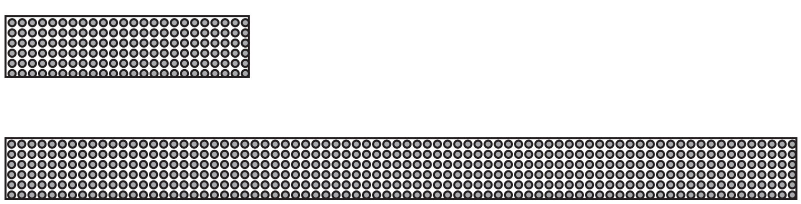


*Figure 4. The Beads task from Interview 2.*

## Conclusion

We are still a long way from illustrating how teachers might better understand individual topics within the MCF by developing a single lens that ties them together; but, the example of Hanna suggests processes through which teachers might construct such a lens. In particular, using the example of emerging facility with fractional multipliers we have illustrated how Hanna used fine-grained coordination and refinement of knowledge resources to extend her understanding of multipliers in the case of whole numbers.

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How Do We Teach Thee? Let Me Count the Ways
A Syllabus Rubric with Practical Promise for Characterizing Mathematics Teaching

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*Good methods to characterize teaching are needed to describe both current status and changes in teaching practice, and to link student outcomes to particular instructional practices. Such methods are understudied and thus the relative merits of different methods are not well understood. As part of a study examining multiple methods for characterizing teaching in college mathematics, we analyzed syllabi using three rubrics. Syllabi are authentic course artifacts that reflect course design and instructor's intentions; they are readily available from instructors. One of these rubrics, an evaluative rubric known as Measuring the Promise (MtP), proved useful in distinguishing courses taught by a sample of seven early-career instructors and a comparison sample of experienced active learning practitioners. Good correlation of MtP scores with observation scores using the well-established Reformed Teaching Observation Protocol suggest that the MtP may be a useful alternative to costly and time-consuming observations.*

*Keywords:* syllabi, observations, measurement of teaching

For many studies of higher education, it is important to characterize teaching: to describe teaching practice within an institution or across a discipline, to relate student outcomes to teaching practices, or to measure change in teaching practice over time. However, we do not yet have a good understanding of what can be learned from different approaches to describing teaching, nor the strengths and limitations of these approaches (AAAS, 2013). Yet descriptions of teaching practice form the foundation for claims about the effectiveness of various instructional practices, and thus also the basis of many current efforts to change such practices.

Our study is motivated in particular by a need for good methods to measure change in teaching after professional development of college instructors (CIPD). Funders and institutions need good evidence to determine whether, how, and in what forms, professional development may be a good investment. While better student learning is the ultimate goal, measuring student outcomes directly is not always possible, and faculty may need time to gain skill in the new techniques before student learning is measurably improved.

An alternative is to measure the degree to which faculty implement evidence-based teaching practices, such as those introduced in CIPD programs, together with a "golden spike" approach that links these practices to prior studies that demonstrate how these teaching practices influence student outcomes (Brown Urban & Trochim, 2009). Because teaching is a complex activity, this measurement, too, is challenging. Observational studies are viewed as most objective but are complicated and costly, while well-validated instructor surveys are not yet available (Felder, Brent & Prince, 2011). Course artifacts offer a wealth of teaching-related material that is readily accessible and authentic in representing the instructor's actual work, rather than her later representation of it—but it is less clear how to make inferences from these materials about the instructor's classroom instruction, teaching decisions or philosophy. To understand whether and how CIPD improves STEM teaching, we need valid and reliable measures of teaching practice that can be used to learn whether and how instructors' practices and choices change after CIPD.

This methods development study was exploratory by design, involving close examination of teaching practices in seven undergraduate mathematics courses taught by early-career instructors. We compared insights gained from a full suite of teaching measures: student and instructor surveys, observations, and coding of course syllabi and assessment items. The study focused on the potential of these measures to detect change in instruction, such as the changes that might result from professional development, but we did not directly study change. Thus the study is a close look at a small sample. We sought to identify methods that are both informative and practical for measuring teaching practice, and to make judgments about when and how these methods may be useful—alone or combined—in characterizing teaching. This work thus offers advice to researchers and evaluators to make intelligent choices for their own studies.

Our broad research questions were:

1. What are the affordances and limitations of behavior-oriented and outcome-oriented observations, faculty self-reports, student reports and classroom artifacts as methods for characterizing teaching in undergraduate mathematics classrooms?
2. What are the differences among characterizations of teaching in undergraduate mathematics classrooms that are based on these distinct types of measures?

Our study explored both descriptive and evaluative measures of teaching. Evaluative or "outcome-oriented" measures examine the aims and effects of instruction rather than the choesn activities, rating instruction against a specific standard for "good teaching," thus differing from strictly descriptive or "behavior-oriented" measures. Here we focus on two evaluative measures, a widely used observation protocol and a rubric for analyzing course syllabi. These two methods, observation and syllabus analysis, represent extremes of simplicity and complication in the logistics and invasiveness of data gathering and the demands of data analysis, so it is interesting to compare their potential as measures for characterizing mathematics teaching.

## Study Sample

We recruited instructors from MAA Project NExT (PN), New Experiences in Teaching, a professional development program for early-career mathematics instructors. Working with early-career instructors whose teaching methods were still developing, we were able to observe a range of teaching behaviors and skill levels that are likely comparable to those encountered in studying professional development outcomes for other instructors of undergraduate mathematics. We also sought to gather data from courses for varied student audiences and at varied curricular levels.

We solicited study participants through selected PN listservs, inviting respondents to read an online description of the study, review the consent form, and complete a pre-screening questionnaire about their courses and academic calendars. Ultimately, seven instructors took part.

This sample included variety across instructors, courses, departments and institutions. The *instructors* included 4 women and 3 men. Six were white and one was multi-racial; none were Hispanic. Five held tenure-track positions, one a long-term instructorship and one a visiting position. Their teaching experience (including TA work) was 3 to 10 years. Their *courses* spanned the early (3), middle (2) and late (2) undergraduate curriculum and a range of mathematics, STEM, non-STEM, and pre-service teaching audiences. Class size was small, 8-28 students. The six semester-based and one quarter-based courses met for 35-56 hours each. The courses were situated in *departments* that granted bachelors (3), masters (2) or doctoral (2) degrees as the highest mathematics degree. The *institutions* were diverse in geography, institution type and student enrollment; two had high minority student populations.

## Study Methods

Working with each instructor, we selected a single target course from which we collected all data. With this small sample we could not generalize about any particular instructional setting, but we could test the applicability of these methods in varied settings. Each instructor contributed data to support six study components:

a) Video observations of ten class periods, coded with descriptive and evaluative protocols
b) Instructor survey, end of course, self-reporting teaching practices
c) Instructor interview
d) Student surveys, end of course, including both descriptive and evaluative items
e) Course syllabus, coded with descriptive and evaluative schemes
f) Course-specific subset of assessments identified from the syllabus.

In this report, we focus on methods (a) and (e), using observation data as a benchmark to evaluate a syllabus rubric as a possible tool to characterize teaching with relatively low effort. The rubric also offers good potential as a tool for formative evaluation or for providing feedback to instructors as professional development. Elsewhere we discuss results from other methods.

### Why Study Syllabi?

As Eberly, Newton and Wiggins (2001) point out, the syllabus is both "the initial communication tool that students receive" and "the most formal mechanism for sharing information with students" (p. 1) about a course. Ideally, it is "a learning-focused document that communicates clearly and compellingly what students will gain from the course, what they will do to achieve the promise it lays out, how they will know whether they are getting there, and how to best go about studying" (Palmer, Bach & Streifer, 2014b). If, as these authors argue, the syllabus serves as a "framework for designing meaningful learning environments," then it follows that we may be able to diagnose the presence of such intentional and student-focused design from syllabi. Syllabus analysis offers advantages for data collection too: the syllabus is already written and widely available, so collecting it requires low instructor effort; it is brief, thus rapid to analyze; and it is public, so gathering it should not require special IRB permission.

### Analysis of Syllabi

We tested three syllabus analysis tools found in the literature. The SPROUT-S protocol is a descriptive protocol developed at UC Irvine to study the relationship of student academic outcomes to the use of "promising instructional practices" in undergraduate STEM courses (Reimer et al., 2016). The Penn State Engineering Education protocol (Zappe et al., 2015, 2016) is also descriptive, a list of 47 research-based practices in engineering education that is drawn from synthetic work by Hattie (2008) and Chi (2009) examining factors related to student achievement and student learning. While in principle a descriptive approach could assist in analyzing the presence or prevalence of certain instructional methods or philosophies, we found neither of these descriptive tools useful for our study, as we will describe in our presentation.

Instead, the analysis presented here focuses on an evaluative rubric, Measuring the Promise (MtP), a validated rubric from faculty developers at the University of Virginia (Palmer, Bach & Streifer, 2014a). As a rubric, it defines a coherent set of criteria and describes different levels of performance quality on the criteria (Brookhart, 2013). It could thus be used in formative evaluation—to guide professional development on course design—as well as a summative assessment tool. It is designed for use in any discipline.

The holistic and evaluative rubric is strongly literature-based (Palmer, Bach & Streifer, 2014b). The full rubric uses 16 items grouped into five categories: learning goals and objectives, assessment, schedule, classroom learning environment, and learning activities. Each item is rated gold, silver or bronze to indicate its relative importance in the scoring rubric—which is in turn based on its expected influence on student outcomes—and the rater classifies the strength of evidence about each as strong, moderate, or low. The authors specify their assumptions about the rater's background knowledge and provide examples of the kinds of evidence used to assess each criterion. Raters must understand Fink's (2013) significant learning goals, distinguish learning goals and objectives, and assess alignment of goals, objectives, activities and assessments in course design. With this background, modest training is required to achieve interrater reliability.

To emphasize the presence and quality of essential features, the scoring system weights both the features (3,2,1) and the evidence for them (2,1,0) (CTE, 2017). The maximum score is 58: a 'learning-focused' syllabus will score in the range of 41 and higher; a 'content-focused' syllabus at 18 or below. Syllabi in between are called 'transitional.'

For the early-career instructors, syllabi from seven courses taught in 2015-16 were coded using the MtP, plus two more syllabi representing earlier versions of the same course. For comparison, 12 syllabi were coded for courses taught in 2016-17 by 'expert' instructors known to use strongly student-centered teaching approaches.

**Observation Coding with RTOP**

The Reformed Teaching Observation Protocol (Sawada et al., 2002) was developed to evaluate the degree of "reform" toward student-centered teaching in science. RTOP's 25 items assess the degree to which the classroom is learner-centered in five categories: lesson design and implementation, propositional knowledge, procedural knowledge, communicative interactions, and student-teacher relationships. RTOP scores in K12 classrooms have been correlated with student achievement and used to assess change as a result of CIPD. It requires significant training and nuanced judgment against externally defined criteria for effective teaching.

We collected observation data for 8-10 class sessions taught by the early-career instructors using a portable video camera shipped to instructors and mounted behind students, facing forward. We followed RTOP data analysis methods outlined by Ebert-May et al. (2011). Five items forming five subscales are scored on a Likert scale 0-4. After initial training, six videos were randomly selected and coded by three raters. We tested interrater reliability by computing intraclass correlation coefficients (ICC) and achieved an acceptable ICC level (>0.80) for overall RTOP scores and for each subscale.

One rater then coded five randomly chosen class sessions for each of the seven courses. We computed RTOP and subscale scores for each class and calculated means of the five observations for each course. Total scores of 0-100 are classified into five categories using score breakpoints of 30, 45, 60, and 75, where scores ≤30 are interpreted as "straight lecture" and scores >75 as "active involvement in open-ended inquiry," with intermediate scores placed along a spectrum of interaction and inquiry. Because the classifications are framed in language common in discussing inquiry-based science, such as carrying out experiments, we re-interpreted the classifications for college mathematics, considering inquiry-oriented processes such as preparing, explaining and critiquing proofs or problem solutions, and explicitly considering alternative solutions.

## Results

Figures 1 and 2 show MtP scores for syllabi from courses taught by early-career (designated PN for Project NExT alumni) and experienced instructors (EX). As a group, total scores for courses of experienced instructors (mean 32 ± 12) were not statistically distinguishable from those of early-career instructors (mean 30 ± 13). However, 3 of 7 courses by early-career instructors, vs. only 1 of 9 courses of experienced instructors, were rated as content-centered. In both figures, arrows link pairs of syllabi for a single course that represent a particular instructor's historical and current practices; these pairs are discussed further below.

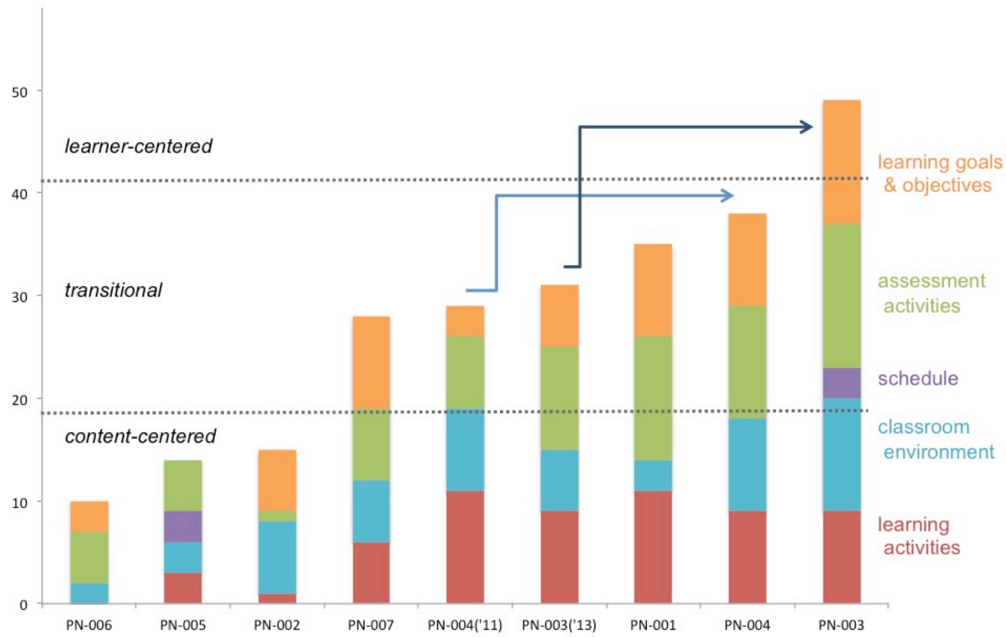


*Figure 1: MtP syllabus ratings for early-career instructors, by item category.*

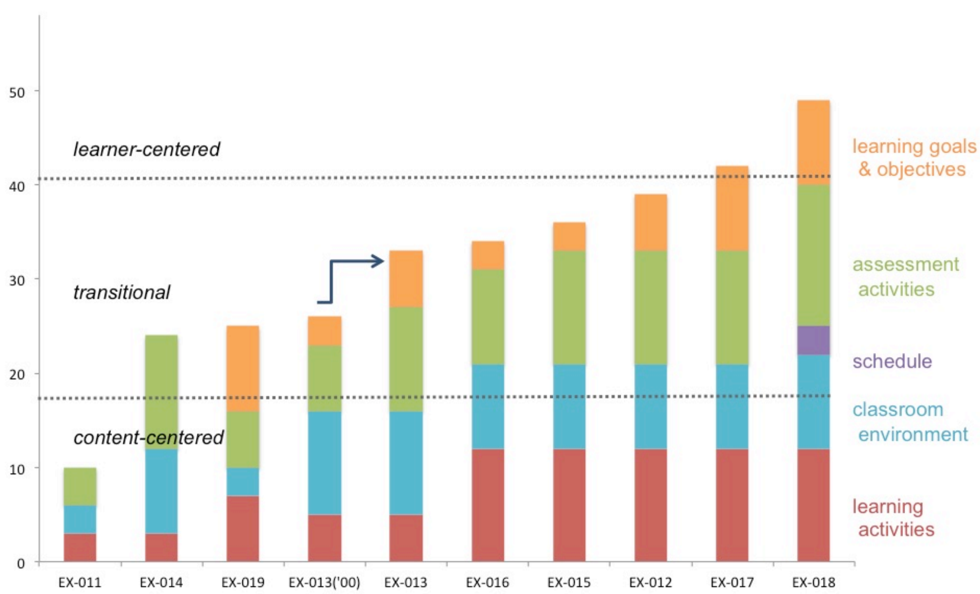


*Figure 2: MtP syllabus ratings for experienced instructors, by item category.*

Comparing subscores in detail reveals some more distinctive patterns of difference between early-career (PN sample) and experienced (EX sample) instructors. Scores on *learning goals and objectives* were slightly higher among early-career instructors (mean of 6.3 for PN vs 4.8 for EX, of 12 points maximum). Scores on *assessment activities* were fairly high for both groups (8.0 PN, 10.1 EX, of 12). In mathematics, homework is frequently assigned and used to give formative feedback. Scores on the *schedule* were low across the board (0.7 PN, 0.3 EX, of 6). Scores for the *classroom environment* were higher among experienced instructors (8.3 EX, 6.1 PN, of 12). Scores for *learning activities* were moderate to high among both groups, but somewhat more consistent among experienced instructors (6.6 PN, 8.3 EX, of 12).

Comparison of current/historic pairs of syllabi (marked by arrows in Figures 1 and 2) for individual instructors shows positive change over time for the three cases available that reflect the start of a teaching career as compared to now. Other data (not shown) suggests that very experienced instructors find a teaching groove and stick to it; their MtP score does not change.

These data suggest that the MtP has good discrimination on aspects of course planning that may differ between instructors of differing experience and/or skill. To relate the syllabus score to a measure based on actual classroom practice, we compared the MtP scores to mean scores on the RTOP (Figure 3) for the PN sample, for which we had both data types.

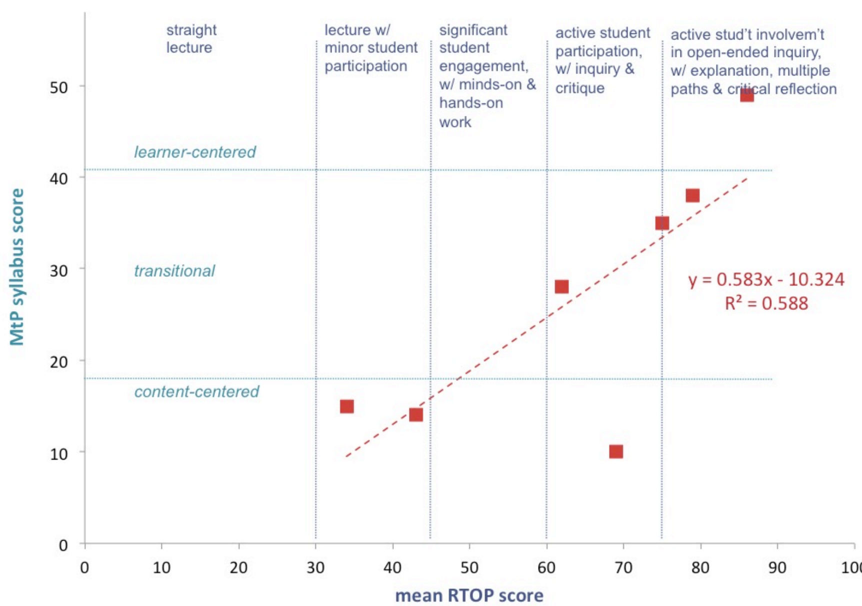


*Figure 3: Correlation of MtP syllabus scores with RTOP scores, including score classifications*

Five of seven courses score in the active learning ranges of the RTOP scale, while two scores are described as interactive lecture. Moreover, MtP syllabus scores correlate well (R=0.59) with mean RTOP scores. In general, low MtP scores reflect underdeveloped or incomplete syllabi; actual classroom practice may be more interactive and inquiry-driven than is shown in the document. For example, the outlying point in Figure 3 represents a course where we observed inquiry activities, peer to peer collaboration, and extensive use of multiple representations of mathematical ideas, yielding a medium-high RTOP score, but the syllabus was disorganized, overly rule-oriented, and uninviting to the learner as an entrée into the discipline.

## Discussion

In general experienced active-learning instructors scored high on the MtP items for *classroom environment, assessment* (especially formative assessment), and *learning activities*, thus the rubric does show evidence of their student-centered orientations. Some of the early-career instructors were also IBL users, and their syllabi reflect aspirations toward the same student-centered practices. The reverse trend for *learning goals and objectives*, where early-career instructors scored higher, may reflect greater exposure of early-career instructors to learning goal-setting through professional development or exposure to RUME work. In addition, interview data revealed that some departments had set common learning objectives for particular courses; thus the learning objectives may be inherited rather than originated by the instructor.

Low scores on *schedule* arose because information on the choice and sequence of topics was commonly missing in syllabi from courses using inquiry-based learning (IBL), which reduced the score on items related to the intellectual organization or conceptual flow of the course and its pacing. It is also possible that college mathematics instructors take course content as canonical, whether or not they use IBL. For instance, with high consensus about what goes into a Calculus 1 course, and with many students required to take it, instructors may not think to justify to students their choices about the selection and sequencing of big ideas. Content sequencing may also be seen as given if it is decided departmentally and used by all who teach the same course.

In the cases where we could compare two versions of a course, the observed changes in MtP score suggest that the rubric is sensitive to change over time in instructors' practice.

The strength of correlation between the MtP and the RTOP is somewhat surprising, given that the MtP is based solely on the written plan for the course, and the RTOP rates instruction as implemented in class. However, both are holistic measures that focus on instructional design and set standards for 'good teaching' that are literature-based and thus aligned in many respects. Coding of a separate observation sample with RTOP will tell us if this correlation is robust.

Both our study groups, early-career and experienced, were more learning-focused than a general university instructor population (CTE, 2017). Mean MtP scores exceeded the median pre-test score for faculty who enrolled in a week-long institute on course design—but were lower than the post-test scores for those faculty. However, our study samples do not represent college math instructors nationally; they were volunteers already participating in educator communities.

## Conclusions and Implications

The MtP emphasizes instructors' design choices, as reflected in their syllabus, and focuses on clarity and alignment of the course design. It does not attempt to judge how well a course is executed but does capture elements of how instructors view students, teaching, and their subject. The rubric offers high face validity, due to its grounding in instructional design literature, and good discrimination, due to the weighted scoring system. Moreover, syllabus analysis with the MtP offers advantages for both gathering and analyzing data. In these ways, we find the MtP rubric a tool with significant potential to be useful in studies of teaching or change in teaching.

The correlation of MtP with RTOP in this small data set is particularly intriguing, because it suggests the potential of MtP to substitute, in some studies, for time-consuming and costly observations. Like the RTOP, MtP does require specialized expertise to apply, but it is well supported with coder training materials. Analysis of a limited data set suggests that the MtP has promise in detecting change in individuals' practice over time, but may be less useful in characterizing entire groups of instructors, due to the variability within groups.

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# Learning Our Way into Effective Professional Development: Networked Improvement Science in Community College Developmental Mathematics

Carlos Sandoval, Haley McNamara, Ann Edwards

*Every year, hundreds of thousands of college students are placed into, and do not complete, developmental math courses. The Carnegie Math Pathways, a nationwide initiative aimed at addressing this problem, is comprised of a student-centered instructional system that forefronts mathematical sense-making and conceptual understanding; structural changes to course offerings; and a system of faculty professional development. This paper reports on the use of Improvement Science, an approach grounded in methods and tools of quality improvement, to design, improve, and scale a professional development program for first-time Pathways instructors. We also report on insights derived from the improvement approach about effective professional development in the Pathways and findings related to common challenges faced when teaching the Pathways. We conclude with implications for professional development in higher education and the use of improvement science to scale effective professional development.*

*Keywords:* Developmental mathematics, professional development, community college, improvement science, continuous improvement

## Introduction

Over 14 million students are enrolled in community college, seeking an educational pathway to a productive career and better life. Between 50 and 70% of incoming community college students must take at least one developmental math course before they can enroll in college-credit courses (Bailey, Jeong, & Cho, 2010; Complete College America [CCA], 2012). However, 80% of the students who place into developmental mathematics do not complete a college-level math course within 3 years (Bailey et al., 2010). The pattern is similar in comprehensive 4-year institutions, where 20% of incoming students are placed into developmental math, and 63% do not complete a college-level math course within 2 years (CCA, 2012). Taken together, roughly 1.7 million first-time undergraduate students are placed into developmental math each year (CCA, 2012). Many of these students spend large amounts of money and long periods of time repeating courses; most simply leave college without a credential or developing a sufficient command of the mathematics needed to engage as productive citizens.

To address this national issue, the Carnegie Foundation for the Advancement of Teaching together with the Dana Center at the University of Texas at Austin developed an innovative, transformative strategy in undergraduate mathematics education: the Carnegie Math Pathways [CMP] program. The CMP consists of two distinct course sequences—Statway and Quantway, referred to collectively as the Pathways—that are designed to accelerate developmental students to and through college-level mathematics in one year. Their instructional design provides students with opportunities to learn mathematics content that is more engaging and relevant to their goals than they would encounter in traditional remediation and do so in pedagogical environments that are student- and problem-centered and that support students' persistence and engagement. The CMP initiative is organized as a Networked Improvement Community (NIC), a collection of institutions centered on addressing a particular problem and disciplined by the rigor of an approach called Improvement Science (LeMahieu, Edwards, & Gomez, 2015). The CMP NIC, currently comprised of over 65 IHE's (largely community colleges), organizes its collective efforts to dramatically improve the outcomes and quality of learning of their developmental math students.

We report here on an examination of a key component of CMP: the Faculty Support Program (FSP), professional development aimed at preparing and supporting first-time Pathways instructors. We address how Improvement Science is used to learn about effective professional development and for continuous program improvement. We begin with a discussion of the community college environment as it pertains to instruction and professional development. We then describe the FSP and our use of Improvement Science in the FSP. We then present a brief description of our findings pertaining to professional development in community colleges and conclude with implications, limitations, and directions for future research.

## Background and Context

### The Developmental Math Challenge

The reasons for the low success rates in developmental mathematics are complex. The structure of the traditional developmental math course sequence (Hodara, 2013) and the complexity of the course options are significant barriers to student retention and completion (Cullinane & Treisman, 2010). Also, developmental math instruction often does not employ research-based learning materials and pedagogical practices that can foster deeper student learning (Bransford, Brown, & Cocking, 1999). Many developmental math classrooms resemble the content-focused, knowledge transmission model so prevalent in undergraduate instruction (Bailey, Jaggars, & Jenkins, 2015; Grubb et al, 1999; Grubb & Gabriner, 2013). Instructional activities tend to focus on factual and procedural knowledge as opposed to conceptual content and mathematical sense-making (Mesa, 2011). Many developmental math curricula do little to engage students' interest and demonstrate the relevance of mathematical concepts to everyday life (Carnevale & Desrochers, 2003). In addition, instructors who may be open to alternative approaches, such as learner-centered models, are often skeptical of their efficacy for developmental students, who they perceive as weakly prepared and resistant to such strategies (Grubb & Grabiner, 2013). Many developmental math students have had negative prior math experiences leading to the belief that they are "not math people." These beliefs often trigger anxiety in students who encounter difficult math problems (Blackwell, Trzesniewski, & Dweck, 2007; Haynes, Perry, Stupnisky & Daniels, 2009).

### Carnegie Math Pathways "Change Package"

To address these long-standing challenges, the CMP NIC developed the CMP instructional system: a "change package" organized around Statway and Quantway. In Improvement Science (IS), a "change package" is a well-defined, evidence-based set of "change ideas" and associated metrics. The CMP change package consists of the following components:

**Accelerated pathways:** Rather than being faced with a maze of possible course options (Zeidenberg & Scott, 2011), students are offered an accelerated pathway that meets developmental math requirements and provides college math credit upon successful completion (Cho, Kopko, Jenkins, & Jaggars, 2012; Jaggars, Hodara, Cho, & Xu, 2015).

**Mathematics content relevant to college, career, and citizenship:** Statistics and quantitative literacy, respectively, are the core college-level content and conceptual organizers for Statway and Quantway, with developmental math learning goals integrated throughout. Both courses emphasize core mathematics skills needed for work, personal life, and citizenship, and stress conceptual understanding and its application in a variety of contexts (e.g., Gillman, 2006; Madison & Steen, 2008; and GAISE College Report ASA Revision Committee, 2016).

**Pedagogy supporting deep and flexible mathematics understanding:** Grounded in research on teaching for mathematical understanding and the development of mathematical practices (e.g., Bransford, Brown, & Cocking, 2000), CMP pedagogy emphasizes productive struggle with challenging problems (Schmidt & Bjork, 1992), making conceptual connections explicit (Hiebert & Grouws, 2007), deliberate as opposed to routine practice (Ericsson, 2008; Pashler, Rohrer, Cepeda, & Carpenter, 2007), opportunities for rich mathematical discourse (Moschkovich, 2007), and the role of collaborative learning in promoting mathematical sense-making (Esmonde, & Langer-Osuna, 2013; Webb, 2009).

**Productive Persistence supports:** Integrated throughout the CMP is an evidence-based package of interventions and practices to increase student motivation, tenacity, and learning skills. Based on research from social psychology, strategies focus on reducing student anxiety (Jamieson, Mendes, Blackstock, & Schmader, 2010), increasing a sense of belonging (Walton & Cohen, 2011), and countering fixed mindset (Dweck, 2006).

**Reducing language and literacy barriers:** Given students' diverse linguistic backgrounds, supports and interventions are interwoven into the curricula and pedagogy to assist students with the complex language and literacy demands of mathematics, with its different forms of representation and grammar (Gomez, Rodela, Lozano, & Mancevice, 2013; Gomez et al., 2015).

**Faculty professional development:** A robust professional development system has been crucial as the CMP have moved from early adopter colleges to institutional contexts with more adjunct and inexperienced faculty and limited institutional capacity—this is the focus of the analysis reported on in this paper.

## Teaching and Professional Development in Community College Mathematics

The shifts in pedagogy that the Pathways demand are challenging for many instructors due in part to their professional backgrounds and the availability of professional learning opportunities. Despite the emphasis on teaching in community colleges (Grubb et al., 1999), community college faculty are no more likely to have completed pedagogical coursework than faculty in research institutions. Heavy teaching loads and the low budget priority given to professional development prevalent in community colleges are not conducive for creating opportunities to learn about and develop instructional practices (Bailey, Jaggars, & Jenkins, 2015; Grubb et al., 1999; Tinberg, Duffy, & Mino, 2007). The problem is exacerbated in developmental education, where adjunct faculty, who often have heavier teaching loads, reduced access to professional development, and are assigned lesser status by their peers (Grubb, Badway, & Bell, 2003), constitute 76% of all developmental instructors (Center for Community College Student Engagement, 2014). Although evidence suggests that high quality professional development is critical for sustaining the impacts of systemic reform (Desimone, 2009; Fishman, Marx, Best, & Tal, 2003; Supovitz, Mayer, & Kahle, 2000), professional development in community colleges is described as ad hoc and lacking significant institutional support (Twombly & Townsend, 2008). Additionally, it typically consists of one-shot workshops that do not provide meaningful opportunities for professional learning (Bailey, Jaggars, & Jenkins, 2015; Huber, 2008).

Faculty professional development remains a critical and underutilized driver for improving student outcomes. Little research details the design of effective professional development centered on instructional improvement for developmental math instructors; nor has prior research documented challenges in trying to implement research-based professional learning experiences in community colleges (Twombly & Townsend, 2008). This paper focuses on efforts to address barriers to meaningful and effective professional development in community colleges, reporting on a specific professional development component, the Faculty Support Program (FSP), to prepare first-time Pathways instructors.

**The Carnegie Math Pathways Faculty Support Program (FSP)**

The design of the FSP is informed by the following set of principles derived from research on effective professional development, primarily from K-12 settings (Hawley & Valli, 2007; Hunzicker, 2010; Guskey, 2002; Garet, et al., 2001; LeMahieu, Roy & Foss, 1995): (1) program structure provides for sustained opportunities for professional learning; (2) learning activities are job-embedded, supporting emergent problems of practice; (3) learning activities are context/discipline specific; (4) learning activities provide opportunities for collaborative reflection; and (5) learning activities are centered around artifacts of classroom practice.

The context of the CMP NIC creates specific challenges for program design. CMP faculty are spread throughout the country and tremendous variability exists in the availability of campus resources. Participation in the FSP is voluntary and outside of professional obligations. Therefore, FSP offerings must be flexible and responsive to the needs of faculty, demonstrate clear value to faculty (and administrators), while also providing meaningful opportunities to develop practice. Thus, the FSP includes multiple modalities for faculty engagement, comprised of online activities and resources; intensive, face-to-face workshops, such as national and locally-based workshops; and one-on-one mentoring from designated Pathways faculty mentors who provide support in planning and teaching, including ongoing, just-in-time support.

The design principles and structural components of the program serve as critical guidelines in the spread, and scale of the FSP, particularly as the NIC grows rapidly. In 2015-2016, student enrollment quadrupled to 6220 students, resulting in 222 faculty members teaching the Pathways across 36 institutions nationwide (Hoang, Huang, Sulcer, & Suleyman, 2017). Additionally, the Pathways is spreading to settings widely variable in policy and culture. What is needed is an approach to learning about effective professional development across settings for program improvement. To do so, we employ an Improvement Science approach, detailed next.

## Approach and Methods: Improvement Science as an Approach to Theory Development, Knowledge Generation and Program Improvement

Educational interventions and programs demonstrate limited efficacy at scale (Elmore, 1996), potentially because traditional approaches to research and development often rely on promising interventions whose evidence for efficacy is limited to experiments conducted in controlled settings (Bryk, Gomez, & Grunow, 2011). As a result, such interventions usually rely on the fidelity of implementation by local actors. While appropriate for simple interventions that are procedural and artifact-centric in nature, complex interventions relying on individuals and their expertise across disparate institutions require a different approach (Bryk & Gomez, 2008). Thus, we employ Networked Improvement Science as an approach to improving and scaling the Pathways' system of professional development. Broadly, Improvement Science (IS) prioritizes addressing complex problems through learning deeply about causal systems; developing theories of action for achieving specific and measurable aims; iterative testing of promising ideas connected to theories of action; the use of measurement to determine performance and improvement; and scaling solutions (Bryk, Gomez, Grunow, & LeMahieu, 2015, p. 7). IS consists of a specific set of methods and tools aimed at improving outcomes through the generation of knowledge of what works, for whom, and under what conditions. In IS, theories—of the nature of the problem, of the local settings and systems, and of improvement—are explicated, tested, and refined over time, using diverse types and sources of data.

The generation of knowledge and development of theory using IS are accelerated through leveraging networks organized around a common aim. Thus, the CMP initiative is organized as a Networked Improvement Community (NIC), a group of institutions that share a common aim, a shared understanding of the problem being addressed, and use IS as a common approach (Russell et al., 2017). This network structure is comprised of individual faculty, institutions, and members of a central, organizing hub. The hub is an organization, the Carnegie Foundation for the CMP NIC, that collects and analyzes data from faculty and institutions to learn about implementation and drive improvement efforts. This affords the initiative the opportunity to accelerate the development, testing, and refinement of theory for and of improvement. The hub manages knowledge generated about problems being experienced and tested interventions so that efforts are not duplicated. We now illustrate one way that IS was utilized in the spreading and scaling of the FSP—the common measurement system we developed to manage and improve the program.

**The Faculty Support Program Measurement System**

Traditional mechanisms for feedback and evaluation of professional development programs are blunt tools; they inform program designers and facilitators about whether or not a program "worked" or how well it was received (Guskey, 2000), but often do not provide actionable feedback that can inform ongoing improvement. They generally do not provide insights into faculty's needs, resources, and constraints relative to the design and implementation of professional development programs.

In contrast, the management of the FSP utilizes Bryk et al's (2015) conceptualization of measurement for improvement. Measurement for improvement (a) creates a common language and cohesive vision of program quality across stakeholders; (b) includes an associated set of routines, protocols, and processes for reviewing the program performance, and (c) allows designers and managers to continuously examine and improve the program for its audience. The FSP leverages this framework to a) design structures and activities around a common vision of effective professional development and b) gauge program performance. The FSP measurement system framework was derived from the Institute for Healthcare Improvement's (IHI) system-level measures approach. IHI devised a suite of system-level outcome measures organized around a set of quality dimensions to assess the quality of healthcare and determine improvement priorities across a network of hospitals (Martin et al., 2007). The FSP's system-level measures (13 in total) are organized around the five dimensions of quality, represented in Table 1. These dimensions, along with our design principles, constitute the FSP design framework.

*Table 1 Faculty Support Program Quality Dimensions*

| Quality Dimensions | Definition |
|---|---|
| Effective | New faculty implement the Pathways with integrity and efficaciously |
| Efficient | Preparation and support structures are not wasteful of time, money |
| Responsive | The specific needs of new Pathways faculty are surfaced and met |
| Community-oriented | New faculty seek support from other faculty, new and experienced |
| Faculty-centered, faculty-owned | Faculty are centrally involved in the process of designing and improving FSP activities |

**Leveraging Network Structure and the Common Measurement System to Improve**

Comprising the FSP improvement infrastructure (in the form of its networked organization and measurement system) are a set of routines, protocols, and social processes for collecting and reviewing incoming data about the performance of the FSP as defined by the quality dimensions. Our current data sources and collection timeline are represented in Figure 2 below.

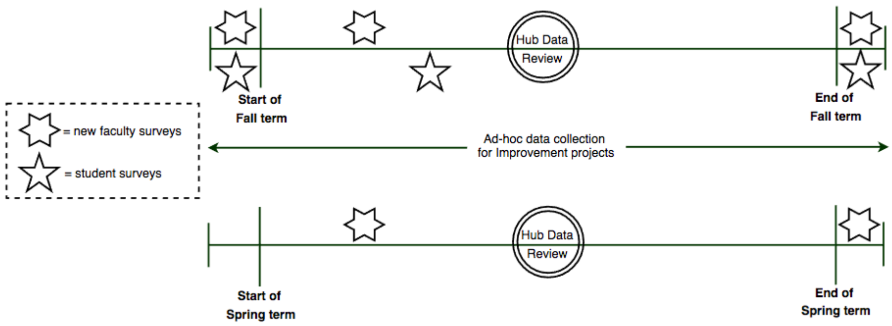


*Figure 1 FSP Measurement Sources and Timeline*

In 2015-16, data were collected from 222 faculty at 36 institutions nationwide. In 2016-17, 462 faculty members from over 65 institutions nationwide are participating. Using the FSP quality dimensions as a guiding framework, data are collected from the network and analyzed by the hub. Data are reviewed twice a year by a diverse set of stakeholders (faculty, institutional leaders, and hub members). During these reviews, stakeholders determine high leverage priorities for improvement, that is, problems that have potential to produce large improvements with relatively lower costs of time and financial resources. Stakeholders then launch improvement projects, often beginning with an investigation into the problems and then progressing towards small tests of changes that may eventually become stable components of the program. The knowledge generated through this process informs the ongoing refinement of our theories of teacher change and the design of professional development driving the work.

## Discussion of Findings

Through this improvement work, three major findings emerged related to the implementation of professional development for mathematics faculty in community colleges. First, the Pathways instructional materials were found to be a critical touchpoint for supporting professional learning. Second, common instructional challenges instructors face in enacting Pathways pedagogy were identified. Third, new instructors' existing relationships at their institutions were often primary sources of support and mentoring. This proposal addresses the first in depth and touches on the others ; if accepted, the final paper will elaborate on all.

### Leveraging Instructional Materials

In our interviews with Pathways faculty in the process of designing and improving the program, the curriculum materials emerged as a core source of instructional support. First-time instructors used the materials as references to better understand lesson tasks and goals and also to guide their implementation of specific pedadgogical moves and decisions within lessons.

CMP instructional materials include student lesson handouts presenting in-class tasks and brief readings and instructor notes, which are instructor-facing materials that contain all content in the student handout along with a) notes about tasks and lesson goals; b) guidance for the implementation of the lesson's activity structures, such as group work or whole-class discussion; c) facilitation guides for whole-class discussion; d) suggested activities or "scripts" for promoting productive persistence; and e) anticipated student responses to rich problems. The manner in and extent to which new Pathways instructors use these materials appears to depend on their familiarity and comfort with the Pathways instructional approach. For those whom the instructional approach is more novel, instructor notes act as a standard protocol to which the instructor adheres for at least the first time. Faculty have reported increased familiarity, comfort, and confidence in teaching the Pathways after using the instructor notes, and they relied on them less in subsequent courses. Instructors more familiar with Pathways pedagogy use the instructor notes as a reference for understanding the lesson objectives, the purpose of each task, and how the lesson is situated within the curriculum broadly. These instructors also annotate the notes with learnings and ideas for future reference, which are often later adapted or omitted upon further trial and reflection.

These findings signaled that, to at least some extent, the materials promoted engagement in some form of reflection on teaching. Although variation existed in how instructors used the materials, we found that nearly all first-time Pathways faculty had studied the instructor notes for each lesson. This finding has broad implications. First, the design of the instructor notes should address specific needs of the faculty. This finding spurred a comprehensive redesign of the instructor notes, in order to better meet the needs of new Pathways faculty and to more effectively surface resources, activities, and opportunities for professional development. Second, while faculty traditionally do not have much pedagogical training in or experiences critically reflecting on their teaching, adjunct and full-time instructors can and often do readily take up opportunities to reflect on their teaching and experiment with instructional moves and practices with which they are not familiar, particularly when those opportunities are accessible.

**Common Challenges When Teaching the First Time**

In a redesign of the FSP, the hub conducted 30 interviews with new faculty in the fall of 2014 to better target resources and design based upon evidence of faculty needs. Five common challenges faced by first-time Pathways instructors emerged: lesson pacing, promoting productive struggle, facilitating group work, sustaining productive persistence beyond the first 4 weeks of a term, and homework completion. If accepted, we will elaborate on this further.

**Leveraging Existing Social Relationships**

Key data collected by the hub as part of program improvement are measures of new instructors' engagement with their assigned Faculty Mentors. A program review in the fall of 2015 revealed low engagement. Through the resulting improvement process, we found that faculty saw local colleagues as a critical source of professional learning and support, and thus resources and structures that leverage existing local support systems were developed, tested, and widely implemented. This finding will also be elaborated upon acceptance.

## Conclusion

The organization of the Carnegie Math Pathways as a networked improvement community has facilitated the design, testing, refinement, implementation, and scaling of the professional development supporting first-time Pathways faculty. Specifically, the NIC organization provided the hub with access to faculty from a wide range of institutions and thus insight into their work processes and needs as Pathways instructors. Additionally, the FSP measurement system aided in specifying areas for improvement, examining how colleges adapt program components to better fit their local context, and determining whether those adaptations resulted in improvement. In sum, engaging users and institutions in collective improvement work around specific problems, and then testing changes to the program to address those problems, provided us with key opportunities to learn about what effective professional development can look like across diverse community college campuses and diverse groups of faculty.

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Future Middle Grades Teachers' Solution Methods on Proportional Relationship Tasks

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*This study examines the solution methods that future middle grades teachers chose when solving a problem on proportional relationships. The examination of the solution methods was framed by a new perspective on proportional reasoning that connects multiplication, division, and proportional relationships into a coherent framework. This framework places emphasis on multiple batches and variable parts. The data were collected from a sample of 22 future middle grades teachers' exams completed as part of a content course at a large university in the Southeastern United States. Findings revealed that future middle grades teachers utilized strategies involving multiple batches and variable parts after completing a two-semester sequence of mathematics content courses on proportional relationships tasks.*

*Keywords:* Proportional relationship, Proportional reasoning, Variable parts perspective, Multiple batches perspective

## Introduction

Skills of multiplicative and proportional reasoning are important because their development, or lack thereof, can greatly influence success for students in later mathematics (Beckmann & Izsák, 2015). First introduced in middle grades mathematics, reasoning proportionally forms a crucial base for further concepts such as functions, graphing, algebraic equations, and measurements (Karplus, Pulos, & Stage, 1983; Langrall & Swafford, 2000; Lobato & Ellis, 2010; Lobato, Orrill, Druken, & Jacobson, 2011; Thompson & Saldanha, 2003). Proportional reasoning is difficult, teachers are often not more advanced than their students and in order to teach effectively, one's own understanding must be deepened (Lobato et al., 2011). In addition, researchers pointed out that teachers need to be "sensitive to the types of reasoning that are most accessible as entry points for students while pushing them to develop more sophisticated forms of reasoning." (Lobato el al., 2011, p. 1). Despite the growing body of research on proportional reasoning, the studies that have explored future middle grades teachers' understandings of ratios and proportional relationships are rather limited. Thus, there is a need for research on how future middle grades teachers reason with proportional relationships because "teachers are among the most, if not the most, significant factors in children's learning and the linchpins in educational reforms of all kinds" (Cochran-Smith & Zeichner, 2009, p. 1).

## Purpose of the Study

This study investigates the performances of future middle grades teachers in understanding proportional relationships from two distinct perspectives and considers the role of multiplication and division in their reasoning. Mathematics educators need new approaches and perspectives to think about how future middle grades teachers' reasoning about proportional relationships can be supported. With this objective in mind, Beckmann and Izsák (2015) developed a new approach to reasoning about proportional relationships comprising two perspectives and four methods. These methods encompass a coherent understanding of proportional relationships that includes both multiplication and division. Their approach was innovative because they connected multiplication, division, and proportional relationships into a single coherent framework that highlights two complementary perspectives on ratios and proportional relationships. These two perspectives are called variable parts and multiple batches. In line with Beckmann and Izsák's (2015) approach, this study specifically focused on future middle grades teachers' solution methods related to proportional relationships according to the two perspectives and four strategies. This research was guided by the question: What solution methods do future middle grades teachers use when solving a problem at the end of a content course involving two perspectives on proportional relationships?

## Theoretical Framework

This study is framed by Beckmann and Izsák's (2015) perspective on proportional relationships, which integrates multiplication, division, and proportional relationships into a coherent whole.

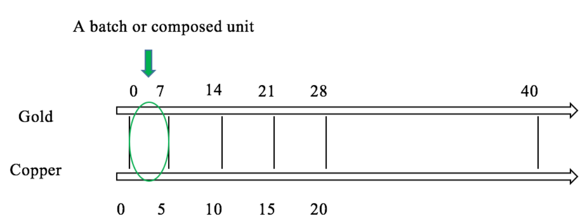
### Equation: M•N= P

Beckmann and Izsák (2015) formalized an equation for multiplication based on equal sized groups as "M • N= P", where M is the number of the groups (multiplier), N is the number of the units (multiplicand) in each whole group, and P is the product amount.

### Perspectives: Multiple Batches and Variable Parts

Beckmann and Izsák (2015) proposed two perspectives, multiple batches and variable parts, by considering the multiplier and multiplicand roles in proportional relationships. In this study, we demonstrate two perspectives by using the following Gold and Copper problem: *To make jewelry, jewelers often mix gold and copper in a 7 to 5 ratio. How much copper should a jeweler mix with 40 grams of gold?*

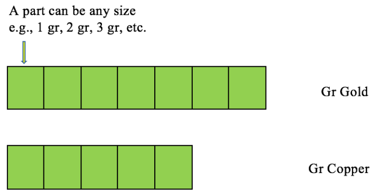
The multiple batches perspective supports at least two solution strategies: multiply-one-batch method and the multiply-unit-rate batch method. For the multiple batches perspective, they stated that "the original batch (A units of the first quantity and B units of the second quantity) are fixed multiplicands, and the multiplier varies; therefore, the proportional relationships can include "all of pairs (rA, rB)", where r > 0 (Beckmann, Izsák, & Olmez., 2015, p. 519). Figure 1 shows one way to represent multiple batches in the Gold and Copper problem.




| Number of Batches | Gr Gold | Gr Copper |
|---|---|---|
| **1** | **1** • 7 | **1** • 5 |
| **2** | **2** • 7 | **2** • 5 |
| **3** | **3** • 7 | **3** • 5 |
| **4** | **4** • 7 | **4** • 5 |

*Figure 1: Multiple Batches Perspective (Beckmann et al., 2015)*

Similarly, the variable-parts perspective supports at least two solution strategies: multiply-one-part method and the multiply-total-amount method. For the variable parts perspective, they considered the two quantities as consisting of A parts and B parts, respectively, where each part contains the same number of units. This time the multipliers are fixed by the numbers of parts, whereas the multiplicand varies with "the number of the measurement units" in every part (see Figure 2). Correspondingly, the multiple-batches perspective, variable-parts proportional relationships include "all of pairs (Ar, Br)" for r > 0 (Beckmann et al., 2015, p. 520). Figure 2 shows one way to represent variable parts in the Gold and Copper problem.




| Gr per part | Gr Gold | Gr Copper |
|---|---|---|
| **1** | 7 • **1** | 5 • **1** |
| **2** | 7 • **2** | 5 • **2** |
| **3** | 7 • **3** | 5 • **3** |
| **4** | 7 • **4** | 5 • **4** |

*Figure 2: Variable Parts Perspective (Beckmann et al., 2015)*

## Methodology

### Research Design

The aim of this study is to explore which solution methods future middle grades teachers used when solving a problem at the end of a content course that introduced two perspectives on proportional relationships. To address the research question, mixed methods were utilized to examine future teachers' solutions. Mixed methods research provides more evidence for studying a research problem than either quantitative or qualitative research alone. Quantitative methods individually provide useful information, however they do not provide an in-depth understanding of the participants approaches and qualitative research makes up for this weakness. Thus, the combination of strength of each approach accounts for the weakness of the other approach. More specifically, we used a sequential explanatory design with a qualitative approach being the first method applied as well as the method of priority (Creswell, Plano Clark, Gutmann & Hanson, 2003). Qualitative research methodologies were used to discover the meanings that participants created in context or in an activity (Wolcott, 2009). When reviewing the student written work qualitatively, we analyzed features of solutions and representations to determine the method they employed. In accordance with sequential explanatory design, it is typical to use qualitative results to reveal additional information and help clarify the primarily quantitative study (Creswell et al., 2003). Thus, we supported our qualitative interpretations with descriptive statistics. This combination of methods provides "multiple ways of seeing and hearing" (Greene, 2007, p. 20). With the priority placed on the qualitative approach, "the researcher builds a complex, holistic picture, analyzes words, reports detailed views of informants, and conducts the study in a natural setting" (Creswell, 2008, p. 15).

### Data Collection

Data for the present study were collected from 22 future middle grades teachers at a large, public university in the Southeastern United States during the Spring 2016 semester of a two-semester mathematics content course. The first semester focused on numbers and operations including multiplication, division, and fractions; the second semester focused on topics related to fraction division, ratio, proportional relationships, and algebra. Both courses emphasized the meaning of multiplication. These courses were intended to help future teachers develop practices outlined in the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The same textbook was used for both courses, *Mathematics for Elementary Teachers with Activities* (Beckmann, 2014). It was standard practice in these courses that future middle grades teachers worked in groups during class, however individually completed homework assignments and examinations.

Tasks on the midterm and final exams that addressed proportional relationships were identified. Then items that allowed future middle grades teachers to choose their own methods as opposed to items that directed them to use a particular method were chosen and analyzed. Ultimately, one task from the final exam of the second semester course was selected (see Figure 3).

| | |
|---|---|
| **Task** | To make jewelry, jewelers often mix gold and copper in a 7 to 5 ratio. How much copper should a jeweler mix with 40 grams of gold? Write two different products A•B for the amount of the copper, where A and B are numbers derived from 7, 5, and 40. Explain each product A•B in detail in terms of the situation using our definition of multiplication and using math drawings as support. |

*Figure 3. Task Item*

**Data Analysis**

Drawing on the theoretical framework, we were able to classify the future teachers solutions. This framework is exemplified in Figure 4, which shows solutions for the Gold and Copper problem that illustrate the two perspectives and four methods and how those methods are coordinated with equations following the multiplier (M) • multiplicand (N) convention. Beckmann and Izsák (2015) indicated that double number lines (DNLs) fit well with the multiple-batches perspective and that the strip diagrams fit well with the variable-parts perspective. DNLs represent quantities visually as lengths and afford such operations as iterating, partitioning, or addition. Strip diagrams represent quantities in terms of variable parts.

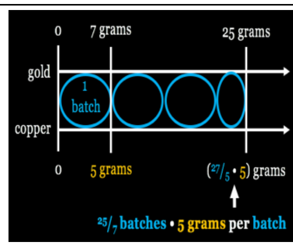
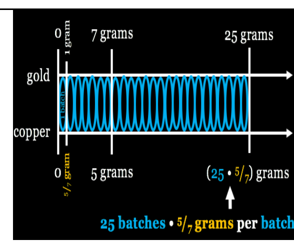
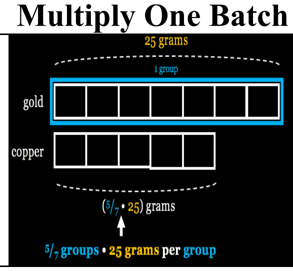
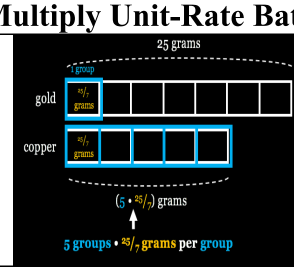
<table>
<tr><td colspan="3">Gold and copper problem: “A company makes jewelry gold using gold and copper. The company uses different weights of gold and copper on different days, but always in the same ratio of 7 to 5. If the company uses 25 grams of gold on one day, how much copper will they use?”</td></tr>
<tr><td>Multiple Batches</td><td>
</td><td>
</td></tr>
<tr><td>Strategy</td><td>Multiply One Batch</td><td>Multiply Unit-Rate Batch</td></tr>
<tr><td>Variable Parts</td><td>
</td><td>
</td></tr>
<tr><td>Strategy</td><td>Multiply Total Amount</td><td>Multiply One Part</td></tr>
</table>

*Figure 4.* Solutions to the Gold and Copper Problem using two perspectives on proportional relationships and four strategies (Reproduced Kulow, 2017)

The data were sorted based on the perspective future teachers chose (multiple batches or variable parts) and then based on methods that fit with those two perspectives. Task analysis focused on the future teachers' solutions according to their drawings, equations, and explanations.

## Results

Future middle grade teachers who completed the two-semester sequence of content courses emphasizing topics related to ratio, proportional relationships, fraction division, algebra, and the meaning of multiplication were able to appropriately use the multiple-batches and variable parts perspectives. When working on a problem that allowed them to select their own method, future middle grades teachers tended to use the variable-parts perspective as opposed to the multiple batches perspective.

Table 1 shows counts for solution classifications to the Gold and Copper problem. Recall that the task asked for two solutions. The counts in Table 1 show that 44 solutions were provided by 22 future teachers: 19 future teachers used two different methods, two future teachers used one method, and one future teacher used four methods, as shown in Table 1. According to these results, the future teachers used the variable-parts perspective in 29 solutions and the multiple batches perspective in 15 solutions.

*Table 1. Frequency of each method*

| Perspective | Total | Method | Total |
|---|---|---|---|
| Variable Parts Perspective | 29 | Multiply Total Amount | 12 |
| | | Multiply One Part | 17 |
| Multiple Batches Perspective | 15 | Multiply One Batch | 8 |
| | | Multiply Unit-Rate Batch | 7 |
| Total | | | 44 |

The total number of solutions in which future teachers used the variable parts perspective with the multiply-total-amount method was 12, whereas the total number of solutions in which future teachers used the variable parts perspective with multiply-one-part method was 17. Additionally, the total number of solutions in which the future teachers used the multiple batches perspective with multiply one batch method was 8. The total number of the solutions in which future teachers used the multiple-batches perspective with multiply unit-rate batch method was 7. Some future teachers used the multiple batches perspective with multiply-one-batch method logically in combination with a strip diagram instead of a DNL. Some future teachers used the multiple batches perspective with multiply-unit-rate-batch method logically in combination with a strip diagram instead of a DNL.

### Variable-Parts Perspective with the Multiply Total Amount Method

Future teachers who used the variable parts perspective with the multiply-total-amount method included an equation which mainly included appropriate values for M and N (i.e., M is 5/7 and N is 40). For instance, the future teacher LM defined M = 5/7 as “# of groups”, N = 40 as “# of grams in one whole group”, and P = 200/7 is “# grams in 5/7 group”. In addition, LM showed the total amount of gold and copper in the math drawing (see Figure 5).

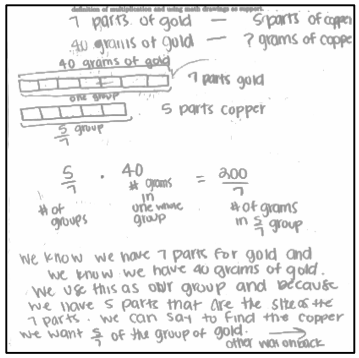



*Figure 5.* LM's solution

## Variable-Parts Perspective with the Multiply One Part Method

Future teachers who used the variable parts perspective with the multiply one-part method included an equation with appropriate values for M and N (i.e., M = 5, N = 40/7, and P = 200/7). The future teacher BM stated M is "# of groups", N is "units per group", and P is "amount of copper needed" (Figure 6).

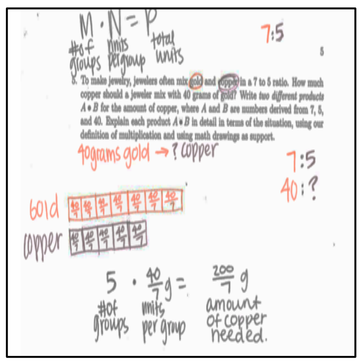



*Figure 6.* BM's solution

## Multiple-Batches Perspective with the Multiply One Batch Method

Future teachers who used the multiple-batches perspective with the multiply one batch method included an equation which mainly included appropriate values for M and N (i.e., M = 40/7, N = 5, and P = 200/7). Figure 7 includes future teacher AH's solution using the one batch method that included explicit descriptions for M, N, and P such as M is "groups gold", N is "grams copper per group", and P is "grams copper per 40 grams gold."

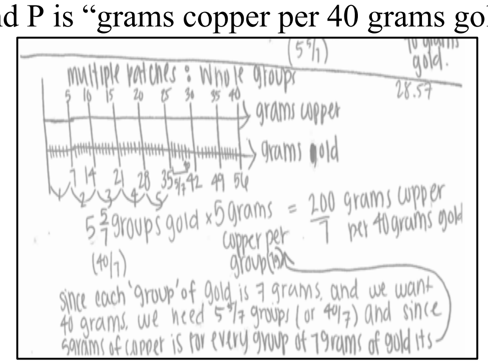



*Figure 7.* AH's solution

**Multiple-Batches Perspective with the Multiply Unit-Rate Batch Method**

Future teachers who used the multiple-batches perspective with the multiply unit-rate batch method included an equation which mainly included appropriate values for M and N (i.e., M is 40, N is 5/7, and P is 200/7). In Figure 8, future teacher KC used the mathematical drawing, showed total amount of gold and copper. More specifically, in KC's solution, DNL indicated target amount (e.g., tick mark for 40 grams of gold) and DNL indicated initial batch (e.g., tick mark for 7 grams of gold and 5 grams of copper) (see Figure 8).

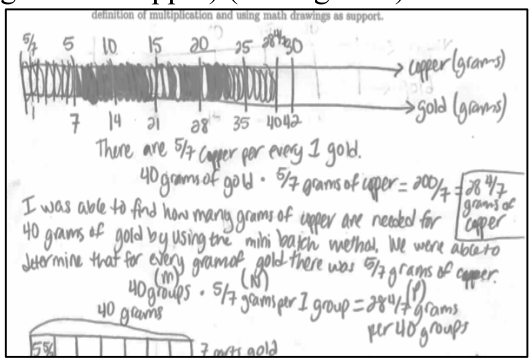



*Figure 8.* KC's solution

## Discussion and Conclusion

Proportional relationships are at the heart of middle grades mathematics, so learning and teaching this concept is crucial. In order to improve learning the concept of proportional relationships, we need to educate future teachers. Thus, there is a need for research on the mathematical training of future middle-grade teachers for better teaching and learning of proportional relationships between co-varying quantities. In order to reach this goal, the education program for future middle grades teachers should be designed to support proportional reasoning. The findings of this study indicated that when topics related to ratio, proportional relationships, fraction division, algebra, and the meaning of multiplication were emphasized in a two-sequence content course, future middle grades teachers were able to use the multiple-batches and variable-parts perspectives and the associated methods in an appropriate way on an exam problem.

This study revealed that two perspectives are important since both have been designed by combining multiplication, division, and proportional relationships. While the sample size of the study is small, more participants are needed in more classes for future work. In addition, studies including interviews are needed to further understand future teachers' solution methods by considering two perspectives.

The instructional approach to topics in the multiplicative conceptual field appeared to support development of future middle grades teachers' understanding of proportional relationships. This approach also supports future teachers' understanding of the meaning of multiplication and division and the use of each perspective's features. According to Beckmann and Izsák (2015), the variable-parts perspective offers students an approach to thinking about variations of quantities in proportional relationship problems. In this study, students used the variable-parts perspective (n = 29) more often than the multiple-batches perspective (n = 15). This result represents the first determination regarding students' tendency when choosing which perspective to work with.

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# Challenging the stigma of a small N: Experiences of students of color in Calculus I

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*Because students of color are underrepresented in undergraduate mathematics classes, their experiences are often ignored in studies drawing on large data sets or are inferred based on the experiences of other underrepresented populations, specifically women. This exclusion and misrepresentation of students of color is often attributed to methodological limitations. In this study, we reexamine the data studied for a previous analysis attending to student race and ethnicity rather than to gender. Due to the smaller numbers of non-white students, we utilize different analytic tools, and draw on students' open-ended responses to a survey question asking about their experiences in Calculus I. In addition to adding to the literature on students from marginalized populations in undergraduate mathematics, this paper argues for a reframing of how we value papers with a small N, and what this value indicates about our value of the students making up the small samples.*

*Keywords:* equity, quantitative methods, calculus

A number of recent studies have been published that draw on a large data set to make strong claims about students' gendered experiences in undergraduate mathematics (see e.g., Ellis, Fosdick, & Rasmussen, 2016; Laursen et al., 2011). In each of these studies, the researchers had access to the students' race and ethnicity, but were unable to conduct the same statistical analyses differentiating by race and ethnicity as they did by gender because of the small sample of students from non-white populations. For instance, in discussing learning gains between the active learning courses (specifically Inquiry Based Learning; IBL) and non-active learning courses, Laursen and her colleagues (2011) state: "We could only compare [the learning gains of] white and Asian students, because the number of other students of color in our sample was very low (see Appendix A3)" (p. 55). Similarly, in a recent publication investigating factors related to students' and instructors' experiences in calculus: Hagman, Johnson, and Fosdick (2017) state: "We do not investigate the association between race or ethnicity and [opportunities to learn] due to the small proportion of non-white students and instructors in our study" (p. 5).

Because of the underrepresentation of students of color in undergraduate mathematics courses, their experiences in these courses are made invisible in studies that draw on large data sets, or are inferred based on the experiences of other underpenetrated populations, specifically women. For instance, researchers use Laursen and her colleagues' (2011) paper as evidence that active learning benefits students from "underrepresented" groups. As a prototypical example of this, Webb (2016) states: "Research has shown that undergraduate students who are involved in active learning techniques can learn more effectively in their classes, resulting in increased achievement and dispositions… particularly so for underrepresented groups (Laursen et al., 2011)" (pp. 1-2). While Laursen et al. (2011) are able to make substantive claims about the benefits of IBL for women and typically low-achieving students, which are both underrepresented groups in STEM, these findings are being generalized to make claims about underrepresented students in general, which is often taken to specifically include students from underrepresented racial and ethnic minorities.

While there are some studies about such students' actual experiences in undergraduate mathematics (see Adiredja & Andrews-Larson, 2017 for a review of this literature), such studies are “limited [in] number” (Adiredja & Andrews-Larson, 2017, p. 451) and, due to the underrepresentation of such students in undergraduate mathematics, draw on a smaller data set and often employ qualitative methods. For instance, McGee and Martin (2011) studied the experiences of 23 Black mathematics and engineering college students, Levya (2016) studied the experiences of five Latin@ engineering college students, and Adiredja and Zandieh (2017) studied the experiences of 8 Latina’s in a Linear Algebra course. In comparison, Laursen et al. (2011) drew on survey data from 1,100 students and Ellis, Fosdick, and Rasmussen (2016) analyzed data from 2,266 students, with about 50% identifying as female in each study.

Studies drawing on large data sets are viewed as more reliable and objective than studies drawing on smaller data sets, are able to use statistics to generalize findings, can seek to identify cause and effect relationships, and aid in testing hypotheses (Creswell & Clark, 2007). However, there are a number of issues that arise when considering a quantitative design in the study of race and ethnicity (Adiredja & Andrews-Larson, 2017; Teranishi, 2007). One such issue is simply that the sample size (and the population itself) of non-white populations may be too small in even very large data sets to utilize these benefits. Another issue, as Adiredja and Andrews-Larson (2017) explain, lies in generalizing:

> [G]eneralizability of findings from quantitative studies as a result of a large sample size is always in tension with their reliance on aggregate outcomes and averages. Attending to this tension means being mindful of the reality that the effects on each student in the study are not the same, despite the closing of any gap between groups…The use of averages unfortunately also has the potential to deemphasize any perpetuated inequities. (p. x).

Taken together, these perspectives indicate that both quantitative and qualitative studies can be valuable as we seek to better understand students’ experience in undergraduate mathematics courses, especially understanding experiences of students of color. Qualitative studies are powerful in understanding the nuanced differences of such students, and are able to discuss the experiences of students as individuals rather than as groups. Quantitative studies are powerful in understanding the strength and prevalence of such experiences.

Because students of color are underrepresented and their numbers are small, their experiences are often ignored or are inferred based on the experiences of other underrepresented populations, specifically women. This “exclusion and misrepresentation of [students of color] in education research” is often attributed to methodological limitations (Teranishi, 2007, p. 38). In this study, we reexamine the data studied for a previous analysis (Ellis, Fosdick, & Rasmussen, 2016) attending to student race and ethnicity rather than to gender. Due to the smaller numbers of non-white students, we utilize different analytic tools, and draw on students’ open-ended responses to a survey question asking about their experiences in Calculus I. In addition to adding to the literature on students from marginalized populations in undergraduate mathematics, this paper seeks to argue for a reframing of how we value papers with a small N, and what this value indicates about our value of the students making up the small samples.

## Related Literature and Theoretical Perspective

In K-12 education, African American, Latin@, and Native students continue to be denied equitable access to a high quality mathematics education (Kitchen, Ridder, & Bolz, 2016), including high level mathematics courses (US Department of Education Office for Civil Rights, 2016). This is problematic in higher education due to advanced mathematics courses frequently being used as gateways (pre-requisites) for college credit mathematics and science courses, as well as entrance and continuation in undergraduate mathematics and science courses and majors.

This racialized disparity in K-12 mathematics education access directly and continually negatively impacts the numbers of students of color in higher education mathematics.

Access to resources however represents only part of the marginalized experiences many students of color face in mathematics education. Students of color continue to face systemic racism and racialized negative narratives in mathematics classrooms (Anderson & Tate, 2008; Jackson, Gibbons, & Sharpe, 2017; Spencer & Hand, 2015). Despite the large and still-growing academic research and literature unveiling these disparities and systems of oppression, dominant mathematics education narratives continue to defend the "objectivity" of mathematics as a color-blind discipline (Martin, 2013; Shah, 2017). While this color-blind narrative put forth in defense of the continued practice of marginalization of students of color has been challenged for many years (Gutiérrez, 2007; Martin, 2003; Nasir & Hand, 2006), the issue remains of low concern and priority in higher education, perhaps largely due to the continued low numbers of students of color in college mathematics courses. This underrepresentation, while itself an area of high concern, has set up those students of color who do gain access to college mathematics to have their experiences and even their entire presence often completely erased in research surrounding college/university mathematics education research. One significant contributor to this phenomenon is the practice of removing would-be outliers and/or subgroups within large data sets that have small Ns.

The issue and outcomes of racialized invisibility (Haynes, Stewart, & Allen, 2016) in mathematics education research has been challenged and countered by two notable collectives or research, one housed in the socio-cultural theoretical framework (Boaler, 2008; Hand, 2010; Nasir, 2008; Stinson, 2008) and the other in critical theories (Gutstein, 2006; Leonard & Dantley, 2002; Martin, 2009). Both approaches have made significant contributions toward challenging the dominant narratives and research practices that have worked to erase the presence and experiences of students of color in mathematics education, as well as to elevate the voices and experiences of students of color in mathematics classrooms. While these research endeavors and traditions have elevated the voices, experiences, and presence of students of color in mathematics education, the stigma of the small N remains a significant (albeit socially contrived) barrier for researchers (and their publication reviewers) in quantitative research. Little work has been done to look closely at and provide suggestions and guidance for quantitative researchers whose research involves large data sets and for whom students of color are not an expressed focal point.

Leveraging the notion that the invisibility of students of color in mathematics education research has become a socially acceptable norm, along with the foundational premise of the sociocultural and critical theory research identified above that racial identity, racial experiences, and the presence of students of color in our classrooms are important and they should not be erased by any measures, including a methodological approach. Thus, we take a pragmatic approach to a re-analysis of the data from a previous study focused on persistence in calculus and gender (referred to as the gender-comparison study) with the goal of making an early contribution toward a re-thinking of how the undergraduate mathematics education research community makes its methodological decisions with consideration to students of color.

## Methods

In this study, we conduct three analyses of the data coming from a 2010 survey of Calculus I students, with decreasingly coarse approaches. By doing so we seek to highlight the benefits and drawbacks of each approach, and take all three together to better understand the experiences of students of color in Calculus I.

## Background and Data Collection

The data used for these analyses draws on a national data set made available by the Mathematical Association of America. This data was collected by surveys sent to Calculus I students at the beginning and the end of the course. Data used for this analysis comes from both surveys, focusing on demographic data, persistence data, and a reflective free response question. To determine students' persistence in calculus, we use data from both the beginning and end of term surveys. Students are identified as Persister if they intended to take Calculus II at both the beginning and end of Calculus I, and as Switchers if they originally did intend to and then no longer intended to by the end of Calculus II. For demographics, we rely on three questions: "What is you gender?", with options given only as male and female, "What is your race? (Mark all that apply)" with options given White, Black, Asian, Pacific Islander, American Indian or Alaska Native, and Other (please specify), and "Are you of Hispanic origin?" with responses Yes or No. A new variable was created identifying the students' race and/or ethnicity, where the race and ethnicity questions were combined. Students who wrote in a real race or ethnicity were either grouped with the most aligned category (for example, Puerto Rican students were identified as "Of Hispanic Origin") and students who wrote in something other than a real race or ethnicity category were grouped together with the students who chose not to report their race or ethnicity (for example "human" or "race is a social construct"). Students who identified as multiple non-white races and ethnicities were grouped together. The last question used in this study was a free-response question on the end of term survey, that asked students "Is there anything else you want to tell us about your experience in Calculus I?". Table 1 illustrates the number of students from each race or ethnicity category in the large data set and in the free response data set.

***Table 1.*** *Race and ethnicity for data used*

| | Large Data Set | | FR Data Set | |
|---|---|---|---|---|
| **Race or Ethnicity** | **Number** | **Percentage** | **Number** | **Percentage** |
| White and/or non-Hispanic/Latin@ | 6674 | 68.2 | 348 | 67.0 |
| Non-white and/or Hispanic/Latin@ | 2921 | 29.8 | 157 | 30.2 |
| *Black* | 431 | 4.4 | 28 | 5.4 |
| *Asian* | 1334 | 13.6 | 63 | 12.1 |
| *Pacific Islander* | 70 | 0.7 | 3 | 0.6 |
| *Native American or Alaska Native* | 107 | 1.1 | 11 | 2.1 |
| *Of Hispanic origin* | 844 | 8.6 | 48 | 9.2 |
| *Multiple non-white identities* | 135 | 1.4 | 4 | 0.8 |
| Race or ethnicity identity not listed | 112 | 1.1 | 6 | 1.1 |
| Chose not to report | 86 | 0.9 | 9 | 1.7 |
| Total | 9793 | | 520 | |

## Data Analysis

The first analysis is the coarsest, and attempts to mimic the analysis of the Ellis, Fosdick, & Rasmussen (2016) analysis as much as possible by grouping students into two race and ethnic categories: (a) White and/or non-Hispanic/Latin@ and (b) Non-white and/or Hispanic/Latin@. For this proposal, we conduct chi-square statistics to test if there are significant difference in the persistence in calculus between these binary categories of students. (Note: At the time of submitting this proposal the statistician involved in both studies is on maternity leave and is actually taking a break for work. We honor this break, and thus for this proposal the analyses are not as robust as they were in the initial analysis, nor as robust as they will be for the presentation of this study in February.) The second analysis is similar but disaggregates by race and ethnicity categories. The third analysis is the least coarse and draws on students' free responses on the end of term survey. To analyze these responses, we draw on a thematic analysis conducted for a previous study (Ellis & Cooper, 2016).

## Results

### Analysis 1: Most Coarse

The output variable used for the gender-comparison study was student persistence through the calculus sequence. In that study, we looked at the relationship between gender and student persistence, controlling for a number of factors that may be related, such as career intentions and previous calculus experience. Through that analysis, we found that female students were 50% more likely to be identified as Switchers compared to male students, after controlling for a number of factors. This result was very statistically significant, which we were able to test for because of the large number of students and, more specifically, the large number of students who identified as male (N=1,236) or female (N=1,030).

For the first analysis in this paper, we attempt to mimic the above analysis as much as possible by identifying students as either white, non-Hispanic/Latin@ (N=2213) or not (N=864). In the data set used for this analysis (N=3077), we do not find a significant different between the persistence of white, non-Hispanic/Latin@ students compared to non-white or Hispanic/Latin@ students [$X^2$ (1, $N$ = 3077) = 0, $p$ = .997]. We note that this result holds when students identifying as Asian are included with the white, non-Hispanic/Latin@ students. Among both groups of students, 19.6% of students were identified as Switchers. For reference, 14.5% of all male students and 25.3% of all female students were identified as Switchers. When looking at the intersection between gender and racial and ethnic identity, 26.1% of female, white, non-Hispanic/Latin@ students and 23.8% of female, non-white or Hispanic/Latin@ students were identified as Switchers.

Such a coarse analysis allows us to make claims about statistical significance and compare the findings to the gender-comparison analysis, but this comes at the cost of identifying all non-white, Hispanic/Latin@ students together. However, what this analysis does tell us is that generalizing the experiences of female students as a marginalized population in mathematics to students from racial and ethnic minorities in mathematics does not work, at least in this setting.

### Analysis 2: Less Coarse

In this second analysis, we disaggregate by race and ethnicity identity. As shown in Table 2, while we saw no numerical nor statistical differences in the first analysis, we do see numerical difference in the persistence rates among different race or ethnicity identity groups of students. However, due to the small sample sizes in some groups we cannot make any claims about the significance of these differences.

This analysis shows that there are differences in the persistence rates among students in our sample with different race or ethnicity identities. The most drastic outliers from the general trend are students who identify as Native American or Alaska Native, with 27.9% of the 43 students identified as Switchers, students who reported multiple non-white identities, with 36.4% of the 33 students identified as Switchers, and students who identify as Asian, with 17.3% of the 398 students identified as Switchers. While these results are not statistically significant since we could not test the significance, they convince the authors that students from these populations are likely not persisting through the calculus sequence at the same rates as students from other racial or ethnic groups.

***Table 2.*** *Persistence by race and ethnicity*

| Race or Ethnicity | Persister % | Switcher % | Total N |
|---|---|---|---|
| White and/or non-Hispanic/Latin@ | 80.4 | 19.6 | 2213 |
| Non-white and/or Hispanic/Latin@ | 80.4 | 19.6 | 864 |
| *Black* | 80 | 20 | 95 |
| *Asian* | 82.7 | 17.3 | 398 |
| *Pacific Islander* | 82.4 | 17.6 | 17 |
| *Native American or Alaska Native* | 72.1 | 27.9 | 43 |
| *Of Hispanic origin* | 80.3 | 19.7 | 249 |
| *Multiple non-white identities* | 63.6 | 36.4 | 33 |
| Race or ethnicity identity not listed | 82.8 | 17.2 | 29 |
| Chose not to report | 83.3 | 16.7 | 24 |

While the above analysis does not group all non-white and/or Hispanic/Latin@ together, it still problematically groups all Asian students together (Teranishi, 2007), for example. Also, while this analysis allows us to identify trends between the different racial and ethnic groups of students, we cannot identify the strength of these trends due to the small N.

**Analysis 3: Least Coarse**

In order to take a more nuanced look at students' reports of their experiences in Calculus I and how this may relate to their race or ethnicity identit(ies), we rely on students' free responses to the question "Is there anything else you want to tell us about your experience in Calculus I?". Of the 9,793 students for whom we had race and ethnicity data, 520 provided responses to the open-ended question. These responses were analyzed using thematic analysis (Clarke & Braun, 2006), and was originally studied in order to explore the relationship between persistence and gender (Ellis & Cooper, 2016). The original two authors each coded subsets of 50 student responses to develop and refine codes. Affect was the most frequently used code, and was defined to include statements about "Student's emotions, attitudes, and beliefs about (a) the calculus course, (b) mathematics, (c) themselves as learners." (p. X). Each code was weighted with the values -1, 0, or 1 to indicate a negative, neutral, or positive connotation, respectively, and each student response was coded with as many codes as appropriate.

In this section, we focus on student responses coded with Affect. White, non-Hispanic/Latin@ students have similar frequency of Affect codes and a similar frequency of positive Affect codes when compared to non-white or Hispanic/Latin@ students, with around 45% of responses from each group identified as related to Affect and around 42% of those comments coded as positive. However, among non-white or Hispanic/Latin@ students there are differences within the race or ethnicity identify group – for instance, of the 28 responses from Black students, 57% were coded as related to Affect but only 37.5% of these were coded as positive. While these numbers are too small to make generalizations, they do inspire curiosity among the authors to better understand the experiences of the students in our sample who provided open-ended responses. We highlight a few of the responses from students who identify as Native American or Alaska Native and students who reported multiple non-white identities because of the higher Switcher rates in the second analysis. We also highlight responses from

Black students due to the high number of responses coded as Affect but the low number of those responses identified as positive.

> *This class made me lose my love for math. The teacher was absolutely awful. I had to learn it on my own, and books were not efficient enough to do so. The tutoring program available is a complete and total waste of time unless you wait in there four hours for the two tutors available to help you. Thank you [University] for such a terrible academic standard of professors.* – Male Switcher; Native American or Alaska Native

> *The class was extremely helpful for trying to further investigate modern mathematical applications. The instructor was genuinely concerned with the students' success, and I thought highly of him and his methods of teaching unfamiliar material.* – Male Persister; Native American or Alaska Native

> *Although the material for Calculus 1 was the same in both high school and college, I had much more trouble learning the concepts this year in college than I did in high school. I do not know if that was because my high school teacher taught in a way that I could better understand the concepts; I do know, however, that I did much better in Calculus 1 in high school than in college.* – Female Switcher; Asian and Puerto Rican

> *This Calculus I experience made me dislike Calculus greatly. I found myself confused and lost throughout most of it. My peers had to constantly reiterate what the professor taught in class and I still did not understand.* – Female Switcher; Black

> *I had a great experience. It was much more fulfilling, satisfying, and doable than I had thought it would be.* – Female Persister; Black

The above quotations are presented to help give student voices to the quantitative data. They do not allow us to generalize student voices, to learn how prevalent these voices may be, or how they compare to white voices. Instead, they help to answer the call put forth by Adiredja and Andrews-Larson (2017): "While our ability to conduct quantitative analyses with large sample sizes may be limited, we can still highlight and prioritize the experiences of these students in research" (p. 459). We position these responses as examples of the kinds of powerful differing voices that can be erased when traditional small N decisions are made. In other words, the results are not the trends in what was said, but show that meaningful things were said that indicate racialized experiences exist and should not be erased.

## Brief Discussion

The goal of this paper is to bring attention to the normative practice in our community of ignoring the experiences of students of color in our quantitative studies. While our qualitative colleagues work to richly understand and document the experiences of students of color in our undergraduate classes, and while we eagerly wait for the representation of students of color to increase in our classes and in our data sets, we must challenge and overcome the stigma of a small N. This paper indicates that the experiences of students of color are (a) different from the experiences of women, (b) not all the same, and (c) are more complex that statistics can indicate.

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# How Does Problem Context Shape Students' Mathematical Reasoning on Calculus Accumulation Tasks?

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*Calculus serves many students from myriad fields of study. Investigations into the ways students from these fields of study reason about calculus concepts are vital, yet lacking (Rasmussen, Marrongelle, & Borba, 2014). The biological and life sciences make up 30% of traditional Calculus I students (Bressoud, 2015) and yet we know very little about how these students utilize context as they reason about calculus ideas like the definite integral. In this study, task-based interviews were conducted with 12 undergraduate students majoring in the biological and life sciences. Data were analyzed via open coding from a constructivist grounded theory approach (Charmaz, 2000) and a new analytic tool, local theory diagrams was developed. Results indicate problem context influenced students' assessment of the viability of their solution strategies as well as enabled them to reason through apparent contradictions in their work.*

*Keywords:* Calculus, Integral, Biology

## Framing the Study

Calculus is at the heart of a great many disciplines. Biology, computer science, economics, engineering, and physics are just a few of the undergraduate programs that require at least one semester of calculus. Enrollment in calculus courses at the secondary and post-secondary levels continues to rise (Bressoud, Carlson, Mesa, & Rasmussen, 2013; Kaput, 1997) and so understanding how students reason about calculus concepts is vital to better serve this growing community. Since the 1980s, research in calculus teaching and learning has blossomed into a field unto itself where researchers have explored several areas including the cognitive development of introductory calculus concepts in students and the potential for new digital tools to change calculus instruction (see Rasmussen, Marrongelle, & Borba, 2014 for a review).

Recent studies have highlighted the service nature of introductory calculus at the undergraduate level, since "very few students in Calculus I - between 1% and 3% of those enrolled in this course - intend to major in mathematics" (Bressoud et al., 2013, p. 691). Most students in these classes are majoring in other fields, what are often called the *client disciplines* of calculus. One popular client discipline of calculus is the biological and life sciences. Researchers have identified that 30% of the students in traditional Calculus I courses intend for careers in the biological and life sciences (Bressoud, 2015). However, the traditional Calculus I course "is designed to prepare students for the study of engineering or the mathematical or physical sciences" (Bressoud et al., 2013, p. 691). Which means a great many students in calculus are not seeing many contextually-based tasks catered to their field of study.

This study specifically addresses students' solution strategies on tasks involving the definite integral and accumulation primarily because integration and accumulation serve an important role in differential equations, which are used extensively in modeling within the biological and life sciences. Researchers have investigated student conceptions of the definite integral and have found that calculus students are good at using the standard antiderivative techniques taught in introductory calculus (Ferrini-Mundy & Graham, 1994; Grundmeier, Hansen, & Sousa, 2006; Mahir, 2009; Orton, 1983) and that while area under the curve dominates instruction of the definite integral in calculus, the multiplicative structure of the Riemann sum is a more powerful way to conceive of the definite integral as seen in both mathematics and physics education research (e.g. Jones, 2015a; Sealey, 2014). Unfortunately, researchers have seen that students struggle to make these meaningful connections between rate of change and accumulation in definite integral tasks (Bajrachara & Thompson, 2014; Thompson, 1994). Furthermore, researchers have found that when solving physics-based tasks, students' problem-solving strategies differ in relation to the context presented (e.g., Bajracharya & Thompson, 2014; Jones, 2015b; Sealey, 2014), and that some of these strategies are productive in a physics context when compared to a decontextualized mathematics context (Bajracharya, Wemyss, & Thompson, 2012; Jones, 2015a).

To better serve students from the myriad client disciplines of calculus, we must understand how students solve calculus tasks set in contexts relevant to those fields and whether those contexts play a significant role in their mathematical reasoning. Rasmussen et al. (2014) end their review of the state of research on calculus teaching and learning with a call for "research that closely examines the ways in which calculus ideas are leveraged in the client disciplines, how these ideas are conceptualized and represented in the client disciplines, and what these insights might mean for calculus instruction" (p. 513). The current study was designed to address this gap in the literature. My specific research question is: What role does context play in how undergraduate students majoring in the biological and life sciences solve calculus tasks involving accumulation?

## Theoretical Perspective

The perspective of learning that influenced the construction and analysis of this study is constructivism, specifically a view of knowledge as cognitive adaptation. In a constructivist theory of learning, the fundamental assumption is that learners build up knowledge for themselves instead of being imbued with knowledge by those around them. In other words, the learner must actively participate in the development and organization of the cognitive structures making up their understanding of the world (von Glasersfeld, 1982). To explore an individual's understanding, one must consider the following three factors: "the individual's current state of development, social and cultural influences of a tribe (group), and environmental/physical factors in relation to the task at hand" (Confrey & Kazak, 2006, p. 317). This perspective on learning, while maintaining focus on the individual learner, acknowledges that social and environmental factors must necessarily play a role in that learning. For this study, such a perspective provides the foundation for analyzing individual's approaches to calculus tasks while framing those approaches within the influence of those individual's backgrounds (in this case, as undergraduate students majoring in the biological and life sciences) and the interview setting itself.

One aspect of constructivism that played a key role in the data analysis in this study is a view of knowledge as an adaptive function. Ernst von Glasersfeld, in his interpretation and extension of the work of Jean Piaget, stresses the connection between the mechanisms of evolution by natural selection and how individuals learn. von Glasersfeld (1982) claims "knowledge for Piaget is never (and can never be) a 'representation' of the real world. Instead it is the collection of conceptual structures that turn out to be adapted, or as I would say, viable within the knowing subject's range of experiences" (p. 4). Viability is the crucial idea. Just as with the evolution of an organism in an ecosystem, what students learn is not driven by matching some objectively true reality, but what the student, within their personal "ecosystem," finds viable. Therefore, for learning, as in evolution, there is an emphasis placed on stability and equilibrium. von Glasersfeld states that "in the sphere of cognition, though indirectly linked to survival, equilibrium refers to a state in which an epistemic agent's cognitive structures have yielded and continue to yield expected results, without bringing to the surface conceptual conflicts or contradictions (p. 5). This is the heart of the concept of viability in constructivism, that learning is the development of stable cognitive structures and forms the foundation for the analytical tool developed herein, local theory diagrams, which were designed to highlight this process of students assessing the viability of their mental schemes.

## Methods

To answer the research question posed, qualitative methods were employed. I utilized task-based interviews with twelve undergraduate students majoring in the biological and life sciences at a large public university in the Southeastern United States that I will call South State University (SSU) in the spring of 2016. Task-based interviews allowed me to investigate students reasoning about calculus tasks involving accumulation and to probe their understanding as they solved the problems. Data were open-coded via methods from constructivist grounded theory (Charmaz, 2000) which led to the development of a new analytic tool, local theory diagrams.

### Participants

The population was all undergraduate students majoring in the biological and life sciences at South State University (SSU). SSU is a large, public university serving approximately 24,000 undergraduates. The students at SSU are of high academic caliber; half of all incoming freshman rank in the top ten percent of their high school class with a GPA of at least 3.75. SSU is considered “very selective” with 46% of applications admitted per year (The College Board, 2017). Students majoring in the biological and life sciences at SSU at the time of this study, were required to take at least two semesters of calculus, either the calculus sequence for life and management sciences or Calculus I and II.

Participants were solicited by visiting second semester calculus courses specifically designed for students studying in the biological and life sciences as well as upper-level courses within the biological and life sciences. Twelve students were interviewed, half of which were freshman or sophomores while the other half were juniors or seniors. The students were predominately female (8 of 12) and Caucasian (11 of 12).

### Interview Protocol

In this study, I utilized task-based interviews in which students completed five calculus tasks concerning accumulation (approximately 50 minutes). In each of the five tasks, the students were presented with a rate of change function of some quantity and asked questions about the accumulation of said quantity over various periods of time. To answer the research question: “What role does context play in how undergraduate students majoring in the biological and life sciences solve calculus tasks involving accumulation?”, the contexts for the tasks were chosen to be diverse but relevant for the students’ backgrounds. In this session, I will discuss the results of two of the tasks, which are reproduced below in Figures 1 and 2.

**Task 3**

The greenhouse effect is the rise in temperature that the Earth experiences because certain gases prevent heat from escaping the atmosphere.

According to one study, the temperature is rising at the rate of: $R(t) = 0.014t^{0.4}$ degrees Fahrenheit per year, where $t$ is the number of years since 2000. Given that the average surface temperature of the earth was 57.8 degrees Fahrenheit in 2000, predict the temperature in 2200.

*Figure* 1. *Interview Task 3*

**Task 5**

A new virus has shown up in a small town. The following table gives the estimated rate of change of the number of infected individuals on a given day (measured in people per day). Describe how the number of infected people is changing and use the information in the table to estimate the total change in the infected population over the given time period.

| Days | **0** | **2** | **4** | **6** | **8** | **10** | **12** | **14** | **16** |
|---|---|---|---|---|---|---|---|---|---|
| Rate (Infected per day) | 18 | 25 | 20 | 4 | -5 | -10 | 16 | 17 | 20 |

*Figure 2. Interview Task 5*

## Data Analysis

Analysis of the interview transcripts followed a constructivist grounded theory approach (Charmaz, 2000). Constructivist grounded theory, like other forms of grounded theory (e.g., Glaser & Strauss, 1967; Strauss & Corbin, 1990), allows the researcher to explore the data without a preconceived framework of what results should emerge from the data. Charmaz notes that objectivist grounded theorists "assume that following a systematic set of methods leads them to discover reality and to construct a provisionally true, testable, and ultimately verifiable 'theory' of it" (p. 524) and therefore that the data collection and analysis procedures should aim to minimize the role of the researcher to be able to make claims about an observer-independent reality. For constructivist grounded theory, Charmaz argues, this is not the case. She argues that "the research products do not constitute the reality of the respondents' reality. Rather, each is a rendering, one interpretation among multiple interpretations, of a shared or individual reality" (Charmaz, 2000, p. 523). Charmaz illustrates this succinctly when she says, "data do not provide a window on reality. Rather, the 'discovered' reality arises from the interactive process and its temporal, cultural, and structural contexts" (p. 523-524). This approach to data analysis fits with the theory of learning described earlier, particularly the focus on viability in learning since both perspectives reject the assumption that we are uncovering some objectively true reality.

The open-coding process led to the development of a new analytical tool, local theory diagrams, which visually represent a student's solution strategy and all its mutations for a given task. Local theory diagrams showcase the "core" of the student's current theory concerning the given task and its solution (e.g. how to interpret the given rate of change function) and is then surrounded by all the hypotheses the student generates based on that assumption and ideas the student believes to be true at the time. The local theory diagrams also illustrate how these theories shift as the student interacts with the task and assesses whether their current assumptions and strategies make sense. In this way, the diagrams show the process of students coming to develop more viable theories of the tasks they solve. Examples of the local theory diagrams are given in the next section.

## Results

There were two primary ways the problem context helped shape students' mathematical reasoning. The first was their use of the context partnered with the given information to refine their local theories of the task to increase the perceived viability of their strategies. Secondly, students would occasionally use the problem context to help explain away apparent contradictions within one of their local theories. I will use the results of the open-coding process as well as a few examples of the local theory diagrams to illustrate each of these findings.

### Using Problem Context and Given Information in Theory Refinement

Whenever the students began working through one of the accumulation tasks, they were continuously revising or replacing a local theory concerning the task. For Task 3, there were a few pieces of information students attached to while generating various solution strategies. Primarily, students knew that because the initial temperature was given to be 57.8 degrees Fahrenheit and the problem concerned climate change and the warming Earth, that their answer must be greater than 57.8 degrees Fahrenheit. Seven of the 12 students interviewed initially assumed that the given function would output the average surface temperature in the year 2200. This assumption runs contrary to the actual problem text in which it is stated that "the temperature is rising at the rate of: $R(t) = 0.014t^{0.4}$ degrees Fahrenheit per year." While many of them read the task out loud prior to beginning their work, they neglected this specific description of the function as a rate and instead assumed it represented the average surface temperature.

With this assumption, each of the seven students then evaluated $R(200)$ and were then faced with contradictory evidence since $R(200)$ equals approximately 0.116. Each of the seven students then realized their current theory was no longer viable, their understanding of what the answer to the task should be overwrote their assumption that the function would output the average surface temperature and so a new local theory was developed to explain this new contradictory evidence. As Tom acknowledged after seeing the result of $R(200)$, "and I said that was wrong because I was, wait, that's so small." It is important to note that this realization does not necessarily lead the students to interpret the output of the function as it was intended, as a rate of change. When contradictory evidence is acknowledged, the student adjusts their local theory or abandons it for another local theory that is more viable to them. In five of the seven interviews in which students acknowledged this contradictory evidence, the student then developed a second local theory with the core assumption that the function is outputting the change in the average surface temperature instead of the average surface temperature itself. Thus, the students tend to suggest adding 0.116 to 57.8 to find the new average surface temperature. This new hypothesis is more viable for the students since it fits within the contextual assumptions they have made. This hypothesis is not mathematically accurate. The students are adding a value of the instantaneous rate of change to the initial temperature instead of using the rate of change to approximate or calculate the change in the temperature over the 200 years. However, the students do not tend to perceive any contradiction here, their current local theory is viable to them since the contextually-based assumptions are now not in any perceived contradiction with the evidence. The fact that their solution is mathematically inaccurate is not a factor in the students' assessment of viability.

In Figure 4 below, we see such an example in Anne's local theory diagram for her work on Task 3. We see in Anne's first local theory that her core assumption is $R(t)$ outputs the average surface temperature. While she describes the function as the "rate of change for the temperature" she then claims that by plugging in 200 into the equation she will get the average surface temperature in the year 2200. After calculating $R(200)$ she notes that this is a very small value and after I ask her what the function tells her about the context she drops her current theory for a more viable one, one with a core assumption that the function outputs the change in the average surface temperature. This theory is more viable for Anne since she is now able to explain her formerly contradictory information that $R(200)$ is a small number. I ask Anne specifically what the units on $R(t)$ are with the intention of seeing if this will cause her to acknowledge another contradiction but she is content in stating the units are degrees Fahrenheit per year without acknowledging any contradiction in her theory. Later in the interview, after she had completed all the other tasks, I again direct her attention to the units and ask her if she can add degrees per year to degrees. Now, based on this question and my desire to return to this task, Anne shifts to another local theory with the core assumption that the function outputs the rate of change of the average surface temperature instead of the actual temperature or the temperature change. Anne now reasons that the rate of change would vary each year and so she would have to add the value each year to the starting temperature of 57.8 degrees. Anne has now utilized the context of the task, the given information from the task, and the interview setting to continually revise her local theory about the task and so her final local theory would be considered the most viable for her at the end of the interview.

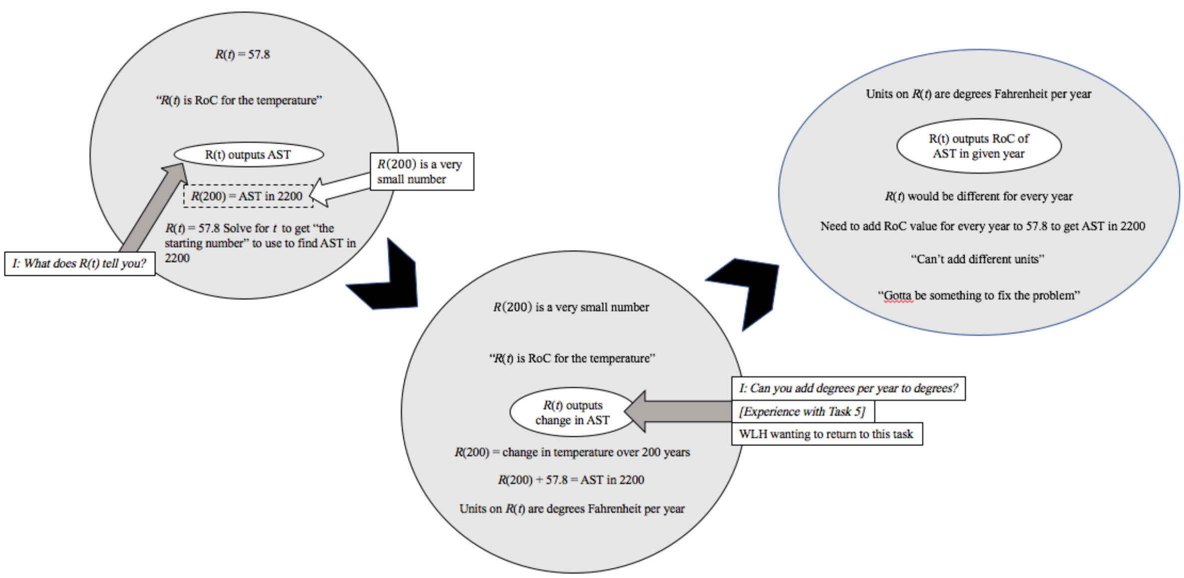


*Figure 3. Local theory diagram for Anne's work on Task 3*

## Reasoning About Contradictions via Problem Context

Another way students utilized the context in reasoning about Task 5 was how they interpreted the negative table values. For some students, like Jake, they could shrug off a potential contradiction. Jake interpreted the table values as representing the number of infected individuals. This means that a negative table value could have served as a contradiction, leading to a theory shift. However, Jake waves away the contradiction by claiming that the negative table values must imply, “there’s like a negative amount of people infected I guess. Um, let’s see, I don’t know just, dropped below the line of infected individuals, I guess maybe they were infected and they died? And they’re still… or maybe they’re people immune.” Jake does not have to settle on any one idea here to disregard the apparent issue. It appears he assumes he must not fully understand the problem and so this allows him to continue with his core assumption that the table values represent the number of infected individuals instead of having to generate a new local theory. This may be related to the difficulties other researchers have identified with how students reason about area under the curve when a function is not strictly positive (e.g., Orton, 1983).

Anne similarly reasons her way through a potential contradiction but instead of attributing the discrepancy to a lack of understanding, she adjusts the problem context entirely to fit within her current theory therefore preserving the viability of her assumption that the table values represent the number of infected individuals. Anne acknowledges that negative people is not a viable interpretation, “so I mean obviously, you don’t have like negative people but, like it’s saying on day zero there was eighteen people…” but instead of altering her theory to increase viability, she alters the problem itself. She claims that the negative five in the table must represent five people who should have been included in the original figures but were not, “I guess if like if they had these people as the original eighteen then they found five people who were sick who weren’t sick anymore they would be like oh, that was five people hadn’t included in the original number that were sick and, but now they’re not sick.” This is a rather sophisticated approach to maintain the consistency of her theory and I believe this creative alteration of the given data is only possible for her because of her confidence with and understanding of the context.

## Implications

There is ample evidence in the current study that problem context influenced how the students reasoned about the tasks. For educators, this means that we need to give students ample opportunities to solve accumulation tasks within various contexts. The results of this study indicate that students may only reason about the accumulation in specific ways when given a specific representation. Additionally, we need to be cautious about what kinds of tasks we use in summative assessments in calculus. Assuming a student’s performance on a contextually-based test question accurately models that student’s ability to solve similar tasks in different contexts may not be warranted. Their experiences in their chosen major and their educational history has given them specific tools they will utilize to reason about these tasks.

Viewing students’ work on calculus tasks through the lens of viability is a meaningful way to approach the data analysis. The local theory diagrams were immensely helpful in my attempt to better understand how the students were solving the tasks by developing and revising local theories concerning the tasks. Creating these diagrams provided me the opportunity to view the data through a different lens and thus I came to understand more about how students interpret calculus tasks and what it takes for them to notice a mathematical contradiction. I believe there is merit in the continued development and use of the local theory diagrams in qualitative data analysis both in calculus and more broadly in mathematics education research.

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Implementation and Impact of a Web-based Activity and Testing System in Community College Algebra

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*Most community college students in the U.S. must complete at least one developmental class, such as elementary algebra, before they can enroll in a college-level mathematics course. Increasingly common in such courses is the use of a web-based activity and testing system (WATS). This report presents initial results of a mixed-methods study of elementary algebra learning among 510 students in the classes of 29 instructors across 18 community colleges. Instructors were randomly assigned to use a particular WATS (treatment condition) or their usual approach (control condition). The focal WATS had adaptive problem sets, hints, and videos. Treatment group instructors had access to online support for implementation. For the study, students completed common pre- and post-tests and instructors regularly provided information about their teaching practices. The early results reported here indicate that greater instructor fidelity to developer intentions regarding frequency of assignments are positively associated with greater student learning.*

*Key words:* College Algebra, Multi-site Cluster Randomized Controlled Trial, Fidelity of Implementation

More than 14 million students are enrolled in community college in the United States. Each is seeking an educational path to a better life. Community college students are more likely to be low-income, the first in their family to attend college, from a group under-served by status quo K-12 education (e.g., from an ethnic, racial, or linguistic minority group; Bailey, Jeong, & Cho, 2012). Most must take at least one developmental class, such as elementary algebra, before they can enroll in a college-level course (Porter & Polikoff, 2012). When it comes to technology and early algebra learning in college, what works? For whom? Under what conditions? When instructors implement technology tools, how are they used? In ways aligned with developer intentions? To what degree? Several web-based activity and testing system (WATS) have emerged for use in college developmental mathematics (e.g., ALEKS®, Khan Academy). Some WATS, like the one at the heart of this study, include adaptive problem sets, videos, and tools for instructors to monitor student learning. Though some research on the efficacy of WATS exists (e.g., Gardenhire, Diamond, Headlam, &Weiss, 2016 and references therein), the study reported here is the first large scale, multi-institution, mixed-methods experimental study of a WATS in community college developmental algebra of which we are aware.

## Research Questions

Funded by the U.S. Department of Education, we are conducting a multi-year large-scale mixed methods study in over 30 community colleges in one U.S. state. We report here on the first year. The study is driven by two research questions:

Research Question 1: What is the impact of a particular WATS learning platform on students' algebraic knowledge after instructors have implemented the platform for two semesters?

Research Question 2: What challenges to use-as-intended (by developers) are faculty encountering and how are they responding to the challenges as they implement the learning tool?

## Background and Conceptual Framing

Regardless of how they might be used, WATS environments vary along at least two dimensions: (1) the extent to which they adaptively respond to user behavior (e.g., static vs. dynamic) and (2) how they are informed by a model of cognition or learning. Static WATS are non-adaptive – they deliver content in a fixed order and contain scaffolds or feedback that are identical for all users. The design may be based on intuition, convenience, and/or a hypothesized common learning trajectory. An example of this type of environment might be online problem sets from a textbook that give immediate feedback on accuracy (e.g., "Correct" or "Incorrect").

Dynamic WATS environments keep track of learner behavior (e.g., errors, error rates, time-on-problem) and use this information in a programmed decision tree that selects problem sets or feedback based on estimates of student learning. An example of a dynamic environment might be a system such as ALEKS® or the approach now used in Khan Academy Missions. For example, at khanacadmy.org, a behind-the-scenes data analyzer captures student performance on a "mastery challenge" set of items. Once a student gets six items in a row correct, the next level set of items in a target learning trajectory is offered. Depending on the number and type of items the particular user answers correctly (e.g., on the path to six in a row correct), the analyzer program identifies and assembles the next "mastery challenge" set of items.

Above and beyond responsive assignment generation, programming in a dynamic environment that is also cognitively-based is informed by a theoretical model that asserts the cognitive processing necessary for acquiring skills (Anderson, Corbett, Koedinger, & Pelletier, 1995; Koedinger & Corbett, 2006). For example, instead of specifying only that graphing should be practiced, a cognitively-based environment also will specify skills needed to comprehend graphing (e.g., connecting spatial and verbal information), and provide feedback and scaffolds that support these (e.g., visuo-spatial feedback and graphics that are integrated with text). In cognitively-based environments, scaffolds themselves can be adaptive (e.g., more scaffolding through examples can be provided early in learning and scaffolding faded as a student acquires expertise; Ritter, Anderson, Koedinger, & Corbett, 2007). Like other dynamic WATS, such systems can provide summaries of student progress. No fully operational cognitively-based WATS currently exists for college students learning algebra. Several dynamic systems do exist (e.g., ALEKS®, Khan Academy Missions). The particular WATS investigated as the treatment condition in our study was designed primarily for use by learners as replacement or supplement to homework or in-class individual seatwork.

The theoretical basis for our approach to examining the instructional implementation of a WATS lies in program theory, "the construction of a plausible and sensible model of how a program is supposed to work" (Bickman, 1987, p. 5). As in many curricular projects, developers of the WATS in our study paid attention to learning theory inasmuch as it shaped the content in standard algebra texts upon which the WATS content was based. Developers articulated their assumptions about what students learned in completing WATS activities, but the roles of specific components, including the instructor role in the mediation of learning, were not clearly defined.

Munter and colleagues (2014) have pointed out that there is no agreement on how to assess fidelity of implementation (how close implementers come to realizing developer intentions; Dusenbury, Brannigan, Falco, & Hansen, 2003). However, there is a growing consensus on a component-based approach to measuring the structure and processes of implementation (Century & Cassata, 2014). Five core components are key in examining implementation: Diagnostic, Procedural, Educative, Pedagogical, and Engagement (see Table 1).

*Table 1. Components and focus in examining implementation.*

| *Components* | *Focus* |
|---|---|
| Diagnostic | These factors say what the "it" is that is being implemented (e.g., what makes this particular WATS distinct from other activities). |
| Structural-Procedural | These components tell the user (in this case, the instructor) what to do (e.g., assign intervention *x* times/week, *y* minutes/use). These are aspects of the *expected* curriculum. |
| Structural-Educative | These state the developers' expectations for what the user needs to know relative to the intervention (e.g., types of technological, content, pedagogical knowledge needed by an instructor). |
| Interaction-Pedagogical | These capture the actions, behaviors, and interactions users are expected to engage in when using the intervention (e.g., intervention is at least *x* % of assignments, counts for at least *y* % of student grade). These are aspects of the *intended* curriculum. |
| Interaction-Engagement | These components delineate the actions, behaviors, and interactions that students are expected to engage in for successful implementation. These are aspects of the *achieved* curriculum. |

## Method

The study we report here used a mixed methods approach combining a multi-site cluster randomized trial with an exploration of instructor and student experiences. Half of instructors at each community college site were assigned to use a particular WATS (treatment condition), the other half taught as they usually would, barring the use of the treatment group's focal WATS tool, though other WATS might be used (control condition). Faculty participated for two semesters so treatment instructors could familiarize themselves with implementing the WATS with their local algebra curriculum. ***Note***: We report here on data collected from the first of two years. Hence, we purposefully under-report some details.

### Sample for this Report

Initial enrollment in the study included 89 instructors across 38 college sites. Attrition of instructors by the end of the year was significant (68%). In the end, 29 instructors at 18 colleges finished the study (i.e., we had sufficient data from them to include them in analyses). This report is based on data from these instructors and their 510 students (see Table 2).

*Table 2. Counts of teachers, students, and colleges in the study.*

| *Condition* | *Teachers* | *Students* | *Colleges* |
|---|---|---|---|
| Control | 17 | 328 | 13 |
| Treatment | 12 | 182 | 11 |
| *Total* | *29* | *510* | *18* |

### Measures

A great deal of textual, observational, and interview data were gathered. These data allow analysis of impact (Research Question 1) and an examination of implementation structures and processes (Research Question 2). Initial indices of implementation fidelity were based on instructor weekly self-reports of WATS use and, for the treatment group, on the WATS audit trail of student use. The primary outcome measure for students' performance was an assessment from the Mathematics Diagnostic Testing Program (MDTP), a valid and reliable test of students' algebraic knowledge (Gerachis & Manaster, 1995).

**Student Mathematics Performance.** One way to estimate student achievement on the MDTP tests is the *raw* score (i.e., proportion of correct answers as a percentage). However, such a calculation does not take into consideration other parameters of interest, such as item difficulty. To address this, in a second analysis we used a multilevel extension of two-parameter logistic item response theory to compute student pre- and post-test *scale* scores (Birnbaum, 1968). Specifically, we computed response-pattern *expected a posteriori* estimates (EAP scores; Thissen & Orlando, 2001) for each student. Also, we created EAP average scores for each classroom (a teacher-level score).

**Instructor Implementation Processes.** The components in Table 1 were operationalized through a rubric, a guide for collecting and reporting data on implementation. Each component has several factors. The research team developed a rubric for fidelity of implementation that identified measurable attributes for each component (Hauk, Salguero, & Kaser, 2016). For this report, we focus on the first element in the "procedural" component (see Table 3). The values and proportions (e.g., at least 2/3 of weeks) were specified by the developer. Data on the aspects in Table 3 was collected from weekly logs in which instructors indicated (a) WATS assignments made, (b) encouragement to students to complete assignments, (c) use of recommended mindset lessons, and (d) nature of attention in-class to student experiences with the WATS.

*Table 3. Example of rubric descriptors for levels of fidelity, Structural-Procedural component.*

| ***Procedural:*** These components tell the user (instructor) what to do regarding instruction. Scaling within unit of instructor.. | | | |
|---|---|---|---|
| | **Low Level of Fidelity** | **Moderate Fidelity** | **High Level of Fidelity** |
| **Assigned WATS** | Instructor rarely or never assigns WATS activities (2 or fewer times per semester). | Instructor sometimes assigns WATS activities (between 3 and 8 times per semester). | Instructor regularly assigns WATS activities (at least 8 times per semester). |
| **Value of WATS** | Instructor rarely or never encourages students to complete assignments (less than 1/3 of weeks/term). | Instructor sometimes encourages students to complete assignments (1/3 to 2/3 of weeks/term). | Instructor regularly encourages students to complete assignments (at least 2/3 of weeks/term). |
| **Effort-based mindset** | Instructor conducts at most 1 session of mindset training. | Instructor conducts 2 sessions of mindset training. | Instructor conducts recommended 3 sessions of mindset training. |
| **Intensity of in-class supports for WATS use** | Explicit mention or attention in class to content in WATS in fewer than 50% of weeks in term. | Explicit mention or attention in class to content in WATS from 50% to 80% of weeks in term. | Explicit mention or attention in class to content in/from WATS at least 80% of weeks in term. |

## Results

The study employed Hierarchical Linear Modeling (HLM), controlling for students' pre-test MDTP scores, to estimate the impact of WATS use on student achievement. The hierarchical modeling approach accounts for the nested structure of the sample (Raudenbush & Bryk, 2002), specifically the nesting of students within instructors. Preliminary analysis indicated that such a hierarchical model was justified: the intra-class correlation in the unconditional model was 0.36, suggesting that the observations were not independent (i.e., scores varied based on classroom – statistically, the teacher mattered – so single-level regression was not appropriate). The exact model and random and fixed effects for it are reported elsewhere (Hauk & Matlen, 2017).

## Intervention Impact

**Baseline equivalence.** The What Works Clearinghouse (2014) considers baseline differences with a Hedges $g > .25$ not to be amenable to statistical correction. The raw pre-test scores were higher in the treatment group, with a marginal effect size ($g = 0.25$) for the difference between groups; however, the difference between treatment and control student pre-test EAP scores was substantive ($g = 0.30$). The EAP pre-test difference is large enough that the analytic sample might be considered non-equivalent at baseline on this variable (below, we discuss this fact).

**Impact analysis.** The aim of impact analysis was to address the question: What is the impact of the WATS intervention on students' elementary algebra knowledge, as measured by the MDTP? Controlling for students' pre-test scores, we found that using WATS corresponded to, on average, treatment student post-test scores 5 percentage points higher than the control group ($p < .05$). The Hedges $g$ value for this effect is 0.32, which is considered a small but noteworthy effect in educational research for studies of this size (Cheung & Slavin, 2015; Hill et al., 2008). The 95% confidence interval of the Hedges $g$ value is .14 - .50 (i.e., entirely above zero). Using EAP instead of raw scores, we obtained similar results. Since baseline differences between treatment and control group student raw scores were within the range of statistical correction, the similarity between the two *models* (raw score and EAP score models) is important, offering more confidence in the estimates of positive impact.

## Implementation Fidelity and Student Learning

Of the 12 treatment instructors, 9 provided sufficient weekly information about the amount of instruction using WATS to determine a level of implementation. Three instructors were coded as high fidelity, 3 as moderate, and 3 as low. We explored whether the level of this category of procedural implementation fidelity in the treatment group correlated with student learning. To estimate *learning* we computed a normalized gain score, calculating $z$-scores for the pre- and post-test EAPs separately, and then subtracting the post-test $z$-scores from the pre-test $z$-scores for every student. These gains represent a difference between a student's relative position on the distribution of pre-test scores to their relative position on the distribution of post-test scores. Thus, a negative gain does not mean that a person (or in the case of Figure 2, a group of people) know less by the end of the course. Rather, it means that students are lower in the standardized distribution at post- than at pre-test. Figure 1 shows the average gain for treatment group students at each of the different fidelity of implementation levels for "Assigned WATS." We report here on this Procedural factor because it had the most notable differences across levels of fidelity.

The results in Figure 1 (next page) suggest that the more regularly instructors assigned WATS lessons, the larger the student gain. However, sample sizes at present are small, so results are not definitive. Nevertheless, these results are consistent with the developer's expectations (and an addition of a second cohort will allow us to examine whether these associations persist).

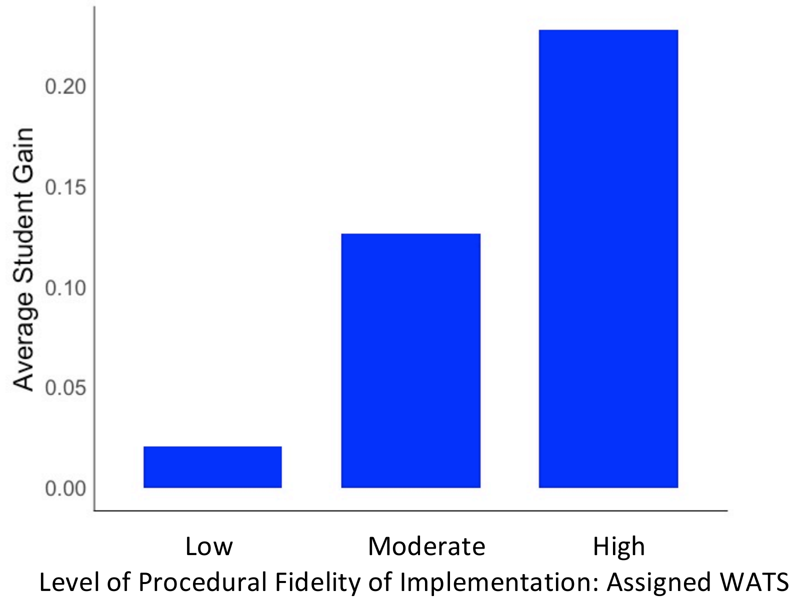


*Figure 1. Average student gains in treatment classrooms according to instructor's level of procedural fidelity of implementation in the factor "Assigned WATS."*

Control group instructors might use a WATS, but were restricted from using the focal WATS under study. Thus, one question was whether use of *any* WATS, regardless of whether it was the focal one, correlated with student learning. To explore this possibility, we examined average student gains for instructors using the treatment WATS ($n$ = 9) compared to those in the control group who used a different WATS ($n$ = 8). Figure 2 suggests that use of the study's focal WATS in particular, not just any WATS, had a positive relationship with student learning.

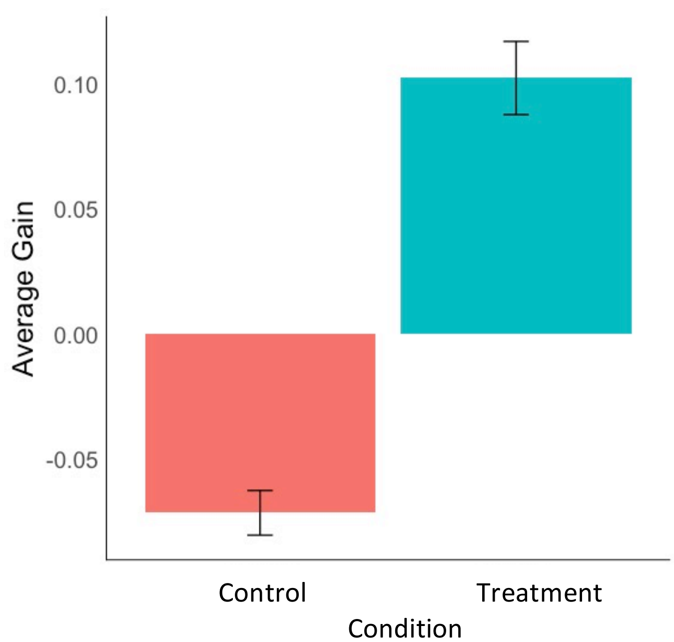


*Figure 2. Average student gains for control group classes using a WATS vs. treatment WATS.*

## Discussion

We continue to explore relationships between fidelity of implementation and student success. While the results to date suggest that the focal WATS had a positive impact on students' elementary algebra achievement, recall there was high instructor attrition. This fact, coupled with moderate to large baseline differences at pre-test, warrant caution in interpreting the results. Still to do is a systematic consideration and testing of alternative explanations for Figures 1 and 2. For example, differential attrition may mean that treatment instructors who stayed in the study were better at incorporating the focal WATS into instruction. Mitigating against this explanation are the exit surveys completed by instructors who left the study in which course reassignment was the primary reason for treatment instructor attrition. Also, while we know the types of WATS used by control group instructors, we do not have a developer-validated fidelity of implementation rubric for each of those other WATS.

The ultimate purpose of a fidelity of implementation rubric is to articulate how to determine what works, for whom, under what conditions. In addition to allowing identification of alignment between developer expectations and classroom enactment, it provides the opportunity to discover where productive adaptations may be made by instructors, adaptations that boost student achievement beyond that associated with an implementation faithful to the developers' view. As we move forward with modeling, implementation indices (or vectors of values, one for each factor) will be used at the instructor level in statistical modeling of the impact of the intervention as part of a "specific fidelity index" (Hulleman & Cordray, 2009).

## Conclusion

As indicated above, we have repeated the study with a second cohort of participants in the 2016-17 academic year. The new data, combined with the first study reported here, may provide additional results and insights by the time of the RUME conference.

**Implications for practice.** For the question: Should faculty use a WATS? The answer is a cautious: It depends. We know that treatment instructors had supports for WATS use in the form of video-based professional development and access to a project consultant who was experienced with the focal WATS in community college algebra. The implementation supports for control group teachers who used another WATS were varied. Taking into account the potentially biased statistical impact results and the exploration of variation in instructor implementation, there is still an open question about what might be the minimal supports needed for an instructor to have high fidelity on procedural components (e.g., Assigns WATS).

**Implications for research.** There were significant challenges in recruiting and retaining community college mathematics instructors for the study. To build community and assist in future research efforts in two-year colleges, we are sharing the processes and results of this work in materials read by community college faculty and administrators (e.g., *MathAMATYC Educator* – a journal of the American Mathematical Association of Two Year Colleges). It is important for potential faculty participants in research and their chairs/deans to be aware of the enormous contributions faculty can make to research. A second pragmatic implication for research is in how to manage the data generated by such projects (Hubbard, 2017)

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Early Undergraduates' Emerging Conceptions of Proof and Conviction

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*Before enrolling in an introduction-to-proof course, undergraduates often hold conceptions of mathematical proof that do not align with those accepted by the mathematics community. These conceptions are informed, in part, by past experiences with proof in mathematics and science courses. In this study, we sought to investigate the influence of these past experiences on students' conceptions of proof. We conducted interviews with nine undergraduates in their first or second year in which we asked them to solve number theory tasks and determine the validity of provided number theory statements. In this paper, we report on the various conceptions of proof these students conveyed and the influence of past experiences on these conceptions.*

*Keywords:* Proof, Conceptions, Student Thinking

## Introduction and Motivation

It has been well-established in the literature that undergraduate students struggle to learn to prove. One challenge that students face is that many of them enter university mathematics courses with conceptions of proof that differ from those accepted in the mathematics community. These conceptions include what constitutes a mathematical proof, what purposes a mathematical proof can serve, and how one constructs a mathematical proof. Students develop these conceptions through their past experiences in mathematics, as well as through experience with the idea of proof in non-mathematical settings. Notably, most students in the United States encounter proofs in high school when studying geometry. They also may encounter proofs in a Calculus course, constructing proofs (e.g. epsilon-delta proofs) or making sense of instructor-provided proofs (e.g. the Mean Value Theorem). These experiences influence the way that students conceive of proof in mathematics.

In order to help students develop more robust conceptions of proof, we need to understand the conceptions they bring in with them. In this paper, we explore these emerging conceptions, the factors that influence these conceptions, and the strategies students already employ when determining the truth of a mathematical statement. We ask the following questions:

- How do early undergraduate students' past experiences in math and science influence their conceptions of proof?
- How are early undergraduate students' conceptions of proof related to their strategies for gaining conviction?

## Relevant Literature and Theoretical Perspective

Following Thompson's (1992) definition of conceptions of mathematics, we use conceptions of proof to refer to one's "conscious or subconscious beliefs, concepts, meaning, rules, mental images, and preferences" concerning mathematical proof. Conceptions of proof have been studied across populations including high school students (Chazan, 1993; Healy & Hoyles, 2000), undergraduate mathematics majors (Harel & Sowder, 1998; Weber, 2010), and mathematics teachers (Knuth, 2002). In recent years, researchers have also investigated the conceptions of proof held by early undergraduate students - students who have enrolled in at least one college-level mathematics course, but have not yet enrolled in an introduction to proofs course or other proof-based mathematics course (Janelle, 2014; Raman, 2001; Stylianou, Blanton, & Rotou, 2015; Stylianou, Chae, & Blanton, 2006). In the largest of these studies, Stylianou, Blanton, and Rotou (2015) conducted a survey of over 500 early undergraduates about their conceptions of proof, including questions about beliefs and past experiences and multiple-choice proof evaluation tasks. They found that most of the students surveyed selected deductive arguments as the *most rigorous*, but that the proofs students selected as *most explanatory* were the arguments they identified as closest to their own approach (split between deductive, empirical, and narrative). They also found that only a quarter of the students reported having past classroom experiences that "emphasized the importance of developing proofs" (p. 112) and more than half of the students reported past instructors using examples to prove mathematical statements. In this paper, we investigate further the influence of these past experiences on students' beliefs about proof.

When identifying students' strategies for gaining conviction, we focus on *ascertaining*, which Harel and Sowder (1998) define as "the process an individual employs to remove her or his own doubts about the truth of an observation" (p. 241). Much of the existing literature focuses on conviction in terms of what participants identify as convincing in the arguments of others (e.g. Janelle, 2014; Knuth, 2002; Healy & Hoyles, 2000; Chazan, 1993) as opposed to how students construct arguments to convince themselves. Each of these studies found that the majority of participants accepted both deductive and empirical arguments as convincing. However, Stylianou et al. (2015) found that the proofs that students identify as the most convincing and the most like their own approach don't always match their actual proof construction. They gave the same four mathematical statements to 60 students first as proof construction tasks, then as proof evaluation tasks two weeks later. They found that the majority of students constructed empirical arguments, but then reported two weeks later that a narrative or deductive argument was most like what they would construct. Considering this finding, we look at conviction in this paper in the context of students' generated arguments.

## Methods

In this study, we conducted hour-long interviews with nine undergraduate students. The participants were all freshman or sophomore students at a small, Hispanic-serving university in the Western United States. The participants were selected because they were enrolled in a college-level mathematics course but had not yet taken an introduction to proof course. Of the nine participants, four were enrolled in Calculus I, three were enrolled in Calculus II, and two were enrolled in Discrete Math. Four of the participants were biology majors, three were marine science majors, and two were computer science majors.

Each student participated in an individual, hour-long, semi-structured interview. During the interview, they were presented with five number theory tasks to explore one at a time. Participants were asked to think aloud as they worked; their speech and writing were recorded using LiveScribe pens, and each interview was videotaped.

On each task, participants were asked if they were convinced by the work they had done and if they considered their work to be a mathematical proof. Depending on their answers, the interviewer asked relevant follow-up questions (e.g. What is missing that would make this a proof? What would you need to do or see to be fully convinced?). After the first task, participants were asked what they believe it means for something to be a mathematical proof. At the end of each interview, participants were asked about their experiences with mathematical proofs in the context of their mathematical careers. Specifically, they were asked if their professors ever show proofs of theorems in class, if they have ever written proofs in their classes or for homework, and what they thought the purpose of proofs in mathematics is.

## Tasks

Each participant was asked to work on five number theory tasks, including the three tasks in Table 1. We chose number theory as the content area because it is one of the first topics that students typically encounter in an introduction to proof class. The five tasks were chosen to be easily accessible to the students, requiring only knowledge of divisibility, factors, and even/odd numbers. Some tasks asked students to determine whether a statement was true or false, while others were more exploratory in nature, asking students to create a conjecture.

*Table 1. Three of the five number theory tasks used in the study*

| Task Number | Task Statement |
|---|---|
| Task 1 | Consider the statement: The sum of any 5 consecutive whole numbers is divisible by 5. Is this statement true or false? Would this statement still be true if 5 was replaced with any other number? |
| Task 3 | A factor of a number is a whole number that divides it evenly. For example, the factors of 10 are 1, 2, 5, and 10. Which numbers have an odd number of factors? |
| Task 4 | If $a$, $b$, and $c$ are whole numbers, is $a$ times $b$ plus $a$ times $c$ always even, always odd, or can it be either? If $b$ and $c$ are required to be odd, will $a$ times $b$ plus $a$ times $c$ always be even, always be odd, or can it be even or odd? |

## Data Analysis

For analysis, each interview was transcribed and images of student work from the Livescribe PDFs were added to each transcript. The transcripts were analyzed using a grounded theory approach (Strauss & Corbin, 1994). The two researchers independently coded each transcript using open coding and then discussed themes arising from the generated codes. Codes capturing aspects of participants' conceptions of proof were refined and a modifier was added to capture whether the code was something the participant was convinced by, not convinced by, considered to be necessary/sufficient for proof, or considered not to be necessary/sufficient for proof. Codes were also developed to capture the references participants made to past experiences when discussing proof and conviction. Once the coding scheme was refined, the second author recoded each of the transcripts and wrote a descriptive narrative for each participant. These narratives outlined each participant's ideas surrounding proof and conviction, providing examples and direct quotes from the transcripts to illustrate what they found convincing and what they believed a proof to be.

# Results

We report on two aspects of students' emerging conceptions of mathematical proof: the sources they draw upon when forming and articulating these conceptions and the implications for their view of the definitiveness of proof. We also highlight one participant, Rosa, to illustrate the relationship we observed between students' conceptions and the strategies they use to gain conviction.

**Influence of Math and Science Experiences on Conceptions of Proof**

While the students in our study discussed the concept of proof in mathematics in diverse ways, they drew upon common themes and past experiences as sources for their understanding and reasoning. The three most common themes participants referred to were *Science, High School Geometry,* and *Discrete Mathematics*.

Among the seven science majors (biology, marine science), five drew connections to the study of science in their discussions of mathematical proof. Rosa, José, and Alicia (all Biology majors) described a mathematical proof as only needing evidence – examples or explanations. José described a proof as *"Evidence. Any kind of evidence. Material, biological, any kind of evidence is proof."* Alicia's description of proof was similar: she shared that a mathematical proof is *"show[ing] evidence that it works."* She also talked about more examples being valuable because they served as replication. On Task 4, she gave two confirming examples of the statement, and when asked why two examples were necessary, she explained it as, *"I guess, like, the ability to reproduce the results. Because [the second example] kind of justifies the prior one."* Unlike Rosa and José, Alicia and the two Marine Science majors, Gabriela and Cecilia, described mathematical proof in contrast with ideas from science. Gabriela drew a distinction between the definitiveness of proof in mathematics and in science:

*Interviewer*: So, in mathematics, what is a proof? What is necessary for something to be a proof?

*Gabriela*: I would say that it's like an absolute thing. And it's been tested many, many times to make sure that there aren't any aren't any exceptions to that one rule. Kind of like how- like it's like a law in science would be.

Alicia drew a similar contrast when comparing proving in mathematics and biology, saying that *"Math is like- I don't know, I feel like once you have it on paper it's pretty much irrefutable, but bio and pretty much it's just, at one point it can be proven wrong."*

Another common theme among the science majors were references to proofs in high school geometry. Four of the seven science majors referred to the two-column proofs they learned in high school, but their interpretations of these proofs differed considerably. Cecilia interpreted the steps in a two-column proof as steps in a deductive argument, describing the second column in terms of logical arguments:

> Well, it's really to prove your logic to get from A to B. It's to show and explain in ways that another person who understands math can look at it, see your work, read that explanation whether it's just, you know, explaining the logic in that one step or actually citing some theorem.

In contrast, Rosa used two-column proofs as justification for why examples and informal arguments were sufficient for proof:

*Rosa*: Proofs, I get them. I think of geometry. We would get the proofs on one side and, like, have to show that it's true. So proof is making a statement and showing through examples or other like rules that this is a true statement.

*Interviewer*: Oh okay, so you're thinking back to your high school geometry when you were proving properties of triangles and circles.

*Rosa*: Yeah, you'd have like all the true statements on one side and then your work and your explanations on the other side.

The two computer science majors in our study, Antonio and Ana, were enrolled in a discrete mathematics course at the time, and both drew primarily from the content of the course when discussing proof. This is unsurprising since the course contains a week-long unit on mathematical proof, but what was interesting were the features of proof that were most salient to these students. Both students described a proof as a deductive argument which shows that a mathematical statement is always true, but they also emphasized the need for formal language, symbols, and certain structure. For instance, on Task 1, Antonio was fairly convinced that the statement was true from patterns he observed in his examples, but to be totally convinced, he would need to write it the "fancy way":

*Antonio*: So, if I did it in the fancy way, with Discrete Math, that's a way to prove how you got the answer.

*Interviewer*: Yeah?

*Antonio*: Pretty much. Uh, cause sometimes proving, you need to require some, like, kind of Discrete Math symbols, I say.

The requirement of formal language was restrictive for Ana on Task 3. She had generated a conjecture and articulated an argument in support of her conjecture, but she felt like she lacked the language necessary to write a proof:

*Interviewer*: Do you think, at this point, based on what you know about this problem, that you would be able to write a proof?

*Ana*: Probably not. [Laughs]

*Interviewer*: Why not?

*Ana*: Because like, I'm just assuming this. I don't know how I would formally write a formal proof. Well, like, when I think proof, it has to be formal, so like, there would have to be, like, if this, then this. And then suppose this. And then you show your proof.

These quotes are representative of a phenomenon we saw broadly in our study of participants using key past experiences as reference points when describing and conceptualizing proof. These references also played a role in how students thought about obtaining conviction, as we discuss more below.

**Conceptions of Proof and Strategies for Conviction**

Although many of the students in our study described conceptions of proof that differ from the accepted norm in mathematics, there was general consistency between students' conceptions of proof and how they sought to convince themselves.

**The case of Rosa.** To illustrate this notion of consistency, we highlight the work of Rosa, a freshman Biology major. As referenced above, Rosa accepted examples and informal explanations as a proof. She described proofs as making a claim and supporting that claim with some evidence. For Rosa, proofs are not definitive:

> Proof is kind of like, I have this idea that, like, you know. […] It would be like [on Task 4], "I think that, you know, when we do the same *a* term with two different *b* and *c* terms, I think we'll always get even" and it could've been true. Like if they were to say I think it's always even and we did these 2 examples right here, we'd be like "Oh, okay." But then I would prove them wrong by saying "Well this one's odd."

For more definitive arguments, she assigned the terms *theory* or *law*. Generalizing from the scientific definitions of the words, Rosa defined theories as, *"things they've experimented and it's been true for the most part. Like, not always, but almost all the time,"* and defined laws as, *"something like gravity, yeah, there's no proving that it's not true."*

Rosa's beliefs about proof, theory, and law were consistent with what she viewed as convincing. Since proof was not definitive for Rosa, a proof was not necessarily convincing. On Task 1, she tried two examples that both worked and she described her two examples as a proof of the statement. However, she wasn't convinced that it would always be true:

*Interviewer*: Okay, are you convinced that it's going to be true for any 5 consecutive numbers?

*Rosa*: I don't think, um...There's not a lot of absolutes in math, like you know? So I don't know. I'm not convinced that this will always be true, but like for right here it was true.

Rosa was more convinced when she could articulate why a statement was true. On Task 3, she formed a conjecture that the numbers with an odd number of factors were the perfect squares, and she checked her conjecture with three confirming examples (16, 4, and 64). She then explained the rationale behind her conjecture, that the numbers with an odd number of factors are *"the ones where you don't have to list the extra factor"* because the *factors "partner up"*. Rosa was quick to describe this as a proof (since *she "made a statement and then showed examples to why [she] made that statement"*), but later upgraded it to a theory, bordering on a law:

> I would go to say that this one is, like, I did theory on this one because it would be different if I said like you know, I gave a couple of examples but I'm saying like specifically the squares, like the perfect square ones, so I'm already going more in depth and, um, if it was the, yeah theory. See, it's almost on the border of a law.

From her work on these two tasks, we see that Rosa is more convinced by arguments that are more deductive and explanatory in nature, classifying these arguments as theory or law. Since Rosa's *law* is closest to what mathematicians would call *proof*, her reasoning is significantly more mathematically sound than her definition of proof would suggest.

**Definitiveness of Proof.** We observed another area of consistency between students' conceptions of proof and whether they viewed proofs to be definitive. Of the nine students in our study, four described mathematical proofs as non-definitive (i.e. a proof does not guarantee that a mathematical statement is always true). However, as we saw with Rosa, all but one of the students' views of the definitiveness of proof were consistent with their conceptions of proof. All four of the students who viewed a proof as non-definitive also accepted examples and informal explanations as a proof, whereas four of the five remaining students required a formal deductive argument as a proof.

## Implications

In this study, we observed students drawing upon common past experiences in math and science when thinking about and describing mathematical proof. However, despite these experiences being common in a broad sense, different students had internalized different meanings from the experiences. For instance, two students used two-column proofs from high school geometry to support the claim that proofs are deductive, but two other students used the same proofs to claim that examples were sufficient. As a result, it seems that identifying the sources students draw upon in their conceptions is too coarse of a unit of analysis for making sense of those conceptions.

We also observed that, while some students classified empirical arguments as proofs, this does not necessarily mean they viewed those arguments as definitive. In fact, the students' notions regarding what made an argument convincing were far more mathematically accurate than their notions of what constitutes a proof. In future studies, researchers should take care not to conflate these two separate sets of conceptions.

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Gestures as Evidence of Assimilation When Learning Optimization

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*Teachers and students often produce gestures during communication about mathematical concepts and processes. Our goal in this study was to determine whether students would produce gestures similar to those used by the teacher. Each of five students in a first semester calculus course was asked to solve two optimization problems based on a video lesson in which the teacher used primarily pointing, primarily depictive gestures, or no gestures at all. Though our data do not show the students' gestures directly imitating the teacher's, they provide support for the claim that frequent gesture use during communication may indicate assimilation of new concepts and that assimilation improves student performance on optimization tasks.*

*Keywords:* gesture, calculus, optimization, assimilation, accommodation

## Background

Optimization problems are frequently difficult for students in first semester calculus. These problems often require the drawing of figures, definition of several variables, coordination of multiple equations, algebraic substitutions, application of derivatives, and ultimately interpretation of the final results. LaRue and Engelke Infante (2015) studied student responses to optimization problems and determined that students have the most difficulty during the early, "set up" parts of the problem. This part of the problem solving process is referred to as the *orienting phase* (Carlson & Bloom, 2005). During this early phase of problem solving, the student "deciphers the problem and assembles the tools he or she thinks may be required" (LaRue & Engelke Infante, 2015, p. 2).

Deciphering the problem may evoke for the student certain *concept images*. The notion of *concept image*, as defined by Tall and Vinner (1981), is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." For instance, one's concept image of the derivative might include things like a prototypical example curve, tangent lines, slope, rate of change, "prime" notation, and processes like the power rule, product rule, and chain rule. Note that the concept image is *dynamic*: it changes in response to new experiences. Prior to learning about optimization, students in calculus will almost certainly have learned about derivatives, maxima and minima (in relation to curve sketching), and the second derivative test for concavity. When optimization is introduced, however, it may become a part of the student's concept image for any or all of these concepts.

A student's concept image may incorporate gesture. Gestures are a naturally occurring part of communication; as such, teachers frequently gesture when teaching. For example, it is not uncommon for teachers to trace the shape of a parabola in the air, or to point to an equation written on the board for reference. Students may internalize these gestures as part of their concept image, and in turn, they may produce these or similar gestures during communication. It has been shown that thinking about an object or an event activates the same regions of the brain that become activated during the actual physical perception of those objects or events, and thus regions of the brain responsible for reacting to these stimuli are also activated (Hostetter & Alibali, 2008). Hostetter and Alibali's (2008) *Gesture as Simulated Action* framework posits that the activation of these regions of the brain in response to simulated (mental) actions will sometimes result in the realization of an overt movement: a gesture.

Studies suggest that students are more likely to produce gestures when communicating difficult information (McNeill, 1992; Radford, 2009; Roth, 2000). Roth (2000) specifically noted, "This and other research documents a high incidence of gestures when individuals deal with unfamiliar situations" (p. 1711). In light of the results of LaRue and Engelke Infante (2015), we expect that when solving an optimization problem, students might produce more gestures during the orienting phase of solving the problem. This study aimed to answer the following question: Do students mimic the teacher's gestures when solving problems similar to what the teacher presented? While we did not see evidence of this, we did observe evidence that students were more likely to produce gestures if they are *assimilating* new information, rather than *accommodating* it.

## Theoretical Perspective

We frame our research using Piaget's (1985) notions of *assimilation* and *accommodation*. *Assimilation* is "the cognitive *process* by which the person integrates new perceptual matter or stimulus events into existing schemata or patterns of behavior" (Wadsworth, 1975, p. 15). During the learning process, an individual is said to have assimilated new knowledge when they have made cognitive connections between the new information and their pre-existing knowledge. However, assimilation may not be possible: the individual may not possess an existing schema into which the new information fits. Under this circumstance, accommodation may take place. *Accommodation* is "the creation of new schemata or the modification of old schemata" (Wadsworth, 1975, p. 16). Piaget posits that cognitive systems exist in a state of dynamic equilibrium involving both processes of assimilation and accommodation (Piaget, 1985).

Piaget's concepts of assimilation and accommodation describe two ways in which learners attempt to reconcile new information with their pre-existing knowledge. This includes the incorporation of sensorimotor input like gestures (Piaget, 1985). As evidence of the assimilation of perceived gestures into existing schemata, we observe the repetition of these or similar gestures during communication. Here, we adopt Sfard's (2001) communicational approach to cognition, which views thinking as a special case of communication, "as one's communication with oneself" (p. 26). With this perspective, gestures that are realized during interpersonal communication, as well as those performed during individualized thought, are taken as evidence of assimilation.

## Methods

The goal of our study was to determine how student understanding is affected by the instructor's gesture use in the classroom. We prepared a lesson on optimization for a first semester calculus course, and three scripts were prepared: one in which the instructor used only *pointing* gestures, one in which the instructor used only *depictive* gestures, and one in which the instructor made no gestures. Using the definitions in Alibali et al. (2014), a *pointing gesture* is one which "indicate[s] objects or locations in the physical world," and a *depictive gesture* is a simulated action or a conceptual action grounded in a physical action, such as simulating the action of collecting objects as a metaphor for the conceptual action of adding numbers. Apart from the differences in gestures, these three scripts were identical. One member of the research team was filmed presenting each script, and three videos were prepared. It should be noted that this lesson used the second derivative to confirm that the answer that was obtained was a maximum/minimum instead of the first derivative.

Interview subjects were assigned one of the above videos to watch based upon the order in which they arrived for interviews. Students were permitted to take notes while watching their video. Immediately after watching the video, students were asked to solve two optimization problems:

*Problem 1: If the perimeter of a rectangle must be 84 inches, what are the dimensions of the rectangle that has the largest possible area?*

*Problem 2: A company wishes to manufacture a rectangular box with an open top whose base length is twice as long as its base width. If the box must contain a volume of 32 $ft^3$, what are the dimensions of the box that will minimize its surface area?*

Students were encouraged to speak aloud as they worked so as to ascertain why they took the steps they did to solve the problem. Interviewers prompted the students when they were quiet for long periods of time and after they had completed certain steps in their solutions. Interviews were filmed to capture students' thoughts and gestures during this process. Students were compensated for their time with a $10 gift card.

There were a total of five interview subjects who were assigned pseudonyms: Ben, Andrew, Eric, Lisa, and Mary. Ben and Lisa watched the "Pointing" video, Andrew and Mary watched the "Depictive" video, and Eric watched the "No Gesture" video. All five students were enrolled in first semester calculus at the time of their interviews. Students Ben and Mary reported having taken a first semester calculus course in the past, while Andrew and Lisa reported that they had not. Ben and Eric self-reported that they were international students.

We employed a thematic approach to the data set (Braun & Clarke, 2006). Each video was watched several times by each member of the research team who made notes about the students' problem solving activity, paying particular attention to the gestures being made. From these notes, it became evident that two of the participants were actively seeking to make connections between the new information that had been presented to them and their existing knowledge of functions and calculus. Hence, complete transcripts (all speech and gesture production) for Lisa and Mary were made and further analyzed to examine how they were assimilating the new ideas.

## Data

Ben, Andrew, and Eric all displayed superficial understandings of the lesson presented in the videos. Evidence of accommodation was present in the form of utterances referring to the instructor's words in the videos, but little evidence of assimilation was demonstrated by any of these three subjects. Most of the actions performed by these subjects during their solution attempts were simply appealing to memorized rules they had learned either from the video or from some other source; little evidence of true understanding manifested. For these reasons, we focus on the results of interviewing Lisa and Mary, which we present here as case studies.

### Lisa

Lisa began the explanation of her solution to Problem 1 with several pointing gestures referring to her written perimeter and area formulas. When asked to explain how she knew she had the correct formulas for area and perimeter, she initially stated that "Teachers have beat those into my brain," and "That's just what I've always been told." However, when asked if these formulas have meaning for her, Lisa immediately explained that the perimeter is the sum of the lengths of the sides of the rectangle she had drawn, and that the area was the product of the side lengths, pointing to the relevant sides of her figure as she spoke about them. She elaborated that she thinks of the area as “tiny squares everywhere, so how many squares on this side [points to one side of rectangle] times how many squares on this side [points to a perpendicular side] will give you how many squares in total [mimes shading in the figure].”

To determine the length needed to obtain the maximum area, Lisa found the derivative of her area function and set it equal to zero. When asked why she chose to do that, Lisa explained that, on the graph of the area function, this would be where the graph changed from increasing to decreasing. While explaining this, Lisa traced a “concave down” shape in the air. While continuing her explanation, Lisa also drew a rough sketch of the curve she pictured in her mind, then also quickly sketched the graph of the derivative of this curve to explain that a maximum would occur when the values of the derivative changed from positive to negative.

Before beginning Problem 2, Lisa indicated that she did not know the formula for the surface area of the shape in question. However, she began to think aloud as she reasoned through what the formula should be. She vocalized that the figure had five faces, and initially suggested “5 times length times width.” At the interviewer’s prompt, Lisa drew and labeled a picture of the figure. She stated that surface area is “kinda the area-perimeter of everything on the outside… all the material on the outside.” When asked to elaborate on what she meant by the term *area-perimeter*, she explained, “It’s kinda both in a way. ‘Cause it’s all around [moves her hand in a circle around an imaginary object] the object, but it’s the area of each face as well [mimes touching the five faces of the box by holding her hands in parallel then rotating them 90 degrees to indicate the next pair of parallel sides].” She was then able to determine the area formula for each face, pointing to the appropriate faces on her figure as she did so, and add them together to obtain the surface area formula for her figure.

Continuing, Lisa used the volume formula and the constraint that the base length is twice the base width to rewrite her surface area function in terms of one variable. After staring quietly at her formula for about 30 seconds, Lisa said, “Well, I’m thinking about taking the derivative of this, but I don’t want to.” She explained that she didn’t like the quotient rule, but then she acknowledged that she could avoid using it by rewriting her formula using negative exponents, and she then differentiated her function:

> *Interviewer:* And why do we take the derivative?
>
> *Lisa:* So that we can find the critical point, which will be our, hopefully our minimum. I mean, it’ll likely be concave up, but…
>
> *Interviewer:* OK, and so, you’re hoping that it’s concave up. Why are you hoping that it’s concave up?
>
> *Lisa:* If it’s concave up, then it will be like a U [puts her thumbs together and extends her index fingers into a U-shape], and then it’ll have a minimum value [puts a fist at the bottom of the U-shape and points at it with her other hand] at the critical point where the slope is zero.

She then set her derivative equal to zero and found the critical number to be the cube root of 24. Initially, she wrote “$\pm\sqrt[3]{24}$,” but she decided that her answer must be positive because it defines a width. After another moment, however, she concluded that her solution must be positive, as 24 is positive, and “negative times negative times negative would be a negative number.”

When asked how she knew that the dimensions she obtained would minimize the surface area, Lisa answered, “’Cause I’m gonna take the second derivative, and if it’s a positive number then I’ll know it’s concave up.” She talked briefly about testing the function for points of inflection, but then decided against it. She then spoke about substituting a value into the second derivative, but she was unsure what value to use. She concluded that she could choose any number in the domain, so she chose to substitute $w=1$ into the second derivative. Lisa seemed unconvinced by her result, so she then used the first derivative test to confirm that the graph of her surface area function was concave up at the critical number. With this information in mind, she then returned to the second derivative and substituted $w=3$ to again confirm that the graph was concave up, as she expected it to be.

Throughout her interview, Lisa utilized a variety of gestures to express various notions relating to work she had done on her paper. When explaining written parts of her work, she used pointing to refer to the relevant portions of her work; when describing more general concepts like slope or maximum and minimum, she often used depictive gestures, as exemplified in the following excerpt:

> *Lisa*: The derivative will be zero when L is 21.
> *Interviewer*: OK, so why, why do we care about that?
> *Lisa*: Um, because that is, uh, that'll be the critical point. So that'll be [traces a concave down arc in the air] when it's changing directions from, um, from increasing to decreasing on the graph, and we care about that because we want the maximum area.

In addition, she frequently referenced the notes she had taken while watching the video any time she felt unsure as to how to proceed.

**Mary**

When Mary began working through Problem 1, she generated the correct formulas for area and perimeter. When asked how she knew that her formulas were correct, she replied in similar fashion to Lisa, initially citing memorization but elaborating a clear conceptual understanding via similar gestures to those used by Lisa.

Throughout solving Problem 1, Mary referred to her notes on the video to confirm her procedure as she worked. Mary expressed frankly that she was unsure what the significance of the second derivative was in solving these problems. Despite this, Mary worked quickly through most of Problem 1, and used her notes on the second derivative to confirm that her solution would yield the maximum area for the rectangle. Like Lisa, Mary utilized a combination of pointing and depictive gestures.

Mary began Problem 2 by sketching a box with an open top and labeling its dimensions with *l*, *w*, and *h*. Similar to the other interview subjects, Mary said "I don't even know the surface area of a cube to be perfectly honest." However, she knew that the surface area represents "the area of, like, all of the outside… let's say rectangles, added together [uses her hands to depict the parallel pairs of faces of a box]." After a brief conversation about this idea, Mary was able to determine the correct formula for the surface area of her open-topped box. She then proceeded to solve Problem 2 using the same method she employed to solve Problem 1. She continued to express doubt about her use of the second derivative, but she followed the rules stated in the video.

Prior to stating that "If it is a max or a min, then the derivative has to be zero," Mary said that if the derivative is equal to zero, "it either has to be a max or a min… well – no, not necessarily." Following the exchange in the previous paragraph, the interviewer returned to this comment to ask Mary how she convinced herself that this original statement was false. She answered, "I was just, um… Because, like, in cubic functions, [sketches a graph similar to that of $y=x^3$] um, you can have a point where the derivative right here [draws a point at the point of inflection] would equal zero, but it's not necessarily an absolute max or a min."

At the conclusion of the interview, Mary asked if it was necessary to test the endpoints in addition to the critical number by substituting them into the original surface area equation, apparently thinking about the test for absolute extrema on a closed interval (the domains for both Problems 1 and 2 are open intervals). However, she correctly identified that, if she were to do this, the input value which yielded the smallest surface area would give the location of the function's minimum.

## Discussion and Conclusions

In response to our research question, we did not observe that students consistently mimicked the instructor's gestures when solving similar problems. However, we argue that assimilation of new information increased the frequency of gesture production and increased subjects' degrees of success in solving the problems in this study. In the data, it is clear that Lisa and Mary performed a significant number of gestures, while Ben, Andrew, and Eric did not. We now discuss evidence of Lisa and Mary's assimilation and its correlation to their success on Problems 1 and 2.

The new information presented in the videos in this study is twofold: the context of the problems (optimization), and the use of the second derivative in the determination of extrema. Data collected from interviews with Ben, Andrew, and Eric are minimally discussed here, as we observed little evidence of assimilation of this new information. These subjects showed some evidence of accommodation. For Ben, Andrew, and Eric, the use of the second derivative in this way appeared to be detached from their prior knowledge of calculus. Rather than assimilating this knowledge, they appear to have simply accommodated it by adding it to a collection of disconnected procedures. For example, Eric initially claimed that both the first and second derivative tests were necessary to confirm the location of the maximum in Problem 1. However, he concluded that the second derivative test alone was sufficient, as the instructor in the video had said this, and because "The second derivative is negative two; negative number is concave down… uh, concave down. The first derivative is equals to a positive number, so that's why we got, like, a maximum value." Eric's response is typical of these three students. While often incorrect, each of these students used what they believed to be appropriate rules in an algorithmic manner with no evidence of attempting to make connections between concepts. These three subjects were largely unsuccessful in their solutions of both Problems 1 and 2. Correspondingly, Ben, Andrew, and Eric gestured minimally when discussing their solutions.

Lisa and Mary both demonstrated significant evidence of assimilation, and we observed significantly more gesture from these subjects, so we focus on the results of their interviews. First, we note that Lisa and Mary were the only two subjects who were able to articulate a conceptual understanding of the perimeter and the area formulas. From their oral descriptions, one has the sense that both of them have a clear concept image of perimeter and area, at least for rectangles. This knowledge assisted them in determining the formula for the surface area of a box. By using gesture to help them visualize the sides of the box in a manner similar to perimeter, both were able to construct an appropriate formula.

Furthermore, both subjects demonstrated a rich concept image of the first derivative as it relates to maxima and minima. Lisa's explanation in Problem 1 for setting the first derivative equal to zero rested on the idea that the function should be increasing to the left and decreasing to the right of this point, and her gesture of drawing a concave down arc in the air is further evidence of her understanding. Moreover, without prompting, she was able to quickly sketch an example curve and its first derivative to support her claim that the derivative should be equal to zero. Mary's concept image of the first derivative as it relates to maxima and minima contains counterexamples to erroneous claims. Solving Problem 2, when trying to explain why she set the first derivative equal to zero, she said that if the first derivative is equal to zero, then "it either has to be a max or a min…" but quickly corrected herself, as she appeared to have internally convinced herself that this statement was false. When probed about this later, Mary was able to provide the example $y=x^3$, a function which she explained contains a point where the first derivative is equal to zero but that point is not an extremum of the function. In these and other examples in the data, we see evidence of very detailed concept images of the first derivative.

The data suggests that Lisa readily assimilated the new information about the second derivative into her existing schema for finding maxima and minima. In solving Problem 2, Lisa expressed not only an intention to find the second derivative of her surface area function, but also her expectation that her calculation should yield a positive result: "If it's concave up, then it will be like a U [puts her thumbs together and extends her index fingers into a U-shape], and then it'll have a minimum value [puts a fist at the bottom of the U-shape and points at it with her other hand] at the critical point where the slope is zero."

Despite Mary's expressed lack of confidence in the use of the second derivative to solve these problems, she tried to use this method. When she did so, her comments reflected an internal struggle in which she sought to reconcile this new knowledge with her existing schema for extrema: "this is… where she took the second derivative, which I didn't really understand the purpose, but… [writing] So A double-prime of l is 2… which means that it's going to be a min… at 2. Um… [looking at her work] … we're trying to maximize the area, yeah, I don't know. This is where I get a little confused," and later, "Is it because of taking the second derivative and getting that max that you know that those are the dimensions that give you the largest possible area?" Though it doesn't appear that Mary had fully assimilated this use of the second derivative during her interview, there is evidence to suggest that she was making a concerted effort to do so.

Ben, Andrew, and Eric showed little evidence of assimilation; rather, we observe only the most basic accommodation. They appear to remember snippets from the videos they watched, but none of them appears to have a complete picture. Of these three students, only one of them obtained the correct solution to Problem 1 via a logically valid procedure, and none of these students obtained a correct solution for Problem 2. None of these students attempted to justify their solution without being prompted to do so, and none of them provided an accurate explanation for how to do so. Lisa and Mary both demonstrated evidence of at least an attempt at assimilation, if not success. Not only were these two the only subjects to obtain complete solutions to both Problems 1 and 2, but they were also the only subjects to attempt to justify their solutions without prompting. They were the only subjects to use logically sound reasoning about the second derivative in their justifications.

The results of this study point to frequent gesture use as a potential indicator of assimilation of knowledge. Future research might investigate: Does student gesture use facilitate assimilation, or might it simply indicate that assimilation has occurred? More research needs to be done to better understand the role gesture plays in assimilation of new concepts.

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# The Next Time Around: Shifts in Argumentation in Initial and Subsequent Implementations of Inquiry-Oriented Instructional Materials

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*Considerable learning is entailed in adopting an inquiry-oriented approach to teaching a class. In this analysis, we examine classroom video data of three instructors' initial implementation of an inquiry-oriented instructional unit and their implementation of the same unit one year later. We document consistent increases in instances of eliciting and building on student contributions across tasks and instructors, and use Toulmin's argumentation scheme to offer an illustration of how classroom discussions became more mathematically robust and student-centered from initial to subsequent implementations. Implications for instructor learning are discussed.*

*Key words:* inquiry-oriented instruction, instructor learning, instructional practice

Enrollments in science, technology, engineering, and mathematics (STEM) programs in the United States must grow to meet projected workforce demands in coming years (PCAST, 2012). Following a growing body of research documenting the positive outcomes related to student-centered approaches to instruction in undergraduate STEM (e.g. Freeman et al., 2014), there is increased institutional and financial support for initiatives that promote this kind of teaching. Student-centered approaches range widely, from approaches that provide opportunities for students to practice things demonstrated by their instructor with groups of peers during class time, to inquiry-oriented approaches that aim to provide students with opportunities to participate in the reinvention of important mathematical ideas by working with peers to solve non-standard problems with many possible solution paths. Inquiry-oriented approaches are instructionally complex in that as students are inquiring into the mathematics, instructors inquire into students' mathematical thinking so it can be leveraged as a resource for moving forward the development of the class's mathematics (Kwon & Rasmussen, 2007).

The difficulties experienced by instructors attempting to implement research-based, inquiry-oriented instructional materials developed by others have been documented to include struggles in making sense of and building on student reasoning (Johnson & Larsen, 2012; Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007). While these findings suggest it is challenging to teach in an inquiry-oriented way for the first time, there is little work at the undergraduate level that examines what one learns as a result of teaching in this way. In this analysis, we draw on video data of three instructors' initial implementation of an inquiry-oriented instructional unit in linear algebra and their implementation of that same unit one year later. The research question is: How does instructors' facilitation of whole-class discussions shift from initial to subsequent implementations of inquiry-oriented instructional materials?

## Literature Review & Theoretical Framework

To conceptualize the kind of knowledge needed for teaching mathematics, Hill, Ball, and Schilling (2008) developed a model of mathematical knowledge for teaching (MKT) which is split into two major domains: subject matter knowledge (SMK) and pedagogical content knowledge (PCK). This distinction builds on Shulman's (1986) argument that there is a distinction between knowledge of mathematics and the specific knowledge about mathematics that is needed to teach it effectively. In this work, we are particularly interested in how PCK might develop as a result of implementing inquiry-oriented instructional materials so we focus on that part of Hill and colleagues' framework. Hill et al. (2008) divide PCK into three subdomains. First, Knowledge of Content and Students (KCS) refers to the knowledge a teacher has about their students' prior knowledge of specific content and how student learn that content. Second, Knowledge of Content and Teaching (KCT) has to do with instructional decisions that require "coordination between the mathematics at stake and the instructional options and purposes at play" (Ball, Thames, & Phelps, 2008, p. 401). Third, Knowledge of Curriculum (KOC) refers to what teachers know about how ideas build in the context of a particular set of curricular materials.

At the elementary level, Remillard (2000) found that instructors can learn from curricular materials when materials focus on mathematical problem solving and analysis of student reasoning. Sherin (2002) conducted an analysis of two high school algebra teachers' work with a reform-oriented unit on linear functions, noting three kinds of learning: their subject matter knowledge, their views of curricular materials, and their ideas about student reasoning. However, little work at the undergraduate level has explored what teachers learn by engaging in inquiry-oriented approaches to instruction. Inquiry-oriented approaches require careful attention to student reasoning as well as skill at facilitating classroom discussion and argumentation, so we seek to document teacher learning by examining shifts in instructional practice as evidenced by shifts in classroom mathematical argumentation.

We draw on Toulmin's (2003) model of argumentation that was originally used to examine how individuals supported claims in front of an audience. Krummheuer (1995) extended the model to examine mathematical argumentation in classroom discussions that involved input from multiple individuals. Others have used this framework as a tool of analysis for both individual mathematical argumentation (e.g. Wawro, 2015), and collective mathematical discussion (Ramussen, Wawro, & Zandhieh, 2014; Conner, Singletary, Smith, Wagner, & Francisco, 2014). In this study, we draw on four core components of Toulmin's argumentation model: claims, data, warrants, and backings. Wawro describes a claim as "the conclusion that is being justified" whereas data is the evidence that supports the claim (Wawro, 2014, p. 320). Warrants are "statements that connect data with claims" (Conner et al., 2014, p. 404) or alternatively "clarification [statements] that connects the data to the claim" (Rasmussen et al., 2015, p. 263). On occasions when a backing is given, it can be defined as the support that gives a warrant authority (Rasmussen et al., 2015). Claims and data can also be a pair; once a claim has been established in a discussion, it can be used as data for a subsequent argument (Conner et al., 2014).

## Study Design (Study Context, Participants, Data Sources, Methods of Analysis)

This work is part of a broader NSF-funded project aimed at developing shareable, research-based resources for instructors interested in using teaching inquiry-oriented linear algebra. The primary data used in this analysis consists of video-recordings of three instructors who implemented a 4-6 day instructional unit on span and linear independence two years in a row. The instructional approach is detailed in Wawro, Rasmussen, Zandieh, Sweeney, & Larson (2012). The instructional sequence consists of four tasks, starting in a context where students are given two "modes" of transportation (a magic carpet and a hoverboard that can travel in particular directions that are represented as vectors) and start at home (the origin). Students work to figure out if they can reach a particular location (to introduce ideas related to linear combinations of vectors), determine if there is anywhere they can't reach (to introduce span), and explore when they can take "non-trivial" journeys that start and end at home (to introduce linear in/dependence). In the final task, which we examine in greater detail, students work to generate examples of linearly dependent and independent sets of specified numbers of vectors in $R^2$ and $R^3$, as well as generalizations that emerged from their efforts to generate these examples.

The instructors participating in this study were 'best case' implementers in many ways: they all expressed interest in implementing the materials and all were situated in departments that were supportive of these efforts. The instructors represent a variety of institutional contexts including two small, private teaching-focused institutions (one religious, one not) and one large research institution. One instructor was a mathematician interested in RUME research, one was a RUME researcher, and one was teaching faculty in a mathematics department at a large research-focused institution. Class sizes ranged from 8-35 students.

Our analysis had four phases: content logging, development of codes for whole class discussions, coding of whole class discussions, and a comparative case study. We began distilling the data by generating a content log for each day of classroom instruction that was video recorded as part of the instructional unit for each instructor for both the first and second year of implementation. The content log was organized in four columns: time stamp and discourse structure (whole group, small group), key events, new language or notation introduced, and other notes. A new row was created when there was a change of discourse structure or shift in topic.

In our second phase of analysis, we drew on data from the analysis of the first year's data to identify four levels of eliciting and building on student ideas: 1. Getting students to talk, 2. Getting students to explain, 3. Using student ideas to explain or formalize, and 4. Using student ideas as the basis for a new mathematical question or task. In our third phase of analysis, two coders separately coded each whole class discussion according to the highest level of eliciting and building on student ideas that was observed. There was difficulty coming to agreement about when one whole class discussion ended and another began based on the criteria of "topical shifts," but by aggregating all scores to the "maximum" score observed in all whole class discussions that took place after students worked on each task in the instructional sequence, agreement was reached.

We noted that in the second year implementation, whole class discussions felt "smoother" in that they seemed to have a clearer mathematical direction, but this distinction was difficult to operationalize. In order to better understand shifts in discussion from the first to the second implementation, we decided to closely examine the mathematics that emerged in whole class discussions following the final task in the sequence in the first and second year. We selected mathematically analogous ten-minute segments from the same instructor, transcribed them, and used Toulmin's (2003) argumentation model to examine the mathematical arguments that emerged, who contributed what to these arguments, and the role of the instructor in the construction of these arguments.

For our final phase of analysis, we transcribed the selected segments, noted important gestures, such as writing on the board or pointing to work, and identified the role group of the speaker (teacher or student). We analyzed the transcripts to identify core claims being argued and from there identified the data and warrants that supported these claims. When they occurred, backings were also noted. Argument components were numbered according to the order in which they occurred in the transcript. Once we had identified and numbered the components of argumentation, we constructed diagrams of the mathematical discussion. We adapted Conner et al.'s (2014) convention of using solid and dotted line in these diagrams to distinguish the primary contributor of the statement. The Toulmin mappings allowed us to count the types of contributions by role to quantify shifts in argumentation relative to who made contributions.

## Findings

To show how instructors' facilitation of whole class discussions shifted from their initial implementation of inquiry-oriented instructional materials, we leverage our coding scheme for eliciting and building on student reasoning to show consistent, quantifiable growth. Table 1 shows the maximum score each instructor received in all whole class discussion after students had worked in small groups on each of task 1 through 4 in year 1 and year 2. Every instructor elicited and built on student reasoning in whole class discussion as much or more in the second year's implementation. However, this does not capture the way in which year 2 discussions seemed to generally hold a clearer sense of direction and seem more mathematically rich while also building on student ideas. To explore this, we examine two mathematically similar ten-minute segments of discussion for instructor A in year 1 and year 2 through a Toulmin analysis.

Table 1: Instructors' Eliciting and Building on Student Reasoning

| | Year 1 | | | | | Year 2 | | | | | Year 1 to 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Task:** | **1** | **2** | **3** | **4** | **mean** | **1** | **2** | **3** | **4** | **mean** | **Mean change** |
| Instr A | 3 | 4 | 2 | 3 | 3 | 3 | 4 | 3 | 4 | 3.5 | +0.5 |
| Instr B | 3 | 3 | 3 | 2 | 2.75 | 3 | 3 | 4 | 4 | 3.5 | +0.75 |
| Instr C | 2 | 2 | 3 | 2 | 2.25 | 4 | 3 | 3 | 3 | 3.25 | +1.0 |
| **Mean** | **2.67** | **3** | **2.67** | **2.33** | **2.67** | **3.33** | **3.33** | **3.33** | **3.67** | **3.42** | **+0.75** |

### Toulmin Analysis of Selected Exchange: Year 1

Figure 1 shows a Toulmin mapping of a whole group discussion in instructor A's class in year 1 related to the claim that any three vectors in $R^2$ must form a linearly dependent set. The class had already concluded (in previous discussion) that two vectors in $R^2$ are linearly dependent when they are scalar multiples of one another or point in the same direction (note the zero vector case had not been teased out). The instructor then pointed out one group's example of a linearly dependent set of three vectors in $R^2$ (marked as Claim 1). Upon the teacher's request, students offered somewhat vague data (2) that alluded to scaling vectors to get a non-trivial solution (without specifying *what* equation would have such a non-trivial solution), supported by the warrant (3) that this was possible because the three vectors were not parallel.

The instructor built on students' ideas by writing the equation $a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$ on the board and asking "how are you sure of that without finding the coefficients?" (Data 4). When the students seemed unsure of how to answer, the instructor asked, "What if we just had these two? [*pointing to two of the three vectors in the original set of vectors*] Is this an independent set?" Students agreed with a choral response (Warrant 5). The instructor continued, "they're not multiples of each other, easy to check with two vectors, throw the third in, now I might be able to write a linear combination of two that gets me the third, that's our triangle… um pretend this is your magic carpet and your hover board cause I claim this is no different from that situation. They look different directions, they're not in the same line, they're an independent set. Where can you get on your magic carpet and hover board?" Students chorally responded, "Everywhere"

(Data/Claim 6 and Warrant 5). The instructor then rearranged the equation in Data (4) to point out that the homogeneous vector equation corresponding to the set of vectors proposed by the students has a non-trivial solution, so that the set is linearly dependent by definition (Warrant 7 and Backing 8).

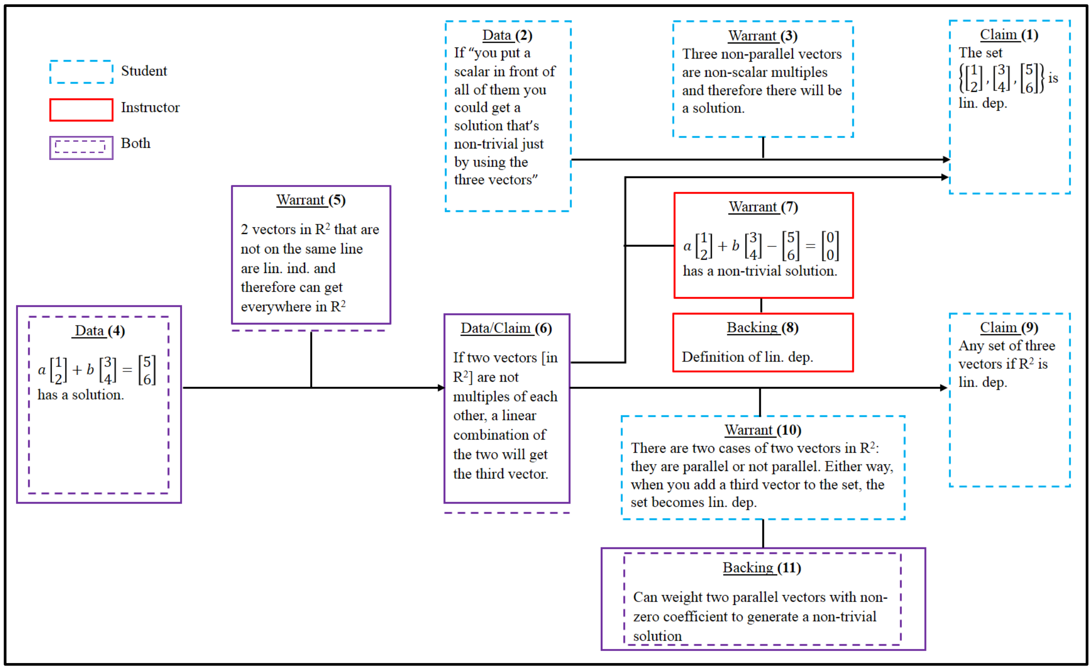


*Figure 1*: Year 1 Toulmin Mapping of Selected Episode for Instructor A

The instructor then asked, "Say I didn't take (1,2) and (3,4) say I take any two vectors that didn't lie on the same line, I throw a vector into that set. What's true about it?" A student suggested, "You can get back to the start… because you can get to that point using those two non-parallel vectors." After the instructor rephrased this idea in terms of span, a student observed, "We just showed that if we have the two [vectors] that are not parallel and then a third one, we can get anywhere. So if we have two that are parallel, then we can just go out one and come back the other. So either they're parallel or not parallel; there's no third case. So in every case they're going to be dependent" (Claim 9 and Warrant 10). The instructor then acknowledged that she set students up to make this observation, and a student asked, "If we have two parallel vectors and the third one, are we allowed to throw a zero on that third one?" The instructor confirmed that that is still considered a non-trivial solution and offered an example (backing 11) linking this case to the definition.

Overall, the two core claims in this exchange (1 and 9) came from students, though the instructor offered significant support that build toward the formulation of claim 9. Specifically, an incomplete data and warrant (2 and 3) were initially offered by students. The instructor added information to these by formulating them as an equation (data 4). She then built those into a new claim (6) and warrant (5), with students contributing to these in the form of choral responses. The instructor then used claim 6 as data to support claim 1, also providing the warrant (7) and backing (8). As such, we argue that the instructor provided the majority of the justification for claim 1 that built on student ideas while making clear efforts to involve students in the development of that justification. On the other hand, claim 9 and warrant 10 were fully articulated by a student after an initial question from the instructor that the instructor later explicitly acknowledged was intended to lead students in this direction.

**Toulmin Analysis of Selected Exchange: Year 2**

In year 2, the conclusion that three vectors in $R^2$ cannot form a linearly independent set was arrived at rather differently (see Figure 2). The class had previously established, as suggested by a student, that two vectors in $R^2$ are linearly dependent when "they are scalar multiples of one another." The discussion began with one group's (incorrect) claim (1) that the set of vectors $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}7\\2\end{bmatrix},\begin{bmatrix}8\\5\end{bmatrix}\right\}$ is linearly independent[1]. The instructor did not correct the students but asked them how they checked they were independent. One student from the group explained, "you can't add vector one and vector two to create…vector three" (marked as Data 2). The instructor then offered what we refer to as an "empathetic" warrant (3): "Because you can't generate uh, by sort of observation, a dependence relation between those, they're independent." Another student then (correctly) noted that any pair of vectors from this set "can reach anywhere in the plane…" (Claim 4) because they are linearly independent (Data 5). The instructor provided the warrant (6) that any two of these vectors are not multiples of each other and "point in different directions". The student continued, noting that you "can reach eight, five" (Warrant 7), so the set is linearly dependent (Claim 8).

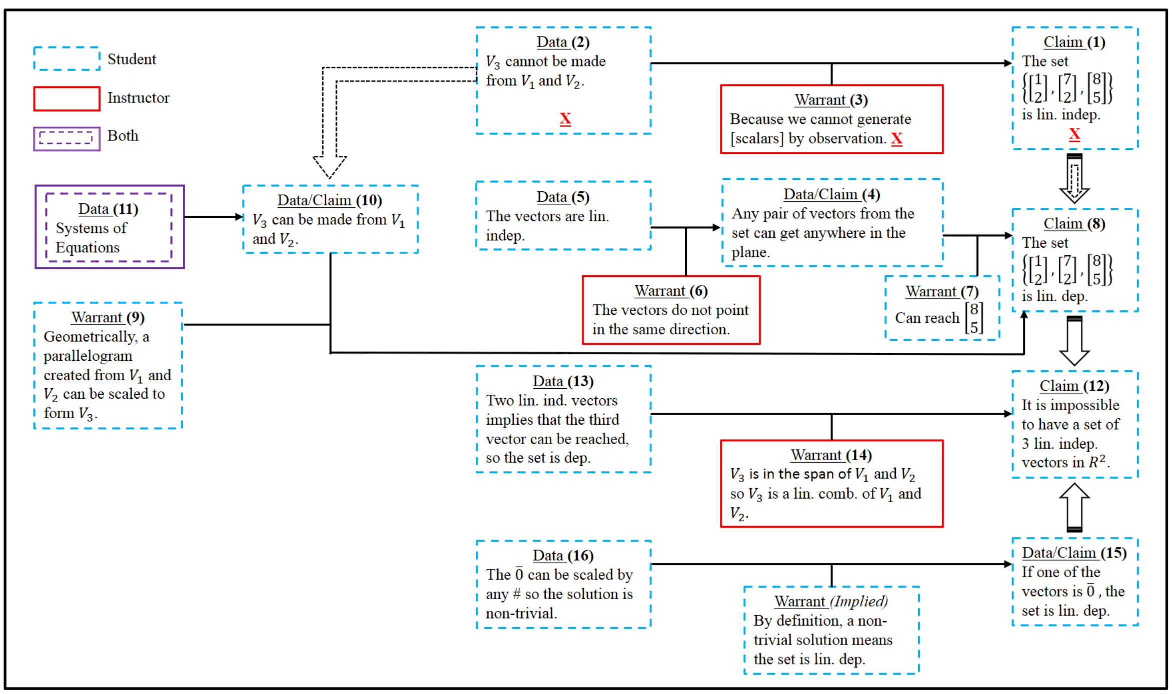


*Figure 2*: Year 2 Toulmin Mapping of Selected Episode for Instructor A

The instructor then revisited the question of how one can tell if the third vector $\begin{bmatrix}8\\5\end{bmatrix}$ can be made from a combination of the other two vectors. A student rectified the previous reasoning from Data 2, arguing that a parallelogram created from the other two vectors could be scaled to

[1] In the Toulmin mapping, statements that are not mathematically correct are marked with an X. Incorrect statements that resolve to correct statements are marked with dotted arrows.

make the third vector (Warrant 9 and Data/Claim 10). Another student suggested you could make a system of equations (Data 11), which the instructor elaborated somewhat extensively.

The instructor then said, "So this isn't an example of an independent set. Can we come up with three [vectors] that are? Who came up with a different example? Back corner, you guys have something written for that." A student claimed that it's "impossible" to create a set of three linearly independent vectors in $R^2$ (marked as Claim 12) because two linearly independent vectors in $R^2$ "can reach wherever the third is going which by definition makes it to become dependent" (marked as Data 13). The instructor elaborated, "Whatever third vector you pick it's already in the span of these. It's a linear combination of these. It's dependent on the previous two" (Warrant 14).

A student-to-student exchange about making the third vector the zero vector follows. A student suggested that if one vector is the zero vector, then the set is linearly dependent (Data/Claim 15), with another student noting "you could scale the zero vector by any number you want, then it's not trivial" (Data 16). The warrant remained implicit, as the instructor confirmed the second student's reasoning that "you can always put something non-trivial there" and ended the class period.

**Comparing Argumentation in Year 1 to Year 2**

In comparing these two exchanges, we noted a shift in the interconnectedness of mathematical ideas discussed, as well as a shift toward students taking authority and contributing more claims and data in year 2. Table 2 identifies how many claims, data, warrants, data/claims, and backings were articulated primarily by the teacher, and primarily by students.

Table 2: Count of Components of Arguments by Primary Contributor

| | Claim | | Data | | Warrant | | Data/Claim | | Backing | |
|---|---|---|---|---|---|---|---|---|---|---|
| | T | S | T | S | T | S | T | S | T | S |
| Y1 | 0 | 2 | 0 | 1 | 2 | 2 | 1 | 0 | 2 | 0 |
| Y2 | 0 | 4 | 1 | 4 | 3 | 3 | 0 | 2 | 0 | 0 |

We argue that in year 1, the teacher elicited correct student ideas, built on, and reshaped them so as to achieve her mathematical goals. In order to accomplish this, the teacher ended up formulating a larger portion of the mathematical argumentation as compared with the following year. In year 2, the instructor made space for students to explore their ideas in a supported way, even when those were incorrect. By engaging an incorrect response as a starting point, and allowing students to articulate and explore the ambiguity and their own uncertainty (supported by both the instructor's empathetic warrant and her invitation to discuss their uncertainty), the students were able to correct their reasoning and provide a larger portion of the mathematical argumentation.

## Discussion

Our analysis offers insight into the shifts that take place between initial and subsequent implementations of inquiry-oriented instructional sequences among instructors with interest in implementing (and some access to conversations with curriculum developers). Our analysis suggests that teachers' knowledge of curriculum, content and students, and content and teaching all increased. Our study also functions to contribute a potential methodological approach of using Toulmin's argumentation model for investigating shifts in instructional practice, an important form of instructor learning.

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# The Generation and Use of Examples in Calculus Classrooms

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*In this paper, we analyze video data of five instructors teaching the Mean Value Theorem in a first-semester calculus course. Throughout the lessons, graphical examples were provided by the instructors and/or the students of functions that satisfied or did not satisfy the conclusion of the Mean Value Theorem. Through the use of thematic analysis, we identified four themes related to emergence and use of examples: who generated the example, who evaluated the example, for what purpose the example was used, and the richness of the example. We emphasize that instruction that leverages student generated examples can provide a great deal of richness in a mathematics lesson and create opportunities to engage students in authentic mathematical activity. This work contributes to an evolving notion of what is entailed in students' active learning of mathematics and the role of the instructor.*

*Keywords:* Example space, Calculus, Mean Value Theorem, Active Learning, Graphical Representations

Although educational research has shown that students develop deeper understanding of mathematics in classrooms where they are actively engaged, lecture is still the primary (and often only) mode of instruction in many collegiate level mathematics courses (Freeman et al., 2014). In this project, we studied five instructors of first semester calculus who were committed to increasing the amount of active learning in their classes. We analyze data from instruction covering the Mean Value Theorem, which provided many graphical examples. While many themes emerged from this data, in this paper we describe instructors' use of graphical examples in covering the Mean Value Theorem. Specifically, we seek to answer the research questions: In what ways are examples generated and used in instruction? What role do these examples play in contributing to an active learning environment?

## Literature Review

While many students view examples provided by teachers and texts as templates for solving homework exercises (Lithner, 2003, 2004), examples can play an important role in developing understanding of concepts. Watson and Mason (2005) introduced the notion of learner generated examples (LGEs) and advocated for their power as a tool for deeper learning. Mason and Watson (2008) elaborated:

> ...when a teacher offers an example and works it through, it is the teacher's example. Learners mostly assent to what is asserted. … When learners construct their own examples, they take a completely different stance towards the concept. They 'assert'; they actively seek to make sense of underlying relationships, properties, and structure which form the substance of the theorem or concept. (p. 200)

Mason and Watson (2008) noted "Learners who are encouraged to be creative and to exercise choice respond by becoming more committed to understanding rather than merely automating behavioural practices" (p. 192). To promote creativity in LGEs, students should be encouraged to consider variation. That is, students need to be comfortable asking and exploring questions such as: “What can vary in this problem?” and “To what extent can this aspect of the problem vary?” Watson and Shipman (2008) note that “if students generate examples, reflection on those examples could, through perceiving the effects of the variations they have made, lead to awareness of underlying mathematical structure. ‘Structure’ here means how elements and properties of mathematical expressions are related to each other.” (p. 98) They further indicated that directed example generation, rather than “directionless exploration,” can be a good way to begin understanding concepts.

Through LGEs, a personal example space (PES) is constructed and developed. A PES is defined to be the set of available examples and methods of example construction a learner has at their disposal for solving problems. Sinclair, Watson, Zazkis, and Mason (2011) examined how personal example spaces are structured, paying attention to the varying degrees of “connectedness” such PESs may have. The more connected one’s example space, the greater the likelihood of having a stronger understanding of the concept. They indicate that slightly different prompts may trigger the use of different examples.

## Theoretical Perspective

We frame our work considering active learning and the role of examples in the undergraduate mathematics classroom from a communities of practice perspective (Wenger, 1998). The mathematics classroom, as a community, should be a microcosm of the broader mathematics community—engaging in similar disciplinary practices such as proof and justification, seeing structure in mathematics, and the collaborative pursuit of mathematical discovery. What makes the mathematics classroom, whether in K-12 or at the undergraduate level, different from the academic discipline of mathematics is that most of the participants (students) are often newcomers to the taken-as-shared practices, norms, and habits of mind of doing mathematics. However, the classroom community does not (or at least should not) exist in a vacuum—the goal should not only be for students to become more central participants in the classroom (for the sake of the classroom itself) but also in the broader discipline of mathematics, specifically the ways of thinking and reasoning about and communicating with mathematics. Viewing the mathematics classroom as a community of practice, as defined by Wenger (1998), has implications for considering what learning consists of and the role the instructor plays in supporting learning. For our purposes, this perspective also helps clarify some of the structural elements and characteristics of supporting “active learning” in the undergraduate mathematics classroom.

As a social theory of learning, learning from the communities of practice perspective integrates four components: meaning, practice, community, and identity (Wenger, 1998). A productive mathematics classroom is one in which students have the opportunity to learn mathematics. From this communities of practice perspective, this means that students have opportunities to: experience meaningful ways of doing and constructing mathematics (meaning), to then engage in those authentic practices (practice), to be positioned in the classroom community as competent participants in mathematical activity (community), and to come to see themselves (and be seen by others) as one who does mathematics (identity). This multi-faceted process by which newcomers learn and become included in a community of practice is referred to by Lave and Wenger (1991) as “legitimate peripheral participation.” This raises questions about conceptions of active learning that only focus on “participation”—such as opportunities for working in small groups or monitoring air time in whole group contexts. The substance of that participation and how students are ultimately positioned in the midst of that participation is equally important. A focus only on participation may support students coming into the community of the classroom from a purely social standpoint, but be divorced from engaging meaningfully in mathematical ways of working and from being positioned as someone whose ideas are worthwhile, worth building on, and contributing to a collective effort. In our work, we have come to focus on students' opportunities to reason about, offer, and make connections among mathematical examples, and how students have a clear sense of the way in which examples serve a collective effort to build mathematical ideas.

## Methods

Subjects in this study were five instructors of first-semester calculus at a large public research university. One of the authors served as the coordinator of the course as well as one of the instructors in the data set. To help preserve confidentiality, we use the term *instructor* to describe the instructor of record of the course, regardless of whether the instructor was a tenured faculty member, a full-time teaching instructor, or a graduate student. We use the pronouns *she, her, and hers* to describe all five instructors, referred to in this paper as Instructor A, Instructor B, etc. All subjects consented to the study, and all but the author received a $500 stipend for their participation at the completion of each semester of the project. Additionally, students in each class signed a media release form granting permission to use their image or voices in our data.

During the first semester of the project, we videotaped class sessions of all five instructors, starting in week three of classes. All sessions that covered new material were recorded, but we did not record sessions when students were reviewing for an exam or taking a quiz or an exam. In each classroom, a video camera was placed in the back corner and was focused on the instructor during the class period. During the second semester, the same five instructors were video recorded when teaching two units, one on the Mean Value Theorem (MVT) which was not coordinated and one on definite integrals, which was highly coordinated. In this paper, we discuss data from the MVT during the second semester. We purposefully selected data from the uncoordinated sessions because this provides an authentic example of instruction in these classrooms without the influence of the coordinated lessons. Three of the instructors covered the material in one day of class, and two of the instructors used two partial days of class.

As part of a larger project, we used thematic analysis, which is a "method for identifying, analysing, and reporting patterns (themes) within data" (Braun & Clark, 2006, p. 79). We employed both theoretical and inductive thematic analysis. Theoretical thematic analysis is "driven by the researcher's theoretical or analytic interest in the area, and is thus more explicitly analyst-driven" (Braun & Clark, 2006, p. 84). Initially, our focus of the analysis was on ways in which active learning was being used in the classroom. As such, we were using theoretical thematic analysis to code for times when students were working in groups or were actively participating in doing mathematics. Moreover, the communities of practice perspective requires that we look not only at the ways in which students are participating, but in the ways that they were meaningfully engaging in mathematics. Thus, we employed theoretical thematic analysis to identify these instances. Additionally, we employed inductive thematic analysis (similar to Strauss and Corbin's (1998) grounded theory) to identify additional themes that were not driven by our own interests. Using both of these techniques, we found instructors' use of examples to be of particular significance. From this, we focused on instances of an example emerging across the five instructors' MVT lessons. Multiple passes through these instances yielded several themes regarding the generation and use of examples—both in isolation and in the context of the full instructional episode.

## Data

Recall that the Mean Value Theorem (MVT) says "If $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a\ b)$, then there exists a number $c$ in $(a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$" (Larson & Edwards, 2015). A special case of the MVT where $f(a) = f(b)$ is stated in Rolle's Theorem, resulting in the existence of a $c$ value where $f'(c) = 0$. In this section, we first provide a general overview of the five lessons. We then summarize the four themes centered around examples that emerged from our data. Finally, we provide a detailed description of two of the classrooms to illustrate these themes.

During the instruction on the MVT, only Instructor A required students to work in groups during the development of the MVT. Instructors B, C, and D asked their students to work in groups to solve problems related to the MVT after lecturing on the topic. The nature of the worksheets was practicing problems similar to what had been done by the instructor and did not introduce any new material. During all of the lectures, there were many times where the instructors asked students to participate in some way, usually by answering a simple question or verifying that they understood something that was said. Instructors B, C, and E chose to introduce Rolle's Theorem prior to the MVT, while Instructors A and D presented the MVT first, with Rolle's Theorem given as a special case of the MVT. In every lesson, at least five graphical examples were utilized.

### Who Generated an Example

The first theme evident in our data relates to *who* generated an example. In all of the classrooms, there were times during a lecture when the instructor would provide an example for the class and write it on the board. We will refer to these instances as *Instructor Generated Examples.* At other times, the instructor asked students to provide an example. In these instances, typically, one or two students provided a response, which tended to be a short one or two-word response. The instructor would then interpret the response and sketch a graph on the board. As such, we call these *Instructor Interpreted Examples.* For example, during Instructor B's lecture, the instructor asked the students if it was possible to draw a graph that was continuous but did not have a horizontal tangent. A student responded with "points," and the instructor drew a graph on the board that resembled $f(x) = |x|$. Often times, *Instructor Interpreted Examples* were in response to questions asked by the instructor that had a very small response space. By this, we mean that the set of possible correct answers is relatively small, and thus, one correct answer often suffices to move the lecture forward. Moreover, it often seemed that the instructors were expecting a specific response to these types of questions and, once the response was given, the instruction proceeded.

Finally, we discuss *Student Generated Examples,* which as the name indicates, are examples that are created by the students. These examples were typically given in response to questions that had a broader response space where there existed several possible correct answers. Most often, these examples were generated when students were given a prompt by the instructor, followed by time to work in groups or to work independently at their seats. For example, during Instructor A's lesson, students worked in groups to create several examples of graphs that did or did not satisfy a list of properties. The students then placed these examples on the board. However, we also saw one case of a *Student Generated Example* given during a lecture, when the instructor asked for an example, and a student responded with $y = x$. While this is still a short response, we claim that the instructor did not need to interpret the meaning of this example, and instead was able to sketch on the board the student's desired graph.

**Who Evaluated an Example**

A second theme centers around whether the instructor or the students were engaged in evaluating the validity of an example. When an example was presented (by a student or an instructor), it seemed to be assumed that if the instructor put the example on the board, then it was a valid example. One can certainly argue that students should always be evaluating the validity of the examples, and that no audible response from the students does not necessarily indicate that students did not do so. There were certainly times when the instructors asked the students, "Does this work?" or "Does this make sense?" However, in our data, we only saw one chunk of video when students audibly discussed whether or not a graph was a valid example. This happened in Instructor A's class. After the students put their own examples on the board, they were given an opportunity to critique each other's examples and argue whether or not they were valid. However, even in this case, the instructor settled the disagreement and explained why the graph was a valid example.

**For What Purpose the Example was Used**

We saw two main ways that an example was used. One was to demonstrate an idea or a property. These examples tended to be along the lines of "existence proofs" where one example was enough to demonstrate that something was possible. Another way an example was used was to build an idea and/or to have students discover a concept. In these cases, there seemed to be several examples that were generated, and connections were made across examples. Or, sometimes a specific example was used to address a common misconception. For example, students often mistakenly thought that a linear function did not satisfy the conclusion of the Mean Value Theorem, and both Instructors A and B used a linear example to address this misconception.

**The Richness of the Examples**

Finally, we note the importance of the richness of the examples that were used in a lesson. Here we consider first if there were any errors in the examples. Instructor D had multiple mathematical errors in her lesson. One small error occurred when a graph she provided did not pass the vertical line test, and thus did not satisfy the basic condition that $f(x)$ be a function. This particular error was not commented on in the class. We also evaluate here examples that may be correct, but perhaps are limited in scope. A deeper discussion of the theme of richness will be provided in the next section.

**Classroom Vignette: Instructor A**

We discuss Instructor A's classroom, as this lesson demonstrates all of the themes discussed previously. At the beginning of the instruction on the MVT, the graph in Figure 1 was provided to the students as an *Instructor Generated Example*. She then asked her students to tell her what a secant line was (several students responded), and she drew the secant line on the graph between the two endpoints. Next, she told her students to work at their seats to see if there was any place on the graph where there was a tangent line with the same slope as the secant line, and if so, to sketch the tangent line at that point on their own paper. For just over two minutes, the instructor walked around the room, looking at the work done by the students and clarifying directions for students who had questions. We noticed that it seemed to be expected that every student would participate. This is in contrast to the lectures of the other instructors, where one or two students would provide an answer, but the rest of the students would not actively contribute.

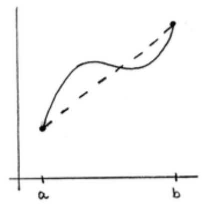


*Figure 1: Instructor Generated Initial Example*

After a brief class discussion about the previous graph, Instructor A told the class to work in groups to see if they could find examples of other graphs where there was or was not a tangent line with the same slope as the secant line between the two endpoints. She instructed her students by saying, "Your next job is to make sure you find some graphs that do have this property and some graphs that don't have the property." Students spent approximately 16 minutes working in groups to create several *Student Generated Examples* that satisfied the property and several that did not satisfy the property. At one point, the instructor put one of the student's examples on the board (a linear function) and told the class to make sure they discussed an example like this one, if they hadn't already done so. She did not tell them whether or not that graph satisfied the conditions, but expected the students to decide on their own.

After it was clear that every group had several *Student Generated Examples*, she instructed each group to send at least one person to the board to sketch an example of a graph that did not have this property (i.e. a graph where there was no tangent line with the same slope as the secant line between the endpoints). Nine graphs were drawn on the board by the students, and Instructor A added one more graph that was used by one of the groups, but was not the one they chose to put on the board. Thus, there were ten *Student Generated Examples* on the board, a few of which are shown in Figure 2. Next, the class was instructed to look at all of the graphs to see if there were any graphs that should not have been on the board, so in other words, to see if any of the graphs on the board had a place where the slope of the tangent line was equal to the slope of the secant line. This created an opportunity for the students to evaluate the validity of the examples.

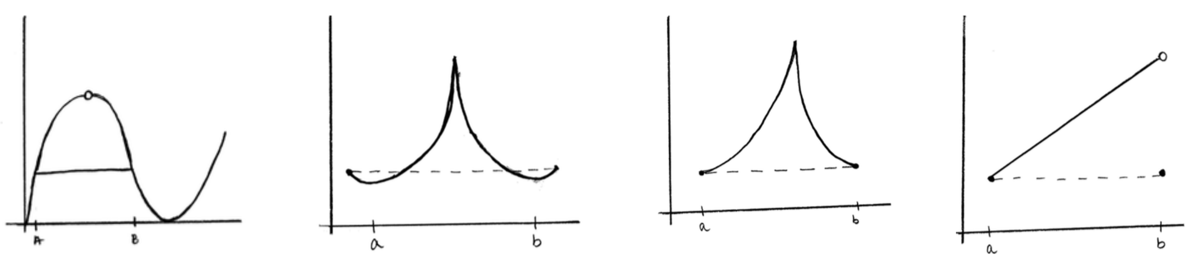

*Figure 2a* *Figure 2b* *Figure 2c* *Figure 2d*

*Figure 2: Student Generated Examples*

When discussing the *Student Generated Examples*, three interesting things happened. First, one student argued that the graph shown in Figure 2a was wrong because there is a place outside of the interval with a horizontal tangent line. The instructor clarified that the task was only to attend to whether or not the property held *on* the interval from *a* to *b*. Next, another student questioned the graph in Figure 2a because he recognized that even though the function was not defined at one point, it looked like the limit would still exist. At this point, the instructor led the class in a nice discussion about the definition of the derivative and why tangent lines do not exist at places where there is a removable discontinuity. Third, Instructor A pointed out that the graph shown in Figure 2b was not quite accurate, even though the students' intent was correct. She cautioned the students to be careful with their graphs and make sure that their examples clearly illustrated the intended properties, then modified the graph to form the example in Figure 2c.

Next, the instructor led the class in a discussion about what the ten graphs on the board had in common. First, she highlighted the seven graphs that had some sort of discontinuity, and asked the students what the other three graphs had in common. At least one student responded that those graphs had a point or a cusp, and Instructor A introduced the term *differentiable* and emphasized that all of the graphs that were drawn without a tangent line parallel to the secant line were either not continuous or not differentiable. Then, the instructor gave the class a short period of time to think about graphs that are both continuous and differentiable on the interval to decide if those graphs had to have a place where the tangent and secant lines were parallel.

The purpose of the examples that had been generated was illustrated as the instructor wrote the MVT on the board and related it to what the students had created. For example, when stating that the function must be continuous on the closed interval [*a, b*], she referred to the example in Figure 2d to illustrate that an open interval would not have guaranteed that the property held. This example was a rich example that nicely illustrated this concept. In contrast, Instructor B had an *Instructor Generated Example* on the board that was extremely similar to Figure 2d, but she did not discuss why this example illustrated the need for the function to be continuous on a closed interval. Furthermore, Instructor D, claimed that a closed interval was required so that it would be possible to compute the average rate of change. As indicated by Figure 2d, Instructor D's statement does not justify the need for continuity on a closed interval.

## Discussion and Teaching Implications

In a classroom that supports students' mathematical learning in a way consistent with the communities of practice perspective, the instructor is also tasked with supporting newcomers in engaging with and becoming more skilled with disciplinary practices. This has implications for the way in which the instructor represents mathematics (for example, the role of examples in developing mathematical ideas) and how the instructor engages students meaningfully in that effort as well. We want to emphasize that this does not simply mean that students should have more opportunity to work in groups or that students should talk more during class; instead, we emphasize that the nature of the task must provide students with the opportunity to *deeply* explore mathematical concepts. In our data, a simple prompt from Instructor A afforded her students the opportunity to deeply engage in developing the Mean Value Theorem. Watson and Shipman (2008) sum this up as:

> ...significant learning can result from the process [of generating examples] because learners generate and explore example spaces related to the ideas, in particular spaces of relations between objects. The importance of normal classroom expectations and teacher guidance cannot be overestimated here. (p. 108)

We also emphasize that the task of generating the examples, while extremely important for student learning, is not all that is necessary. The instructor also needs to be skilled in leading a discussion about the examples in a way that moves the lesson forward. He or she needs to know which examples to highlight in order to provide richness as well as to demonstrate concepts. Our focus on the generation and use of examples contributes to a sense of what is entailed in students' active learning in mathematics. These findings have implications for how instructors can be supported—through materials, coordination, or instructional support—to create classroom environments that actively engage students in doing mathematics.

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**Peter's Evoked Concept Images for Absolute Value Inequalities in Calculus Contexts**

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*Statements involving absolute value inequalities, such as the definition of continuity at a point, abound in Advanced Calculus. In textbooks, such statements are frequently illustrated with graphical representations. Despite their abundance, how students think about absolute value inequalities and their representations in these contexts is not widely known. This study examines one undergraduate mathematics student's evoked concept images (Tall & Vinner, 1981) for absolute value inequalities in various contexts, including those from Advanced Calculus. The student's evoked concept image differed based on the context of the statements involving absolute value inequalities. Notably, the student's evoked concept image did not support his understanding of the visual representation of the formal definition of continuity. The results of this study suggest that some students may not conceive of absolute value inequalities in ways that are productive for understanding the formal definitions of Advanced Calculus concepts.*

*Keywords:* Absolute Value Inequalities, Calculus, Visual Representations

Absolute value inequalities are used in numerous formal definitions and theorems central to advanced Calculus, including statements involving limits, continuity, and sequence convergence. For example, the formal definition of continuity at a point, historically attributed to Weierstrass, may be stated as: "A function $f$ is continuous at a point $c$ in its domain if, for each real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all $x$ in the domain of $f$ with $|x - c| < \delta$, $|f(x) - f(c)| < \varepsilon$." Not much is known about how students conceive of absolute value inequalities in such statements from advanced Calculus. While research has examined students' understanding of absolute value inequalities, most studies have addressed students' procedural fluency and their common errors at lower levels (Almog & Ilany, 2012). Additionally, many high school algebra textbooks that introduce absolute value inequalities treat them procedurally, instructing students to consider cases of inequalities (Boero & Bazzini, 2004). Conceiving of absolute value inequalities primarily in terms of the algebraic procedure for finding a solution may be insufficient for making conclusions from statements involving absolute value inequalities, such as those commonly used in Advanced Calculus. Furthermore, a procedurally-oriented conception may not be sufficient to support students in understanding graphical representations of statements such as the definition of continuity at a point. For example, several Analysis texts introduce the formal definition of continuity of a function at a point along with an image like the one shown in Figure 1 (Gaughan, 1997).

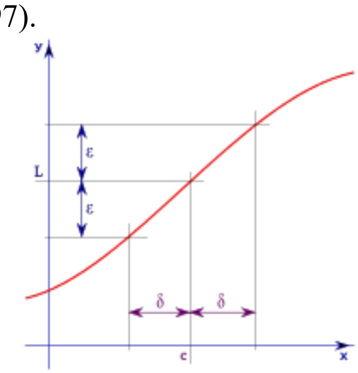


*Figure 1. A visual representation of continuity at a point*

A student that only has a procedural meaning for absolute value inequalities, such as $|x - c| < \delta$, may not necessarily associate the values of $x$ that satisfy this inequality, with the values of $x$ within a distance of $\delta$ from $c$ on the $x$-axis in Figure 1. In graphical representations of advanced Calculus statements, solutions to absolute value inequalities typically refer to a region of space in the rectangular coordinate system with points whose coordinates are within a certain distance from a point. Several studies have found that conceptualizing an absolute value as a distance on a number lines helps students visualize the solutions of an absolute value inequality, thus developing a critical conception of absolute value statements at lower levels (Curtis, 2016; Sierpinska, Bobos, & Pruncut, 2011). The aim of this study is to extend the research in this area by characterizing students' meanings for absolute value inequalities like those found in statements from advanced Calculus, particularly with regard to associated visual representations.

Specifically, the research question for this study is as follows: *What meanings for absolute value inequalities are elicited for students in the context of advanced calculus statements?*

## Theoretical Perspective

In this report, I adopt a constructivist perspective, consistent with von Glasersfeld's (1995) view that students' knowledge consists of a set of action schemes that are increasingly viable given their experience. In this view, students construct knowledge for themselves, and words and images do not inherently contain meaning. This viewpoint also implies that I, as a researcher, do not have direct access to students' knowledge and can only model student thinking based upon their observable actions and behaviors. To characterize student meanings for absolute value inequalities in this study, I also adopt Tall and Vinner (1981)'s constructs of *concept image* and *evoked concept image*. By *concept image*, Tall and Vinner (1981) refer to "the entire cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). Thus, a student's concept image for absolute value inequalities may include numerous cognitive processes and images built on various experiences with the topic over time. While a student may have a concept image that contains many properties and processes for absolute value inequalities, in a given context, only parts of this concept image are activated at a given time. Tall and Vinner (1981) thus define *evoked concept image* to refer to the aspects of the concept image accessed within a particular context. They also note that different aspects of a students' concept image may be in conflict with one another, without the student's awareness.

### Hypothesized Productive Meanings for Solutions to Absolute Value Inequalities in Advanced Calculus Statements

Based on how absolute value inequalities and their solutions are currently utilized in communicating ideas of advanced Calculus, such as the definition of continuity at a point, one productive way of thinking about absolute value inequalities is in terms of bounded distances. For instance, students may understand that solutions to an absolute value inequality of the form $|x - c| < \delta$ can be determined by finding all values of $x$ that are within a distance of $\delta$ from $c$ on a number line. In two dimensions, this set of solutions is a region of points whose $x$ values are within a distance of $\delta$ from $c$ on the $x$-axis. Coming to such an understanding involves connections between a relationship represented algebraically and a set of solutions represented geometrically. Acquiring this level of understanding can be complex, requiring understandings of many foundational ideas, such as variable and difference.

The solution to $|x–c| < \delta$ can be represented analytically as $c–\delta< x < c+\delta$ or geometrically as all $x$ values within a distance of $\delta$ from $c$. Connecting this inequality successfully to the graphical representation involves students viewing both the algebraic inequality and graph as representing an upper bound on how much $x$ can differ from $c$. That is, they must see that the solutions can be represented by an interval on a number line that includes all values (represented by the letter $x$) within $\delta$ of a value represented by $c$. They must conceptualize the letter $c$ as representing a central value, and the $\delta$ symbol as representing an upper limit on the solutions' distance away from $c$. For example, the solution set to an inequality like $|x –(–1)| < 3$ can be represented as follows:

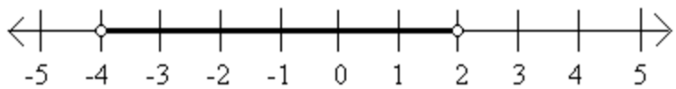


*Figure 2. One-dimension representation of solutions to $|x –(–1)| < 3$*

In this representation, values within a distance of 3 from –1 are included in the set of solutions to the inequality.

In the Cartesian plane, rather than an interval on a number line, this solution set is represented by a region marked by vertical lines representing the boundaries of this solution set. Thus, the region would include all points whose $x$-coordinate is at most a distance of 3 away from –1. Similarly, with inequalities of the form $|f(x)–f(c)| < \varepsilon$, the two-dimensional representation of $f(x)$ values (represented on the vertical axis) that satisfy this inequality can be represented by a horizontal region. This solution set is represented by a region marked by horizontal lines representing the boundaries of this solution set. Thus, the region would include all points whose $y$-coordinate is at most a length of 1 away from 3.

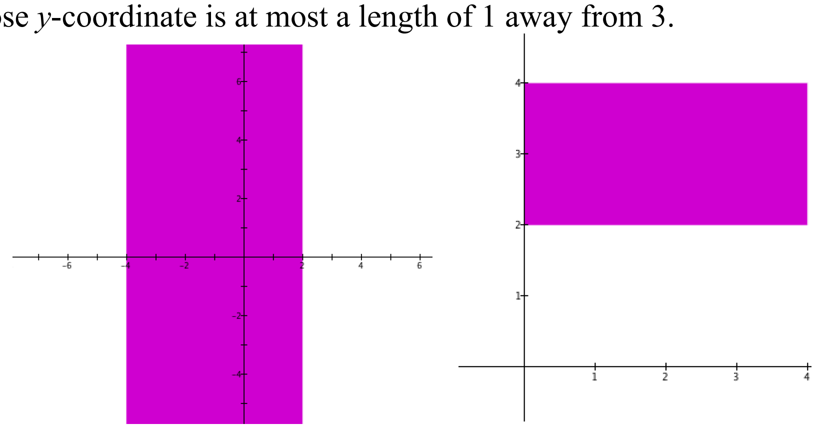

*Figure 3. Two-dimensional representation of solutions to $|x +1| < 3$ and $|f(x)–3| < 1$*

## Methods

For this study, I conducted one 120-minute clinical interview (Clement, 2000) with an undergraduate mathematics student, Peter. Peter was a math major who had completed the Calculus sequence and an Introduction to Proof course, but had not yet taken an Advanced Calculus course.

In the interview, Peter was given tasks that were designed to elicit his meanings for absolute value, absolute value equations, absolute value inequalities, including associated visual representations, such as representing solutions on a number line. Because of the hypothesized evoked concept images for each task, the tasks were ordered in such a way that earlier tasks would not be influenced by later ones. One of the earlier tasks involved a statement about a function $f$ that was the formal definition for continuity at the point $x = 1$ as shown below:

> For each real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all $x$ in the domain of $f$ with $|x - 1| < \delta$, $|f(x) - f(1)| < \varepsilon$.

After asking the student to explain the statement in his own words, as well as each portion of the statement, I presented him with two graphs, first Figure 4 (left) and then Figure 4 (right). After presenting each graph, the student was asked to evaluate whether the statement was true or false for the function $f$ shown in each graph and explain his reasoning.

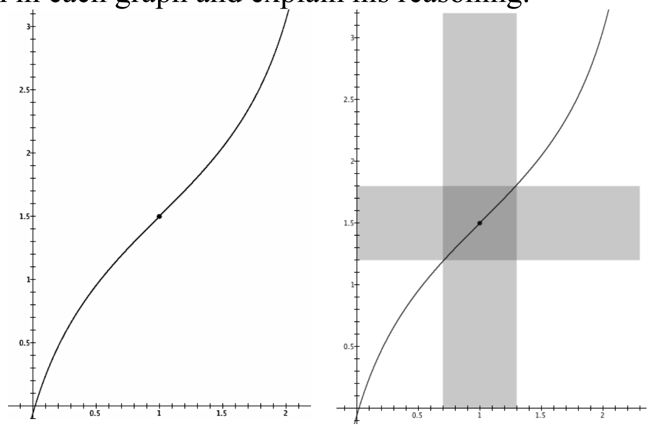

*Figure 4. Graphs used with statement of continuity at x=1*

The final task given to the student is shown in Figure 5.

> a. Find a pair of values, $a$, $b$ such that $|a - b| = 3$.
> b. Find another pair of values $a$, $b$ with $a < 0$ such that $|a - b| = 3$.
> c. Find another pair of values $a$, $b$ with $a < b$ such that $|a - b| = 3$.
> d. If $a = 1$, how many possible values of $b$ satisfy the statement?
> e. What must be true about these pairs of values?
> f. Can you use a number line to explain what $|a - b|$ represents?
> g. Can you use a number line to explain what $|a|$ represents?

*Figure 5. Final task presented to student in interview*

The purpose of this task (Figure 5) was to examine how the student solves absolute value equations, and how he explains the meaning of solutions to absolute value equations involving multiple variables. After the interview, I analyzed the data by modeling Peter's evoked concept image of absolute value (inequality) in each task, especially looking for distinctions in the types of images evoked between contexts.

## Results

In this section, I report several key responses to tasks that revealed Peter's evoked concept image for absolute value and absolute value inequalities. Early in the interview, Peter's written work and utterances suggested that his initial evoked meaning for absolute value was "that a value is positive." When presented with the formal definition of continuity at a point, Peter expressed some confusion, and acknowledged that he was not sure what the statement meant. When presented with the first associated graph (see Figure 4, left), he labeled the graph as shown in Figure 6 below. Peter's procedural meaning for absolute value, that is, making values positive, led to him representing the absolute value of a difference on each axis as a single value.

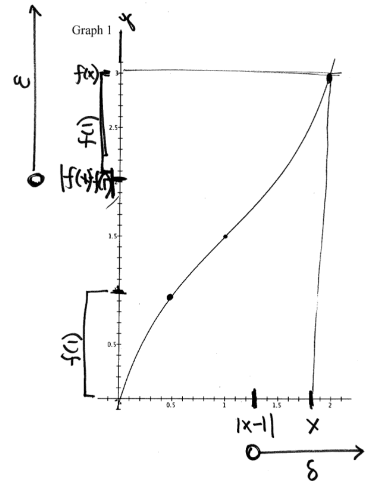


*Figure 6. Peter's labels on the graph of a function related to the formal definition of continuity at a point*

When the interviewer asked Peter to explain "$|x–1|<\delta$," on this graph, Peter responded by saying,

> "So let's say I just chose some value of $x$ here (*labels x on the x-axis as shown in Figure 6*), then $x$–1 (*labels |x–1| on the x-axis to the left of x*), (*pauses*) then δ would have to be larger than that (*draws ray with open circle and labels it "δ"*), so uhh all the values…delta could possibly be any value along this interval (*points to ray he just drew*)."

Peter's words and labels suggest that he was conceptualizing $|x–1|$ as a value on the $x$-axis to the right of zero and one unit to the left of $x$. When prompted to explain what the inequality "$|x–1|<\delta$" represented graphically, Peter provided a literal interpretation of the symbols in his response, stating that δ had a value greater than the value of $|x–1|$. He illustrated this on the number line by constructing a ray with an open circle at $|x–1|$ extending to the right on the $x$-axis. Notably, when shown the graph in Figure 4 (right) and asked to explain the statement relative to the image, Peter paused for a long period of time and acknowledged that he was not quite sure how the statement related to the graph of the function and the shaded regions.

Later in the interview, working on Task 8, Peter's work indicated a different evoked concept image for absolute value that included a distance from zero. In this task, Peter was asked to compare the values of "$|a+b|$" and "$|a|+|b|$" using a number line. Peter produced the following illustration:

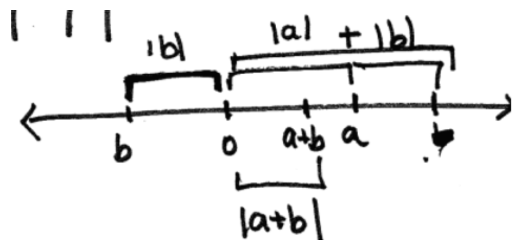


*Figure 7. Peter's work on Task 8*

In Peter's work (Figure 7), he label *a, b,*and *a*+*b* at different locations on the number line, and then labeled line segments from 0 to respective places on the number line with absolute values. Peter placed $|a+b|$ on the segment starting at 0 and ending at $a+b$. The labeling suggests that he considered $|a+b|$ as the distance $a+b$ was from 0 on the number line. Additionally, Peter's work on this task shows his attention to distances. Peter independently chose $b$ to be to the left of 0, and $a$ to be to the right of 0, farther away than $b$ was from 0. When considering $a+b$, Peter attended to the placement of this value relative to the distance $a$ and $b$ were from 0. That is, since $a$ is farther to the right of 0 than $b$ is to the left of 0, $a+b$ would have a positive value less than the value of $a$, and Peter placed $a+b$ to the left of $a$ but to the right of 0. Peter's work on this task was the first indication that he was using absolute value to represent a distance from 0 on a number line.

In the final task, Peter's meaning for absolute value of a difference shifted from his previous meaning. Earlier (as shown in Figure 6), Peter labeled the absolute value of a difference at a location on the number line, indicating he was thinking of a single value that was the result of taking the absolute value of a difference. In the final task, in answering part e) "What must be true about these pairs of values [that satisfy $|a–b|=3$]," Peter's evoked concept image shifted.

To answer this question, Peter first wrote out two equations, "$a–b = 3$" and "$a–b = –3$" and solved them in terms of $b$ and then in terms of $a$. Peter then explained "If I were to choose $a$, then $b$ would be either 3 away from $a$ or 3 on the other side of $a$" and drew a number line to illustrate his idea, as shown below in Figure 8.

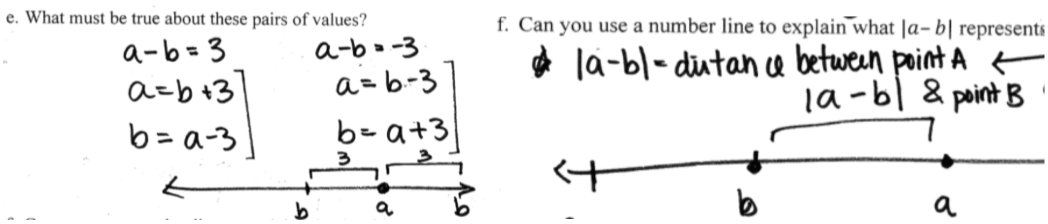


*Figure 8. Peter's work showing what must be true about a and b when |a–b|=3*

Peter illustrated his algebraic interpretation of the relationship between $a$ and $b$ using a number line. He labeled the segment he drew between $a$ and $b$ in either direction with "3," indicating that he recognized the distance between $a$ and $b$ was 3 units. This was the first time that Peter interpreted a difference as a distance between two points, neither of which were 0. When Peter encountered absolute values of differences when responding to earlier tasks, he considered them to be single values on the $x$-axis, or measuring a distance from 0. Rather than treat an absolute value of a difference as a single value whose reference point is 0, Peter treated an absolute value of a difference as a distance between the two values, without reference to 0. In the next sub-question in the final task, Peter confirmed that he was conceptualizing the absolute value of a difference $|a–b|$ as the distance between point $a$ and point $b$ (Figure 8, right).

To check to see if his image for absolute value inequality that had been evoked in the task above influenced his understanding of continuity at a point, I again showed Peter the formal definition of continuity at a point, and associated graph, and asked him to re-label the graph.

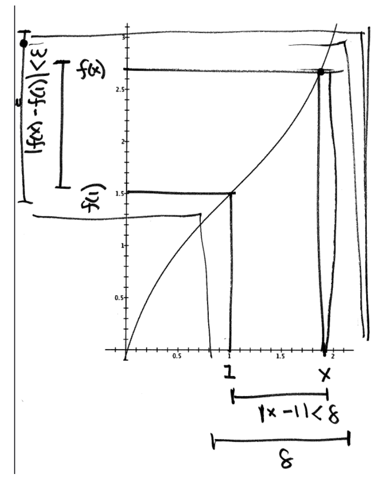


*Figure 9. Peter's updated labels on graph after completing final task*

This time, Peter labeled the distance between $x$ and 1 on the $x$-axis with the label $|x–1|$ rather than labeling $|x–1|$ as a value on the $x$-axis itself. He recognized that the statement "$|x–1|<\delta$" was a statement about a comparison of a distance between two values, and a value $\delta$. While Peter did not attend to $x$ values to the left of 1, his evoked image for absolute value of a difference as measuring a distance (Figure 8) supported Peter in connecting the image of a graph with the definition of continuity at a point.

## Conclusion & Discussion

Peter's work suggests his concept image for absolute value and absolute value of a difference contains several distinct meanings and processes. In different contexts throughout the interview, Peter's work indicated different evoked concept images for absolute value, consistent with findings by Tall and Vinner (1981). In the beginning of the interview, Peter's meaning for absolute value inequalities elicited by the initial tasks included a procedure of making a value positive. Later in the interview, Peter's evoked concept image for absolute value included a meaning of distance from zero on a number line. In the final task, Peter's evoked concept image for absolute value of a difference was a distance between two points.

Most notably, Peter's initial meaning for absolute value elicited by the continuity at a point statement and associated graphs did not include a difference between two points on axes, but rather were of absolute value as an operator that makes values positive. Peter's initial evoked image is consistent with the way absolute value inequalities are introduced in high school textbooks (Boero & Bazzini, 2004). Due to his evoked meaning for absolute value as an operator, Peter was unable to explain the continuity statement relative to the graphs in Figure 4. However, through other various tasks, different aspects of Peter's concept image for absolute value were evoked, which allowed Peter to conceptualize the absolute values in the continuity statement differently than he had previously. The findings from this study suggest that students entering advanced Calculus courses may interpret absolute value inequalities and their visual representations differently than intended. Specifically, their evoked concept image for absolute value may not support their attempts to connect such statements to associated graphs. Instructors of courses utilizing statements involving absolute value inequalities may consider including tasks to evoke different meanings for absolute value. Instructors and curriculum developers should not assume that students' evoked concept image for absolute value inequalities will align with how their solutions are represented in illustrations on graphs.

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# A Preservice Mathematics Teacher's Covariational Reasoning as Mediator for Understanding of Global Warming

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*I examine one preservice mathematics teacher's (PST's) covariational reasoning in relation to two functions involved in modeling global warming. I also discuss how her covariational reasoning mediates her understanding of important concepts related to global warming. Jodi, the PST, completed a mathematical task I created for the study during an individual, task-based interview. The analysis of Jodi's responses revealed that: (a) the level of covariational reasoning and conceptions regarding quantities can constrain/facilitate the understanding of concepts related to global warming, (b) overreliance on discrete variation can led to conflicting notions regarding global warming, and (c) reasoning about rate of change is necessary to make sense of mathematical models for global warming based on energy balance.*

*Keywords:* Covariational Reasoning, Global Warming, Preservice Teachers, Modeling

## Introduction

In recent years, there have been several calls to include global warming in school and college instruction (McKeown & Hopkins, 2010; UNESCO, 2012). Global warming is a contemporary and pressing issue affecting different people around the globe (Intergovernmental Panel on Climate Change [IPCC], 2013). Moreover, global warming provides a motivating scientific context to study important scientific and mathematical concepts. Mathematics teachers, however, are likely not prepared to incorporate global warming into their instruction. Researchers have demonstrated that the public have many problematic conceptions about important concepts related to global warming (Leiserowitz, Smith, & Marlon, 2010; Pruneau, Khattabi, & Demers, 2010). Also, teachers and students without sufficient scientific and mathematical literacy can have difficulties understanding concepts related to global warming (Barwell, 2013; Lambert & Bleicher, 2013). Therefore, there are both societal and cognitive needs for studies regarding global warming and mathematical reasoning.

In my research, I investigated how preservice mathematics teachers (PSTs) make sense of introductory mathematical models for global warming. By introductory models, I mean those for which the mathematics can be accessible to high-school students. The models require PSTs to think about a dynamic situation in terms co-variation between quantities. Existing research in mathematics education has demonstrated that students and future mathematics teachers can have persistent difficulties comprehending and mathematically expressing co-variation between quantities (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2012; Oehrtman, Carlson, Thompson, 2008; Thompson, 2011). In this paper, I focus on one PST's covariational reasoning in relation to two functions: *the planetary energy imbalance function*, N(t), and *the planet's mean surface temperature function*, T(t). I also discuss how her covariational reasoning mediates her understanding of important concepts related to global warming.

## Background Information

Earth's climate system is powered by the sun and there is a continuous flow of energy between the sun, the planet's surface, and the atmosphere. This continuous flow of energy is known as the *Earth's energy budget* (Figure 1). The sun warms the planet's surface (S). As the surface warms up, it radiates (infrared) energy to the atmosphere (R), the majority of which is absorbed by *greenhouse gases* (GHG) such as water vapor ($H_2O$), carbon dioxide ($CO_2$), and methane ($CH_4$) (B). The atmosphere re-radiates the absorbed energy in both directions toward space and toward the surface (A). This continuous energy exchange between the surface and the atmosphere is known as the *greenhouse effect* and influences the planet's mean surface temperature. The energy flows S, R, B, L, and A (Figure 1) are all magnitudes of energy flux density, while the abundance of GHG is a magnitude of concentration. *Energy flux density* is a flow of energy per unit of area per unit of time incident to a surface, usually measured in Joules per square meter per second ($J/m^2/s$). *Concentration* is the volume of a gas relative to the total volume of the mixture in which the gas is contained, usually measured in the same units of volume (e.g., $m^3/m^3$) or in parts per million by volume (ppmv). The parameter $0 < g < 1$ (Figure 1) is related to the greenhouse effect. Quantifying changes in the energy flows due to changes in the abundance of GHG is central to accurately model global warming. My study focuses on how variation in the atmospheric concentration of $CO_2$ produces variation in the energy flows over time, and how that variation affects the planet's mean surface temperature.

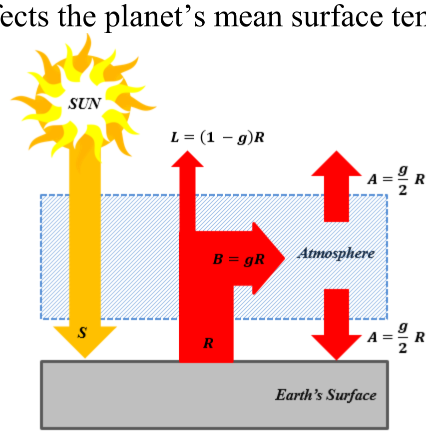

*Figure 1: The Earth's energy budget, assuming a one-layered atmosphere*

The *planetary energy imbalance function* N(t) is a measure of the energy imbalance in the Earth's energy budget over time. In particular, N(t) can be defined as a difference between the downward radiation and the upward radiation at the planet's surface, or mathematically $N(t) = (S + A(t)) - R(t)$. The Earth's energy budget is said to be in radiative equilibrium when $N(t) = 0$ (downward radiation equals upward radiation), which implies that the *planet's mean surface temperature function* T(t) remains constant. However, there are factors or *forcing agents* that can push the energy budget out of equilibrium, producing $N(t) \neq 0$. The present study focuses on how N(t) and T(t) vary over time after a *positive forcing by $CO_2$* occurs at $t = 0$. An instantaneous increase in the concentration of $CO_2$ results in an atmosphere with more capacity to absorb surface radiation R(t). This translates into a value for A(0) such that $N(0) = (S + A(0)) - R(0) > 0$, which means that the downward radiation exceeds the upward radiation. As a result, the planet's surface starts warming up (i.e., an increasing T(t)); a hotter surface produces more radiation (i.e., an increasing R(t)). The atmosphere absorbs even more radiation, increasing its own radiation back to the surface (i.e., an increasing A(t)), further warming the surface. The expression $N(t) = (S + A(t)) - R(t) = S - \beta R(t)$, where S is the solar constant and $\beta = 1 - g/2$, indicates that R(t) continues to increase until the upward radiation equals the downward radiation since $N(t) \to 0$ as $t \to \infty$. This in turn indicates that T(t) increases at a decreasing rate as it approaches to a new equilibrium temperature. In fact, mathematical models for global warming commonly known as Energy Balance Models (EBMs) rest on the idea that $\frac{dT}{dt} = \alpha N(t)$ for a constant $\alpha > 0$ (Widiasih, 2013).

## Conceptual Framework

Carlson et al. (2002) defined covariational reasoning as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Based on this definition, Carlson and colleagues developed the Covariation Framework as a theoretical instrument to examine and assess a student's covariational reasoning abilities relative to a mathematical task showing two co-varying quantities. Their framework describes five mental actions involve in reasoning about quantities that vary together. *Mental Action 1 (MA1)* involves coordinating the value of one variable with changes in the other (e.g., labeling the axes with verbal indications of coordinating the two variables such as "y changes with changes in x"). *Mental Action 2 (MA2)* involves coordinating the direction of change of one variable with changes in the other variable (e.g., constructing an increasing straight line or verbalizing an awareness of the direction of change of output while considering changes in the input). *Mental Action 3 (MA3)* involves coordinating the amounts of change in one variable with changes in the other (e.g., plotting points, constructing secant lines, or verbalizing an awareness of the amount of change of the output while considering changes in the input). *Mental Action 4 (MA4)* involves coordinating the average rate of change of the function with uniform increments in the input variable (e.g., constructing contiguous secant lines or verbalizing an awareness of the rate of change of the output while considering uniform increments of the input). *Mental Action 5 (MA5)* involves coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function (e.g., constructing smooth curve with clear indications of concavity changes, verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function, or correctly interpreting concavities and inflexion points). The collection of mental actions inferred from the student's responses is examined to determine the student's overall level of covariation reasoning relative to the task. There are five levels of development, each more sophisticated than and built upon the previous one: *dependency of change* (L1: *y* changes when *x* changes), *direction of change* (L2: *y* increases as *x* increases), *amounts of change* (L3: a change $\Delta y$ in *y* correspond to a change of $\Delta x$ in *x*), *average rate of change* (L4: *y* increases more rapidly for successive changes $\Delta x$ in *x*), and *instantaneous rate of change* (L5: *y* increases more rapidly as *x* continuously increases). If a student's covariational reasoning is classified at a particular level, then it is implied that the student's covariational reasoning supports the mental action associated with that level *and* the mental actions associated to all previous levels.

## Methods

This paper is part of a larger study that investigated how PSTs make sense of introductory mathematical models for global warming. That larger study consisted of two parts: (1) exploring PSTs' conceptions of intensive quantities commonly used to model global warming, and (2) examining PSTs' covariational reasoning relative functions commonly used to model global warming. To address these goals, I created an original sequence of six mathematical tasks involving intensive quantities, functions, and concepts related to global warming.

Three secondary PSTs enrolled in a mathematics education program at a large Southeastern university participated in the larger study. The PSTs have completed three mathematics content courses (calculus I, calculus II, and introduction to higher mathematics) and were completing a mathematics education content course (connections in secondary mathematics). In this paper, I focus on the case of Jodi, one of the three PSTs who participated in the larger study. Specifically, I focus on Jodi's responses to the sixth mathematical task in my sequence. Her case is interesting for two reasons. First, Jodi's responses were markedly different from her peers, which represent a unique case for discussion. Second, her case shows clear examples of how covariational reasoning can mediate the understanding of scientific concepts related to global warming.

I started by showing Jodi a 7-minute long video introducing the *Earth's energy budget*, *radiative equilibrium*, and *greenhouse effect*. The video was retrieve from the NASA YouTube channel *NASAEarthObservatory*. Then, I answered any questions she may have had concerning the concepts discussed in the video. Next, I presented her with a diagram of the energy budget (Figure 2a) and the following task:

> *An increase in the atmospheric concentration of $CO_2$ results in an energy imbalance in the Earth's energy budget. This initial imbalance is known as forcing by $CO_2$. We want to examine how the planetary energy imbalance N(t) and the planet's mean surface temperature T(t) vary over time after the forcing. Use what you learned about the Earth's energy budget, the greenhouse effect, and the definition $N(t) = (S(t) + A(t)) – R(t)$ to determine: (a) how N(t) varies over time and sketch its graph and (b) how T(t) varies over time and sketch its graph.*

Jodi completed the mathematical task during a 60-minute, semi-structured, task-based interview (Goldin, 2000). The interview was video recorded and transcribed for analysis. All of Jodi's work on paper was collected as well.

Videos and transcripts were analyzed through Framework Analysis (FA) method; this method five inter-related stages of data analysis: familiarization with data, developing an analytic framework, indexing and pilot charting, summarizing data in analytic framework, and synthesizing data by mapping and interpreting (Ward, Furber, Tierney, & Swallow, 2013). Through these stages, the researcher creates and refines framework analysis' distinctive feature: the *matrix output*, a table arrangement into which the researcher systematically reduces, summarizes, and analyzes the data. I utilized the mental actions in the Covariation Framework (Carlson et al., 2002) as themes for coding interview transcripts. Then, I re-read all transcript texts categorized under a particular mental action. I selected and summarized those transcript texts that were more representative of that particular mental action. I repeated this process until I selected representative texts for each mental action. Then, I organized the selected texts into a matrix output containing five columns (one for each mental action) and two rows: one for N(t) and another for T(t). The matrix output allowed me to develop an idea of Jodi's: (a) overall level of covariational reasoning, (b) understanding of N(t) and T(t), and (c) conceptions of the energy budget and radiative equilibrium.

## Results

Jodi's responses to the first part of the task suggest covariational reasoning abilities at the direction of change level (L2) when her object of reasoning was the situation (i.e., how the energy budget evolves after a positive forcing). When her object of reasoning was the graph of N(t), she demonstrated abilities at the amounts of change level (L3). To start the task, I told Jodi to imagine that an instantaneous increase in the atmospheric concentration of $CO_2$ produces an imbalance of energy equal to $N(0) = 5$ J/m$^2$/s (positive forcing by $CO_2$). Jodi is then given a diagram of the Earth's energy budget showing the initial values: $S = 240$ J/m$^2$/s, $R(0) = 390$

J/m$^2$/s, B(0) = 310 J/m$^2$/s, L(0) = 80 J/m$^2$/s, and A(0) = 155 J/m$^2$/s (Figure 2a). Notice that N(0) = (S + A(0)) – R(0) = (240 + 155) – 390 = 5. Jodi was expected to visualize how N(t) varies as time t increases. Jodi imagined energy moving from R to B, then to A, and finally back to R, what she labeled as *cycles*. Using these cycles, Jodi determined the following values for the energy flows R, B, and A: R($C_1$) = 395 J/m$^2$/s; B($C_1$) = 313 J/m$^2$/s, and A($C_1$) = 157 J/m$^2$/s, and R($C_2$) = 397 J/m$^2$/s; B($C_2$) = 315 J/m$^2$/s, and A($C_2$) = 158 J/m$^2$/s (Figure 2a), where $C_i$ represents *cycle i* after the positive forcing. When I asked Jodi whether N(t) was increasing or decreasing, she stated "I guess it would increase? [*Pauses*] but, I don't see an argument for why it wouldn't stay the same." I then asked her to determine the values of N(t) for each one of her cycles. Jodi determined the values N($C_0$) = 5 J/m$^2$/s, N($C_1$) = 2 J/m$^2$/s, and N($C_2$) = 1 J/m$^2$/s, where $C_i$ represents *cycle i* after the positive forcing. Jodi stated that she was not expecting N(t) to decrease over time ("I though N would be larger"). When I asked her to interpret this decreasing N(t), Jodi replied "[it means] that we are going back to an equilibrium, or we are not as far from equilibrium as we were." When Jodi was able to establish the direction of change of N(t), she began to conceive N(t) as a measure of the energy imbalance. Also, the direction of change helped her develop the idea that the energy budget moves towards (radiative) equilibrium after a positive forcing. These represent foundational concepts to understand introductory mathematical models for global warming.

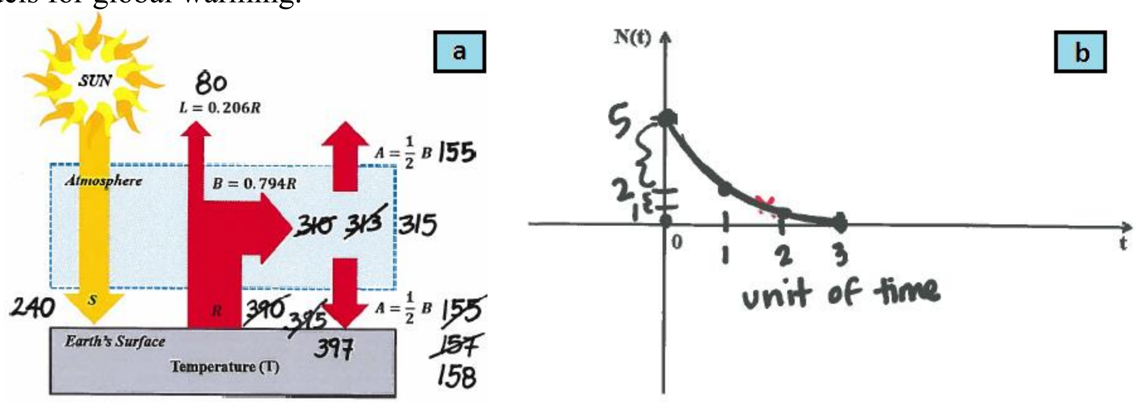


*Figure 2: (a) Jodi's work on the diagram of the energy budget. (b) Jodi's final graph of N(t).*

Jodi constructed the graph of N(t) by plotting the points ($C_i$ , N($C_i$)), and then joining them by a concave-up, decreasing curve (Figure 2b). Jodi looked at the curve and stated that "we are decreasing at a decreasing rate." When asked to elaborate, Jodi said

> Each time we are increasing t, we are decreasing N by smaller amounts. Like here, we decrease 3 [curly brackets on Figure 2b], and then we decrease 1 … I am trying to make sure I know what the graph looks like. OK, when you have a graph and you do like this [*draws a concave-up, decreasing curve*], this is one and this is two [*makes two equally-spaced marks on the horizontal axis*]; you would be decreasing by smaller amounts each time. The same thing what we are doing here [*draws the curly brackets on Figure 2b*], so I want to say that the graph looks like this: decreasing at a decreasing rate

Jodi's responses regarding the rate of change of N(t) and how N(t) decreases by smaller and smaller amounts were a result of reasoning about the graph of N(t). Jodi did draw a concave-up curve, but the concavity was the result of joining all points by a curve. Notice that she need not reason beyond L2 to accomplish that. Jodi did not notice that R, B, and A were also increasing at a decreasing rate. This suggests that she was not attending to the situation when thinking about amounts of change. It was by using the graph as her object of reasoning that Jodi attended to the variation in amounts of change in N with respect to changes in time. This appears as a version of L3 covariational reasoning, a version that makes use of the graph as an object of reasoning. It did not seem that this version of L3 helped Jodi understand the energy budget since the latter was not the object of reasoning. Also, Jodi's verbalization regarding the rate of change of N(t) must be taken with caution. Jodi's responses suggest that she was reasoning in terms of amounts of change rather than rate of change. It is, therefore, unlikely Jodi's covariational reasoning was at the rate of change levels L4 or L5.

Jodi provided an interesting interpretation of N(t) in relation to the variation in energy (or heat) in the surface. Jodi stated that the surface was losing heat because N(t) was decreasing. When asked to elaborate, Jodi stated that "we would need to be losing energy so that we can go back to equilibrium." For Jodi, a decreasing N(t) represented an energy budget moving towards equilibrium, but in the sense that *thing were going back to their original state* (i.e., a budget before the positive forcing). Jodi conceive N(t) as a measure of energy imbalance as in measuring *how far* the budget was from its *original radiative equilibrium*. Jodi's conception of energy imbalance did not involve N(t) as a difference between downward radiation and upward radiation. Jodi's conception of N(t) shaped her understanding of radiative equilibrium.

Jodi's responses to the second part of the task suggest covariational reasoning abilities at a discrete version of the amounts of change level (L3) when her object of reasoning was the situation (i.e., how the energy budget evolves after a positive forcing). For this task, Jodi attended to the way R and A were changing between cycles as shown in Figure 2a. Specifically, Jodi attended to the amounts of change in R and A with respect to changes in time.

> It increased by two (A changes from 155 J/m$^2$/s to 157 J/m$^2$/s), and then it decreased by two (R changes from 395 J/m$^2$/s to 397 J/m$^2$/s) [*pauses*]. So, it is almost as if there was no change in temperature because I associate energy as kind of having a relationship with temperature. So, if the energy increases, then the temperature increases. But, in this scenario an equal change in energy was an equal change in output [*simultaneously points at A and R*]

Jodi saw that any increase in A, or radiation from the atmosphere towards the surface, was match by the same increase in R, or radiation from the surface towards the atmosphere. She interpreted it in the following way: "the Earth would heat up because it got more energy [*points at A*], but then it would release it within the same cycle [*points at R*]." This suggests that the discrete approach to estimate the values of R, B, and A was shaping Jodi's thinking about the situation. Jodi conceived time varying in discrete units, or *cycles*. For *cycle i*, A instantaneously increased by an amount $\Delta_i A$ (at the beginning of *cycle i*), while R instantaneously increased by an amount $\Delta_i R = \Delta_i A$ at the end of *cycle i*. Following this reasoning, Jodi concluded that the surface energy was oscillating over time, which led her to conclude that T(t) was also oscillating over time. She represented this oscillatory variation by two periodic curves (Figure 3). Jodi drew two different periodic curves (*arcs* curve and *dashes* curve) because she was not sure whether the energy, and consequently the temperature, was increasing and decreasing within each cycle (*arcs* curve) or increasing within a cycle and instantaneously decreasing at the end of it (*dashes* curve). Her responses showed evidence of L2 covariational reasoning since she described the direction in which T(t) was changing over time (i.e., as t increases, T(t) increases and decreases).

Interestingly, Jodi constructed a third graph for T(t) by attending to the variation in the amounts of change in the energy flows in the budget. She attended to the variation in the amounts of change in B with respect to changes in time (Figure 2a). Since B was increasing by smaller and smaller amounts, Jodi thought that T(t) was still oscillating, but its *amplitude* was decreasing between *cycles*. Jodi probably saw the decreasing increments in B as consistent with her idea of a budget returning to the original radiative equilibrium. She represented this quasi-periodic variation by drawing a quasi-periodic curve whose *arcs* were decreasing in size (Figure 3). Her response and graph suggest that Jodi, in a way, was reasoning about how T(t) was changing in relation to time. Since she attended to the variation in amounts of change, I consider Jodi's covariational reasoning a version of L3, which was shaped by a discrete conception of time variation. Notice that her L3 covariational reasoning led her to conclude that T(t) was decreasing over time (i.e., the planet's surface was cooling down). This may become an obstacle to understand the link between $CO_2$ pollution and global warming.

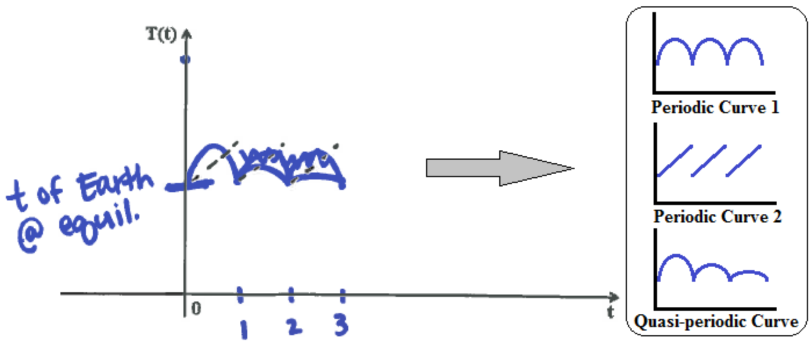


*Figure 3. Jodi drew three different curves for T(t): two periodic curves and one quasi-periodic curve*

## Conclusions

The study's findings suggest that Jodi's covariational reasoning mediates her understanding of concepts related to global warming. Covariational reasoning at the direction of change level (L2) appears to facilitate the understanding of the budget moving towards radiative equilibrium after a positive forcing by $CO_2$. This is a foundational understanding for introductory mathematical models for global warming since it highlights the impact of $CO_2$ pollution over the planet's flow of heat. Jodi's case also shows the importance of developing covariational reasoning at the amounts of change level (L3) by using the situation as object of reasoning. Without this connection, L3 covariational reasoning can be of little use to understand global warming. Moreover, L3 covariational reasoning based on a discrete conception of variation can led to misunderstanding regarding the energy budget. In the case of Jodi, her discrete L3 led her to conclude that the planet was cooling down after a forcing, which contradicts the link between $CO_2$ pollution and global warming. Additionally, Jodi did not make use of N(t) to construct the graph of T(t). This suggests that Jodi did not see N(t) as a measure of the rate of change of T(t). This may be explained by Jodi's inability to reason about co-variation at the rate of change levels (L4 or L5). Another explanation involves Jodi's conception of N(t). She did not see N(t) as a difference between downward radiation and upward radiation. Without such understanding, it is unlikely to see the relationship between N(t) and T(t). Also, her conception of N(t) led her to think that the planet's surface was cooling down. This contradicts the long-term impact of $CO_2$ emissions on the planet's average surface temperature.

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# The Authority of Numbers: Fostering Opportunities for Rational Dependence in a Mathematics Classroom

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*This cross-case analysis of quantitative literacy instruction at the undergraduate level compares three different settings where activities were introduced that required students to seek out and make use of information outside of the classroom. These activities provided students with opportunities to engage with quantitative claims made by experts and by comparing these cases I was able to identify several axes of variability that affect the extent to which the problems supported the practice of rational dependence, or the reasoned dependence on the knowledge of others. These variables include the extent to which students are held accountable for their choices of information sources, the way in which the teacher frames what it means to critically appraise a quantitative claim, and the role that mathematics plays in the activity.*
*Key Words*: Quantitative Literacy, Information Literacy, Statistics, Comparative Case Study

Rational dependence (Erickson, 2016) is the reasoned dependence on the expertise of others. If mathematics instruction is to help prepare students for the quantitative claims that they may expect to encounter in their everyday lives, then they need to be given the opportunity to develop rational dependence. In order to explore what happens when such opportunities are created, I collaborated with several teachers of quantitative literacy-focused undergraduate mathematics courses in order to introduce information-based problems (Walraven, Brand-Gruwel, & Boshuizen, 2008), or those problems that require students to seek out and evaluate information sources outside of the classroom. I developed case studies for each location which informed cross-case observations about how these teachers used information-based problems. The resulting multiple case analysis allowed me to answer questions about whether and how opportunities for rational dependence arose in the context of the activities.

My analysis of classroom work suggests that opportunities for rational dependence were associated with the way that teachers used information-based problems and, in particular, the structure of the academic tasks through which the problems were implemented. All of the teachers prioritized their students' development of a critical stance towards quantitative claims over direct assessment of the credibility of sources. I found that the classroom tasks that contributed to opportunities for rational dependence included (a) how students were held accountable for the sources that they found, (b) how the teachers operationalized their students' development of a critical stance towards quantitative claims, and (c) the role that mathematics played in the tasks.

## Review of the Literature

### Quantitative Literacy

Mathematics instruction has long had an instrumental role with respect to training in the STEM fields, but there has been a more recent push targeting the development of the general mathematical skills and attitudes that might best serve students in their daily lives. This aggregate of skills and dispositions is sometimes referred to as quantitative literacy (Steen, 2004), mathematical literacy, quantitative reasoning or statistical literacy (Cullinane & Treisman, 2010; Watson, 2013). Accordingly, the development of quantitative literacy has become an important goal of courses offered by many colleges for non-STEM majors who need to fulfill a mathematics requirement as part of their liberal arts education. Examples of quantitative literacy are usually directed at quantitative claims made by journalists, politicians, or advertisers rather than experts, but the question of how students should relate to experts does occasionally arise. For example, Gal (2002) in his analysis of statistical literacy, suggests that critical skills, i.e., knowing which questions to ask about sources of information and their biases, is a component of statistical literacy, and that this must also be accompanied by a disposition to maintain a critical stance towards statistical claims. However, one of the most thorough attempts to understand how students can come to interact productively with expert information can be found in the field of scientific literacy.

**Learning to Live with the Expertise of Others**

Norris (1995) breaks down what science education might look like if it served to prepare students to attain an intellectual independence tempered by their dependence on expert communities. He stressed three components of this program: a) learning science in the sense outlined by Moje's (2007) "usable disciplinary knowledge", b) learning about the history and philosophy of science, and c) "learning to live with science" (Norris, 1995, p.214).

This last component is elaborated by Norris, "the only access to scientific truth for most of us is through the efforts of scientific experts [...] therefore, students need to acquire the disposition to question, and to seek other opinions on scientific issues that matter in their lives and in their community" (Norris, 1995, p.215). But this questioning disposition should not, per Norris, be indiscriminate,

> A skeptical disposition is not sufficient if one does not know how to exercise wisely that skepticism. […] [Students] should be taught how to use criteria for judging experts: the role and weight of consensus; the role and weight of prestige in the scientific community; the role and weight of publication and successful competition for research grants; and so on. As part of learning to live with science, students need practice in judging the credibility of scientific experts. This practice should be based on real-world problems that currently affect their lives. (Norris, 1995, p. 216)

Gaon and Norris (2001) go on to argue that there are *content-transcendent* modes of inquiry into claims made by experts and that a non-expert can, and should, ask questions about scientific claims:

> Does this scientific belief embody or support any particular social hierarchies such as those based on race, on gender, or on class? If so, what normative assumptions have been made? Have these norms been thematised and justified scientifically, or are they simply assumed? Have alternate accounts of the same phenomenon been developed? By whom? What were the grounds for choosing one account over another? Are these grounds themselves free of normative assumptions; are they as certain as they appear? Who decided? (Gaon & Norris, 2001, p.200)

These questions apply equally well to quantitative claims and serve as a road map for thinking about what it looks like to live with expertise in any disciplinary area. This gives rise to a further question: How could activities be created that would provide students with the opportunity to develop these skills?

**Information-Based Problems**

The seeking out of information on the internet can become a dilemma for the instructor once they allow this activity to take place in their classroom. At this point, the question becomes not so much about classroom management (e.g., "What should the smartphone policy be?", "How do I keep students from surreptitiously texting?") but rather about managing the classroom's *didactical contract*. I borrow this last term from Guy Brousseau (1997) who refers to the division of labor and system of accountability that specifies how a classroom activity provides evidence that the envisioned learning has in fact occurred. In particular, this type of activity amounts to a modification of the traditional terms of the mathematical *tasks* (Herbst, 2006; Doyle & Carter, 1984), or the actions associated with a given problem along with an established set of resources. This may help explain why, even though the information-seeking behavior of academics differs across disciplines (Palmer & Cragin, 2008), there are few attempts to educate students in the discipline about how that discipline-specific information-seeking is carried out (Grafstein, 2002). This would not necessarily be a problem if students were able to learn how to seek out information through generic instruction that could then be used to support quantitative reasoning, but research has shown that content knowledge is deeply tied to successful information-seeking (Walraven et al., 2008).

## Theoretical Framework

**Rational Dependence as an Educational Goal**

Successful engagement with real-life quantitative claims is predicated on our *epistemic dependence* (Hardwig, 1985) on others, or the fact that much of what we know is dependent on our trust in the expertise of others. This observation must be tempered by the fact that an individual can rely on others in a more or less rational way (Siegel, 1988). What does all of this mean for mathematics instruction? Although the importance of information-based problems for disciplinary literacy is easy to justify as long as one accepts that information-seeking is an important part of practice in the disciplines, it requires a little more unpacking to explain why this type of instruction might have a place in mathematics instruction. One way to begin such an explanation is to imagine an applied mathematics problem -- say students are given an editorial in which the author argues that federal guidelines on fuel efficiency will end up costing the country more money than it will save (Diefenderfer, 2009). Students are asked to read the editorial and then provided with several guiding questions that encourage the students to analyze the numerical argument contained in the article while noting some of the additional information that might be required prior to coming to a final verdict on the validity of the editorial's argument. If a reader were to actually want to determine whether a quantitative claim was true or not, they would want to locate the relevant *epistemic community* (Haas, 1992), i.e. that community that possesses the expertise to tentatively rule on the truth of the claim. In other words, they would need to engage in the practice of rational dependence by finding experts on whom the students have good reason to rely. An *information-based problem* (Walraven et al.,

2008) provides such an opportunity by requiring students to seek out and evaluate sources outside the classroom. In order to come to a better understanding of an information-based problem, the student must "identify information needs, locate corresponding information sources, extract and organize relevant information from each source, and synthesize information" (Walraven et al., 2008, p.2) in a process called *information-problem solving*. My inquiry can be framed, then, as a question about how mathematics teachers and their students cope with the introduction of information-based problems, and whether and how these problems afford opportunities for rational dependence in the classroom.

Accordingly, this study seeks to answer the following research questions:

1. How can opportunities for the practice of rational dependence be introduced to a quantitative-literacy focused mathematics class?
2. What aspects of information-based problems in a quantitative-literacy focused mathematics classroom most influence students' opportunities to practice rational dependence?

## Research Methodology

I investigated the research questions outlined above through a multi-case analysis (Stake, 2013) of collaborations with three teachers of terminal undergraduate mathematics classes targeting non-STEM majors. We worked together to design activities in which information-based problems would be introduced to their students. Table 1 provides more information about the sites where this research took place. At Phi University students were assigned to argue one side in a classroom debate. To prepare, they were required to research their topic and provide some statistical evidence supporting their side of the issue. At Rho University, we developed a two-part activity where students were asked to look for articles in which a conjecture about causation was being studied (e.g., vaccines and autism). They were asked to locate the quantitative evidence used to claim that the two variables were or were not correlated, and then engaged in a small-group discussion with their peers about the topic. Their groups tried to come to a consensus on the issue at stake and then shared their verdict with the rest of the class. The students at Delta University also worked in small groups, but here they were asked to create a presentation in which they would analyze the way that statistics were used in a research article for the rest of the class. The focus of this analysis would be on the sampling methodology, but they were free to talk about other facets of the article if they chose to do so.

The *quintain* (Stake, 2013), or the phenomenon of interest for this cross-case analysis, is the introduction of information-based problems to an undergraduate mathematics course. The data for this study includes pre- and post-interviews with the instructors at each of the three sites, supplementary interviews with teaching assistants and students, field notes taken while observing instruction prior to the introduction of the information-based problems, video and audio-recordings of the in-class component of the activities, and copies of the work that the students submitted. These data sources informed the writing of individual case reports which were, in turn, used to develop the cross-case analysis. Following Stake (2013), I developed themes based on my research questions that I then used as an analytical lens for the development of case reports for each of the three sites. After writing up the case reports, I cross-referenced case-specific with the themes of the larger study. This allowed me to warrant theme-based assertions and used those to inform the final cross-case assertions (see Figure 1) about the introduction of information-based problems to undergraduate mathematics classrooms.

| University Name[a] | Course Name[a] | Students | Topics | Structure |
|---|---|---|---|---|
| Phi University (Research) | Topics in Mathematics | 22 entering Freshman, Liberal Arts Majors | Gun Control, Marijuana Legalization, Single-Sex Education, Death Penalty | Debate Format |
| Rho University (Regional) | Quantitative Reasoning | 14 Juniors and Seniors, many are prospective Nursing students | Autism and Vaccination, The Mozart Effect, Gun Control, Health Care Reform | Small-group Discussions |
| Delta University (Doctoral) | Mathematics in Today's World | 24 Juniors and Seniors, many are prospective Nursing and Education students | Autism and Vaccination, Gun Control, Murder Rate, Vehicular Accidents, Employee Prospects | Small-group Presentation |

[a]These are pseudonyms

**Table 1.** Description of research sites and student population

## Results

The three cases provided instructive examples of how and why the actions of a teacher can open up or limit opportunities for rational dependence. I have broken down these observed axes of change into three categories. First, the degree to which students are made accountable for their choice of sources to draw from; second, the way in which a critical stance to quantitative claims is framed by the instructor; third, the role of mathematics in the information-based problem.

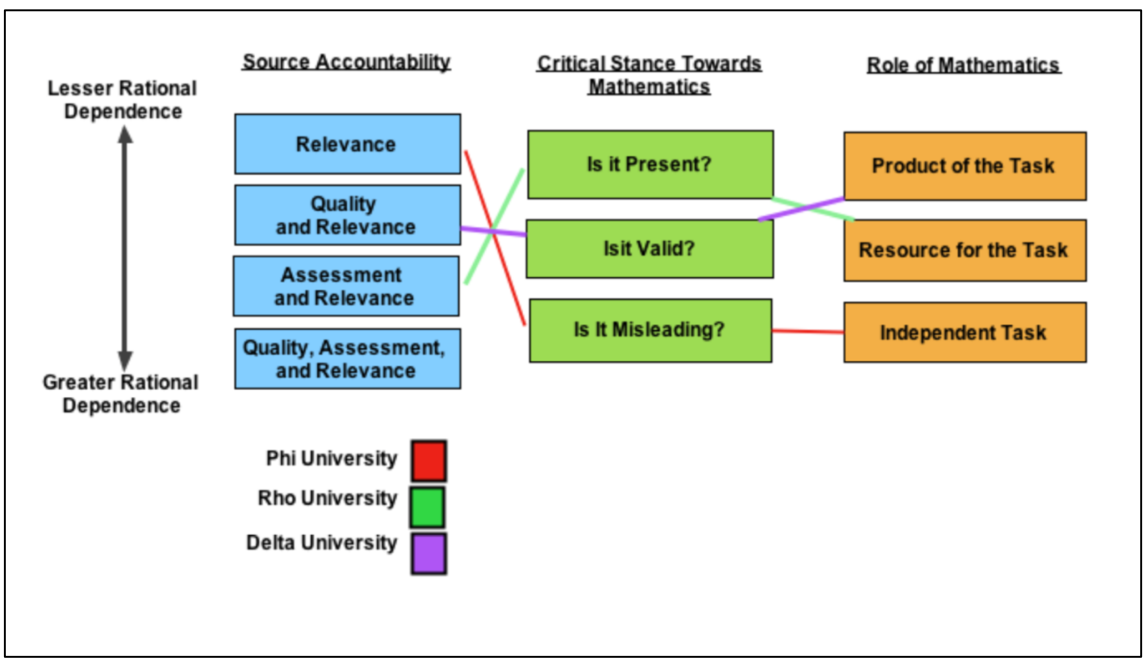


**Figure 1.** Opportunities for rational dependence in each of the three cases.

## Source Accountability

The information-based problems, as specified by these teachers, differed with respect to the way in which the students were held accountable for the sources that they found. This feature appears as the leftmost column in Figure 1 and contains four categories that I have ordered based on the degree to which the feature presents opportunities for rational dependence. The least conducive to rational dependence are those problems for which students are held accountable for finding sources that are *relevant* and nothing more. For example, students participating in the debates at Phi University did not need to say anything about the quality of their sources nor did they have to account for the process through which they decided on those sources. At Delta University, students were required to assess the quality of their sources through an analysis of the researchers' sampling methodology – this also provided students with an opportunity to bring to bear the statistics that they had been learning. The greatest opportunity for engaging in rational dependence with respect to sources of information, however, occurred at Rho University where students were accountable for the process that they used to find sources and were asked to compare sources to one another. Thus, students engaged in the type of work described by Gaon and Norris (2001) by comparing different accounts of the same phenomenon and the grounds by which one account might be prioritized over another. For example, students prioritized research published in scholarly journals by relevant experts in the field over reports by journalists who had a record of supporting one side of the debate over the other.

## Critical Stance Towards Mathematics

In the second column of Figure 1, I refer to the manner in which the teacher intends their students to take a critical stance towards mathematical claims. As these were mathematics classes, the students were expected to be critical of mathematical content specifically but, perhaps surprisingly, this expectation took a different form in each of the three cases: the students at Phi University were encouraged to watch for biased mathematical content, at Delta University they were asked to assess the validity of the mathematical argument supporting the claims, and at Rho University they were simply told to check for the presence of mathematical backing in the form of a correlation coefficient or a confidence interval. This last approach is positioned as least conducive to rational dependence in Figure 1 because the presence or absence of mathematical content does not say anything consequential about either the validity of the arguments being made by a source or whether a source's claims are supported by a broader epistemic community. Assessing the validity of the mathematical argument used by a source is more conducive to rational dependence because answering that question gives a better sense of whether a sources' claims are supported. However, that approach fails to account for the possibility that the author of an article might present a mathematically invalid argument due to their lack of mathematical knowledge even if the claim is held to be true by those with expertise in the area. Indeed, it is commonly held that popular science reporting falls prey to this exact problem (Bubela et al., 2009). Thus, the approach most consistent with the development of rational dependence is to determine whether a source has a bias and to stay conscious of how the relevant research is being framed by the source in question (Bubela et al., 2009).

### Role of Mathematics

Finally, in the third column of Figure 1, I address the role of mathematics in these information-based problems. Ideally, this role would be consistent with the manner in which rational dependence arises in out-of-school contexts where the goal is to educate oneself more broadly about a specific claim rather than a mathematical concept in isolation. Information-based problems where the mathematical task is independent are the most consistent with rational dependence since this approach best reflects the reality that mathematics is one tool among many for approaching such problems. This is how the debates at Phi University treated the use of statistical charts, it was a component of the debate, but not treated as the determining factor for deciding on the credibility of a source. It is less consistent with rational dependence to give the identification of mathematical content a central role in credibility assessment, as occurred at Rho University. For example, by directing students to focus on the presence or absence of a correlation coefficient, the teacher led students to use this one element of a source as a marker of credibility over and above other important elements such as its institutional affiliation or the presence of corroborating sources. The activity at Delta University positioned the role of mathematics in a way that was least consistent with the practice of rational dependence as the product of the task prioritizes mathematics as the sole determinant of credibility.

## Discussion and Conclusion

Although this cross-case analysis is only a first look at the introduction of information-based problems to quantitative literacy-focused instruction, the present findings provide some guidance for future iterations of these activities. First, they suggest the importance of holding students accountable for both how and why they choose sources to address the problems that they are assigned. This was deftly achieved at Rho University by the teacher when she facilitated small-group discussions in which students had to defend their choices to one another and come to a consensus about which sources were the most credible. Second, pushing students to reflect on the larger context in which their sources are providing information will provide them with an opportunity to identify the existence of competing explanations for the phenomenon under investigation as well as possible sources of bias. Finally, even though mathematical concepts must necessarily remain the focus of a mathematics class, it may be productive to provide students with a perspective on their problem-solving work that foregrounds the limited role of mathematics in establishing the credibility of sources. This does not have to mean that the mathematics itself is given short shrift, after all, the students need to be provided with the opportunity to develop the quantitative literacy skills that will help them engage in information problem-solving in their everyday lives – however, it does mean that students should be reminded that there are other facets to the information problem-solving process and that quantitative reasoning is only one, albeit important, element of their repertoire.

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Sensemaking in Statewide College Mathematics Curriculum Reform

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*We draw on the theory of sensemaking and sensegiving to characterize the social cognitive aspects of transformative organizational change in the context of statewide college mathematics curriculum reform efforts with the goal to understand the barriers to implementing the forms. In order to further understand the change process and the challenges these changes present, we conducted interviews with institutional, state, and national leaders of these efforts.*

*Keywords:* Sensemaking, Corequisite Remediation, Math Pathways, Faculty Change

The public institutions of higher education in Oklahoma are currently in the process of designing and implementing corequisite remediation and mathematics pathways. The goal of these efforts is increased student success in introductory mathematics courses and, ultimately, increased persistence to certificate and degree completion. In contrast to the traditional remediation model which places undersparred students into non-credit barring courses, corequisite rededication places underprepared students directly into college-level courses with assistance. The goal of math pathways is to create alternatives to the college algebra/calculus sequence to better serve students in primarily non-STEM degree programs. The alternative pathways provide students with mathematics courses which are more applicable to these other majors, incorporating relevant skills and applications. Preliminary data show promising results from these reforms (Wilson & Oehrtman, 2017). In this paper, we aim to characterize the social cognitive aspects of the reform process drawing on constructs of sensemaking and sensegiving in order to characterize the barriers or challenges in implementing systemic reforms.

## Theoretical Perceptive

In this section, we give a brief overview of the theory of organizational change. Van de Ven and Poole (1995) define organizational change as, "an empirical observation of difference in form, quality, or state over time in an organizational entity" (p. 512). However, as Kezar (2001) notes, this characterization ignores the individual's perception, which can be just as important. Therefore, we use Kezar's definition that amends Van de Ven and Poole's definition to include, a perceived "difference in form, quality, or state over time in an organizational entity." We define *a change process* to be a series of change events. An individual attempting to invoke or direct a change process is called a *change agent*.

Organizations are continually changing. The question is, to what degree or to what scale is that change occurring. First-order change or organization development is continual change or change within normal operating practices. First-order change does not require a substantive shift in participants' beliefs. However, second-order change (transformational change or deep change) does require a substantive shift in beliefs or values of the members of the organization. Second-order change is a substantive divergence from previous operating practices. As such change requires a shift in beliefs, the process is often studied by using theories that emphasize cultural or social cognitive aspects. (Kezar, 2014).

One such theory is *sensemaking*, so-named by Weick in 1979, which has been employed by many theorists and researchers throughout a large body of interdisciplinary literature.

Sensemaking is how individuals continually create and understand their reality and role in the organization. Sensemaking is inherently both an individual and social process. Organizations are made up individuals who have to make sense of their reality which is, in-turn, shaped by their social context in the organization. Therefore, the organization changes as a response to the sensemaking activities of its members then members must then make sense of new realities. Through such a dialectic, Weick (1993) explained, “the basic idea of sensemaking is that reality is an ongoing accomplishment that emerges from efforts to create order and make retrospective sense of what occurs” (p. 635). The key to sensemaking is its retrospective nature (Weick, 1995). In a transformational change process, sensemaking is used to plan for the future by reflecting on “errors” of the past, “tight implications formed in hindsight, are wrong because the future is actually indeterminate, unpredictable” (Weick, 1995, p. 28). Weick goes on the say that, “the past has been reconstructed knowing the outcome.” That is, any errors of the past are only errors now that we know the outcome. For individuals involved in a change process, “sensemaking is about understanding a change and making it meaningful” (Kezar, 2013, p. 775). In summary, sensemaking is how individuals involved a change understand the change, in particular how they make sense of their role in the changing organization and how they understand and interpret past events.

Often, sensemaking is studied in conjunction with the idea of *sensegiving* from Gioia and Chittipeddi (1991). Sensegiving describes how the change is disseminated and framed to those involved or affected by the change. That is, sensegiving is done to help others engage in sensemaking. Kezar (2013, p. 763) defines sensegiving as “influencing the outcomes” of a change strategy. Gioia and Chittipeddi, (1991) devolved a four-stage process for change linking, sensemaking and sensegiving. The stages are envisioning (an aspect of sensemaking), signaling (sensegiving), re-envisioning (sensemaking), and energizing (sensegiving). In the envisioning phase, change agents develop a strategic vision for the change. Overlapping the envisioning phase, *signaling* involves communication about the change by its agents to those it affects. The individuals affected by the change process voice feedback stemming from either support or opposition in the re-envisioning phase, which may cause a modification of the change process. Finally, in the energizing stage of effective large-scale reform, a broad coalition is built as the changes begin to be implemented.

Gioia and Chittipeddi describe these phases in terms of cogitation and action. Sensemaking is a mental understanding that one must engage in while sensegiving is an action that one does to affect or influence the change.

In 2013, Kezar found that, when looking at a campus wide-effort to implement interdisciplinary learning, the change process did not follow a linear path. In particular, they found that change agents must transition back and forth between the sensemaking and sensegiving often when implementing the change process.

## Methods

As discussed in the previous section, the purpose of the study is to understand the change process as well as the challenges in implementing corequisite remediation and mathematics pathways. In order to answer this question we conducted semi-structured interviews with institutional, state, and national leaders involved in the Oklahoma effort. The purpose of conducting the interviews was to understand the affordances and barriers in implementing the reforms in Oklahoma, including obstacles faced by faculty, mathematics departments, and institutions. The aim was not only to situate the reforms in a national context, but to understand how the efforts of different states across the county can inform the efforts here in Oklahoma.

The participants of the study were individuals involved in the reforms in Oklahoma and individuals involved in the national reforms who were engaged in supporting the Oklahoma reform efforts. Interviewees consisted of co-chairs from state task-forces on corequisite remediation and math pathways, instructors involved in implementing pilot courses at OSU, and administrators/faculty at institutions across the state. The interviews were conducted during the spring of 2017 which included a total of 10 participants. The interviews were conducted in person or over the phone and were audio recorded. Each of these participants has a unique perspective from their role in the change efforts.

The interviews were transcribed and coded using open coding. The aim was to understand the barriers to implementation and analyze the change process. Therefore, in coding we looked for challenges that change agents faced in the process of implementing the reforms and how the participants engaged in sensemaking through the barriers. That is, we looked for instances where change agents would be engaged in sensemaking, in particular, where one would try to make retrospective sense of past events or engage in envisioning changes. Narratives describing the challenges for each corequisite remediation and math pathways are given in the results sections as well as a characterization of the change process through the lens of sensemaking.

## Findings

Over the course of conducting the interviews, several themes emerged. First, we discuss the change process using the sensemaking perspective and then discuss the challenges unique to each of these change efforts.

### Change Process

The change process began in 2011 when the state signed on to CCA's agenda and specified the following three goals though the state regent's office: (1) improve mathematics preparation of students entering college, (2) reform mathematics remediation to be more effective and (3) strengthen mathematics preparation for all majors.

In order to address the second goal, in April of 2012, the Oklahoma State Regents held the first of several statewide meetings, *The Remedial Reform Summit.* This summit, facilitated by presentations from individuals outside the state, included discussions about the current remedial landscape. This summit led to the regents holding the *Mathematics Faculty Conference* in September of 2012 with 150 mathematics faculty and administrators to identify "a systemwide strategic approach of encouraging and implementing innovation to improve student success" (Oklahoma State Regents for Higher Education, 2015, p. 5). One the outcomes was the creation of the working group called the *Math Success Group.* The Math Success Group held a planning meeting in September of 2013, which lead to strategies to address three goals. One of the strategies to address the second goal was to offer corequisite courses.

The *Math Pathways Taskforce,* formed in 2014 following Oklahoma joining the Charles A. Dana Center's New Math Pathway's Project, consists of mathematics faculty from each public institution. The taskforce published recommendations in February of 2017. In March 2016, CCA and OSRHE hosted the Corequisite at Scale Conference. In April of 2017, the Dana Center and the OSRHE hosted the math pathways meeting.

These meetings allowed for collaboration and for attendees to engage in sensemaking as a social activity. In particular, attendees we able to engage with the idea of corequisite remediation reflecting on the past remediation model.

The taskforce and workshop attendees are able to gather information and envision the changes while engaging with other attendees. As one of the faculty members and co-chair noted:

> I think one of the challenges is to make every feel that they have a voice at the table that they are involved. I think the workshops are tremendous in that…. everybody expresses what their opinion, their idea is of what we're doing, and then it's discussed and then things are sorted. And tables get together and pick these particular topics. And then tables discuss them in small groups all the topics get discussed. Then the tables report back and that's refined over a period of time. And so you get a lot of buy-in, because everybody's had a voice and a chance to say what they think is important what the challenges are and how they think it should be addressed and sit there and discuss with other people.

Another faculty member had similar sentiment:

> We've gone through such a slow and deliberate process where we've engaged everybody in the conversation and it's clearly not been a conversation 'you do this and that' right it's been even from the beginning let's clearly identify what problems we're trying to solve right and what we think are promising solutions and explore those...

Moreover, the process and open-ended nature of the workshops afforded attendees the opportunity to engage in sensemaking and construct their own sense for the changes. In particular, the attendees can understand the change and envision the change on their campus and what the challenges will be for their particular campus. As each of these campuses has a unique mission and goals, its challenges will be unique as well. Statewide meetings can facilitate understanding the challenges entailed in these changes, which was highlighted by one faculty member:

> And so there's a little bit of kinda mismatch of perspective right when you think of the goals of each institution what they're trying to implement. That's required some navigating and frankly just even understanding. You know we don't even know there are issues like that until we all sit down at the table together and say 'why are you trying to do this?', well because our students have these needs and they don't already have a stat course right. Oh that's not even part of our perspective because we're not dealing with associates degrees.

After the workshop, the attendees were able to return to their campuses to engage in sensegiving with others, helping others understand what the change will entail on their individual campuses and creating buy-in. In the language of Gioia and Chittipeddi (1991), these meetings constituted an envisioning phase of the reform, laying the ground work for what the change will entail across the state. In a signaling phase, the taskforce presented recommendations and engage others on at their institutions in the change process. On their respective campuses, faculty and administrators can engage in the re-envisioning phase by framing the reforms in the context of their campus. The institutions then began to implement these changes in the energizing phase. The process here is iterative as the implementation has involved pilot programs.

## Challenges in Corequisite Remediation

In the efforts to implement corequisite remediation, many of the challenges are related to policy and implementation. One of challenges articulated by the participants of this study was assessment and placement of entering students. The goal is to place students into mathematics courses in which they will succeed without unnecessarily extending their sequence of required courses. In order to place students, an assessment mechanism is needed, however, some are costly. As one community faculty member noted, "I know that [one of the research institutions] uses ALEKS but that's little cost prohibitive for us." One administrator noted,

> The fact that we have the placement test with the online learning modules that's something that we are big enough that we have an economy of scale we can pay for by imposing fees

> and we have an apparatus to deal with it with the testing center… Having that large apparatus and this large infrastructure makes those sorts of things easier in some respect.

These challenges highlight the different experiences between two-year and four-year institutions. Additionally, there are logistical challenges of placing students in corequisite sections. These logistical challenges are highly localized. For example, the implementation at the research institution was so that the course did not appear any different on the transcript between corequisite and non-corequisite students. As the administrator responsible for implementing the corequisite course at the institution noted:

> Our big challenge right now [the enrollment management system] can't really test for this population of students. So we have the corequisite sections set up as department permission only.

Because of this, the department had to manually add each of the students in the corequisite courses. The administrator continued,

> That's a labor intensive sort of thing, I haven't figured out a better way to do it. It's worth it to me to use that labor to help the students.

While the model had its challenges, the university did not want to offer 0-credit hour courses nor did it want to give students five credits for the course. Using a 0-credit hour course in some form to deliver the supplemental instruction also comes with its own challenges. Depending on whether the instructor is the same for the regular course as for the supplemental course can lead to challenges either way. If the instructor is the same, scheduling can be difficult. However, the benefit can be that the instructor is familiar with where the students are at and what they know. These challenges in scheduling lead one community college to attempt to intermix the students. That is allowing students regardless of preparation to enroll in any college algebra section and using 0-credit hour supplement course which are not tied to any particular section. However, this approach introduces its own challenges with communication and as faculty member at the college noted "making sure students get enrolled in both," which is particularly challenging as the course is not linked to a supplement course. Moreover, not all the students in the college algebra sections need to enroll in a supplemental section as some are college ready.

**Challenges in Math Pathways**

The most prevalent challenge to emerge concerning math pathways was on course transferability, particularly from two-year to four-year institutions. The issue is ensuring courses with the same or equivalent content transfer between institutions, so students who transfer are not taking duplicate courses. In Oklahoma, the *Course Equivalency Project* or CEP[1] is a matrix, which codifies course transfer between public Oklahoma Institutions. That is, the matrix guaranties that a course *will* transfer and *how* the course transfers. Currently, College Algebra transfers between every public institution as it is listed on the CEP, and the course can be used to satisfy the degree requirements. However, course transfer is not guaranteed for alternative pathways. The focus of the change agents and change leaders to address this challenge is to add the new pathways to the CEP. By adding these alternative pathways to the CEP, students taking for example the course for the modeling pathway at an Oklahoma institution will transfer to any other institution as the equivalent modeling course.

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[1] The CEP matrix can be found http:/www.okhighered.org/transfer-students/course-transfer.shtml

The issue of course transferability is particularly relevant to the two-year colleges where students intend to transfer to a four-year university. Faculty at a two-year college explained they will not offer a course until they can know it transfers, that is until it appears in the CEP matrix,

> We've also got a new modeling course, but I haven't put it in the schedule yet because I'm not sure it's going to transfer. Once we get that pathway developed and in the matrix, I'll put it in the schedule.

Also, important for course transferability is ensuring the course satisfies the desired degree requirements at the transfer institution, otherwise a student would need to take another mathematics course. This issue came up in context of the statistics pathway:

> …when talking about statistics pathways right, so our two-year schools their associates in something like psychology currently has no statistics in it, so adding statistics as part of the pathway kinda makes sense for them, cause they are giving them some content and experience that's useful. Now at four-year school where the bachelors program in psychology students are taking already have lots of statistics built in adding another statistics course that repeats some of the most of the content maybe at a lower level that's already in their program. It doesn't really help right. You're trying to put a course in there they've already got.

Solving the transfer problem and offering courses is only the first challenge. One also has to get students enrolled in these courses. This change has multiple parts, one needs to get the degree programs to allow alternative pathways and have advisors place the students into the courses:

> These are real challenges. The client disciplines getting them on board and doing it. The math departments may be on board, but you've got to get the client disciplines on board. Then you've got to get the advisors on board. The advisors are going to be the one that when you come in say what you want to be, they're going to tell you which pathway to go to. Well a lot of them are scared. Cause they've always told people you better take college algebra, it transfers anywhere. If you take modeling, you may not be able to transfer that and get your requirement, so you'll have to take college algebra when you get there, and so that's a real problem.

Mathematics departments will need to engage other departments and advisors in sensegiving. Specifically, they will have to help client departments and the advisors make sense of the College Algebra alternative and what these courses would mean for their students. Advisors need to understand the goals of each student and which course will best help the students achieve those goals. This challenge is compounded by the fact that many students enter college without a declared major. Placing these students into the correct course can be particularly challenging. However, the pathways are intended for broad groups of majors, hereby enabling students to select a field instead of a more specific major when entering.

## Conclusion

The change process to design and implement statewide reform of corequisite remediation and new mathematics pathways is multileveled and highly collaborative, involving individuals from the regent's office to campus administrators to advisors to mathematics faculty, with many of these individuals involved in leading and shaping the change efforts. The numerous participant meetings, workshops, and taskforces helped to facilitate and guide the change process. The statewide meetings helped attendees to engage in sensemaking and sensegiving activities which enabled broader buy-in. Additionally, the statewide meetings helped others make sense of the implementation process on their campus, but also allowed individuals to see the changes faced by other institutions.

In this paper, we have identified a few important challenges; however, there are many more challenges. Some of these may be implementation or institution specific. Additionally, there will undoubtedly be challenges in the classroom.

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Adapting an Exam Classification Framework Beyond Calculus

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*This paper reports a methods-building project that seeks to make inferences about mathematics instructors' teaching practices from their exams. We adapt and revise a framework by Tallman et al. (2016) and expand its applicability across the undergraduate curriculum, beginning with a sample of seven exams from early-career mathematics instructors. We describe the rationale for the adaptation process and patterns of differences between exam sets. Future work includes coordinating this analysis with results from other data sets from the same instructors.*

*Keywords:* Exams, Teaching Practice, Methods-building

This paper is part of a larger research project that seeks to detect changes in the teaching of individual instructors. To accomplish that, we are working to develop and coordinate methods that capture aspects of instructors' teaching from diverse data sets including syllabi, classroom video, instructor and student surveys, and classroom artifacts such as assessments. We expect that these data sources capture different aspects of instructors' teaching, and they vary in how invasive or expensive they are to collect and analyze. Rather than hoping to argue that one of these data sources and methods is best, we seek to understand the affordances of each method and the kinds of questions that are appropriate for each for the purposes of detecting change through professional development. Additionally, we hope to help clarify the desired outcomes of professional development, the change trajectories of participants, and the kinds of evidence that demonstrate that change is occurring. The results of this project could be used to help assess and improve professional development for mathematics instructors, which may in turn support the shift towards active and inquiry-based pedagogies advocated by professional organizations for collegiate mathematics instructors (CBMS, 2016).

This paper focuses on course exams to ask the following question: What can an instructor's exams tell us about that person's teaching? Exams are generally high-stakes assessments, so presumably they represent the values, beliefs, practices, and theory-in-use (Argyris, 1976) of the instructor who authored them (Black & Wiliam, 1998). However, timed exams also put constraints on the kinds of activities that are possible for students, so we do not expect exams to capture all of an instructor's perspective in general. Exams are easy to collect; they are also authentic artifacts in the sense that they are created as part of the course, for the students. Thus, we articulate the detailed research question:

- Can we build or adapt a coding scheme that detects patterns and differences among instructors' exams in order to support inferences about their teaching?

This phase of the project is methods-building, so this paper emphasizes the development of a scheme for coding exam items from undergraduate mathematics courses across the curriculum. We include some results from coding of a small study sample as evidence that the resulting scheme captures patterns and differences among instructors' exams that in turn may offer evidence about their instructional choices.

Later stages of this program could develop a theoretical perspective on the aspects of teaching and their hypothesized relationship to professional development, but we are not yet at the stage where we can articulate such a framework. Instead, we seek to develop methods that focus our attention on aspects of instructors' teaching, seek patterns and connections within or across these data and methods, and use both our theoretical sensitivity as researchers and our experience as teachers and professional developers to identify potentially meaningful observations.

## Literature Review

We are building a scheme for coding exams to learn about instructors' perspectives on teaching, so we focus on the requirements that they make of students in exam items. Subsequent to this paper, we will coordinate this scheme with analyses of other aspects of these instructors' practice, so our scheme must be independent of specific knowledge of other elements of the course or the students' backgrounds, though it can depend on a coder's more generic knowledge of undergraduate mathematics.

We draw on work of Tallman et al. (2016) to summarize some prior research that has examined individual mathematical tasks. Li (2000) built a three-dimensional coding scheme to assess whether the item required one or more mathematical procedures, whether the item was purely abstract or set in an illustrative context, and what format and cognitive demand were required for a response. It is difficult to determine the grain size of a single procedure without information about the specific course context, but the other dimensions of this scheme align with our goals. Lithner (2004) focused on the potential student strategies for seeking a solution to an exam problem; our project focuses on what is expected of all students in common rather than on potential differences. Smith et al. (1996) and Anderson and Krathwohl (2001) produced coding frameworks that modify and update Bloom's taxonomy. Bloom's taxonomy has been critiqued because the actual cognitive demand of any task depends on the individual student's prior experience, but we accept that an instructor can have a well-defined intended cognitive demand for a task, and that a coder with mathematical expertise could assess this intent from the exam. Mesa et al. (2012) used Charalambous et al. (2010) to incorporate information about representations and metacognition in their coding framework. These dimensions align with our goals, but we focus on how they are required by the instructor rather than on possible student understandings and approaches they support.

As part of a project to determine characteristics of successful programs in post-secondary calculus, Tallman et al. (2016) developed a scheme for coding individual items on Calculus I final exams, called the Exam Characterization Framework (ECF). The ECF has three dimensions: *item orientation*, which captures the cognitive demand required to respond successfully to the item; *item representation*, which captures representations and other objects in both the task and required response to the item; and *item format*, which captures the structure and scope of the expected response to the task. Consistent with their critique of prior work, we observe that the ECF aligns with our own approach except for its exclusive focus on calculus. They applied the ECF to a large corpus of exams from 2010/11 to develop a summary of the expectations of calculus courses, and they contrasted these exams with a sample from 1986/87 to describe the impact of 25 years of reform efforts. Based on this work, we determined to start our scheme-building process by trying to adapt or generalize the ECF to a broader context.

## Methods

Our data set is the exams (or mastery quizzes) from seven instructors who had completed a professional development program for early-career mathematics instructors. We expect this population to exhibit a range of teaching behaviors, styles, and skill levels. This sample is small because we collected multiple other kinds of data, including classroom video, from the same instructors (not discussed here). These seven instructors are teaching abstract algebra, discrete mathematics or introduction to proofs, content courses for future elementary teachers, introductory statistics, or calculus II/III. The first author, who is the main coder, has the credentials to teach all of these courses and has experience teaching courses similar to 6 of them. The data set includes 208 items from 13 distinct assessments from these seven courses.

The first author familiarized himself thoroughly with the ECF as described in Tallman et al. (2016) and then attempted to use his understanding of this framework to code the seven sets of exams, along the way adapting and revising the ECF into a new but related scheme. The goal was to develop a coding scheme that was applicable across undergraduate mathematics courses, that captured all aspects of exam items that seemed to speak to larger patterns in the instructor's teaching, and that was articulated in an internally coherent way that supported reliable coding and distinction between codes. As he coded, the first author noted items for which his current understanding of the framework was not sufficient to assign definitive codes; he also noted aspects of items that were not captured by the codes. He later repeated this process and then compared the codes and comments as an indicator of intra-coder reliability. He then revised his interpretation of existing codes and defined new codes based on repeated comments; these revisions required overt articulations or re-articulations of the hierarchical structure of the codes in each dimension. Finally, he repeated this process of coding and revision until the framework and its interpretation stabilized.

For two of these cycles and for the stable framework, the first author presented examples of coded items, rationales for changing the framework, and descriptions of the hierarchical structure of each dimension of the scheme to the second author as sense-making checks; these checks were an initial effort to establish face validity in our study context. These discussions emphasized consistency in interpreting individual code definitions and coherence and discrimination across the framework components.

## The Item Characterization Framework

The resulting framework, which we call the Item Characterization Framework (ICF), contains three broad dimensions: item orientation, item format, and item components. These dimensions are analogous to those in the ECF, but include new categories and codes (Table 1).

*Table 1: Dimensions, Categories, and Codes in the Item Characterization Framework*

| **Item Orientation** | | | **Item Format** | | | | **Item Component** |
|---|---|---|---|---|---|---|---|
| Cognitive Demand | Familiarity | Certainty | Breadth | Format | Formality | Other Support | Task/Response |
| Remember | Recreate | Low | Single | Multiple choice/TF | No support | Neither | Applied/Modeling context |
| Recall and apply Procedure | Adapt | Medium | Forked | Fill in the blank | Informal support | Interpretation/ Context | Symbolic representation |
| Recall and reproduce argument | New | High | Delineated | Short answer | Formal support | Control/ Evaluation | Verbal representation |
| Understand | | | Open | Long answer | Unclear | Both | Graphical representation |
| Apply understanding | | | | | | | Tabular representation |
| Analyze | | | | | | | Statement (Thm/Dfn) |
| Evaluate | | | | | | | Claim (Conj/Arg) |
| Create | | | | | | | (Counter-) Example |

### Item Orientation

This dimension captures the assumed cognitive demand of producing a successful response to the item. The category *Cognitive Demand* uses an expanded version of Bloom's Taxonomy. *Recall and reproduce argument* is the only novel code here; this code is analogous to *Recall and apply procedure* but applicable to the context of proof construction.

The cognitive demand of a task depends heavily on the student's past experience with the task (Anderson & Krathwohl, 2001). Both ECF and ICF assume that the coder holds an understanding of the generic undergraduate curriculum and student; in ICF, these assumptions are made explicit by coding how novel the coder believes the task to be for the intended student in *Familiarity* and their confidence in this assessment (and thus of *Cognitive Demand*), in *Certainty*.

### Item Format

This dimension captures the structure of a required response. *Breadth of Responses* captures whether there is a single or multiple acceptable form(s) for a successful response, as well as whether that form is overt in the task statement for the student. *Format of Responses* captures the extent to which the structure of a successful response is provided for the student. *Formality of Response* captures the explicit requirements for justification and support for a successful response. *Other Support* captures the extent to which the item requires the student to corroborate conclusions with secondary evidence or metacognition (e.g., checking work).

The ECF also has a dimension called *Item Format* that is significantly revised in the ICF. The ECF *Item Format* codes blend ideas of breadth, format, and support; additionally the ECF *Item Representation* code *Explanation* contained ideas that blended formality of support and other support with item components. Splitting and rearranging these ideas in this fashion represents the largest revision from ECF to ICF. The shift is from questions about the overt structure of the task and response to questions about how much of the structure of the task and response is made explicit or unknown for a student.

### Item Components

This dimension captures the representations, objects, and statements in the item. These codes apply separately to both the task statement and the required response. For example, *Statement* is applied to tasks that contain a statement, such as a theorem or definition, whose truth-value is (framed as) known, while the same code applied to a response means that the student is required to produce such a statement. The *Claim* code is applied to tasks containing statements with unknown truth-value and to responses that require students to decide on the truth-value of a statement or to generate a statement with unknown status, such as a conjecture.

## Data and Results

Tables 2 and 3 summarize the frequencies and ranges seen for the exams in this data set. Averages are computed from the percentages of each exam set, rather than from the total collection of items, to give the same weight to each participant.

### Comparing Courses

**Item Orientation and Item Format.** To compare exams, we label each as average (within 10% of the group average), or otherwise high/low frequency for each code. For example, P1 and P3 have low frequencies of items with single answers; P1 has correspondingly high frequency on questions with forked responses, while P3 is high on delineated and open items. Similarly, P1 and P3 are both low in items asking students to remember declarative facts, but P1 is high in terms of asking students to reproduce arguments, and P3 is high in tasks that ask students to apply understanding or analyze.

*Table 2: Observed frequencies and ranges for Item Orientation and Item Format codes*

| Item Orientation | | | Item Format | | |
|---|---|---|---|---|---|
| Code | Average | Min-Max | Code | Average | Min-Max |
| Remember | 14% | 0% - 36% | Single | 64% | 16% - 100% |
| Recall and apply proc | 30% | 0% - 66% | Forked | 22% | 0% - 79% |
| Recall and reproduce argument | 12% | 0% - 47% | Delineated | 7% | 0% - 36% |
| Understand | 0% | 0% | Open | 7% | 0% - 18% |
| Apply understanding | 38% | 19% - 69% | Multiple choice/ TF | 16% | 0% - 38% |
| Analyze | 2% | 0% - 8% | Fill in the blank | 11% | 0% - 39% |
| Evaluate | 5% | 0% - 20% | Short answer | 44% | 11% - 62% |
| Create | 0% | 0% | Long answer | 28% | 2% - 74% |
| Recreate | 37% | 7% - 79% | No support | 40% | 0% - 93% |
| Adapt | 60% | 21% - 93% | Informal support | 33% | 3% - 57% |
| New | 3% | 0% - 19% | Formal support | 27% | 0% - 74% |
| Low certainty | 3% | 0% - 14% | Unclear | 0% | 0% - 3% |
| Medium certainty | 24% | 2% - 46% | Neither | 87% | 62% - 100% |
| High certainty | 73% | 54% - 98% | Interpretation/ Context | 1% | 0% - 5% |
| | | | Control/Evaluation | 9% | 0% - 38% |
| | | | Both | 2% | 0% - 14% |

*Table 3: Observed frequencies and ranges for Item Component codes*

| Item Components | Task | | Response | |
|---|---|---|---|---|
| Code | Average | Min-Max | Average | Min-Max |
| Applied | 13% | 0% - 46% | 8% | 0% - 32% |
| Symbolic | 71% | 12% - 100% | 64% | 16% - 100% |
| Verbal | 15% | 0% - 64% | 12% | 0% - 56% |
| Graphical | 15% | 0% - 40% | 12% | 0% - 56% |
| Tabular | 9% | 0% - 31% | 6% | 0% - 15% |
| Statement | 14% | 0% - 38% | 9% | 0% - 46% |
| Claim | 33% | 4% - 79% | 33% | 0% - 79% |
| Example | 8% | 0% - 31% | 16% | 0% - 44% |

Of potential interest for detecting instructors' authentic instruction practices (Gulikers et al. 2004) through exams are the codes that capture uncertainty and open-ended tasks. In Table 4, we summarize the data by participant for four such codes or combinations: analyze, evaluate, and create (A+E+C); new; forked, delineated, and open tasks (F+D+O); and claim.

*Table 4: Observed frequencies for combined uncertainty/open-ended codes by course*

| | P1 | P2 | P3 | P4 | P5 | P6 | P7 |
|---|---|---|---|---|---|---|---|
| A+E+C | 5% | 8% | **28%** | 0% | 7% | 0% | 0% |
| New | 0% | **19%** | 0% | 0% | 0% | 0% | 0% |
| F+D+O | **84%** | 0% | **80%** | 31% | 29% | 13% | 12% |
| Claim | **79%** | **54%** | **48%** | 38% | 4% | 7% | 36% |

**Item Component.** We separate the task and response component codes. Table 5 shows that P3 is high frequency in five *Item Component* subcodes, which is more than the other exam sets, and is also the only course to be high frequency for more student response subcodes than task codes.

The ICF appears to capture the distinctive demands of teaching subfields of mathematics. P5 is the statistics course, and it has the highest frequency of applied components. P1, P2, and P4 are proof-based courses that have symbolic representations in every task and response; P2 and P4 are introductions to proof, with high frequencies of theorem and definition statements in tasks. P3 and P6 are courses for pre-service elementary teachers with lower than average use of symbolic representations and higher than average use of graphical and geometric representations. The high frequency of *Recall and apply procedure* in P5 and P7 may encode the fact that they are lower-division, computational courses.

*Table 5: Item Component code frequencies (light/medium/dark represents low/average/high frequency)*

| **Task** | P1 | P2 | P3 | P4 | P5 | P6 | P7 |
|---|---|---|---|---|---|---|---|
| Applied | 0% | 0% | 20% | 0% | 46% | 23% | 0% |
| Symbolic | 100% | 100% | 12% | 100% | 75% | 23% | 86% |
| Verbal | 5% | 0% | 64% | 0% | 0% | 20% | 12% |
| Graphical | 0% | 0% | 40% | 0% | 14% | 36% | 12% |
| Tabular | 0% | 3% | 0% | 31% | 25% | 7% | 0% |
| Thm/Dfn | 5% | 38% | 12% | 38% | 0% | 0% | 2% |
| Claim | 79% | 43% | 40% | 23% | 4% | 7% | 36% |
| Example | 0% | 8% | 20% | 31% | 0% | 0% | 0% |
| **Response** | | | | | | | |
| Applied | 0% | 0% | 20% | 0% | 32% | 2% | 0% |
| Symbolic | 100% | 100% | 16% | 100% | 43% | 36% | 52% |
| Verbal | 0% | 5% | 56% | 0% | 21% | 0% | 2% |
| Graphical | 0% | 3% | 56% | 0% | 0% | 18% | 5% |
| Tabular | 11% | 3% | 4% | 15% | 0% | 7% | 0% |
| Thm/Dfn | 0% | 46% | 0% | 0% | 18% | 0% | 0% |
| Claim | 79% | 54% | 48% | 38% | 4% | 0% | 7% |
| Example | 26% | 0% | 44% | 23% | 7% | 11% | 2% |

## Discussion

Our evidence tentatively supports the claim that the ICF also detects differences among teaching practices in similar courses. For example, P3 and P6 are both courses for future elementary teachers, but P3's exams include more higher-order and open-ended summary codes in Table 4, and more item components, especially those required in the response, while P6 is average or below in each of these indicators. We suggest that P3 is asking more, or perhaps more authentic mathematics, of students than P6, which could indicate that its instructor holds a more-developed teaching perspective. There are similar, if weaker, patterns of difference between P2 and P4 (introduction to proofs, discrete) and P5 and P7 (lower-division, computational, applied).

These analyses also highlight the ways P3, and to a lesser extent P1 and P2, are asking students to do mathematics that is potentially more authentic (Gulikers et al., 2004) on exams than P4, P5, P6, and P7. P1 accomplishes this by asking students to prove or disprove statements, P2 by asking students to work with new definitions, and P3 by asking students to evaluate arguments and to coordinate multiple goals simultaneously. The evidence and analysis above is consistent with the assertion that the ICF captures dimensions of teaching that are of interest to professional developers of mathematics instructors.

We would predict that items that use different representations in their task and response would be more complex and demanding for students. Tallman et al. (2016) use statistical methods to determine if task components correlate with response components or other codes, but this kind of analysis is not possible on our small sample. In a larger sample, this might highlight another aspect of more-developed teaching practice.

The next phase of the project will involve coordinating the analyses of these seven participants' teaching using other data sets, including their syllabi, video recordings of their classrooms, and surveys of both the participants and their students, for which method development and coding have proceeded independently. Initial conversation indicates that these different data sets generally highlight a similar subset of courses as exhibiting valued teaching aspects, but these data will also highlight different aspects of their teaching, such as espoused theory (Argyris, 1976) from instructor surveys to contrast with theory-in-use from exams.

We have not yet tried to code and test for inter-rater reliability. Thus far, reliability rests on three points of content validity. The first author re-coded the data repeatedly until the framework and codes stabilized, and justified codes and changes to the framework to the other researchers. Restructuring the item format dimension around epistemological questions in particular helped the team agree on their understanding of codes. Finally, the *Certainty* codes serve as a measure of confidence in the coding. The majority of items were coded as medium (24%) or high (73%) certainty. If higher certainty is desired, items coded as medium certainty could be resolved either by scanning the course textbook to see if the question was familiar or asking the instructor to complete a simple survey declaring the familiarity of each item on their exams. These approaches are both more invasive than simply collecting exams, but could be ways to gather this information easily in future rounds of data collection. Next steps for this project must include reliability testing across multiple raters.

Thus far, claims about the utility of the ICF rest on the analysis of a small sample, which is intertwined with the researchers' experience with professional development of mathematics instructors, including advocacy for active and inquiry-based pedagogies. The local goal is to develop a method for coding exams so that we can understand whether and how analyzing exams may be helpful to characterizing teaching. If this method proves useful, the larger goal is to detect change in the instructors who participate in these kinds of professional development. The target teaching outcomes for this kind of professional development are often broad; developing a measure that is focused on assessments and that can be applied widely across course topics may contribute to detecting change in dimensions not currently studied in other ways. We do not claim that an ideal exam is entirely higher-order cognitive tasks, but we do believe that high quality teaching would include requiring students to engage *some* higher-order tasks on exams and that ask students to work in uncertainty. We do not think that an ideal exam completely avoids symbolic representations, but we have valued those exams that avoid using only symbolic representations and that ask students to reason with multiple representations. We also need to connect the ICF to existing research to solidify these utility claims.

Future research could explore these questions, some of which are analogous to those explored by Tallman et al. (2016). Are exams for courses other than calculus changing across time in the discipline? What correlations exist among the codes in the ICF (in a sample large enough for statistical analyses), and how do these correlations depend on course level/domain? To what extent are mathematics students asked to use and translate between multiple representations in their (high stakes, timed) assessments? With additional instructor data, how do instructors' perspectives about their exams related to researcher analyses of the items, and what are the relationships between instructors' stated values and their assessment practices? Using a coordinated and longitudinal data set from professional development, do changes in exams lead or trail other teaching changes in response to professional development?

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# When "Negation" Impedes Argumentation: The Case of Dawn

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*Abstract: This study investigates one student's meanings for negations of various mathematical statements. The student, from a Transition-to-Proof course, participated in two clinical interviews in which she was asked to negate statements with one quantifier or logical connective. Then, the student was asked to negate statements with a combination of quantifiers and logical connectives. Lastly, the student was presented with several complex mathematical statements from Calculus and was asked to determine if these statements were true or false on a case-by-case basis using a series of graphs. The results reveal that the student used the same rule for negation in both simple and complex mathematical statements when she was asked to negate each statement. However, when the student was asked to determine if statements were true or false, she relied on her meaning for the mathematical statement and formed a mathematically convincing argument.*

*Key words:* Negation, Argumentation, Complex Mathematical Statements, Calculus, Transition-to-Proof

Many studies have noted that students often interpret logical connectives (such as *and* and *or*) and quantifiers (such as *for all* and *there exists*) in mathematical statements in ways contrary to mathematical convention (Case, 2015; Dawkins & Cook, 2017; Dawkins & Roh, 2016; Dubinsky & Yiparki, 2000; Epp,1999, 2003; Selden & Selden, 1995; Shipman, 2013, Tall, 1990). Recently, researchers have also called for attention to the logical structures found within Calculus theorems and definitions (Case, 2015; Sellers, Roh, & David, 2017) because students must reason with these logical components in order to verify or refute mathematical claims. However, Calculus textbooks do not discuss the distinctions among different connectives nor do they have a focus on the meaning of quantifiers or logical structure in algebraic expressions, formulas, and equations, even though these components are used in definitions and problem sets (Bittinger, 1996; Larson, 1998; Stewart, 2003).

Undergraduate students frequently evaluate the validity of mathematical conjectures that are written as complex mathematical statements. By *complex mathematical statements*, I mean statements that have two or more quantifiers and/or logical connectives. Other work has investigated students' understanding of complex mathematical statements (Zandieh, Roh, & Knapp, 2014; Sellers, Roh, & David, 2017). However, these studies focus on students' understandings of statements as written, and do not address students' meanings for the negation of complex mathematical statements. Several studies have investigated student meanings for negation (Barnard, 1995; Dubinsky, 1988; Lin et al., 2003), but these studies do not explicitly address complex mathematical statements from Calculus. In order for students to properly justify why Calculus statements are true or false, and for students to develop logical proofs, they must understand a statement in both its written form and its opposite (Barnard, 1995; Epp 2003). For example, students at the Calculus level are asked to determine if sequences are convergent *or* divergent, if functions or sequences are bounded *or* unbounded. Thus, we also must explore student meanings for negation in the Calculus context—both the negation of an entire statement, and the negation of its logical components. In this paper, I will investigate one student's meanings for the negation of various types of mathematical statements as well as how these negation meanings affected her justifications for several Calculus statements. Thus, I seek to investigate the following research questions for this student:

1. *As mathematical statements become increasingly complex, will a student keep the same negation meanings? If some or all of her negation meanings change, which meanings change and how do they change?*
2. *How do the student's meanings for negation affect her evaluations of complex mathematical statements from Calculus and her justification for these truth-values?*

## Literature Review & Theoretical Perspective

Both colloquialisms for quantifiers and logical connectives as well as mathematical content may affect students' logic in mathematics courses. If I claim, "Every book on the shelf is French," the statement may be viewed as false if there are no books on the shelf. However, in mathematics, this statement would be vacuously true if there were no books on the shelf (Epp, 2003). If I claim "I'll get Chinese or Italian for dinner" one would assume that I was going to *either* get Chinese or Italian, but not both. We often use an *exclusive or* in our use of the English language, but in mathematics, we would consider that this statement would be true if both propositions were true (Dawkins & Cook, 2017; Epp, 2003). Students may also change their use of mathematical logic depending on the content of a mathematical statement (& Cook, 2017; Durand-Gurrier, 2003). Dawkins & Cook (2017) presented students with the statements "Given an integer number *x*, *x* is even or odd" and "The integer 15 is even or odd." Some students claimed that the first statement is true, but the second statement is false because they already knew that 15 is an odd integer.

Even if students correctly interpret a mathematical statement, their negation of parts or all of a mathematical statement may follow different conventions. If a statement contains more than one quantifier, students often negate only one of these quantifiers (Barnard, 1995; Dubinsky, 1988). Students may also leave disjunctions or conjunctions alone in a negation (Epp 2003; Macbeth et al., 2013). For example, some students negated statements of the form $P \wedge Q$ as $\sim P \wedge \sim Q$. In general, for all negations, Dubinsky (1988) claims that students often use *negation by rules*. The rules they use may or may not be correct rules of negation.

There may be other student meanings for negation that have yet to be discovered. My goal in this study is to describe my best perception of one student's own meanings for negations of complex mathematical statements at different moments. I use the phrase "student meaning" throughout this paper the same way in which Piaget views that each individual constructs his own meanings by assimilation and accommodation to schemes (Thompson, 2013). A *scheme* is a mental structure that "organize[s] actions, operations, images, or other schemes" (Thompson et al., 2014, p. 11). I cannot see a student's schemes, but can only do my best to create a model of students' negation schemes by attending to their words and actions throughout the clinical interview process. Schemes are tools for reasoning that have been built in the mind of the student over time. If a student repeats the same type of reasoning repeatedly, they begin to construct their *negation scheme* until the scheme is internally consistent. If students face inconsistencies, then they may adapt, or accommodate their schemes.

Some student meanings may be stable, but other meanings may be "meaning(s) in the moment" (Thompson et al., 2014). Thompson et al. (ibid) describe a meaning in the moment as "the space of implications existing at the moment of understanding" (p. 13), so students could be assimilating information in the moment by making accommodations to their current schemes. A student's thoughts may begin to emerge or different meanings may be elicited in different moments. Thus, I consider several different moments of interaction for each student because different moments of interaction may result in different types of student negation.

## Methods

This study is part of a larger study that will seek to answer these research questions with undergraduate students from various mathematical levels. For this particular study, I conducted clinical interviews (Clement, 2000) with one student, Dawn, who is currently enrolled in a Transition-to-Proof (T2P) course. Dawn completed two clinical interviews that were each two hours long. Both interviews were video-recorded. One camera was used to zoom in on her work, while the second camera was zoomed out to capture her gestures. Different levels of tasks were chosen to determine if Dawn's negations stayed the same or changed across different levels of complexity. Clinical interview questions were used that would help me to determine why Dawn's negations stayed the same or changed across different tasks.

**Interview tasks.** I first presented Dawn with thirteen statements with one quantifier or logical connective to address my first research question. Two of these statements are shown in Figure 1 (left).

| **Statements with One Logical Component** | **Statement with Two Logical Components** |
|---|---|
| 1. Every integer is a real number.<br>2. 12 is even and 12 is prime. | There exists a real number $b$ such that $b$ is odd and negative. |

*Figure 1*. Selected items with either one logical component or two logical components.

Dawn was asked to evaluate (i.e. provide a truth-value) and negate each statement, as well as evaluate her negations. After she completed these tasks, I presented her with a list of "other students' negations." I created these hypothetical negations based on variations of changing different parts of each statement. These hypothetical negations allowed me to test a wider range of possible negations that Dawn might accept as valid negations.

I also conducted a follow-up interview with Dawn to compare her negations of one logical component with her negations of complex mathematical statements in an attempt to begin to answer the second research question. I first presented Dawn with two statements, like the one shown in Figure 1 (right), which involves two logical components (an existential quantifier and either a conjunction or disjunction). I asked Dawn to evaluate and negate these statements in the same manner as she did in the first interview.

I later compared Dawn's negation of the more complex statements with her negations for the statements with one logical component to try to answer my first research question. I presented Dawn with three complex mathematical statements from Calculus, one of which is shown in Figure 2, in order to address my second research question.

There exists a $c$ in [-1, 8.5], such that for all $x$ in [-1, 8.5], $f(c) \geq f(x)$ and there exists a $d$ in [-1, 8.5], such that for all $z$ in [-1, 8.5], $f(d) \leq f(z)$.

*Figure 2.* Complex statement from Calculus.

I asked Dawn to evaluate if the statement in Figure 2 was true or false for seven different graphs. The statement given in Figure 2 is based on the conclusion of the Extreme Value Theorem (EVT) and the intervals shown in the graphs. Some of these graphs had only one of either an absolute maximum or absolute minimum, some graphs had neither an absolute maximum nor an absolute minimum, and some graphs had both an absolute maximum and an absolute minimum.

These graphs were selected in hopes that Dawn's data would include some moments where I could explore Dawn's negations in the context of her justifications for statements that were false (in her opinion). The Extreme Value Theorem (EVT) only holds for continuous functions. Since I omitted the hypothesis of the EVT, there are cases where this statement I present is false. I was then able to compare the negations that were part of her justifications with her previous negations with other statements.

**Data analysis.** My analysis was conducted in the spirit of grounded theory (Strauss & Corbin, 1998) using videos of the student interviews as well as the students' written work. Hence, the consistencies and inconsistencies in Dawn's negation meanings emerged from the data. I identified moments where distinctions could be made about Dawn's negation meanings. ew moment began when Dawn was presented with a new question or task, she changed her evaluation or interpretation of a given statement, or if she changed her argument or negation of a statement in any way. After identifying these moments of interest, I compared Dawn's one-component negations with the two-component negations. Finally, I compared her negations in the context of her justification for the Calculus statement with all previous negations.

## Results

A consistent pattern emerged from Dawn's negations when I directly asked her to provide negations. However, when I presented graphs and asked Dawn to evaluate the validity of the mathematical statements for each graph, Dawn's negations in her argumentation did not always match her previous patterns of negation. A difference in interview questions appeared to influence Dawn's patterns for the negation of logical connectives and quantifiers.

### Consistencies Across Negations

Dawn stated that in order to determine a valid negation, she could negate one part of the original statement, but not both parts of the original statement. I asked Dawn to explain why she believed she should only change one side of a statement for its negation. She stated, "In general, it's just some kind of rule that I follow, like you only negate one side." She also stated that negations for the same statement could have a variety of different truth-values (i.e. for the same statement, one negation that she deemed valid could be true while another negation that she deemed valid could be false). Since Dawn relied on a negation by rule and accepted negations with various truth-values, the evidence suggests that her overall meaning for the word "negation" was related to a constructed procedure rather than a statement that could prove or disprove the original statement.

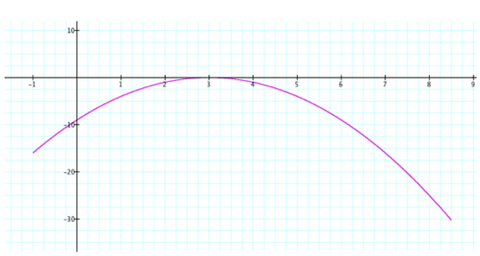
I first noticed Dawn's use of this procedure for statements of the form $\exists x, P(x)$. For statements with an existential quantifier of the form "There exists an $x$ such that $P(x)$," she referred to "There exists $x$" as one part and "such that $P(x)$" as another part of the statement, and claimed that she "could only negate one part of the statement." Dawn would not accept negations of the form $\forall x, \sim P(x)$. She said that changing the "there exists" to "for all" would be "changing too much." Dawn usually preferred to start with the negation of the form "There does not exist an $x$ such that $P(x)$," which is a valid negation. However, she also stated that statements of the form "There exists an $x$ such that *not* $P(x)$" were valid negations. For example, for the statement, "There exists a whole number that is negative," Dawn wrote both "There does not exist a whole number that is negative" and "There exists a whole number that is not negative" as negations.

Dawn's algorithm for negating one part of a statement was also consistent with her negation of statements with a conjunction or disjunction because she still claimed that she could negate one part of a statement, but not both parts of the statement. For the statement, "12 is even and 12 is prime," Dawn wrote the negations, "12 is odd and 12 is prime" and "12 is even and 12 is not prime." For both of these negations, Dawn retained the logical connective and changed one part of the original statement in each negation. (Dawn usually kept the disjunction or conjunction from the original statement in her negations, but she sometimes accepted hypothetical negations that altered the logical connective if she felt as though a negation had the same meaning as the original statement.)

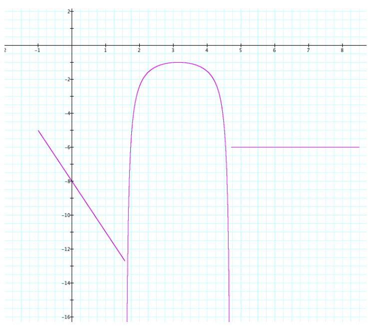
**A combination of negation meanings.** The statement "There exists a real number *b* such that *b* is odd and negative," has two logical components. Dawn interpreted the negation of both the quantifier and the conjunction in this statement in a similar manner as her earlier negations, as seen in her two negations: "There does not exist a real number *b* such that *b* is odd and negative" and "There exists a real number *b* such that *b* is even and negative." These negations are similar to the negations she preferred for "there exists" statements in the first interview, as they are also of the form "There does not exist an $x$ such that $P(x)$," and "There exists an $x$ such that *not* $P(x)$" (even though her negation of $P(x)$ is incorrect). Yet again, she did not consider the use of a universal quantifier in her negations and only changed one part of the statement. Dawn also negated the proposition within the statement that contained a conjunction in the same manner that she did with the first set of statements. The phrase "*b* is odd and negative" has its own parts that Dawn also considered. She negated "*b* is odd and negative" as "*b* is even and negative." She verbalized that she could have also used "b is odd and positive" for this part of her second negation. I asked her to consider explaining to a friend why her negation for the first complex statement was valid, to which she replied, "I would tell them that [my negation is correct] because I changed the second half of the statement." This reply indicates that Dawn is assessing the validity of her negation on her rule for negating one part of the statement, rather than comparing the *meaning* of the negation with the original statement.

**Negation in Argumentation: When Negation Isn't Viewed as Negation**

In the previous examples, I detailed Dawn's treatment of negation when I asked her to provide a negation. In the last set of tasks, I did not ask her to negate, but rather asked her only to determine if the statements were true or false on a case-by-case basis and to justify her claims. For the statement shown in Figure 2, Dawn interpreted the original statement as intended. Dawn explained why the statement shown below is true for the given graph:

| Statement & Graph Presented | Transcript |
|---|---|
| There exists a $c$ in [-1, 8.5], such that for all $x$ in [-1, 8.5], $f(c) \geq f(x)$ and there exists a $d$ in [-1, 8.5], such that for all $z$ in [-1, 8.5], $f(d) \leq f(z)$ .<br> | D: There is a maximum $y$-value at 3 [x=3] and a minimum $y$-value here (points to (8.5, $f$(8.5)) . So no matter what $x$ is, this [$f$(8.5)] is going to be the least $y$-value.<br>I: So what part of the statement tells you [that] you need to focus on the least $y$-value and the largest $y$-value?<br>D: Because we want to pick values for $c$ and $d$ strategically so that they are going to be the maximum and minimum $y$-value.<br>I: What part tells us we're going to pick the max and min?<br>D: Here, for all $x$, you want it to, no matter what the value of $x$, the value of $f(x)$ is going to change. And you want this statement here, this inequality, to hold true, and there's only one instance where that can be true—at the max or min. |

Dawn appeared to have a conventional interpretation for this statement. She expressed that she needed to choose *the* maximum or minimum that works *for all x*. Dawn's meaning for this statement and its negation was also revealed in her explanation of when the statement is not true. In the following example, Dawn claimed that the same statement is false for this case.

| **Statement & Graph Presented** | **Transcript** |
|---|---|
| There exists a *c* in [-1, 8.5], such that for all *x* in [-1, 8.5], $f(c) \geq f(x)$ and there exists a *d* in [-1, 8.5], such that for all *z* in [-1, 8.5], $f(d) \leq f(z)$ .<br> | D: The minimum *y*-value is $^{-}\infty$ , so you couldn't pick a value for… *d*, that would always make this inequality true (points to $f(d) \leq f(z)$ ).<br>I: Let's say your friend said, "For all the values that I look at, for all the *y*-values that I look at, if I chose *any* value for *d*, then I can always find a smaller value..." Would you agree with your friend's argument?<br>D: Yeah, I would agree with his argument.<br>I: Would you say that his argument is the same as your argument?<br>D: Yeah, because I said there isn't a value for *d*, where there's the smallest *y*-value. I think that's kind of the same thing. It isn't the smallest because you could always find one smaller. |

Dawn said that she *could not pick* a value for *d* such that this value of *d* would *always* satisfy the inequality. This response is similar to the negation "there *does not exist a d* such that $f(d) \leq f(z)$ ." Dawn's response was consistent with her prior approach to negate one part of a statement in her negation. Also recall that Dawn stated in the first interview that changing the second part of a statement and adding a universal quantifier would "change too much." Thus, I responded by asking Dawn to consider an alternative negation that used a universal quantifier and changed more than one part of the statement.

In the context of this statement where Dawn was asked about her argument rather than for a negation specifically, she accepted a negation that involved changing more than one part of a statement and she did not mention having an issue with the universal quantifier changing too much of the statement. Her original denial aligns with the argument, "there *does not exist* an *x* with a corresponding minimum *y*-value," but she also recognized that my proposed argument, "for *any x*-value, a smaller *y*-value than *f*(*x*) can be found," was equivalent to her original denial. Thus, she accepted the argument that aligned with the negation "for *any* value of *d,* there exists a *z* such that $f(z) \leq f(d)$ " by stating that this argument was "kind of the same thing" as her argument. She even explained why the logic for the two negations is equivalent: the *y*-value "isn't the smallest because you could always find [a *y*-value] smaller." Even though she had previously rejected alternate negations in the first interview that involved a universal quantifier, in the context of justification for this Calculus statement, she recognized that an alternate negation with a universal quantifier was valid.

In instances when my question or request omitting the word "negation," Dawn considered the *meaning* of the statement rather than her memorized rule to negate one part of the statement. Her interpretation of a statement and her negation for that statement varied based on the context of each mathematical statement. These moments in the second interview were characterized by the question, "Is this statement true or false for this graph?" rather than the command "Negate this statement." The word "negation" appeared to alert Dawn to negate only one part of the statement. However, when asked to think about the validity of a statement in a particular context, Dawn's approach was to use her reasoning to apply logical argument.

## Conclusion & Discussion

When responding to negation tasks in the first interview, Dawn negated one part of a given statement, but not both parts of a given statement. This finding is similar to Dubinsky's (1988) finding that students tend to use rules (which may or may not be correct) to negate a statement. Dawn's meaning for the command to "negate" was to change any one part of the given statement, no matter what type of statement that was given. Her procedural approach for negating a statement could help explain why students only negate one of two quantifiers when statements contain multiple quantifiers (Barnard, 1995; Dubinsky, 1988) and why they often retain disjunctions and conjunctions in their negations (Epp, 2003; Macbeth et al., 2013). In Dawn's words, changing two quantifiers or changing a logical connective might be "changing too much" in the student's view. Dawn's negations are also consistent with other literature that has claimed that students' logic can change across different tasks (Dawkins & Cook, 2017; Durand-Gurrier, 2003). Evidence from this study indicates that we may have undergraduate students in our classes who only *negate by rules in certain mathematical contexts*. The negation scheme evoked with tasks that used the word "negation" did not appear to be elicited with tasks that did not use the word "negation."

For many students, the word "negation" may be associated with a *procedure* rather than using logical arguments based upon their own *reasoning*. Dawn applied the same negation meaning even as statements became increasingly complex as long as I asked her to negate. However, the command to determine if a statement was true or false actually led her to negate according to mathematical convention. Students who apply a memorized rule to negate a mathematical statement may have the ability to negate appropriately if the word "negation" does not hinder their argumentation. Students may benefit from tasks that use the command, "Explain why the following statement is true or false" and from negating quantifiers and logical connectives in different mathematical contexts. Then, the students may be asked questions that may help them construct their own rules for negation that are consistent with their argumentation. The word "negation" may be more appropriate to use *after* students have already used negation and constructed their own rules for negation based on their own reasoning about multiple mathematical statements.

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# Insights into Students' Images of a Geometric Object and its Formula from a Covariational Reasoning Perspective

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*In covariational reasoning, when a student conceives of a situation as composed of measurable attributes that vary in tandem, discussing the relationship between quantities represented in a formula requires an interplay between a students' image of the situation and their conception of a formula. In this study, I categorize four pre-service teachers' images of both the situation and the formula as they describe the relationship between a given triangle's height and area. The results indicate how students' images of the situation and conceptions of a formula influence reasoning about the relationship between two quantities, specifically the role of numerical values and the development of a sophisticated dynamic image of the situation from which the student is able to draw conclusions.*

*Keywords:* Covariational Reasoning, Cognition, Geometry, Pre-Service Secondary Teachers

## Introduction

Researchers have identified the importance of covariational reasoning – conceiving of a situation as composed of measurable attributes that vary in tandem – in numerous K-12 topics including ideas surrounding rates of change (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995; Ellis, 2007; Johnson, 2015; Moore, 2014; Thompson, 1994). In undergraduate mathematics courses, students often invoke ideas of rate of change when reasoning about formulas (e.g., using formulas for basic shapes to find the area under a curve, deriving formulas for areas and volumes using derivatives and antiderivatives). However, following the Common Core Standards Initiative (2010), students' middle and high school experiences with formulas mostly involve informal proofs using manipulations of static objects (e.g., using Cavalieri's Principle); these types of experiences do not give students the opportunity to explore how covariational reasoning can be invoked when reasoning about geometric objects. In other words, this treatment of formulas does not consider the variables involved in the formula as *varying* as described by Thompson and Carlson (2017). With these ideas in mind, I explore how four pre-service secondary teachers (heretofore referred to as students) who have successfully completed an undergraduate calculus sequence reason about the area of a geometric shape, a triangle, and its height. Thus, the research question for this report is the following: (How) do students explore the covariational relationship of the height and area of a well-known shape, an isosceles triangle, using a formula?

## Background and Theoretical Perspective

In this study, I attend to students' images of a situation and students' conceptions of formulas when describing the relationships between quantities—measurable attributes of objects (Thompson, 1994). Throughout the study, I assumed the idiosyncrasy of individual's conceptions of quantities and images of a situation because I approach quantities as actively constructed by an individual (Steffe, 1991; von Glasersfeld, 1995). Moreover, I assumed when an individual constructed a relationship between quantities, the individual relied on their understanding of the quantity and their image of the relevant situation, which may have evolved over the course of the interview. Because of these two assumptions, I did not assume that students conceive the situations I provided to them in the same way I did; that is, they conceptualized the task using different quantitative structures. I noticed some of these differences, for example, when one student discussed changing the height of the triangle by drawing new triangles beside one another while other students described a smooth image of one triangle whose height was varying.

Moore & Carlson (2012) and Thompson & Carlson (2017) indicated how students' images of a situation impacted their reasoning, and other researchers have observed students reasoning quantitatively about real-world situations leading to their successful construction of equations (e.g., Ellis, 2007; Izsák, 2000; Moore & Carlson, 2012). Ellis (2007) differentiated this type of interplay between a student's image of a situation and their understanding of an equation in terms of covariational reasoning based on a student's attention to quantities. She, and other researchers who have prompted students to construct formulas via covariational reasoning using *area* situations, (Matthews & Ellis, In Press; Panorkou, 2016; Stevens et al., 2015) relied on the covariational reasoning defined by Carlson et al. (2002). In this view of covariation, giving the students a situation with which to operate is crucial. One popular context is that of a growing rectangle, which Thompson (1999) proposed and researchers have implemented with elementary and middle school students (Matthews & Ellis, In Press; Panorkou, 2016).

Alternatively, in the covariational reasoning described by Confrey and Smith (1994), students identify patterns in tables and use those numerical patterns to construct an equation. In this case, students are likely reasoning about values abstracted from operations of measurement rather than quantity's measurements (Thompson & Carlson, 2017). I use these two ways of covariational reasoning to differentiate students' reasoning with formula values versus those students who reason using their image of the situation.

I also situate my discussion around the results of two studies from researchers who have used growing area contexts. Matthews and Ellis (In Press) used a growing triangle context in their work. In their situation, the triangle's base remained on the left side of a square and the third vertex of the triangle began at the bottom left corner and traveled counterclockwise around the square. Although the two middle schoolers in their teaching experiment eventually successfully produced a normative graph of the distance the moving vertex traveled and the area of the triangle, never explicitly referencing a formula, the authors offer a caveat that the students may have reached their conclusion of a constant rate of change based on perceptual features, such as the constant speed of the traveling vertex. The authors also mentioned the difficulty of measuring area in their context. I extend this study by offering a point of comparison by working with students who have had vast experience with symbolization, area, and rates of change.

Stevens et al. (2015) also used an area context in a study with pre-service teachers; the pre-service teachers watch a dynamic image of a growing cone whose slant stayed at a constant angle and whose height grew and then shrank at a constant speed. The pre-service teachers described the relationship between the surface area and height of the cone. Half of the ten students in the study attempted to create and use a formula to determine the relationship between the quantities, and only two students constructed a graph using images of covariation rooted in the situation. None of these students produced a normative formula for the surface area of a cone. These results highlight the difficulty students seem to have relating a covariational relationship between two quantities they have constructed using their image of a situation with a formula that represents that relationship. It is important to note that in this task, both the 3-D nature of the cone and the formula for the surface area of the cone may have contributed to the students' difficulties with the task. This study extends this work by examining the relationship between the two when the student is given a simpler 2-D image and can produce a correct formula.

## Methods

In an effort to focus this study on exploring ideas of how students connect ideas of rates of change and geometric objects, I chose a population of students who have had vast experiences with both. The four participants of the study were either in their first or second semester of secondary mathematics teacher program at a large public university in the southeastern U.S. Each student had completed a Calculus sequence and at least two other upper level mathematics courses (e.g., linear algebra, differential equations) with at least a C in the course. The students had all been enrolled in a spring semester content course exploring secondary mathematics topics through a quantitative and covariational reasoning lens using the *Pathways Curriculum* (Carlson, O'Bryan, Oehrtman, Moore, & Tallman, 2015). I interviewed all four students who expressed interest in the study after contacting the entire class about the study. This particular study focuses on the second task of a semi-structured clinical interview style (Clement, 2000) pre-interview in a series of 3-5 interviews; the interview was exploratory in nature. These pre-interviews lasted about two hours each, and for two of the interviews, there was an observer present. I encouraged the students to think aloud (Goldin, 2000) and attempted to ask only questions that would enable me to construct viable models of the *students' mathematics* (Steffe & Thompson, 2000).

Each interview was videotaped and these videos were digitized for analysis. Using an open (generative) and axial (convergent) approach (Strauss & Corbin, 1998), I offer distinguishing features of students' approaches to the *Growing Triangle* task based on my models of their images of the situation, their conception of their formula, and the role of the two in their description of a relationship between quantities. Students interpreted the relationship between the quantities differently; three of the four wanted to draw a conclusion about the directional change between quantities, and only one of the four attempted to consider amounts of change in one quantity with respect to the other. The results highlight further distinctions.

## Task Design- *Growing Triangle*

In *Growing Triangle*, the student views a sketch on *Geometer's Sketchpad* of a static isosceles triangle. Students can drag a vertex of the triangle to increase or decrease its height and base while maintaining an isosceles triangular shape (i.e., $\overline{AB}$ stays constant but point $C$ can be dragged along the perpendicular bisector of $\overline{AB}$) (Figure 1). Only Charlotte dragged $C$. I asked each student to "describe the relationship between the *height of the triangle* and the *area of the triangle*." All students considered the height to be the perpendicular distance from point $C$ to $\overline{AB}$.

I purposefully designed the task to be a static image to see if asking about the relationship between two quantities would invoke a sense of change independent of watching a dynamic image. Also, not providing a dynamic image enabled me to avoid students concluding a constant rate of change based on a perceptual feature of the constant movement of a dynamic vertex. I also chose not to identify a specific height on the triangle for two reasons. First, I did not want to restrict their thinking if a student were to imagine rotating the figure at any point and wanted to consider a different height (Charlotte did, but quickly abandoned the idea). Second, by not identifying two particular quantities in the situation, I gained insight into what about the given situation they thought would stay constant and what would change in order to make a conclusion about the two given quantities. For instance, two students (Kimberley and Charlotte) considered different bases and categorizations of triangles (e.g., changing base with constant height, equilateral triangle shape maintained). For those students, I let them make a conclusion before directing them to consider specifically the case when $\overline{AB}$ stays constant and point $C$ changes.

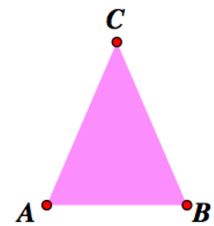



*Figure 1. Original position of triangle ABC.*

## Results

I describe how each student's image of the situation and use of the formula played a role in how they described the relationship between the height and area of the triangle.

### The Case of Jordan

Jordan was quick to discuss both a formula for the area of a triangle and to relate her conclusion to her image of the situation. When given the prompt, she said, "My first thought is the formula for the area of a triangle is one-half the base times the height, so if everything else is staying constant, except for height, which is increasing, then I would think the area would increase." Although her language may have seemed to indicate that she was reasoning about the formula, she says that she is imagining "*C* just like being pushed up," indicating that she was imagining the triangle varying as she was reasoning. Jordan's follow-up statement is further evidence she was reasoning by imagining at least one other image of a triangle:

> *Jordan*: So I'm thinking if it goes the other way. So, if you take C and you drop it [*motions finger downwards*], then it would decrease. Then there's just not as, like the base is staying the same, so [*makes pinching motion with fingers*]… if you have a squat triangle [*reaches to sketch and makes pinching motion smaller than the given height of the triangle*], like if I took *C* and dropped it, you can draw inside of this one [*motions where the two legs of the shorter triangle would be given the height of the squat triangle*] and see that it's taking up less space as this one [*makes circling motion around original triangle*].

After making this statement, Jordan justified her conclusion about her gross comparisons of the areas of the triangles by noticing that as *C* "goes up", the angles from the base and the legs increase, and said that the smaller triangle would fit inside the bigger triangles. After making this statement, she drew Figure 2 to illustrate her thinking. From this drawing, she concluded, "As *h* increases, *A* increases, and as *h* decreases, *A* decreases (see Figure 2)," once again supporting the idea that she imagined a dynamic image of a triangle. Moreover, her dynamic image of the triangle had quantitative entailments and invariant properties upon which she could operate in order to make a conclusion about the relationship between the two given quantities.

When asked if she could make any other conclusions about the relationship between the quantities, she stated, "I don't know by how much the area is changing when the height is changing." She added that she was not sure whether "it [would] be a constant change…I just can't picture like the height changing consistently, how that would change the area." This statement indicates that although her image of the situation had quantitative entailments and invariant properties sophisticated enough to imagine directional change of the area of the triangle with respect to height, it did not entail images of amounts of change in the quantities' magnitudes she could quantitatively compare to one another. I also note that Jordan did not return to her formula to attempt to make conclusions about how much the area is changing. In conclusion, Jordan relied only on her image of a dynamic situation to make conclusions about the relationship between quantities; she expressed knowledge of a formula for the area of a triangle, but she did not assimilate reasoning with the formula as a way to make conclusions about the rate of change between the height and area of the triangle.

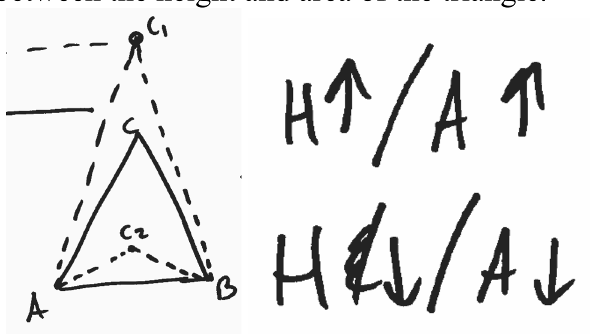
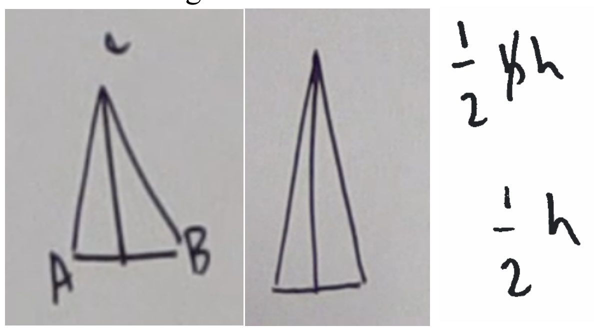



*Figure 2. Jordan's illustration of the triangles resulting after moving point C up and down and her conclusions about the relationship between the height and area of the triangle.*

**The Case of Kimberley**

Kimberley drew and considered a different instance of the triangle's growth when asked to consider the case when $\overline{AB}$ stays the same and *C* changes, but she was unsure how to make gross comparisons of the areas of the resulting triangles she drew. She drew a triangle (Figure 3b), noted that, in this case, the height has increased and said, "I'm not sure that the area has increased" when comparing it to the image on the screen (Figure 1a) "'cause we're getting [*makes narrowing motion with hands by bringing palms of hands closer to one another*], well [*pause*]". This statement indicated she was attempting to make a gross comparison of the area of the original triangle and the triangle with an increased height, but her image of the two triangles' areas, like Jordan's, did not afford her a way to make a conclusive comparison.

After this statement, she paused before saying, "Then the area of a triangle is one half base times the height [*writing* "½*bh" on her paper*]. So we keep-kept that part the same *[highlighting* $\overline{AB}$ *in her drawing of the original triangle* (Figure 3a)], that part [*highlighting bottom of triangle in Figure 3b*], but then the height increased [*motioning upwards from base of triangle*]." After making these comparisons between the bases and heights of the two triangles, she immediately concluded, "So then the area did increase." She justified her conclusion by saying, "So we know base is the same [*crosses out "b" in* "½*bh"* (Figure 3c)], so we can just look at what my height [*writes* "½*h"* (Figure 3c)], one half the height is, so if we know that the height is this here [*traces along height in original triangle* (Figure 3a)], but we know it got bigger here [*traces along height in Figure 3b triangle*], then that would have to be bigger [*pointing to* ½*h*] in this scenario." When asked to clarify how that discussion related to area, she said, "The area would be bigger because the height would be bigger." This discussion indicated a shift in how Kimberley was analyzing the situation. When her image of the situation did not afford her to make a conclusion about the relationship between the given quantities, she recalled and relied on a known formula for a triangle's area. Specifically, she compared her two cases and noticed that only *h* would have a different value in her formula. She did not use specific values, but rather unknown values such that one was greater than the other. By also noting the letter *b* in her formula was inconsequential to the resulting values for comparison because the base of the triangle stayed constant, she realized that by making a gross comparison of the values of ½*h*, she could also make gross comparisons of the area. Thus, she concluded that an increased height implied an increased area. In summary, Kimberley's conclusions about the areas of the two triangles resulted from a comparison of unknown values in a formula after a comparison of her image of the areas of the triangles was insufficient for her to make a definitive conclusion. She connected her conclusions back to the situation, but unlike Jordan, Kimberley's justifications relied on her conclusions from reasoning with the formula.




*(a)* *(b)* *(c)*

*Figure 3. (a) Kimberley's representation of the original triangle given in the sketch with height labeled, (b) drawing in the case when C is "higher" than the initial point, and (c) reasoning with the formula for a triangle's area.*

**The Case of Charlotte**

Unlike Kimberley, Charlotte immediately considered using a formula for the area of a triangle; when Charlotte is first presented with the task, she asked if it can be "something I've already been taught about triangles and areas and heights." Also unlike Kimberley, Charlotte was unsure about her recollected formula, stating that area equals "base times height" but that the formula may only be true for right triangles. Being unsure of her formula, she turned to reasoning with the static image of the triangle because, as she later reflected, "I tried to throw it [her formula] away, because I thought it only applied to right triangles for a moment" and that she did not know a formula for the triangle in front of her. She claimed that an increase in height implied an increase in area, noting that she was "picturing [her]self dragging this *C*". Thus, like Jordan, Charlotte had a mental image of a dynamic triangle in mind to try to make a conclusion about the relationship between the given quantities. However, her dynamic image was not as sophisticated as Jordan's image because Charlotte was unable to justify her claim as Jordan did. For instance, Charlotte dragged point *C* up in the sketch to illustrate to me how she was thinking of the situation, and remarked that "it made a bigger pink space" but later returned to the situation to drag *C* again and said, "If I increase and decrease that [height], wait, am I changing the area? Yea, definitely. At some points, it's easier to tell than others, but I feel like I'm changing the area." Like Kimberley, Charlotte had difficulty comparing areas with different heights, relying on what she *felt* was happening, rather than being able to draw conclusions using her image of the situation. Charlotte gave some insight into her image of the situation when we returned to this task later on in the interview when I asked her to draw how she was imagining "moving C up." She drew in Figure 4a and says, "I don't know. I can't draw that. Can I see it [the animation] again?". Upon doing so, she drew Figure 4b. From those drawings, we see an idea of narrowing sides (perhaps similar Kimberley's image) (Figure 4a) and a notion of stacking (Figure 4b), neither of which support an image of an isosceles triangle with increasing height that would enable her to identify amounts of change in area for equal changes in height.

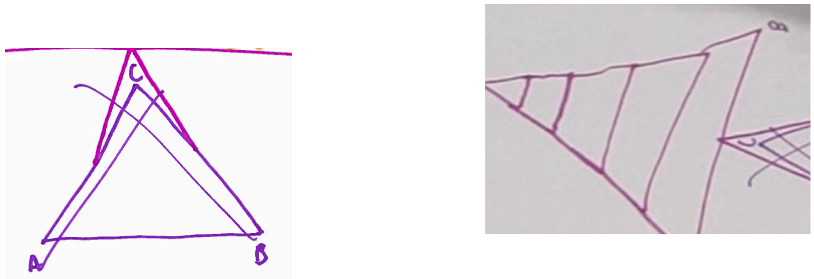



*Figure 4. Charlotte's first (left) and second (right) attempt to draw her image of the dynamic triangle.*

**The Case of Alexandria**

Like Charlotte and Jordan, Alexandria immediately referenced a formula and writes " ½*bh*=*A*". She represented the triangle on the sketch in a way similar to Kimberley's triangle (*Figure 3a*), pointing out the base ($\overline{AB}$), the height (*vertical line*), and the area (*region inside triangle*). However, unlike the other students, she did not consider changing the size or shape of the triangle. To her, the formula itself *was* the relationship between the triangle's height and area. Each variable in the formula represented an unknown value she could identify in the situation. Thus, she had no intellectual need to consider different values for height or area to plug into her formula to make a conclusion about covariational relationships. Her image of the situation remained static through the discussion.

## Discussion

From these results, I argue that a student who assimilates a formula to a given situation will not necessarily assimilate a task asking to describe the directional covariational relationship between quantities by using numerical values. None of the students in this study did. Moreover, if a student attempts to reason about the relationship between two quantities by focusing on the situation instead, their image of the situation plays a crucial role in their ability to justify their conclusions about the relationships between quantities. For instance, only Jordan's image of the situation had quantitative entailments and invariant properties that enabled her to justify the directional relationship between the quantities by seeing that one instance of a triangle fit completely inside another. Conversely, Kimberley and Charlotte were only able to provide intuitions about changes in area based on their image of the situation. Kimberley was able to justify her claim by reasoning with her formula. She did not use specific values and so we cannot say she reasoned numerically as described by Confrey and Smith (1994), but she did illustrate a separation from reasoning about quantities' measurements in the situation in order to make gross comparisons between two hypothetical unknown values that had a specific relationship to one another. Afterwards, she reconnected her conclusion to the situation. Lastly, only Jordan made an attempt to reason using amounts of change, but her image of the situation was insufficient for her to make a definitive conclusion. These results call for a way to support and scaffold students' images of change quantitatively that they might be able to make conclusions about rates of change using amounts of change.

## Acknowledgements

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The Emergence of a Prototype of a Contextualized Algorithm in a Graph Theory Task

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*Research has described the necessity and dangers of prototypes in mathematical learning, without offering explanations for what makes prototypes appropriate or inappropriate, or indeed how prototypes emerge in the first place. We explore one part of the emergence of a prototype: how a feature of a concept's example becomes predominant in subsequent generated examples. We describe how three students developed what they regarded as four examples and one non-example of an algorithm suitable for a client with a contextualized graph theory problem. The students engaged in a 'patching process' that preserved an inappropriate feature of the initial example in the other examples that were generated. We argue that the development of appropriate prototypes may depend on the types of processes (like the 'patching process') that students use to abstract and preserve features of the concept examples.*

*Keywords:* Concept, Prototype, Algorithm, Graph Theory.

## Introduction

Prototypes—those examples of a concept that are said to be *popular* or *typical*— play a significant role in mathematics learning (Hershkowitz, 1989; Tall & Bakar, 1992). On the one hand, prototypes make formal and abstract concepts more accessible (Tall & Bakar, 1992). On the other hand, prototypes can become obstacles when their properties that are unnecessary from the perspective of the formal concept, are perceived as something that any concept example must have (Hershkowitz, 1989). For instance, studies show that students are more likely to classify objects as examples of a concept when they "seem closer" (e.g., visually) to the prototype(s) (e.g., Presmeg, 1992). Accordingly, Tall and Bakar (1992) suggest that educators should help students to develop prototypes that are "as appropriate as possible" (p. 13), thereby implying that some prototypes are more appropriate than others. We propose that the appropriateness of a prototype is not its innate quality, but something that depends on the situation in which it emerges and is used. The study reported in this short paper comes to contribute to classic research that identifies prototypes that students have already developed (e.g., Hershkowitz, 1989; Rosch, 1979; Tall & Bakar, 1992), by exploring how prototypes emerge in the first place.

## Theoretical Underpinnings

### How have prototypes been conceptualized in the literature?

Rosch (1973) introduced the term *prototypes* to refer to examples of a category that were more 'central' or 'popular' (among a group of people) than others. The notions of centrality and popularity arose from the observation that humans perceive examples of a category as not having equal status—an example's 'closeness' to the prototype(s) influences its status (Rosch & Mervis, 1975). Hershkowitz (1989), conducting research in learning geometry, made a similar claim to Rosch's: "All the concept examples are mathematically equivalent [...] they satisfy the concept definition, but they are different from one another visually and psychologically. There are super examples which tend to be much more popular than all others" (p. 63). Tall and Bakar (1992) also observed that students, when asked if an object is a function, tended to answer "yes" if the object resonated with their prototypes, and “no” otherwise. We observe three different usages of the term *prototype* in this classic research of Rosch (1973), Hershkowitz (1989) and Tall and Bakar (1992). First, a prototype may refer to a concrete example of a concept, for example, a robin seen flying outside is a prototype of a bird. Second, the “category of a robin”, that is, a set containing the defining features of a robin, is a prototype of a bird. Third, a prototype of a bird is a set of features that are predominant among all birds.

We use the third sense of the term prototype in our research by looking at how features of one example of a concept come to appear in other examples students generate. Specifically, the definition of *prototype* that we follow in this paper is: an abstract representation (as opposed to a concrete example) that possesses the most predominant features of examples of a concept (note our definition aligns with the way it is used in Dean, 2003; Rosch & Mervis, 1975). By this definition, the emergence of a prototype is equivalent to the emergence of predominant features among examples of a concept.

Consequently, our research question is: what processes are involved when a particular feature of an example appears in other examples of a target concept? We explore this question by analyzing the work of three students who worked together on a contextualized graph theory task, in which they were asked to develop an algorithm to satisfy a client’s needs. After establishing their first example of the target concept, the group used a ‘patching process’ to generate their other examples. This particular patching process preserved a feature of the initial example in all the other examples that the group generated. We propose that this patching process is one instance of the processes that students might be engaged in when they abstract and preserve the features of the concept examples that become predominant.

**What do we know about how prototypes emerge?**

Research provides several explanations to how prototypes emerge. One explanation offered in mathematics education literature points to the role of our visual-perceptual limitations (Hershkowitz, 1989). That is, features that we “see” frequently among the examples of a concept are the ones that emerge to form our prototypes (but these frequently seen features are not necessarily equivalent to the defining features of the concept). This aligns with a common explanation in the cognitive psychology literature whereby prototypes arise out of frequent use (Taylor, 2003): if an example is repeatedly activated with the concept, then the example becomes a prototype. But Rosch (1999) argued that the frequency explanation falls short in some cases (e.g., even though children see blue and black skies equally, they almost always draw a blue sky when asked to associate the sky with a color). Another explanation suggests that an example becomes a prototype if it bears properties that are most common among other popular examples. In this case, prototype status is granted to the example by already existing prototypes (Taylor, 2003).

**In what sense is an ‘algorithm’ a ‘mathematical concept’?**

We are aware that our use of the terms *concept* and *prototype*, and indeed our characterization of an algorithm as a concept may seem unconventional (not prototypical), so we offer a conceptual argument to justify this usage. Vinner (2014) refers to a *concept* as a generalization of instances that share certain things in common. Thomas (2014) defines an *algorithm* as “a step-by-step set of instructions in logical order that enables a specific task to be accomplished.” Under these two definitions, an algorithm can be viewed as a concept because it is a generalization of structured instances that enable solving a particular task. Furthermore, in our study, we do not look at students’ notions of an algorithm as an abstract entity. Instead, we are interested in the contextualized algorithms that students develop, algorithms that are proposed for a particular client with specific needs that can be derived from a contextualized narrative provided by the task.

## Method

### Participants, Research Setting, and Data Collection

The participants in this study were three students—Chad, Gil and Lome (pseudonyms)—who were enrolled in a pre-degree mathematics course in a large New Zealand university, and knew each other well as friends. None of the students had studied graph theory before, and they were studying high school level algebra when they participated in our research. Chad, Gil and Lome were recruited as part of a larger research project (Yoon, Chin, Griffith Moala, Choy, 2017) that involves over fifty undergraduate, secondary, and post-secondary/pre-degree students, and which explores student-mathematizing in tasks that present discrete mathematics concepts in contextualized narratives. Chad, Gil and Lome worked together on four discrete mathematics tasks in four one-hour sessions over the course of three weeks. These four sessions took place outside of class time and course requirements, and were audio recorded and video recorded. The group worked in the presence of an interviewer (the second author), who answered clarification questions about the wording of the task, but did not offer any mathematical hints.

### Task

We report on Chad, Gil and Lome's mathematical activity in the third discrete mathematics task they worked on: "The Jandals Problem" (Yoon, Griffith Moala, & Chin, 2016). The task begins with some warm up questions that familiarize students with diagrammatic representations of graphs (networks) within the context of friendship associations, where a node represents a person, and an edge between two nodes represents a friendship between two people. After the warm-up questions a scenario is posed: Xanthe, an American exchange student in New Zealand learns that the locals use the word *jandals* to refer to what she commonly calls *flip-flops*. Upon returning home, Xanthe wants to spread the word *jandals* throughout different networks of friends like the ones shown below in Figure 1 and Figure 2. Students are asked to:

> Create an algorithm (method) that Xanthe can use to figure out the first person whom she should share the word with first in each friendship network to ensure that the word gets passed on to everyone in the network as rapidly as possible. She assumes that a person will share the word with all of his/her friends on one day, and each of those friends will share it with their friends the next day. Ensure that your algorithm will work for any friendship network, not just the one given [Figure 1]. (Yoon, Griffith Moala, & Chin, 2016, p. 12)

Only Figure 1 was initially given to the students; Figure 2 was given to them at a later stage.

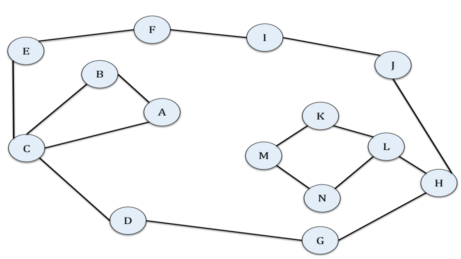


*Figure 1. Friendship Network 1*

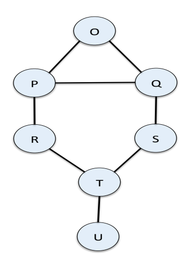


*Figure 2. Friendship Network 2*

This task asks students to develop “an algorithm (method) that Xanthe can use”. Throughout the session, the students and interviewer switched between ‘algorithm’ and ‘method’, and we preserve both when describing and analyzing their work.

**Data Analysis**

Due to the exploratory nature of the study, the aim of the analysis was not to confirm existing constructs but rather to explore aspects of the data that may be used to construct plausible explanatory models (Clement, 2000) for how features of an example come to be predominant in subsequent generated examples. Thus, the analysis involved an “open interpretation of the data” (Clement, 2000, p. 548), which is “useful for constructing initial explanatory models of cognitive processes” (Koichu & Berman, 2005, p. 171) inferred from the data.

Following the task description, we regard the target concept that guides the students’ mathematical activity to be “an algorithm (method) that Xanthe can use” with two defining properties: (1) it identifies the *quickest starting person*; (2) it must work for any friendship network. We searched the data for examples of the target concept that the students created, establishing the presence and predominance of a common feature among the examples. Then we followed the development of the examples (individually and collectively), looking for particular aspects of the group’s work that may have contributed to the emergence of the predominant feature.

## Findings

We present three episodes from Chad, Gil and Lome’s activity during the Jandals problem in which they create an example of the target concept of “an algorithm (method) that Xanthe can use”, and where a particular feature of this first example also appears in further examples that the students generate. Each episode begins with our *account of* (Mason, 2002) the group’s work (i.e., addressing what happened) followed by our analysis (addressing *why* particular things happened).

**Episode 1: A valid example of the target concept emerges**

After the group reads the task instructions, Gil says they need to find the person in the friendship network (Figure 1) that would spread the word quickly. Lome suggests they choose a person, count how many days it would take for the word to spread starting from the chosen person, repeat the process for all other persons, then share the word with the person that gives the least number of days. Lome refers to this entire process as “the elimination method.”

The students use the elimination method on the following persons in the first friendship network: *C, I, L, J, H, G, M*. They determine that the quickest of these is *H*, which yields four days, having incorrectly calculated that *G* yields six days, when in fact it also yields four days, making it another quickest starting person. Lome remarks, “I reckon we’ve solved it!” He then looks back at the task instructions, turns to the interviewer and says:

*Lome*: What’s an algorithm? This [*points to written parts of their elimination method*] is not an algorithm is it?

*Interviewer*: An algorithm is like a method. So it’s not your solution, it...

*Gil*: It’s like the way you got it.

*Interviewer*: Yeah, so that she can use it for any other network, because this is just one of many different friendship networks across the campus.

*Lome*: Can we say we just did elimination method?

*Interviewer*: You've got to explain it as well as you can so that Xanthe can use it for a different one that she is given.

When Chad says he is still unsure what they need to produce, Lome says, "she needs to be able to figure out the solution to any network, from our method." Gil then suggests a method: "Yeah, so it would be like, your method would be like, the [*starting*] person should tell three people because [*points to Figure 1*] if you told *H*, *H* would tell *L*, *G*, and *J*. And then, it spreads." Lome and Chad both nod their heads, and Lome says "Yeah, cool!"

**Analysis.** Two different methods emerge for the group in this excerpt: *the elimination method,* which is the exhaustive search procedure that the group uses to find the quickest starting person in the first network; and *Gil's method* (share the word with someone who tells three people) which is the method Gil suggests giving to Xanthe. Lome's question to the interviewer, "Can we just say that we did elimination method?" can be interpreted as asking whether the elimination method qualifies as a valid example of the target concept. The group's subsequent decision not to share it with Xanthe suggests they do not consider it to be a valid example (although it is indeed a mathematically valid algorithm for Xanthe's purposes). On the other hand, the group's enthusiasm towards Gil's method, indicated by head nods and "yeah cool!" suggests they regard Gil's method to be a valid example of the target concept. Thus, although both methods are put forth as potential examples of the target concept, only one of them (Gil's method) is accepted by the group as a valid example of the target concept.

**Episode 2: A feature of the first example is preserved in the generation of a second example and subsequent examples**

After Gil proposes his method at the end of Episode 1, the interviewer points to the task instructions and says:

*Interviewer*: Can I get you to read what the method needs to do?

*Lome*: So [*looks at Figure 1*] she should share it with someone who tells at least three people. But then mind you, if she starts at *L*, *L* tells three people but it doesn't work as fast.

*Gil*: Yeah, that's true.

*Lome*: So maybe [*points to H*] the starting person needs to tell three people [*points to L, J, and G*] but one of those three people [*points to L*] has to tell two other people.

*Gil*: Yeah [*nods head*].

*Lome*: Because this person [*points to C*] tells four people, but none of those people [*C's friends*] are connected to two other people. That's a method. I'll write it.

Lome writes down: "Share the word with a person who tells three people, and one of those three people must tell two other people." The interviewer asks, "Are you happy?" Chad, Gil and Lome reply, "Yes."

Later in the task, Gil revises Lome's method to "share the word with someone who tells three people, and each of those three people must tell one other person." Then, Chad revises Gil's second method to "Share the word with someone who tells three people, and two of those three people must each tell one other person." These methods are presented in Table 1 below.

**Analysis.** After the group notices a flaw in Gil's method, all of the methods they subsequently suggest nonetheless preserve a feature of Gil's method: *the quickest starting person tells three people*. What may be a plausible explanation for the preservation of this feature? We propose that the group may have noticed that their method needed to perform two functions: (i) it had to find the quickest starting person(s), and (ii) it had to *not* find non-quickest starting persons in a given network. In light of these two functions, the process whereby Lome's method above is obtained by building on Gil's, can be described as: keep the part of the current method that satisfies the first function, and change the part that violates the first function (note that *'change'* also includes adding other parts to it) so that the second function is also satisfied. We refer to this process as a 'patching process" due to its change-only-what-needs-to-be-changed nature. We observed the students using this patching process to generate the other examples (see Table 1).

**Episode 3: The elimination method is judged to be a non-example of the target concept**

After Lome writes down his method—share the word with a person who tells three people, and one of those three people must tell two other people—the interviewer hands the group a sheet of paper on which is printed a new friendship network (see Figure 2 above), and asks them to show how Lome's method would work on this new network. Rather than apply Lome's on this new network, the group uses the original elimination method to find five solutions: *P*, *Q*, *R*, *S*, and *T,* which all give three days (note, their solution set is incorrect; *R* and *S* are the quickest starting persons as they only give two days). Then Lome says:

*Lome*: Can you get four days? [*Chad demonstrates that it takes four days starting from person U*]. OK, so why wouldn't she tell *U* but tell *Q* instead? What's the method? Obviously *Q* will be quicker but why would she tell *Q* and not *U*?

*Gil*: Because, read your thing [*points to Lome's method*]. She has to tell someone who tells three people, so *Q* tells three people. If you tell *Q* first, *Q* tells *S*, *P*, and *O*. Then, the second person must tell at least two other people.

*Lome*: Yeah.

The group again agree that they need to produce a set of instructions and a method that Xanthe can use on any network. Then Lome remarks:

*Lome*: Say she has hundreds of these [*networks*] she doesn't want to do elimination method for every one. What if there's a network with a thousand people? She'll be there for ages counting!

**Analysis.** In total, the group created five methods for the task, which are summarized in Table 1, four of which they regard as examples of an algorithm they could give Xanthe.

*Table 1: The five methods created by Chad, Gil and Lome during the Jandals problem*

| *Name* | *Description of the method* | *Validity as example of target concept* |
|---|---|---|
| The elimination method | Choose a person, count how many days it would take for the word to spread starting from the chosen person, repeat the process for all other persons, then share the word with the person that gives the least number of days. | Non-example |
| Gil's method | Share the word with someone who tells three people. | Valid example until flaw is found |
| Lome's method | Share the word with someone who tells three people, and one of those three people tells two other people. | Valid example until flaw is found |
| Gil's second method | Share the word with someone who tells three people, and each of those three people tells one other person. | Valid example until flaw is found |
| Chad's method | Share the word with someone who tells three people, and two of those three people each tells one other person. | Final valid example |

While Lome was comfortable using the elimination method to find person *Q* as one solution for the friendship network in Figure 2, his questioning of why Xanthe should choose *Q* over *U* suggests he did not regard the elimination method as adequate justification for this choice: "obviously *Q* will be quicker than *U*, but why"? Gil cites Lome's method to justify choosing *Q* over $U$, which seems to satisfy Lome. For Lome then, the elimination method was prescriptive without being explanatory. This perceived feature, together with his characterization of the elimination method as tedious for large numbers of people may have dissuaded the group from perceiving the elimination method as a valid example of the target concept, even though it is indeed a mathematically appropriate algorithm. Rather, the group perceived the elimination method as a non-example of the target concept, which, in giving the group an idea of what a valid example should (not) look like, may have contributed to inclusion of the feature "tell someone who knows three people" in the examples they generated afterwards.

## Discussion and Concluding Remarks

The episodes that were presented in this paper provide an account of the recursive process that a group of students went through when engaging with a concept of algorithm. First, the group considered a method and decided whether it was an example or non-example of the targeted algorithm. The consideration was made against two functions that the group wanted their method to perform. Second, the group recognized that the method under consideration performed one of the functions, but not both. Lastly, a new method was generated in which feature from the previous method was preserved and a new feature was introduced so as to ensure that the resulting method performed both functions. We refer to this recursive process as 'a patching process' due to its change-only-what-needs-to-be-changed nature. This patching process eventuated in the preservation of a feature of the initial example in all the other examples that group generated, and hence the emergence of a predominant feature. We propose that this patching process is one instance of the processes that students might be engaged in when they abstract and preserve features of the concept examples—features that become predominant.

The patching process that we identified in our study puts forward the crucial role of the first concept examples that students encounter. This aspect aligns with the existing research on prototypes (e.g., Hershkowitz, 1989; Tall & Bakar, 1992). We, in our study, show that merely encountering examples is not necessarily sufficient, and that recognition of the example's status (as *an example of the* concept) is necessary. Indeed, the students in our study generated the elimination method, an algorithm that we, as researchers, wanted them to develop. Furthermore, to the best of our knowledge, there is currently no more efficient algorithm to cope with the tasks that were handed to our students. Yet, the group almost immediately labeled the elimination method as "not an algorithm," and its inappropriateness was not questioned in the data that we presented. Thus, it seems reasonable to propose that developing appropriate prototypes may come down to preserving appropriate features (and rejecting inappropriate features) of the examples; which in turn may depend on the processes (such as the patching process we found here) that students go through when abstracting and preserving certain features of the concept examples.

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# Teaching Linear Algebra: Modeling One Instructor's Decisions to Move between the Worlds of Mathematical Thinking


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*In this article, we report results from a year-long study in which a linear algebra instructor worked with the research team to document his instructional decision-making via journals and interviews as well as to code and analyze the data. This work supports the development of a more general model of the instructor's decision-making and provides a lens with which to make sense of the instructors shifts between representations from each of Tall's Three Worlds. With the introduction of the model, we include an example to show how the various codes interact in the instructor's decision-making. We also provide a detailed description of one incident that provides a second perspective on the instructor's decisions, helping to support a more robust understanding of the data.*



*Keywords*: Linear Algebra, Tall's Worlds, ROGs, decision-making


## Theoretical background

Over the past decade, research on linear algebra has revealed that many students struggle to grasp the more theoretical aspects of linear algebra which are unavoidable features of the course and are focused on students' thought processes (e.g. Stewart & Thomas, 2009; Hannah, Stewart &Thomas, 2013; Britten & Henderson, 2009; Wawro, Zandieh, Sweeney, Larson, & Rasmussen, 2011; Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015). The research in recent years have mainly concentrated on students' difficulties and with a few exceptions (Hannah, Stewart & Thomas, 2011; 2013; Zandieh, Wawro, & Rasmussen, 2017; Andrews-Larson, Wawro, & Zandieh 2017), research on instruction in linear algebra is still scarce.

Research in instruction at the university level is fairly new. As Dreyfus (1991) suggested, "one place to look for ideas on how to find ways to improve students' understandings is the mind of the working mathematician. Not much has been written on how mathematicians actually work" (p. 29). Two decades later, Speer, Smith, and Horvath (2010) declare that "very little research has focused directly on teaching practice and what teachers do and think daily, in class and out, as they perform their teaching work" (p. 111). In recent years some mathematics professors have been more willing to examine and reflect on their own teaching styles, leading to a growing body of research in this area (e.g. Paterson, Thomas, & Taylor, 2011; Hannah, Stewart, & Thomas, 2011). The overarching goal of this study was to contribute to this gap in the literature by examining a linear algebra instructor's thought process and teaching decisions over an entire semester.

The theoretical aspects of this study are based on Schoenfeld's (2010) Resources, Orientations and Goals (ROGs). He claims that "if you know enough about a teacher's knowledge, goals and beliefs, you can explain every decision that he or she makes, in the midst of teaching" (2012, p. 343). By resources Schoenfeld focuses mainly on knowledge, which he defines "as the information that he or she has potentially available to bring to bear in order to solve problems, achieve goals, or perform other such tasks" (2010, p. 25). Goals are defined simply as what the individual wants to achieve. The term orientations refer to a group of terms such as "dispositions, beliefs, values, tastes, and preferences" (2010, p. 29). Although, the theory was originally considered as applying to research on school teaching, (Aguirre & Speer, 2000; Thomas & Yoon, 2011; Törner, Rolke, Rösken, & Sririman, 2010), it clearly has applicability to research on university teaching (e.g. Hannah, Stewart & Thomas, 2011; Paterson, Thomas & Taylor, 2011).

As a part of the theoretical framework described in this paper, we also employed Tall's three-world model of embodied, symbolic and formal worlds of mathematical thinking. Tall (2010) defines the worlds as follows: The embodied world is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns...and other forms of figures and diagrams" (p. 22). Embodiment can also be perceived as giving body to an abstract idea. The symbolic world is the world of practicing sequences of actions which can be achieved effortlessly and accurately. The formal world "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22). In Tall's view (2013, p. 18), "formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used". He believes that the formal mathematics is "future-proofed in the sense that any system met in the future that satisfies the definitions of a given axiomatic structure will also satisfy all the theorems proved in that structure. The formal mathematics can reveal new embodied and symbolic ways of interpreting mathematics." (p.18).

We believe that in many cases teachers and text books move between worlds of mathematical thinking very naturally and rapidly, not allowing students time to discuss and interpret their validities. They assume that students will pick up their understandings along the way. As Dreyfus (1991, p. 32) declares "One needs the possibility to switch from one representation to another one, whenever the other one is more efficient for the next step one wants to take… Teaching and learning this process of switching is not easy because the structure is a very complex one." We hypothesize that most students do not have the cognitive structure to perform the switch that is available to the expert. For example, Duval (2006) noted that to construct a graph, most students have no difficulties as they follow a certain rule "but one has only reverse the direction of the change of register to see this rule ceases to be operational and sufficient". (p. 113)

In this study, we employed Tall's three-world framework of embodied, symbolic and formal to follow a linear algebra instructor's movements between the worlds. Our research questions are: How did the instructor's ROGs inform his movements in the three worlds? When did he decide to move between the worlds and why?

## Methods

This narrative qualitative study examined an instructor's teaching journals. The study took place over an entire semester, during which the instructor (David) was teaching two sophomore-level linear algebra courses using the IOLA curriculum (Wawro et al, 2012). With some exceptions, the instructor kept a journal of teaching reflections throughout the semester and met with the research team (lead investigator, senior investigator, and undergraduate assistants) each week. The reflections and team meetings allowed for triangulation of data and gave multiple chances for the instructor to share his reasoning about his teaching decisions. The team then conducted a retrospective analysis of the journals following the methodology of narrative study (Creswell, 2013). Specifically, the team iteratively coded the data, beginning with open coding that each member of the team conducted and brought together to compare. Through comparison of open codes, the team developed a set of focused codes that were iteratively refined through collective discussion. The team then used these focused codes to categorize each sentence from the journals, disputing conflicts through an open discussion until each member of the team was satisfied. This process further refined the focused codes. These discussions resulted in a spreadsheet with each sentence from the journals coded for as many categories (themes) as the group deemed necessary for that section of transcript. Some of these codes are listed in Table 1.

The research team grouped similar codes with each other based on which aspects of the pedagogical process the instructor was discussing. The broad categories included: Teaching, which describes codes in which the instructor is describing what occurred in class; Math, which differentiates instances in which the instructor is explicitly talking about either the students' mathematics (Ms.) or his own (Mi); Reflection, which focused on the successes and failures of implementation toward the desired learning goals; and Tall's Three Worlds, which focused on which of the three worlds the instructor was drawing on in the moment. The codes that we focus on in this section are when the instructor discussed: teaching, focused on the tasks implemented in class (IOLA); teaching, focused on developing specific ideas in the class; teaching, when pedagogical decisions are made; statements about the instructor's mathematics; statements about the students' mathematics; and reflections specifically addressing the students' successes and struggles in developing the intended mathematics.

*Table 1. Some focused codes from the iterative coding of the instructor's reflections.*

| | | |
|---|---|---|
| Teaching (T) - Describing what instructor did in class | | |
| | Focus on **tasks** | Tt |
| | Focus on **developing** ideas | Td |
| | **Responding** to student thinking (Formative Assessment) | Tr |
| | Making **pedagogical** decisions | Tp |
| Math (M) | | |
| | Instructor | Mi |
| | Students | Ms |
| Reflection (R) - Reflecting experiences and on the success/failure of implementation | | |
| | **Students** | Rs |
| | **Implementation** | Ri |
| | **Comparing** to Prior Experiences | Rc |
| Tall's Three Worlds (W) - | | |
| | Embodied | TWe |
| | Formal | TWf |
| | Symbolic | TWs |

## Results and Discussions

After coding the instructor's journals, the team focused on identifying narratives that the codes supported. Through an examination of the codes, we identified several patterns that help explain specific instances of how the instructor made decisions in planning his lessons. One such pattern is diagrammed in Figure 1. In this pattern, statements about developing specific ideas in the classroom (Td) and making pedagogical decisions (Tp) inform the tasks (Tt) in which the instructor engaged the students. The instructor then reflected on the students' activity (Rs), and explicated the insight this allowed him to gain about their mathematics (Ms). Following this, the instructor then drew on his own mathematical understanding (Mi) to make sense of the students' mathematics in the context of his reflection on his own instruction (Ri). This act, in turn, informed the instructor's pedagogical decisions (Tp) and focus on which ideas to develop (Td) as well as a means of developing them through specific task (Tt). Although it is consistent with a few examples from the instructor's journal, this cycle is a generalization of how such a process might unfold and so we expect this process might be different for other instructors. Further, we find that shifts among these codes might provide some insight into how instructors make decisions regarding shifts between Tall's Three Worlds in their instructional decisions.

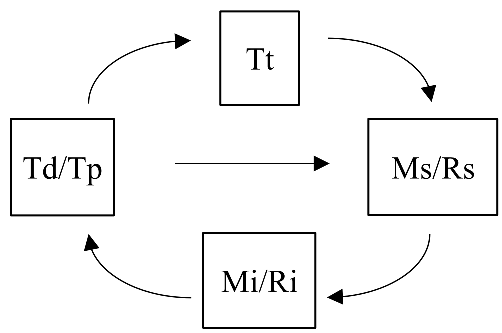


*Figure 1. Possible diagram of codes describing instructor's decision-making.*

We now provide an example (see Table 2) to show how this pattern of decision-making might unfold. The instructor began this episode by referring to the task that he wanted the students to complete, which comes from the IOLA curriculum intended to support students' development of linear independence, span, and basis (Tt). The instructor then reflected about the students' engagement in the task (Rs) and the mathematical understanding of linear in/dependence and span that they exhibited in their work (Ms). In these lines, he described what he had observed as the students' limitations in completing the table and conjectured why this might be the case, citing limitations of his own instruction in preceding class sessions (Ri). The instructor then responded to this by anticipating an approach that might address what he saw as an issue in his students' understanding. Specifically, he relied on an activity he had developed for himself (Mi) to make sense of the notion of basis. He then concluded that he would implement this task in the next class session (Tt, Td, Tp) and drew on his prior experiences using this task (Mi, Ri). Altogether, this sequence resulted in a shift from focusing on students' generation of examples set in the symbolic world to a focus on the students' embodied notions of linear independence. The instructor saw value in a different way of understanding and anticipated that focusing on this facet of understanding linear in/dependence would support the desired ways of thinking from the students.

*Table 2. An excerpt from the instructor's journal.*

| Excerpt | Tt | Td | Tp | Mi | Ms | Rs | Ri | TWe | TWs |
|---|---|---|---|---|---|---|---|---|---|

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| I wanted students to complete U1T4 from IOLA. | 1 | | 1 | | | | | | 1 |
| Each class finished the task, though some groups had some pretty serious reasoning deficiencies. | | | | | 1 | 1 | | | |
| For instance, very few groups in the first class realized the impossible cases. | | | | | 1 | 1 | | | |
| I think this is a result of rushing through the definition of LI/LD and my failure to support deep geometric thinking about linear dependence. | | | | | 1 | | 1 | 1 | |
| I think I can help fix this on Monday by having the students do the "building set" task while focusing on LI/LD. | 1 | 1 | 1 | 1 | | | | | |
| We've done this task when talking about span and so I think they'll be comfortable with it, I just need to give them time to feel comfortable with thinking about linear dependence spatially. | | | | 1 | | 1 | | 1 | |

**Moving between the three worlds**

In teaching two sections of linear algebra, David wished to get three fundamental points across. First, he wanted to help students discover that linear independence means there will be infinite solutions to the homogeneous equation. Second, he wanted to demonstrate that linear combinations of linearly independent vectors are unique. Finally, he wanted to help students reason about when and why matrices are invertible by connecting to the importance of basis and forming an appropriate connection between matrices and linear combinations.

David's teaching began smoothly enough, despite some minor complaints about time. Because of this shortage on time, he was unable to introduce the Elementary Row Operations (EROs) that the students had previously asked him about at this time, which disappointed him. He did however, introduce the vector space axioms (formal) and went over a few examples of vector spaces, at which point students finished their prescribed task quickly(Tt). David remarked that the second class seemed much less involved than the first class, perhaps because of the streamlining he did as a result of his experience from the first class (Rs). All groups converted a system of equations to obtain the same solution, though different groups chose different variables and David felt they were unaware of scaling solutions(Mi/Ri). He therefore synthesized the groups' classwork (Td/Tp), helping them focus on free variables (symbolic) and tying their work back to the "getting back home" problem (embodied). He utilized embodied reasoning here, envisioning traveling in a triangle, though perhaps due to rushing through some definitions, the students did not achieve the deep geometric thinking David would have preferred. In attempting to point out that linear combinations of linearly independent vectors are unique, David turned to symbolic reasoning by writing two separate equations for linear combinations of linearly independent vectors(Tt). This achieved the desired effect as students realized that the only way to obtain the zero vector from linearly independent vectors was to make the coefficients all zero, and some students, in David's words, had "really cool ways of thinking about why a=c," (Rs/Ms) and mentions that he wishes he had video of these in-class conversations.

In one particular interview from October 11, David described ways in which he helps his students learn the notion of linear independence. He leveraged the power of the embodied and symbolic worlds as resources to help students understand the formal world more completely. He stated his main goal was to help students understand that matrices are invertible if and only if their column vectors form a basis, that is, if and only if their column vectors are linearly independent (and span the field).

It seems that throughout the interview David utilized embodied reasoning as a resource to help students understand difficult concepts. Early on, he mentioned when students “can’t see” some concept he is teaching, he “takes it to a more familiar analog,” and furthermore shows pictures of vectors that challenge student intuition. At one point, David talked about helping a student in office hours refine their reasoning about linear independence through embodied use of markers pointing in particular directions. That is, he asked the student if she could get to various places on the desk traveling only in lines parallel to the two markers he had laid out pointing in specific directions. It seems David utilized the embodied world, supported by his example with the markers, to inform his student’s understanding about what the more formal terminology of “linear independence” meant.

Earlier in the interview, David stated his wish to help students develop math based on the representations they’re already using, as well as their own natural tendencies (orientation). To this end, he asked students in class for their thoughts on a formal idea they’re discussing as a class. For example, he wrote “What does it mean to be invertible?” on the board at one point, but rather than having students memorize the formal definition, he asked them for input and wrote the facts they suggested on the board. In this process, he guided them toward the idea of linear transformation which they had already covered, in order to use their previous embodied and symbolic reasoning to inform this new formal idea they were moving towards.

Finally, David suggested he used symbolic work to help reinforce formal ideas. In answer to an interviewer probe, David agreed that he felt that “pictorially” students were fine, but had trouble with representations that “were not just a symbol” but were more strongly “rooted in the definition”. In response to this question, David gave an example of another student he asked, “If $\begin{bmatrix}1\\0\end{bmatrix}$ goes to $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ goes to $\begin{bmatrix}1\\1\\2\end{bmatrix}$, where does $\begin{bmatrix}x\\y\end{bmatrix}$ go? The student took an algebraic approach and felt this was intuitive, but still missed the connection to linear dependence. With some help from the instructor, however, sometimes this symbolic reasoning can still inform this formal definition. In doing similar problems, this student answered quickly but her work was disorganized, and therefore confused her. David reorganized the symbolic calculations into a single line with reasoning about linear dependence and transformations informing this symbolic flow of logic. It seemed that David once again utilized symbolic reasoning to help the student cross a bridge from the symbolic world into the beginnings of formal reasoning.

Thus, David builds upon students’ prior mathematical knowledge, and utilizes first embodied reasoning to inform the formal definition. Once students are comfortable with embodied reasoning, David begins utilizing tools from the symbolic world to further inform student reasoning, switching back to embodied reasoning (see markers example, or Geometric Sketch Pad (GSP) usage) when students experience further difficulty in reasoning. In class, David successfully led students to symbolically reason about the uniqueness of linear combinations of linearly independent vectors, his second of three main goals.

Not all such teaching was smooth however. One serious reasoning deficiency was that students did not realize some of the cases were impossible, such as writing down three linearly independent vectors in $\mathbb{R}^2$. David first attempted to fix this by helping students think about a matrix times a vector as a linear transformation via a “building set” task they had previously completed while thinking about span rather than linear independence (Tt). He also wanted students to be able to think about the product as a linear combination of column vectors(symbolic), so that students would realize the importance of basis in conjunction with the invertibility of matrices, as well as in conjunction with linear transformations. In particular, column vectors must form a basis for the matrix and the associated transformation to be invertible(formal). However, students had a hard time understanding David's point that a matrix isn't a transformation until you do something with it, and that bases have to be named. He was just trying to explain to them the difference between a simple symbol and the activity that is sparked by that symbol(Ri/Mi)—the difference between symbolic and embodied reasoning. As he felt this point was lost on them, in his second attempt he used GSP (embodied) to actually demonstrate the distinction between symbols and the action the symbol is taken to signify. Asking students to name a vector led them to realize they needed to see axes in order to name a vector(Rs/Ms). David was then able to start with the standard basis axes, then distort them with GSP to a different basis and ask students to come up with a linear combination for the same vector once again. Through embodied work with GSP, students came to understand that with a different basis, a completely different matrix can stand for the same linear transformation, and that a matrix isn't a linear transformation until the bases are established. Thus, GSP in conjunction with David's directed teaching successfully effected a transition in the students from embodied to symbolic reasoning. Showing students (embodied) the difference proved much more effective than just trying to explain the difference(symbolic/formal).

## Concluding Remarks

In both accounts, it appears that David roughly followed the cycle laid out in Figure 1, where he utilized developed tasks to draw out or further refine students' existing mathematics. This affords him the opportunity to utilize his own knowledge of mathematics to reflect on his instruction and make inferences about his students' mathematics. Armed with this knowledge, he can then make more appropriate pedagogical decisions and ideas within the classroom. While David notes that students were not typically comfortable shifting between Tall's three Worlds (i.e., formal was most difficult for them while embodied was most accessible, with symbolic somewhere in the middle), in contrast, David himself shifted between worlds regularly in an effort to teach his students. As seen in the October 11 interview, it appeared that David also utilized the embodied world as a resource to help develop the symbolic world, and then used both of these as resources to begin to approach the formal world. Thus, Tall's three worlds functioned as a resource to help David meet his goal of instructing students about the relationship between linear independence and invertibility.

This work provides a foundation for further investigations into linear algebra instructors' decisions, especially those decisions about shifting between representations in each of Tall's three worlds. This research also provides insight into how we might frame instructional decision making more generally – beyond the context of linear algebra instruction. For instance, the decision-making diagram in Figure 1 could be used in any context, though we found it useful in this study to help focus on shifts between Tall's three worlds.

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# Could Algebra be the Root of Problems in Calculus Courses?

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*Calculus serves as the gateway for most STEM degrees. Due to students' challenges successfully completing calculus, more than half of students are deterred from a career in STEM. Our preliminary investigation indicates that students' difficulties with algebra cause significant problems in many first-year math courses. The aim of this paper is to investigate in what ways the difficulties with algebra impact students' success in calculus.*

*Keywords*: Algebra, Calculus, common errors, accommodation

## Introduction

Calculus occupies the position of gatekeeper to disciplines in STEM since at least one calculus course is required for all STEM majors. "For too many students, this requirement is either an insurmountable obstacle or—more subtly—a great discourager from the pursuit of fields that build upon the insights of mathematics" (Bressoud, Mesa, & Rasmussen, 2015, p. v). Research has shown that negative experiences encountered in gatekeeper or introductory math and science courses are a major factor in the national problem of significant attrition (more than half) of declared STEM majors (Crisp, Nora, & Taggart, 2009; Mervis, 2010). Studies by Stewart & Reeder (2017a; 2017b) suggest that college students' weaknesses with high school algebra play a major role in their success in their first-year math courses.

Although research on students' difficulties with algebra in school has been well documented (e.g. Kieran, 1992; Hoch & Dreyfus, 2004), research on these difficulties and their impact at university level are scarce. Stacey, Chick, and Kendal (2004) discussed the main problems of algebra in school algebra, little was mentioned in the way of consequences for college level mathematics. Research has catalogued common errors in computation and algebra (Ashlock, 2010; Booth, Barbieri, Eyer, & Pare-Blagoev, 2014; De Morgan, 1910). Our findings parallel these categorizations and document that these errors continue to persist in college level mathematics work, potentially complicating student success in college mathematics courses (Stewart & Reeder, 2017b). As Author (2017, p. vii) points out: "Many college instructors are facing this dilemma every day. Students who seemingly follow more complex mathematical concepts, are unable to proceed as problems, for example involving fractions, will soon let them down." We suggest that challenges students have with the high school algebra content that is embedded in calculus problems are a major cause of failure for many Calculus students.

The goal of our research is to understand how students' difficulties with algebra impact their work in calculus problems. For this study, Calculus students were given algebra tasks and calculus tasks with algebra embedded to help answer the following research questions: (a) What were the most common algebra problems in both the algebra and calculus tasks? (b) What were the students' perceptions of their challenges with algebra and calculus related to these tasks?

## Theoretical framework

Piaget's (1952) theory of accommodation and assimilation as a theoretical framework was employed for this study. A schema (mental structure) serves two purposes: "It integrates existing knowledge, and it is a tool for acquisition of new knowledge" (Skemp, 1971, p. 39). When new situations and experiences are encountered, the human brain deals with it by either accommodation or assimilation; the structure of the schema must change to adapt to the new situation, "this may be difficult; and if it fails, the new experience can no longer be successfully interpreted, and adaptive behavior breaks down- the individual cannot cope" (p. 44). In this way, how we understand concepts is constantly changing and adapting as we are presented with new information, experience things, and learn new concepts. While assimilation is easier and often produces a feeling of mastery, accommodation is difficult. Vinner (1988) stated that "very often (and specially in mathematics) the cognitive structure of the learner is not suitable for incorporating the new material" (p. 594). He believed that acquisition of new mathematical concepts in more advanced settings requires accommodation, since "a concept which seems quite simple to the mathematician can be difficult for the student to accommodate" (p. 606). He further believed that the lack of attention to accommodation will lead into situations where "certain concepts are not conceived by the students the way we expected" (p. 593). Skemp (1979) introduced two further notions: *expansion* and *reconstruction*. He clarified that "our schemas grow by expanding existing concepts and by forming new ones" (p. 126). Sometimes, however, we may encounter a situation for which we have a relevant schema which is not adequate. If we are unable to avoid such situations, we need to re-construct our schema. This is "disruptive, unwelcome, and difficult: because while this is going on, we are unable to use our schemas effectively for directing our actions" (p. 126). We suggest that success in calculus requires expanding and reconstructing schemas about algebra in order to make sense of the calculus contexts in which they appear.

## Method

This qualitative research study involved 275 Calculus I students at a university in the Southwest US at the end of their 16-week course. Students were asked to solve three common Calculus I tasks and four algebra tasks, identify what caused them the most challenge, algebra or calculus, and provide a brief discussion about what challenged them while solving the tasks (see Table 1). The algebra tasks were designed such that they focused on the algebra students would encounter while solving the calculus tasks. Students were asked to solve the calculus tasks first (30 min) and were given the algebra only after their calculus problems were completed (20 min). The open response question was provided last.

Once all data were collected, it was de-identified and incomplete data sets were removed. The result was N = 84 complete sets of data. The research team (four individuals) met to analyze each problem to develop an initial codebook. The initial codebook was used by researchers to code ten sets of data independently for both calculus mistakes in the calculus problems and algebra mistakes in both the calculus and algebra problems. A second meeting of the research team focused on establishing the code book and inter-coder reliability. With an established codebook (see Table 2), each set of problems were analyzed and coded independently by two members of the research team. Each team met to review the codes and establish 100% agreement.

*Table 1. The Calculus and Algebra tasks.*

| Calculus tasks | Algebra tasks |
|---|---|
| 1. Implicitly differentiate. $\sqrt{xy} = 1 + x^2y$<br>2. Find the critical numbers of the function $f(t) = t\sqrt{4 - t^2}$<br>3. Evaluate the limit. | 1. Solve for y. $5 + xy = 10 + x^2y$<br>2. Solve for y. $\frac{1}{2\sqrt{5x}}(5 + xy) = 10x + x^2y$<br>3. Solve for t. |

| | |
|---|---|
| $\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}$ | $\frac{1}{\sqrt{t+1}}-\frac{1}{t}=0$<br>4. Solve for y.<br>$\frac{2y^2}{2\sqrt{y^2-9}}+\sqrt{y^2-9}=0$ |
| My main problem with the test was: Algebra ☐ Calculus ☐ | |
| Please write a comment relevant to your experience in taking this test. | |

*Table 2. Potential errors for Calculus and Algebra contexts.*

| Possible Calculus Errors | Possible Algebra Errors | Other Possible Error |
|---|---|---|
| 1. Power Rule<br>2. Product/Quotient rule<br>3. Chain Rule<br>4. Process of Implicit<br>5. Interpret Critical Numbers (set =0)<br>6. Undefined points are Critical<br>7. Taking the limit | 8. Convert radical to exponent<br>9. Exponent Operations<br>10. Balance Points<br>11. Distributive Property<br>12. Combining Like Terms<br>13. Cancelling<br>14. Factoring<br>15. Simplifying nested fractions<br>16. Sign error<br>17. Operations with radicals<br>18. Finding Common Denominators<br>19. Recognizing undefined values<br>20. Conjugating Rational Fractions<br>21. Quadratic Functions<br>22. Operations with Fractions | 23. Blank<br>24. Incomplete Calculus<br>25. Incomplete Algebra<br>26. Computation<br>27. Avoiding Algebra<br>28. Avoid Calculus<br>29. Miscellaneous<br>30. Isolating Variables |

## Results

Our first research question focused on determining the most common errors students made while completing the algebra tasks and calculus tasks while our second question focused on the students' perceptions of their challenges with algebra and calculus. As such, the results are presented in two sections: Research Question 1 and Research Question 2.

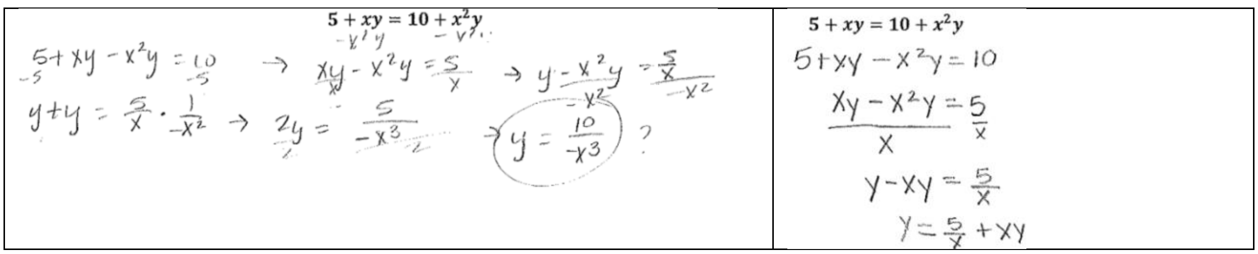
### Research Question 1

Analysis of the algebra and calculus tasks revealed that the student errors were numerous and significant with algebra in both sets of tasks and calculus related errors were frequent in the calculus tasks as well. The most common algebra errors made in both sets of tasks were problems working across the balance point in equations, cancelling, operations with radicals, appropriate application of the distributive property, and incomplete algebra (work that was not completed due to confusion). While the students' work with the calculus tasks were replete with algebra errors, they also made many calculus errors. The most common among these were correctly taking the derivative implicitly, using the product rule properly, failing to identify undefined points as critical, incorrectly taking the limit, and avoiding algebra.

#### Analysis of Algebra tasks

The first algebra task directed participants to solve for y. This problem required collecting like terms and then factoring to isolate the variable *y*. The most common mistakes illustrated that participants had an incomplete conceptual understanding of what it meant to solve an equation, either because they did not isolate the *y* variable or because they did not recognize factoring as a strategy that could help isolate the variable. Figure 1 illustrates two examples of these types of mistakes by different students.

The initial mistake by the first student occurred when s/he attempted to divide each side of the equation by - $x^2$ (see Figure 1(a)). Clearly, the student was attempting to rewrite the left side of the equation in a form which would allow the terms containing *y* to be combined; based on the incorrect work, the student combined these terms. The mistake was failing to recognize that factoring would accomplish this goal while performing operations on both sides of the equation would not. The second student compounded the errors as s/he tried to find a way to combine the two *y* terms. In other words, the student either failed to recognize that manipulating terms was no longer a viable option, or was unable to determine another viable strategy for solving equations. Likewise, the student whose work is presented in Figure 1(b) reached the point where s/he should have shifted strategies from manipulating both sides of the equation to factoring the left side of the equation, but continued to manipulate both sides of the equation instead, which resulted in an equation that was not solved for *y*.

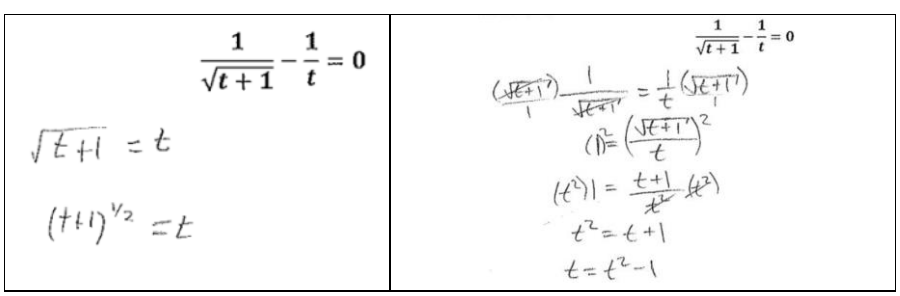



(a) (b)

*Figure 1: (a) Student did not recognize factoring as a strategy for solving equations, (b) Student did not isolate y.*

In the second algebra task, students had similar issues determining what strategies to use and when to move between strategies to solve the equation. Even when students successfully solved the problem, it sometimes appeared as if strategies were chosen at random and students seemingly solved the equation through determination and perseverance. Because of the multitude of technically correct, but unhelpful strategies that can be employed for task two, more mistakes were made with this task than any of the other algebra tasks.

The difficulty students faced in the third algebra task centered around points at which they needed to change strategies. Determining a strategy that allowed the equation to be rewritten without a radical and a strategy to use to solve the resulting quadratic equation challenged many students (see Figure 2). Interestingly, students were much more likely to simply stop working on task three when they reached one of these decision points than they were to stop working on task one or two.



(a) (b)

*Figure 2. (a) Student was not able to work with the radical, (b) Student was unable to determine a strategy for solving the quadratic equation.*

Operations with radicals proved to be a major difficulty for students in problem four. Not only did students have difficulty determining how to eliminate the radical, but students were also more likely to make mistakes in earlier algebraic concepts when wrestling with them in conjunction with radical. For example, one student (see Figure 3) properly eliminated the radical through multiplication, but failed to distribute the negative through the resulting binomial; a mistake fortuitously corrected by his/her next mistake. In addition, the student failed to recognize that the two $y^2$ terms could be combined, and incorrectly assumed that a radical could be, for lack of a better term, distributed to each term within it. It is important to note that many students who correctly applied the distributive property and correctly combined like terms in earlier problems routinely misapplied these procedures in task number four when radicals were involved. As students learn mathematics they build new schema, assimilate and accommodate new information, and expand and reconstruct existing schema. If these schemas are formed around misconceptions or incomplete understandings of mathematical concepts then as they are expanded and reconstructed through the ongoing process of accommodation, students will have continual problems in mathematics.

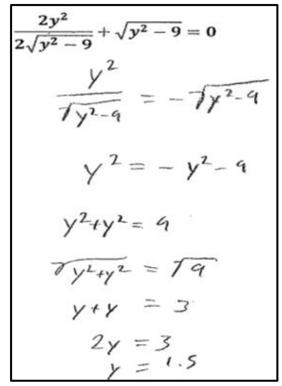
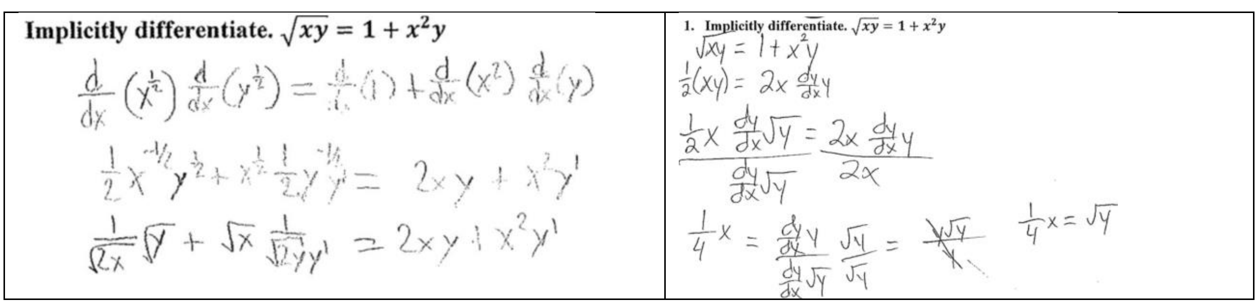



*Figure 3. Difficulties with radicals.*

### Analysis of Calculus tasks

In the first calculus task, the most common errors students made were correctly taking the derivative implicitly, using the product rule properly, and failing to complete the necessary algebra correctly. For example, one student (see Figure 4 (a)) incorrectly differentiated each variable separately on both sides of the equation in the first line, and then did not finish solving for $y'$. Note, that despite the incorrect notation of the first line, the second line appears to contain the correct derivatives.




(a) (b)

*Figure 4. (a) Incomplete Algebra (b) Incorrect Chain Rule and Product Rule with disappearing derivative.*

In contrast, another student (see Figure 4(b)) did not apply the chain rule properly on the left side or the product rule properly on the right side. Also note the strange algebra in the second and third lines lead to the $\frac{dy}{dx}$ term disappearing, making it impossible to solve for $y'$ as required, so this again is incomplete algebra. Four students total were completely correct for Question 1 as

61 of the 84 students either could not differentiate implicitly did not correctly apply the product rule, or did not solve for $\frac{dy}{dx}$.

In the second calculus task, the typical errors were failing to identify undefined points as critical, incorrectly applying the product rule or failing to apply it altogether, and failing to complete the calculus portion of the task. For example, one student could not complete the calculus due to difficulty with the product rule (see Figure 5). However, it is notable that s/he successfully identified the mistake and gave reasonable instructions for how the problem *should* be solved. Note s/he finds two of the critical points ($\pm 2$), but this is somewhat accidental, as s/he finds these by setting his incorrect derivative to 0, when the points $\pm 2$ should be obtained from finding the points at which the derivative is undefined. The points obtained from setting the correct derivative to 0 should be $\pm\sqrt{2}$. Only two students correctly solved task two.

The two most common errors in the third calculus task were incorrectly taking the limit and avoiding algebra (i.e., actively avoiding rationalizing the numerator). One student (see Figure 6 (a)) used the quotient rule in an inappropriate scenario (perhaps conflating with L'Hopital's Rule) to simplify the limit. S/he followed this very well executed quotient rule with an improper cancellation of one of the *t*'s, which led her/him to assume that the limit does not exist, despite still having *t*'s in both numerator and denominator. A few students (see Figure 6 (b)), incorrectly utilized a limit law to separate the two terms to separate the limit. While this strategy works well if both resulting limits converge, it does not here because the two separate limits both diverge. Note that while this student did not claim that the limit does not exist, s/he appears to have stalled out and never attempted to evaluate the limit. This task resulted in the more correct work from students (7 of 84) but provided the most variation in the types of errors students made.

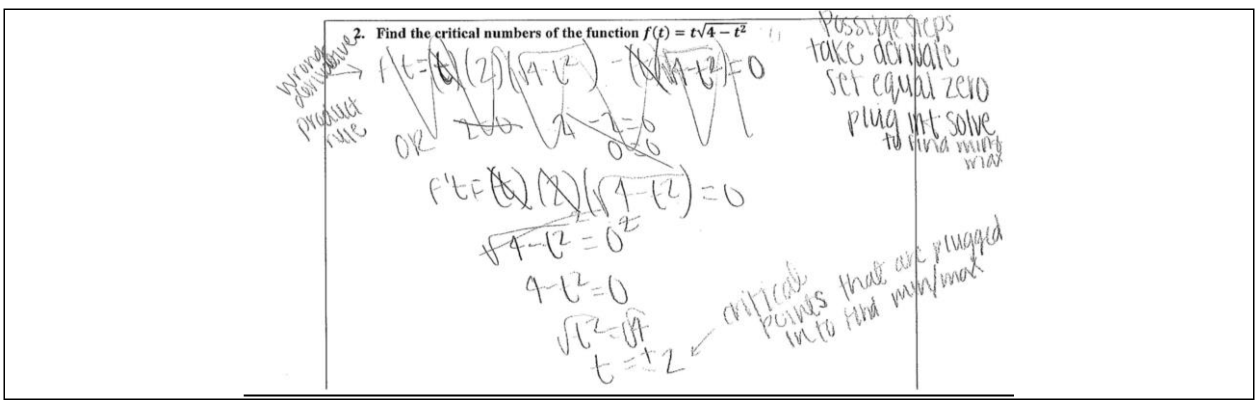
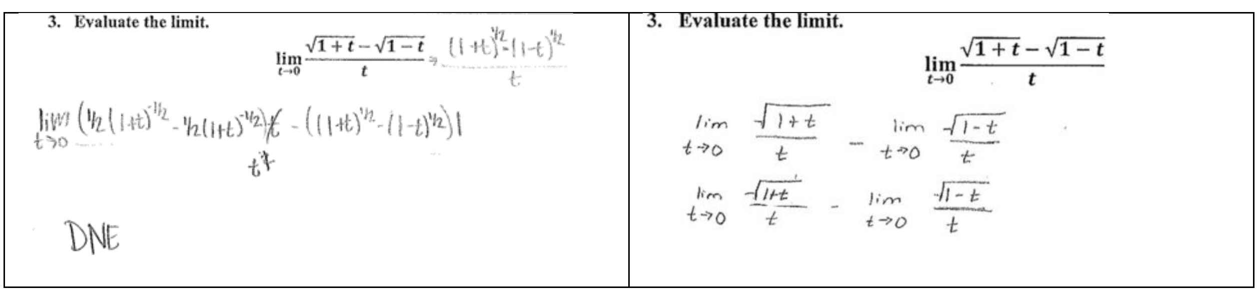



*Figure 5. Instructions for how problem should be solved.*




(a) (b)

*Figure 6. (a) Out-of-context Quotient Rule (b) Incorrect limit law.*

As students encounter new concepts in calculus they are no doubt building new schema to accommodate for the news ideas and new mathematics. However, in the midst of dealing with new ideas they must also rely on schemas they developed for algebraic manipulations in the setting of the new concepts they are learning in calculus. If the schema for their algebra understanding are incomplete, then they may present significant challenges for the students as they rely on them to develop understandings of new concepts.

**Research Question 2**

**Analysis of Students' Comments**

Our second research question aimed to provide insight on the students' perceptions about their abilities with algebra and calculus as presented in the tasks they were asked to solve. During the one-hour data collection session, students solved three calculus tasks and four algebra tasks and while pressed for time, 73 of the 84 chose to provide a response to our short-answer item. When asked to simply select which gave them more challenge, algebra, calculus, or both, 57% indicated algebra, 31% indicated calculus, while 12% indicated both. Their comments overwhelmingly expressed recognition that algebra causes them difficulties, frustration, anxiety, and in some cases, hopelessness about their abilities to succeed in mathematics. An excerpt of student comments below capture this well:

- *Square roots and fractions can make algebra difficult and confusing. Calculus can be difficult too but there are more steps either before or after the calculus that involve algebra and that can either "make or break" the problem and solution.*
- *I've had a very weak base in Algebra, ultimately leading to a dysfunction in Calculus.*
- *I struggled the most on the algebra portion of the test. However, I struggled with both portions of the test. I felt as if I hadn't learned anything or retained anything in my course of math. I want to be better at math, but I don't know how.*

## Concluding Remarks

We hypothesized that students would solve algebra problems largely correctly when these problems were in isolation from calculus, but make predominantly algebraic mistakes in the context of calculus problems with algebra problems embedded. However, we found that our sample of students had difficulty in all aspects of both the algebra and calculus tasks. Students routinely struggled with the isolated algebra tasks as well as the calculus tasks. While the work with the tasks presented students challenges with both calculus and algebra the student responses overwhelmingly indicated they had frustration and concerns with their algebra abilities. In the words of one student "I knew how to start the problem, but couldn't finish because of the difficulty of the algebra involved." This presents a challenge for those of us teaching undergraduate mathematics. Our students may have the prerequisite knowledge, but it may not be strong enough to function as a versatile tool in calculus as expected or required. Certainly, further research is needed to examine students' abilities with algebra and its' impact on their success in undergraduate mathematics. We are in the process of designing further studies by interviewing students and mathematics professors in order to gain a better appreciation of students' difficulties. Ultimately, we would like to create a model of intervention to remedy calculus students' struggles with algebra.

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# The Counter-storytelling of Latinx Men's Co-Constructions of Masculinities and Undergraduate Mathematical Success

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*While Latinxs complete undergraduate engineering degrees at lower rates than Whites and Asians, Latinx men trail behind Latinx women who recently earned over half of engineering and science degrees conferred to Latinxs. With multiple semesters of mathematics required in engineering majors, qualitative analyses of undergraduate Latinx men's strategies of persistence and success in engineering can illuminate ways to inform more socially-affirming postsecondary educational opportunities and thus increase retention in STEM (science, technology, engineering, and mathematics). This report presents findings from a phenomenological study that characterized variation in two undergraduate Latinx men's negotiations of their masculinities with pursuits of mathematical success as engineering majors at a large, predominantly White four-year university. Findings illuminate the Latinx men's strategies of managing risks of mathematics classroom participation, building academically and socially supportive relationships with faculty members, and negotiating pursuits of STEM higher education with their gendered sense of commitment to family.*

*Keywords:* equity, gender, identity, intersectionality, Latinx

Analyses of academic success among Latinxs[1] in undergraduate STEM education have shed light on disparities between Latinx women and Latinx men (Chapa & De La Rosa, 2006; Cole & Espinoza, 2008; Simpson, 2001). Cole and Espinoza (2008), for example, highlighted how undergraduate Latinx women in STEM have higher grade point averages and degree completion rates than Latinx men. At the same time, Latinx women demonstrated lower levels of confidence and weaker academic self-concept often shaped by the masculinized nature of undergraduate engineering and mathematics spaces perpetuated through issues of representation and valued norms of engagement (Camacho & Lord, 2014; Cole & Espinoza, 2008). Despite the masculinization of engineering spaces, Latinx women outnumber Latinx men as recipients of undergraduate engineering degrees in the United States (U.S.; NSF, 2017).

Engineering is a mathematics-intensive field of study. The socially exclusionary nature of mathematics, therefore, raises considerations about how different constructions of masculinity are privileged or marginalized in undergraduate mathematics, including those among Latinx men pursuing engineering majors. In mathematics education, Latinx students have "seldom been asked for their perspectives on their classroom mathematics experiences" (Varley Gutiérrez, Willey, & Khisty, 2011, p. 27), especially in relation to how they negotiate mathematical success with multiple intersections of their race, gender, and other identities. This points to the promise of intersectional analyses of mathematical success among undergraduate Latinx men pursuing engineering degrees that focus on their negotiations of academic pursuits with constructions of their masculinities.

This report presents findings from a study that detailed the variation of mathematical success

[1] The term *Latinx* decenters the patriarchal nature of the Spanish language that groups Latin American women and men into a single descriptor *Latino* denoting only men. The "x" in *Latinx* allows for gender inclusivity among Latin Americans (including those identifying as gender-nonconforming) compared to *Latina/o* implaying a gender binary.

among two Latinx men pursuing engineering majors at a large, predominantly White four-year university. A three-tiered analytical framework was adopted to address the following question: What institutional structures, interpersonal relationships, and ideological discourses shaped the two undergraduate Latinx men's co-constructions of masculinities with mathematical success?

## Relevant Literature

Two bodies of literature are reviewed in this section. The first body of literature details insights from intersectional studies on Latinx students' co-constructions of mathematics and social identities in undergraduate mathematics as a socially exclusionary space. The second explores undergraduate Latinx men's constructions of masculinities in their pursuits of higher education. Insights across these bodies of literature provide conceptual points of consideration for the study's exploration of how undergraduate Latinx men co-construct mathematical success as engineering students with their sense of masculinity.

### Undergraduate Mathematics as a Socially Exclusionary Space for Latinx Students

Mathematics has been well documented as a gendered and racialized space for marginalized populations, including women as well as Black and Latinx students (Boaler, 2002; Leyva, 2016; McGee & Martin, 2011; Mendick, 2006; Oppland-Cordell, 2014; Stinson, 2008; Varley Gutiérrez et al., 2011). Issues of gender and race, however, have largely been studied separately in extant mathematics education research and with conceptualizations of gender as a female-male binary rather than socially constructed (Leyva, 2017). This leaves the field with minimal insight on varying forms of mathematics experience among underrepresented student populations at different intersections of gender and other social identities.

Intersectional analyses, thus, allow for the detailing of within-group differences in how individuals make meaning of gendered, racialized, and other socially exclusionary experiences in their pursuits of mathematical success (Martin, 2009). Much of the foundational mathematics education research on Latinxs largely focuses on the importance of validating Latinx students' cultural backgrounds through use of the Spanish language and home experiences as resources for mathematical learning (Khisty & Willey, 2013; Moschkovich, 2013). Thus, there is room for exploring how other social identities including gender intersect with Latinx culture to shape variation in Latinx students' experiences of navigating mathematics as a socially exclusionary space. Below I present findings from studies in undergraduate mathematics education that adopted such intersectional analyses of Latinx students' co-constructions of mathematics and social identities at predominantly White universities.

In a study focusing on the experiences of two undergraduate Latinx women in their first year of pursuing mathematics-intensive majors (Leyva, 2016), I examined self-report data (including interviews, a focus group discussion, and mathematics autobiographies) to capture variation in strategies for negotiating their *familismo* (Marín & Marín, 1991; Suárez-Orozco & Suárez-Orozco, 1995), or a sense of loyalty or responsibility to the Latinx family unit, with pursuits of STEM higher education. Both Latinx women discussed managing gendered cultural discourses of Latinx women becoming young mothers and wives rather than being college-bound.

Oppland-Cordell (2014) coupled self-report and classroom observation data to detail how a Latinx woman's and Latinx man's emerging mathematical and racial identity constructions (EMRICs) contributed to shifts in their participation as learners in an undergraduate calculus workshop. While intersectional analysis revealed how gender was only relevant in the Latinx woman's participation shift related to perceptions of her and her peers' mathematical ability, Oppland-Cordell (2014) detailed how socioeconomic status, particularly having access to more meaningful and socially-affirming mathematical learning opportunities in the workshop than K-12 education, played a role in the Latinx man's workshop experience. These different social influences on the Latinx woman's and Latinx man's identity constructions and workshop participation illustrate the "complex intersectional nature of Latina/o students' EMRICs in mathematics classrooms" (Oppland-Cordell, 2014, p. 51). Considering how such variation of experience exists even within intersectional subgroups as noted in my study involving Latinx women, the coupling of self-report and observation data like in Oppland-Cordell's (2014) analysis allows for more situated insights into variation of how Latinx students, at different intersections of social identities, make meaning of their classroom experiences to inform the future design of more socially-affirming undergraduate mathematics learning opportunities.

**Social Constructions of Latinx Masculinities**

Latinx men is an example of a marginalized subgroup in STEM whose mathematics experiences have been minimally explored, especially using intersectional analyses of gender. However, research insights from higher education and psychology on Latinx masculinity ideologies and constructions of manhood can be leveraged to understand how Latinx men negotiate their masculinities with pursuits of mathematics-intensive STEM degrees like engineering. Compared to Black and White men, Latinx men more readily internalize and endorse traditional, culturally-specific norms of masculinity (Abreu, Goodyear, Campos, & Newcomb, 2000; Vogel, Heimerdinger-Edwards, Hammer, & Hubbard, 2011). Latinx masculinities are largely shaped by notions of *machismo* from Mexican culture that has a negative side (or *traditional machismo*) and a positive side (or *caballerismo*) (Arciniega, Anderson, Tovar-Blank, & Tracey, 2008; Torres, Solberg, & Carlstrom, 2002). Traditional machismo is associated with aggression, emotional restrictedness, hypermasculinity, avoidance of the feminine, and sexism including gender-role dominance (Arciniega et al., 2008). Caballerismo is characterized by ethnic acceptance, chivalry, family-centeredness, nurturing qualities, and problem-solving coping strategies (Arciniega et al., 2008).

It has been documented that Latinx men with high levels of caballerismo and high levels of perceived academic racism produce motivation for success to achieve their aims of protecting and providing for their families (Liang, Salcedo, & Miller, 2011). Latinx men with high levels of caballerismo and low levels of perceived academic racism placed less restrictions on emotional behaviors with other men, thus reflecting a reduction in feeling the need to validate their sense of Latinx masculinity in academic contexts (Levant & Fisher, 1998; Liang et al., 2011). In addition, self-confidence and traditional gender norms were commonly observed among Latinx men (particularly Mexican-American) of more recent generational status and lower socioeconomic status respectively (Ojeda, Rosales, & Good, 2008). Family plays a major role in Latinx men's college persistence as a source of motivation (e.g., parental encouragement) or distraction (Ojeda et al., 2011, Sáenz, Bukoski, Lu, & Rodriguez, 2013; Sáenz, Mayo, Miller, & Rodriguez, 2015). In alignment with the notion of caballerismo, Latinx men's pursuits of higher education can be interpreted as being framed by notions of *familismo* (Marin & Marin, 1991; Suarez-Orozco & Suarez-Orozco, 1995) with aims of ultimately supporting their families.

Undergraduate Latinx men minimally engage in help-seeking behaviors in times of struggle to avoid being perceived as vulnerable, less self-reliant, and thus feminine (Cabrera, Rashwan-Soto, & Valencia, 2016; Gloria, Castellanos, Scull, & Villegas, 2009; Sáenz et al., 2013; Sáenz et al., 2015). Cabrera and colleagues (2016), for example, detailed constructions of self-defeating masculinities among Latinx men at a predominantly White university that brought them to downplay the significance of academic and racial stressors, internalize responsibility for managing these struggles, and refuse seeking help because it was an affront to their masculine pride as well as a manifestation of fear and vulnerability. Such avoidance of help-seeking was a performative strategy that protected the undergraduate Latinx men's masculinity while also jeopardizing their academic success. Focusing on persistence and success in community college, Sáenz and colleagues (2013, 2015) documented how Latinx men managed gender role conflicts with their pursuits of higher education. Machismo operated as both a barrier to academic success and a "quasi-positive" source of motivation for success, mainly by way of competition with Latinx women (Sáenz et al., 2013). Caballerismo guided Latinx men's management of fears about academic failure by positioning full-time employment after high school and higher education at odds with one another, the former representing a "cultural marker of manhood" (p. 91) for the advancement of supporting their families (Sáenz et al., 2013). The positioning of employment as a form of successful Latinx masculinity shaped the discouragement that Latinx men received about going to college from hometown peers as well as discourses of Latinx women as smarter and more destined for higher education than Latinx men (Sáenz et al., 2015).

## Theoretical Framework

This study synthesized various perspectives into a theoretical framework that guided data analysis. Critical race theory (CRT) in education is a perspective that "foreground[s] and account[s] for the role of race and racism" (Solórzano & Yosso, 2002, p. 25) to disrupt racism and other intersecting systems of societal oppression (e.g., sexism, classism) in schools and classrooms. Intersectionality (Crenshaw, 1991), a tenet of CRT, refers to the constitution of unique systemic forms of oppression experienced at intersections of race, class, gender, and other identities. As a "close cousin" to CRT, Latinx critical race theory (LatCrit) was adopted to examine the intersectionality of experience among Latinxs in relation to culture, immigration, and language that often go unaddressed in CRT (Bernal, 2002). The intersectionality tenet of CRT and LatCrit focused this analysis by exploring variation in participants' strategies for negotiating mathematical success with different intersections of their social identities.

## Methods

This yearlong study took place at a large state university in the northeastern U.S. Less than 15% of the 2011-2012 graduating class was Latinx. These Latinx graduates earned only 10% of the university's conferred STEM degrees. Latinx study participants were purposefully recruited based on criteria informed by extant scholarship on successful underrepresented students in STEM (Cole & Espinoza, 2008; McGee & Martin, 2011; Stinson, 2008). Five Latinx participants (2 women and 3 men) were recruited from the university's chapter of the Society of Hispanic Professional Engineers (SHPE), a national organization aimed at empowering the Hispanic community in realizing its potential in engineering through STEM outreach and professional networking. The analysis presented in this report focused on two Latinx men: Brian (a third-year, Peruvian electrical engineering student who had transferred from a community college and immigrated to the U.S. when he was twelve years old) and Daniel (a fourth-year, Dominican- and Ecuadorean-American mechanical engineering student).

Phenomenology informed the study's methodology of collecting and critically examining multiple "texts of life" (Creswell, 2013) to detail the phenomenon of mathematical success among the two Latinx men as engineering majors at the university. Under the CRT perspective, these "texts of life" informed the analytical construction of the two Latinx men's *counter-stories* (Solórzano & Yosso, 2002). Counter-storytelling is a methodology used to tell the stories of marginalized individuals in society that "aims to cast doubt on the validity of accepted premises or myths, especially ones held by the majority" (Delgado & Stefancic, 2001, p. 144). The coupling of CRT with LatCrit framed the study's cross-case, phenomenological analysis of mathematical success as an intersectional endeavor across the two Latinx men's counter-stories.

Four types of data were collected: (i) mathematics autobiographies, (ii) fields notes from classroom observations, (iii) semi-structured interviews, and (iv) a focus group. Observations in the participants' mathematics classrooms and engineering department offered situated insights to complement participants' reflections of experience captured in other data sources for the study.

The mathematics autobiography, completed prior to the first interview, allowed participants to adopt a storytelling role by writing a story of 3-4 paragraphs chronicling major experiences in mathematics. Field observations were completed in participants' college mathematics classes, including three 80-minute lectures and three 80-minute recitations or problem-solving workshops per semester. These observations detailed the instructional and relational spaces of the mathematics classrooms as well as participants' engagement noted in terms of interactions and participation (e.g., answering and asking questions) or lack thereof.

Throughout the academic year, participants completed three 60-minute, semi-structured individual interviews. All interviews were audiotaped and transcribed verbatim. The interviews were opportunities for participants to share and explore what being Latinx and mathematically successful meant to them across different contexts (e.g., classroom, home, SHPE meetings). Interview questions were structured in an open-ended manner, allowing participants to describe varying levels of consciousness of their different social identities across these contexts such as the mathematics classroom (Bowleg, 2008).

In addition, participants completed a focus group centered on three stimulus narratives of events from their mathematics lectures and recitation/workshop sessions. These narratives related to dynamics explored in extant literature of students taking up classroom space (Hand, 2012), stereotypes of mathematical ability (Shah, 2017), and faculty-student relationships (Battey, Neal, Leyva, & Adams-Wiggins, 2016). Participants were probed on the extent to which they observed such dynamics in mathematics classrooms and whether or not they saw themselves in similar situations. The focus group was audiotaped and transcribed verbatim.

Phenomenology guided data analysis by focusing on patterns across participants' mathematics experiences to detail the phenomenon of mathematical success and how it was negotiated with their social identities (Creswell, 2013). Open codes were used to identify the institutional, interpersonal, and ideological influences on mathematical success while axial codes examined the intersectionality across participants' mathematics experiences (Bowleg, 2008; Creswell, 2013). While some axial codes were specific to individual social identities (e.g., race, gender), other axial codes corresponded to different intersections of these identities such as race-gender (Bowleg, 2008). Implicit instances of intersectionality were made explicit through analytical consideration of subtexts in participants' narratives of experience (Banning, 1999).

Validity was reinforced through triangulation of collected data, memoing, and member checking. I brought awareness of my positionality to pursue data analysis with strong subjectivity to develop nuanced understandings of the undergraduate Latinx engineering students' mathematical success. In addition, I developed positive rapport and mutual trust with participants supported by our mutual identification as Latinx STEM majors.

## Findings

Three themes emerged in the cross-case analysis of Brian's and Daniel's counter-stories: (i) managing risks of mathematics classroom participation, (ii) building academically and socially supportive relationships with faculty members, and (iii) negotiating pursuits of STEM higher education with their gendered sense of commitment to family. This section presents the variation in the two Latinx men's co-constructions of their masculinities and mathematical success as engineering students with respect to these three analytical themes.

First, the Latinx men's experiences capture how the university's mathematics classrooms constructed racialized hierarchies of ability (Martin, 2009) along which Latinxs were positioned lower than their White and Asian classmates who were more regularly invited to participate. The Latinx men discussed how such racialized positioning resulted in managing risks associated with classroom participation to protect their status of mathematical ability from negative judgment. Brian described the classrooms' "tense and competitive" atmosphere where he "didn't have the guts" to respond to professors' questions like his White classmates did and avoided possibilities of "feel[ing] embarrassed" if professors thought he did not know the content well. Daniel reflected on "closed off" opportunities from connecting with higher-status peers "who go above and beyond" in participation, unlike him who remained silent because "no one want[ed] to be wrong." Unlike Daniel, Brian felt ease connecting with classmates of any race because they were in the "same position" as him in being successfully admitted to the university. Brian "got used to" being underrepresented in mathematics classrooms and viewed all classmates as "experiencing the same stuff [he was] experiencing" as university STEM students. Limited classroom participation and perceptions of sameness among peers capture the Latinx men's strategies for managing the racialized dynamics of undergraduate mathematics classrooms. However, such nonparticipation and erasure of social differences are problematic as they perpetuate ideologies of whiteness in mathematics that position Latinx students as less mathematically able and mathematics as a neutral, cultureless domain (Battey & Leyva, 2016).

Secondly, the Latinx men reflected on the importance of building academically and socially supportive relationships with professors in and out of the classroom. Brian and Daniel valued professors who established relational spaces in classrooms that welcomed student participation, prioritized mathematical understanding, and were characterized by supportive teacher-student interactions. These influential faculty members' support went beyond coursework assistance, including office hour conversations that were emotionally-reaffirming "turning points" in the Latinx men's academic trajectories as engineers. Brian recalled a meaningful "big talk" with an Argentinian engineering professor at the community college who acknowledged his ability as a mathematics minor and encouraged him to pursue an engineering major. As a fellow Latinx man, the professor's advice was informed by his awareness of Latinxs' marginalized position in society, thus describing an engineering pathway as an opportunity for Brian to apply his ability in challenging deficit views by "be[ing] one of those persons who tries to make yourself look good and also your community." Daniel's relationship with a Honduran calculus professor, Benjamin, played a role in his "metamorphosis" as a calculus student when he began sitting toward the front of lecture halls, voluntarily attending office hours, and feeling like he "could become an engineer." He described Benjamin as a "uncle-grandfather hybrid" who, in speaking Spanish with him during office hours and sharing childhood stories during class, brought him to feel "more comfortable" than with other university professors who "felt like robots." These professors' blending of academic and social support can be likened to notions of apoyo (moral support, Auerbach, 2006) and consejos (culturally-specific forms of advice; Delgado-Gaitan, 1994) that Latinx children receive from family members for educational advancement. I argue that, while both professors in these examples were Latinx men, such family-like forms of support can be adopted by faculty members from other backgrounds with a critical awareness of Latinx men's marginalized positions in higher education and society to inform more equitable educational practices.

Lastly, Brian's and Daniel's counter-stories captured how the Latinx men's familismo shaped how they made meaning of their pursuits of mathematical success and engineering careers as masculine endeavors. Brian's low-income, immigrant family background shaped his perceptions of STEM higher education as an opportunity to pursue a "good career" as an engineer, allowing for social mobility in the U.S. and to "help [his] parents out with economic problems." He viewed his STEM pursuits at a four-year university as being tied to a sense of responsibility of becoming "someone to look up to" in his hometown community. Daniel approached his engineering degree pursuits by associating academic failure with a sense of guilt about letting down his family. Graduating and becoming an engineer were ways that Daniel saw himself "represent[ing]" for his family and meeting his brother's gendered expectations that "you're not a man until you live alone [and] pay your bills." Brian and Daniel, therefore, similarly engaged in forms of caballerismo -- a construction of Latinx masculinity characterized by family-centeredness – through their views of undergraduate mathematical success and engineering career pursuits as ways to contribute to the advancement of their respective family situations. The Latinx men also reflected on having encountered implicit forms of racism in relation to their mathematical ability as engineering students. While Brian reflected on his encounter with a hometown police officer who appeared "a little bit shocked" after learning about his engineering degree pursuits, Daniel interpreted Asian American peers asking for his grade on a mathematics exam as them adopting a "subtle change of words" to essentially ask if he failed. Being mathematically successful, as a result, served as a way for the Latinx men to show that "we're [Latinxs] not stupid" (Brian) as well as "not fall victim to the stereotypes" (Daniel) of racialized mathematical ability. They saw their STEM higher education pursuits as ways of them not becoming a "delinquent or deviant person" (Brian) as well as "not be[ing] a statistic" (Daniel), thus challenging discourses about Latinx men as criminals and underrepresented in higher education respectively. Brian's and Daniel's counter-stories, therefore, illustrate how their family-centered sense of caballerismo shaped their persistence as engineering students and coping strategies for managing interpersonal slights about their academic ability as Latinx men.

## Implications for Educational Practice

Findings from this study raise implications for educational practice. The Latinx men's strategic management of risky classroom participation and limited opportunities for establishing classmate connections highlight undergraduate mathematics educators' important role in designing instruction and participation structures that disrupt racialized and other socially exclusionary status of mathematical ability. With public forms of help-seeking perceived as an affront to successful constructions of Latinx masculinity, it is important for mathematics educators to consider the extent to which they extend opportunities for student support rather than solely expect students to initiate contact. Brian's and Daniel's appreciation of faculty support, likened to notions of apoyo and consejos in Latinx families, illustrates the value of culturally-affirming teaching in undergraduate mathematics toward increasing retention and inclusion among underrepresented groups in STEM. With Brian and Daniel left largely on their own in negotiating engineering pursuits with their commitment to family as well as oppressive discourses about Latinx men, it is important for higher education institutions to carve spaces that bring Latinx men together for collective forms of coping and support in managing the racialized-gendered burdens of such experiences.

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# Schema Development in an Introductory Topology Proof

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*This is an exploratory study into schema development of introductory topology students. We discuss Skemp (1987) and Dubinsky and McDonald's (2001) definitions of schema and how they fit with Piaget and Garcia's (1989) triad framework. We employed these theoretical instances on the idea of schema to analyze students' responses to a final exam problem about a basis for the product topology on a product space. Our analysis indicates that the majority of the students were still in the beginning stages of schema development by the end of the semester in a topology course.*

*Keywords:* schema, Topology, basis

## Theoretical background

Advanced mathematics courses are often difficult for undergraduate students to transition into and research on student difficulties on advanced courses, especially on topology, are scarce. The overarching goal of this project is to build a theoretical framework investigating the differences between expert mathematicians and novice undergraduate students' schemas in topology. We also would like to be able to investigate how students' schemas develop (Piaget's accommodation) and how interactions with peers and instructors affect that development. In this case study, we embark on this journey by examining undergraduate students' proof attempts involving a basis for the product topology on $X \times Y$.

We will employ the idea of schema to gain more insight into the transition towards advanced mathematics, specifically towards topology. Although there are multiple definitions of schema currently in the literature, in this study we will mostly focus on Skemp's version. In 1962, Skemp argued for the need of a valid learning theory that was developed in classrooms:

> A theory is required which takes account (among other things) of the systematic development of an organised body of knowledge, which not only integrates what has been learnt, but is a major factor in new learning: as when a knowledge of arithmetic makes possible the learning of algebra, and when this knowledge of algebra is subsequently used for the understanding of analytical geometry. (p. 133)

Skemp (1962, p. 133) defines schema as the "organised body of knowledge" that integrates existing knowledge and is a major factor for new learning. Additionally, he defines and compares schematic learning to rote learning (non-schematic learning). Unsurprisingly, he finds that "Schematic learning has a triple effect: more efficient current learning, preparation for future learning, and automatic revision of past learning." (p. 140)

Skemp (1987) gives a more detailed definition of schema in his chapter, "The Idea of a Schema". He describes a system where concepts are embedded in a hierarchical structure of other concepts, where levels in the structure are classifications of concepts. For example, a train can be classified as a mode of transportation and can contribute to one's concept of transportation. We can also pair concepts together, giving a relation between them, which we can also classify. Additionally, we can look at transformations of concepts, which can be combined to make other transformations. What makes this hierarchical structure of concepts, relations, and transformations so deep and complex is the fact that these classifications are not unique, giving way to multiple hierarchical structures, which can be interrelated. When components of these conceptual structures come together to make a structure that would not be realized by only looking at the individual components, we call this resulting structure a schema. Skemp (1987) claims that a schema integrates existing knowledge, serves as a tool for future learning, and makes understanding possible. Without a suitable schema, students will have difficulty in understanding or making sense of new concepts. Skemp (1987) used topology in his work for the reason that "the relevant schema can be quickly built up, whereas most mathematical ones take longer." (p. 30) Although this study focuses on a more advanced topology question than Skemp did, we still believe that topology offers ideal topics to observe schema development with since most students do not encounter topology until late in their undergraduate work.

Another definition of schema is embedded in APOS Theory (Dubinsky & McDonald, 2001). Actions, processes, and objects are used to define a schema. Actions are external transformations of objects that become processes once internalized. After an individual becomes aware of a process and the transformations that can act on it, the process has become an object itself. Dubinsky and McDonald (2001) continue on to define schema:

> Finally, a schema for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept. This framework must be coherent in the sense that it gives, explicitly or implicitly, means of determining which phenomena are in the scope of the schema and which are not. (p. 3)

Clark et al. (1997) discussed an application of Piaget and Garcia's (1989) triad framework, Intra, Inter, and Trans, to the chain rule in Calculus. This triad is a theory for schema development within the context of APOS. Before a schema is coherent, it must go through these three stages. In the Intra stage, an object is thought of in isolation from other actions, processes, or objects. Once relationships are seen between the object and other actions, processes, objects, and schemas, the individual is in the Inter stage, also known as a pre-schema. In the Trans stage, a coherent structure begins to underlie the relationships from the Inter stage, and there now exists a schema for the original object in question.

As an example, consider the development of a schema for a topology. Working purely within the definition of a topology and considering basic examples is in the Intra stage. The schema enters the Inter stage once connections between the definition and previous knowledge are made. This includes more complex examples and possibly basic proofs. Viewing a topology as how open sets are defined for a topological space and being able to apply that in more complicated proofs demonstrates ideas in the Trans stage. This triad will be used as a place to begin analyzing schema development for a proof in an introductory topology course.

We view Piaget and Garcia's (1989) triad framework as a continuous spectrum for developing a schema. Dubinsky and McDonald's (2001) definition of schema overlaps with only the Trans stage since that is when a coherent structure appears. In comparison, Skemp's (1987) definition of schema not only overlaps with the Trans stage, but all stages of the triad framework. In our view, an idea does not have to be fully developed or correct in order to be a part of a schema. Our research question for this project is "With respect to the triad spectrum, how developed are introductory topology students' schemas for a basis for a topology?"

## Method

This is a case study into introductory topology students' thinking about a basis for a topology. Eleven final exams were collected and de-identified from a senior-level undergraduate topology class at a research university in the Southwest US. This study focuses on the first of the nine exam questions, shown in Figure 1.

1. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Define the product topology $\mathcal{T}$ on $X \times Y$.

   (b) Show that the projection map $p_X : X \times Y \to X$ defined by $p_X(x, y) = x$ is an open map.

*Figure 1. Question 1. Define and use the product topology on a product space.*

We chose this question for a couple of different reasons. First, it is structured such that students who are in-between the Intra and Inter stages of their schema development for a topology generated by a basis can still answer part a. Then part b requires students to be at least in the Inter stage of schema development. This question quickly reveals students whose schemas are still in the Intra stage.

Compared to other questions on the exam, this problem is more consistent with content from a typical introductory topology class. It would be unusual if the product topology on $X \times Y$ and the use of a basis did not appear in a beginning topology course, and therefore this problem is one that can be considered for use in future expansions of this study. This problem was also the first on the exam and therefore all of the students made an attempt on it.

The data was initially coded by identifying the types of errors made in each part of the problem (see Table 1). We then went through a second round of coding for consistency and grouped the responses together based on these errors and attempted to analyze them with the triad spectrum.

*Table 1. Types of errors.*

| Code | Description | Percentage of Students with Error |
|---|---|---|
| B | Left blank or contributed no original thoughts | 9.1% |
| IN | Issues with notation | 36.4% |
| IL | Issue of beginning proof with conclusion/other incorrect logical statement | 45.5% |
| NB | No reference to a basis | 63.6% |
| LC | Lacking clarity | 72.7% |
| LL | Lacking logical flow | 18.2% |
| LD | Lacking direction | 9.1% |

## Results and Discussion

The product topology on $X \times Y$ can be defined using the collection $\beta = \{U \times V | U \in \mathrm{T}_X, V \in \mathrm{T}_Y\}$ as a basis. The proof for part b involves three main components:

A. Noting that all open sets can be written as a union of basis elements (this part may be considered part of the definition of a basis depending on how it was presented in class)
B. Noting that the projection of a union is a union of projections
C. Showing the projection map is an open map for basis elements

We understand it is up to each instructor as to how detailed students' proofs should be, but these three components should at least be noted somehow in the proof. Figure 2 gives an overview of the proof schema. The arrows in the figure indicate previous knowledge that is needed in order to complete parts of the problem.

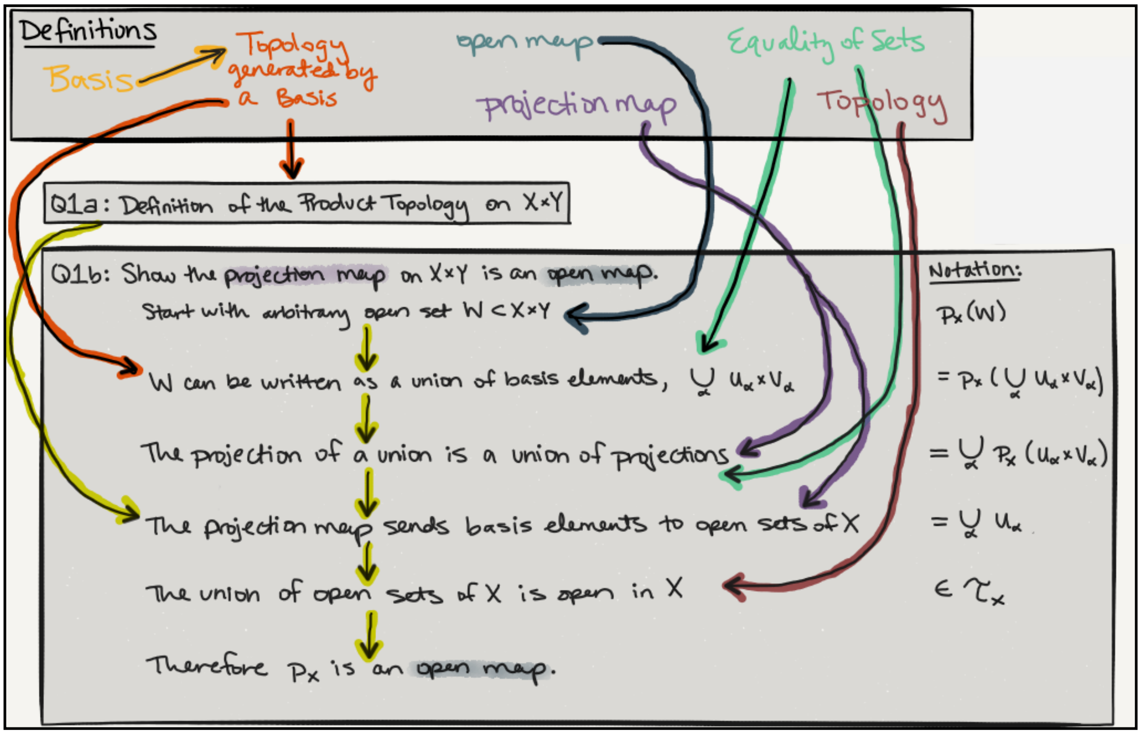
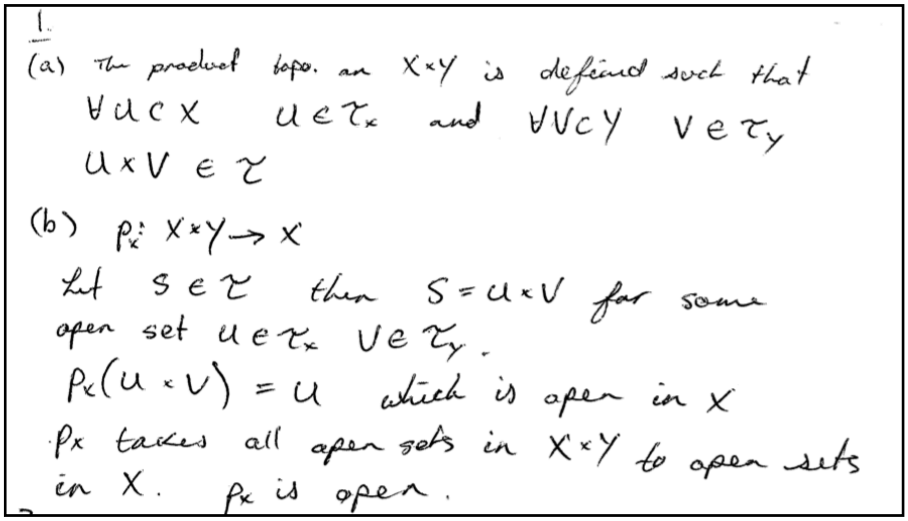



*Figure 2. A proof schema for the problem.*

Seven of the eleven students did not use a basis to define the product topology on $X \times Y$ and six of those seven students claimed that the topology on $X \times Y$ is $T_{X \times Y} = \{U \times V | U \in T_X, V \in T_Y\}$. A typical response of this type is shown in Figure 3.

The following argument demonstrates why this response cannot be the topology on $X \times Y$ and why a basis is needed. Let $\beta = \{U \times V | U \in T_X, V \in T_Y\}$ be the basis for $T_{X \times Y}$. $U_1 \times V_1$ and $U_2 \times V_2$ are both elements of $\beta$ and therefore are also elements of $T_{X \times Y}$. By the definition of a topology, $(U_1 \times V_1) \cup (U_2 \times V_2)$ is also an element of $T_{X \times Y}$. Note, however, that the union is not of the same form as elements of $\beta$ and cannot be in $\beta$, as shown in Figure 4. So $\beta$ cannot be the entire topology on $X \times Y$.

Since the proof for part b depends on the use of a basis, the seven students who did not use a basis in part a were unable to write a complete proof for part b. They often showed component C of the proof but did not include components A or B. The students who had this type of response may not see the need for a basis, when it is appropriate to use one, or how to make use of it. There is a disconnect between this problem and the definition of a topology generated by a basis. Therefore these students' basis schemas are, at best, in the Intra stage of schema development. They have not reached the Inter stage since they are unable to connect a basis with other knowledge.



1.

(a) The product topo. on $X \times Y$ is defined such that
$\forall U \subset X$ $U \in \tau_x$ and $\forall V \subset Y$ $V \in \tau_y$
$U \times V \in \tau$

(b) $p_x: X \times Y \to X$
Let $S \in \tau$ then $S = U \times V$ for some
open set $U \in \tau_x$ $V \in \tau_y$.
$p_x(U \times V) = U$ which is open in $X$
$p_x$ takes all open sets in $X \times Y$ to open sets
in $X$. $p_x$ is open.

*Figure 3. A response that is not past the Intra stage.*

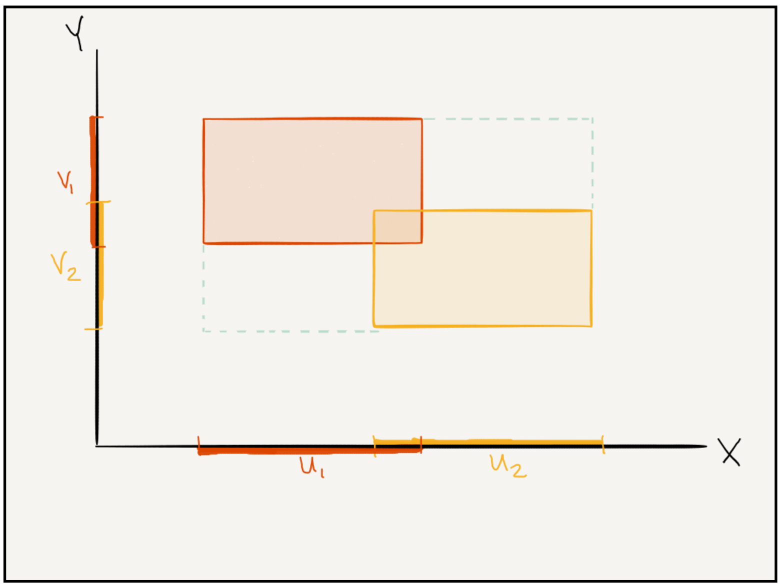
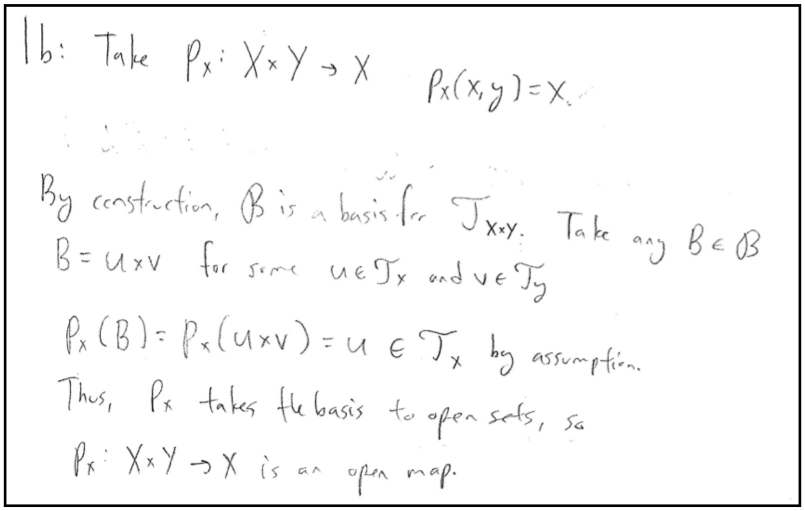



*Figure 4. A visual representation of the need for a basis.*

The four students who did make use of the basis had problems with incomplete proofs and notation. They would write the proof for basis elements only and then immediately jump to the conclusion of the proof without addressing components A or B of the proof. Such an example is in Figure 5. Whether or not the proof is considered to be correct depends on the instructor and the classroom norms. For this study, however, we are not as concerned about the validity of the proof as much as what it does (or in this case, does not) tell us about the student's schema of a basis. The use of the word "basis" can be used as a substitute for component A of the proof, but we cannot assume that the student did or did not understand this. The same goes for component B, which may or may not have been considered trivial in the class. We can say that this student has reached the Inter stage of basis schema development since they could relate a basis with other actions, processes, and objects, but due to the minimal amount of details in their proof, we cannot make any conclusions past this stage about their level of understanding.

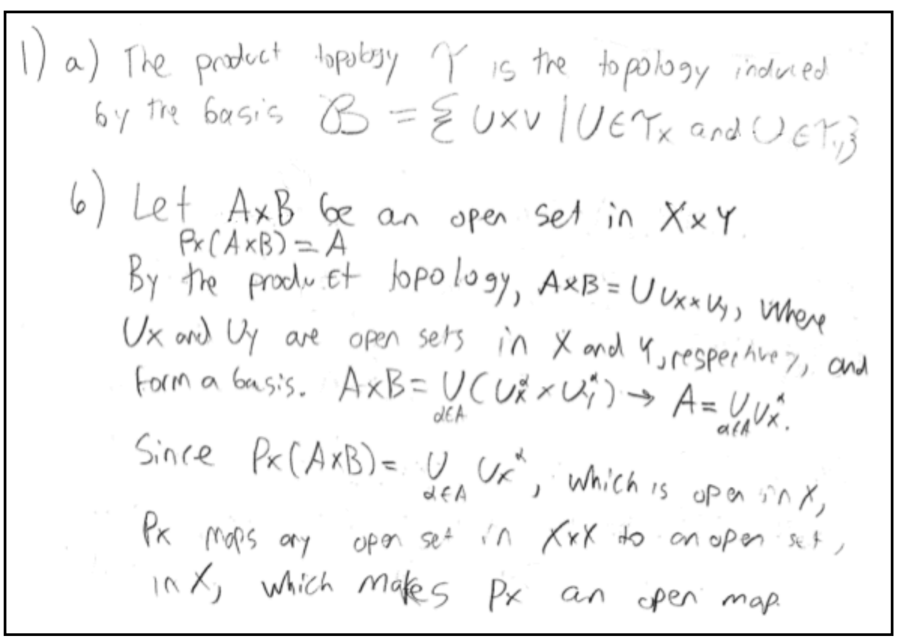


1b: Take $p_x : X \times Y \to X$ $\quad p_x(x,y) = x$.

By construction, $\mathcal{B}$ is a basis for $\mathcal{T}_{X\times Y}$. Take any $B \in \mathcal{B}$
$B = U \times V$ for some $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$
$p_x(B) = p_x(U \times V) = U \in \mathcal{T}_X$ by assumption.
Thus, $p_x$ takes the basis to open sets, so
$p_x : X \times Y \to X$ is an open map.

*Figure 5. A response that has reached the Inter stage.*

There were four students who had notational issues and nearly all students could have made their arguments more clear. An interesting example of this is in Figure 6. The student in this example incorrectly used $A \times B$ as their arbitrary open set of $X \times Y$, yet still included component A of the proof by saying that $A \times B$ is a union of basis elements. This indicates that the student had an understanding of the need for component A in their proof schema, but they did not understand how to denote the arbitrary open set. The student has a coherent proof structure here, but their argument could be improved with some corrections in notation. This student's response shows that they have reached the Trans stage, but there are still some notational gaps to fill in in their overall schema.



1) a) The product topology $\Upsilon$ is the topology induced by the basis $\mathcal{B} = \{U \times V \mid U \in \Upsilon_X$ and $V \in \Upsilon_Y\}$

b) Let $A \times B$ be an open set in $X \times Y$
$P_x(A \times B) = A$
By the product topology, $A \times B = \bigcup U_x \times U_y$, where $U_x$ and $U_y$ are open sets in $X$ and $Y$, respectively, and form a basis. $A \times B = \bigcup_{\alpha \in A} (U_x^\alpha \times U_y^\alpha) \to A = \bigcup_{\alpha \in A} U_x^\alpha$.
Since $P_x(A \times B) = \bigcup_{\alpha \in A} U_x^\alpha$, which is open in $X$, $P_x$ maps any open set in $X \times X$ to an open set, in $X$, which makes $P_x$ an open map

*Figure 6. A response that has reached the Trans stage.*

## Concluding Remarks

The three examples discussed in this study demonstrate three different places along the triad spectrum where student's schemas could be. Even though this problem came from a final exam at the end of the semester, a majority of the students surprisingly were still `at the Intra stage or lower in their schema development for a basis. We cannot comment on why this is since we did not collect any data regarding the norms of the class that these participants were in. This also means that we cannot know what was considered to be trivial in the course, making it difficult to analyze student's responses that are similar to Figure 5. These schemas may or may not include the components that were replaced with equivalent, but highly simplified, statements. We also do not know how much the instructor emphasized the need for a basis for certain topologies or whether or not the students had seen this problem on a previous homework assignment, both of which would affect the students' schemas.

The other limitation to this study is that it is impossible to physically see the schema of another person, so at best we can only make conjectures about participants' schema development, especially since we analyzed written proofs. Interactions with participants will be more informative in future work.

The next steps for expanding our project include interactions with the participants, data collection that occurs at the beginning and the end of a semester, and interactions between participants in either a partner or group setting. We hope to have participants explain their schemas out loud to us or a peer and to observe progress in the development of their schemas over time. We also will be asking a wider variety of questions over introductory topics to gain a better sense of which topics are more challenging for undergraduate students.

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Generalizing in Combinatorics Through Categorization

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*In this report we discuss students generalizing within a combinatorial setting. To facilitate reflection on prior activity, we prompted students in a teaching and a design experiment to categorize a myriad of problems they had previously engaged in. We will discuss the combinatorial underpinnings behind the students' generalizations according to Lockwood's (2013) model for combinatorial understanding. We saw that the students were able to produce generalizations of various basic combinatorial problems while each maintaining different understandings of the combinatorial structures. We conclude by discussing uniformity in the students' reasoning pertaining to combinations and the productive nature of such discussions.*

Key words: generalization, combinatorics, combinations, permutations

## Introduction

The activity of generalization is integral to mathematical thought, reaching all education levels (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Peirce, 1902). While there is a growing body of literature on student generalization, we still have much to learn about fostering productive generalizing activity in various contexts. Through a multi-phase study, we sought to better understand students' generalizing activity in a combinatorial setting. Combinatorics provides a natural setting for generalization, as counting problems are often accessible yet challenging (Kapur, 1970; Tucker, 2002). These accessible problems provide a natural structure from which students may generalize. In this report, we discuss the results of student engagement in a categorization task designed to facilitate reflection on prior work with various counting problems. The students collectively produced sophisticated generalizations while individually maintaining unique combinatorial understandings. We discuss the various nuances of their understandings as well as some affordances of attending to certain combinatorial structures.

We will discuss the students' generalizing activity in accordance with Lockwood's (2013) model for combinatorial thought. Such an analysis provides a deeper understanding of the potential source material for students' generalizations in combinatorics. We seek to answer the following research question: What do students attend to combinatorially as they generalize?

## Literature Review

### Generalization

Generalization has been recognized as a key aspect of mathematical activity by both researchers (Amit & Klass-Tsirulnikov, 2005; Davydov, 1990; Ellis, 2007b; Vygotsky, 1986) and policymakers (Council of Chief state School Officers, 2010). While much of the literature on student generalization focuses on algebraic contexts (Amit & Neria, 2008; Becker & Rivera, 2006; Carpenter, Franke & Levi, 2003; Ellis, 2007a/2007b; Radford, 2006/2008; Rivera, 2010; Rivera & Becker, 2007/2008), more recent studies have looked at undergraduate student generalizations in calculus (Dorko, 2016; Dorko & Lockwood, 2016; Dorko & Weber, 2014; Fisher, 2007; Jones and Dorko, 2015; Kabael, 2011) and combinatorics (Lockwood & Reed, 2016). Lockwood and Reed (2016) first investigated generalization in combinatorics by demonstrating two students that produced similar generalizations while holding vastly different meanings for their constructs. This report contributes to the growing body of literature by providing instances of generalization being rooted in various nuanced combinatorial understandings.

### Combinatorial Reasoning

Though combinatorics provides accessible and deep tasks (Kapur, 1970; Tucker, 2002), students struggle reasoning combinatorially (Batanero, Navarro-Pelayo, & Godino, 1997; Eizenberg & Zaslavsky, 2004; Hadar & Hadass, 1981; Lockwood, Swinyard, & Caughman, 2015b). Our hope is that through investigating how students reason combinatorially, we may discover ways to foster productive thinking in combinatorics. Studies that have been conducted in this spirit include multiple reinvention studies (Lockwood, Swinyard & Caughman, 2015a; Lockwood & Shaub, 2016) where students generated basic counting principles and formulas. One such productive way of thinking that emerged from research is a set-oriented perspective (Lockwood, 2014), where students consider the set of outcomes as integral to the solving of counting problems. Other studies have developed and tested instructional interventions (Lockwood, Swinyard & Caughman, 2015b; Mamona-Downs & Downs, 2004). Our study contributes to this literature base by implementing generalization as a means to develop deep understanding of basic counting phenomena. We offer analysis of a task through which students construct the general formulas for basic counting operations such as arrangements and combinations. By analyzing their combinatorial understandings as they generalize, we learn more about the nature of students' combinatorial thought in these basic settings.

## Theoretical Perspectives

### Generalization

For purposes of describing students' activity as they generalize, we adopt Ellis' (2007a) taxonomy of generalizing activity. Ellis describes three main categories of generalizing actions, those of *relating, searching* and *extending*. *Relating* occurs when "a student creates a relation or makes a connection between two (or more) situations, problems, ideas, or objects" (p.235). Through *relating*, students organize mathematical phenomena they experience based on commonalities. The commonalities may be nuanced, and can take on different forms such as symbolic, structural, activity and more. The relationships formed may then become the source material for further generalizations.

Students also may engage in *searching*. *Searching* occurs when students perform "the same repeated action in an attempt to determine if an element of similarity will emerge" (p. 238). In this activity, students are seeking out regularity in the mathematical operations they perform. While *searching*, students may have some potential regularity they seek to verify, but also they may have yet to discover the regularity they hope to emerge via repeated action.

Finally, students generalize through *extending*. When *extending*, a student "not only notices a pattern or relationship of similarity, but then expands that pattern or relationship into a more general structure" (p.241). This may indeed be the activity most closely associated with generalization. While *extending*, students draw upon known mathematics and then apply them in a more abstract setting. This activity accounts for the learning of more abstract mathematics than previously encountered. This taxonomy allows us to organize students' activity as they generalize.

## Combinatorial Reasoning

We also wish to describe the nuances of students' combinatorial understanding as they generalize. To do this, we utilize Lockwood's (2013) model for the different kinds of reasoning in combinatorics. Lockwood describes three unique and separate ways students reason about combinatorics problems. First, students may attend to the *formula* or *expression* of the combinatorial problem. Attention of this manner includes the symbolic form of the final answer, rather than the final numerical value. Students may also attend to the *counting processes* of a combinatorics problem. Students who reason via *counting processes* attend to the carrying out the process described by the problem. In doing so, they carry out the activity (either mentally or physically) to generate a solution to the problem. Finally, students might attend to the *sets of outcomes*. An outcome is a specific collection of the objects being counted. In considering this set, students attend to the particular structural organization of the outcome-set as a whole. This final way of reasoning is in line with Lockwood's (2014) *set-oriented perspective*, where the set of outcomes becomes a cornerstone of reasoning about any particular counting problem. Students reason in this way by viewing "atten[tion] to sets of outcomes as an intrinsic component of solving counting problems" (p.31). Further research has identified the productivity of attending to the sets of outcomes as well (Lockwood, 2013; Lockwood & Gibson, 2016).

## Methods

The data for this report draws from two larger studies in which we investigated the nature of student generalization in combinatorial settings. To do this, we conducted one paired teaching experiment consisting of fifteen hour-long sessions followed by a design experiment consisting of nine ninety-minute sessions with four students. The students from both studies were recruited from vector calculus courses, and were selected from an initial set of applicants based on a selection interview process. Each of the students in these studies had not taken a discrete or combinatorics course before so that their activity and generalizations were indeed spontaneous rather than implementations of extant schemes.

This study reports on the generalizing activity of the students during the third session of each experiment. The goal of this session was to facilitate reflection on the students' prior activity from the previous sessions, culminating in the construction of general statements of certain combinatorial structures such as the permutation and combination. To do this, we presented the students with various problems they had previously solved either in the first two session or in the selection interviews and prompted them to separate the problems into groups. They were not given any specific instructions on how to group the problems so that distinctions they found relevant would be revealed. The problems included those whose solution methods were arrangement of $n$ objects ($n!$), permutation of $k$ objects from $n$ objects $\left(\frac{n!}{(n-k)!}\right)$, selection of $k$ objects from $n$ objects $\left(\frac{n!}{(n-k)!k!}\right)$, and $k$ repeated selections from a set of size $n$ ($n^k$). Once the students agreed on the categories for the problems, they were asked to describe a general formulation of each category, and then to construct a general formula for the solution to each problem type. Both groups successfully categorized all problems into their respective four groups.

The sessions were video and audio recorded so that the records could later be reviewed for data analysis. The audio was also transcribed. Analysis consisted of reviewing the transcripts and the video files for episodes of generalizing activity. Relevant segments were further analyzed and coded according to Ellis' (2007a) framework and Lockwood's (2013) model.

## Results and Discussion

The categorization task allowed students to both reflect on combinatorial structures and to meaningfully generalize from prior activity. In terms of generalization, we saw students relate and extend commonalities in the combinatorial situations with which they had previously engaged. Further, we saw students demonstrate a fluid ability to reason with and communicate across various components of Lockwood's (2013) model. Through this categorization, the students were able to generate multiple abstract combinatorial situations that demonstrated inherently different structures. Moreover, the students showed understanding of nuanced differences between the combinatorial structures. These understandings resulted from various generalizations rooted in reflection on activity. In this results section, we will both discuss the students' generalizing actions and combinatorial reasoning. This allows for a discussion of generalizing actions motivated by underlying combinatorial understanding.

The students engaged in meaningful relating activity while categorizing the different combinatorics problems. Through the relationships created, they were then able to make general statements reflecting the combinatorial structures. For instance, when first categorizing the problem types, Carson described a collection of problems involving selection with repetition that he had just arranged. Note that in this quotation he is describing multiple problems in front of him:

> This [referring to a specific collection of problems] is independent events. So, [first problem he describes] there are eight questions but the outcome of one doesn't affect the others. [Second problem he describes] There are six characters, but the outcome of one doesn't affect the others.

Here he was relating that each question described a combinatorial structure in which there was no dependence between selections. Indeed, while his language was in terms of outcomes, he described the outcomes not affecting other outcomes in the process. From this we infer Carson held a *process*-oriented perspective. Similarly, Josh then identified two more selection with repetition problems still not categorized:

> *Instructor*: . . . and why did those two go with those [the original collection Carson grouped together]?
> *Josh*: Those two also deal with independent events and finding all the possibilities in those events depend on something raised to some power.
> *Instructor*: Okay, okay, good.
> *Josh*: Like the number of choices that you have raised to the number of choices that you make.

Note here the difference between Josh and Carson's language as they engaged in the generalizing activity of relating. While Josh was responding to Carson's attention to independent events, Josh chose to identify these problems as similar according to the formula for the answer. According to Lockwood's (2013) model, Josh is attending to *formulas/expressions* while Carson is attending to *counting processes*. This diversity in combinatorial language was common during these discussions. Indeed, students often collaboratively generated the categories while appealing to individually different combinatorial details. For instance, while categorizing the same type of problem, the students in the teaching experiment had the following exchange:

*Sanjeev*: And then you want to paint 6 different houses on your block and there are 3 acceptable paint colors you can pick —
*Rose*: Would that one come down here? Because that would be —
*Sanjeev*: You have 6 houses and —
*Rose*: 3 to the power of 6?
*Sanjeev*: you have 3 different paint colors for each, yeah. So this [problem] would be this one [referring to the collection of selection with repetition problems]?

Notice that Sanjeev and Rose were attending to different components of Lockwood's (2013) model during this exchange. Sanjeev adopted a *process*-oriented perspective by attending to the process of picking paint colors. Rose, in turn, attended to the *formula* of the answer as a means of relating the houses problem to other selection with repetition problems she experienced. This further demonstrates the students' abilities to communicate and generalize across varying combinatorial language. While it may not be surprising that students are able to communicate efficiently while demonstrating various combinatorial understandings, we can witness a variety of cognitive material as the source for generalizations. For instance, Rose and Josh both demonstrated attention to common representation (*formula/expression*) of the solutions to the problems. As a contrast, Sanjeev and Carson demonstrated *process*-based relating amongst combinatorial situations. These students continued to attend to such nuances throughout the categorization task.

While there was, as noted, variety in the students' generalizations and combinatorial understandings throughout the task, we found a surprising uniformity of language pertaining to combinations. Indeed, all students demonstrated attention to the structure of the *sets of outcomes* when discussing combinations. The discussions about differentiating combinations from other combinatorial processes revolved around taking care not to count two similar outcomes as different. For instance, when separating the permutations and the combinations, Rose and Sanjeev said the following:

*Rose*: It's [referring to the collection of permutation problems they categorized]— it's how many — it's basically how many ways to put certain amount of items into fewer spots where 1, 2, 3 and 3, 2, 1 are different. And this [the collection of categorized combination problems] is how many ways you put a certain amount of things into fewer spots where 1, 2, 3 and 3, 2, 1 are the same.
*Sanjeev*: On these ones [referring also to the collection of permutation problems] you've got combinations [not referring to the combinatorial sense of the word. Literal combinations of outcomes]. So 1, 2, 3 - 3, 2, 1 would be different combinations. With this one [a combination problem], for example, if you have identical lollipops you can label them 1, 2, 3 or you can just label them 1, 1, 1. So 1, 2, 3 and 3, 2, 1 would be the exact same thing, because 1, 2 and 3 are all the same.

We note that while they mentioned permutations during this exchange as well as combinations, their previous discussion of permutations involved only discussing either the *formula* or the *process* involved in their construction. The distinction of making $\{1, 2, 3\}$ and $\{3, 2, 1\}$ the same was brought up as a means of separating the combination from permutation. Indeed, while there was a variety of combinatorial language used during categorization, students would always use set-based language when discussing combinations. We find attention to outcomes in this way as productive, as it allows for careful consideration of what is being counted. Further, it is interesting that combinations were the only problems in which there was uniformity in the combinatorial language that the students used.

As another example, we saw similar discussions of combinations emerge from the design experiment. Initially, when describing the difference between combinations and permutations, Ann-Marie remarked:

> Yeah, so in those two problems [a pair of combination problems] you divide by two factorials to cancel out the duplicate answers whereas in the other ones you don't have to do that.

Notice that her response also included *formula*-driven language. Indeed, Ann-Marie confessed that she primarily thought of the formula representation when thinking of the problem types. Ann-Marie made the distinction of "two factorials" in this case to contrast division by "one factorial" in the permutation group. What we see here is that within her formula driven remarks, she also used outcome-based language to describe the need for the extra division by a factorial. Also, later when explaining why the formula for $\binom{n}{k}$ adds on a division by $k!$, Aaron explained:

> Well, because you're trying to get rid of all the combinations that you're not looking for that you can make out of those three slots because they're all the same. So, that just accounts for it.

Indeed, most descriptions of combinations involved *outcome* - based language so that they could be differentiated from permutations. Often, the design experiment students described "dividing by redundancies" when performing combinations. It is interesting that among the students we worked with, combinations were uniformly a source of *outcome* - based language. Returning to the teaching experiment, we see Rose also using *outcome* - based language when describing why a subset selection problem is grouped with other combinations. After negotiating the particulars of the problem involving finding subsets of a set of numbers, Rose said the following:

*Rose*: and if that was the case then we'd want to put it over into this group [the collection of combination problems].

*Int*: Okay. And how come?

*Rose*: Because now you don't want — you just want unique combinations. And if you're getting rid of all the — the repeated subsets, then you're just finding the unique combinations.

Here, we see Rose clarified that the desired outcomes were indeed "unique combinations". The uniqueness was generated by getting rid of repeated subsets, which indeed would emerge from a standard permutation. Thus, we see that Rose diverged from her typical *formula*-centered language to attend to unique outcomes. Finally, when also discussing the subset selection problem, the design experiment students had the following exchange:

*Carson*: . . . and these [their initial collection of permutation problems] you're arranging a given number of things in smaller number of spaces than there are things.

*Josh*: Is this one [the subsets problem] really the same as the others though because you're only looking for the four number set?

*Carson*: So, every number in the four limit subset is unique, right? So, there's no repeated

numbers.
*Josh*: There can be repeated numbers.
*Ann-Marie*: But like zero, one, two would be the same as one, zero, two.
*Carson*: Right and you can't have zero, zero, zero.
*Josh*: Oh, yeah.
*Carson*: So, that would be an arrangement one as well just with the caveat that there are only four of them.

Here we see two types of outcome - differentiation occurring. We see the students describing that ordering the numbers in the subset should not create a different outcome. This is consistent with the set-oriented perspective the students took on combinations. We also see Carson noting that an outcome cannot have multiples of a number in the subset. While this is not unique to combinations, it is another example of *set-oriented* language. The above discussion naturally centered around whether or not certain outcomes would be considered as distinct. Indeed, the students in both groups consistently attended to the set of outcomes while discussing combinations.

While much of the above discussions centered around the activity of relating, we also saw students engage in extending. The students were prompted to generate statements and formulas that reflected the categories they had created. The following statement the teaching experiment students wrote for the collection of combinations further reflects the *outcome*-centered underpinnings of their generalizations. Rose and Sanjeev produced the following characterization of combination problems:

3) $\uparrow_2$ *... and divide by the factorial of the given spots to delete repeated sequences because any arrangement of the same given elements is considered the same combination.*

Note that the arrow marked with a 2 at the beginning is referring to their previous statement of a permutation process. This characterization suggests that the structure of a combination involved constructing a permutation followed by the further operation of division as described above. We bring attention to their *outcome*-oriented justification for their addition to the permutation. This further demonstrates that their understanding of a general combination process involves accounting for multiple arrangements of a particular outcome. We again note that such distinctions are productive, and allow for deeper understanding of the combinatorial objects.

## Conclusions

We see that the categorization task allowed the students to generalize their prior work on individual counting problems into more general contexts in which different combinatorial structures could be illuminated. The students productively engaged in relating and extending, both activities underpinned by the nuances of the combinatorial settings, as described by Lockwood's (2013) model. We saw meaningful generalizations being underpinned by all three aspects of Lockwood's model. Moreover, there was a uniformly *set-oriented* approach to generalizing combinations. Further, such distinctions between permutations and combinations demonstrated productive understandings of the combinatorial objects. Such a perspective allows for specific criteria with which students can evaluate whether a combination or permutation applies to a counting situation. Thus, we see that through engagement in categorizing and reflecting on prior work, students meaningfully generalized while gaining a better understanding of the combinatorial objects.

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# Generalizations of Convergence from $\mathbb{R}$ to $\mathbb{R}^2$

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*Sequential convergence is a powerful tool in the field of real analysis. Though its structure persists throughout various metric spaces, students initially understand sequential convergence as it manifests on the real line. Students often do not encounter more generalized forms until advanced analysis courses. As part of multiple teaching experiments, students were given the opportunity to generalize sequential convergence from $\mathbb{R}$ into the $\mathbb{R}^2$. This report will demonstrate various generalizations rooted in reflective abstraction of convergence in $\mathbb{R}$. We will also discuss students generalizing by reduction, reflecting on the utility of distance as a map between spaces.*

Key words: generalization, real analysis, convergence, limits, advanced calculus, vectors

## Introduction

Convergence is a phenomenon encountered at all levels of mathematical practice. A utility of sequential convergence is its persistent structure throughout metric spaces. Students studying introductory real analysis encounter the convergence of real number sequences and also of continuous functions. These contexts for convergence may be leveraged to facilitate understanding of convergence in more abstract spaces through generalization. These spaces create unique opportunities for students to generalize their understandings in productive ways.

During the selection interviews for multiple teaching experiments, students were given an opportunity to generalize convergence of real numbers to the convergence of real vectors in two dimensions. Their work revealed multiple instances of generalization rooted in abstraction of real number convergence. In this report I seek to answer the following research question: How do undergraduate students leverage convergence of real numbers when defining convergence in more abstract spaces?

## Literature Review

### Student Understanding of Convergence

While student understanding of limits and convergence has been thoroughly investigated, there is still much to learn about how students understand convergence beyond introductory contexts. Early studies investigated student initial understanding of limits, problems that may result from students' initial understandings, and intuitions behind the limit concept (Bezuidenhout, 2001; Cornu, 1991; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996, Davis & Vinner, 1986; Oehrtman, 2003/2009; Roh, 2008/2009; Tall, 1992; Tall & Vinner, 1981; Williams, 1991). Many of these studies focused primarily on student understanding of informal limiting processes, leaving room for investigations of formal limiting processes.

In 2011, Swinyard began investigating students' formal understanding of limits via a teaching experiment in which two students reinvented the formal definition of a limit. Useful constructs for describing student formal understanding of limits emerged from this experiment. Studies that followed expanded on his work, proposing strategies for fostering useful understanding of limits and also reinventing the formal definition for sequential convergence (Swinyard & Larsen, 2012; Oehrtman, Swinyard, & Martin, 2014).

Other studies have also examined student understanding of formal limiting processes. Adiredja investigated how students make sense of the relationship between the multiple limit-controlling variables (Adiredja, 2013/2015; Adiredja & James, 2013/2014). Also, Roh and colleagues (Dawkins & Roh, 2016; Roh, 2009; Roh & Lee, 2016) implemented interventions such as the "Mayan Activity" and the "$\epsilon$ -strip activity" designed to illuminate the logical structure of formal convergence. Finally, Reed (2017) examined a student's understanding of the logical structure of point-wise convergence for functions. These studies examine the nuances that accompany the logical statement of convergence.

### Generalization

While the activity of generalizing has been investigated in many contexts, such studies have not yet examined students generalizing formal mathematics. Generalization has been deemed a relevant mathematical activity both by researchers (Amit & Klass, 2005; Lannin, 2005; Pierce, 1902; Vygotsky, 1986) and educators (Council of Chief State School Officers, 2010). Indeed, generalization has been thoroughly investigated in algebraic and other elementary contexts (Amit & Neria, 2008; Becker & Rivera, 2006; Carpenter, Franke, & Levi, 2003; Ellis, 2007a/2007b/2011; Radford, 2006/2008; Rivera, 2010; Rivera & Becker, 2007/ 2008).

More recent investigations have begun to explore student generalizations at the undergraduate level. Researchers have studied student generalizations in both single and multi-variable calculus (Dorko, 2016; Dorko and Lockwood, 2016; Dorko & Weber, 2014; Fisher, 2008; Kabael, 2011; Jones and Dorko, 2015) as well as combinatorics (Lockwood and Reed, 2016). For instance, Jones and Dorko (2015) considered different ways in which the multivariable integral is understood as a generalization of notions that students held for single variable integrals, such as generalizing from an "area under the curve" model in single variable calculus. While these studies investigate the nature of generalizing activity in various advanced contexts, the research so far has not investigated generalization of formal mathematics.

This report contributes both to the literature on convergence and to the literature on generalization by observing students generalize the concept of convergence in a formal context.

## Theoretical Perspectives

### Generalization

We wish to characterize the activities students engage in while generalizing. To do this, we consider student activity according to Ellis' (2007a) taxonomy of students' generalizing activity. This examines generalization from an actor-oriented perspective (Lobato, 2003).

Ellis described three broad categories of generalizing activity in which students engage: *relating, searching* and *extending*. In *relating*, "a student creates a relation or makes a connection between two (or more) situations, problems, ideas, or objects" (p. 235). Generalizing activity can manifest as an organizing of similar situations which then become the source material for further generalizations.

The next activity students engage in is to search for a pattern or relationship. This is where students perform "the same repeated action in an attempt to determine if an element of similarity will emerge" (p.238). The distinction here is that the student repeats an activity to uncover some regularity.

Finally, students engage in *extending*. This occurs when a student "not only notices a pattern or relationship of similarity, but then expands that pattern or relationship into a more general structure" (p.241). This extension can be done in multiple ways that expand the source material to new abstraction. *Extending* moves beyond the observance of relationships or patterns, and involves the creation of new mathematical objects that reflect the source of the generalization in some way. These three categories provide us language with which to observe and discuss the generalizing activity of students in any mathematical setting.

**Abstraction**

We find Piaget's notion of *abstraction* (Piaget, 2001; Glasersfeld, 1995) to be complementary to studying student generalization. Indeed, through *abstraction* Piaget describes the cognitive mechanisms through which activity is reorganized and extended. Specifically, we are concerned with facilitating *reflective abstraction* (Glasersfeld, 1995). In *reflective abstraction*, an operation (mental activity) "developed on one level is abstracted from that level of operating and applied to a higher one" (Glasersfeld, 1995, p. 104). Indeed, *reflective abstraction* accompanies generalization as it can describe mathematical activity being organized at higher levels of thought. *Reflective abstraction* is characterized in two parts. The first is a réflexion, or reflection, of the operations from their original context (p. 104). This indeed highlights the importance of salient activity from which to abstract. The second part of reflective abstraction is a réfléchissement, or projection, of the borrowed operation to a higher level of thought (p. 104).

Thus we see *reflective abstraction* involving the borrowing of activity to then be applied at higher levels of thought. In mathematical contexts, this indeed can be used to characterize the generalization of operations. Using *reflective abstraction* as an underpinning for generalizing activity allows us to use mathematical activity as a direct source of generalization. This indeed will be useful in describing the generalizations of students as they engage in extending their mathematical understandings.

**Mathematical Discussion**

Convergence is a generalizable concept that obeys the same structure in various real spaces. Convergence of real number sequences and other seemingly more complex objects, such as uniformly convergent function sequences, are indeed the same because of the metric structure associated with each space. Consider the formal definition for convergence of real numbers: A sequence $\{x_n\}$ of real numbers converges to a real number $x$ if $\forall\epsilon>0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, we have $|x_n - x| < \epsilon$. The mathematical structure of such convergence stems from the definition of convergence within any general metric space: A sequence $\{x_n\}$ in a metric space $(M, \delta)$ with a metric $\delta$ converges to $x$ if $\forall\epsilon>0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, we have $\delta(x_n, x) < \epsilon$. Such is the case for real vectors in $\mathbb{R}^2$. Indeed, the only alteration to make in each metric space is the notion of distance. On the real line, distance is measured using the absolute value norm. While many equivalent metrics may be applied in the plane, perhaps the most natural is the metric given by the Euclidean distance. Indeed, when the distance between vectors is measured using the Euclidean distance, the convergence of a sequence of real vectors may be characterized as follows: A sequence of vectors $\{\vec{x_n}\}$ in $\mathbb{R}^2$ converges to a vector $\vec{x} \in \mathbb{R}^2$ if $\forall\epsilon>0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, we have $\sqrt{(x_n^1 - x^1)^2 + (x_n^2 - x^2)^2} < \epsilon$. Note that in this notation $x^k$ represents the $k$-th component of the vector $\vec{x}$.

Similarly, the characterization of Cauchy sequences is uniform throughout metric spaces: A sequence $\{x_n\}$ in the metric space $(M, \delta)$ with metric $\delta$ is Cauchy if $\forall\epsilon>0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, we have $\delta(x_n, x_m) < \epsilon$. This allows for Cauchy sequences to be characterized on the real line by: A sequence $\{x_n\}$ of real numbers is Cauchy if $\forall\epsilon>0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, we have $\delta(x_n, x_m)<\epsilon$, and on the real plane by: A sequence of vectors $\{\vec{x_n}\}$ in $\mathbb{R}^2$ is Cauchy if $\forall\epsilon>0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, we have $\sqrt{(x_n^1 - x_m^1)^2 + (x_n^2 - x_m^2)^2}<\epsilon$.

## Methods

This study reports on the selection interviews and the first sessions of multiple teaching experiments (Steffe and Thompson, 2000) being conducted for my dissertation. Two of the teaching experiments conducted were for pilot purposes, and consisted of six hour-long sessions each with one student. The other two teaching experiments consisted of ten 90-minute sessions, one with a pair of students and one with an individual student. The students reported on were each mathematics majors at a large university. Each were recruited out of the advanced calculus sequence, and had finished at least one course in the sequence before the interviews were conducted.

The sessions were semi-structured and task-based (Hunting, 1997) so that student activity could be observed and understanding could be inferred. The goal of these episodes was to facilitate reflection on real-number convergence and the characterization of the absolute value as a measurement of distance. Thus, the interviews began with discussions of real number convergence and distance measurement on $\mathbb{R}$ and $\mathbb{R}^2$. The students were prompted to write down and explain their definitions of real number convergence, and then to demonstrate that a specific sequence converges. Once details were discussed, and distance measurement was thoroughly discussed, the discussion turned to convergence of vectors. The students were prompted to characterize convergence of a sequence of vectors, and similarly demonstrate that a specific sequence of vectors converges. Any further discussion negotiated nuances of their multiple characterizations of convergence.

The interviews were video recorded, and the records have been reviewed multiple times looking for episodes of student generalization to be analyzed using Ellis' (2007a) framework and Piaget's (2001) construct of reflective abstraction.

## Results

### Convergence on $\mathbb{R}$

Each interview began with a review of the known distance and convergence concepts on the real line. I will describe an understanding commonly held by the students that was relevant for their generalizations of convergence.

When prompted to characterize convergence of a sequence of real numbers, each student gave the standard $\epsilon$-N definition described above in the mathematics section. The discussion that followed allowed the students to further explain their understanding of the characterization. For instance, Kyle said the following while describing distance measurement via the absolute value function while characterizing a Cauchy-convergent sequence:

> Well we end up measuring the distance between two subsequent points. That's what we're doing here-we're saying that the absolute value of $a_n - a_m$ is the distance between these two points in the sequence. So you're trying to measure, as your $n$ is getting arbitrarily large, what's happening between this point in the sequence [$a_n$] and this point in the sequence [$a_m$].

The absolute value as a distance measurement on the real line similarly emerged in all of the interviews. For instance, using $1/n$ as an example, Jake said the following while explaining why we take the absolute value to characterize Cauchy-convergent sequences:

> If we had this instead of $1/n$ be $-1/n$ ... we're just concerned with the width between the numbers not where they are relative to the $x$-axis ... 'cause we're always concerned with the relative distance of the two. Not if it's, you know, below of above the $x$-axis. The absolute value takes care of that.

Similar discussions of real number convergence were had with all students in the study. Each student displayed a sophisticated understanding of sequential convergence on the real line. These discussions then influenced the students' generalizations of convergence to $\mathbb{R}^2$

**Convergence on $\mathbb{R}^2$**

Following discussions of convergence on the real line and characterizations of distance in $\mathbb{R}$ and $\mathbb{R}^2$, the students were prompted to develop a characterization of vector convergence. Two distinct generalizations emerged from the students' characterizations of vector convergence. Both generalizations result from reflections on real number convergence.

The first generalization involves considering vectors on the real plane component-wise and isolating real number sequences in each component. This generalization manifested differently among the students, each instance demonstrating unique understandings. Laura and Kyle considered vectors in terms of their components separately, and described a sequence of vectors as a pair of real-number sequences. This allowed for convergence of the sequences to rely on convergence of the components. Below, Laura described the sequence of vectors converging in the following manner:

> If this [a sequence of vectors $\{\vec{v_n}\}$] converges, then that means your x-component has to converge and your y-component has to converge. Which is realistically seeing if - two independent sequences converges - you have some sequence $\{x_n\}$ and some sequence $\{y_n\}$ that make up your vector, then it's basically like doing the convergence thing twice but you have to fit it for both $x$ and $y$.

Here Laura extends convergence of real numbers to a vector setting by making note that a sequence of vectors forms two sequences of real numbers, and asserts that controlling the convergence of each component will result in a convergent sequence of vectors. Note that here, Laura is using the known structure for convergence of real numbers.

Jerry, Jake and Christina produced similar component-wise characterizations, however their approaches differed subtly from Kyle and Laura. Specifically, Jake and Christina attempted to bound the sequence of difference vectors by an "error vector". When generalizing convergence to two dimensions, Christina initially wrote out again the definition of real number convergence, and used the same notation as real-number convergence, while noting the caveat that in this case $|A_n - A|$ denoted a distance between vectors. Moreover, she described making the distance vector "smaller" than some error vector $(a, b)$:

> So, this [$A_n^1 - A^1$] is describing the horizontal distance that will be traveled, and then this [$A_n^2 - A^2$] is describing the vertical distance that will be traveled. And the whole entire thing describes a vector that would create that translation, and it's going to be - I guess less than would actually be smaller than - the $\epsilon$ vector.

Christina went on to describe that in this context “less than” does not necessarily mean an ordering, but instead referred to the sizes of the vectors. From this, she reduced convergence to comparing the components of the difference vector $\{A_n - A\}$ and the ”error vector”. She then reduced the vector comparison to a component-wise comparison and arrived at a similar characterization as above. Similarly, Jake wrote a characterization for Cauchy-convergence of vectors that was identical to Christina’s up to notation. Motivated by the behavior of a sequence when the y-axis is constant, Jake said the following:

> And you do the same with the $x$-axis. You get an analogous statement with the $x$-components [where keeping the x-components constant yields a real number sequence in the y-components]. But we have the difference between the $x$-components and the $y$-components both decrease below some arbitrary $\epsilon$. Otherwise you could have convergence with respect to the $y$-axis but having it oscillate back and forth in the $x$, or increase without bound on the $x$ or vice versa.

So for Jake, while the problem of convergence was to capture varying vectors, the variation could be simultaneously handled in both components.

Thus we see these students were able to reduce the problem of vector convergence into a form that they are familiar with, namely convergence on the real line. In this case, they reflected that they could iterate real number convergence a finite amount of times (in this case twice), and that convergence of each component implied convergence of the vector as a whole. Mathematically, these characterizations are interesting because multiple geometries can result from considering the component-wise absolute value distances, including the $\ell_1$ and $\ell_\infty$ distances.

The final generalization we will discuss differs from the generalizations above in that it involves reflection on the role of measuring distance in convergence, rather than taking advantage of the repeatability of real number convergence. It involves reflection on structures that are more consistent with a general metric. After being challenged to find a characterization of convergence that involves a single calculation rather than multiple calculations, Jerry and Christina together used the Euclidean distance formula to create a sequence of the distances between vectors in the sequence and the convergent vector. These distances formed a sequence of real numbers that would converge to 0. After formalizing the convergence of the sequence of distances, Jerry made the following statement:

> I like this ‘cause it seems like we reduced the problem to something that was like, that we already know, so I feel like this is on the right track So we now have this number that we can check for every single vector in our sequence and that generates a sequence of real numbers which we already know the rules of convergence for. And that’s something we can check.

Jake constructed a similar generalization of a Cauchy-convergent sequence of vectors. These generalizations allow for direct comparison between vectors involved in the convergence process via a distance calculation. As the students indicated, this calculation allows for the convergence to be stated in terms of a varying set of real numbers, namely the sequence of distances between the elements of the set of vectors converging. This transforms the problem into one of a known operation, namely the convergence of real numbers. This, in fact, is the logical structure of convergence in abstract metric spaces. In contrast, the students’ first generalization is indeed also generalizable, but only within finite dimensional vector spaces, as the requirement of checking convergence in each component is only possible a finite number of times.

## Discussion and Conclusion

### Reflective Abstraction

Within these episodes we see multiple instances of reflective abstraction when generalizing sequential convergence from the real line to $\mathbb{R}^2$. While both generalizations involve reflection on the structure of convergence along the real line, the réfléchissement of such reflection is manifest in two qualitatively different ways. I infer that the component-wise construction involves abstractly projecting the action of taking a limit on the real line. In observing that vectors can be expressed via components of real numbers, and formulating the sequences of vectors to reflect real number sequences, the students project convergence of real number sequences to two simultaneous iterations of real number sequences that converge in conjunction with one another. This involves reflection on the process of real number convergence, and then projection of this to each component in the vector structure.

### Generalizing through reduction

While the second generalization also may be characterized via reflective abstraction, it also reveals a new form of generalizing by extending. Within this episode we also see instances of students generalizing a more abstract phenomenon by reducing aspects of the problem to a known setting. Jerry and Christina, as well as Jake, utilized the Euclidean distance formula as a map from $\mathbb{R}^2$ to $\mathbb{R}$ that reduced the problem from one that involved 2-dimensional geometry back to the convergence of real numbers. Thus, the students manipulated the structure of the problem at hand to match a familiar structure. The use of a mapping in this way is entirely consistent with productive activity in multiple areas of mathematics. As an example, the integral can be similarly used to reduce problems of functional variation to problems of varying real numbers via special limiting processes. These are interesting instances of generalization, as they involve coordinating of an abstract process via simplification. This is indeed reminiscent of Jerry's final comment. Moreover, while Jerry and Christina were challenged to perform a single calculation, their characterization of the problem in terms of a sequence of real-number distances involved constructing an explicit relationship between the vectors that varied in the sequence and the known structure of $\mathbb{R}$.

### Conclusions

In this report we see two distinct generalizations rooted in sophisticated understandings of real number convergence. By reflecting on convergence as an activity, the students generated two generalizations unique from each other mathematically and cognitively. The generalization characterizing vector sequences as pairs of real numbers reflectively abstracts the repeatable operation of checking real number convergence. Further, the generalization utilizing the Euclidean distance involves reducing the more abstract mathematical phenomena of vector convergence to the simpler and more familiar setting for convergence, that of the real numbers. These findings begin to illuminate the nature of student thought an generalization in more formal mathematical settings. Specifically, we see students attending to natural relationships in real space to facilitate meaningful generalizations of known analytical phenomena. Further research will investigate student generalization in more abstract spaces.

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Examining a Mathematician's Goals and Beliefs about Course Handouts

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*In this qualitative narrative study, we employed Schoenfeld's theory of Resources, Orientations and Goals (ROGs) to analyze a mathematician's beliefs and goals in creating handouts for his students. Some of the instructor's primary goals in creating the handouts were: (1) to help students gain an intuition about Algebraic Topology, (2) to provide a resource for students to revisit the difficult material outside of class, and (3) to prompt students to complete exercises so that they could monitor their own mastery of the course content. As part of this study, one of the students in class took daily journals. These journal entries revealed that he appreciated the time and careful preparation that was necessary to create the handouts, particularly the pictures that the instructor drew in the margins to help students gain an intuition. However, one obstacle that the student faced was struggling to appreciate the instructor's goal of expecting students to monitor their mastery of content outside of class time through completion of ungraded exercises in the handouts.*

*Keywords*: Algebraic Topology, handouts, beliefs, goals

## Theoretical background

Giving out handouts is a common practice in many mathematics classrooms. In his book, *Mathematics Teaching Practice: A Guide for University and College Lecturers*, Mason (2002) shares a variety of reasons why mathematics instructors give handouts to students (see Table 1). In his view, "people have a mixture of aims, and so use different approaches at different times" (p. 64). Mason differentiated among handing out complete notes, writing everything on the board, and giving no notes during the lectures. In the case of providing complete notes, students may not see the need to attend lectures; in the case of writing everything on the board, students will turn into transcribers; in the case of providing no notes, students have nothing to fall back on to make sure they are gleaning the important points from the lecture. Teaching is a complex activity and clearly designing handouts and successfully implementing them in lectures requires careful thought.

*Table 1. Pedagogical insights in using handouts in lectures (Mason, 2002, p. 64).*

| Aims | Expected Actions by Students | Styles of Handouts |
|---|---|---|
| Cover (explain, teach, transmit, or convey) the definitions, theorems, proofs, and techniques | Study (not just read) notes mathematically; work mathematically on set exercises | Complete notes as if in a book (available in advance or after the lecture)<br>Definitions, theorems, and sample worked examples |
| Inspire | Appreciate overall flavour; pick up details from carefully working on notes and perhaps texts, not just working through exercises | Notes with headings but details left as spaces for students to fill in as you work through the exposition |

| | | |
|---|---|---|
| | | Explicit references to standard texts<br>Extra suggestions not mentioned in the lecture |
| Work at understanding, making connections between topics or theorems | Re-construct topics for themselves from lecture notes and text, and, increasingly, independently from text alone | A succinct mathematical summary, perhaps with worked examples or challenging questions to explore |
| Teach how to carry out required techniques and solve sample problems | Work on ‘worked’ examples; justifications and theory found in text | Elaborated worked examples displaying choices, wrinkles, and use of theorems |

The current qualitative narrative study is a part of a larger study with the main goal of understanding the mind of a working mathematician as he made pedagogical decisions (Stewart, Thompson, & Brady, 2017). In this paper, we describe what motivated a Geometer to design and employ 35 handouts in an Algebraic Topology course. In our holistic approach to investigate this instructor’s teaching, we examined his handouts, his weekly teaching journals, and discussions that occurred during weekly meetings with a team of researchers. Additionally, we examined one student’s daily journals to get the student’s perspective on the instructor’s handouts. We do not aim to prove or disprove whether handouts are ideal educational resources, rather, our ultimate goal is to understand the mind of the working mathematician by investigating what motivated him to create detailed handouts. To analyze the Geometer’s motivations, we employed Schoenfeld’s (2010) Resources, Orientations, and Goals (ROGs) theoretical framework. This theory helped us identify the knowledge and materials at the instructor’s disposal, his values and beliefs, and what he wanted to achieve with the handouts. Schoenfeld claims that “if you know enough about a teacher’s knowledge, goals, and beliefs, you can explain every decision that he or she makes, in the midst of teaching” (2015, p. 229). Resources, or knowledge, include “the information that he or she has potentially available to bring to bear in order to solve problems, achieve goals, or perform other such tasks” (Schoenfeld, 2010, p. 25). Orientations are “dispositions, beliefs, values, tastes, and preferences” (Schoenfeld, 2010, p. 29). Goals are what the individual wants to achieve. Although, the theory was originally applied to middle and high school teaching, (Aguirre & Speer, 2000; Thomas & Yoon, 2011; Törner, Rolke, Rösken, & Sririman, 2010), it has more recently been applied to university teaching (e.g. Hannah, Stewart, & Thomas, 2011; Paterson, Thomas, & Taylor, 2011).

Our current research questions are: What were the instructor’s ROGs in making the handouts? Was the student aware of the instructor’s goals for creating the handouts, and what were his reactions toward the handouts?

## Method

In this qualitative narrative study (Creswell, 2013), our research team analyzed a Geometer’s thought processes and pedagogical decisions while he taught a course in Algebraic Topology. The research team consisted of four members: a mathematics education researcher; a Geometer (the course instructor); a cognitive psychologist; and a mathematics postdoc. The Algebraic Topology course was the first in a two-semester sequence of courses; eight students were enrolled. During class meetings, the instructor (Noel Brady) passed out handouts to help students follow along with the topic of the day. Students actively solved problems together in groups, or individual students were called to the board to complete problems.

One source of data we analyzed was a series of teaching journals that contained the instructor's reflections on his preparations for class, what happened during class, as well as some descriptions of the events that took place during office hours. The research team read his daily journal entries and discussed them during weekly research meetings. During these meetings, the research team asked the instructor further clarification questions, and he often drew additional pictures as he described the course content. These meetings were audio recorded, and the meeting transcripts were also used as a source of data. Our team also analyzed data from a student in the instructor's class, who wrote daily journals. These student journals provided an additional perspective into the events that took place in class. The final source of data was 35 handouts that the instructor created for his students.

The data were analyzed thematically, meaning we mainly considered the key issues that emerged in this study. One of the main themes that emerged from this instructor's journals was "teaching". Forty-six percent of all instances from his journals were coded with the "teaching" code and 20% of those codes fell into the sub-category of "handouts/notes." More details about data coding and analysis is described in Stewart, Thompson and Brady (2017).

## Results and Discussion

The instructor's resources included: (1) his knowledge of mathematics and the subject area, (2) many years of teaching experience, (3) course notes from when he was a student, (4) the textbook (Hatcher, 2001), and (5) many hand-drawn images. Analysis of the instructor's 35 handouts illuminated his motives. These handouts gave the research team a more authentic glimpse into the mind of the mathematician than his teaching journals. The instructor noted that he was self-aware when he wrote the journals, as he knew the research team would subsequently analyze and discuss them. On the other hand, he created the handouts solely for his students.

Apart from the usual dose of definitions, theorems, and proofs, the instructor's mostly handwritten handouts included headings such as, "intuition," "motivation," and "application," which are often lacking in textbooks. Table 2 summarizes the essence of the instructor's goals and beliefs as indicated in his teaching journal entries.

*Table 2. The instructor's pedagogical goals and beliefs about handouts.*

- Presented the information in a conversational tone
- Accompanied class activities
- Inspired by notes from his own graduate courses
- Contained relevant examples and exercises that referenced his published research
- The chosen textbook (Hatcher, 2001) gave a "muddled discussion" when the topic was "highly non-trivial and non-obvious" for the students
- Referenced alternative discussions of difficult topics that other mathematicians had posted on their websites
- Inspired by assigned homework
- The instructor wanted to present his students with "ultra-detailed" arguments that had taken him at least an hour to develop.
    - The handouts helped the instructor feel organized and not scattered.

|  |
| --- |
| - He was able to add onto existing handouts from semester-to-semester.<br>- If he wanted to be particularly precise, he used LaTeX to type up the handout, but those handouts were time-intensive to create.<br>• Some handouts synthesized information for the students. The instructor synthesized this material on his own when he was a student.<br>• When the instructor was unable to cover all of the material that he wanted to in his lectures, he created handouts on the topic.<br>- Handouts gave the instructor the license to go through the material more quickly because the students could revisit the information at their own pace outside of class time.<br>• Helped students build intuition<br>- The instructor drew images by hand that represented the complex mathematics.<br>- The instructor noted that some students may be learning about the topics for the first time.<br>• Handouts were helpful when the material was particularly complex.<br>• The proof of the E-S axioms (for singular homology) was a component of the course. The textbook withholds the axioms until the end of coverage for singular homology, but the instructor decided to present them first. He believed that the axioms could serve as a framework, or table of contents, for the topic.<br>• The instructor created a summary handout to give students a preview of what was to come in the second course on Algebraic Topology. He expected that the students would attempt to solve some of the problems when they were on break between the Fall and Spring semesters. |

In this section, we will discuss some of the instructor's goals and beliefs in more detail and match them to the student's comments.

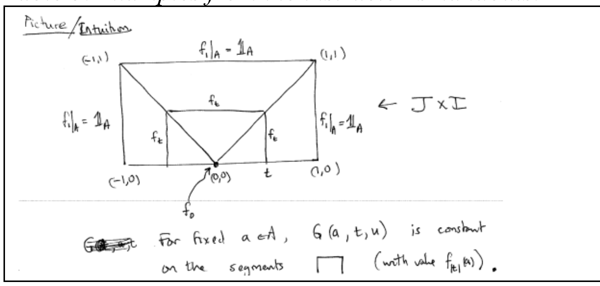
**Helping Students to Build Intuition**

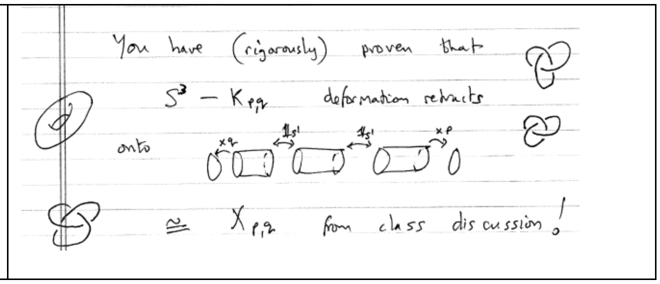
One of the instructor's main pedagogical goals was to help students build intuition. The instructor mentioned this goal often during the research meetings, which indicated his strong belief in and the importance he placed on helping his students build intuition. He drew images by hand (see Table 3) that represented the complex mathematics to help students who were learning about the topics for the first time. Research team members noticed that the instructor often used phrases such as "carefully and slowly" and "careful proof" in his teaching journals. These phrases indicated that he wanted to make sure students followed the arguments as they unfolded.

| The instructor's comments | The student's comments |
| --- | --- |
| "Did some examples carefully and slowly, but told them that they have to get used to computing boundaries quickly and efficiently."<br><br>"They still had difficulty going from an intuition to a formal proof, and I gave some | "I've so far enjoyed reading Dr. Brady's notes in the handout, which are rife with helpful commentary and ultimately very user-friendly. I've noticed how careful he is with building some of these mathematical concepts from the ground up." |

| | |
|---|---|
| handouts that sort of went through stuff fairly carefully." | "His pictures in the margins are abundant and very helpful, and there's always a nice subject line or topic sentence for each section along the lines of "here's what our ultimate goal is for the next few pages and here's how we're going to do it." |

*Table 3. Examples from the instructor's handouts.*




## Giving Students a Resource to Revisit

The instructor's orientation was to place a value on providing students with detailed notes that they could revisit outside of class time. The instructor viewed the textbook as wonderful in many ways, but was "a bit fast and loose" with the coverage of some topics, so he decided to create handouts to supplement the textbook.

| The instructor's comments | The student's comments |
|---|---|
| "Even with the handout here, I have to say this is cool, but me telling you it is cool, or me going in there and writing very quickly on the board and showing this is not going to work. You need to go and figure out why it is cool yourself."<br><br>"There are topics here that I think they should read, and they should read it carefully enough—meaning maybe a couple of paragraphs they need to spend several hours on teasing them out and then present it to their peers for 50 minutes."<br><br>"We need to get our hands dirty to do this." | "Overall, I'm happy with the handout, as it works through several examples entirely, with plenty of marginal remarks by Dr. Brady as always. I always know that, given a handout, Dr. Brady has license to cruise through the material in that lecture even quicker than usual, but that I have it right there with me to review by myself later." |

## The Instructor Believed the Students Should Master the Material on their Own Time

The instructor noted that some of the handouts helped students draw connections across course content, and these were connections that took him quite some time to realize when he took the course himself. Therefore, the instructor's goal was to provide these connections for his students to facilitate their "a-ha" moments. This goal is aligned with Mason's (2002, p. 64) statement that students should "re-construct topics for themselves from lecture notes and text, and, increasingly, independently from text alone". Hence, the handouts were no substitute for the students putting in the time and effort outside of class to master the content. In the comment below, we noticed that the student did not realize the instructor's motivation for asking his students to complete ungraded assignments in the handouts.

| The instructor's comments | The student's comments |
|---|---|
| "It had exercises for them to verify things. I am not grading it. It is up to them to make sure they understand it, and they come up and chat with me if they want." | "I can't remember exactly when our last homework assignment was, but it must have been weeks ago. Since then the course has consisted only of lectures and unofficial "I suggest you do these" exercises assigned by Dr. Brady. I have to admit that I haven't attempted all of them, partly because there are so many and partly because the lack of incentive (that they won't be graded)." |

## Concluding Remarks

This qualitative narrative study investigated a mathematician's ROGs through his handouts, teaching journals, and conversations that occurred during weekly research team meetings. Additionally, we analyzed one of his students' journals to investigate whether the student understood the instructor's goals for creating the handouts.

Analysis of the instructor's teaching journals and transcripts of the weekly team meetings revealed that the Geometer noted a myriad of reasons why he created handouts for his students. We focused on three goals: (1) helping students to build intuition, (2) giving students a resource to revisit, and (3) assisting students with mastery of the course content outside of class time, and we also provided quotes linking the instructor's and students' thoughts about the handouts.

The student appreciated the detailed handouts, particularly the hand drawn images that helped students understand the gist of the topics before the formal proofs were introduced. In one of his journals, he mentioned: "The immense amount of effort Dr. Brady must put into class preparation shone through again with this handout." However, the student was keenly aware that the handouts allowed the instructor to "cruise" through the material faster than he might otherwise. Further, the student may not have recognized the value in completing the ungraded exercises that the instructor suggested in the handouts. To put this into the instructor's words, the students had to grapple with the handout. It was not enough just to read it over once. They had to "get their hands dirty" and work on it on their own time.

We suspect that many other mathematics instructors share similar pedagogical goals. Creating handouts can be time-intensive, but once the handouts are created, they can be adapted from semester-to-semester. We would like to make some pedagogical recommendations for the inclusion of handouts in advanced courses. First, instructors may consider alerting their students to their goals for creating the handouts. For instance, instructors could let their students know that they will be covering the material more quickly than if they did not have a handout prepared, but the students can revisit the handouts outside of class to more fully grasp the content. Second, handouts can provide an avenue that helps instructors show how concepts unfold step-by-step to help their students grasp an initial intuition about the content. The instructor that we studied also noted that when he created handouts, it helped him to feel like he had a plan and was not scattered for his lecture. Third, if instructors recommend that students complete ungraded exercises, they could explicitly state why they believe these exercises will benefit the students' learning, even though they are not incentivized with course points.

As members of the mathematics community, we are constantly faced with challenges of finding the best ways to maximize our students' understanding. Making the right pedagogical decisions that are aligned with our goals and beliefs are not always trivial. Although, we recognize that we only examined one student's journals in this study, it is rewarding to know that some of our instructional decisions and efforts are effective and appreciated by our students.

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# Factors Supporting (or Constraining) the Implementation of DNR-based Instruction in Mathematics

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*DNR-based instruction in mathematics (Harel, 2008a, 2008b, 2008c) is a theoretical framework for the learning and teaching of mathematics. DNR-based professional development is a long-running program spanning seven years with multiple cohorts of in-service secondary mathematics teacher participants. This report investigates teacher change among five key variables: facilitating public debate, using holistic problems, attending to students' intellectual need, attending to the meaning of quantities and use of students' contributions. Commitment to change and perseverance, the nature of available curricular materials and teachers' view of the role of curriculum, a view of students as partners in knowledge construction, institutional context, collaboration and content knowledge were identified as factors that afford or constrain DNR implementation. This work has implications for the design of future professional development efforts and the development of a more robust theory of teacher change.*

*Keywords:* Teacher change, DNR-based instruction, Intellectual Need, Professional Development, In-service

In their review of 106 articles reporting findings on mathematics professional development programs between 1985 and 2008, Goldsmith, Doerr and Lewis (2014, p. 21) point out that "existing research tends to focus on program effectiveness rather than on teachers' learning," while much less has been said about "*how* teachers develop knowledge, beliefs, or instructional practices". Understanding how teachers' development entails also understanding factors that facilitate or inhibit desired forms of development. In this report, we also strive to answer similar research questions: *How effectively are aspects of a professional development guided by a particular theoretical framework adopted by teachers and integrated into classroom teaching? And what factors challenge or support teachers in implementing this framework in the classroom?*

For the purposes of this report, the first question is seen as merely a starting point. The second questions, is in line with Goldsmith et al's (ibid) observed gap in the literature base. However, we argue that a crucial step in understanding how teachers develop is identifying factors that afford or constrain implementation of the targeted forms of knowledge, beliefs or practices. We are not alone in this goal. Indeed, Hill et al (2008) also sought to identify factors that support or hinder teachers' use of mathematical knowledge for teaching in practice as part of a larger research agenda. Similarly, Schoenfeld (2010), seeking to construct models of teachers' decision-making, found it "necessary and sufficient" to characterize teachers' knowledge, goals, orientations (i.e., belief, values, preferences, etc.) and decision making in order to construct their models. A natural extension to this work is understanding factors that afford and constrain teachers' ability to reach their goals or enact their stated beliefs.

## Theoretical Perspective

*DNR-based instruction in mathematics* (*DNR,* for short; Harel, 1998, 2000, 2008a, 2008b, 2008c, 2013a, 2013b) is a theoretical framework for the learning and teaching of mathematics—

a framework that provides a language and tools to formulate and address critical curricular and instructional concerns. *DNR* can be thought of as a system consisting of three categories of constructs: *premises*—explicit assumptions underlying the *DNR* concepts and claims; *concepts*—constructs defined and oriented within these premises; and *claims—*statements formulated in terms of the *DNR* concepts, entailed from the *DNR* premises, and supported by empirical studies.

As the above list of references indicates, *DNR* has been discussed extensively elsewhere, and so in this paper we only reiterate briefly the definitions of the concepts pertaining to the concern of this study.

*Assigning Holistic Problems:* A holistic problem is one where a person must figure out, from the problem statement, the elements needed for its solution (Harel and Stevens, 2011). It does not contain hints or cues as to what is needed to solve it. In contrast, a non-holistic problem is broken down into small parts, each of which attends to one or two isolated elements. Often each of such parts is a one-step problem.

*Intellectual need:* Do students have a need for understanding the mathematics the teacher intends to teach? Does the teacher appeal to a problematic situation that puzzles students when introducing new mathematics?

*Attention to meaning*: When a problem has a context, unknown quantities have meaning with respect to that context (e.g. units related to quantities). Does the teacher attempt to attend to the meaning of quantities within the context of the problem?

*Public debate*: Is there evidence to believe that the whole class is following the discussion? Is the teacher making a successful effort to engage the whole class in debate through questioning and solicitation of contributions? Public debate also includes the need to evaluate mental images and their validity and efficiency.

*Taking student contributions seriously*: A student's contribution is considered to be taken seriously when it is allowed to live in the public space for discussion without immediate teacher evaluation. When taking contributions seriously, teachers solicit ideas and mental images from students, and facilitate public debate about these ideas to highlight and critique both underlying mathematics.

DNR-based professional development (DBPD) is a long-running program spanning seven years with multiple cohorts of in-service secondary mathematics teacher participants. This report investigates teacher change among five key variables: facilitating public debate, using holistic problems, attending to students' intellectual need, attending to meaning, and use of students' contributions. Collectively, these key variables have previously been identified as crucial teaching practices in student-centered classrooms targeting the development of students' mathematical content knowledge compatible with the Common Core's standards for mathematical practice (e.g., University of Michigan, 2006; Harel, Fuller and Soto, 2014; Shoenfeld, 2013). In his attempt to articulate the complexities constructing the TRU Math Framework and presenting the framework itself, Schoenfeld (2013, p. 613)) identified similar teaching actions, identifying them as, "known in the literature to be important" (ibid, p. 610).

## Methods

DBPD consisted of two related support structures: (1) summer institutes and mid-year follow-up sessions and (2) on-site professional development. Both efforts targeted teachers' knowledge of mathematics, knowledge of student learning, and knowledge of pedagogy. This report examines DNR implementation for 33 teachers in a major urban area of the southwestern United States. DBPD included a focus on the five teaching practices: *public debate*, *holistic* *problems*, *intellectual need*, *attention to meaning*, and *taking contributions seriously* as defined above.

Repeated classroom observations of teacher participants were conducted and used to evaluate participants' implementation of DNR and to chart changes in participants' teaching over time. Two forms of data were generated using these observations. First, researchers examined whether or not a particular teaching practice was demonstrated in each participant's classroom during an entire classroom observation across two later years of the program's existence. Rather than relying exclusively on quantitative results for groupings, the authors triangulated available data sources (including their own experiences with the participants as well as formal debrief interviews) in order to maximize reliability associated with the classifications, thus reflecting the change in each of the participating teachers over time. Second, researchers looked more closely at interview data with participants conducted after classroom observations that could be used to give insight into factors that afford or constrain implementation. A summary of findings follows.

## Results

### How effectively are aspects of DBPD adopted by teachers?

Teachers were observed at least twice over the course of a four-year period, yielding data that describe change over time for each participant. In order to explore overall changes in teaching over time, the evaluator created a factor score that aggregately considered implementation of each of the five DNR parameters (public debate, use of holistic problems, attention to intellectual need, taking student contributions seriously, and attention to meaning) as one normally distributed score. This factor score, calculated using the principal factors method across all five DNR parameters, yielded a clear one factor structure, and thus served as an estimate of DNR implementation. Observed levels of implementation of each of the five DNR parameters were coded as absent or present during observations, and coded in duration by seconds, yielding a data structure that was capable of examining proportion of class periods where various levels of DNR implementation were observed. Across the sample for this study, the mean level of DNR implementation was calculated at 1.51, with a pooled standard deviation of 0.34. Results of a panel regression analysis examining whether or not average implementation of DNR increased over time for the entire group yielded a non-significant $p$ value for the variable "time", indicating that average level of DNR implementation did not increase. However, this lack of aggregate results masks important variation within the population.

Analysis of change within subjects/by teacher revealed five distinct groups:

*Evolvers:* These teachers exhibited noteworthy increases in their DNR implementation score over time, an increase of one standard deviation or more over time (specifically, from first recorded observation to last observation).

*Decliners:* These teachers exhibited a decreased DNR implementation score of one standard deviation or more over time.

*Consistent high implementers:* These teachers exhibited DNR implementation score changes of less than one standard deviation, but were consistent in their high DNR implementation score, scoring at least one standard deviation above the mean at some time.

*Consistent moderate implementers:* These teachers exhibited DNR implementation score changes of less than one standard deviation, but were consistent in their moderate DNR implementation score, scoring within one standard deviation of the mean at both times.

*Consistent low implementers:* These teachers exhibited DNR implementation score changes of less than one standard deviation, but were consistent in their low DNR implementation score, scoring at least one standard deviation below the mean at some time.

Triangulated classifications are used for this publication, and are presented below as a table of frequencies across the five categories.

*Table 1. Frequency table for categories of DNR implementation by DBPD participants.*

| Categorization | Number of Participants |
|---|---|
| Consistently Low | 6 |
| Decline | 8 |
| Consistently Medium | 4 |
| Evolvers | 9 |
| Consistently High | 6 |

Closer examination of these groups (and specifically, of the stories these teachers provided through interviews) revealed important details about factors that both facilitate and inhibit DNR implementation. Finally, we noticed that affordances were simply the opposite of the constraints. Therefore, we report findings thematically, describing how a particular side of each theme (such as presence of absence of collegiality in the work environment) related to DNR implementation. In this same vein, we look at conditions that facilitated successful teachers, and conversely, those that facilitated disengagement with the DNR mission and a failure to implement in the classroom.

**What factors challenge or support teachers in implementing DNR in the classroom?**

Participant interview data identified commitment to change and perseverance, the nature of available curricular materials and teachers' view of the role of curriculum, a view of students as partners in knowledge construction, institutional context, collaboration and content knowledge as factors that afford or constrain DNR implementation. However, given the current space constraints and the qualitative nature of the data analysis, we report here one example of an observed factor, commitment to change and perseverance.

**Commitment to Change and Perseverance**

Across groups, risk-taking and willingness to take initiative emerged as common traits among consistently high implementers and evolvers. Similarly, the absence thereof characterized consistently low implementers and decliners. Three participants describe sharply contrasting points of view which explain their rationale for choosing to implement (or not) two different aspects of DNR-based instruction, holistic problems and public debate.

P1: They're trying to get us all to do the Euclid unit, and I'm just not on board - I was going to do it, but my school was not into it! Then I decided against it. There is a lot of construction and proofs, and that's what our students struggle with the most. The problems in there are geared for teachers, so you'd have to re-write everything to be more appropriate for students.

DBPD participants experienced the unit P1 refers to as "The Euclid Unit" (see Harel, 2014 for a description) over two summers. P1 described an initial willingness to implement this curriculum. Ultimately, she decided against implementing the unit for the following reasons: (a) lack of support at her "school" – where "school" may refer to departmental colleagues, students, parents or administrators, (b) a belief that construction and proving problems are beyond what her students' are capable of appreciating and productively struggling with, and (c) a need to lower the level of the questions, generating work beyond the scope of her capacity. We save constraints (a) and (b) for discussion in subsequent sections while focusing on constraint (c) here.

In contrast to P1, P2 and P3 illustrated how a commitment to change supported implementation, regardless of their experience with the same kinds of constraints faced by P1. That is, these participants knowingly chose to search for ways to overcome existing constraints because of their belief systems.

P2: The students were not buying into public debate – they always just want to know the steps. That's a struggle, especially with the advanced classes. Same story over and over. I believe in what I'm doing, but the resistance is always there. Most students are buying in, doing great, will publicly debate and challenge themselves. They understand that the methods and strategies we're applying [are] good for them… A lot of times the students work in groups of four where they talk to one another a lot already. I don't do direct instruction as much as any other teachers around here. That takes a lot of coaching every time – basically the months of September and October where I really have to drill the process into them, including training the kids and talking to the parents about why I do it like this.

P2 discussed how her belief in the role of public debate drove her to persevere in the face of student and parental pressure. P1 also cited a constraint regarding lack of support among important stakeholders (i.e., her "school"). Also in contrast to P1, P2 describes an actual, rather than a perceived, form of resistance. However, her belief in the benefits of public debate in the learning process led her to persist in her implementation.

P2 also describes a time frame in which she knows she will have to endure this resistance along with an understanding that communication with parents and powerful student learning experiences can combat this resistance despite the amount of effort it will take. We note that at one point in time P2 confronted and overcame these constraints before developing confidence that resistance eventually fades, and usually after the first two months of class. In the end P2 developed a sense for how long resistance will last and the benefits of persistence.

Also in contrast to P1, P3 describes below how a valuable experience at the DBPD summer institutes led to implementation of a particular holistic problem.

P3: Anytime I get a holistic problem, I try and spot some of the techniques they've used…like, the teacher-researcher gave us a problem where a sweeping line was covering some area…and I used that type of thing with my students. I thought, I can make that

> simpler and thus appropriate for 9th graders… You have to have faith that the kids will learn something valuable.

We first note potential differences in beliefs between P1 and P3. P1 reported feeling pressure to implement an entire DNR-based curriculum. P3 described a desire to bring a particular problem to his students, an approach emphasized during the summer institutes. P3 also described a desire to find and implement a general principle. Both participants felt a need to translate problems introduced at the DBPD to their students' level. P3's choice was supported by a belief in students as partners in knowledge construction saying, "You have to have faith that kids will learn something valuable." Consequently, P3's commitment to change surpassed constraints of the amount of work needed to translate a DNR concept (holistic problems) in his classroom.

Regardless of implementation level, participants reported that generating holistic problems was difficult and time consuming. On many occasions we observed, and participants reported, that they needed to try several versions of these problems before finding one that elicited productive student images useful in advancing the teacher's mathematical agenda. This was expressed nicely by one high implementer who noted after several years of implementation, "Almost all the problems I use I make up…". Another high implementer reported, "I'm always writing the material on my own. The book that we have doesn't support DNR-type instruction at all … I have very limited time. So, curriculum is a big need." While both of these comments also point other factors to be discussed later, we emphasize that this factor, commitment to change and perseverance, was a necessary ingredient in these participants' *eventual* shift toward the development of a curriculum consisting of a large number of holistic problems.

Assigning holistic problems, facilitating public debate and attempting to attend to students' intellectual need are high risk behaviors with the potential for high rewards or catastrophic failure. A common theme among low implementers and decliners was the observation that that assigning holistic problems could lead to high student frustration, especially from students who had little experience with them. This was often cited as a reason they did not implement key aspects of DNR, while high implementers and evolvers demonstrated a commitment to change and perseverance in implementation. Nearly all participants expressed fear of complaints from parents (and students) who might not understand the intention of a problem, why teachers were not providing algorithmic approaches in advance or why teachers might record an incorrect solution on the board. As one participant said, "A challenge is the resistance from parents and students when DNR or Common Core Standards are implemented since they are used to being told what to do."

In addition to targeted DBPD, change in teaching practice or sustaining the forms of teaching practice compatible to DNR-based instruction requires a concerted effort. Successful implementation of DNR requires teachers to see teaching as a mission – a mission of implementing and evangelizing innovative mathematics instruction – rather than a job.

## Discussion/Summary

We summarize what we have learned, what remains open and what implications for instruction/future programs.

### What have we learned?

Echoing the findings of other mathematics educators (e.g., Schoenfeld, 2010; Harel, Fuller & Soto, 2014; Ball, Hill & Bass, 2005), we are still far from understanding the inner workings of how teacher change their practice, especially "*how* teachers develop [and change existing] knowledge, beliefs, or instructional practices". It takes a professional with an exceptional commitment to change and perseverance in the face of the many obstacles that constrain teaching practice. Choosing a holistic problem for students is a non-trivial task, especially in light of the current culture of textbook school mathematics where the decisive majority of problems are non-holistic. We have found that successful implementers of DNR believe that students are partners in knowledge construction. Even when teachers do hold this belief, they may still struggle to solicit student thinking, anticipate student difficulties and bring the whole class conversation to a meaningful closure in a reasonable amount of time. There are many institutional pressures that constrain DNR-compatible teaching.

A constant theme among DBPD participants, regardless of implementation level, was a desire for a DNR-based curriculum. This large task requires a large set of resources, both human and fiscal. Nevertheless, DBPD participants repeatedly pointed out that curriculum would be a practical way to make inroads with colleagues, administrators, parents and students.

Another lesson learned is the importance of collaboration among teachers involved in DBPD. In order for DNR-based professional developers to influence instructional practice, participants must have access to first-hand experiences in which DNR-based instruction is demonstrated first-hand with actual students at the level of the participants. One important finding was that DNR-implementation was actually independent of the level of student being taught. While this result seems to contradict our finding that teachers needed to experience DNR-based instruction with students, in actuality it is more of a statement about teacher perception rather than fact.

Finally, we note that our participants enjoyed doing mathematics together. Successful implementers found ways to parlay their content knowledge into the selection or refinement of holistic problems for their students, the goals they set for public debate, the ability to make better sense of what students were saying and meaning, and the ability to make something out of those statements. Thus, giving their students a sense of ownership over mathematical ideas in the classroom.

**Questions for further research**

These findings point out that high implementers demonstrated a set of beliefs about learning and teaching coupled with particular dispositional traits (e.g., sees teaching as a mission, demonstrates perseverance/adherence to the belief that learning can only be accomplished through problem-solving, views content knowledge as central to good teaching). Were these participants selected for these traits or did DBPD influence them in some ways? If so, how?

We cited many institutionally related constraints here (and there are certainly others). How can DBPD providers attend to these in the future? For example, two participants at the same site, with the same preparation period, taught the same content and they seemed to benefit most from the DBPD. Another question concerns curriculum. How much will teachers feel is sufficient to support DNR implementation? An entire year? A particular grade band? All grades?

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Relationships between Precalculus Students' Engagement and Shape Thinking

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*This study examines relationships between community college precalculus students' understanding and engagement to link mathematical success to a malleable construct, and offer new insights for addressing consistently poor success rates in community college precalculus (Barnes, Cerrito, & Levi, 2004). Two-part interviews, consisting of a task and debriefing, were conducted with 8 students to investigate their shape thinking (Moore & Thompson, 2015), and engagement, conceptualized through flow theory (Csikszentmihalyi, 1975). Results suggest that students can be highly engaged in mathematics tasks regardless of understanding and that students exhibiting different ways of thinking about graph construction tended to experience different forms of engagement.*

*Keywords:* Student engagement, Shape thinking, Precalculus, Community college

This study represents a portion of the author's dissertation in which community college precalculus students' engagement, understanding of precalculus concepts (e.g., covariation) and relationships between the two were investigated to address consistently poor success rates in community college precalculus (Barnes, Cerrito, & Levi, 2004). Others have demonstrated that student engagement is positively associated with academic achievement (e.g., Finn & Rock, 1997; Finn & Zimmer, 2012; Newmann, Wehlage, & Lamborn, 1992; Reschly & Christenson, 2012; Skinner & Belmont, 1993) and success in mathematics (e.g., Barkatsas, Kasimatis, & Gialamas, 2009; Lan et al., 2009; Martin, Way, Bobis, & Anderson, 2015; Rimm-Kaufman, Baroody, Larsen, Curby, & Abry, 2015; Robinson, 2013), where achievement and success are typically measured by students' performance on high-stakes assessments and student engagement is frequently reported by teachers or observers. Studying student engagement remains a focal point for educational research; this study contributes to that body of literature by investigating community college students' engagement and associations between engagement and understanding – as opposed to achievement on standardized tests. This study focuses on exploring relationships between student engagement and understanding of covariation. Information on such relationships would extend our understanding of the importance of fostering student engagement in community college precalculus classrooms.

## Framework

### Student Engagement

Student engagement is a metaconstruct consisting of emotional, behavioral, and cognitive components (Fredricks, Blumenfeld, & Paris, 2004). Flow theory (Csikszentmihalyi, 1975, 1990) is a valid framework for conceptualizing student engagement because both flow and engagement are comprised of similar components, both are described as states of intense concentration and investment in a task or activity, and both are intrinsically motivating (Steele & Fullagar, 2009). From the perspective of flow theory, student engagement is comprised of interest, enjoyment, and concentration (Shernoff, Csikszentmihalyi, Schneider, & Shernoff, 2003), where interest and enjoyment are elements of emotional engagement and concentration constitutes behavioral and cognitive engagement.

## Covariation

Among other precalculus concepts (e.g., quantity/quantizing and function), covariational reasoning is paramount for success in future undergraduate mathematics courses (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Moore and Thompson (2015) suggest that shape thinking provides perspective for describing students' covariational reasoning in the context of graphs. They describe students' shape thinking as static or emergent, where "static shape thinking involves operating on a graph as an object in and of itself" (p. 784). Monk (1992), defined iconic translations as a way of thinking in which perceptual features from a situation are associated with the shape of a graph for that situation. Also, Thompson (2015) explains that students may use thematic associations by regarding features of a situation as being necessary elements of the corresponding graph. Stevens and Moore (2016) elaborate that both iconic translations and thematic associations are examples of static shape thinking because both ways of thinking rely on perceptual features of an event for reasoning about a graphical representation; hence, operating on a graph as an object. On the other hand, "emergent shape thinking involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation)" (Moore & Thompson, 2015, p. 785, emphasis in original).

# Methods

The purpose of this study is to explore and describe any relationships between community college precalculus students' engagement and shape thinking in the context of a task-based interview. The following research question is addressed. Is there a relationship between student engagement and shape thinking, and if so, what are characteristics of this relationship?

## Setting and Participants

Data collection for the dissertation study took place during Fall 2016 at two Southeastern community colleges, where 15 precalculus instructors and 101 students participated. As part of the larger study, three instructors were selected for classroom observations, and their students were considered to take part in task-based interviews. Eight students from these three classrooms were selected based on their self-reported levels of engagement (i.e., interest, enjoyment, and concentration) during the first five weeks of the semester. These students were consistently more engaged during class time than their peers. Students with relatively high levels of engagement were selected to take part in interviews to increase the researcher's opportunity to "see" student engagement so any relationships between engagement and shape thinking could be explored.

Seven of eight students were enrolled full-time and all but two (Richard and Suzy, pseudonyms) were taking precalculus for the first time as college students.

## Data Collection and Analysis

Data collection took place during interviews with students. Each interview consisted of two parts. The first part was task-based, where students worked through the *Taking a Ride* task, which has been used by Moore and colleagues to investigate students' shape thinking and other graphing activities (e.g., Stevens & Moore, 2016). The second part of each interview was a debriefing, where participants were asked to reflect on their interest, enjoyment, and concentration (i.e. engagement) while working the task. The data consist of video recordings, transcripts, and student artifacts.

Interview transcripts and artifacts produced during the task-based portion of interviews were coded for indicators of students' shape thinking based on definitions provided above. Transcripts from the debriefing portion of interviews were open coded to describe themes in participants' descriptions of their interest, enjoyment, and concentration while working the task. Finally, students' various ways of thinking coupled with themes emergent in their engagement are used to identify and characterize any relationships between student engagement and shape thinking.

**Task description.** In the Taking a Ride task, students are prompted to watch an animation of a Ferris wheel perpetually rotating clockwise and "graph the relationship between a rider's total distance traveled *around* the wheel and the rider's distance from the ground" (emphasis in original). Following this, participants view a second animation of a Ferris wheel; however, in this animation the ride stops periodically. Participants are then asked to discuss the relevance of their original graph to the new situation.

## Results

This section is organized to first present results on students' shape thinking, and then describe themes in students' engagement.

### Shape Thinking

Three students exhibited static shape thinking: Richard, Paula, and Sally. Though, all three students did not associate perceptual features of the animation and their graphs in the same way. Richard interpreted the image of the Ferris wheel as a coordinate system and explained how he envisioned such a coordinate system working. He explains the image/graph was structured with x's that were all zero "because the fact that every um lines over there [*pointing to the image in the animation*] kind of direct me straight to the middle [of the ride]." He used the image of the ride to establish radial coordinates, like those of a polar coordinate system, which he described as heights. Richard labeled his horizontal axis "time," which was constant at zero "because everything points at zero." Two visuals of Richard's coordinate system are provided in Figure 1.

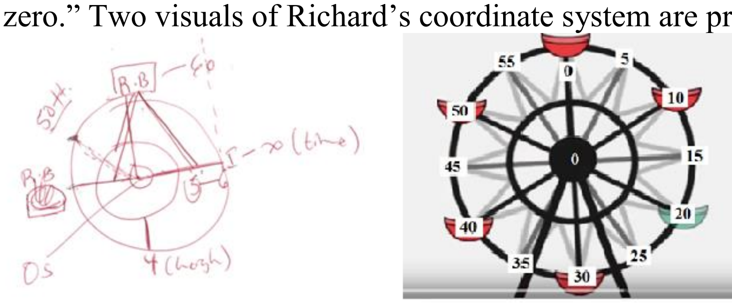


*Figure 1. Richard's sketch (left) and a reconstruction (right) of his coordinate system.*

Paula and Sally also demonstrated static shape thinking in their work on this task, by sketching circular graphs depicting the path of a rider on the Ferris wheel. Their justifications for circular graphs are exemplified by the following interview excerpt.

*Paula*: he's not going straight up or like going straight to the side, he's going in a circular motion so that is what I put it like that [a circular graph].

*Interviewer*: So, how would that change [*pointing to Paula's graph*] if the wheel were rotating the other way?

*Paula*: If it were rotating the other way, it would start here [*pointing to the right-most side of the wheel*] and then go around that way [*tracing around the wheel counterclockwise*]. So, it would go this way [*tracing the same path on her graph*].

Figure 2 shows these students' graphs.

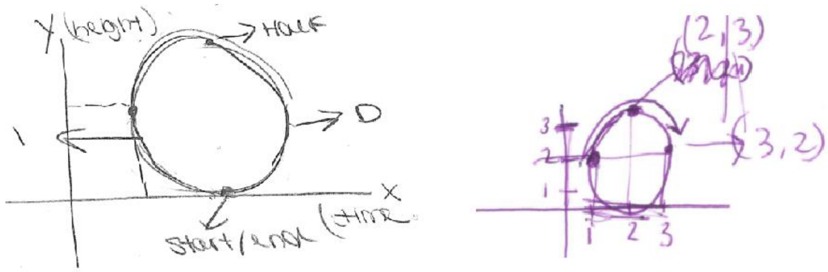


*Figure 2. Paula (left) and Sally's (right) graphs of circular paths.*

The remaining five students, Beverly, James, Suzy, Marianne, and Patricia exhibited emergent shape thinking while working the task. Beverly describes her graph as depicting the rider's position at a given time during the ride. She considers the horizontal axis to be "distance from the middle" or how she describes the rider's lateral displacement from a starting point. Her vertical axis is constructed similarly to reflect the rider's height above the ground. In this regard, she is coordinating simultaneous vertical and lateral displacements in a bounded space, both with respect to time, to produce her graph as the emergent path a rider travels around the ride. Her graph is shared in Figure 3.

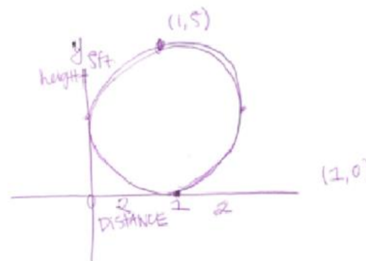


*Figure 3. Beverly's sketch of the path a rider travels as an emergent trace.*

Suzy and James both interpreted the prompt from the task to require two graphs: one for the relationship between a rider's distance traveled around the wheel over time, and a second for the rider's height over time. James' graph(s) in Figure 4 reflect how both students thought about this task. James sketches both relationships on the same plane, where the green graph depicts the rider's total distance traveled over time and the purple graph reflects the rider's height over time. James explains how he coordinates changes in time and height to construct his graph for that relationship, "at no time, you're at zero assuming it starts with the rider at the bottom…[then] halfway through that [height] would be halfway… so here's a graph." He continues, "for the total distance around it would be similar, but… it would extend forever."

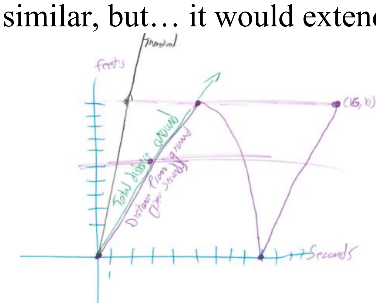


*Figure 4. James' graph(s).*

Marianne also interprets the prompt to be about a relationship between a rider's height and time, but she sketches a "sine or cosine curve" to represent the situation. Her emergent shape thinking is demonstrated in her explanation of how her graph changes for the second animation.

*Marianne*: So it stops about every like quarter of the way, so you would just have to like scrap your graph where it stops and draw a straight line. Um, but still have it like connect to the curve. So I guess it would stop like here [*sketching Figure 5*], so you would just straight line and it would resume. And then it would stop here, so straight line…

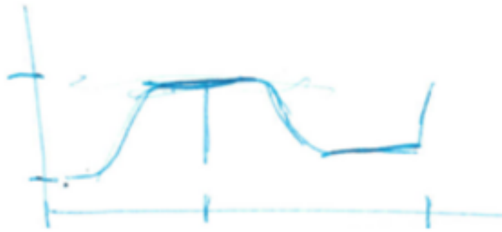

*Figure 5. Marianne's graph of the second animation.*

Lastly, Patricia coordinates changes in the rider's total distance traveled around the wheel with distance from the ground. In the following excerpt, Patricia demonstrates understanding her graph as an emergent trace created as the rider's distance traveled and distance from the ground covary by physically tracing her graph (Figure 6) as she explains its behavior associated with stopping in the second animation.

*Patricia*: …if we're traveling now [*tracing her graph with her pencil while watching the animation*] and I pause [*stops tracing*] I'm like, it doesn't affect [sic], like I'm not still going straight with my distance from the ground, and I'm not going down with my distance traveled because I am just standing there. Like I'm still, but then I keep going.

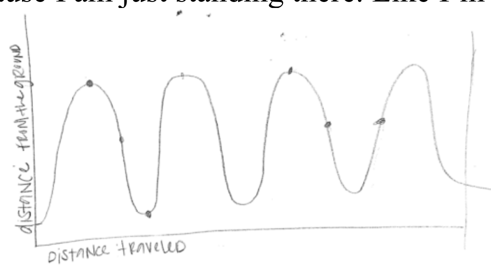


*Figure 6. Patricia's graph.*

The next section shares results from debriefings, to report on these students' engagement.

**Student Engagement**

**Concentration.** To begin, the average amount of time spent working the task was about 48 minutes, ranging from about 26.5 minutes (Patricia) to 76 minutes (Sally). This persistence with working and explaining their thinking evidences high levels of concentration in all students. Further, when asked about their concentration during debriefings, two themes emerged in responses regardless of shape thinking. First, students explained their focus on elements of the task they found confusing. For example, Patricia (emergent shape thinking) was "throw[n] off that time's not in there." Second, students discussed needing to concentrate on their explanations while working the task. Paula (static shape thinking) exemplifies this theme by reflecting on how evaluating her thoughts inhibited her work on the task, "just like the fact that maybe I was just wrong, so I would think about something and then I'd be like, no that is wrong, don't say that." Students concentrating on their own thinking is possibly due to the task-based interview setting, but does reflect high levels of concentration while working a mathematical task.

**Interest and Enjoyment.** All eight students did not describe similar feelings towards interest and enjoyment on the task. In fact, there appear to be differences in the self-reported levels of interest and enjoyment among groups of students whose shape thinking was categorized differently. Specifically, those who exhibited emergent shape thinking tended to enjoy the task because it was challenging, promoted problem-solving and thinking, and the context was relatable; these students also tended to find the task interesting because it was challenging, promoted problem-solving, and allowed for autonomy. For instance, Suzy discussed that she found the task interesting because "it makes a person think… this is very important to try to think logically and solve problems in real life."

Though, not all students demonstrating emergent shape think expressed high levels of enjoyment and interest. For example, Marianne indicated that she enjoyed the task because she "enjoyed thinking things out, like trying to make sense of the wheel and drawing it on paper." However, she mentioned that the open-ended nature of the task "put[ting] me back in that place where like I was unsure of myself." Further, Patricia found the task uninteresting and unenjoyable, both of which she attributed to the open-ended nature of the task prompt and not knowing what to do. Thus, Marianne and Patricia associated low confidence while working on the task to lower levels of enjoyment (and interest in Patricia's case).

On the other hand, students who demonstrated static shape thinking while working on this task did not enjoy the task but tended to find it interesting. They associated their lack of enjoyment with finding the task challenging, confusing, and allowing for too much autonomy. Paula and Sally also discussed low confidence being associated with their low level of enjoyment. For example, when asked about her enjoyment, Paula stated, "pretty bad… because I think it is just all wrong… Especially with this one [*referring to the first animation*] because I never saw this before, like the whole circle in just one little section." However, these students did report that working the task was interesting because it was challenging and open-ended. Sally explicitly states this apparently contradictory result "so interesting because difficult; not enjoyable because difficult."

Table 1 presents a matrix of themes associated with student engagement (i.e. concentration, enjoyment, and interest) discussed by students during debriefings based on shape thinking.

*Table 1. Themes associated with engagement by shape thinking*

| | Concentration | Low Enjoyment | Low Interest | Enjoyment | Interest |
|---|---|---|---|---|---|
| Static Shape Thinking | Persistence<br><br>Focus on confusing elements<br><br>Concentrate on explaining | Challenging<br><br>Confusing<br><br>Autonomy<br><br>Low confidence | | | Challenging<br><br>Open-ended |
| Emergent Shape Thinking | | Low confidence (Patrician & Marianne) | Autonomy (Patricia) | Challenging<br><br>Promotes problem solving<br><br>Context | Challenging<br><br>Promotes problem solving<br><br>Autonomy |

## Conclusion

There appears to be a relationship between community college precalculus students' engagement and shape thinking. Specifically, students exhibiting differences in shape thinking described differences in their interest and enjoyment, such that those exhibiting static shape thinking tended to be interested while working the task but experienced low enjoyment. On the other hand, students exhibiting emergent shape thinking tended to find the task both interesting and enjoyable, except for Patricia, who expressed struggling with confidence. Regardless of students' understanding of covariation and shape thinking, these students demonstrated and discussed high levels of concentration.

This study has shed light on factors community college precalculus students associate with their levels of engagement while working through a challenging mathematical task. Researchers have showed that student engagement is positively associated with academic achievement (e.g., Finn & Rock, 1997; Finn & Zimmer, 2012; Newmann et al., 1992; Reschly & Christenson, 2012; Skinner & Belmont, 1993) and success in mathematics (e.g., Barkatsas et al., 2009; Lan et al., 2009; Martin et al., 2015; Rimm-Kaufman et al., 2015; Robinson, 2013). Results presented in this study suggest that community college precalculus students can be highly engaged in mathematics tasks regardless of understanding and that students exhibiting different ways of thinking about graph construction tended to experience different forms of engagement. These results demonstrate the importance for establishing learning environments that foster student engagement.

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# Observable Manifestations of A Teacher's Actions to Understand and Act on Student Thinking

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*This study produced a framework that describes different levels of teacher-student interactions during teaching. The framework characterizes observable teacher behaviors that are associated with each of the four levels of decentering that emerged from analyzing the teacher-student interactions of three teachers when teaching.*

*Keywords:* Decentering, teacher-student interaction, teacher education, radical constructivism

## Introduction

In recent decades, the importance of teachers' attending to and understanding their students' mathematical thinking, and building their instructional decisions on this understanding is highlighted in many research studies and publications in mathematics education (Ball & Cohen, 1999; Sowder, 2007). In *Principles and Standards for School Mathematics* (NCTM, 2000), one of the principles of effective teaching says "Effective teaching involves observing students, listening carefully to their ideas and explanations, having mathematical goals, and using the information to make instructional decisions" (p. 19). Therefore, it is crucial to characterize what is involved in attending to and understanding student thinking, and to illustrate how instructional decisions can be influenced by this understanding, especially in the context of teaching.

Studies have highlighted that understanding students' thinking and deciding how to act based on this understanding are not traits that are inherently possessed by teachers and they should be considered as types of expertise that need to be developed (Jacobs, Lamb, & Philipp, 2010). This position is supported by studies that have described teachers' difficulties in attending to, anticipating, and understanding students' mathematical thinking (Kazemi & Franke, 2004; Rodgers, 2002; Wallach & Even, 2005). In recent years, researchers have introduced and used theoretical constructs as a way to conceptualize the nature and development of this expertise. As one example, the construct of noticing has ben used to make inferences about teachers' ability to focus on students' mathematical thinking in a classroom environment (Sherin, Jacobs, & Philipp, 2011). Similarly, Jacobs et al. (2010) introduced the notion of professional noticing of children's mathematical thinking. They also recommend further research that "can connect teachers' professional noticing of children's mathematical thinking with the execution of their in-the-moment responses" (p. 197).

Piaget's (1955) construct of *decentering* has also been used as a theoretical lens to extend research on teachers' execution of in-the-moment responses based on student thinking (Carlson, Bowling, Moore, & Ortiz, 2007; Marfai, Moore, & Teuscher, 2011; Teuscher, Moore, & Carlson, 2016). In his work on child development, Piaget (1955) introduced the idea of decentering and described it as an action of adopting a perspective that is not one's own. More recently, Steffe and Thompson (2000) and Thompson (2000, 2013) extended Piaget's idea of decentering and conceptualized a meaningful human communication from the perspective of radical constructivism. Teuscher et al. (2016) state that, even though the construct of decentering has rarely been used to investigate interactions between a teacher and student(s), it has the potential to provide researchers with a framework for characterizing how a teacher's attention to (or lack of attention to) student thinking might impact the teacher's in-the-moment instructional decisions.

The purpose of this study was to characterize the degree to which a teacher attempts to make sense of and use student thinking when teaching. It was also our goal to describe the different levels of student-teacher interactions in terms of both the teacher's mental actions (i.e., decentering) and his or her observable behaviors. We have extended previous studies that described different levels of decentering (Carlson et al., 2007; Marfai et al., 2011) by presenting a framework of observable behaviors associated with teacher decentering. The framework has the potential to contribute to the research on effective teacher-student interactions by illustrating behaviors of a teacher that are associated with both non-decentered and decentered interactions between a teacher and her students.

## Theoretical Framework: A Conceptualization of Human Communication in Radical Constructivism

According to Thompson (2000), people interact with others reflectively or unreflectively. In the case of teacher-student interactions, if the teacher acts reflectively, she then can act as an observer and be aware of the student's contributions to the interaction. Otherwise, the teacher acts as an actor, which prevents her from attempting to adopt the student's perspective (i.e., decentering).

A reflective interaction between two people is described as "the process of mutual interpretation and accommodation" (Thompson, 2013, p. 64). In this process, each participant attempts to understand what the other has in mind by building second-order models of the other's mental structures. Second-order models are "the hypothetical models an observer may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (Steffe, von Glasersfeld, Richards, & Cobb, 1983, p. xvi). During the reflective interaction, each participant continuously adjusts his or her second-order models of the other's knowledge by comparing the other's responses with the responses that he or she anticipates (Thompson, 2013). Besides attempting to understand the other, each participant also makes an effort to have the other understand what he or she has in mind. In the case of teacher-student interaction, for example, the teacher considers how the student could interpret his or her utterances when attempting to convey his or her ways of thinking to the student based on a second-order model of the student's thinking. By continuously updating second-order models of the student's thinking through decentering, the teacher makes better decisions about how to convey his or her intended meaning to the student (Teuscher et al., 2016; Thompson, 2013).

If a teacher interacts with the student unreflectively, he or she is an actor rather than an observer of the student's thinking in this interaction. Thus the teacher is constrained to use his or her first-order model when making decisions about how to act (Teuscher et al., 2016). First-order models are "the models an individual constructs to organize, comprehend, and control his or her own experience, i.e., their own mathematical knowledge" (Steffe, et al., p. xvi).

## Method

### Subjects of the Study

The subjects of the study were three graduate teaching assistants (GTA) at a large public university in the United States. Two of the subjects were PhD students in mathematics and one was a PhD student in mathematics education. They were using the research-based and conceptually oriented Pathways curriculum (Carlson, Oehrtman & Moore, 2016). Prior to the beginning of the semester the subjects attended a 2-day workshop and during the semester when teaching they attended a weekly 1.5-hour seminar, both which focused on supporting the course instructors in developing deep meanings for and critical connections among the key concepts of the course.

### Data Collection and Data Analysis

As the main sources of data, classroom observations were made during the spring semester 2017. Each subject's class was videotaped with a lapel microphone used to capture the teacher's explanations and conversations with students. Moreover, the first author of the study observed each lesson and took field notes.

We began our data analysis by identifying video excerpts in which the teacher was interacting with one of her students. We followed by transcribing these excerpts and began the process of studying the interactions carefully for the purpose of characterizing the degree to which the teacher exhibited decentering behaviors. We also took note of the mathematical meanings displayed by the teacher and the degree to which the teacher exhibited mathematical goals aligned with the mathematical goals of the Pathways curriculum. Our study of the videos led to our constructing codes to characterize the teacher's decentering actions, i.e., the degree to which they were constructing models of his/her students' thinking during interaction. In order to check the inter-coder reliability, two researchers independently coded randomly selected interactions (approximately 20% of the whole data set). We reached 85% agreement in our coding of these pieces of data. Discrepancies between the codes assigned by the two coders were discussed and a consensus on these codes was reached. The first researcher then coded the entire data set.

Following the coding process, we compared the collection of interactions and clustered interactions that were similar relative to the teacher's decentering actions. This analysis led to our identifying four types of student-teacher interactions. We then described observable behaviors and attempted to draw inferences about the mental actions (i.e., decentering) associated with each of the four levels of interactions. In the following paragraphs, we introduce the framework that emerged and then we illustrate how this framework can be used to characterize and describe teacher-student interactions.

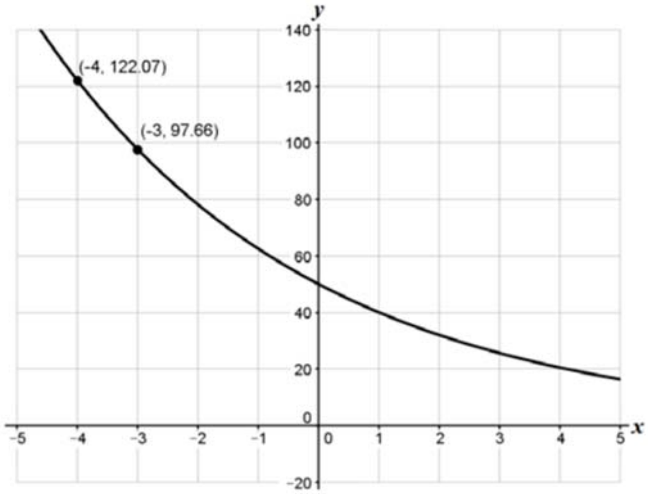
## A Framework for Analyzing Student-Teacher Interactions

The framework illustrates four different levels of student-teacher interactions. Level 1 and Level 2 are considered low-level interactions in terms of the teacher's decentering actions. We see that the teacher is acting from her/his mathematical meanings and is not considering how the student is thinking (Table 1). The primary difference between Level 1 and Level 2 interactions is that a teacher who is classified to be exhibiting Level 2 mental actions poses questions that probe students' thinking, while in a Level 1 interaction the teacher is only interested in students' answers and calculations, or getting students to echo the teacher's phrases. A Level 2 interaction is further characterized by the teacher posing questions and giving explanations aimed at moving students to his or her way of thinking.

Level 3 and Level 4 of the framework are considered to be higher-level interactions in terms of the teacher's decentering actions. In these levels, the teacher attempts to understand the student's perspective and makes general instructional moves based on the student's current thinking when interacting with the student around the course's key ideas. The primary distinction between Level 3 and Level 4 interactions is that during a Level 4 interaction, the teacher exhibits behaviors that suggest that she both respects students' idiosyncratic ways of thinking and makes moves to support students in making connections.

*Table 1. Framework for analyzing student-teacher interactions in terms of the teacher's mental actions and observable behaviors*

| Mental actions | Levels | Description of the behaviors |
|---|---|---|
| • Not reflecting on aspects of the interaction that are contributed by students (i.e., interacting unreflectively)<br>• Creating first-order models of the interaction to organize, comprehend and control his/her own experience<br>• Operating entirely from first-order models (i.e., his/her own mathematical knowledge) | *Level 1:* Shows no interest in students' thinking but shows interest in students' answers and takes actions to get students to say the correct the answer | • Asks questions to elicit students' answers<br>• Listens to students' answers<br>• Does not pose questions aimed at understanding students' thinking<br>○ May pose questions focusing on procedures or calculations<br>○ May evaluate how students' responses compare to his/her own way of thinking<br>○ May pose questions to get students to echo key phrases and/or complete steps to get an answer |
| • Assuming that students' thinking is identical with him/her. In the case of recognizing students' thinking is different then him/her, not attempting to discern the students' mental actions driving the student's behaviors | *Level 2:* Appears interested in students' thinking, does not pose questions focused on students' thinking and attempts to move students to his/her thinking or perspective without trying to understand or build on the expressed thinking and/or perspectives of students | • Poses questions to reveal student thinking but does not attempt to understand students' thinking (e.g., Why? What does that mean? What does that term represent?)<br>• Guides students toward his/her own way of thinking.<br>○ Poses questions for the purpose of getting students to adopt his/her way of thinking<br>○ Gives explanations aimed at getting students to adopt his/her way of thinking |
| • Reflecting on aspects of the interaction that are contributed by students (i.e., interacting reflectively)<br>• Creating second-order models of students' thinking to explain his/her experience of students' states and activities | *Level 3*: Appears to make sense of students' thinking and/or perspectives, and makes general moves based on the expressions of the students | • Asks questions to reveal and understand students' thinking<br>• Follows up on students' responses in order to perturb students in a way that extends their current ways of thinking<br>• Attempts to move students to his/her thinking or perspective<br>○ Poses questions and gives explanations informed by students' current thinking and his/her understanding of the mathematical ideas |
| • Operating from second-order models of student thinking<br>• Assuming that students have idiosyncratic thinking and attempting to discern the students' mental actions driving the students' behaviors | *Level 4:* Takes action to understand students' thinking, appears to understand the expressed thinking and/or perspective of students and takes actions that build on and respect the rationality of these expressions | • Prompts students to explain their idiosyncratic ways of thinking<br>○ Poses questions to gain insights into students' thinking<br>• Draws on students' idiosyncratic ways of thinking to advance students' understanding of key ideas in the lesson<br>○ Poses questions and/or gives explanations that are attentive to students' thinking and/or aimed at advancing students' understanding of an idea<br>○ Poses questions and/or gives explanations to support students in making connections between different viable ways of thinking of a mathematical idea |




1. Find the ratio of output values that correspond to increases of 1 in the input value in order to determine the growth or decay factor.
2. Determine the 1-unit percent change by comparing the change in the output values to the function value at the beginning of a 1-unit interval for x.
3. Identify or determine the value of the function when x=0.
4.Use the information from parts (a) through (c) to define a function formula for the relationship.

*Figure 1. The task used in the classes where the excerpts in the illustration1 and illustration 2 come from (Carlson et al., 2016)*

**Illustration 1:**

The task in Figure 1 requires that students understand that the growth factor in an exponential function represents the relative size of two output values in terms of both multiplicative and percent comparisons (Carlson et al., 2016).

Before the conversation in Excerpt 1 began, the teacher discussed the task by describing how he expected students to think when they see this type of question. He stated, "I know it is a decay factor because it looks like as I move my input up my outputs are going down. I want you to look at it and be thinking these kinds of thoughts". He followed by describing how he determines a 1-unit decay factor. The teacher appeared to be focused on getting the students to imitate how he approaches this type of question, in contrast to showing any interest in the students' thinking. He then turned his attention to finding the function's initial value (Excerpt 1, Line 1).

Excerpt 1

[Line1] *Teacher:* I need to find my initial value. How might I find it? However, I do know a way to find it because I know that every time to get my new output at -2, what do I multiply 97.66 by?

[Line2] *Student1:* .8

[Line3] *Teacher:* .8; the decay factor. So at -2, I have .8 times 97.66. Ok, to get my value at -1 what do I multiply this number by?

[Line4] *Student2:* .8

[Line5] *Teacher:* .8 again, right? So I'm just a kind of walking my way down the graph to figure out what my value is at 0. So I know that I'm gonna have to multiply by .8 once to get the -2, twice to get the -1, and three times to get the zero, right?

We classified this interaction between the teacher and students in Excerpt 1 at Level 1 since there is no evidence that the teacher was interested in the students' ways of thinking about exponential growth or the idea of growth factor. He posed questions and listened to students' responses. However, the teacher's questions were directed at getting students to express the computation to get the correct answer (Lines 1, 3). After one student suggested a factor for multiplying (Line 4), the teacher failed to acknowledge her response; instead he proceeded to explain how the decay factor could be used to find the initial value of the exponential function. This explanation was a presentation of the teacher's way of thinking with no regard for whether his explanations were relevant to the student (Line 5). During this exchange the teacher did not attempt to reveal and understand students' meaning of a 1-unit growth/decay factor, nor did he build a second-order model of his students' thinking. The teacher's questions and explanations were based on his first-order model (his understanding), instead of models he built of students' thinking/meanings.

**Illustration 2:**

Before the conversation in Excerpt 2 began, the teacher asked students to express their ways of thinking about how they could determine the initial value of the function. One of the students expressed that the initial value could be determined by finding the 1-unit growth factor first. The teacher followed by asking the student to express how she determined the 1-unit growth factor. This response suggests that the teacher was interested in understanding the student's meaning of a 1-unit growth factor. The student then explained that she determined the 1-unit growth factor by dividing 122.07 by 97.66. The teacher probed the student about the fraction ($\frac{122.07}{97.66} \cong 1.25$) by saying, "Take a second and looked at this fraction. Is there anything standing out about this fraction?" The teacher's decision to focus students' attention on the value of the growth factor appeared to be for the purpose of getting students to see that a growth factor of 1.25 is not reasonable. The teacher's question led the students to realize that the ratio ($\frac{97.66}{122.07} \cong 0.8$) would produce a reasonable growth for scaling. During this interchange the teacher's interest in supporting students' thinking resulted in him helping the student confront her weak meaning for growth factor. His questions appeared to be based on his understanding of the student's problematic way of thinking. Then the teacher prompted the student to consider how to approach finding the function's initial value (Excerpt 2).

Excerpt 2

[Line1] *Teacher*: How do we go from having a 1-year growth factor to confirming that our initial value is 50?
[Line2] *Student1*: What I did was I just divided .8 the 97.66 so then I kept going down three, three downs until my input is 0.
[Line3] *Teacher*: Ok. So you said you multiplied or divided by a 0.8?
[Line4] *Student1*: Divided
[Line5] *Teacher*: So you did like 97.66/0.8. What are you computing with that?
[Line6] *Student1*: The initial value when the input is -2.
[Line7] *Teacher*: So, to find $f$(-2) we take $f$(-3), which is 97.66 and divide by 0.8. What do you think?
[Line8] *Student2*: I don't know why this happened but you plug this in you get to 122.07
[Line9] *Teacher*: So you're saying that if you compute this value, you get 122.07?
[Line10] *Student2*: Yeah.

[Line11] *Teacher*: Ok. So, if we put that in your calculator we should all get this 122.07. Why is that happening? You should recognize that number, because it's $f$(-4), right? Why we're getting $f$(-4) back when we do this computation? Student3, what do you think?

[Line12] *Student3*: When you divide the output by the growth … bigger…so instead we need to multiply.

We classified this interaction between the teacher and students at Level 3. We observed that the teacher initially prompted students to explain how to find the initial value of the exponential function using the 1-unit growth factor (Line 1). The teacher prompted one student to explain his approach; he then asked the student to provide a rationale for his approach (Line 5), demonstrating that he was interested in understanding how the student was thinking. When the student replied by saying that he was finding the initial value when the input is -2, the teacher followed by re-expressing the student's explanation; his explanation (Line 7) suggests that he understood how the student was thinking. He continued by posing questions to reveal how the student was thinking (e.g., Why are we getting $f$(-4) back when we do this computation?, [Line 11]).

## Discussion and Conclusion

Prior research has characterized teachers' attempts to understand students' thinking, including how they respond when students express their thinking and whether they take student thinking into consideration during teaching. Researchers have used the idea of decentering as a theoretical lens to make inferences about a teacher's ability to make sense of student thinking (Teuscher et al., 2016). Piaget's construct of decentering has been considered as a powerful lens for researchers when focusing on how teachers build models of students' thinking and to what degree they use student thinking to make instructional decisions (Teuscher et al., 2016). Moreover, Thompson's (2013) conceptualization of a productive interaction between two people (i.e., the interaction where "each participant is oriented to understand what others have in mind and is oriented to have others understand what he or she intends" (p. 63)) extends the idea of decentering. There are also studies in which different levels of a teacher's decentering actions during his or her interaction with students are characterized based on this theoretical perspective (Carlson et al., 2007; Marfai et al., 2011; Teuscher et al., 2016). This study extends these research efforts by introducing a framework that provides a fine-grained characterization of teacher-student interactions. The framework describes two levels of a teacher's mental actions (i.e., non-decentered and decentered) and four levels of the teacher's observable behaviors that are associated with both non-decentered and decentered actions. The levels in the framework will be useful for both researchers and teacher professional developers by illuminating subtle and productive ways in which a teacher can leverage student thinking when interacting with students.

Studies also point out that all teaching actions are strongly related to the teachers' mathematical meanings for teaching (Thompson, 2013; Thompson, Carlson & Silverman, 2007). In Thompson's (2015) view, teachers' mathematical meanings for teaching are the main sources of their instructional decisions and actions. While this study does not examine how a teacher's meanings for a mathematical idea influence the quality of his or her decentering actions, we will investigate the relationship between teachers' mathematical meanings and their decentering actions in future research.

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# Developing Strategic Competence With Representations for Growth Modeling in Calculus

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*Using inquiry based modules centered around growth modeling, we study the development of strategic competence and representational fluency in undergraduate calculus. Building on student experiences and using multiple representations with discrete and continuous methods, we discuss the emerging substantial and problematic practices with representational fluency, communication, and strategic competence for modeling growth.*

*Key words:* Representational Fluency, Strategic Competence, Calculus, Differential Equations, Modeling

It has been suggested that mathematical modeling should be taught at every level of mathematics education (GAIMME, 2016), however successful modeling of realistic problems, like population dynamics, in STEM related fields requires students to achieve high levels of mathematical proficiency. The National Research Council defines mathematical proficiency as having five components, or interwoven strands: 1. conceptual understanding - comprehension of mathematical concepts, operations, and relations. 2. procedural fluency - skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. 3. strategic competence - the ability to formulate, represent, and solve mathematical problems 4. adaptive reasoning - capacity for logical thought, reflection, explanation, and justification. 5. productive disposition - habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (NRC, 2001).

A crucial part of mathematical literacy, representational fluency refers to the ability to represent mathematical ideas with different representations, to translate these ideas across representations, to gain understanding about the underlying entities that are being represented, and to generalize across representations (Zbiek et al. 2007). It requires a metacognitive perspective requiring knowledge and synthesis beyond the representations themselves. This perspective was expressed by Sigel and Cocking as the ability to comprehend the equivalence of different modes of representation (Sigel and Cocking, 1977) after one can transfer information from one representation to another.

Despite the need for and benefits of representational fluency (e.g., Kaput, 1989), there is relatively little known about the calculus student's ability to solve problems when presented with different representations, or to translate ideas among different representations. Studies have reported that students have difficulties linking different representations and moving flexibly between representations (Even, 1998; Janvier, 1987). For example, researchers observed that calculus students were often comfortable with different results in different representations without realizing the inconsistency of the results (Ferrini-Mundy and Graham, 1993). Some researchers noted that students may link representations without an understanding of the deeper conceptual links between them (Greer & Harel, 1998). Beyond an equivalence perspective amongst representations, there is a need for a deeper look into how representational fluency translates to improved mathematical proficiency and strategic competence. More pointedly, little is known about how fluency among representations across discrete and continuous mathematics contribute to mathematical proficiency. Even (1998) highlighted that there is "not much known about the nature of the processes involved in working with different representations," despite agreement among mathematics educators about their importance in learning mathematics.

## Objectives and Research Questions

In this paper we discuss the collaborative action research of two mathematics faculty members with the goal of improving the practice of teaching calculus(Stinger, 2014). We infused collaboratively planned and purposefully designed inquiry based activities into a two semester freshman calculus sequence. Our activities were designed to provide opportunities for students to experience fluency with multiple representations from both a discrete and continuous perspective while investigating population growth modeling. Ultimately, we hope to further the development of both the strategic competence and the representational fluency in our students, and in doing so, to make the content of growth modeling more accessible for our student population. We both observed that this content is otherwise problematic with the traditional integration methods. Our main research questions are:

1. How do calculus students develop representational fluency when modeling population dynamics with an enriched instruction on discrete methods?
2. How do calculus students develop strategic competence when modeling population growth, specifically when they learn to connect complimentary discrete and continuous concepts, such as differential and difference equations?

## Conceptual Framework for Representational Fluency in Growth Modeling in Calculus

Multiple external representations traditionally associated with mathematics have been outlined by many authors (Lesh, Post, and Behr, 1987; Kaput 1998; Kendal 2003); in this paper, we will refer to five different modes: Graphical, Algebraic, Verbal, Manipulative Models, and Real Life Scenarios (see Figure 1). Aligning with Kaput and Lesh, we take special care to incorporate real life scenarios and manipulative models, extending beyond just the big three representations.

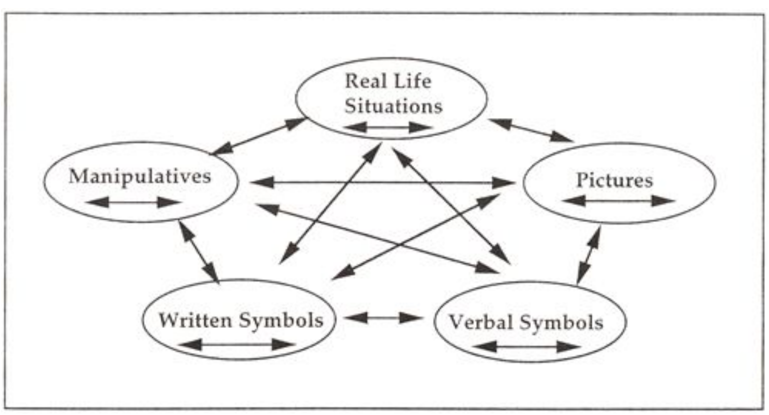


*Figure 1.* Lesh et al.'s model depicting five representational modes with Real Life Situations, Pictures/graphs, Written Symbols, Manipulatives/digital/concrete models, Verbal Symbols.

Each of the different representational modes affords the student different opportunities for mathematical insight. Advancements in technologies, the ease and availability of graphing calculators, and computer algebra systems now allow differentiation and integration to be easily calculated using numerical and graphical representations. Of course, these numerical and graphical solutions are primarily at a point or within an interval, rather than a global solution, as can be often found with the traditional analytical approach that relies on symbolic representations and algebraic manipulations.

In the context of calculus, and more specifically, growth modeling, students can demonstrate strategic competence by formulating modeling problems, by representing them with multiple representations, and by choosing flexibly among discrete or continuous methods to suit the demands of the mathematical content. Adaptive reasoning, on the other hand, refers to the capacity to think critically about the relationships among concepts and situations. Adaptive reasoning is the meta-cognitive leap to assess the fitness of the method and the adequacy of representations to provide the insight into problem in its realistic context.

We build on Rasmussen and Kwon's (2007) approach to inquiry based undergraduate mathematics by engaging our students in cognitively demanding tasks that prompt the exploration of important mathematical relationships and concepts, by orchestrating mathematical discussions in class and in small groups, by developing and testing conjectures, and by having students explain and justifying their thinking. Following an inquiry approach, we continually build upon, refine and expand our questions on population dynamics as we introduce new concepts in calculus. For example, we revisit population dynamics and present modeling opportunities at each step as we progress through major topics such as rates of change, anti-differentiation, and differential equations.

## Methods and Setting

In Calculus I and II, we integrated both differential and difference equations as major components with instructors devoting approximately four weeks in each semester to these topics. Realistic scenarios were built around population growth, which was used as a cross-cutting theme that permeates across courses for the same group of students. The inquiry based modules that we infused into the calculus sequence emphasized discrete approaches to problems traditionally approached from a continuous perspective. The researchers collaboratively designed the modules used for this study since 2013. Our students were tasked with solving difficult problems in small groups by utilizing visual, analytical and verbal representations. Activities were purposely designed with the main goals of i. creating a more balanced approach to calculus with discrete and continuous methods; ii. Enhancing representational fluency; iii. Developing strategic competence.

We used multiple data sources, including analyzing student work, student reflections, and student discussions in an attempt to observe the student's representational fluency and strategic competence during the activities. The researchers also noted their observations and reflections on student behaviour and practices in follow-up discussions. Data was collected from students during the Fall and Spring Semesters of 2016 and 2017; in total, there were 23 students in Calculus I and 19 students in Calculus II.

## An Integrated Calculus Instruction

As previously mentioned, the instructors spent approximately four weeks each semester engaging in inquiry based modeling activities focusing on discrete and continuous representations of population growth. For illustrative purposes, we offer a short description of two of the modeling activities we used, one from Calculus I and one from Calculus II. Aligning with recommendations from GAIMEE, we encourage our modeling problems to be approached in an open-ended manner to allow for the possibility of student conjecturing, exploration and investigation.

### A Growth Modeling Activity in Calculus I

Students are presented with a modelling scenario involving the growth a fruit fly population, which was inspired by a similar problem in Thomas' Calculus(2014) that builds the idea of derivative from the rates of change of a logistic model given visually and numerically.

> *Imagine that one day a rotting apple in your kitchen has attracted some fruit flies. Suppose that on that day you count two fruit flies. You (unwisely) leave you home for 50 days, leaving the apple on your counter. When you return, the fly population has grown by 350 flies.*

We introduce alternative growth models before discussing the rate of change behavior for a logistic curve, not only with continuous but also with discrete methods. Our goal is to have students explore the given real world scenario and develop various models that can represent the growth of the population over the 50 day period, based on the assumptions that they formulate. The instructor ensures that the students represent their idea using multiple representation modes. In this case, most students are initially drawn to familiar continuous representations of linear growth, such as the (continuous) algebraic representation: $y = 7x+2$, the (continuous) graphical representation: a linear graph, and the verbal description of "a growth of 7 flies each day." If they choose this continuous approach, they are required to demonstrate their model using graphing technology (Geogebra or similar). The instructor asks questions which require manipulation of their model under different conditions (different initial population or growth rate, etc.). In our case, all students began the activity using this continuous approach.

Once they have successfully modeled linear growth with continuous methods, they are challenged to represent the growth using discrete methods. The students must now transfer ideas laterally among the same representation modes; for example, the represent growth algebraically with a difference equation: $P_{n+1} = P_n +7$, graphically with a scatter plot, and they are asked to use a manipulative model such as Microsoft Excel to experiment with different parameters.

Students become aware of the limitations of the linear model and initiate investigations into other models, which we direct towards exponential and logistic growth. Once again, students must represent their ideas using algebraic equations, graphical images (see Table 1 below for more detail), and they must utilize manipulative models that can account for the changing of initial conditions. They are free to initiate either a discrete or continuous approach to their models, but through group collaboration, discussion, and reflection, all groups eventually see how these ideas can be modeled from both perspectives. Unifying questions relating the rates of change and the changes in the rates of change emphasize the complementary nature of the discrete and continuous approaches, and discussions involving the difficulties encountered by some approaches emphasize the importance of flexibility and strategic choice.

**Continuing Growth Modeling Activity in Calculus II**

In the second semester of Calculus, while studying first-order differential equations, students are presented with another modeling scenario involving a locally relevant invasive lionfish population:

> *Biologists have determined that a coral reef can safely sustain a population of 350 or fewer lionfish; however, once the population exceeds 350, irreversible damage will be done to the ecosystem.*

Once again aligning with GAIMEE recommendations, we allow students to formulate their own questions and ideas to investigate these scenarios. In this case, the instructors steered the students towards suggesting a harvesting strategy to keep the fish population below the threshold of 350. In previous modeling activities, students discovered a carrying capacity of 850 lionfish, and they proceed under that constraint. They make assumptions, such as an initial population, and the frequency of their harvesting expeditions, and proceed to answer questions like: *How many fish do we need to harvest if we send an expedition once every 6 months?* A continuous approach leads to representations like algebraic differential equations: $\frac{dy}{dx} =- 0.25y(1 - \frac{y}{850})$, continuous solution curves and slope fields, and manipulative models like slope field generators in GeoGebra. A discrete approach has students transfer between the difference equation: $P_{n+1} = 1.25P_n - \frac{0.25}{850} P_n^{\ 2}$, and the graphical scatter plots made using Microsoft Excel (or similar), with which they can experiment with different parameter values. Ideas are summarized and presented to the class, so that discussion can ensue on the pros and cons of the different approaches.

Without including the graphical representations, we provide descriptions of the basic growth models introduced in modeling the population dynamics.

*Table 1.*
Summary for the Models for Population Dynamics

| *Underlying Math Models for Growth* | *Contextual/ Verbal* | *Symbolic- Discrete* | *Symbolic Differential* |
|---|---|---|---|
| Linear | Constant Change | $\Delta P_n = P_n - P_{n-1} = k$ | $\frac{dP}{dt} = k$ |
| Exponential | Unbounded | $\Delta P_n = (a-1)P_{n-1}$ | $\frac{dP}{dt} = (a-1)P$ |
| Logistic | Limited Capacity | $\Delta P_n = mP_{n-1}(1 - \frac{P_{n-1}}{C})$ | $\frac{dP}{dt} = mP(1 - \frac{P}{C})$ |

## Observations and Results

Our observations suggest that the enhanced treatment of growth modeling with a balanced focus on discrete and continuous methods can improve the development of representational fluency and strategic competence in participants. On several occasions, we observed what we perceived as higher than usual student growth in the ability to transfer ideas across traditional representational paths, such as from continuous equations to continuous graphical representations. For instance, the majority of students (72%) that were unable to correctly connect a differential equation to its direction field prior to our modeling activities were able to successfully do so afterwards. The assessment question used in this case is seen in Figure 2.

*Which of the following equations is the differential equation whose slope field is shown below?*

- *a) y'=3-y-x*
- *b) y'=3+y-x*
- *c) y'=3-y+x*
- *d) y'=3+y+x*
- *e) y'=3-yx*

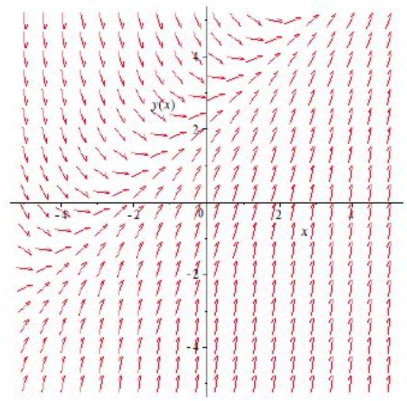

*Figure 2.* Slope field assessment task.

In addition, we also observed that upon the completion of our course, students were demonstrating an enhanced flexibly in choosing among discrete or continuous methods that best suited the problem at-hand. In our initial assessments, students would largely prefer continuous approaches, regardless of the comparative difficulty of discrete approaches. For example, in our unit on arc length, students were tasked with the well-known problem of finding the length of the Golden Gate Bridge, which is modeled with the equation $y = \frac{x^2}{8820} - \frac{10x}{21} + 500$, but only to within 10 feet of accuracy. We observed that 4 of 5 groups pursued an exact solution via the continuous integration formula, whereas the remaining group solved the problem using a line segment approximation. We note that students had practiced such approximations recently. In fact, all four groups were unable to solve to the continuous integral, and resigned the problem rather than switch approaches. After completing our modeling exercises, students attempted the following question:

> *The population of lionfish in a water column above a coral reef near Buck Island is given by* $\frac{dy}{dx} =- 0.15y(1 - 0.06375y + \frac{y^2}{12800})$ *where y is the population in lionfish and x is in years. Biologists determine that the reef can safely sustain a population of 350 or fewer lionfish, but once the population exceeds 350 irreversible damage will be done to the ecosystem. A diving survey team estimates a current population of 180 lionfish. After approximately how many months will the population equal 350?*

This time, the majority of the groups (4 out of 5) used a discrete approach (Euler's method) for their initial strategy; whereas the remaining group began with a continuous approach, but were able to switch the the discrete method after some initial failure. We further make note of our observation of what we perceived to be better than expected results in the student's ability to formulate and solve modeling problems. Groups engaged in the harvesting exercises demonstrated more mathematical autonomy and independence in completing their assigned tasks. It was clear that student's strategic competence became amply evident in growth modeling tasks when the instruction allows student experimentation with manipulatives, such as the dynamic spreadsheets that blend the numerical or graphical representations. Our students performance exceeded our expectations with their problem formulating skills, their critical thinking in the creation of their models, and their suggestions for harvesting schemes for population control. They also seemed to become more productive and reflective after strategic choices of visual representations, such as flow diagrams, substantially empowered them towards a dynamic sense of the global behavior of solution curves under different initial conditions.

By the culmination of the activity sequence, we observed students development in both the cognitive and content related skills in calculus, such as representational fluency and building connections between discrete and continuous methods in modeling growth. Our final remark is that the additional fluency involving the discrete representational forms emerged in a critical capacity as providing certain students access to deeper mathematical ideas that were inaccessible to them from a continuous standpoint. We observed that several students had difficulty solving growth problems analytically, in particular, when modeling logistic growth, however, they ultimately overcame their earlier problems producing solution curves algebraically when asked to use traditional integration techniques in calculus. Most of our students who struggle with difficult concepts in topics like differential equations were more able to experience success with this approach, as exemplified in the harvesting activity outlined above. Their exercised ability to use spreadsheets allowed even the weakest students to see the impacts of harvesting at set time periods clearly.

## Acknowledgments

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# Building on Covariation: Making Explicit Four Types of "Multivariation"

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*Covariation and covariational reasoning have become key themes in mathematics education research. In this theoretical paper, I build on the construct of covariation by considering cases where more than two variables relate to each other, in what can be called "multivariation." I share the results of a conceptual analysis that led to the identification of four distinct types of multivariation: independent, dependent, nested, and vector. I also describe a second conceptual analysis in which I took the mental actions of relationship, increase/decrease, and amount from the covariational reasoning framework, and imagined what analogous mental actions might be for each of these types of multivariation. These conceptual analyses are useful in order to scaffold future empirical work in creating a complete multivariational reasoning framework.*

*Key words:* covariation, multivariation, reasoning, mental actions

The construct of covariation and the cognitive activities involved in reasoning about it have become important themes within mathematics education research (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, Paoletti, & Musgrave, 2013; Moore, Stevens, Paoletti, & Hobson, 2016; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994). Yet, work on *co*-variational reasoning has essentially been limited to examining two variables changing in tandem with each other, perhaps with time as a mediator to that relationship (explicitly or implicitly). By contrast, as students continue to higher levels of science and mathematics courses, they encounter contexts in which there are more than two variables potentially changing in relation with one another. Note that I use the term "variable" in this paper to generally mean any potentially varying numeric value, including values of real-world quantities and mathematical function inputs and outputs. This theoretical paper is meant to build on the construct of covariation by explicitly considering cases where more than two variables (in addition to time) relate to and change with one another, in what can be termed "*multivariation*." In particular, I share the results of a conceptual analysis in which I identified four different types of multivariation, each with its own potential set of mental actions for reasoning about it. I also share the results of a second conceptual analysis examining what "*multivariational reasoning*" might possibly look like for each type, in terms of analogous mental actions corresponding to those already documented for two-variable covariational reasoning. The results of these conceptual analyses are meant to scaffold future empirical work, by helping to inform study design and data analysis, which can be used to establish a complete multivariational reasoning framework.

## Covariation and Multivariation

Over the past couple decades several researchers have been contributing to a carefully developed sense of what "covariation" means (Carlson et al., 2002; Castillo-Garsow, 2012; Confrey & Smith, 1995; Johnson, 2012; Saldanha & Thompson, 1998). The central theme to this work is that covariation consists of imagining "two quantities [i.e., variables] changing together" (Castillo-Garsow, 2012, p. 55) in which "they are changing simultaneously and interdependently" (Johnson, 2012, p. 315). The specific term "quantity" often has additional meaning beyond being only a numeric value, and usually implies a measurable quality of an object (Thompson, 1994). However, covariation has also been applied to purely mathematical functions that are not necessarily contextualized as relationships between physical quantities (Oehrtman et al., 2008; Thompson & Silverman, 2008). In this paper, I consider covariation of variables both in terms of physical quantities and mathematical functions.

An important part of covariation, regardless of the variables involved, is the concept of time (Castillo-Garsow, 2012; Oehrtman et al., 2008; Thompson, 2011). Sometimes time can be explicitly present in the covariation as one of the two real-world quantities, such as distance and time. However, even for two non-time variables, or for a mathematical function, $y = f(x)$, if one applies "smooth" covariational reasoning (see Castillo-Garsow, 2012), time must necessarily be involved in imagining the change in progress. The necessity of time resonates with the assertion in Oehrtman et al. (2008) that, "The idea of covariation is fundamentally that of parametric functions" (p. 38). Thus, for my purposes, if a variable "A" is said to be varying (or covarying with another variable), it can be thought of as changing in time, $\mathrm{A}(t)$. However, this change does not have to happen linearly in "real" time, but can be conceptualized to move forward quickly or slowly, or to move in reverse, or to pause at a given instant.

**Multivariation in the Current Literature and Conceptual Analyses**

I began thinking about the construct of "multivariation" during a study involving the limits of complicated expressions (Jones, 2015) and another study involving multiple, line, and vector integrals (Jones & Naranjo, 2017). These ideas were further stoked when I encountered the "partial derivative machine" at a RUME conference, in which it is not always possible to hold certain variables "constant" in order to use basic covariation (see Roundy et al., 2015). I also began to see in mathematics and science textbooks how many instances there were in which multivariation could be involved. I wish to make clear that I am in no way claiming to be the inventor of the notion of multivariation, and that ideas surrounding multivariation have, in fact, been present in the mathematics education research literature, including studies on the graphs of multivariate functions (e.g., Dorko & Weber, 2014; Martinez-Planell & Trigueros-Gaisman, 2012; Weber & Thompson, 2014), on partial and directional derivatives (Bucy, Thompson, & Mountcastle, 2007; Martinez-Planell, Trigueros-Gaisman, & McGee, 2014, 2015), and on multiple integrals (McGee & Martinez-Planell, 2014). However, the main reason I believe this paper is needed is that while ideas pertaining to multivariation are present in the literature, multivariation as a construct in and of itself has essentially been implicit. Thus, there is still a need to explicitly discuss what *multivariation* and *multivariational reasoning* might consist of.

This theoretical report consists of the products of two conceptual analyses (see Thompson, 2008) meant to form the basis of future empirical work. The first conceptual analysis, presented in this section, focuses on what possible types of multivariation might exist. (The second analysis is described in the next section). To perform it, I looked through a large set of mathematics, science, and engineering functions and formulas, found mostly inside textbooks (e.g., Hibbeler, 2012; Serway & Jewett, 2008; Stewart, 2015), and considered how the variables in them could be conceptualized as changing with respect to one another. This conceptual analysis led to the identification of four distinct types of multivariation: *independent, dependent, nested,* and *vector*.

**Four Types of Multivariation**

Here I describe the four ways I identified that more than two non-time variables might be "changing together" in a potentially "simultaneous and interdependent" way (Castillo-Garsow, 2012; Johnson, 2012). I have stipulated non-time variables precisely because time is inherent in all types of variation, as discussed previously, whether univariation, covariation, or multivariation. Thus, time-parametric equations are not considered a separate type of multivariation, since they are already inherent in all types.

**Independent multivariation**. The first type of multivariation I describe, *independent multivariation*, is probably the most commonly imagined type of multivariation in mathematics because of how we often work with multivariate functions, like $z = f(x,y)$. In this type, there are multiple “input” variables (e.g., $x$ and $y$) that each individually covary with an “output” variable (e.g., $z$), but where the “input” variables need not covary with each other. In other words, the covariations between each input variable with the output variable can be conceptualized as *independent* from each other. In contrast to covariation, a change in the output does not necessarily imply a change in one particular input, since the change in output could have happened as a result of covariation with a separate input variable. Next, I note that what counts as “input” and “output” does not necessarily need to be fixed (e.g., solving to get $x = f(y,z)$), so long as the covariations between each of the input variables and the output variable remain independent. I also note that this type of multivariation could be extended to include as many input variables as desired, such as for the function $z = f(x_1,x_2,\ldots,x_n)$.

Since each input variable covaries with the output variable, it might be tempting to think of this type of multivariation as simply basic covariation by holding all but one of the input variables constant at a time. While that can be true, what makes this distinct from two-variable covariation is that it is, in fact, possible to imagine *all* of the input variables changing at the same time, each having their own impact on the output variable. This is similar to the idea of directional derivatives (see Martinez-Planell et al., 2015), or to taking a surface defined by $z = f(x,y)$ and tracing out a curve on it by parameterizing $x(t)$ and $y(t)$ over the interval $a \leq t \leq b$.

This type of multivariation is present in many science contexts involving real-world quantities. The key is whether it is realistically and conceptually reasonable to hold certain variables constant while varying others. For example, force (an output variable) can be defined as the product of mass and acceleration (the input variables), as in $F = ma$. In this case, one can imagine holding $m$ constant and changing $a$ to produce changes in $F$, or holding $a$ constant and changing $m$. Yet, what makes this “multivariation” rather than “covariation” is that $m$ and $a$ could be imagined to be changing simultaneously, yet independently, each producing concurrent changes in $F$. Note that $m$ and $a$ do not have to be the input variables, since one could imagine holding $F$ constant and changing $m$ to produce changes in $a$.

**Dependent multivariation.** The second type of multivariation, *dependent multivariation*, more commonly arises in real-world contexts, since input variables for mathematical functions are typically conceptualized as, literally, “independent variables.” However, for certain scientific contexts it might not make sense to conceive of holding some variables constant while the others vary. In fact, some science educators have already brought up this idea, since “holding constant” is not always possible (e.g., see Bucy et al., 2007; Roundy et al., 2015). The main idea for this type of multivariation is that, rather than having several independent covariations between multiple “input” variables and a single “output” variable, a change in any variable produces simultaneous changes in *all* other variables. Further, as those other variables change, they also immediately induce changes in all other variables in the system.

For example, if a fixed amount of gas is contained in a flexible balloon, the ideal gas law models the relationship between the volume, $V$, pressure, $P$, and temperature, $T$, of the gas through the equation $PV= kT$. However, unless certain laboratory conditions are imposed, it might not be realistic to hold $P$ constant while $T$ and $V$ change with respect to each other. More realistically, if the temperature increases, the pressure and volume both increase simultaneously and their changes can feed back into the system immediately. Or, to pull from a rather different context, suppose an economist is examining how price, affected by demand and supply, is changing for a particular good in a market that is in flux. Again, it might not be realistic to imagine holding demand constant in order to manipulate supply and measure the corresponding changes in price. As the supply changes, both price and demand may change simultaneously as the market approaches a new equilibrium.

To be clear, in this type of multivariation, I am not saying that it is not *mathematically* possible to hold one of the variables constant in order to enact calculations. However, my point is that these types of contexts cannot conceptually be fully accounted for *only* through multiple independent two-variable covariations. Rather, one would have to use mental actions that involve multiple variables all having simultaneous impacts on each other in order to reason accurately about the real-world processes.

**Nested multivariation.** The third type of multivariation I describe, *nested multivariation*, comes from how one might conceptualize changes when the relationships between variables are based on the structure of function composition, such as $z(y(x))$ (for more on student understanding of function composition, see Ayers, Davis, Dubinsky, & Lewin, 1988; Breidenbach, Dubinsky, Hawks, & Nichols, 1992). For $z(y(x))$, if one imagines changes in $x$, then there are corresponding changes to $y$. Yet those changes in $y$ now correspond to changes in $z$. While it is true that one can, in fact, think of direct two-variable covariation between $x$ and $z$, nested multivariation conceptualizes the relationship as having intermediary variables. Thus, the difference between whether it is two-variable covariation or nested-variable multivariation is *not* inherently dependent on the structure of the formula or function. Rather, it is necessarily a product of how one *conceptualizes* the changes taking place. For example, for the equation $y = \sin^2(x)$, it is true that one can imagine $x$ and $y$ changing directly with each other. However, it is also possible to imagine that as $x$ increases, from say 0 to $\pi/2$, it produces corresponding increases in the value of "$\sin(x)$," and that as the value of $\sin(x)$ increases, it in turn generates increases to the "$\sin^2(x)$" values. In other words, as one variable changes it induces a change in a second, which induces a change in a third variable (and potentially so on to include as many variables as desired).

To describe an example from science, consider the formula from relativity relating velocity, $v$, with the relative mass of an object, $m$, given by $m = m_o \Big/ \sqrt{1-(v/c)^2}$ ($c$ is the speed of light and $m_o$ is the relative resting mass). When I have asked students to describe what happens to mass as $v$ approaches $c$, they tended to think through this formula piece by piece. They would first discuss how an increasing $v$ made the ratio between $v$ and $c$ approach one. They would then discuss how that corresponded to $\sqrt{1-(v/c)^2}$ shrinking to zero, which lastly made the value of the entire expression tend toward infinity. To represent their thinking in mathematical notation, they essentially thought of the mass equation broken down into a ratio function, $\beta(v) = v/c$, which became an input for the Lorenz factor, $\gamma(\beta) = 1\Big/\sqrt{1-\beta^2}$, which in turn became the input for the mass, $m(\gamma) = m_o\gamma$. As explained previously, it is true that one can think of direct covariation between $v$ and $m$. If one does so, then in that case they are employing covariational reasoning. However, if they imagine nested changes from $v$ to $\beta$ to $\gamma$ and finally to $m$, then I argue they are employing nested multivariation reasoning.

**Vector multivariation.** The last type of multivariation I describe, *vector multivariation*, may be the most cognitively complex and gets its name because it deals with multiple independent inputs each simultaneously associated with multiple independent outputs (i.e. a vector function).

Thus, vector multivariation is essentially a generalized version of independent multivariation in that it consists of several *independent multivariations* each happening *independently* of each other. For a vector function, $\vec{F}(x,y)=\langle u(x,y),\, v(x,y)\rangle$, like with independent multivariation, one can think of holding, say, $y$ constant and letting $x$ vary, but in this case that variation corresponds to changes in both $u$ and $v$ at the same time. Further, imagining both $x$ *and* $y$ varying simultaneously leads to four pairs of independent covariations that could potentially need to be cognitively managed all together. As with all other types of multivariation, vector multivariation could be extended to include as many variables as desired, including several input or several output variables.

For examples of vector multivariation, consider a vector field mapping $\mathbf{R}^2$ to $\mathbf{R}^2$. If one were to take a starting point $(x,y)$ and increase the $x$-coordinate, tracing a horizontal line through the vector field, both the horizontal and vertical components of the vector field could be changing simultaneously. Similarly, if one increased the $y$-coordinate and traced vertically through the vector field, both components of the vector field could change. Now, for full vector multivariation, if one traced out a curve, $C$, in the $x$-$y$ plane along which both $x$ and $y$ are changing simultaneously, one would have to coordinate how much $x$ and $y$ are each changing, and what the resulting changes in the horizontal and vertical components of the vectors are. This type of multivariation shows up in vector integrals, $\int_C \vec{V}\cdot d\vec{r}$, and also occurs for functions with complex inputs and outputs, $f: C \rightarrow C$. For complex functions, if the input complex variable changes along a curve in the complex plane, one would have to simultaneously attend to changes in the real and imaginary parts of the input variable, as well as the changes in the real and imaginary parts of the output variable. In science, this type of multivariation could be present in any context involving vector spaces, such as gravitational fields or electrical fields. One could imagine a particle tracing some path through those fields, with changes happening in each of the vector components as the path is traced out.

**Comparison of structures**. To summarize this first conceptual analysis, Figure 1 shows the distinct conceptual structures for the four different types of multivariation (and covariation). Of course, each type of multivariation could be extended to include as many variables as desired.

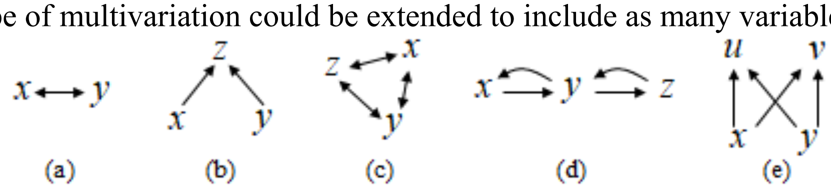


*Figure 1. Comparison of structures for (a) basic covariation, (b) independent multivariation, (c) dependent multivariation, (d) nested multivariation, and (e) vector multivariation, where (b)–(d) could each be extended to include as many variables as desired.*

## Covariational Reasoning and Multivariational Reasoning

To describe the second conceptual analysis, I briefly return to basic two-variable covariation. Carlson et al. (2002) described five mental actions that pertain to increasingly sophisticated levels of covariational reasoning. The first three mental action levels are given as (p. 357): (1) "Coordinating the value of one variable with changes in another," (2) "Coordinating the direction of change [i.e., increase or decrease] of one variable with changes in the other variable," and (3) "Coordinating the amount of change of one variable with changes in the other variable." For my purposes, I label these three mental actions as "relationship," "increase/decrease," and "amount." The fourth and fifth mental action levels then progress to changing rates of change, marking a shift from reasoning about the two variables directly to reasoning about how a rate of change itself varies. For my conceptual analysis, I focused on what mental actions for each type of multivariation might be analogous to the *relationship*, *increase/decrease*, and *amount* mental actions from covariation. I do not include changing rates of change in this conceptual analysis at this point because of the potential complexity of multiple simultaneous changing rates of change. Rather, my conceptual analysis focuses on providing an initial step into how one might imagine the variables themselves in the system and their direct relationships with each other.

**Analogous Multivariational Reasoning Mental Actions**

Here I describe the potential mental actions of multivariational reasoning that might be analogous to relationship, increase/decrease, and amount from covariation. This "thought experiment" is intended to scaffold possible empirical methods aimed at examining the nature of multivariational reasoning, by imagining beforehand what cognitive activities might specifically be targeted in empirical research.

First, what might be the mental actions in *independent multivariational reasoning* analogous to relationship, increase/decrease, and amount? The first mental action would likely consist of a realization that multiple input variables may impact a single output variable, and that each change may be happening in isolation or simultaneously. In thinking of the surface defined by the graph of $z = f(x,y)$, it would be the realization that one can trace a path along this surface in *any* direction, freely. The next mental action may consist of coordinating each individual change in the input variables to an overall net *directional change* for the *inputs*. In the case of $z = f(x,y)$, this would be congruent to imagining a "change vector," $\Delta V$, whose components are $<\Delta x,\Delta y>$, though I use the word "vector" for convenience and note that a student would likely not conceptualize it as an actual "vector." In contrast to the second level of covariational reasoning, where there is already attention to whether the output increases or decreases, I hypothesize that this is a required preliminary mental action for independent multivariational reasoning, not yet involving "increase/decrease." That is, it may be required to simply identify the direction of the change vector *before* determining whether the output increases or decreases along it. It would then be a separate mental action in which one would coordinate this *change vector* with whether the output variable increases or decreases. Thus, we can see additional sophistication in independent multivariation reasoning above what is required for two-variable covariational reasoning. Only after these three mental actions would a fourth mental action coordinate the *amount* of change in the output variable along the direction of this change vector.

Next, consider *dependent multivariational reasoning*. Here, the first mental action would likely consist of the coordination of a change in one variable with simultaneous and interdependent changes in all other variables in the system. That is, it would be the realization that some variables cannot be held constant in a realistic way and that a system may only be understood by imagining all variables changing interdependently. The second mental action might then consist of coordinating the change in one variable with whether each of the other variables increases and/or decreases. This mental action is quite sophisticated, since one must coordinate interdependent increases and decreases, meaning it may even consist of separate mental actions. For example, in the balloon context, increasing temperature would mean an increase in pressure, but the fact that the volume also increases means that the pressure would not increase by as much as would be predicted if volume were able to be held constant. The next mental action would consist of coordinating the change in one variable with the *amount* by which each of the variables in the system interdependently change as a result.

For *nested multivariational reasoning*, each mental action essentially deals with *chained* reasoning. The first mental action would involve coordinating a chain of changes from one variable to the next. It would be the realization that a change in one variable would have effects on a *sequence* of other variables. The second mental action may consist of coordinating the change in the first variable with whether the second variable increases or decreases, and coordinating whether increases or decreases in the second correspond with increases or decreases in the third, and so on. A possible metaphor is a sequence of gears where one attends to how a rotation in the first induces rotations on the others. Again, this may actually represent several separate mental actions. The next mental action would follow this same chain, but would coordinate *how much* each variable in the sequence increases or decreases.

For *vector multivariational reasoning*, the first mental action may consist of coordinating changes among several input variables with changes among several output variables. It would be the realization that multiple input variables can impact multiple output variables, in isolation or simultaneously. The second mental action, like with independent multivariation, would likely not deal with whether the output variables are increasing or decreasing, but would consist of a preliminary mental action of coordinating the changes in the input variables to form an overall "input change vector," $\Delta V_{in}$. This input change vector defines the direction along which the change is happening. I believe that it may then require several mental actions to achieve complete analogs to the increase/decrease and amount mental actions. The first of these would be to coordinate whether each output variable increases or decreases in the direction of the input change vector. The second would be to coordinate the amount of change in each of the output variables. The third would be to coordinate the changes in each of the output variables to create an overall "output change vector," $\Delta V_{out}$. These may then culminate into a fourth mental action that directly coordinates the *input* change vector, $\Delta V_{in}$, with the *output* change vector, $\Delta V_{out}$.

Lastly, I note that for independent and vector multivariation, it is possible to consider the direction of change first, such as imagining tracing along a curve, $C$ (like in line and vector integrals). In this case, some of the mental actions may reverse, and rather than construct the input change vector from changes in the inputs, the mental actions might consist of *decomposing* the change vector into changes in the inputs.

## Conclusion

In this paper I described conceptual analyses into different types of multivariation. I also described mental actions potentially associated with each type of multivariational reasoning and how they might be different from each other and from two-variable covariational reasoning. The usefulness of this report is in producing a conceptualization of multivariation that can provide the basis and framing for empirical studies into the nature of multivariational reasoning, such as ensuring that each hypothesized mental action is targeted during the study. I claim that the different types of multivariation described here are far more than theoretical curiosities. Students encounter, both in mathematics and in science, many contexts in which one of these types of multivariational reasoning might be needed. In fact, any context that involves more than two variables, which can even show up in pre-collegiate mathematics, may inherently require at least some of the more basic multivariational mental actions. As such, I believe this paper to be a useful step in understanding how reasoning about these contexts may be developed.

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## E-IBL, Proof Scripts, and Identities: An Exploration of Theoretical Relationships[1]

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*The purpose of this theoretical report is to further current discussions of the relationships between Equity-Oriented Instruction (EOI) and Inquiry Based Learning (IBL) pedagogies. Specifically, it proposes a framing of Equity-Oriented Inquiry Based Learning (E-IBL) that foregrounds equitable practice, as opposed to viewing equitable practice as a gratuitous outcome of IBL pedagogies. Drawing on data from teaching experiments conducted in IBL-Introduction to Proof courses, the inter-relationships between knowledge, identity and practice (Boaler, 2002), Pickering's 'dance of agency,' Gutiérrez's dimensions of equity, and Bourdieu's notion of habitus, this paper explores why intentional attention towards the critical axis of equity – that which links identity and power – is necessary, if IBL pedagogies are to promote equity.*

*Key words*: Inquiry based learning, equity oriented instruction, identity, agency

### Introduction

The purpose of this theoretical report is to further current discussions about the relationships between *Equity-Oriented Instruction* (EOI) and *Inquiry Based Learning* (IBL) pedagogies. Specifically, this report proposes a framing of *Equity-Oriented Inquiry Based Learning* (E-IBL) pedagogies that foregrounds issues of equity, as oppose to viewing equity as a gratuitous outcome of IBL. To understand this position, current framings of EOI and IBL are considered and used to explore rationales for viewing IBL as a pedagogy that promotes equity. Then, drawing on excerpts from teaching experiments in IBL courses, I examine why IBL pedagogies may not gratuitously promote EOI. The paper concludes with a framing of E-IBL.

### A Framing of Equity-Oriented Instruction

Over the past two decades, researchers interested in student learning in school contexts have begun to reconceptualize equity in mathematics education. These researchers (Gutiérrez, 2008; Martin, 2009) have challenged our practice of "gap gazing" and argue for the *de-essentialization* of disparities in students' academic achievement; i.e., against "the framing of mathematics achievement …(as) a kind of individualistic accomplishment" (Gutiérrez, 2008, p. 361) Indeed, drawing on Bourdieu's notion of habitus[2] (Bourdieu, 1984), researchers are illustrating the ways in which practices of structural exclusion enacted in students' mathematics education function to marginalize working-class and culturally diverse students (Jorgensen, Gates, & Roper, 2014). This marginalization occurs through schooling practices that align with the habitus of some students but not others by requiring the linguistic capital and practices of particular classes. Working in ways that align with arguments both for de-essentialization and attention to habitus, Boaler (2002a) has sought to describe the situated nature of learning in schools and argued not only that students' knowledge, practices and identity are inter-related (Figure 1) but that these inter-relationships "constitute the learning experience." This model of the inter-relationship

[1] This paper came about, in part, from conversations with Aditya Adiredja, Luis Leyva, and William Zahner. I would like to thank them, as well as the members of the RUME 2016 Equity Working Group for their insightful comments and feedback during the early stages of this work.
[2] The term *habitus* refers to the informal knowledge and skills that are developed through one's socialization "within the family, home, and immediate environment" so that one learns how to "act in and interpret their worlds" (Jorgensen, Gates, & Roper, 2014, p. 223). It is best thought of as the dispositions that position the individual to operationalize class status.

between identity, practice and knowledge emerged during Boaler's studies of learning in diverse school settings. It posits that one's knowledge is interactively constituted with one's practices. In particular, Boaler found that, "practices such as working through textbook exercises, in one school, or discussing and using mathematical ideas, in the other, were not merely vehicles for the development of more or less knowledge, they shaped the forms of knowledge produced" (p. 43). Speaking to the different instructional practices employed in schools, Boaler notes that direct instruction places the student in a hierarchical relationship with the teacher, where the teacher is the authority and the students are "received knowers" (Boaler, 2002). In contrast, in discussion oriented classrooms students are called on to engage in acts of interpretation, expression, and agency. These practices do not promote students' passive acceptance but rather called on them to "contribute to the judgment of validity, and to generate questions and ideas." And is so doing they foster distinct relationships between students' identities and the "knowledge to be taught." Hence, as Boaler argues, the findings exemplify Wenger's (1998) claim "learning transforms who we are and what we can do, it is an experience of identity" (Wenger, 1998, p. 215).

Identity, however, is not influenced by practices alone. A key component of identity is one's sense of agency. Moreover, as argued by Pickering (1995), working in mathematics requires a *dance of agency*: an interplay of human and disciplinary agency. Disciplinary agency refers to the ways that established practices and artifacts (e.g., proving practices, linguistic conventions, syntax, etc.) interact with and affect the work of mathematics. While individuals express human agency – generating ideas, symbols, terms, and practices – and impact the discipline, the products of human agency must also "surrender to the 'agency of the discipline'" (Boaler, 2002, p. 49). In other words, human agency shapes and is shaped by one's discipline.

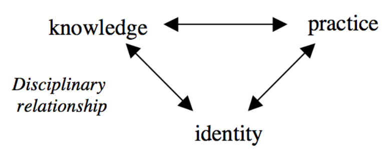


Figure 1. Adapted from Boaler (2002a).

Taken together the works of Gutierrez (2008), Jorgensen, Gates, & Roper (2014), Bourdieu (1984), Boaler (2002) and Wenger (1998) collectively point to the key characteristics of Equity-Oriented Instruction (EOI). EOI necessarily disrupts the reproduction of the structural inequities that are shored up and replicated through students' mathematics education. It intentionally attends to and broadens the forms of habitus that afford participation in schooling by valuing, among other things, the practices and "linguistic repertoires"– that is the capital (Bourdieu, 1984) – of those who are further marginalized by schooling (Jorgensen, Gates, & Roper, 2014). It affords the development of identities that enable rather than inhibit participation in the dance of agency and, therefore, students' engagement in authentic mathematical practices. As practices are enacted in discourses (Gee, 2001), EOI requires students be afforded opportunities to engage in collaborative work that forestalls the impact of one's social capital while also affording access to rich mathematics. It requires instructors avoid essentializing students while working to provide students with "opportunities to draw upon their cultural and linguistic resources (e.g., other languages and dialects, algorithms from other countries, different frames of reference) when doing mathematics, paying attention to the contexts of schooling and to whose perspectives and practices are 'socially valorized' (Abreu & Cline, 2007; Civil, 2006)" (Gutierrez, 2009, p. 5).

## A Framing of Inquiry Based Learning Pedagogies

Inquiry Based Learning (IBL) pedagogies have been defined in a variety of ways. Often IBL pedagogies are defined as any form of instruction in which students actively pursue knowledge through activities and discussions (Rasmussen & Kwon, 2007). According to the *Academy for inquiry based learning,* IBL is a "big tent" term for, "Teaching methods in mathematics courses … where students are (a) deeply engaged in rich mathematical tasks, and (b) have ample opportunities to collaborate with peers (where collaboration is defined broadly)."[3]

IBL pedagogies differ in (at least) two key ways from traditional, lecture-based mathematics instruction.[4] First, curricular activities are often inverted. By this I mean that rather than introducing institutionalized knowledge and having students practice using that knowledge, IBL curricular tasks elicit students' ways of understanding and then through task sequences provide opportunities for students to accommodate their understandings and develop disciplinary practices. The introduction of institutionalized knowledge is the final rather than first step in learning. Second, students are expected and encouraged to act with intellectual autonomy within collaborative settings. In other words, they are called on to demonstrate specific forms of human agency: (a) generating and proposing problem solving strategies; (b) comparing and contrasting approaches; and (c) engaging in acts of justification and validation.

## Why researchers have argued IBL promotes EOI

The association between active learning and equity has a long and well warranted history. The results of the Treisman (1992) studies demonstrated to many in the mathematics community that opportunities to collaborate around rich mathematical tasks could change the outcomes of students who are disadvantaged by structural inequities. More recently, Freeman et al. (2014) conducted a meta-analysis of 225 studies that compared active learning pedagogies to lecture-based instruction. They found that active learning pedagogies significantly decreased failure rates and that "active learning confers disproportionate benefits for STEM students from disadvantaged backgrounds and for female students in male-dominated fields." In a study that specifically focused on IBL pedagogies, Laursen et al. (2014) found not only did enrollment in IBL classes positively impact student success in subsequent courses but also that the IBL courses reduced the gender gap, with female students not only showing equal or greater learning gains but also higher levels of intention to persist than those in non-IBL courses.

Beyond these empirical studies, supports for IBL's potential to promote equitable outcomes can be found in recent theoretical analyses. Tang, Savic, El Turkey, Karakok, Cilli-Turner, and Plaxco (2017) provided an analysis of IBL and its relationship to the dimensions of equity proposed by Gutierrez (2009). Specifically, Tang et al. argue that in collaborative learning environments, *all* students are invited to engage in the "doing, discussing, and presenting" of mathematics. The implication here is that IBL pedagogies increase access to rich mathematics, while also promoting achievement (Freeman et al., 2014; Laursen et al., 2014). Building on the findings of Hassi's (2015) qualitative study, Tang et al. also discuss how collaborative learning environments in which students assert agency, foster growth in self-esteem and self-confidence and, therefore, students' sense of power. Thus, according to Tang et al., IBL pedagogies act not only along the dominant axis of equity but also the critical axis.

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[3] Retrieved from http://www.inquirybasedlearning.org on July 29, 2017.
[4] Kuster and Johnson (2016) proposed a four-component model of IBL that aligns with that proposed here. Cook, Murphy and Fukawa-Connelly (2016) have proposed a six-component model. Due to space limitations, these models are not discussed in this theoretical report.

### Why IBL might not gratuitously promote EOI.

*identity has as much to do with others as it does with self … A large part of who we are is learned from how others interact and engage with us. (Pierson Bishop, 2012, p. 38)*

It is not the purpose of this section to argue that IBL pedagogies do not promote more equitable learning *outcomes* than traditional lecture-oriented pedagogies. Certainly, it would be a fool's errand to do so given recent research (e.g. Freeman et al., 2014). Instead the purpose is to argue that IBL pedagogies are not necessarily EOI pedagogies and, consequently, do not produce equitable learning *environments* "for free." Instead, intentional attention to equity is required.

To explore the ways in which IBL might fail to function as a form of EOI, I will discuss two data excerpts drawn from field notes and proof scripts collected during a series teaching experiments. These experiments occurred in IBL-*Introduction to Proof* courses taught at a designated Hispanic-serving university, where the majority are first generation college students eligible for need-based financial assistance. The classes were majority-minority classrooms: on average 67% were ethnic minorities and approximately one-third were students who identify as female. Students classified as Hispanic by institutional categories were the dominant minority group, with many preferring the terms Latino/Latina or Chicano/Chicana rather than Hispanic.[5]

**The first example.** The first data excerpt is drawn from field notes. It concerns an event of *othering*: viewing or treating an individual as distinct from or alien to oneself or one's group (possibly without intent).

**The Vignette.** The class begins with a whole class discussion about the theorems the class will focus on proving that day and a target time for discussing their proofs. Students are asked to move into their small groups, which have been assigned by students counting off the numbers 1 through 7. Mariella[6], a Latina, begins to move her desk towards her group. She stops a few feet short of her group because the other members of her group (three male students) have already moved their desks together and left no space for her desk. (The pre- and post-grouping of the desks is shown in Figure 2, with Mariella's desk shown as a circle.) She quietly works on her own, occasionally looking at the male students who do not appear to notice her exclusion.

The instructor observes Mariella's situation for approximately 20 minutes in an effort to provide adequate time for the male group members (or Mariella) to rectify the exclusionary situation. The instructor speaks with Mariella to confirm that the group of three male students is, in fact, her assigned group. Mariella requests of the instructor that she be allowed to work alone. The instructor respects her request, observing that she is uncomfortable. The classroom learning assistant (an advanced undergraduate) is asked by the instructor to check in with Mariella periodically. Several extended mathematical conversations are observed between them. After the class, the instructor asks two other female students from the class to speak with her individually outside of class. The instructor asks each student how she would prefer instructors respond in similar situations. Unprompted, both women share similar experiences where they were either physically excluded or "invisible" during group work. Both suggest moving Mariella to a group with another female. The next day Mariella is asked to change her group and, shortly thereafter,

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[5] Following Gutierrez (2013), I use the terms Chicano and Chicana to refer to people with indigenous ancestry in the western United States. I recognize its use by students (and researchers) as intentional and political. Hence forth, I will use the gender neutral terms, Chican@ and Latin@.

[6] All names are pseudonyms.

observed assisting the other female student. Instances of Mariella actively engaging with her new group while engaging in proving efforts are observed in several subsequent classes.

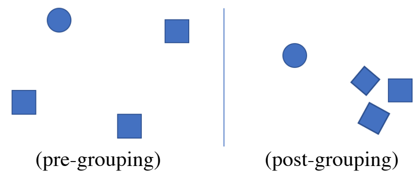



Figure 2. Pre- and post-grouping desk arrangements

**Vignette Discussion.** Why is this an instance of IBL not gratuitously promoting EOI? To be certain, some might argue that the students described in the vignette were not engaging in IBL because a central tenet of IBL is collaboration and the students weren't collaborating. There are two issues with this response. First, the male students were collaborating. Second, Mariella had tried to join the group to collaborate but had been excluded. Another critique might center on the fact that the instructor could have remedied the situation by reminding the male students of the participation norms which were discussed extensively at the beginning of the course or that Mariella should have acted to end her exclusion, since participation is an expectation of *all* IBL students. Such responses, however, assume that the tenets of IBL should be privileged to such an extent that they are enacted in lieu of EOI practices. They ignore the costs marginalized students pay to participate when they are called on to enforce IBL practices and (potentially) act against their own identities, dispositions, or cultural practices. Moreover, privileging collaboration while ignoring these costs does little to mitigate marginalized students' sense of exclusion or the potential for such practices to create the *illusion of participation*. And it is here that the problem lies. Even if *all* IBL students are expected to advocate for their own participation it is not the case that *all* students are called on to do so. More importantly, it is not the case that *all* will have identities, dispositions, or a cultural habitus that are at odds with such actions. Indeed, a post-class discussion with Mariella confirmed that she felt extremely uncomfortable "forcing" herself into the group, preferring instead to work alone after having been publicly *othered*.

**The Second Example**

*Mom, how do you say quesadilla in Spanish?*
*-Sebastian, Age 7*

As noted earlier in the paper, EOI requires students be afforded opportunities to engage in collaborative work that forestalls the impact of one's social capital while simultaneously supporting and empowering students' identities. The position taken in this paper is that one's linguistic practices are not secondary to one's identity but rather are an integral component (Bishop, 2012). The extent to which one's language, culture and practices are valued in an environment determines the extent to which one's identity is valued. Since 1998, Latin@ and Chican@ students have had to deal with the educational fallout of California Proposition 227. This proposition codified a stance towards bilingualism that views students' use of non-English languages as a deficit rather than an asset to the students and their communities. It is one of the reasons Californian dialects that heavily integrate Spanish words are often practiced without users recognizing their use of another language – a point exemplified by Sebastian's remarks.

Gee (2001, 2005) and Sfard and Prusak (2005) argue that identities are constructed through discourse. Others, such as Bishop (2012), argue that discourses “play a critical role in enacting identities” (p. 44). Most who have taught university mathematics courses in environments where the majority of students are first generation urban students can readily attest to the varied and at times colorful slangs currently used. These languages stand in stark contrast to that employed with great continuity for thousands of years among the practitioners of the discipline of mathematics, especially when writing proofs. To illustrate this continuity, I ask the reader to consider the resemblance between the two proofs in Figure 3, the first from *Euclid’s Elements* (c. 350 BC, T.L. Heath’s 1909 translation) and the second from *Mathematical Proofs: A Transition to Advanced Mathematics* by Chartrand, Polimni and Zhang (2008) (see pp. 145-6).

| | |
|---|---|
| ***I.6 If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.***<br><br>***Proof:*** Let ABC be a triangle having the angle ABC equal to the angle ACB; I say that the side AB is also equal to the side AC. For if AB is unequal to AC, one of them is greater. Let AB be greater; and from AB the greater let DB be cut off equal to AC the less; let DC be joined. Then since DB is equal to AC, and BC is common, the two sides DB, BC are equal to the two sides AC, CB respectively; and the angle DBC is equal to the angle ACB; therefore, the base DC is equal to the base AB, and the triangle DBC is will be equal to the triangle ACB, the less to the greater: which is absurd. Therefore, AB is not unequal to AC; it is therefore equal to it. | ***6.17 For every nonnegative integer n, $3 \mid (2^{2n}-1)$.***<br><br>***Proof***: Assume, to the contrary, that there are nonnegative integers $n$ for which $3 \nmid (2^{2n}-1)$. By Theorem 6.7, there is a smallest nonnegative integer $n$ such that $3 \nmid (2^{2n}-1)$. Denote this integer by $m$. Thus $3 \nmid (2^{2m}-1)$ and $3 \mid (2^{2n}-1)$ for all integers $n$ for which $0 \leq n < m$. Since $3 \mid (2^{2n}-1)$ when $n = 0$ it follows that $m \geq 1$. Hence, $m = k+1$, where $0 \leq k < m$. Thus $3 \mid (2^{2k}-1)$ which implies that $2^{2k} - 1 = 3x$ for some integer $x$. Consequently, $2^{2k} = 3x + 1$. Observe that $2^{2m}-1 = 2^{2(k+1)}-1 = 2^{2k+2}-1 = 2^2 \cdot 2^{2k}-1 = 4(3x+1) - 1 = 12x + 3 = 3(4x + 1)$ Since $4x+1$ is an integer $3 \mid (2^{2m}-1)$, which produces a contradiction.<br>■ |

Figure 3. Proofs from Euclid (c. 350 BCE; 1909 translation) and Chartrand et al (2008).

Now consider the following thought experiment: Imagine that humans had not invented mathematical proof well over 2000 years ago and that despite not having invented proofs, enough mathematics developed for some modern technologies (e.g., cell phones, twitter, and texting). What would our proving practices look like if they were invented by our culturally-diverse, economically-disadvantaged, urban youth? Would answers like that shown in Figure 4 be considered normative rather than examples of norm breaching (Herbst & Chazen, 2011)?

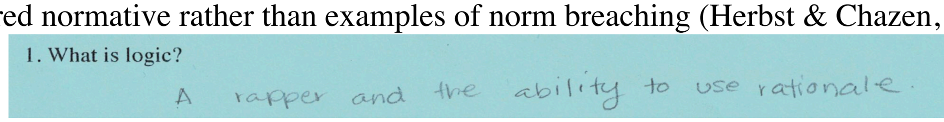
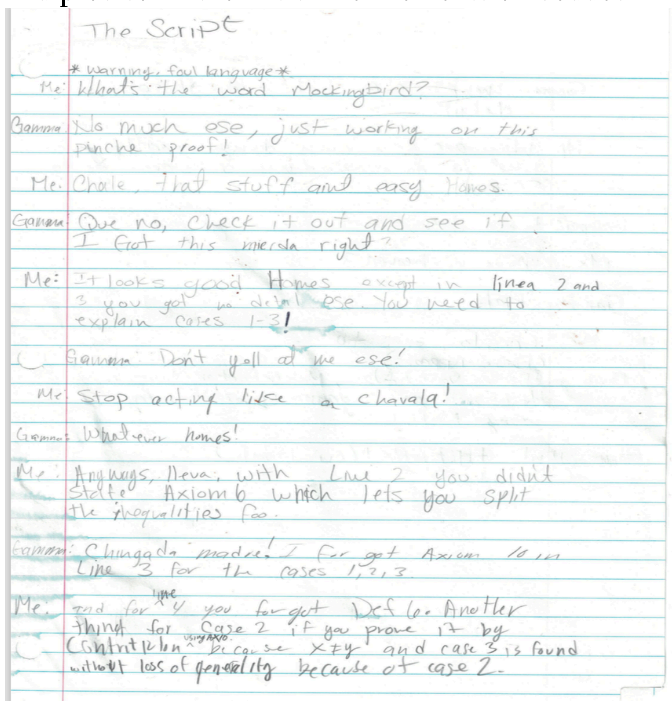



Figure 4. Student Survey Response

Moreover, would students’ proof scripts, like that shown in Figure 5, be viewed as an instance of authentic mathematical discourse rather than as something written in another dialect? Would the pervasive code-switching that occurs in the dialog be seen as exemplifying a student’s masterful blending of two dialects – the urban and the mathematical – rather than as indicative of a lack of participation in unspoken, yet implicitly demanded, disciplinary dialectic practices?

The student’s script was drawn from a set of 43 *proof scripts*: written dialogs in which a student and a fictional peer discuss a proof so as to promote the peer’s understanding of any gaps or key points in the proof. It was chosen as an example of one of many instances of students describing deep mathematical issues using their normative discursive practices. Indeed, field notes indicate that throughout the IBL *Introduction to Proof* course, students had grown increasingly accustom to intensely discussing proofs in their everyday vernacular. It is included in the paper to demonstrate a tension between IBL and EOI. A key tenet of IBL is that students’ move towards institutionalized knowledge (and therefore, normative disciplinary discursive practices) through their collaborative activities. It privileges rather than challenges normative practice by calling on instructors to enact discourse hierarchies in lieu of attending to the critical role discourses play in identity formation and student agency. Consequently, enacting IBL pedagogies means working to curtail rather than recognize (or value) students' discourses. In contrast, practitioners who privilege EOI practices over those central to IBL must attempt to navigate the tension between students' means of expressing identity and disciplinary discursive practices. They must recognize that privileging EOI means rejecting discourse hierarchies while simultaneously providing opportunities for students to become knowledgeable of disciplinary discourses. In other words, drawing on Gutierrez (2009), this paper argues that privileging EOI when enacting IBL, means valuing instances in which students "change the game" (e.g., by seeing value in the student's bridging of his own and disciplinary vernaculars) while also valuing the student's success "playing the game" (e.g., by valuing the mathematical sophistication which underlies the detailed and precise mathematical refinements embedded in the student's remarks).


The Script

*warning, foul language*

Me: What's the word Mockingbird?

Gamma: No much ese, just working on this pinche proof!

Me: Chale, that stuff ain't easy Homes.

Gamma: Que no, Check it out and see if I Got this mierda right?

Me: It looks good Homes except in linea 2 and 3 you got no detail ese. You need to explain cases 1-3!

Gamma: Don't yell at me ese!

Me: Stop acting like a chavala!

Gamma: Whatever homes!

Me: Anyways, lleva, with Line 2 you didn't State Axiom 6 which lets you split the inequalities too.

Gamma: Chingada madre! I forgot Axiom 10 in Line 3 for the cases 1,2,3.

Me: And for line 4 you forgot Def 6. Another thing for Case 2 if you prove it by Contradiction using Ax10 because x≠y and case 3 is found without loss of generality because of case 2.

Figure 5. Joseph's Proof Script Excerpt

## A framing of E-IBL

In this paper, I call into question the assumption that IBL pedagogies gratuitously promote EOI and argue E-IBL requires intentional attention to equity. I posit that intentional attention to equity calls on practitioners to employ EOI as a lens when viewing IBL learning environments. Applying such a lens necessarily entails foregrounding issues of structural exclusion and acting to disrupt the social mechanisms that result in their reproduction in institutional spaces (Jorgensen, Gates, & Roper, 2014; Battey & Leyva, 2016). It means privileging students' identities and habitus when IBL practices call on students to act against either; e.g., by valuing varied forms of social capital (e.g., linguistic resources (Zahner & Moschkovich, 2011)) or addressing instances of *othering* by first attending to students' identities and habitus; i.e., the costs some pay to participate. At its core, this framing posits E-IBL instructors must be willing to recognize that, as argued by Wenger (1998), learning is "an experience of identity" and that identity and power are negotiated in institutional contexts (Adiredja & Andrews-Larsen, 2017). Thus, privileging the demands of EOI over the tenets of IBL, requires instructors navigate the tensions present in spaces that support students not only "playing the game" but also "changing the game" (Gutierrez, 2009) as they develop expertise in mathematics.

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# Generalisation, Assimilation, and Accommodation

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*This paper builds theory by connecting Piaget's assimilation and accommodation constructs to Harel and Tall's (1991) framework for generalisation in advanced mathematics. Based on what they imagined to be the cognitive processes underlying generalisation, Harel and Tall proposed that generalisation might be* expansive *(occurring when a student expands the applicability range of an existing schema without reconstructing it),* reconstructive *(occurring when a student reconstructs a schema to widen its range of applicability), or* disjunctive *(occurring when a student constructs a new, disjoint schema to deal with a new context). I contend that expansive and reconstructive generalisation align with assimilation and accommodation, respectively. I provide 'proof of concept' using data from a study of students' generalisation of graphing from $R^2$ to $R^3$. Further, I show how linking Piagetian constructs to Harel and Tall's work provides a theoretical explanation for other empirical findings about generalisation.*

*Key words:* generalisation, multivariable function, graphing, assimilation, accommodation

## Introduction

Generalisation is a key component of mathematics. Mathematicians seek general formulae; kindergarteners generalise when they seek the next shape in a pattern; and multivariable calculus students generalise their notion of function to include functions of more than one variable. Because generalisation is so critical to mathematical thought, research that investigates how people generalise supports student learning. Moreover, students often struggle to form correct generalisations (e.g. Dorko & Weber, 2014; Jones & Dorko, 2015; Kabael, 2011; Martínez-Planell & Gaisman, 2013, 2012; Martínez-Planell & Trigueros, 2012). Generalising is important to many science, technology, engineering, and mathematics (STEM) courses. For example, students must be able to generalise their mathematics knowledge to chemistry, physics, and upper division mathematics. Difficulty generalising may contribute to students switching out of STEM studies. Efforts to better understand how students generalise and how instructors can support their generalisations could help solve the problem of retaining STEM majors (c.f. Bressoud, Carlson, Mesa, & Rasmussen, 2013; Rasmussen & Ellis, 2013; Uysal, Ellis, & Rasmussen, 2013).

Descriptions of how people generalise often come in the form of frameworks. Frameworks provide language to describe and account for qualitative differences in students' thinking and activity. Knowing what students attend to when generalising can inform instruction and the development of mathematical activities to support productive generalisation. In this paper, I connect Harel and Tall's (1991) framework for describing the "different qualities of generalisation in advanced mathematics" (p. 1) to Piaget's assimilation and accommodation constructs. This came about from my use of Harel and Tall's (1991) framework to classify empirical data, during which I often struggled to distinguish between the expansive and reconstructive generalisation categories. While the definitions of the categories seem clear, I struggled to operationalize them so they could be applied to my data. Because the descriptions of these categories seemed similar to the definitions of assimilation and accommodation (respectively), I wondered if there existed connections between the framework and the Piagetian constructs.

There are two reasons it felt worthwhile to tease apart any possible connections. The first is that such an investigation could provide insight into the cognitive processes involved in generalisation. Harel and Tall (1991) propose three ways students might generalise, but do not explain why a student might engage in one type of generalisation instead of another. Thinking about generalisation in terms of assimilation and accommodation could provide a tenable explanation. A second reason is to situate the framework in widely-understood language. This is useful because while many researchers cite Harel and Tall's (1991) definition of generalisation (e.g. Ellis, 2007; Mitchelmore, 2002) or offer hypothetical examples of Harel and Tall's three categories (e.g. Greer & Harel, 1998; Mitchelmore, 2002), the framework has been used in only three empirical studies (Fisher, 2008; Jones & Dorko, 2015; Zazkis & Liljedahl, 2002). My experience is that Harel and Tall's framework provides a powerful way to think about generalisation, but that it can be difficult to distinguish between the expansive and reconstructive categories. This difficulty may explain the lack of empirical use. I thought that if the expansive and reconstructive categories could be linked to assimilation and accommodation (respectively), it might be easier to use the framework for classifying empirical data. Moreover, the ability to talk about generalisation in terms of assimilation and accommodation affords communication about theory and results in terms of a widely-understood learning theory. This has been the case for research that has connected assimilation and accommodation to transfer (Wagner, 2010) and backward transfer (Hohensee, 2014).

This paper is structured as follows. First, I present Harel and Tall's (1991) framework and connect it to Piaget's assimilation and accommodation constructs. Then, I describe the data set and methods I used to tease apart possible connections. I follow this with an example of the utility of these connections in the context of a student generalising her thinking about graphing from $R^2$ to $R^3$. Then, I discuss how assimilation and accommodation explain other researchers' empirical findings about students generalising graphing and their notion of function from the single- to multivariable case. Finally, I offer suggestions for further research and for instruction.

## Harel and Tall's (1991) Framework, Assimilation, and Accommodation

Harel and Tall (1991) proposed that generalisation in advanced mathematics fell into three categories, termed expansive, reconstructive, and disjunctive generalisation. Table 1 (next page) provides definitions of these categories and an example in the context of vector addition.

I argue that when students engage in expansive and reconstructive generalisation, they do so via assimilation and accommodation (respectively). Piaget proposed assimilation and accommodation as the mechanisms by which people learn. He discussed them in the context of schemes, or "organi[s]ation[s] of mental and affective activity" (Thompson, 2016, p. 436). *Assimilation* is defined as "the integration of new objects or new situations and events into previous schemes" (Piaget, 1980, p. 164 as cited in Steffe, 1991, p. 192). Assimilation "comes about when a cogni[s]ing organism fits an experience into a conceptual structure it already has" (von Glasersfeld, 1995, p. 62). In contrast, *accommodation* is a modification of a scheme. Accommodation occurs when a person's attempt to assimilate a situation to a scheme has an unexpected result, causing a perturbation and disequilibrium. To re-attain equilibrium, the person modifies the scheme (accommodation). In the next section, I describe the data set I used to explore the framework and Piagetian constructs.

## Data Set and Methods

The data excerpts in this paper come from a longitudinal study of calculus students' generalisation of function from single- to multivariable settings (AUTHOR, 2017). I conducted

Table 1. Harel and Tall's (1991) Framework

| Category | Example[1] |
|---|---|
| **Expansive generalisation** "occurs when the subject expands the applicability range of an existing schema without reconstructing it" (Harel & Tall, 1991, p.1). The original schema is "included directly as [a] special case in the final schema" (ibid, p.1). | A student understands vector addition <a,b> + <c,d> as performing addition twice. The student generalises her understanding of addition in R by "repeating it across more terms" (Jones & Dorko, 2015, p.156). |
| **Reconstructive generalisation** "occurs when the subject reconstructs an existing schema to widen its applicability range" (Harel & Tall, 1991, p.1). The original schema "is changed and enriched before being encompassed in the more general schema" (ibid, p.1). | A student understands vector addition <a,b> + <c,d> as performing addition twice. The student generalises her understanding of addition in R by repeating it across more terms. The student learns the geometric interpretation of vector addition as placing vectors head to tail and finding the resultant vector. The idea of vector addition "may not exist for the student in basic addition in R, and consequently the underlying idea of "addition" itself is reconstructed for the new $R^2$ context" (Jones & Dorko, 2015, p. 156). |
| **Disjunctive generalisation** "occurs when, on moving from a familiar context to a new one, the subject constructs a new, disjoint schema to deal with the new context and adds it to the array of schemas available" (Harel & Tall, 1991, p.1). | A student understands vector addition <a,b> + <c,d> as completely separate from addition in R. Jones and Dorko (2015) describe this hypothetical student thinking as "we might still use the same word 'addition,' but it is not the same thing as 'regular' number addition" (p. 156). |

four task-based clinical interviews (Hunting, 1997) with each of five students over the span of their differential, integral, and multivariable calculus courses. The total interview time ranged from 4.25 to 5.67 hours per student. Students answered questions about single- and multivariable topics while they were enrolled in differential and (later) multivariable calculus. This design provided insight into both students' initial sense-making of how ideas from $R^2$ might generalise to $R^3$, and the sense students made of those ideas after instruction. Space constraints prohibit listing all the tasks students answered; this paper focuses on a student's response to the tasks (1) Graph $y = x$ in $R^3$; (2) Graph $y = 2x + 1$ in $R^3$; and (3) Graph $z = 4$ in $R^3$. Students answered these questions at the beginning of their multivariable calculus course before instruction about graphing in $R^3$. This timing provided insight regarding students' initial generalisations of how to graph equations. Prior to their answering the tasks, I provided students with a blank copy of $R^3$ axes and explained the axes' positions.

The first step in my data analysis was to review the data and identify instances of generalisation. I followed Harel and Tall's (1991) definition of generalisation as "the process of applying a given argument in a broader context" (p. 1). I then attempted to code these instances

[1] Harel and Tall (1991) offer two hypothetical examples of their categories, a detailed example of generalising how to solve systems of equations and a brief example of generalising addition in R to vector addition. Jones and Dorko (2015) offer a more detailed description of the vector addition generalisation.

as expansive generalisation, reconstructive generalisation, or disjunctive generalisation based on the definitions from Harel and Tall's (1991) framework (Table 1). I did not find any instances of disjunctive generalisation, and as such, any connections between Piagetian constructs and disjunctive generalisation are not discussed here; this is an area for future work. Finally, I took each instance and sought to code it as assimilation or accommodation based on the definitions provided above.

I gave the data presented in this paper to another researcher, who at the time was studying generalisation in real analysis from a Piagetian perspective and hence was knowledgeable about and experienced with identifying assimilation and accommodation in practise. This person coded the data separately and their codes were the same as my own. In the next section, I provide an example that illustrates the connections between assimilation and expansive generalisation and accommodation and reconstructive generalisation.

## An Example: Line or Plane?

Based on my analysis, I concluded that students may engage in expansive generalisation when they assimilate a new context to an existing scheme, and they may engage in reconstructive generalisation when a new context triggers a perturbation that causes them to modify a scheme. In the following example, Wendy (pseudonym) first draws $y = x$ and $y = 2x + 1$ in $R^3$ as lines and $z = 4$ in $R^3$ as a plane. I argue that these were expansive generalisations, occurring as a result of assimilating $y = x$ and $y = 2x + 1$ in $R^3$ to a scheme for graphing linear functions in $R^2$ with $m \neq 0$ and assimilating $z = 4$ to a scheme for graphing linear function in $R^2$ with $m = 0$ (that is, a scheme for $y = b$). After drawing the three graphs, Wendy said she found it "interesting" that she had drawn both lines and planes. She compared the graphs and equations and reasoned that all three should be planes. That is, she engaged in reconstructive generalisation. I contend that Wendy's initial observation that she had drawn lines and a plane served as perturbation, which caused her to accommodate her scheme for graphing $y = \mathrm{m}x + \mathrm{b}$ equations ($m \neq 0$) in $R^3$.

### Assimilation and Expansive Generalisation

Excerpts 1 and 2 below provide what I take as evidence of Wendy's assimilating $y = x$ and $y = 2x + 1$ in $R^3$ to a scheme for $y = mx + b$ ($m \neq 0$) in $R^2$. "Int." is short for "interviewer."

Excerpt 1. Assimilating $y = x$ in $R^3$ to a scheme for graphing linear functions ($m \neq 0$) in $R^2$

*Wendy:* So if you just plug in values for $x$ and then pull out values for $y$, you're gonna get like 0, 0, 1, 1, 2, 2 [plots these on the $xy$ plane as she says them] and then it's just going to continue being a straight line like this… you could choose any $x$ value, really. I chose like 1. So if $x$ is 1, then $y$ is equal to $x$, so that's also 1.

*Interviewer:* Can you label some of the coordinates that you plotted?

*Wendy:* Okay, so this is going to be like 1, 1, 0 and then 2, 2, 0.

*Interviewer:* Why do we get a line here?

*Wendy:* The way I think of it is it's just like having a 2D graph and plotting $y = x$ and that'll give you a line, you're just taking it and adding and then ignoring the $z$ component… if $y = x,$ you can just always assume that $z$ is 0.Excerpt 2. Assimilating $y = 2x + 1$ in $R^3$ to a scheme for graphing linear functions ($m \neq 0$) in $R^2$

*Wendy:* I'm thinking that it will be like the same kind of concept where we're just ignoring $z$ so you can say like $+0z$ here and that will give you the same equation [writes $y = 2x + 1 + 0z$]. So if you went $2x + 1$ that would be 0, 1 and then 1, 3… basically you would just take the same line that you would have with your $x$ and $y$.

*Interviewer:* And do we get a line there?

*Wendy:* Yeah, that's a line… like I said we're ignoring the $z$ component, but you can think of it as there, you're just, have it, 0 set to it.

I argue that Wendy assimilated these equations to a scheme for graphing in $R^2$. Wendy talked about the coordinate points as $(x, y)$ tuples (e.g., "0, 0, 1, 1, 2, 2") as she was plotting the points (Excerpt 1). Though she described the points as $(x, y, z)$ tuples when asked to identify points, I posit that her thinking of the points as $(x, y)$ tuples during the act of graphing indicates that she had assimilated the question about creating a graph in $R^3$ to a schema for graphing in $R^2$. My inference is supported by Wendy's explicit statement that she saw $y = x$ in $R^3$ as "just like having a 2D graph and plotting $y = x$".

I contend Wendy's treatment of $z$ facilitated her assimilation. We know that Wendy considered $z$ because she said in both graph tasks that she was "ignoring $z$" (Excerpts 1 and 2) or setting it to 0 (Excerpt 3)[2]. Further, when asked what points she had plotted on her $y = x$ graph, Wendy gave $(x, y, z)$ tuples. However, Wendy's statements about $z$ provide evidence that she (a) explicitly considered $z$ and (b) treated it in a way that allowed her to assimilate the $y = x$ and $y = 2x + 1$ in $R^3$ tasks to a scheme from $R^2$. This is in accordance with assimilation as "reduc[ing] new experiences to already existing sensorimotor or conceptual structures" (von Glasersfeld, 1995, p. 63). The result of Wendy's assimilation and expansive generalisation was that she drew these graphs as lines.

I argue that Wendy assimilated $z = 4$ to a different scheme. Specifically, Wendy appeared to have a scheme for $y = b$ in $R^2$. She drew $z = 4$ as a plane (Excerpt 3).

Excerpt 3. Assimilating $z = 4$ in $R^3$ to a scheme for graphing $y = b$ in $R^2$

*Wendy:* I'm thinking that whenever, no matter what $x$ and $y$ equal, $z$ is always going to equal 4. So you get a plane here at 4. That's a really bad drawing of it, but, no matter what these [gestures to $x$ axis] equal, you're always just going to get 4.

*Interviewer:* Can you tell me a little bit more about the 'no matter what these equal'?

*Wendy:* So if you're graphing, so $z = 4$, it's like saying $y = 4$ on a normal graph you get a line at $y$, or 4. You just get that [sketches $y = 4$ in $R^2$]. Because it doesn't matter what $x$ equals. So here I'm kind of thinking that it's the same concept, that no matter what $y$ or $x$ equals, $z$ is always going to equal 4.

*Interviewer:* Do you, as you graphed that $z = 4$, so you pretty immediately said oh, this is a plane. Did you think about this $y$ and $x$ graph? [points to Wendy's graph of $y = 4$ in $R^2$].

*Wendy:* I basically, I took the concept of it and applied it.

*Interviewer:* And what's the concept of it?

*Wendy:* Yeah, the concept of it is like I said even though there's no $x$ in this equation, like we always know that $y$ is going to be equal to 4 so it really doesn't matter what $x$ is, so that's why there's no $x$ in the equation.

*Interviewer:* How come $z = 4$ isn't just a line?

*Wendy:* Because you're in 3D, so if say like $x$ was 1 and $y$ was 2, you're always, $z$ is going to equal 4.

I take Wendy's comment "it's the same concept" as evidence that she assimilated $z = 4$ to an already-existing scheme. What Wendy appeared to see as the "same concept" was that $y$ equaled 4 in $R^2$ "[no] matter what $x$ equals", so $z$ would equal 4 in $R^3$ "no matter what $y$ or $x$ equals." Wendy argued that $z = 4$ was a plane using the example of (1, 2, 4) as a point on the graph.

[2] It is important to note that for Wendy, writing $+0z$ meant that she was setting $z$ to 0 (Excerpt 3). This contrasts a normative interpretation of $y = 2x + 1 + 0z$, in which one sees $z$ as varying.

I contend Wendy assimilated $z = 4$ to a different scheme than the scheme to which she assimilated $y = x$ and $y = 2x + 1$. That is, Wendy appeared to have a scheme for constant functions in $R^2$, and an element of which was $x$ as free. She expanded this to the $R^3$ case by viewing $x$ and $y$ as free. In contrast, she appeared to have a scheme for non-constant linear functions, an element of which was that such functions' graphs are lines. Wendy expansively generalised this scheme to the $R^3$ case by choosing to "ignore $z$" or, equivalent in her mind, 'set it to 0'.

**Accommodation and Reconstructive Generalisation**

The result of Wendy's assimilations to two different schemes resulted in two different graphs, triggering a perturbation that subsequently caused Wendy to reconstruct her scheme for non-constant linear equations in $R^3$ (Excerpt 4).

Excerpt 4. Perturbation

*Wendy:* It's interesting to me… That I think of that [$z = 4$] like that, and then the other ones [$y = x$ and $y = 2x + 1$] I don't think of like that. So if I, if I applied what I did in [$z = 4$] to [$y = x$ and $y = 2x + 1$] I would get planes again, which would look like this… because $y$ is going to equal $x$. I feel like I'm confusing myself.

Wendy then compared her work on the three graphs, which led her to reconstruct her notion of a free variable (Excerpt 5).

Excerpt 5. Accommodation

*Interviewer:* Okay, so do you want to look at these again? [puts $y = x$ and $y = 2x + 1$ graphs in front of Wendy]

*Wendy:* So if I think about it like this [points to $z = 4$ graph], so if I thought of this [$z = 4$] like I think of this [points to $y = 2x + 1$], then this [$z = 4$] would just be a point.

*Interviewer:* Can you say that sentence [again]… the word 'this' gets hard when I do the audio, when I transcribe it.

*Wendy:* Okay so on the previous ones I was thinking of, I was thinking of this [$y = x$] as – this – the $y = x$ as just like $y = x$ and then I was thinking of it as $+0z$. And so out of that you get a line. But instead of thinking of this $+0y + 0x$, I thought of it as more of the $y = 4$. That no matter what the, no matter what the $y$ and $x$ values are here, the $z$ is always going to equal 4… so if I, if I applied what I did in [$z = 4$] to [$y = x$ and $y = 2x + 1$] I would get planes again, which would look like this… because $y$ is going to equal $x$. I feel like I'm confusing myself.

*Interviewer:* So, so do you think $y = x$ in $R^3$ is a plane or a line?

*Wendy:* My initial thought was that it was a line, but now I'm unsure… my initial thought process of it's a line is because I was thinking that you didn't change this $x$ and $y$ coordinate, you just laid it flat, and that is the only thing you did to make it 3D here. And so you could just graph it in 2D and then just lay it flat and put a $z$ axis in it and that wouldn't change the $y = x$. But that was if I was thinking $+0z$ which there isn't a $+0z$. So I think that no matter what $z$ is, $y$ is always going to equal $x$. So whatever $x$ and $y$ are, you're going to have that plane.

I interpret the change in Wendy's graph from a line to a plane as occurring as a result of the following cognitive acts. Wendy's statement that she found it "interesting" that she had drawn a line for two of the graphs and a plane for the third suggests that she expected the graphs to look similar. The unexpected results (the graphs did not look similar) caused a perturbation. Wendy sought to re-equilibrate by comparing how she approached the $y = ...$ equations and the $z = 4$ equation. In doing so, she noticed that in the $y = …$ equations she had assumed $z = 0$, while in the $z$ = 4 equation she had assumed $x$ and $y$ could take on any value. Wendy accommodated her scheme for $y = x$ in $R^3$ as meaning $z$ equaled 0 meaning $z \in R$. Her initial (expansive) generalisation had been that these were lines. When she realised they were similar to $z$ = 4 in that they had a free variable, she engaged in reconstructive generalisation because she widened the applicability range of her notion of free variables. That is, she applied her argument about free variables to the $y = x$ and $y = 2x + 1$ context. This allowed Wendy to "change and enrich" (Harel & Tall, 1991, p.1) her graphing schema for $R^3$.

## Discussion

It appears that assimilation and accommodation explain a variety of empirical findings about what students generalise from $R^2$ to $R^3$. For example, researchers have observed student difficulties with graphing in $R^3$ (Dorko & Lockwood, 2016; Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010). One finding is that students may draw $f(x, y) = x^2 + y^2$ as a cylinder or a sphere because they are accustomed to $x^2 + y^2$ representing a circle in $R^2$ (Martínez-Planell & Gaisman, 2013). I posit students assimilate the $f(x, y) = x^2 + y^2$ to a scheme for circles in $R^2$, causing them to draw "circle-like" shapes in $R^3$. In support of this, Moore, Liss, Silverman, Paoletti, LaForest, and Musgrave (2013) have documented that students often create graphs based on *shape thinking*, or "conceiving of graphs as pictoral objects" (p. 441). That is, a possible explanation for students' graphing difficulties in $R^3$ is that they assimilate $f(x, y)$ equations to their schemes for the shapes of graphs in $R^2$, which allows them to expansively generalise by creating similar shapes on $R^3$ axes. As an example in a different context, researchers have found the function machine model to support students correctly generalising their notion of function from the single- to multivariable case (Dorko & Weber, 2014; Kabael, 2011). I contend that such a model allows students to assimilate multivariable functions to their function machine scheme for single-variable functions, and as such, expansively generalise their notion of function.

These examples illustrate the explanatory power of thinking about generalisation in terms of assimilation and accommodation. More broadly, they demonstrate how identifying connections between frameworks can help researchers better understand phenomena of interest. There are many generalisation frameworks, and one area for future research is to tease out relationships among them and their links to underlying theory.

Finally, Harel and Tall's stated aim in developing their framework was to "suggest pedagogical principles designed to assist students' comprehension of advanced mathematical concepts" (p. 1). One pedagogical suggestion stemming from linking the framework to Piagetian constructs is related to students' tendency to overgeneralize, such as Wendy's initial thought that the $y = \ldots$ equations would be lines in $R^3$ as they are in $R^2$. When instructors notice students overgeneralising, they might consider if students are assimilating when they should be accommodating. Instructors can then help students discern features of the new context that will result in the student becoming perturbed, leading to accommodation.

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# Conceptualizing Students' Struggle with Familiar Concepts in a New Mathematical Domain

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*This article is concerned with cognitive aspects of students' struggles in situations in which familiar concepts are reconsidered in a new mathematical domain. Examples of such cross-curricular concepts are divisibility in the domain of integers and in the domain of polynomials, multiplication in the domain of numbers and in the domain of vectors. The article introduces a polysemous approach for structuring students' concept images in these situations. Post-exchanges from an online forum were analyzed for illustrating the potential of the approach for indicating possible sources of students' misconceptions and meta-ways of thinking that might make students aware of their mistakes.*

*Keywords:* concept image, conceptual change, cross-curricular concepts, epistemological obstacles, polysemy

## Introduction

The multidimensional nature of mathematical concepts has been addressed in a number of frameworks. For example, Sfard (1991) and Gray and Tall (1994) distinguished between approaching a concept as a process and as an object. In the former approach, $\sqrt{9}$ is an operation of extracting the square root from the number 9; in the latter approach, it is a number – an object with particular properties. Research suggests that students' fluency with concepts' dimensions and flexibility with switching among them are necessary for developing a deep understanding and for successful problem solving (e.g., Gray & Tall, 1994; Sfard, 1991). Consequently, considerable effort has been invested in supporting students' linkage among concepts' dimensions through stressing their similarities and compatibility (e.g., Moreno & Waldegg, 1991; Sandoval & Possani, 2016). In these studies, the researchers often focused on a particular concept (e.g., infinity, line and vector) and considered them in a singular mathematical domain (e.g., sets, 3-D).

However, in the landscape of students' mathematical education some concepts are reconsidered in different domains. The domains can be rooted in different axiomatic systems and contain different or new objects. Accordingly, a domanial shift of these *cross-curricular* concepts is often accompanied by a substantial change in familiar dimensions (i.e. definitions, properties, procedures, connections with other concepts, etc.). For instance, when extracted in the field of real numbers, $\sqrt{9}$ equals 3; an application of the De Moivre theorem in the field of complex numbers yields 3 and -3 (Kontorovich, 2016a)

The domanial shift and the substantial change in concept dimensions are potential sources for students' difficulties and mistakes (e.g., Kontorovich & Zazkis, 2016). Accordingly, the study reported in this article is concerned with students' struggles with cross-curricular concepts in a new domain. Specifically, my focus is on cognitive aspects of situations in which students, who are relatively fluent with some dimensions of a concept in one domain, encounter its incompatible dimensions in another domain. The aim of the article is to introduce a polysemous approach for analyzing this phenomenon and to illustrate its usage for indicating possible sources of students' mistakes and affordances that might make students aware of them.

## Theoretical Foundations

This section presents the structures of concept image and polysemy, which are then used for presenting the developed approach.

### Concept Image, Terminology and Symbols

The notion of *concept image*, introduced by Tall and Vinner (1981), remains one of the most utilized constructs in mathematics education literature until today (e.g., Panaoura, Michael-Chrysanthou, Gagatsis, & Elia, 2016). The notion refers to the accumulative cognitive structure that a learner associates with the concept, which includes all the mental pictures, properties and processes. Tall and Vinner suggested that it is unlikely that a learner operates with the whole concept image at once and they assumed that various stimuli evoke partial concept images. Accordingly, mathematics education research has been often concerned with exploring tensions among the evoked concept images (e.g., Bingolbali & Monaghan, 2008; Tall & Vinner, 1981).

The terminology and symbols that one associates with a concept can also be considered as a part of her concept image. Extensive research on the tight connections between language and thinking show that discourse shapes our understanding of mathematical concepts (see Austin & Howson, 1979, for an elaborated review). In our case, it seems reasonable to assume that students can identify cross-curricular concepts based on the same terminology and symbols which are used in different domains. This assumption brings up the constructs of homonymy and polysemy.

### Polysemy of Mathematical Concepts

Durkin and Shire (1991) use the notions of *homonymy* and *polysemy* for referring to words with multiple meanings. The meanings of a homonymous word are different and not related, for example "volume" in the sense of a measure of a 3-dimensional object as opposed to intensity of sound. A word is called polysemous if its meanings are related. In terms of this article, it can be proposed that a concept is homonymous if its dimensions in different domains are barely related, for examples a graph of a function in calculus and a graph in graph theory. A polysemous concept, in its turn, can be characterized with dimensions that are valid in different domains and dimensions that hold in particular domains only. Let us take the square root, for instance: The statement "if $b$ is a square root of $a$ then $b^2=a$" is valid for real and complex numbers; however, thinking of $b$ as a non-negative number is appropriate in the former domain only.

A considerable amount of research has been invested in exploring students' understanding of polysemous words with daily and mathematical meanings (e.g., Shire & Durkin, 1989). The polysemy of meanings within the mathematical register is less acknowledged. Zazkis (1998) exemplified the ambiguity of "divisor" with the exercise $12 \div 5 = 2.4$ where in the domain of rational numbers, the number 5 can be addressed as a divisor, since it is defined as the denominator of a fraction. However, if the exercise is considered in the domain of integers, 5 is not a divisor of 12 because there exist no integer that when multiplied by 5 equals 12. Mamolo (2010) focused on the polysemy of symbols '+' and '1'. Her analysis accounted for the changes in the definitions and, consequently, in symbols' meanings in the contexts of modular arithmetic, transfinite mathematics, et cetera. Based on their analyses, Mamolo (2010) and Zazkis (1998) argued that polysemy in mathematics is a potential source of struggle for learners. This study can be considered as an examination of their argument.

## The Study

In terms of IES and NSF (2013) this is an early-stage exploratory research that aims to contribute to core knowledge in education by refining and developing theories for teaching and learning. Thus, the research was approached with the *abduction* methodology (Peirce, 1955). The methodology requires identification of a phenomenon of interest (see Introduction) and gives rise to an initial theory. Then the theory is supported and refined through a purposeful corpus of evidence (Svennevig, 2001). Svennevig (2001) argues that while being a less than certain mode of inference, abduction compensates with a vengeance by providing new ideas and developments. Moreover, the methodology relies on contextual judgements, which are necessary for analysing conceptual development.

Ideas and evidence emerged from a project that involved 25 high-achieving ninth-graders who participated in a linear algebra course (see Kontorovich, 2016b for more details). The course was aimed at preparing for and engaging school students in undergraduate education in parallel with their regular school studies. The course instruction could be described with an often criticized "definition-theorem-proof" structure, which was applied in the topics of polynomials, matrices and vector spaces. When introduced, polynomials and matrices were approached as not being connected to each other, but were later reconsidered as instances of a vector space.

After each lesson the students were provided with a list of problems to solve at home. The solutions were not intended for submission, but variations of some of the problems appeared in a quiz in the following lesson. This led the students to active engagement with course materials and with each other. The students were encouraged to collaborate in a special closed-for-public asynchronous web-forum. Forum post-exchanges were reviewed in a search for evidence of students' identification and struggle with cross-curricular concepts. The two illustrations presented in the article were chosen to highlight various aspects of the developed account.

## Polysemous Concept Images

This section introduces the theoretical account of *polysemous concept image* that was developed in the study. I start with an illustration that stimulated the appearance of the account and continue with another illustration of its various aspects. The illustrations comprise abbreviated post-exchanges between students.

### Divisibility in the domains of polynomials and integers

*Johnny*: [1] Hi guys, I think there is a mistake in question 1d: it asks to show that $q(x)|p(x)$ when $p(x) = 3x^3 - 19x^2 + 38x - 24$ and $q(x) = 6x - 8$. I did the division and got $0.5x^2 - 2.5x + 3$, which has fractions so $q(x)$ can't be a divisor.

*Student 1*: I'm not sure that I got you. Why isn't $6x - 8$ a divisor?

*Johnny*: [2] Think about $3|7$, you divide and get $2\frac{1}{3}$, a fraction right? So 3 is not a divisor of 7. Same here.

*Student 2*: What about the question 1c?

*Johnny*: [3] It's ok. You divide $0.5x^2 - 3x - 4$ by $1 - 0.5x$, get $-x + 4$ and everyone is happy.

In Tall and Vinner's (1981) perspective, the illustration sheds light on Johnny's concept image of divisibility (or divisor). His image is an ontologically distinct category containing (at least) two types of conceptions: the ones that regard divisibility in the domain of polynomials and the ones that regard divisibility in the domain of integers. For instance, Johnny's utterances [1] and [3] show that he was aware that the two domains contain different elements (i.e. polynomials and integers) and different division procedures. Accordingly, I propose that Johnny's concept image was compartmentalized into *domain-valid conceptions* – ways of thinking that he perceived as valid in one domain but not in another. Furthermore, Johnny was successful with choosing conceptions from the domain that was intended in problem situations in [1-3].

The connections that Johnny drew between the divisibility in each domain allow proposing that both of them were regulated for him by some common set of conceptions. For example, in both domains Johnny used the '|' symbol for denoting divisibility. The element on the left of '|' was the one by which the element on the right was divided. I use *overarching conceptions* to refer to ways of thinking that can be assigned to one's concept image as a whole. Overarching conceptions make one's concept image *polysemous*: consisting of distinct domains that are connected through conceptions which are valid in each of them (see Figure 1 for a schematic representation of Johnny's concept image).

Johnny's reasoning in the domain of polynomials (see [1] and [3]) can be rephrased as "*r(x)* is a result of dividing *p(x)* by *q(x)*. If *r(x)* contains non-integer coefficients, then *q(x)* is not a divisor of *p(x)*". This reasoning resonates with a variation of conventional definitions of a divisor in the domain of integers: "*r* is a result of dividing *p* by *q*. If *r* is not an integer then *q* is not a divisor of *p*". Accordingly, Johnny's mistake in [1] can be explained with a misclassification of a conception, which is valid in the domain of integers, to the set of overarching conceptions regulating the whole image of the divisibility concept.

Johnny's doubt in [1] in the correctness of the assigned problem suggests that he was convinced in the validity of his misclassified conception. What way of thinking might have helped Johnny to question his solution or maybe even to avoid the mistake? A formal definition of divisibility of real polynomials did not seem to help although he was exposed to it in the classroom[1]. Allow me to address the question with a speculative proposal: in the course of his mathematics learning, Johnny engaged with a variety of concepts and domains. In many cases, if a concept was used to manipulate with elements from the domain, the result was also an element belonging to the same domain (e.g., union, intersection and exclusion of sets is a set; calculations with numbers result in a number; operations with functions create a function). It is very likely that Johnny was not aware that he engaged with instances of closure under operation. We can only wonder whether thinking in terms of "operation with elements from the domain often results in an element belonging to the same domain" would have made a difference for Johnny in the problem situation under discussion. Potentially, this way of thinking might have led him to an observation that his reasoning [1-3] was not consistent: in [3], he operated with polynomials having coefficients of a half and negative half, which indicates that he accepted polynomials with non-integer coefficients as elements of the domain of polynomials. However, in [1] he did not accept such a polynomial as a legitimate result of the division operation between two polynomials, which deviates from the presented way of thinking in terms of closure.

I consider a closure as an instance of a *meta-premise* – a generalized way of thinking, which is conceptualized as valid for various mathematical concepts and domains. Additional examples of meta-premises could be formulated as "a contradiction often indicates a flaw in preceding reasoning", "the same symbol usually denotes the same concept", "if a concept has different

[1] The standard definition that was provided in the classroom stated: Let $p(x)$ and $q(x)$ be polynomials in $\mathbb{R}[x]$. If there is a polynomial $r(x)$ such that $p(x) = r(x) \cdot q(x)$ then $p(x)$ is said to be divisible by $q(x)$ and we denote $q(x)|p(x)$.

definitions then they are likely to be equivalent". Note that while these meta-premises are valid in many cases, they are not always valid. Accordingly, Johnny could only notice that his reasoning in [1-3] deviates from the meta-premise but there would be no reason for him to interpret the deviation as a univocal indicator of a mistake. Indeed, there are concepts for which the described closure-under-operation premise does not hold, for example, a scalar product of vectors is not a vector but a number.

The presented post-exchange does not contain evidence of Johnny's engagement with a meta-premise. Next illustration demonstrates how a student can attend and interpret a deviation of their domain-valid and overarching conceptions from meta-premises.

**Multiplication in the domains of vectors and numbers**

In the following post-exchange, students discussed the problem:

In an isosceles triangle *ABC* (*AB*=*AC*) the medians to the legs are perpendicular (*BB'*⊥*CC'*). Find the value of cos *A* (see Figure 2).

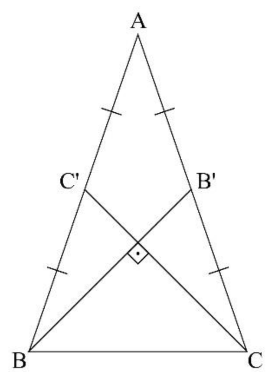


*Figure 2. A triangle from Molly's problem.*

*Molly*: [1] Please find my mistake: I say that $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AC} = \vec{v}$.

[2] So I know that $\overrightarrow{BB'} = \overrightarrow{BA} + \overrightarrow{AB'} = -\vec{u} + 0.5\vec{v}$. In the same way I know that $\overrightarrow{CC'} = -\vec{v} + 0.5\vec{u}$.

[3] Now they are perpendicular so: $\overrightarrow{BB'} \times \overrightarrow{CC'} = 0$.

[4] $(-\vec{u} + 0.5\vec{v})(-\vec{v} + 0.5\vec{u}) = 0$

[5] $\vec{u}\vec{v} - 0.5\vec{u}^2 - 0.5\vec{v}^2 + 0.25\vec{u}\vec{v} = 0$ (actually ignore this part)

[6] $-\vec{u} + 0.5\vec{v} = \vec{0}$ or $-\vec{v} + 0.5\vec{u} = \vec{0}$

[7] and I get $\vec{u} = 0.5\vec{v}$ or $\vec{v} = 0.5\vec{u}$.

[8] But when I come to plug it in the formula I get $\cos A = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{0.5\|\vec{v}\|^2}{0.5\|\vec{v}\|^2} = 1$. So it means that it is a right-angle triangle, which can't be.

*Student*: What's the problem with a right-angle triangle? Then you get cos*A*=*1* and we are done.

*Molly*: [9] I wish… Then we have two different perpendiculars from C to BB' :(

[10] I probably messed up with vectors.

In terms of the introduced approach, the excerpt shows that Molly holds a polysemous image of the multiplication concept, which she considered in the domains of vectors and numbers. While the former domain was new to her, she demonstrated a high fluency with it: she introduced vectors into a problem (see [1]), added and subtracted them (see [2] and [7]), and manipulated with an inner product for determining angles between vectors (see [3] and [8] where Molly erroneously extracted a right angle from $\cos A = 1$). However, her set of overarching conceptions is a mixture of mathematically correct and invalid concept dimensions: distributivity of multiplication is valid to vectors and numbers indeed (see [4-5]) but the symbols of '$\times$', '$\cdot$' and an empty space are not tantamount in the vectors domain (see [3-5]). Also, in [6] she presumed that if a multiplication of two vectors equal zero, then one of them is the zero vector. This is another example of a conception that was misclassified from the domain of numbers, in which it is valid, to the set of conceptions overarching the whole image of the multiplication concept.

In contrast to Johnny, Molly was convinced that her solution contained a mistake [8-9]: after determining the measure of angle $A$ in the domain of vectors, she reconsidered the obtained triangle with geometry and spotted a contradiction. Accordingly, Molly shifted the assigned problem situation between two domains and interpreted a contradiction in one of them as an indicator of a flawed reasoning employed in another. As a result of engaging with a contradiction, which I consider as an instance of a *meta-premise*, Molly pointed out correctly that she "messed up with vectors" (see [10]). The forum contained no additional data to suggest how Molly continued her work, if at all. However, engaging with a meta-premise helped Molly to become aware of a flaw in her reasoning, which I consider to be a milestone towards restructuring the concept image that she developed.

It is worth mentioning that similarly to the case of Johnny, Molly's engagement with a meta-premise cannot be considered as a dogmatic strategy for verification of developed conceptions and solutions. Molly seemed to take for granted that the concepts of angles, triangles and perpendicularity preserve their dimensions after being shifted from the domain of vectors to the domain of geometry. Clearly this was correct in the particular case. However, spotting an incompatibility of concept's dimensions in axiomatically different domains (e.g., Euclidean and Hyperbolic geometries) requires further work for identifying its source: polysemy of a concept or a flaw in one's concept image and reasoning.

## Summary and Discussion

This article is concerned with cognitive aspects of students' engagement with cross-curricular concepts, and particularly with struggles that can emerge when familiar concept dimensions become invalid in a new mathematical domain. A polysemous approach was introduced for systematizing some sources of students' misconceptions including the affordances that might make students aware of their mistakes. The approach is a theoretical development consisting of overarching conceptions governing one's concept image as a whole and conceptions that are valid in one domain but not in another. A concept image of such a structure was referred in this article as polysemous – fragmentized into compartments which are distinct but related.

The approach was introduced for capturing students' ways of thinking in situations that require a conceptual change. Then, it is not surprising that some aspects of the findings that emerged from data analysis with the approach can be reviewed from the theoretical perspectives of Chi (1992), Vosniadou (2014) and others. For instance, the struggles of Johnny and Molly with separating between domain-valid and overarching conceptions bear resemblance to what Vosniadou (2014) calls "fragmented and synthetic conceptions". These conceptions reflect students' attempts to incorporate new and incompatible knowledge into familiar ways of thinking.

The article contributes to the literature on cognitive change by documenting cases in which students employ meta-premises for indicating the existence of mistakes in their ways of thinking, mistakes that follow from domain-valid and overarching conceptions. Notably, Molly indicated an existence of flaws in her ways of thinking without assigning the flaws to a particular conception. These indications can be associated with an intermediate stage in the process towards a conceptual change, at which learners acknowledge the need for it. Analogously, reviewing a solution or an approach to a problem from the perspective of meta-premises can be interpreted as a metacognitive mechanism that is necessary for carrying out a cognitive change.

An important feature discerning the introduced approach from the literature on a cognitive change is that it acknowledges the existence of epistemological obstacles in mathematics teaching and learning – obstacles that are ingrained in mathematics as a discipline (cf. Brousseau, 1997), polysemy for example. In this way, a polysemous perspective on students' concept images does not only account for the gaps between one's concept image and formal definitions but it also distils the domains in which the developed conceptions are valid. The importance of gaps has been addressed with extensive research conducted with the constructs of concept image and concept definition. Domains of validity are instrumental for recognizing the reasonableness in ways of thinking that students bring to the classroom. Analysis of one's ways of thinking in terms of the approach is aimed at explicating their domains of validity rather than at abandoning them.

Students' struggle with epistemological obstacles of mathematics is unavoidable by definition. In the case of polysemy two features should be considered: First, while progression through mathematical topics can yield opportunities for engaging with cross-curricular concepts, it is still up to the students to identify concepts appearing in different domains as *the same*. Second, fluency with polysemy requires acknowledgment of its existence, experience in making connections between ideas studied in different domains, as well as proficiency in axiomatics and formalism. It is not easy to find educational settings in which a combination of these forms of knowledge is systematically promoted.

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Key Memorable Events During Undergraduate Classroom Learning

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*This paper presents a theoretical construct termed Key Memorable Events (KMEs): classroom events that are perceived by students as memorable and meaningful in support of their learning, and are typically accompanied by strong emotions, either positive or negative. As such, the proposed concept implies a hierarchy between different events and their contribution to the learning by focusing on those moments perceived by students as most significant. The concept is exemplified in context of large-group undergraduate-calculus tutorials. Theoretical and pedagogical implications are discussed in terms of lesson design, data analysis, and conceptualization of learning in the undergraduate classroom.*

*Keywords:* Memorability, Classroom Learning, Instruction, Affect

## Introduction

Undergraduate mathematics courses are typically taught in a frontal teaching style (henceforth referred to as FTS), and are often attended by large amounts of students (≥ 100), especially in first-year courses (e.g., Cooper & Robinson, 2000; Pritchard, 2010). The common practice of the traditional FTS at university has been criticized repeatedly, specifically regarding the one-directional non-responsive mode of communication this teaching style promotes. In this regard, it has been argued that FTS is focused on transmitting information (Biggs & Tang, 2011), and does not promote independent student thought (Bligh, 1972). Considering that emotions have been recognized to take an integral part in mathematical problem-solving behavior (e.g., Op't Eynde, De Corte, & Verschaffel, 2006), it has additionally been claimed that FTS does not reveal the human struggle for reaching mathematical discovery, and treats students as "non-emotional audience" who are granted no room for individual difficulties (Alsina, 2002).

While alternative teaching styles have been explored and their benefits acknowledged (e.g., Larsen, Glover, & Melhuish, 2015), FTS remains to be widespread, not prone to change, and will most likely not disappear in the near future (Cooper & Robinson, 2000; Koichu, Atrash, & Marmur, 2017; Nardi, Jaworski, & Hegedus, 2005). Therefore, it is important that research efforts will additionally be put into improving the system from "within", theoretically and practically. This includes gaining a better understanding of the following: how students learn during frontal undergraduate lessons, whilst additionally examining how the learning is shaped by student affect and the teaching that took place; how frontal undergraduate lessons can be designed to create and support a positive and meaningful learning-experience for the students; and how to identify the learning opportunities in class that enable students to be actively engaged learners during the lesson itself and in support of their subsequent independent learning at home. These goals are in line with Lester's (2013) call for attention to be given to the teacher's role in problem-solving instruction and how *large groups* of students learn problem solving in real classroom situations.

The current paper addresses these goals by suggesting a theoretical construct termed *Key Memorable Events* (KMEs): classroom events that are perceived by *many* students as memorable and meaningful in support of their learning, and are typically accompanied by strong emotions, either positive or negative. It should be noted that the *key* aspect here refers to the significance of the event to the *many* at classroom level. While the construct originated out of empirical evidence (see Marmur, 2017, for further detail), the focus of this paper is theoretical, discussing the potential contributions of the notion of KMEs for lesson design, data analysis, and understanding of classroom student-learning. Nevertheless, some data are presented to illustrate the construct.

## Theoretical Background

In the literature there is a variety of concepts that employ different frameworks and terminologies for addressing what we may globally refer to as *key events* during the process of student learning. Such events are situated in the affective and/or cognitive domain, and may have a substantial impact on both the short- and long-term learning. Though the separation between cognition and affect is not always clear-cut, addressing the literature according to these two perspectives can shed light on the nature of these events.

From an affective perspective, Goldin (2014) refers to *key affective events* as events "where strong emotion or change in emotion is expressed or inferred" (p. 404). Rodd (2003) claims that "undergraduate learning is frequently triggered by those unique events which contribute to an individual's agency or self-motivation" (p. 20). In line with this claim, Weber (2008) demonstrated how a single and strong positive experience of success may have a considerable effect on a student's success in a high-level calculus course, by altering the student's attitude and type of engagement with the material for the continuation of the course. This "direct path" to attitudes and beliefs through a single powerful experience has also been reinforced by Liljedahl's (2005) discussion of "Aha!" experiences during problem-solving activities.

From a cognitive perspective, different researchers have focused on crucial moments in student thought-processes during problem-solving activities. Nilsson and Ryve (2010) focus on what they refer to as *focal events*, i.e. those parts of student reasoning that are noticeably salient. They explain that such events steer our educational attention towards "the problems that stand out as central in the students' thinking in a certain phase of a learning activity" (pp. 245-246). In an analysis of a collaborative-learning situation, Damsa and Ludvigsen (2016) identify *key moments* in the group interaction, i.e. "an action or sequence of actions at the epistemic level that triggered subsequent actions and leading to a particular relevant development regarding the shared object" (p. 5). Their analysis of such moments was based on the more general theoretical approach provided by Webster and Mertova (2007) of considering *critical events* as an analysis tool in research on teaching and learning.

## Conceptualizing KMEs

In continuation of the theoretical approaches presented above, this paper wishes to put emphasis on the aspect of the *memorability* of a classroom event, as the memorability of an event may shed "unified" light on both dimensions of cognition and affect related to student learning. The New Oxford American Dictionary (Stevenson & Lindberg, 2010) defines the adjective *memorable* as "worth remembering or easily remembered, especially because of being special or unusual". It additionally suggests the following as possible synonyms: unforgettable, significant, notable, noteworthy, important, special. These definition and synonyms suggest that *memorable events* are not merely events that can be recalled from memory upon request, but that these are events that additionally hold significance and meaning for a person who experienced them. For example, one can imagine the moment when "the penny dropped" (i.e., the moment when some important aspect of the material became understood and things fell into place) to be a memorable event for a student, and that for him/her this event was also filled with emotions, such as the excitement of success in understanding a complex concept, followed by a raise of self-confidence.

The suggested focus on memorability of events finds further support in the neuroscientific domain, which informs us that the brain does not store all information it encounters. As articulated by Wolfe (2006), the brain is in fact "designed to forget" (p. 36). Focusing on memorable events may supply insight into student short- and long-term learning processes, since, as claimed by Wolfe, memorability means that information is stored in permanent and rich networks, thus enabling its future retrieval. In this regard, "emotional events often attain a privileged status in memory" (LaBar & Cabeza, 2006, p. 54), taking a crucial physical part in filtering which information from the environment will be "saved" for future use (McEwen & Sapolsky, 1995). For example, neuroscientific experiments reveal that the triggering of negative emotions may jeopardize the functioning of the working memory during mathematics problem-solving (e.g., Ashcraft & Krause, 2007), or even induce physical reactions that can alter the memory altogether (e.g., McEwen & Sapolsky, 1995). On the other hand, lessons designed to evoke student emotion may lead to stronger memories, and can consequently serve as a hook for learning (Wolfe, 2006). In summary, this demonstrates neuroscientific reinforcement for the link between memory, emotions, and learning, and its relevance to education research (see also Hinton, Miyamoto, & Della-Chiesa, 2008).

When considering student learning-experiences in the undergraduate classroom, I propose to imagine a two-dimensional representation, where on one axis there is a detailed list of consecutive classroom events, and on the other axis a list of the different students attending the lesson. Accordingly, we may treat the location (X, Y) in the resulting table as the meaning and importance given to event X by student Y in terms of his or her learning experience at that moment in time. However, as supported by the theoretical background presented above, if we conceptualize the flow of a lesson as a sequence of classroom events, these events will not all be at the same level of importance to students, and some events may be more significant than others. These will be referred to as *memorable events* (see Figure 1: Key Memorable Events).

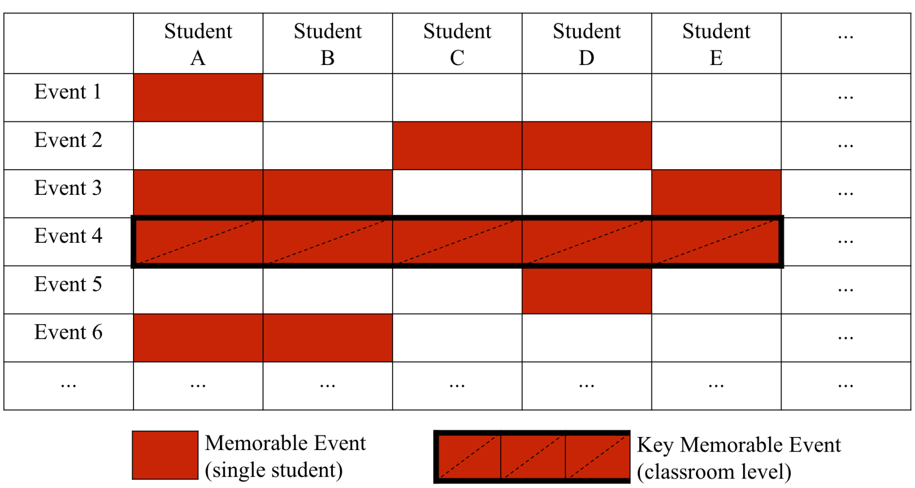


*Figure 1: Key Memorable Events*

Continuing this representation, I suggest two complimentary approaches that may be used for addressing and analyzing the nature of the learning in class. We may consider the learning of a *single student* through the progression of classroom events, leading us to a conceptualization of a single student's learning trajectory, in Simon's (1995) framework, or the student's affective pathway, in Goldin's (2000) framework. On the other hand, and what is here emphasized, is the focus on a *single event* and how this event affects the different students in class in similar ways. This approach may supply insight into how a specific pedagogical act impacts the students as a *group* by recognizing repetitive themes students report on regarding this event. Repeating this approach with different events during a lesson may result in an overall categorization of how various instructional acts relate to the students' learning as a whole. Accordingly, this approach emphasizes the immediate relation between the teaching and the overall classroom learning.

Focusing on the memorability of events on the group level, it should be acknowledged that what may be memorable to one student, may not be memorable nor significant to another. The term *key memorable events* is thus used to refer to those events in the lesson that are perceived by *many* students in class as memorable and meaningful in support of their learning (see Figure 1: Key Memorable Events). Whereas an operational definition should additionally quantify the phrase "many students", this conceptual definition could nonetheless be easily implemented methodologically by means of stimulated-recall interviews (see Marmur, 2017) in order to identify classroom events that are memorable and key.

## Two Examples

The two examples presented here are taken from a wider research project examining student learning in undergraduate large-group calculus tutorials. The first exemplifies the utility of KMEs as a guide for lesson design, whilst the second example demonstrates its utility as a methodological tool in explaining classroom phenomena.

### Example 1: A "Designed" Setting

Marmur and Koichu (2016) presented an iterative process of lesson design aimed at creating an aesthetic experience for students in a "traditional-instruction" calculus tutorial. In the final successful iterations of the lesson, the incorporation of two surprising events of reaching a dead-end in the solution, prior to the surprising presentation of a non-routine solution, managed to serve as a teaching method leading to an aesthetic experience for many students.

Revisiting the data with the suggested terminology of KMEs, the students' strong and emotional responses to these "surprising events" revealed that these events indeed served as KMEs of affective nature for the students (as expressed in stimulated-recall interviews, as well as observed in class). The data suggest that this combination of KMEs not only supported the creation of an aesthetic experience, but was also most influential on the students' learning process in both cognitive and affective terms (see also Koichu et al., 2017). In affective terms, the students reported heightened involvement and enjoyment during the lesson, as well as a rise in self-confidence by being encouraged not to give up when initial attempts at solving a challenging problem are unsuccessful while working independently. In cognitive terms, the KMEs raised the students' focus, attention, and interest during the lesson; they exposed the students to an expert's thought process of how to reach a solution; and they enhanced the memorability of the material taught. Moreover, students reported to be actively engaged learners by attempting to accomplish a range of self-imposed tasks. These included independently testing alternative ideas to the solution; attempting to predict the next step in the solution; identifying difficult places in the proof to come back to; looking for connections between the problem taught in class and the corresponding homework assignment; and formulating problem-solving strategies from their current experience that they could use in the future. It should be noted however, that the students reported that this was not part of their routine behavior in class. Additionally, some students reported that this lesson impacted their after-class learning activity in a non-standard way. This included deciding to independently re-solve the problem, as well as go through all material from the beginning of the semester. The latter was encouraged by the students' newly recognized need for what Schoenfeld (1985) refers to as *resources* – knowledge in support of solutions for non-routine types of problems.

**Example 2: A "Regular" Instructional Setting**

Marmur and Koichu (in press) juxtaposed two similar large-group undergraduate-calculus tutorial-lessons as a contrastive basis to examine how students' emotional states relate to the type of mathematical discourse conducted in class. Though both lessons contained a similar challenging problem the students did not understand, the students evaluated the lessons in opposite manners. While the lack of understanding in one of these lessons (Lesson-N) was (unsurprisingly) accompanied by negative student emotion of anger and frustration, the second lesson (Lesson-P) was (surprisingly) perceived by the students as special and good, even though they admitted key parts of the proof to be incomprehensible, and showed disbelief in their ability to solve such problems on their own.

In KME terminology, the difference was analyzed by the identification of a single KME of affective nature per lesson, that shaped the students' overall learning experience. The difference was related to the type of discourse initiated by the instructor during the identified KME. In the KME of Lesson-P, the instructor initiated a type of meta-level discourse on how to approach a challenging problem (termed *heuristic-didactic discourse*). According to the data, this KME demonstrated expert problem-solving heuristic-behavior in an approachable way to students, shaped the learning experience during the rest of the lesson, and additionally served as a neutralizing factor for possible negative emotions as a result of not understanding the solution. However, in the KME of Lesson-N, the instructor made a promise for heuristic-didactic discourse, yet did not fulfill this promise in the remainder of the lesson.

## Discussion

The proposed concept of KMEs may supply insight into student learning at group-level, and could be utilized both as pedagogical tool for the improvement of undergraduate teaching (as in Example 1) and as methodological tool for analyzing real classroom situations (as in Example 2). As the second use requires a presentation of additional data outside the scope of this paper (see Marmur, 2017), I expand the discussion focusing on the first proposed use for teaching improvement, as well as a theoretical reflection on classroom student learning.

**KMEs as a Guide for Lesson Design**

Being that KMEs are conceptually regarded as events that are perceived as memorable and meaningful by *many*, I suggest that they may be utilized by instructors as indicators for events in the lesson that will most likely remain "invariant" in future "same" lessons taught to other students. Building on variation theory (Marton & Booth, 1997; Runesson, 2005) for lesson design, Watson and Mason (2009) claim that variation in lessons should be "foregrounded against relative invariance of other features" (p. 98). As suggested by their argumentation, the significance of understanding the invariant aspects of lessons is to be able to utilize them as "anchors" on top of which variation is created. Accordingly, KMEs may indicate a likely *invariance* in a lesson design, which provides crucial information for its further development and refinement. In practice, utilizing the notion of KMEs as such "anchors", may additionally allow lecturers and instructors to each time "fill" them with varying mathematical content.

In this regard, it should be noted that the creation of hierarchy between different classroom events in regard to student learning, as suggested by the KME concept, is not foreign to the way some researchers address the mathematics itself. For example, according to Leron (1983), the common practice of presenting proofs linearly in undergraduate lessons is unsuccessful in its support of student learning, as it lacks the dimension of communicating how such proofs are generated and thought of. Rather, Leron's notion of a "structural proof" suggests to first supply students with the general structure of a proof, and only then start filling in the missing details.

Continuing the analogy, while this paper focuses on *key events* as experienced by many students, in the literature we may find research focusing on *key mathematical ideas* (e.g., Gowers, 2007; Hanna & Mason, 2014; Raman, 2003). Raman (2003) states that: "For mathematicians, proof is essentially about *key ideas*; for many students it is not" (p. 324). Gowers (2007) emphasizes the relevance of key ideas to mathematical activity, by claiming that a focus on the key ideas of a proof may increase its memorability and promote its mathematical understanding. This naturally comes with the pedagogical implications of how to reveal to students what these key ideas are, and how these may serve their learning. In relation to lesson design, I suggest that the notion of KMEs may be considered in combination with key mathematical ideas that could be learned in context of the problem (as in Example 1, a KME around a non-routine solution method for a challenging mathematical problem).

An additional use of KMEs for lesson design relates to the affective domain, and more specifically to negative student emotions. As we know, negative emotions are a natural part of mathematical learning, and as Goldin (2014) suggests, may even lead to greater satisfaction and pleasure when "overcoming" challenging problems. However, it is our responsibility as educators and researchers to support students' meta-affect in relation to such experienced emotions. As claimed by Goldin (2000, p. 218):

> "The challenge to the educator is to interrupt the incessant negative feelings, a first and necessary step in the needed cognitive and affective reconstruction. The challenge to the researcher is to find ways to do this."

I suggest that the concept of KMEs may be utilized in lesson design as a tool to "steer" student emotions towards specific segments of a lesson, and thus be able to reduce the overall experienced negative emotions. This is illustrated in Lesson-P (see Example 2), in which the creation of a very positive event at the beginning of the lesson (on how to approach the problem), managed to neutralize possible negative student emotions stemming from not understanding the subsequent solution. Such a case demonstrates that, even though we may not be able to stop students feeling frustrated when dealing with challenging mathematics, we may utilize the idea of KMEs in order to design lessons that could shape what would ultimately be perceived as a more positive memorable learning-experience.

## KMEs and Theoretical Considerations on Learning

> "The line, even in science, between serious theory and metaphor, is a thin one—if it can be drawn at all. [...] There is no obvious point at which we may say, 'Here the metaphors stop and the theories begin.' " (Scheffler, 1991, p. 35)

In regard to existing literature and theory, I suggest that the KME concept may provide a theoretical contribution to our views on learning, as well as on the evolvement of student affect during the process of learning. Sfard (2015) argues that we conceptualize and examine human learning by utilizing metaphors, and that "what often appears as but an innocent figure of speech may in fact inform how we think about the topic, what we are able to notice, and what pedagogical decisions we are likely to make" (p. 635). Furthermore, Sfard (1998) emphasizes that in order to produce a critical theory on learning, we must be willing to lean on more than one "metaphorical leg" (p. 11), even if the different metaphors induce some level of contradiction.

Though learning is not a linear process, I argue that some of the very useful learning metaphors we find in the literature at least hint towards some level of linearity. Conceptualizing student learning over the course of a lesson in metaphors such as "paths", "pathways", "trajectories", or "tracks", indeed has great pedagogical value in terms of lesson design. As suggested by Simon's (1995) notion of *learning trajectories*, a teacher may hypothesize on learning paths of students, whilst aiming to "match" these with preconceived teaching goals. However, at least on a theoretical level, this metaphor may imply an unrealistic linearity in the process of learning, where students have to go through a series of consecutive steps, each one supporting the following. It is interesting to note that even when discussing emotions, which are clearly nonlinear, in context of learning, we find it convenient to conceptualize these into *affective pathways* (Goldin, 2000) that progress from one emotional state to the next.

I suggest that the concept of KMEs may supply a "horizontal" approach to learning (see also Figure 1: Key Memorable Events), enriching the more dominant "vertical" view on learning as pathways. The KME notion suggests a hierarchy between different events and their contribution to learning, whilst putting emphasis on what we may regard as "*snapshots*" or *highlight moments* in a lesson as identified by many. I do not wish to imply that other moments in a lesson are insignificant for the learning, or even that they do not play a contributing role in the creation of a KME. However, I suggest such a hierarchal approach is not only a useful tool for lesson design, but also examines learning in closer relation to what we may refer to as our "human experience". If for example we imagine a musical piece, it is reasonable to assume we will not remember all individual notes. Rather, our overall experience is shaped by certain highlight moments during its course (Huron, 2006). Bringing the analogy back to KMEs, the findings presented in Example 1 and 2 suggest that a student's overall learning-experience in a lesson is shaped by several key moments during its course.

Lastly, I suggest that the KME concept may supply an added layer to Goldin's (2000, 2014) theory on local and global affect. While the KME construct addresses emotion experienced during a lesson, the dimension of strong memorability points towards a possible affective phase situated in an "intermediate zone" between local emotional states and longer-term attitudes and beliefs. Though there seems to be a consensus in the literature, that the more stable attitudes and beliefs are a result of a lengthy and slow process of experiencing repeated emotional states (e.g., Goldin, 2000; Zan, Brown, Evans, & Hannula, 2006), not all these experienced states are of equal importance. Focusing on those experienced emotions during what is here referred to as KMEs, may supply us with a more accurate indication of how this transitional process possibly takes place.

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# Toward a Functional Grammar of Physics Equations

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*An area of student difficulty in introductory physics courses is how they use and reason with equations. We propose that part of this difficulty is due to meaning that is embedded in the structure of equations. As equations are manipulated, their structure and concomitant meanings change. As mathematics is considered the "language of physics," our starting point will be to propose that it has a grammar. As equations change form and meaning, they are doing so within a certain grammatical system. We will show how physics equations can be categorized and mapped to ideational clause types as devised by Halliday (1994). This mapping could be useful in relating the mathematical "language" used in physics to "natural language," benefitting physics instructors who are trying to understand the struggles of their students, and helping students to understand the rich meanings embedded in physics equations.*

*Keywords:* physics, mathematics, linguistics, ontology, grammar

It is often stated, to the point of cliché, that "math is the language of physics." This is intuitive, and readily accepted by most. The concepts of physics can be explained without any mathematics whatsoever, but this approach results "...in an understanding of physics that is fundamentally different from physics as understood by physicists" (Sherin, 2001, p. 524). Certainly, for most who are looking for any kind of practical aptitude in these concepts, it is essential to be able to work with equations.

If, indeed, mathematics is the language of physics, what kind of a language is it? What is its system of grammar? Knowing this could be useful, especially for educators whose competence in this language has surpassed the need to think of its underlying structure. At such a high level of expertise, it can be difficult to truly understand what is causing students difficulties as they learn how to communicate and do physics with mathematics. If instructors could see how conceptually complex it really is to know what equations *mean*, perhaps they could better understand the struggles of their students and be better equipped to help. Research has been conducted into the lexical meaning of symbols in physics and how those diverse meanings pose both interpretative and epistemic difficulties for students (Torigoe & Gladding, 2011; Redish & Kuo, 2015). Others have examined the structure of equations themselves and how that structure facilitates or constrains physical reasoning (Sherin, 2001; Landy, Brookes, & Smout, 2014). Weinberg, Dresen and Slater (2016) have examined mathematics as a semiotic system used productively by students for meaning-making. But to the best of our knowledge, no real attempt has been made to develop a grammar of physics equations. Our goal in this paper is to lay the groundwork for this process. We are going to suggest that equations have fundamental spatial structure, ordering, and function that encodes underlying meaning and it is in this area that additional challenge arises for students. It should be noted that our focus is on physics equations, in particular those seen by students in lower-division undergraduate physics courses. The broader discussion about the grammar of mathematics as a whole is beyond the scope of our work.

This project began with the observation that common physics equations can be separated into different categories based on their meanings. As equations are rearranged or manipulated, these meanings change. For instance, $a=F/m$ (Newton's second law) is what we would call a *causal* equation; it has an effect - acceleration - on the left side, and a cause - force - on the right. Mass is an inhibitive contribution to the cause, as it is inversely proportional to the effect. Research has already empirically shown that equation users are sensitive to the ordering of causal equations (result left of the equals sign, cause on the right), and reversing the order is confusing or changes the meaning of the equation (Mochon & Sloman 2004). We claim this is prima facie evidence of equations having a grammatical structure.

## Is Mathematics a Language?

Before proceeding, we must ask a simple question that - it appears to us - has no simple answer: Is mathematics legitimately a *language*? To address this, we will start in the broader realm of semiotics. Mathematics is most certainly a kind of semiotic system; it is vehicle for making meaning and communicating. Semiotics, to put it simply, is the study of signs. There are two predominant models of signs: the dyadic and the triadic. A dyadic sign would consist of a "signifier" and a "signified." The signifier could also be called the "sign vehicle" and the signified the "referent" (Noth, 1990). Essentially, there is a thing being represented and a way of representing it. The chief limitation of the dyadic model is that it lacks context. There are semantics and syntax, but no pragmatics (Ongstad, 2006). A triadic model, as devised by C.S. Peirce, would add what he called the *interpretant* to the previously described schema. A more commonly used term for this is *sense*, meaning there is someone or something "receiving" the sign and interpreting it. The triad is thus *sign vehicle*, *sense*, and *referent* (Noth, 1990).

Both of these models have played out in existing analysis of the semiotics of mathematics. A kind of dyadic model is proposed by Rotman (1988), in which the mathematical sign has the components *thought* and *scribble*. The two cannot be separated and be considered a true mathematical sign. In Rotman's mathematical semiosis, a person in essence *creates* a Mathematician, who then creates an Agent. Each of these take on the firstness, secondness, and thirdness, as devised by Peirce, as they proceed through the creation of a proof (Rotman, 1988). Ongstad advocates for a triadic model in which a sender and receiver are involved in the interpretation of content. The sign itself in this case is made up of the elements Symptom, Symbol, and Signal. These correspond to senders, "objects or states of affairs," and receivers, respectively (Ongstad, 2006).

The semiotics of mathematics is a rich topic. It is clear that in doing mathematics, we are engaging in some form of communication. But what of our treatment of mathematics as a language? Leibniz attempted to develop a universal language involving a "...*calculus ratiocinator*, a system of rules for the combination of semantic primitives" (Noth, 1990, p. 274). Frege's mathematical symbolism "embodies fundamental principles of reasoning based on an analysis of language" (Bouissac, 1998, p. 249). Rotman describes mathematical texts as being a "...mixture of natural and artificial signs...conventionally punctuated and divided up into what appear to be complete grammatical sentences…" (Rotman, 1988, p. 7). Ongstad gets more specific in proposing that mathematics could be "...a set of interrelated, semiotically different languages or sign systems" (Ongstad, 2006, p. 248). If one takes this perspective, it is reasonable to suggest that physics equations might be a language in their own right, semiotically different than the languages of pure mathematics or statistics (Redish & Kuo, 2015).

What is it that makes physics equations unique? At least part of it is that they do not consist of pure, abstract mathematical objects; rather, they use these objects to describe patterns in nature. This distinction is what shall characterize the ontology - and, as we shall see, the grammar - of physics equations.

## Systemic Functional Grammar

If we are to devise a grammatical system of physics equations, it must - if it is to be useful - be analogous to one that is familiar to us. We will use our native language of English, but we will attempt to minimize applications of grammatical concepts that are not also applicable to other languages. For this kind of universality, we look to Functional Grammar, as devised by Halliday. It is a systemic theory, a "...theory of meaning as choice, by which language, or any other semiotic system, is interpreted as networks of interlocking options…" and it is "...functional in the sense that it is designed to account for how the language is **used**" (Halliday, 1994, p. xiii). Our focus has been primarily on the experiential aspects of the grammar, which look mostly at the clause. The clause in this framework has three different metafunctions (sometimes called components); ideational ("clause as representation"), interpersonal ("clause as exchange"), and textual ("clause as message") (Halliday, 1994). The ideational metafunction is the most appropriate for application to our mathematical "clauses" of interest. Physics equations do, after all, *represent* - or model - objects, interactions, systems, and states (Etkina, Warren, & Gentile, 2006).

The ideational metafunction models the clause as a process, within which there is an internal process (typically a verb), a participant (typically a noun), and, optionally, a circumstance (Halliday, 1994). For instance, in the sentence, "The girl caught a fish from a lake," "The girl" and "a fish" are participants, "caught" is a process, and "from a lake" is a circumstance. These classifications are quite broad, so different process types are used depending on the function of the clause. This type of categorization is called a *transitivity system*. Structure is "explained in terms of meaning." The three primary process types within the ideational component are material, mental, and relational processes. These are processes of doing, sensing, and being, respectively (Halliday, 1994).

For example, a material process might be our previous example (omitting the circumstance for brevity), "The girl caught a fish." In this case, we call "the girl" the Actor and "a fish" the Goal. These describe processes of *doing*.

In the case of a mental process, it is necessary to have a personified participant - something that can "sense" something else. In the sentence, "He likes it," "he" is classified as a Senser and "it" is a Phenomenon. "Likes" is the process. We sometimes use this language in physics to personify things like electrons, which we might say "want to be in the ground state" (Brookes & Etkina, 2007).

Relational processes are the most varied and complex, as they describe relationships in which things are identified, symbolized, or otherwise related to other things. The two main types of relational processes are attributive and identifying. The participant types associated with these are Carrier/Attribute and Identified/Identifier (sometimes called Token/Value), respectively. The former treats a participant as a member of a category, while the latter *identifies* the participants as each other, and is thus reversible ("Alice is wise" vs. "Alice is the wise one") (Halliday, 1994). In addition, the relational process has three subcategories: Intensive ('$x$ is $a$'), circumstantial ('$x$ is at $a$'), and possessive ('$x$ has $a$'). These can each be combined with either "attributive" or "identifying" to form such combinations as "circumstantial identifying" or "possessive attributive." Distinctions like this will be quite useful in formulating a kind of transitivity system for physics equations.

Finally, our brief summary of some important aspects of functional grammar must include a discussion of what goes on "below the clause." At this level, the ideational component splits into two categories: Experiential and logical. These turn out to be two different ways to examine phrases and groups within a clause, and the ordering of functional elements within a group. For instance, let's look at the experiential structure of the nominative group "those two old diesel trucks." It exhibits the typical ordering of elements: Deictic, Numerative, Epithet, Classifier, and Thing. To arrange the sentence in any other way would not make sense. If we look at the same group's logical structure, we would call "trucks" the Head and everything else the Modifier. Each word is then assigned a Greek letter, starting on the right with "trucks" and moving to the left. We would thus *read* this group's logical structure from left to right: Modifier (ε, δ, γ, β), Head (α). Conceptually, this ordering is characterized as moving "...from the kind of element that has the greatest specifying potential to that which has the least…" (Halliday, 1994, p. 187) This type of analysis could be effective in characterizing the order of elements in mathematical "groups" as well. There are clearly certain consistent tendencies, like putting numerals before constants, which are then put before variables. Our focus here is less on the group and more on the clause, as we aim to set up a transitivity structure for our equations. However, the ways in which mathematics is at least "like" a language continue to unfold; it does not appear to be a superficial connection.

## Ontology, Grammar, and Interpretations of Equations

An important concept in this discussion is that of ontological "trees," as devised by Chi, Slotta, & de Leeuw (1994) and later modified by Brookes and Etkina (2007). Chi et al. proposed that people separate the world into three primary ontological categories (trees), each having its own subcategories (branches). These are Matter, Processes, and Mental States. When an idea or entity is initially conceived to belong to one of these categories, and then must be moved to another, this is called *conceptual change*. Topics that require this kind of shifting exhibit a kind of "incompatibility" of conception and tend to be more difficult to learn. This is part of what is called the Incompatibility Hypothesis. Many science concepts require the learner to continually alternate between categories, which creates exceptional difficulty (Chi et al., 1994).

The version of this model adapted specifically for the language of physics by Brookes and Etkina changes the category of Mental States to the more general States (Brookes & Etkina, 2007). Etkina, Warren, & Gentile devised a taxonomy of physical models, which comprised of models of objects, interactions, systems and processes (split into causal and state equations) (Etkina et al., 2006). These were mapped to the ontological categories of matter processes and physical states, in part to understand the prominent use of metaphorical language in how physicists talk about physical ideas (Brookes & Etkina, 2007). For example a physicist might say "Energy flowed into the system by heating." In this sentence "energy" is the *matter*, "flowed" is the *process*, and "by heating" is a circumstance that elaborates the nature of the *process*. On the other hand, a physicist could say "Heat flowed into the system." In this case, "heat" has shifted in its grammatical function (from circumstance to participant) and likewise has shifted its ontological category from elaborating the *process* to being categorized as *matter*.

Another important precedent for our work is the concept of *symbolic forms*. Symbolic forms are what Sherin (2001) describes as "knowledge elements" with two components: A symbol template and a conceptual schema. The symbol template component is primarily how Sherin distinguishes the forms. This is an abstraction of a mathematical expression in which symbols are replaced primarily with shapes ($\square$), generalized variables ($x$) and ellipses (...), so that the focus is on the structure of the expression, rather than its specific content. For example, the symbolic form "balancing" is represented with the symbol template $\square = \square$. "*Identity*" is represented with $x = […]$. Sherin presents a "semi-exhaustive list" of these forms, and suggests that its organization into "clusters" is "...primarily for rhetorical purposes - not to reflect any psychological grouping of the elements. However, within a given cluster, the various schemata tend to have entities of the same or similar ontological type" (Sherin, 2001, pp. 505-506). We suggest that the reason these clusters are not clearly defined is because of the level of abstraction used in the model of symbolic forms. The meanings of the mathematical structures cannot be adequately understood when their "participants" have been generalized and removed from their processes. Sherin seems to be primarily analyzing the constituent structure (Halliday, 1994) of the equations' *orthography*, not their grammar. That is, we see how this "language" is typically written down, but not how it is used. For example: $\vec{p} = m\vec{v}$ fits into the *identity* template and is recognized as such in physics, but so does $N = mg$, an equation that does not represent *identity*.

Among Sherin's references is Anna Sfard, who has written at length about the meaning of the equals sign. She has suggested that although "*...there is a deep ontological gap between operational and structural conceptions…* they are in fact *complementary*" (Sfard, 1991, p. 4). This duality of object vs. process is found in many forms in the mathematics education literature, but Sfard's comparison of this distinction to the "complementarity" of waves and particles in quantum mechanics is unique. This illustrates the subtlety and ambiguity of the equals sign, our grammatical "process."

## Mapping Grammar to Equation Types

Taking into consideration all of the literature reviewed above, the most potent for analyzing the meanings of physics equations has been Halliday's functional grammar. It works surprisingly well to simply map ideational process types to equation categories. This mapping is certainly not one-to-one, but it offers useful distinctions and is remarkably consistent with the way these equations are used in (for example) Knight's popular undergraduate physics textbook (Knight, 2004).

We started the mapping by dividing equations in physics into three broad categories based on three distinct meanings of the equals sign. Building on prior work (Keiran, 1981; Sherin, 2001; Redish and Kuo 2015) we recognized that an equals sign in physics can mean "is," "is equal to," or "is a result/consequence of." The second key to mapping equations to grammar is to examine how the equation functions in relation to the words that surround it. In other words: equations in physics *cannot* be separated from the surrounding (English) sentence if we want to understand their full meaning. We will present our analysis based on a commonly used University-level, introductory physics textbook (Knight, 2004).

In our framework, equations in which the equals sign means "is" belong to a category we called "Operational Definition," similar to Sherin's *identity* template. For example, acceleration $\vec{a} = \frac{d\vec{v}}{dt}$, or momentum $\vec{p} = m\vec{v}$ operationally define useful physical quantities in terms of how they are measured. These correspond to grammatical relational intensive processes. We take this to generally also mean these processes are identifying rather than attributive, because mathematics does not have quite the same issues with reversibility and active vs. passive voice that we have in English. For example, when dealing with momentum, Knight writes, "The product of a particle's mass and velocity is called the momentum of the particle: momentum = $\vec{p} = m\vec{v}$" (Knight, 2004, p. 262). This is an intensive identifying clause; it is reversible, and it serves to assign an identity to the signs $p$ and $mv$. The equation itself does essentially the same thing. The two signs are interchangeable - the equation serves to identify.

Next, we identified a category of equations where the equals sign reads "is equal to". Equations in the "Is Equal To" category map to relational circumstantial processes. Each of these is true only within certain specific circumstances. One example from Knight: "If the angle $\theta$ is such that $\Delta r = d\sin\theta = m\lambda$, where $m$ is an integer, then the light wave arriving at the screen from one slit will be *exactly in phase* with the light waves arriving from the two slits next to it" (Knight, 2004, p. 938). In this sentence $d\sin\theta = m\lambda$ functions as a grammatical circumstance. On the other hand, that equation, presented on its own on (for example) an equation sheet, lacks any indication of its circumstantial nature. We hypothesize that an expert seeing this equation implicitly sees the surrounding context as well, even when it is absent. The sentence quoted doesn't come out to be a single circumstantial clause, but it doesn't need to. As we have noted, this mapping from the grammar of language to the grammar of physics equations is not one-to-one. The distinguishing characteristic of these equation types is their being tied to circumstance, sometimes in a subtle way. For example, $N = mg$ (another equation that falls into the "is equal to" category) is only applicable when an object of mass $m$ is resting on a level/horizontal surface, close to the earth's surface and ignoring the fact that our rotating (Earth's) reference frame is slightly non-inertial. Halliday writes of circumstantial identifying processes in which the participants act as the circumstantial element: "The relation between the participants is simply one of sameness; these clauses are in that respect like intensives, the only difference being that here the two halves of the equation - the two 'participants' - are, so to speak, circumstantial elements in disguise" (Halliday, 1994, p. 131). This true of an equation like $\lambda = v/f$. In order for this to be true, there has to *be* a frequency $f$ to create a wave with a wavelength $\lambda$ constrained to be traveling at a velocity $v$ through a medium.

The third meaning of the equals sign ("is a result/consequence of") defines a category of equations we have called "Causal." These equations represent material processes - processes of *doing*. Much as we have an Actor and a Goal in such grammatical processes, causal equations have what we shall call a Result (or Change) on one side of the equals sign, with an Agent and, optionally, an Inhibitor on the other. In the ubiquitous equation that represents Newton's second law, $\vec{a} = \frac{\sum \vec{F}}{m}$, "$a$" is our Result (*defined* to be $\frac{d\vec{v}}{dt}$: a rate of change in velocity with respect to time), while "$\sum \vec{F}$" is the Agent and "$m$" is the Inhibitor. These kinds of relationships are some of the clearest, as we can see in Knight's words: "A force applied to an object causes the object to accelerate" (Knight, 2004, p. 120). It is interesting to note that some controversy about the causal form of Newton's second law exists. Although many textbook authors readily acknowledge forces exerted on an object cause the object to accelerate, Newton's second law is frequently written in a form that contradicts that causality: $\sum \vec{F} = m\vec{a}$. Future work needs to examine whether there is indeed a student difficulty that arises from the apparently inconsistent ways in which Newton's second law is presented and understood by experts. Another example of a causal equation with the Inhibitor absent would be the first law of thermodynamics: $\Delta U = Q + W$. Here $Q$ and $W$ represent the two possible agents (heating and work) that result in the change in internal energy $\Delta U$ of the thermodynamic system.

## Implications and Future Work

Students in beginning physics courses face many challenges, but perhaps the most daunting for them is the use of equations. A substantial contributing factor to this area of student difficulty is a sense of what equations *mean*. This problem is both lexical *and* ontological in nature, and if we are to understand the ontological challenges that students face, we need to understand the grammar of physics equations. We suggest that in order to understand equations in action we need to understand how their meaning shifts as they are rewritten and manipulated. For example, a student might start out with a causal form of Newton's second law: $\vec{a} = \frac{\sum \vec{F}}{m}$ and at some later point rewrite the same equation as $F = ma$ to find the value of a particular force in order to solve a specific physics problem. In manipulating the equation this way, force has shifted grammatically (and consequently, ontologically) from being an Agent to an entity that can be determined from other physical quantities. Grammatically the equation has shifted from the material process category to being a circumstantial clause. We hypothesize that, in a way that is analogous to spoken or written language, the meaning and function of entities in an equation are constantly shifting as the equation is re-organized, and manipulated. A key part of reasoning with mathematics (Redish and Kuo, 2015), is being comfortable with these shifts, just as a native speaker of a language is comfortable turning verbs into nouns and noun into verbs in a way that is communally understood by other native speakers of that language. In short, the mark of native speakers of a language is their ability to play "fast and loose" with the lexico-grammatical interaction of that language and still engage in meaningful communication. A detailed exposition of this should be followed up upon in future work.

Educators often take the reasons for students' difficulty with equations for granted because of their experience and expertise. If educators could see how complex the meanings of common physics equations are, they could perhaps be more equipped to help students make sense of them. If we are willing to look at these equations as a part of the "language of physics," as most already do, we can treat them as clauses in this language. Halliday's Functional Grammar is a useful tool in making meaningful comparisons between different types of equations and types of clauses in English. Mapping between our proposed categories of physics equations and Halliday's transitivity system for ideational clauses works exceptionally well as a theoretical framework to understand meaning-making with equations

The theory as presented is a sketch. It has the potential to be fine-tuned with more analysis of physics textbooks, as well as deeper research in linguistics and perhaps other fields. Involving experts in other fields, such as linguists, educators, psychologists, mathematicians, and more, could be of immense benefit to the theoretical framework.

Finally, it will be necessary to devise experiments from which we can extract data to help determine the useful applications of this idea. Pedagogical strategies involving the theory must be developed and then tested, perhaps by surveying large groups of students and/or examining smaller groups working and reasoning with equations. The implementation of this theory into curriculum could be subtle, where the instructor could simply repeatedly emphasize the meanings of equations, or more overt, where the students are explicitly made aware of the equations' grammatical structure and the implicit meaning associated with that structure.

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# Conceptual Analysis in Cognitive Research: Purpose, Uses, and the Need for Clarity

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*This theoretical paper discusses conceptual analysis of mathematical ideas relative to its place within cognitive learning theories and research studies. In particular, I highlight specific ways mathematics education research uses conceptual analysis and discuss the implications of these uses for interpreting and leveraging results to produce empirically tested learning trajectories.*

*Keywords:* Conceptual Analysis, Cognitive Research, Hypothetical Learning Trajectories

Cobb (2007) argued that mathematics education "can be productively viewed as a design science, the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes" (p. 7). This requires that researchers' work leverages scientific methods to inform design (at the instructional, curricular, or institutional level) that positively impacts student learning.

Thus, a useful way to characterize cognitively-oriented research goals is the production of empirically tested learning trajectories that provide opportunities for students to construct ideal ways of reasoning about mathematical ideas within a coherent trajectory spanning their entire mathematical careers.[1] *Conceptual analysis* (Thompson, 2008a) plays an important role in this work, yet researchers are not always explicit about how they use conceptual analysis, nor are they clear about how conceptual analysis of an idea contributes to both the design and refinement of interventions that contribute to the broader goal of advancing knowledge in the field.

In this paper I will discuss conceptual analysis of mathematical ideas relative to its place within cognitive learning theories, highlight different ways that conceptual analysis is used in specific research studies, and explore how these uses contribute in different ways to achieving the overall goals of cognitively-oriented mathematics education research.

## The Importance of Theory in Mathematics Education Research

Conceptual analysis focuses on defining mental activity characterizing both real and epistemic individuals' meanings, and as such derives from general constructivist principles. diSessa and Cobb (2004) and Thompson (2002) both describe theoretical perspective hierarchies starting from broader background theories like Piaget's (1971) genetic epistemology to more narrow domain-specific theories that "entail the conceptual analysis of a significant disciplinary idea…with the specification of both successive patterns of reasoning and the means of supporting their emergence" (diSessa & Cobb, 2004, p. 83). Background theories serve "to constrain the types of explanations we give, to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem" (Thompson, 2002, p. 192). Domain-specific theories address "ways of thinking, believing, imagining, and interacting that might be propitious for students' and teachers' mathematical development" (p. 194).

That conceptual analysis originated from radical constructivism has implications for its character and purpose. A description of what it means to understand a mathematical idea should be phrased in terms that reflect a researcher's epistemology, and not in a faint or elusive way. This is why conceptual analysis, as defined by Glasersfeld (1995), Steffe & Thompson (2000),

[1] The use of "ideal" here is framed as a goal even if consensus is never reached. Refinements to improve the effectiveness and coherence of students' mathematical experiences is the manifestation of this goal in practice.

and Thompson (2008a), is a description of cognitive states and processes. Grounding conceptual analysis in descriptions of mental actions and schemes attunes us to focusing on important ways of understanding foundational ideas that influence students' abilities to construct and leverage productive images of sophisticated ideas articulated by a researcher's learning goals and *hypothetical learning trajectory* (Simon, 1995).

**Conceptual Analysis, Hypothetical Learning Trajectories, and Teaching Experiments**

Thompson (2008a) defined conceptual analysis as a description of "what students must understand when they know a particular idea in various ways" (p. 42) and outlined four uses: 1) to build models of students' thinking by analyzing observable behaviors, 2) to outline ways of knowing potentially beneficial for students' mathematical development, 3) to outline potentially problematic ways of knowing particular ideas, and 4) to analyze coherence in meanings among some set of ways of knowing. From a Piagetian-constructivist perspective, understandings are organizations of mental actions, images, and conceptual operations. Describing an understanding—either actual or intended—therefore involves specifying the mental actions, images, and operations that constitute it. Conceptual analysis provides clarity on the mental actions that characterize particular understandings, their potential origins, and their implications for subsequent mathematical learning. Conceptual analysis does not produce a list of mathematical facts or particular learning objectives. Conceptual analysis is about articulating the cognitive processes that characterize particular understandings, which serves as a basis for task design and shapes researchers' identification of students' mathematical thinking and learning. Thus, conceptual analysis is a form of theory itself—an operationalization of what diSessa and Cobb (2004) call an *orienting framework* in the context of mathematics education research.

Ellis, Ozgur, Kulow, Dogan, & Amidon (2016) joined others (e.g., Clements & Samara, 2004; Sztajn, Confrey, Wilson, & Eddington, 2012) in stressing the importance of learning trajectory research. There is no consensus definition for hypothetical learning trajectory yet. Most descriptions are refinements of Simon's (1995) original definition as "[t]he consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Hypothetical learning trajectories, as indicated by their name, should be framed as hypotheses to be tested in empirical studies, which often employ the teaching experiment methodology (Steffe & Thompson, 2000). As such, each of the three components of a hypothetical learning trajectory must be clearly articulated in enough detail so that during a teaching experiment, and in retrospect, it is possible for the researcher to provide empirical support for accepting or rejecting any part of the hypothesis. A teaching experiment, as described by Steffe and Thompson (2000), is the means by which to assess and refine hypothetical learning trajectories informed by a conceptual analysis. Teaching experiments have three parts, and different uses of conceptual analysis contribute to each part in different ways (see Figure 1).

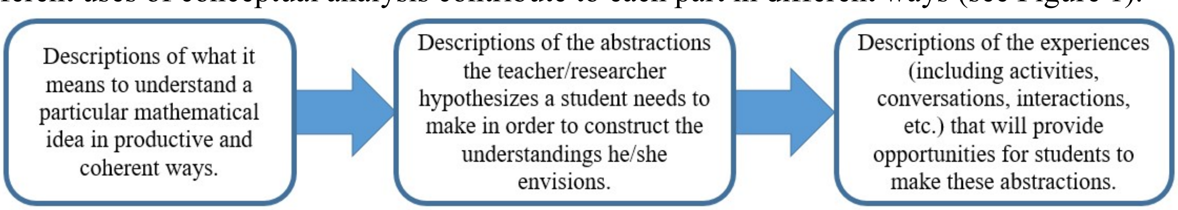


*Figure 1. Parts of a teaching experiment.*

Thinking in these terms, we can clarify how the results from different research studies contribute to the goal of creating empirically tested ideal mathematical learning trajectories.

## Examples of Different Uses of Conceptual Analyses

Since researchers' contributions to learn trajectory research depend on how they used conceptual analysis, their conceptual analyses constitute an interpretive lens to make sense of their data and indicate the specific ways that others should leverage and interpret their work. The following three examples will help to illustrate this point. Each are drawn from compelling, influential research related to the teaching and learning of exponential growth.

### Confrey and Smith's Retrospective Conceptual Analysis: Modeling Student Reasoning

Confrey (1994) and Confrey and Smith (1994, 1995) developed robust descriptions for students' images of multiplication, ratio, covariation, function, and rate based on retrospective conceptual analysis of teaching interviews. Student working through tasks like paper folding and predicting future values for an item retaining 90% of its value each year leveraged meanings for multiplication, rate of change, and function that often differed from conventional meanings. By carefully modeling students' schemes, Confrey and her colleagues described productive images that they claimed could be a powerful foundation for understanding exponential growth.

**Images of multiplication, covariation, function, rate of change, and exponential growth.** Confrey (1994) described thinking about multiplication via *splitting*. A *split* is the action of creating equal copies of an original amount or breaking an original amount into equal-sized parts. She then defined multiplication as the result of some $n$-split on an original whole and division as examining one of the equal parts of the split relative to the whole. Ratios rather than differences are then the natural means of comparison when conceptualizing splitting. Confrey and Smith (1994, 1995) described students engaging in *covariational reasoning* when coordinating splits and defined *covariation* discretely as a process of synchronizing successive values of two variables. A function relationship is then "the juxtaposition of two sequences, each of which is generated independently through a pattern of data values" (1995, p. 67) with specific function characteristics emerging from repeated actions during this coordination.

Confrey and Smith argued that students reasoning covariationally developed notions of *rate* that differed from conventional definitions. Some students coordinating arithmetic and geometric sequences to reason about exponential growth described the relationship as having a constant rate of change, meaning that thinking about rates as a ratio of additive differences is not an inevitable choice for students. They proposed defining *rate* in a way that respects students' intuitions. A *rate* is a unit per unit comparison where *unit* refers to what remains constant in a repeated action (Confrey, 1994). Thus, changes (and rates) can be conceived of additively or multiplicatively. Confrey and Smith argued that coordinating repeated addition to move through an arithmetic sequence with repeated multiplication to move through the geometric sequence and interpolating values by coordinating arithmetic means with geometric means is productive foundation for understanding exponential growth.

**Commentary.** Confrey and Smith's work modeled students' constructed schemes from empirical data and theorized about the utility of specific meanings for multiplication, covariation, function, and rate of change for understanding exponential growth. This kind of retrospective conceptual analysis is very useful for characterizing the way that some students productively reasoned about specific tasks spontaneously, including novel ways of thinking not typically emphasized in curricula. Confrey and Smith were not focused on generating detailed learning trajectories,[2] nor did they consider the implications for their specific meanings on understanding

[2] Weber (2002a, 2002b) and Ström (2008) both studied the implications of Confrey and Smith's conceptual analysis, as did Amy Ellis and her colleagues. I will say much more about Ellis et al.'s work later in this paper.

sophisticated mathematical ideas students will encounter in the future such as the Fundamental Theorem of Calculus (FTC). Their work was limited to modeling students' meanings for mathematical ideas within a fairly narrow scope of mathematical tasks and considering implications of these meanings for what they conceived as related ideas.

There are some limitations in studies using conceptual analysis in this manner, and understanding these limitations is critical to putting their results in perspective. In teaching interviews and experiments, results are always impacted by a researcher's choice of tasks and initial assumptions. For example, Confrey and Smith assumed that repeated multiplication is a useful foundation for defining exponential growth, and all of the tasks could be solved by (and perhaps encouraged) images of repeated multiplication. Since they were attuned to looking for productive ways of reasoning in these tasks, their conclusions depended on this initial assumption. Since results are influenced by the researchers' initial assumptions and task selection, their work does not compare the relative strengths of various potential meanings and learning trajectories. That requires a different use of conceptual analysis that looks more broadly at issues of coherence in mathematical ideas at all levels, which is not what Confrey and Smith sought to achieve. Scientific and mathematical progress throughout history is almost entirely a story about breakthroughs in understanding that defy human expectations and intuition. Thus, we should expect that classifying students' productive schemes for an idea will give us powerful insights into how individuals construct internally coherent schemes but not necessarily uncover ideal meanings we may want students to construct.

**Thompson's Conceptual Analysis: Coherence of Mathematical Ideas Leading to Calculus**

Thompson's (1994a) unpacking of the key ideas in calculus, particularly the FTC, motivated and informed his conceptual analysis for exponential growth (Thompson, 2008a). Thompson imagined a broadly coherent trajectory for students' mathematical experiences focused on quantitative reasoning, covariational reasoning, and representational equivalence that could unite most topics from grade school mathematics through calculus (Thompson, 2008b). Thus, his conceptual analysis considers exponential functions as just one of many opportunities for students to develop and apply particular ways of thinking.

**Quantitative and covariational reasoning, rate of change, accumulation, and the FTC.** Thompson's meanings for covariation, function, and rate of change are different from Confrey and Smith's because his goals are different. His work is grounded in *quantitative reasoning*, which describes conceptualizing a situation to form a quantitative structure that organizes relevant *quantities* (measureable attributes) and *quantitative operations* (new quantities representing a relationship between other quantities) (Thompson, 1988, 1990, 1993, 1994b, 2011, 2012). If someone sees a situation as composed of quantities that change together and attempts to coordinate their variation, then she is engaging in *covariational reasoning* (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Sophisticated covariational reasoning involves linking two continuously varying quantities to create a *multiplicative object*, a unification that combines the attributes of both quantities simultaneously (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017).

Thompson (1994a, 1994b) and Thompson and Thompson (1992) outline an image of constant rate as a proportional correspondence of two smoothly covarying quantities. When one quantity's magnitude changes by *any* amount, the other quantity's magnitude changes proportionally. This was Newton's image of rate that allowed him to conceptualize the relationship between accumulation and rate of change expressed formally in the FTC (Thompson, 1994a, 2008a). Over small intervals, he imagined that any two covarying quantities change together in a proportional correspondence. This can be modeled by a piecewise constant rate of change function and its corresponding piecewise linear accumulation function. The FTC describes how these two functions are related as the interval sizes tend to zero. See Figure 2.

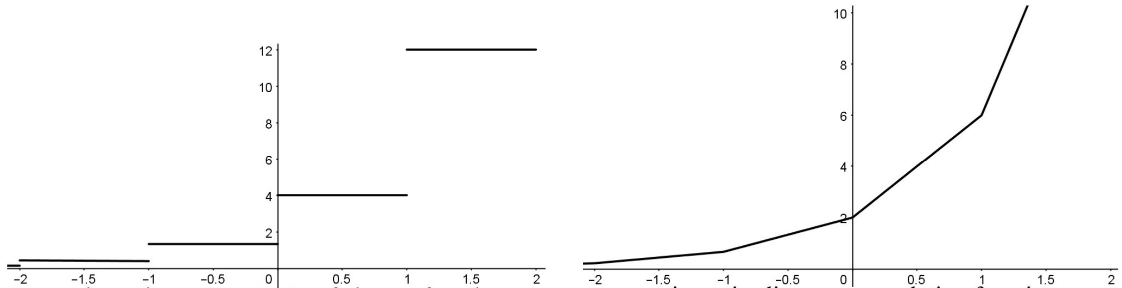


*Figure 2. Piecewise linear accumulation function and piecewise constant rate of change function.*

**Exponential functions.** Building from his images of constant rate of change and the FTC, Thompson's (2008a) conceptual analysis involved thinking about classifying functions based on similarities in their rate of change functions and imagining a function as emerging through accumulation. Specific to exponential functions, he conceptualized a relationship with a rate of change on some interval that is always proportional to the function value at the beginning of the interval. As the interval size decreases, the piecewise linear accumulation function converges to an exponential function. Thompson (1994a, 2008a) argued that this way of understanding allows a person to conceptualize both change and accumulation as happening simultaneously, makes it natural to imagine the function value growing continuously and producing outputs for all real number inputs, is consistent with a coherent way of reasoning about all function relationships, and leads to a productive operational understanding of the FTC.

**Commentary.** Much like Confrey and Smith, Thompson's work is not a detailed hypothetical learning trajectory.[3] Thompson's conceptual analysis is part of a broader, idealized web of ideas stretching from students' first mathematical experiences through calculus. It does not consider students' actual mathematical background experiences in modern classrooms, the cognitive load it places on students, or whether the ideas reasonably coincide with common ways students may attempt to spontaneously reason about tasks. It also depends on a different meaning for function relationships, how functions are categorized, and the foundational criterion for a relationship to be exponential. In Thompson's conceptual analysis, exponential growth is related to repeated multiplication almost by coincidence and is not the foundational meaning.

### Ellis and Colleagues: From Exploratory to Hypothetical Learning Trajectory

Ellis and her colleagues (Ellis, Ozgur, Kulow, Williams, & Amidon, 2012, 2015; Ellis et al., 2016) mostly leveraged Confrey and Smith's images of covariation, rate, and exponential growth to construct a rough exploratory learning trajectory surrounding a single context. Ellis et al. extended and clarified how Confrey and Smith's ideas might productively support students' understanding of exponential relationships and chose a situation where they conjectured students could easily justify that the function's domain and range were not restricted to a set of discrete values. They built a Geogebra applet showing the image of a plant (the Jactus) with a height that varied exponentially with elapsed time. The applet's user can vary the elapsed time by sliding the plant along the horizontal axis and its height would update in real time. The applet also displays the time elapsed and the plant's height as an ordered pair as the user slides the plant horizontally.

[3] Castillo-Garsow (2010) did produce a learning trajectory and empirical study based on this conceptual analysis.

In designing their study, Ellis et al. anticipated, and later confirmed, that students' initial models for exponentiation involved an informal image of repeated multiplication. Ellis et al. wanted students to leverage covariational reasoning to build a more robust image of exponential growth focused on coordinating multiplicative changes in one quantity with additive changes in another quantity. With this understanding, students might understand $b^x$ as both the possible height of a plant at some moment in time and as representation of a (multiplicative) change in height. Students working through the activities exhibited key shifts in their thinking reflecting increased attention to how the two quantities changed together over intervals of varying size. "[These] results…offer a proof of concept that even with their relative lack of algebraic sophistication, middle school students can engage in an impressive degree of coordination of co-varying quantities when exploring exponential growth" (Ellis et al., 2012, p. 110).

**Commentary.** Ellis et al. used conceptual analysis in three ways. First, they further unpacked Confrey and Smith's conceptual analysis of exponential growth as students might construct it from images of coordinating additive and multiplicative changes. Second, they continuously modified and updated their exploratory learning trajectory and tasks throughout the study based on models of students' schemes. These analyses, coupled with retrospective analysis on the empirical data, allowed them to craft highly detailed descriptions of students' meanings at various points in time and how those meanings developed through interactions with tasks and teaching interventions (Ellis et al., 2016). The result is the foundation for a powerful hypothetical learning trajectory. Ellis et al. now have empirical grounding for theories on how students may come to construct specific meanings related to exponential growth and related ideas. The refinements from the exploratory research and their model for how students construct specific meanings in specific contexts is now a fully realized hypothesis for systematic testing.

Ellis et al.'s work is an impressive example of critical work in developing empirically tested learning trajectories and demonstrates how initial exploratory work in developing an understanding of students' scheme construction, like the work of Confrey and Smith, can be refined and expanded to contribute to important work on learning trajectories. However, as they note, "Our learning trajectory is an attempt to characterize the nature of the evolution of students' thinking in a particular instructional setting" (2016, p. 153) and is thus only one of many possible learning trajectories. Like Confrey and Smith, their work assumes that repeated multiplication is the starting point from which to develop an understanding of exponential growth. In fact, the initial activities in their exploratory learning trajectory encouraged and then attempted to modify this reasoning. Their work does not extend to considering the long-term implications for students who develop their intended meanings compared to students with other potential meanings for exponential growth, nor does it (as of yet) seek to explain persistent challenges students encountered. This was not the role of the described study but does describe critical future research.

## Summary and Theoretical Implications

A teaching experiment is a method of testing a research hypothesis (a carefully detailed hypothetical learning trajectory) informed by conceptual analysis that analyzes the degree to which (and aspects of) tasks and interactions that promoted specific abstractions. None of the research studies described in this paper satisfy these criteria of a formal teaching experiment because the empirical work, when present, was more exploratory in nature. However, each of them contribute to the goals of cognitively-oriented mathematics education research in powerful ways. Confrey and Smith described students' schemes related to repeated multiplication based on spontaneous reasoning about particular mathematical tasks. Ellis et al. further unpacked these schemes and, based on retrospective analysis of empirical data, produced a well-defined hypothetical learning trajectory for specific meanings using specific tasks that now has the clarity and specificity necessary to be a scientific hypothesis. Thompson's work takes a broader view and suggests ways of understanding exponential growth situated within a coherent body of mathematical ideas extending beyond a single topic.

Currently there is no consensus on the exact meaning of a hypothetical learning trajectory. Ellis et al. (2016) have an excellent literature review detailing the different interpretations. In addition, reflecting on their work suggests that the field may benefit from greater clarity in defining different types of learning trajectories with the definitions influenced by the role of conceptual analysis. A potential starting point is given below.

- *Exploratory learning trajectory* – Conceptual analysis (either based on a researcher's analysis of mathematical ideas or based on empirical data) can suggest potentially useful ways of understanding particular ideas. A researcher then creates tasks and a rough exploratory trajectory for gathering empirical data on how students reason about specific contexts in specific settings. Since the enacted learning trajectory is continually modified based on modeling students' emerging meanings, this is not yet a scientific hypothesis.
- *Enacted learning trajectory* – An actual learning trajectory unfolded based on the exploratory learning trajectory. Conceptual analysis is used retrospectively to describe how students' schemes changed as a result of their mathematical activity.
- *Hypothetical learning trajectory* – This describes a specifically stated research hypothesis outlining specific targeted mental actions and schemes, specific tasks and a task sequence, and descriptions of how those tasks will contribute to students accommodating their schemes. The teaching experiment that tests this hypothetical learning trajectory seeks to accept or reject particular aspects of the hypothesis, and will ultimately result in refinement. Conceptual analysis is critical to the design of the learning trajectory and retrospectively in analyzing outcomes in the more formal teaching experiment.
- *Empirically supported learning trajectory* – After potentially several rounds of refinement and testing with hypothetical learning trajectories, a researcher can articulate an empirically supported learning trajectory. In comparing the results and implications of competing empirically supported learning trajectories, researchers can move closer to a learning trajectory that supports the development of ideal ways of understanding.

Any of these learning trajectories could be narrow in scope (focused on a particular mathematical idea) or grand in scope (focused on students' learning as an arc from grade school through graduate level mathematics). Researchers' questions of interest and how they use conceptual analysis dictate the type of learning trajectory they are developing and studying, and the scope of their work dictates their contribution to the field from models of students' schemes relative to particular ideas to coherent mathematical experiences across many topics and grade levels.

As researchers, we are obligated to not only produce scientifically-valid findings but also to communicate our work in ways that allow others to leverage our results to advance the collective mission of our design science. Being more explicit about the role of conceptual analysis in our work and having greater clarity on how our learning trajectory research contributes to design research can help us achieve this. I hope that my articulation of how different uses of conceptual analysis are relevant to developing different kinds of learning trajectories facilitates relevant and productive communication among cognitively-oriented, qualitative mathematics education researchers.

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# Computational Thinking in University Mathematics Education: A Theoretical Framework


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| Brock University | Brock University | Brock University | Cinvestav-IPN |



*In recent years "Computational thinking" has become a trending topic among teachers who have seen their curricula include the term, and researchers who seek to pinpoint both what it means and how it can be implemented in a meaningful way in classrooms. We see a crucial need in mathematics education to understand how students could be empowered to participate in the computational thinking that is now becoming an integral part of the mathematics and broader community. In our research, we are interested in examining how university mathematics students may come to appropriate programming and engage in computational thinking for mathematics, as mathematicians would do. In this paper, we present the theoretical framework that grounds our research.*



*Keywords:* computational thinking, instrumental genesis, programming, third pillar of scientific inquiry, undergraduate mathematics


## Introduction

Before the advent of the personal computer, Papert (1971) envisioned a world in which children fluently program computers, using them as a tool to act as young mathematicians. Nearly half a century later, we've witnessed a widespread resurgence of interest in that vision, manifested in educational reforms (e.g., in Europe: Bocconi, Chioccariello, Dettori, Ferrari, & Engelhardt, 2016) and research regimes (e.g., Computational Thinking in Mathematics Education, n.d.) in the name of computational thinking, which now is deemed a 21st century skill. We see a crucial need to understand how students can be empowered to participate in such computational thinking that has become an integral part of the mathematics and broader community.

This paper focuses on the theoretical framework that grounds our recently launched 5-year study, funded by the Canadian Social Sciences and Humanities Research Council (SSHRC), that seeks to examine how postsecondary mathematics students learn to use programming as a computational thinking instrument for mathematics. It is a naturalistic study that takes place in a sequence of three programming-based mathematics courses implemented in the mathematics department at Brock University (Canada) since 2001, where undergraduate mathematics majors and future mathematics teachers learn to design, program, and use interactive computer environments to investigate mathematics conjectures, concepts, theorems, or real-world applications (Buteau, Muller, & Ralph, 2015; Muller, Buteau, Ralph, & Mgombelo, 2009). The objectives of our research include: (a) describing students' instrumental geneses of using programming as a computational thinking instrument for mathematics; (b) exploring whether or not students appropriated it and, if so, have sustained it beyond course requirements; and (c) identifying how instructors create a learning environment to support students' instrumental geneses. This study builds on our past and ongoing research (e.g., Buteau & Muller, 2014; Buteau, Muller, & Marshall, 2015; Buteau, Muller, Marshall, Sacristán, & Mgombelo, 2016).

## Proposed Theoretical Framework

We start by discussing computational thinking and programming from a broad perspective based on the work of Wing (2008) and others. We then turn our attention to computational thinking in mathematics portrayed by mathematicians' research practices—for example, as stated by the European Mathematical Society (2011). This leads to a discussion of computational thinking in mathematics education, which in turn is informed by the work of Weintrop et al. (2016) and the constructionist paradigm (Papert and Harel, 1991). Next, we elaborate on our view of learning mathematics by engaging in computational thinking drawing on some ideas from the work of Lave and Wenger (1991). Finally, we discuss Guin and Trouche's (1999) instrumental approach framework to inform our understanding of technology integration in mathematics teaching and learning.

**Computational Thinking**

Wing (2014) describes computational thinking as "the thought processes involved in formulating a problem and expressing its solution(s) in such a way that a computer—human or machine—can effectively carry out" (para. 5). Thus, computational thinking is an underlying process to computer programming. And as Grover and Pea (2013) state, computer programming "is not only a fundamental skill of [computer science] and a key tool for supporting the cognitive tasks involved in [computational thinking] but a demonstration of computational competencies as well" (p. 40). Wing (2008) explains that "the essence of computational thinking is abstraction" (p. 3717) and elaborates:

> Computational thinking is a kind of analytical thinking. It shares with mathematical thinking in the general ways in which we might approach solving a problem. It shares with engineering thinking in the general ways in which we might approach designing and evaluating a large, complex system that operates within the constraints of the real world. It shares with scientific thinking in the general ways in which we might approach understanding computability, intelligence, the mind and human behaviour. (p. 3717)

The relationship of computer programming and computational thinking with mathematical and scientific thinking and learning has been recognized since the development of the Logo programming language (cf., Papert, 1980a; Feurzeig & Lukas, 1972). This relationship is also highlighted in Brennan and Resnick's (2012) proposed three-dimensional framework characterizing "computational thinking" in terms of

> *computational concepts* (the concepts designers engage with as they program, such as iteration, parallelism, etc.), *computational practices* (the practices designers develop as they engage with the concepts, such as debugging projects or remixing others' work), and *computational perspectives* (the perspectives designers form about the world around them and about themselves). (p. 1)

In the following sections we discuss computational thinking in mathematics and computational thinking in mathematics education.

**Computational Thinking in Mathematics**

In terms of the development of mathematics itself, the European Mathematical Society (2011) recognized an emerging way of engaging in mathematical research: "Together with theory and experimentation, a third pillar of scientific inquiry of complex systems has emerged in the form of a combination of modeling, simulation, optimization and visualization" (p. 2). The notion of a third pillar had been raised previously in a 2005 report by the United States' President' Information Technology Advisory Committee (2005) highlighting the role of digital technology: "Together with theory and experimentation, computational science now constitutes the 'third pillar' of scientific inquiry, enabling researchers to build and test models of complex phenomena" (p. 1). In 2016, mathematicians who led a 6-month long thematic semester on *Computational Mathematics in Emerging Applications* at the Centre de recherches mathématiques (CRM) in Montreal (Canada) indicated that:

> A fundamental change is taking place in the role of applied and computational mathematics. The relationship between the modelling, analysis, and solution of mathematical problems in applications has changed. … In emerging applications, the choice of models goes hand in hand with the computational tools and the mathematical analysis. (CRM, 2016, para. 1)

These emerging practices in mathematics research, we argue, fall under the umbrella of computational thinking for mathematics and are grounded on programming technology. Indeed, Weintrop et al.'s (2016) taxonomy (see Figure 1) gives insights into the computational thinking engagement by mathematicians and scientists, which encompasses the activities described by the European Mathematical Society (2011) and by the organizers of the computational mathematics session at CRM. Weintrop et al.'s work was based on an extensive literature review, an analysis of mathematics and science learning activities, and interviews with "biochemists, physicists, material engineers, astrophysicists, computer scientists, and biomedical engineers" (p. 134); the authors also outline what they believe to be the integral computational thinking practices for mathematics and science. Broley, Buteau, and Muller (2017) exemplified, through concrete research of mathematicians' work, the different forms of integral computational thinking practices proposed by Weintrop et al.

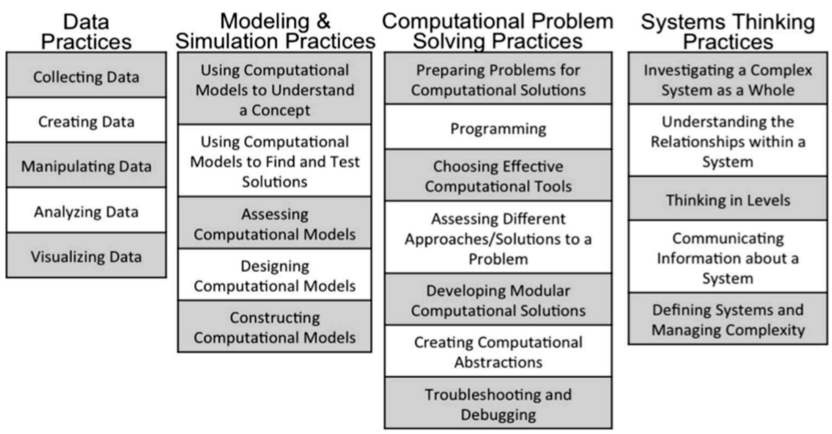


*Figure 1*. Taxonomy of computational thinking in mathematics and science (Weintrop et al., 2016, p. 135).

Adopting Brennan and Resnick's (2012) framework in the context of mathematics, the work of Weintrop et al. (2016) not only provides discipline-specific details for the computational practices dimension, but also foregrounds computational perspectives—that is, perspectives the mathematicians have come to recently develop about mathematics as a discipline "in line with the increasingly computational nature of modern science and mathematics" (Weintrop et al., 2016, p. 127).

**Computational Thinking in Mathematics Education**

Furthermore Weintrop et al. (2016) argue that "the varied and applied use of computational thinking by experts in the field provides a roadmap for what computational thinking instruction should include in the classroom" (p. 128). Their detailed taxonomy thus provides us with what it means, in the mathematics classroom, to engage in computational thinking for mathematics as mathematicians would do.

As mentioned earlier, computational thinking in mathematics education has a legacy of over 45 years in the Logo programming language and in the theory of constructionism (Papert & Harel, 1991). The fundamental premise of the constructionist paradigm is to create student-centered learning situations for students to consciously engage in constructing (e.g., program) shareable, tangible objects, through meaningful —usually computer-based– projects: “People construct new knowledge with particular effectiveness when they are engaged in constructing personally meaningful products ... [that is] something meaningful to themselves and to others around them” (Kafai & Resnick, 1996, p. 214).

Studies of constructionism at higher-level mathematics education show how programming supports students’ understanding of mathematical concepts (e.g., Leron & Dubinsky, 1995; Wilensky, 1995) and how it contributes to the development of critical thinking skills (e.g., Abrahamson, Berland, Shapiro, Unterman, & Wilensky, 2004; Marshall, 2012). In fact, Noss and Hoyles (1996) stress that a learner, when engaging in modifying a program, articulates relationships between concepts involved in a microworld “and it is in this process of articulation that a learner can create mathematics and simultaneously reveal this act of creation to an observer” (p. 54). In our work we concur with the constructionism approach for classroom implementation of programming and the computational thinking that it involves, and conceive mathematical learning by drawing from ideas found in situated learning theory, as described next.

**Learning Mathematics by Engaging in Computational Thinking**

Our view of learning draws from Lave and Wenger’s (1991) work on communities of practice. Hoadley (2012) points that two definitions of community of practice stem from Lave and Wenger’s work: (i) a feature based definition that derives from the words themselves meaning a community that shares practices and (ii) a process based definition which focuses on the process of learning whereby communities of practice are seen as groups in which a constant process of “legitimate peripheral participation” takes place. In our work, we rely on the process-based definition. Lave and Wenger use the concept of legitimate peripheral participation to describe how learners enter a community and gradually take up its practices. We use this idea of legitimate peripheral participation to understand how students learn mathematics through computational thinking. “Mathematics” is not seen as a body of knowledge to be acquired by the student, but rather as a process of participation through which the student gradually gains membership to a community (of mathematicians). Also, we do not see computational thinking from a cognitive point of view (e.g., seeing a computer as an interactive learning tool in illustrating concepts). Instead, we focus on how students create and use computer tools to engage in opportunities to participate peripherally in practices considered to be integral to the mathematical community as outlined by Weintrop et al. (2016). In other words, we focus on how students (newcomers) engage in computational thinking for mathematics as mathematicians (elders) would do.

This view on learning concords with the constructionism paradigm. Papert (1971) argued that “being a mathematician, … like being a poet, or a composer or an engineer, means *doing*, rather than knowing or understanding” (p.1), and that through programming mathematics, learners engage in “computational mathematics” (p.25) through which they mathematize. For Papert (1980b), the computer provides the learner a means for constructing “objects to think with” and “allow[s] a human learner to exercise particular powerful ideas or intellectual skills” (p.204) through exploration and discovery in a knowledge domain. This resonates with how many mathematicians and scientists use the computer in the 21$^{st}$ Century as described earlier.

The work by Broley et al. (2017), cited earlier, exemplifies how undergraduate students learned mathematics through the construction of interactive computational objects (i.e., ‘objects to think with’), and how these practices align with those of working mathematicians: for example, a first-year undergraduate’s engagement in computational problem-solving practices –where she had to design, program, and use an interactive environment to explore, graphically and numerically, the behavior of a dynamical system based on a two-parameter cubic– shared similarities with a mathematician’s engagement in his research on permutation of subsequences (see Figure 2).

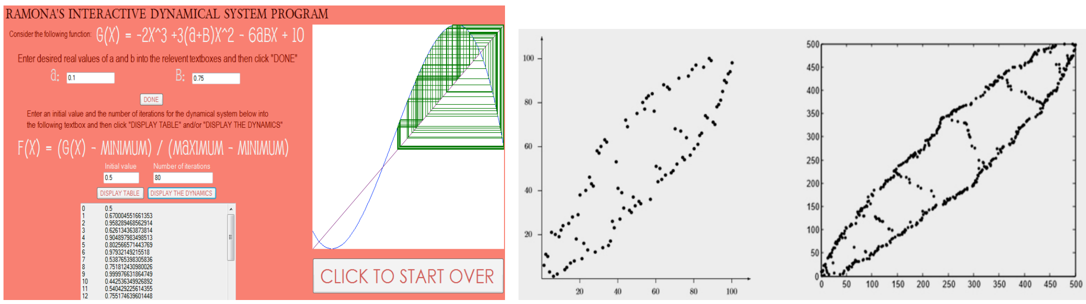


*Figure 2*. Examples of computational problem-solving practices. Left: screenshot of an undergraduate’s exploratory work of a dynamical system. Right: screenshot of a mathematician’s exploratory work on a permutation structure (Broley et al., 2017, pp. 4, 6).

When students become proficient at using programming to engage in computational thinking for mathematics “as mathematicians would do” (i.e., engaging in the computational practices as well as taking on the computational perspectives similar to how a mathematician would do), we consider that this technology has been integrated or that appropriation has occurred. We now turn to discussing this and how it can be assessed.

**Students’ Appropriation of Programming as a Computational Thinking Instrument**

Cook, Smagorinsky, Fry, Konopak, and Moore (2002) explain that appropriation is a developmental process involving socially formulated, goal-directed, and tool-mediated actions through which learners actively adopt (i.e., what we could call “make their own”) conceptual and practical tools, thus internalizing ways of thinking related to specific settings in which learning takes place. The instrumental approach (Rabardel, 1995/2002) is a useful framework for analyzing technological integration (Artigue, 2002; Guin & Trouche 1999) and gaining insights into how students appropriate a (technological) tool, and such an approach is used increasingly at the university level (cf., Gueude, Buteau, Mesa, & Misfeld, 2014).

The instrumental approach describes how artifacts (whether material or symbolic) are appropriated when they are transformed into instruments through schemes of usage and action by what is called *instrumental genesis* (Artigue, 2002). Trouche and Drijvers (2010) suggest that an instrument has been appropriated when a “meaningful relationship exists between the artifact and the user for a specific type of task” (p. 673). Thus, in order to assess the appropriation and technological integration, it is necessary to look at the instrumental genesis, by looking at both the artifact and its attached schemes. One way to do so is to look at the traces that students leave in their activity and what they do with an artifact (Trouche 2004). Parallel to this, it is also necessary to take into account the teacher’s activity: his/her conceptions, design, and orchestrations of the teaching resources (Trouche, 2004) and the *instrumental integration*, which is "how teachers organise the conditions for instrumental genesis of the technology proposed to the students and to what extent (s)he fosters mathematics learning through instrumental genesis" (Goos & Soury-Lavergne, 2010, p. 313). Instrumental integration describes four stages of growing technology use in the classroom (Assude, 2007): (a) instrumental initiation (stage 1)—students engage only in learning how to use the technology; (b) instrumental exploration (stage 2)—mathematics problems motivate students to further learn to use the technology; (c) instrumental reinforcement (stage 3)—students solve mathematics problems with the technology, but must extend their technology skills; and (d) instrumental symbiosis (stage 4)—students' fluency with technology scaffolds the mathematical task resulting in an improvement of both the students' technology skills and their mathematical understanding.

We associate these stages to a student's computational thinking development dimensions from Brennan and Resnick's (2012) framework: stages 1 and 2 to computational concepts, stages 2 to 4 to computational practices, and stages 3 and 4 to computational perspectives. And it is in stage 4 where we argue that the student has appropriated programming as an instrument for mathematics "as mathematicians would do" (both in terms of computational practices and perspectives) as mentioned in the previous section, which we term "programming as a computational thinking instrument for mathematics."

## Next Steps for the Research

In this paper, we presented the theoretical framework underlying our study focused on how undergraduate mathematics students come to appropriate programming as a computational thinking instrument for mathematics. Brennan and Resnick (2012) suggest ways of assessing computational thinking development, including project portfolio analysis and interviews. Accordingly, in our research we will collect student participants' programming-based mathematics projects (14 in total over the three courses) together with their corresponding reflective journals, and students' lab reflections. We will also conduct semi-structured individual interviews with each of the participants in order to gain insights into students' creation process (including decision-making) and traces of their ongoing work. This is planned for two cohorts of 10 students each, followed over 3 consecutive years. Final interviews and questionnaires will be used at the end of the participants' 4- or 5-year program studies, to examine the sustainability of their programming use. Aligned with Trouche's (2004) recommendation, semi-structured interviews with course instructors, field notes of computer lab session observations, as well as course material will provide insights into the instructors' didactical aims and participants' learning environment. The latter data will also shed light on the instructors' pedagogical decisions and to what extent these are in accordance with the constructionist paradigm.

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# Scaling-Continuous Variation: A Productive Foundation for Calculus Reasoning

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*This paper introduces a new mode of variational and covariational reasoning, called scaling-continuous reasoning. Scaling-continuous reasoning builds on Leibniz' ideas of increments and infinitesimals and does not rely on images of motion. Instead, it entails (a) imagining a variable taking on all values on the continuum at any scale, (b) understanding that there is no scale at which the continuum becomes discrete, and (c) re-scaling to any arbitrarily small increment for x and coordinating that scaling with associated values for y. We present one clarifying example of this type of reasoning and argue that scaling-continuous reasoning can support a robust understanding of foundational ideas for calculus, including rates of change, differentiation, and the definite integral.*

*Keywords:* covariation, rate of change, infinitesimal

*A curved line may be regarded as being made up of infinitely small straight-line segments.*
– The Marquis de L'Hôpital, 1696

*When I think of a curve, I think of a bunch of really tiny lines.*
– Wesley, research participant

## Introduction

Researchers argue that continuous covariational reasoning is critical for students' development of a robust understanding of function, rates of change, and the foundational ideas of calculus (e.g., Carlson et al., 2003; Kaput, 1994; Thompson & Carlson, 2017). Further, students need opportunities to reason covariationally throughout their K-12 schooling in order to be positioned to make meaningful sense of introductory calculus courses at the undergraduate level. Thompson and Carlson (2017) emphasize, in particular, the importance of smooth continuous reasoning, while also acknowledging the challenges in supporting students' ideas of smoothness. In response to these challenges, we introduce a new mode of reasoning, *scaling-continuous variation / covariation*. Building on Leibniz' notion of infinitesimal increments, scaling-continuous reasoning entails an image of the continuum as infinitely zoomable, coupled with the understanding that one can re-scale to any arbitrarily small increment for *x* and coordinate that scaling with associated values for *y*. We argue that this mode of reasoning can support productive ways of thinking about key calculus ideas, including varying and instantaneous rates of change, limit and differentiation, and the definite integral.

## Background: Variation and Covariation as Foundational Ideas for Calculus

Developing a conception of quantities' values varying continuously – and consequently understanding functions as processes of covariation – is central to the emergence of calculus understanding (e.g., Carlson et al., 2003; Kaput, 1994; Rasmussen, 2000; Thompson & Carlson, 2017; Zandieh, 2000). Research suggests that students both enter and emerge from freshman calculus courses with a weak understanding of the function concept, struggling to conceptualize a function as a mapping, to use functions to model dynamic situations, and to develop robust understandings of varying and instantaneous rates of change (Breidenbach et al., 1992; Carlson, 1998; Carlson et al., 2002; Dubinsky & Harel, 1992; Monk & Nemirovsky, 1994; Thompson, 1994). One factor contributing to these difficulties is the lack of emphasis on variation in secondary mathematics; students typically do not have access to exploring functions as a way to measure variation before calculus (Cooney & Wilson, 1996; Ellis, 2011; Roschelle, Kaput, & Stroup, 2000; Thompson & Carlson, 2017; White & Mitchelmore, 1996). Thompson and Carlson (2017) argue that continuous covariational reasoning is epistemologically necessary for students to develop the foundational ideas of calculus, and moreover, students are unlikely to succeed in calculus without this foundation already in place. Students must therefore build ideas of continuous variation in secondary school in order to develop the ways of thinking necessary for meaningful calculus learning at the undergraduate level.

Researchers have addressed covariational reasoning in a variety of ways, but for the purposes of this paper we focus on work that considers the imagistic foundations that can support students' abilities to think covariationally (e.g., Castillo-Garsow, 2012; 2013; Saldanha & Thompson, 1998; Thompson, 1994; Thompson & Carlson, 2017; Thompson & Thompson, 1992). These researchers describe covariational thinking as the act of holding in mind a sustained image of two quantities' values varying simultaneously; students imagine how one quantity's value changes while imagining changes in the other. A person thinking covariationally can couple two quantities in order to form a multiplicative object (Thompson & Saldanha, 2003), subsequently tracking either quantity's value with the immediate understanding that the other quantity also has a value at every instance (Saldhanha & Thompson, 1998).

Castillo-Garsow (2012; 2013) distinguished between two types of continuous variation, which he termed chunky and smooth; Thompson and Carlson (2017) subsequently built on these distinctions to create a covariational reasoning framework. Chunky continuous variation is similar to thinking about values varying discretely, except that one has a tacit image of a continuum between successive values. This image entails intermediate values without imagining the quantity actually taking on those values (Thompson & Carlson, 2017). Instead, one imagines change occurring in completed chunks, without imagining that variation occurs within the chunk. In contrast, smooth continuous variation entails an image of a quantity changing in the present tense; one can imagine a value varying as its magnitude increases in bits while simultaneously anticipating smooth variation within each bit (Thompson & Carlson, 2017). An image of a quantity's value varying from $a_1$ to $a_2$, one will also include an image of that value passing through all intermediate measures between $a_1$ and $a_2$.

Thompson & Carlson (2017) emphasize Castillo-Garsow and colleagues' point that smooth variational thinking requires thinking about motion (Castillo-Garsow, 2012; Castillo-Garsow, Johnson, & Moore, 2013). They note that this argument "is reminiscent of Newton's description of fluents – the flowing quantities that were at the root of his calculus" (Thompson & Carlson, 2017, p. 430). Further, they point to the importance of motion to smooth covariational reasoning as well, explaining that this is akin to defining a function parametrically in terms of an underlying time variable, in which a parameter is used in the sense of a variable that is not assigned to an axis in a coordinate system. They describe the act of coordinating quantities' values as similar to forming the pair $[x(t), y(t)]$, in which the parameter "$t$" represents conceptual time, which is distinguished from experiential time in that it is an image of measured duration: "We are speaking of someone imagining a quantity as having different values at different moments, and envisioning that those moments happen continuously and rhythmically" (p. 445).

Smooth-continuous reasoning, which relies on this underlying image of time-parametrization, reflects one mode of robust variational and covariational reasoning. We propose an additional mode of reasoning, which we call scaling-continuous reasoning, and suggest that scaling-continuous reasoning may be both distinct from and equally robust to smooth continuous reasoning.

## Scaling: An Alternative to Motion

Motion is an essential image for Newton's reasoning with variation and covariation. For Newton, a variable quantity was a "fluent," which depended on and changed with time. A "fluxion" was an instantaneous speed of this fluent's motion, and what we call a derivative is a ratio of two fluxions (Edwards, 1979). In contrast with Newton, G. W. Leibniz, the other inventor of calculus, seldom described variation, functions, and ideas of calculus in terms of motion. Instead, Leibniz attended to *differences* (differentiae) or *increments* between two values of a quantity, and he distinguished among types of these differences based on their relative *scales* or orders. For instance, Leibniz began with the notion of a "function[1]," an algebraic relationship between the values of a variable quantity such as $x$ and the values of another variable quantity, $y$ (Bos, 1974). Leibniz's differential calculus was then a way to derive from such a function a new equation describing the relationship between infinitesimal increments of the two quantities, $\mathrm{d}x$ and $\mathrm{d}y$. Integral calculus simply went the other way, enabling the determination of a function from a given differential equation.

Here is a brief characteristic example of Leibniz' discourse about the product rule:

> $d(xy)$ is the same as the difference between two adjacent $xy$, of which let one be $xy$, the other $(x+dx)(y+dy)$. Then $d(xy) = (x+dx)(y+dy) - xy$, or $xdy + ydx + dxdy$, and this will be equal to $xdy + ydx$ if the quantity $dxdy$ is omitted, which is infinitely small with respect to the remaining quantities, because $dx$ and $dy$ are supposedly infinitely small. (From Leibniz' *Elementa*, quoted in Bos, 1974, p. 16.)

There are several things to notice about Leibniz' ideas. Firstly, he began by creating a new variable quantity, $xy$, and then sought to derive an equation describing the *correspondence* between an infinitesimal increment of this quantity, $d(xy)$, in terms of infinitesimal increments ($dx$ and $dy$) of the other two quantities $x$ and $y$. These increments were static entities, although they were variable because their values depend upon where on the curve they are taken. Although Leibniz did not appeal to motion, he relied on an underlying image of every increment of one quantity, no matter how small, corresponding to an increment of an associated quantity. This is one of the crucial ideas entailed in continuous covariation (Thompson & Carlson, 2017).

Secondly, Leibniz dismissed the quantity $dxdy$ because it is infinitely small even in comparison to other infinitely small quantities such as $dx$ and $xdy$. Leibniz developed a scheme of orders of the infinitesimal and the infinite in order to systematize an idea of *scaling*. For instance, at the finite scale, infinitesimals such as $dy$ are negligible, but at the first-order infinitesimal scale they become significant, with second-order differences still negligible. The idea of imagining covariation and correspondence different scales was crucial to a coherent system of calculus for Leibniz. It is also part of successful formalizations of this system, such as nonstandard analysis (Keisler, 1986). The manner in which infinitesimal differences in the continuum become significant at different scales is illustrated by Keisler's image of a

[1] It appears that Leibniz actually coined this term. His usage differs from our modern idea in that this relationship need not be uni-valued, but must be represented algebraically.

microscope with an infinite scale factor (Figure 1).

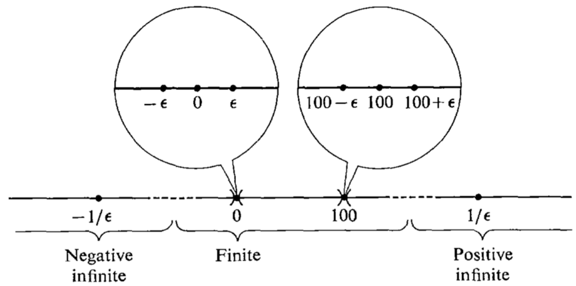


*Figure 1: Infinite microscope on the continuum reveals infinitesimal increments (Keisler, 1986)*

It is also notable that Leibniz' sense of variation, although not explicitly illustrated in the quote above, contained *no atomic level* of scaling. One may imagine zooming in on the continuum to smaller and smaller increments. Regardless of how far one scales, even zooming in by higher orders of infinity, at no level will the continuum ever reveal itself to be discrete points.

Finally, Leibniz emphasized "that a curvilinear figure must be considered to be the same as a polygon with infinitely many sides" (1684, p. 126). It is not that the curve *is* such a polygon, but that it can be *treated* as one. According to Bos (1974), this idea reflects the close relationship between Leibniz' idea of variation and of the infinitesimal, in which successive terms of sequences have infinitely small differences. Thus, "the conception of a variable and the conception of a sequence of infinitely close values of that variable, come to coincide" (p. 16).

## Scaling-Continuous Variation and Covariation

With Leibniz' reasoning in mind, we propose a new category for variational and covariational reasoning that is distinct from the smooth-continuous category, but yet plausibly just as robust for supporting coherent and powerful reasoning with continuous quantities, functions, and rates of change. We call this category *scaling-continuous* for variational and covariational reasoning. Scaling-continuous reasoning entails the following:

1) *Variation*: Imagining that at any scale, the continuum is still a continuum and a variable takes on all values on the continuum. There is no scale at which the continuum is discrete or one reaches a point. One can conceive of the continuum as infinitely "zoomable", in which the process of zooming will never reveal any holes or atoms.
2) *Covariation*: Conceiving of a re-scale or "zoom" into any arbitrarily small increment for $x$ and coordinating that scaling with associated values for $y$. One can imagine a window of $x$-values growing or shrinking, and the window of $y$-values simultaneously growing/shrinking, as a correspondence between increments of $x$ and $y$.
3) This way of thinking does not fundamentally rely on an image of motion or an underlying time parameter.

Scaling-continuous reasoning reflects only one aspect of Leibniz' way of thinking about calculus. We do not treat it as necessarily entailing Leibniz' other ideas; for instance, it does not explicitly entail any particular conception about the infinite or the infinitesimal. It is possible to appeal to the image of scaling or zooming to discern or to describe covariation at different levels, without also having encapsulated an image of an infinite scale factor revealing infinitesimal increments. This latter image would require another cognitive act beyond just employing scaling-continuous reasoning. Although Bos (1974) points out that for Leibniz the idea of infinitesimal and variable are closely wed, we do not wish to presume them wed, a priori, for students who are in the process of developing covariational reasoning.

## An Example: Wesley's Scaling-Continuous Reasoning

Here we introduce an episode from a teaching experiment in which the second author worked with two students who reasoned about constant and varying rates of change. One of the students, Wesley[2], communicated ideas consistent with scaling-continuous reasoning. Because this is a theoretical report, we present the following only as a motivating and clarifying example. We use as a guideline that new theoretical work should emerge through encounters with research episodes, and it was through a few key examples from students at both the secondary and undergraduate levels that we developed the theoretical category of scaling-continuous reasoning. We chose this excerpt with Wesley, a secondary student, because his descriptions of scaling-continuous ideas were both spontaneous and clearly articulated.

One aim of the teaching experiment was to introduce contexts in which students could explore situations that, to us, entailed two continuously covarying quantities. The students investigated linear, quadratic, cubic, and higher-order polynomial functions in settings emphasizing rates of change. The following episode was the result of a task in which a triangle dynamically swept out from left to right (Figure 2); students observed a movie of the sweeping action and then produced a sketch of the total accumulated area compared to the length swept.

It is important to note that placing students in situations that we as researchers conceive of as continuous does not guarantee that students will reason with those situations continuously. In fact, on the day prior, both students drew piecewise linear graphs. During this episode, however, the students produced graphs that they described as "smooth curves" (Figure 3). Wesley (W) explained why the graph should be curved, stating that on the prior day, his graph had looked piecewise linear because he had used big increments, but "if you add the tiny increments, like in between, then it curves out," indicating that straight segments were a vestige of a rough graphing process.

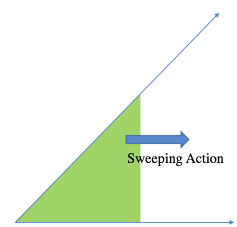


*Figure 2: A static image of the triangle's area swept from left to right*

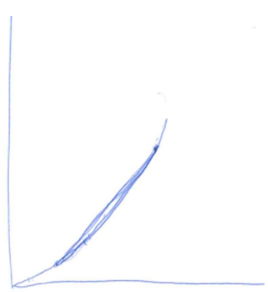

*Figure 3a*

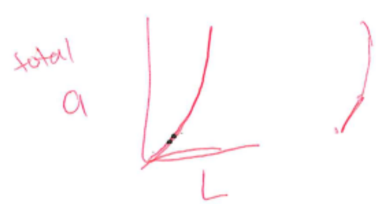


*Figure 3b*

*Figure 3: Wesley (3a) and Olivia's (3b) sketches comparing accumulated area with length swept*

To clarify, the teacher-researcher (TR) asked what the graph would look like between two points that were "super close together," marking two small black dots on Olivia's graph in Figure 3b.

[2] Gender-preserving pseudonyms were used for all participants.

Would it be curved or straight? The following dialogue ensued:

W: I think it would, like, like these two points here (his points on Figure 3) and if you add them, do them exactly it's kind of like that and it kind of goes not straight to the curve and I think it would be more of a, a little bit more of a curve.

TR: You think it'd be a curve?

W: Yeah.

TR: And how come?

W: Because, like, there's tiny points *in between* those tiny points.

TR: Ah. There's tiny points in between those tiny points. (To Olivia): Does that make sense?

O: Yeah.

TR: What if I picked two points that were so close together that I couldn't, you couldn't even like see the difference? They were just so close together there's like an infinitesimal difference in between them. Would the connection between them be a straight line or a curve still?

O: Like the tiniest ones? [TR: Mm-hm.] Then it would be a straight line.

TR: Hmm. (Turns to Wesley). What do you think?

W: I think it'd be more of a curve because I think like it goes on infinitely kind of the points. So if you zoomed in really close on those it would like look like that and then *in between those there's still more points and it goes on forever*.

TR: Hmm. (Turns to Olivia). What do you think?

O: I still think it'd be a straight line because to me it's just a whole bunch of little straight lines and so like to me it would eventually stop because you're graphing the triangle's, like, placing, and so if you like had to choose a place to graph it from each time, then you would connect the points straight like, just straight but a whole bunch of those makes a curve, you know? (She sketches the graph on the right in Figure 3a). And so, I think this or even smaller would be the straight line. The smallest one.

TR: Hmm. What do you say to that? (Turns to Wesley).

W: I think, like, because like if they're two really tiny points right here and you zoomed in a ton it would kind of look something like these two points (in Figure 3) and then *there's still like really little points in between those points*.

The above episode reveals a contrast between Wesley and Olivia's reasoning. For Olivia, the graph is composed of straight segments. The size of those segments does not seem to be absolute, but rather to depend on a choice that is made during the graphing process. The rest of the graph is made by connecting the endpoints, but she does not treat the graph locations on the straight segments as representing quantities in the same way that the endpoints do. Here, and elsewhere in the teaching experiment, Olivia does not conceptualize variation within the straight bits of her graphs. Thus, she seems to be reasoning with, at best, chunky-continuous covariation.

In contrast, Wesley's reasoning is characterized by scaling-continuous variation and covariation. He describes variation happening within bits on his graph, and on each interval he treats the quantities' values as varying continuously, taking on all possible values within the interval. Thus, he imagines the graph to be curved on every interval, no matter how small, appealing to the idea that there are points in between the "tiny points". Wesley does not appear to rely on chunky continuous reasoning, because he fluidly rescales and imagines points in between the points, at any scale, even zooming in infinitely, explaining that this can go on forever.

However, Wesley's reasoning is not smooth-continuous variation either, because he never speaks of, nor does his imagery appear to rely on, a variable moving and tracing out values as it moves. In fact, consistent throughout the teaching experiment is Wesley's lack of reference to movement. Instead his imagery entails zooming and scaling. He explains that if you take any small increment "and you zoomed in a ton" you would see variation, "and then there's still like really little points in between those points." Furthermore, Wesley is explicit that this ability to zoom in, to rescale, "goes on forever," "goes on infinitely," and never grounds out at some atomic level. This is a crucial element of scaling-continuous reasoning, that there is continuous variation at every scale and it never becomes discrete. This entails the *recursion* Thompson and Carlson describe with smooth-continuous variation: "…the person, while reasoning variationally, is alert to the *potential need* to think about smaller intervals in precisely the same way as they are thinking about the interval that is currently in their reasoning" (2017, p. 440). This recursion extends to covariational reasoning for Wesley also; the fact that he sees the graph as curved at each new level of scaling indicates that there is covariation between the two quantities even at the new scale, and that this covariation is non-constant.

## Supporting Calculus

We propose that scaling-continuous covariational reasoning may provide a robust foundation for student thinking in calculus; after all, it was instrumental in Leibniz' invention of calculus. Scaling-continuous reasoning can support an understanding of the ideas of rate of change, limit and derivative, and definite integrals, among others. For instance, Thompson & Carlson (2017) note that the idea of a function's rate of change being non-constant occurs by thinking of a function having constant rates of change over infinitesimal intervals of its argument, "but different constant rates of change over different infinitesimal intervals of the argument" (p. 452). As evidenced by Wesley's explanations, this image is a direct outcome of scaling-continuous covariation, through which one imagines zooming to an infinitesimal scale to imagine a tiny interval on which the function's rate of change is constant. Further, it can provide a foundation for developing an image of instantaneous rate of change, in which the rate of change at a point can be imagined as an average rate of change over an infinitesimal interval. This offers a natural motivation for the limit definition of the derivative.

Scaling-continuous reasoning can also support the concept of definite integral. In an undergraduate calculus course taught by the first author (Ely, 2017), students developed the idea of a definite integral as an accumulation of infinitely many infinitesimal bits of a quantity, each bit corresponding to an infinitesimal increment of the independent variable. This interpretation of definite integral, in turn, provided a robust support for meaningful modeling with integrals.

We do not suggest that smooth-continuous reasoning is unimportant for the development of key ideas about function and calculus. Indeed, we agree that it is a critical aspect of understanding the mathematics of change, including the ideas of calculus, and we support instructional efforts at all grade levels to develop conceptions of continuous covariation. Instead, we suggest that an additional form of reasoning, scaling-continuous variation / covariation, may also plausibly foster productive understandings to support learning in calculus. Given the potential for this form of reasoning to support key calculus ideas, we advocate for additional research to better understand the nature of scaling-continuous variation and covariation and its affordances for productive mathematical thinking.

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Learning Progressions in Mathematics and Physics: An Example for Partial Derivatives

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*We describe the creation of a learning progression about partial derivatives that extends from lower-division multivariable calculus through upper-division physics courses for majors. This work necessitated three modifications to the definition of a learning progression as described in the literature. The first modification is the need to replace the concept of an upper anchor with concept images specific to different (sub)disciplines. The second modification is that rich interconnections between ideas is the hallmark of an expert-like concept image. The final modification is using representations in several ways to support the development of translational fluency in emerging experts. These theoretical changes are supported by examples of research and curriculum in the use of differentials in thermodynamics.*

*Keywords:* Learning progression/trajectory, partial derivatives, multiple representations, representational fluency, thermodynamics.

## Learning Progressions

Science education has recently focused on describing learning progressions (LPs) for content that spans multiple years of instruction (Duschl, Schweingruber, & Shouse, 2007; Lemke & Gonzales, 2006); a similar idea, known as a learning trajectory, has been used in mathematics education (Clements & Sarama, 2004, p. 83). Though many of the LPs described in the literature have focused on K-12 instruction, there are science topics at the university level for which a similar model may prove valuable for educators. LPs are typically characterized by a sequence of qualitatively different levels of knowledge and skills. One goal in the development of LPs is to refocus instruction from concepts that are less consequential to those that are more central to mathematics and science (Plummer, 2012). In particular, LPs are not based solely on a logical analysis of mathematics and science ideas—they are sequences that are supported by research on learners' ideas and skills.

Although the research literature includes various definitions for what constitutes a learning progression (Lemke & Gonzales, 2006; Sikorski & Hammer, 2010; Sikorski, Winters, & Hammer, 2009), the National Research Council defines a learning progression to be "the successively more sophisticated ways of thinking about a topic" (Duschl et al., 2007). The range of content addressed by an LP is defined by a lower anchor, which is grounded in the prior ideas that students bring to the classroom, and by an upper anchor, which is grounded in the knowledge and practices of experts. These anchors are identified by research on the thinking of both novices and experts. An LP hypothesizes pathways that students may follow through content, pathways that are then tested empirically (Corcoran, Mosher, & Rogat, 2009). Individual students might follow one of many such pathways, which may be influenced by a variety of factors, including the educational environment.

Some have noted limitations with learning progressions. LPs tend to place students in definite levels of sophistication, when students might in fact give different answers to different questions, making it difficult to place students on a single level. LPs also tend to identify only one scientifically correct upper anchor. We agree with the assessment of Sikorski and Hammer (2010, p.1037) that "rather than describe students as 'having' or 'not having' a particular level of knowledge" recent learning research "conceptualizes students' knowledge as manifold, context-sensitive, and coupled to and embedded in the social and physical environment." In this paper, we describe a perspective on learning progressions that embraces this manifold view of knowledge by incorporating the idea that it is a learner's *concept image* (Tall & Vinner, 1981) that progresses in a way that broadens or enriches a learner's understanding of a topic.

In the next section, we describe three implications of thinking about LPs in terms of concept images: (1) upper anchors must be generalized in a way that allows experts from different content areas to be different from each other, (2) the strength of the interconnections within an individual's concept image are indicative of expertise, and (3) the role that representations and representational fluency play in illuminating the LP must be elaborated. Then, we illustrate our suggested theoretical changes with an example from an LP we are developing on student understanding of partial derivatives, spanning the collegiate curriculum from lower-division multivariable calculus courses through upper-division physics courses in thermodynamics.

## Theoretical Additions to Learning Progressions

### Experts' Concept Images as "the" Upper Anchor

Interviews with faculty experts (Kustusch, Roundy, Dray, & Manogue, 2012, 2014; Roundy, Weber, et al., 2015) have demonstrated, for example, that physicists and engineers have several ways of reasoning about small quantities that are not shared by mathematicians. These studies, along with our own internal group discussions, have shown that the ways in which experts approach complex problems vary from person to person and from field to field—mathematics experts and physics experts are not the same! We identify the rich and varied understandings of experts with the *concept image* of Tall and Vinner (1981, p.152), *i.e.,* "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." Thus, we see the goal of an LP not as a definite, idealized upper anchor, but rather as a richer understanding more akin to the concept images of experts from varied fields.

### Connections as Indicative of Expertise

Hiebert and Carpenter (1992, p.67) suggest that understanding a mathematical idea requires it to be part of an internal network and that "the degree of understanding is determined by the number and strength of the internal connections." From a concept image perspective, we view a learning progression as describing the enrichment and the increased interconnectivity of a learner's concept image. As developing professionals, middle-division students need to develop such connections rapidly. Yet Browne (2002) found that middle-division students tend not to go back and forth between elements of a concept image spontaneously. To help students increase the strength of their connections, our LP emphasizes opportunities for students to translate between such elements.

Students' ability to transfer knowledge in these ways offers an important means for the empirical validation of our LP. Some of our data (Bajracharya, Emigh, & Manogue, 2017) shows that, while students readily develop a broad concept image, the separate pieces within such a concept image are not necessarily well connected. In contrast, the research discussed above indicates that experts have a rich set of tools that they can use fluently. This representational fluency is itself a key attribute of the upper anchor; achieving such fluency is one of the primary goals of our curricular materials. We regard a learning progression as leading to the enrichment of students' concept images.

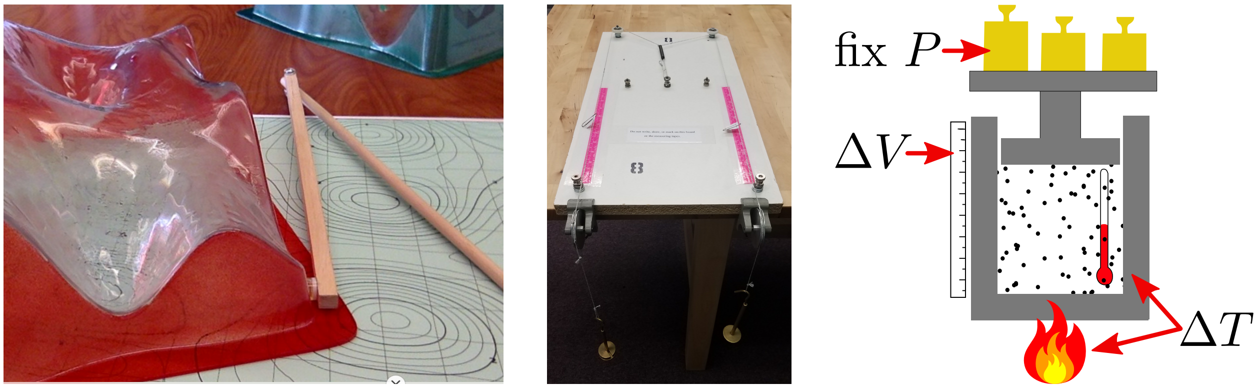
**Representations and Representational Fluency**

In addition to conceptual knowledge, an important aspect of a learner's concept image of a topic is knowledge of (external) representations, such as graphs, equations, experimental configurations, *etc*. Representations are tools that communicate information between learners and instructors, and that also aid learners with thinking and learning (Hutchins, 1995; Kirsh, 2010). Therefore, representations are centrally featured in our learning progression, both in our instruction and in our research about expert and student reasoning.

In our curriculum, we think about external representations in three ways: as languages for doing mathematics/physics, as disciplinary artifacts, and as pedagogical tools. First, we consider different types of representations to be different languages for doing mathematics and physics. For example, one might calculate a partial derivative at a given point in the domain from an equation, a table of data, or a contour plot. These three different ways of expressing a multivariable function have different features and therefore require different procedures for making the calculation. Starting with an equation requires acting on the equation with a differential operator, thereby transforming one algebraic expression into another, and then the evaluation of the new expression at the desired value in the domain. Starting with a table of discrete data requires reading off values, taking differences, and finding a ratio. In this case, it is necessary to include checks to insure that the differences come from a sufficiently linear regime, with the definition of "sufficiently" depending upon the experimental context (Dray, Gire, Manogue, & Roundy, 2017). We want students to be fluent with each of these representations, and also to be able to coordinate or move between representations. The language metaphor suggests that it should be possible for students to achieve some fluency with representations, which would be consistent with an interconnected concept image.

Second, we think of some external representations as disciplinary artifacts. We use the term artifact to emphasize that they are tools of cultural interest within the discipline. Continuing the metaphor of representations as language, these particular representations play the role of technical vocabulary. This distinction is particularly productive when a representation is commonly used in the professional community but is pedagogically problematic. We want students to be able to communicate with the broader community of mathematicians or physicists, so we make a point of introducing these representations in our instruction. For example, a physicist describing a thermal system might plot, on a single graph, data from two (or more) distinct processes. In cases where the resulting curves intersect, an expert interprets this plot as two smooth functions that describe two different experiments, but some students interpret such plots as a single function with a "cusp," and therefore a discontinuous first derivative (Emigh & Manogue, 2017). Plotting multiple experiments (functions) on a single set of axes is common in physics and physics courses but atypical in mathematics courses.

Third, we use representations as pedagogical tools. In particular, we introduce some representations for their pedagogical affordances even if they are not normative (*i.e.,* used by professionals




*(a) Surfaces* *(b) PDM* *(c) Name the experiment*

*Figure 1: Three representations with pedagogical affordances. (a) A plastic surface and matching contour map. (b) The Partial Derivative Machine (PDM), a mechanical analogue of thermodynamic systems. (c) An experiment to measure $\left(\frac{\partial V}{\partial T}\right)_p$ in which the temperature of a gas in a piston is changed using a burner, and the change in volume is measured with a ruler while the pressure is held fixed by weights on the piston.*

while doing their work). For example, professionals do not make plastic surfaces (Wangberg & Johnson, 2013) to represent functions of two variables. However, these surfaces (see Figure 1a) are useful tools for helping students understand many multivariable calculus concepts, including partial derivatives, level curves, the gradient, and line/surface/volume integrals. Similarly, the Partial Derivative Machine (Figure 1b) is a mechanical system that was invented to help students understand thermal systems because the two systems have the same underlying mathematical structure (Sherer, Kustusch, Manogue, & Roundy, 2013). However, those who study thermal systems do not use Partial Derivative Machines in their research.

## Example from a Partial Derivatives Learning Progression

In this section, we describe selected elements from a learning progression for partial derivatives that spans advanced undergraduate courses in mathematics and physics. We focus on partial derivatives because, to physicists, partial derivatives are physically meaningful quantities. We begin by describing an instructional activity that is part of our overall LP and that focuses on key elements of the concept image for partial derivatives. Then, we highlight several results from a research project that has informed our LP and has suggested new curricular changes.

### The "Name the Experiment" Instructional Activity

A typical example of a thermodynamic system is a gas in a piston (Figure 1c). Such a system has a number of physical properties that may be measured and controlled, such as temperature $T$, pressure $p$, volume $V$, and entropy $S$. These properties are not independent, as the *state* of the system (when in equilibrium) is defined by just *two* of these quantities. Each of these four quantities—as well as any other measurable property of a gas—is referred to as a *function of state*, meaning that its value is fully determined by the *state* of the system, which itself may be determined by (*i.e.,* may be a function of) any pair of state variables. Physicists denote such dependencies by algebraic statements such as $T = T(S, V)$ which is to be interpreted as "we are currently thinking of the physical temperature $T$ as depending on the physical quantities entropy $S$ and volume $V$."

We note that this notation is not identical to the function notation commonly taught and used in mathematics.

When encountering partial derivatives in thermodynamics, students have difficulty understanding the significance of the quantity that is being held fixed—a quantity physicists denote using a subscript, as in $\left(\frac{\partial V}{\partial T}\right)_p$ to hold the pressure fixed. The quantity to be held fixed needs to be specified because it is not physically possible to "hold everything else fixed," and there is no unique pair of independent variables describing the system. Roundy, Kustusch, and Manogue (2014) introduced an instructional activity aimed at improving students' overall understanding of thermodynamic variables and what is meant by holding a variable fixed. In the activity, students are prompted to design an experiment that could be done to measure a given partial derivative. One goal of the "Name the Experiment" activity is for students to recognize an experiment as a representation of a particular partial derivative. Linking the experiment—a type of conceptual story—to the algebraic symbols goes beyond simply assigning a physics meaning to each symbol. The experimental story includes a relationship among these physical quantities over time. Figure 1c shows an example of how one could measure $\left(\frac{\partial V}{\partial T}\right)_p$ by heating a gas in a piston, while holding the pressure fixed using unchanging weights on the piston. Determining this derivative requires measuring the small changes $\Delta V$ and $\Delta T$ and then computing their ratio. This procedure reflects the ratio layer of Zandieh's (2000) framework for concept image for the derivative, as embodied in the experimental representation introduced by Roundy, Dray, Manogue, Wagner, and Weber (2015). This framework for *ordinary derivatives* is the starting point for our concept image for *partial derivatives*.

**Research on Representational Fluency with Partial Derivatives**

In this section, we present some of our research and describe how it has influenced our LP. This research focused on how students coordinate information from different types of representations. We gave a problem-solving task (see Figure 2a) involving the calculation of a partial derivative from a table of data and a contour graph, neither of which is sufficient on its own to solve the problem. Each of these representation types is commonly used by professional scientists; therefore this task is an appropriate probe of the students' representational fluency. This task was given as a think-aloud interview to students (N=8) who had completed an upper-division thermodynamics course (Bajracharya et al., 2017).

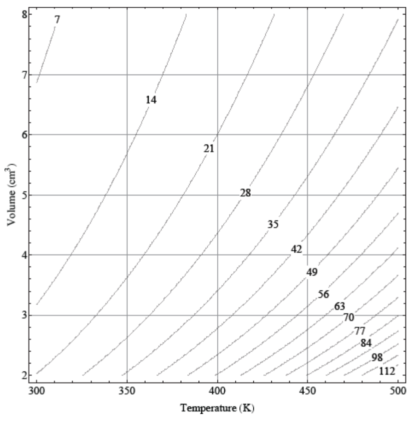
The interview task is a challenging problem with a solution requiring the coordination of many different aspects of the concept image. The analysis suggested that, in order to identify where students are having trouble, it is necessary to examine the individual steps in a solution method at a high level of detail. To facilitate our analysis, we developed a visual means of displaying these steps, which we call a transformation diagram. An example is shown in Figure 2b for one idealized solution to the interview prompt using the method of differential substitution. (Other solution methods, such as sketching a constant-pressure path on the contour map, are also valid and were attempted by students.) In the diagram, boxed items refer to individual representations, arrows refer to transformations between representations, and the transformation steps are numbered for convenience. The transformation diagram is a research tool; we do not (yet) use it as a pedagogical tool. Below, we briefly discuss the interview results pertaining to the solution shown in the diagram, and describe what these results tell us about our curriculum.

The top row of the diagram shows the three different representations of given information: a symbolic expression, a table, and a graph. Each representation gives information about a relationship between three different variables, and this information can be translated (step 1) into a purely

**Evaluate $\left(\frac{\partial U}{\partial T}\right)_P$ at $P$ = 10 atm., $T$ = 410K using the information below.**

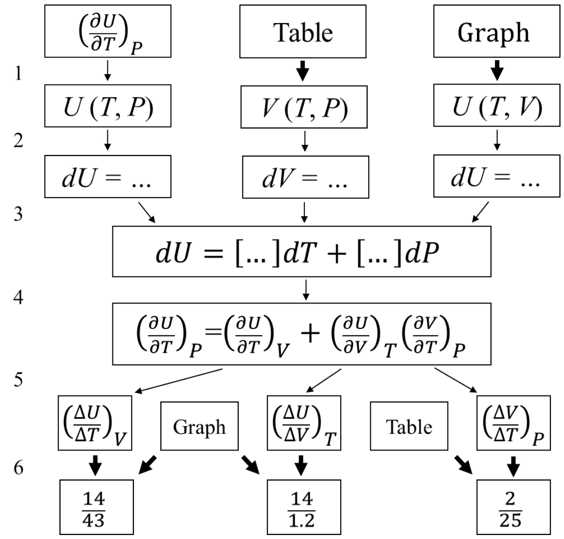
| P(atm.) | T(K) | V(cm³) |
|---|---|---|
| 10 | 300 | 1.32 |
| 10 | 310 | 1.44 |
| 10 | 320 | 1.57 |
| 10 | 330 | 1.71 |
| 10 | 340 | 1.85 |
| 10 | 350 | 2.00 |
| 10 | 360 | 2.15 |
| 10 | 370 | 2.32 |
| 10 | 380 | 2.49 |
| 10 | 390 | 2.67 |
| 10 | 400 | 2.86 |
| 10 | 410 | 3.05 |
| 10 | 420 | 3.25 |
| 10 | 430 | 3.47 |
| 10 | 440 | 3.69 |
| 10 | 450 | 3.91 |
| 10 | 460 | 4.15 |
| 10 | 470 | 4.40 |

Pressure $P$, Temperature $T$, and Volume




Internal Energy $U$ ($T$, $V$).

*(a) Interview Prompt*




*(b) Transformation Diagram*

*Figure 2: An interview task (a) focused on coordinating representations, and a diagram (b) showing the transformations between representations in one idealized solution.*

symbolic form, such as $U(T, V)$, that explicitly identifies the dependent and independent variables associated with that information. It is then possible to determine the total differential for each representation (step 2). This pair of steps proved surprisingly challenging for some students. We believe this is partly due to the fact that jumping directly from the given information to the total differentials is too big a jump for many students to make. Experts often go through the symbolic representation mentally, but students are rarely taught to use it as an intermediate step. We aim to design instructional sequences that can leverage this result to help students identify and use such stepping stones while solving complicated problems.

Once the total differentials have been found, they can be combined using substitution to eliminate $dV$, which does not appear in the desired partial derivative (step 3). This expression is compared to the differential form of the multivariable chain rule, $dU = \left(\frac{\partial U}{\partial T}\right)_p dT + \left(\frac{\partial U}{\partial p}\right)_T dp$, to identify the desired partial derivative as the coefficient of $dT$ (step 4). This pair of steps was particularly difficult for the interviewees—they were consistently unable to consolidate information from three separate representations into a single expression. This finding has suggested a new curricular goal for our LP, to help students learn when, how, and why it is necessary to consolidate information in this fashion.

Once a multivariable chain rule has been determined, each of the three new partial derivatives can be approximated (step 5) as a ratio of small changes and then read from the graph or the table (step 6). In practice, we found that few students struggled with either of these steps, once they had a symbolic expression for partial derivatives that were individually calculable from only a single representation of information. This result validates this piece of our learning progression and suggests that elements of our curriculum that focus on finding derivatives from data have been successful and should continue to feature prominently in future instruction.

## Conclusion

We have described an expansion of the theory for learning progressions in undergraduate courses, illustrated by the specific example of partial derivatives in mathematics and physics. Our learning progression focuses on students' development of a rich, expert-like concept image involving multiple layers and representations, informed by extensive research on both students and experts. This perspective has led us to develop curriculum that fosters students' ability to go back and forth between many fine-grained representations fluidly and spontaneously.

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# Revisiting Reducing Abstraction in Abstract Algebra


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|---|---|---|
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*In this paper, we revisit Hazzan's (1999) fundamental work on reducing abstraction in abstract algebra tasks. As we analyzed hundreds of students' activity related to abstract algebra tasks, we identified many ways students reduced abstraction that did not align with the original framework. We leverage additional theories of abstraction to expand and refine Hazzan's framework to reflect new aspects of familiarity, contextualization, complexity and connectedness, and formality. For each of the new categorizations, we provide illustrations of students engaged in the relevant reduction of abstraction. We conclude with consideration to how the expanded framework may highlight productive types of abstraction reduction.*



*Keywords:* Abstraction, Abstract Algebra, Student Activity


It is well-documented that abstract algebra is a challenging course for students (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Leron, Hazzan, & Zazkis, 1995; Weber & Larsen, 2008). For many students, this is the first time they engage with mathematical objects that are brought into existence via formal definitions. These stipulated concepts are general, complex, and often unfamiliar to students. Hazzan (1999) created the *reducing abstraction* framework to document how students engaged with the generality, complexity, and unfamiliarity of concepts in abstract algebra tasks. She leveraged a number of theories of abstraction to categorize various ways students reduced abstraction when engaging these tasks. This work is foundational and remains one of the more nuanced treatments of student activity in abstract algebra.

In our recent work exploring hundreds of students responses to abstract algebra tasks (Melhuish, 2015), we similarly observed students reducing abstraction. However, we identified a number of ways students reduced abstraction beyond the classifications in Hazzan's (1999) work. In this paper, we synthesize additional theories of abstraction to expand Hazzan's framework in order to better reflect the nuances and variety of approaches found in our students' activity. We share our expansions and provide illustrations of students engaged in reducing abstraction in both productive and unprodctive ways.

## Theories of Abstraction in Mathematics Education

In the field of mathematics education, we have many treatments of the abstraction construct stemming from Piaget's comprehensive work to von Glasersfeld's constructivism and Freudenthal's *Realistic Mathematics Education*. As Piaget noted (1980), "All new knowledge presupposes an abstraction..." (p. 89). However, what scholars mean by an abstract concept, and what we mean by abstraction varies according to a given theory of learning. Hazzan (1999) originally identified three treatments of abstraction: relationship between the object of thought and the thinking person, process-object duality, and complexity of concept of thought. We see these three categorizations as essential, but not exhaustive for exploring student task engagement in the setting of abstract algebra. We discuss several theories of abstraction that ultimately inform our expanded framework.

Before we begin the discussion, we acknowledge an important dimension along which theories of abstraction differ: activity-based versus cognitive. In our overview, we condense features of the theories with little attention to whether the theory was meant to describe cognition or activity. Rather, our purpose is to identify the means through which abstraction is posited to occur.

**Abstracting via Apprehending Properties**

A number of abstraction theories focus on students apprehending properties from a set of known objects. Piaget's (2013) theory of empirical abstraction provides the foundation of much of this work. For empirical abstraction, properties are observed through empirical investigation. If you view a set of white objects, you can abstract the idea of whiteness. Skemp (1986) further expanded this theory explaining, "Abstracting is an activity by which we become aware of similarities ... among our experiences. Classifying means collecting together our experiences on the basis of these similarities" (p. 21) Skemp presented a two-part process of recognizing similarity and then creating a class of object based on similarities. Scheiner (2016) built on this idea further by introducing structural abstraction. Rather than purely empirical (abstracting from empirical objects), abstraction can occur through exploration of mental objects. This exploration may be focused on similarity, but may also occur through focusing on complementary aspects. In each of these theories, a concept is abstracted through collecting a relevant set of properties.

**Abstracting via Building Connections and Complexity**

An alternative lens for abstraction focuses on building connections between or within concepts. Connections play a fundamental role in a number of abstraction theories such as within Dubinksy and McDonald's (2001) schemas or Hoyles, Noss, and Kent's (2004) webbing. Abstraction occurs through the correct coordination of various concepts. This may be internal such as in Dayvdov's (1990) theory where understanding a concept involves unity amongst its connected parts. Alternately, an assembly metaphor (e.g. Ohlsson and Lehtinen, 1997) may underlie a connection focused abstraction theory. Ohlsson and Lehtinen explained that new knowledge structures are developed via assembling "previously acquired ideas" (p. 42). In this sense, a concept is abstracted via coordination of various properties and/or concepts that compose the finalized object.

**Abstracting via Decontextualization**

Decontextualization theories tend to focus on moving from a familiar context to building something abstract that is independent of the context. This type of abstracting can be found in the school of Realistic Mathematics Education and Hershkowitz, Schwarz, and Dreyfus' (2001) *abstraction in context.* These theories distinguish horizontal mathematizing, "the process of describing a context problem in mathematical terms – to be able to solve it with mathematical means" (Gravemeijer & Doorman, 1999, p.117), from vertical mathematizing, where this activity is mathematized through abstracting, generalizing and formalizing (Rasmussen et al., 2005). This type of abstraction occurs when *model of* a specific context or problem (one which is mathematically real to a student) transition to a *model for* additional mathematics that does not rely on the underlying context. These task-based theories align themselves with two views of abstraction related to familiarity. First, a concept can be thought of as abstract if it has moved from a model of a familiar situation to a model for other contexts. Alternately, a concept within a context is more or less abstract depending on how mathematical real it is to an individual. This is roughly equivalent to a student's familiarity with it (cf. Wilensky, 1991).

## Abstracting via Delineation and Refinement

While many theories posit that concepts move from concrete to abstract, Dayvdov (1990) introduced an alternative view where the abstraction process concretizes an abstract kernel of an idea. His theory posits that an object begins as an undeveloped (potentially inconsistent) basic form. This form can be analyzed, and refined until a coherent model is developed. This theory of abstraction can be thought of as moving from a vague idea of concept to a concretely delineated defined concept. The delineation may be more fundamental in advanced mathematics where stipulated definitions form the basis of mathematical structures. Zandieh and Rasmussen (2010) provide insight into this sort of refinement through illustration of students' concept images and definitions of triangle developing. In some senses, this type of abstraction connects to pseudo-empirical abstraction (Piaget, 2013) where abstraction can occur via interacting with an object. In Dayvdov's sense, an object may be a mathematical model rather than a purely empirical "real-world" object. Tall and Pinto (2002) provide such an example where a student moves from a generic visual representation of limit to build to the formal definition. From this theoretical lens, a concept is abstracted when a stipulated definition is abstracted from imprecise models.

## Abstracting via Encapsulating Processes

The final treatment of abstraction is that of process-object duality. This type of abstraction has been explicated through a number of theories including Dubinsky and McDonald's (2001) Action-Process-Object-Schema theory, Sfard's (1991) object reification, and Gray and Tall's (1994) procept theory. Each of these theories operationalizes Piaget's work in the context of various mathematics settings. The underlying feature is the encapsulation of or reification of some particular process into an object. These theories break into three stages: a process that requires individual steps, a holistic view of the process, and a view of the process as an object itself to be used in other processes. For example, Asiala et al. (1997) illustrated this duality in abstract algebra where students may rely on the canonical procedure for creating a coset rather than or in conjunction with treating a coset as an object itself. In this sense, a concept is abstracted when it is no longer treated exclusively as a process, but rather can be used as an object for other processes.

In synthesizing the preceding theories, abstraction has a dual nature: it can be seen both as a cognitive activity and as the concept resulting from that activity. When viewed as a cognitive activity, *abstraction* is a process that transforms a concept via given means. The resulting concept is said to be an abstraction (or "abstracted"). In what follows, we use the term "level of abstraction" to refer to the means by which the student carries out the abstracting activity. Thus, in *reducing abstraction*, an individual is acting cognitively via specific means in order to reduce for herself the level of perceived abstraction. Reduction is tied to the specific context in which the student is working. The Expanded Reducing Abstraction framework in Table 1 presents the levels of abstraction and operationalizes the means by which the activity is carried out.

# Reducing Abstraction: An Expanded Framework

We leverage the prior discussion of abstraction theories to introduce an expanded classification of reducing abstraction. As in Hazzan's (1999) work, we do not claim that these ways of reducing abstraction are mutually exclusive or exhaustive. Rather, we introduce the framework as a tool for making sense of the many ways students engage with tasks containing abstract concepts. We illustrate categorization with data from several of our studies (Melhuish 2015; Melhuish & Fagan, 2017), Hazzan's original paper, and outside literature.

*Table 1. Expanded Reducing Abstraction Framework*

| Abstraction Level as: | Operationalization of Reducing Abstraction |
|---|---|
| Relationship between the object of thought and the thinking person | • Moving from an unfamiliar concept/context to a familiar one[1]<br>• Using familiar concept to bridge between unfamiliar concepts<br>• Moving from decontextualized to familiar context |
| Reflection on the process-object duality | • Moving from an object to a algorithm<br>• Moving from an object to a process[1] |
| Complexity of concept of thought | • Moving from a set to an element[1]<br>• Moving from cohesive concept to disjoint parts<br>• Moving from connected concepts to isolated concepts |
| Precision/formality of concept of thought | • Moving from formal definition to informal definition<br>• Moving from definition to metaphor<br>• Moving from formal definition to generic model |

[1] Aligned with Hazzan's (1999) operationalization.

**Relationship between the object of thought and the thinking person**

Hazzan (1999) operationalized reducing abstraction in this sense via moving from an unfamiliar to familiar situation. She introduced an example where students engaged in tasks related to modular arithmetic groups and instead used properties and knowledge of familiar groups like the real numbers. This type of reducing abstraction occurs not when a particular concept is more general, but rather when it is new and unfamiliar. In many ways, a specific modular arithmetic group is just as concrete as a group like the reals. In our work, we similarly found students reverting to properties of familiar groups such as desiring identities to be either "0" or "1" regardless of binary operation.

We argue that Hazzan's (1999) second example, misapplying Lagrange's theorem to determine that $\mathbf{Z}_3$ is a subgroup of $\mathbf{Z}_6$ constitutes a parallel, but different way of reducing abstraction. In this application, a student does not replace an unfamiliar concept with a familiar concept, but rather uses a familiar concept (divisibility) as a bridge between the unfamiliar setting and an unfamiliar concept (Lagrange's Theorem). We found many students engaging in this type of abstraction reduction. For example, when students identified the size of cosets- rather than attend to the order of the subgroup used to build the coset- they provided the index as their answer. This illustrates a lack of familiarity with cosets bridged via a familiar concept divisibility to an unfamiliar concept, index.

In our third category, a student moves from unfamiliar (general) to familiar (specific) contexts. We saw students do this in a number of places in our data. For example, when students were asked to determine if the equation $(ab)^2=a^2b^2$ holds in groups generally, students returned to a number of familiar contexts including integers under multiplication (a misleading reduction of abstraction) or permutation groups (a productive reduction of abstraction). We see this activity as related to Gravemeijer and Doorman's (1999) referential activity in Realistic Mathematics Education designed tasks. During the process of reinvention of mathematical ideas, students often reduce abstraction and return to a specific context to productively explore ideas. (For an example, see Larsen and Lockwood's (2013) teacher-student exchange about left and right coset equivalence (p. 14).)

## Process-Object Duality

As in the previous category, we subdivide Hazzan's (1999) process-object duality category. Process-object theories often distinguish between holistic processes and step-by-step actions (Tall et al., 1999). The use of "I" statements as highlighted by Hazzan (1999) may reflect algorithmic approaches where procedures are carried out step-by-step. Hazzan presented such an example where a student makes sense of the definition of a quotient group by explaining the canonical procedure for creating a coset using such language as "each one [element] by itself" (p. 81) as she walks through the relevant product creation.

We see this individual algorithm or action as one way to reduce abstraction. However, students may also go from an object and de-encapsulate (productively) to a process or inappropriate replace an object with a process (unproductive). For example, when a student was asked to find the kernel of a specific mapping, they responded, "The kernel of the homomorphism is what is inputted in $\boldsymbol{Z}$ [domain] to output the identity in $\boldsymbol{H}$ [codomain]." The student continued to treat the homomorphism holistically and identify the correct kernel set. This was a productive reduction in abstraction as de-encapsulating the kernel allowed the student to leverage the holistic process to correctly identify the kernel. In contrast, many students reduced abstraction to an action and provided incomplete kernel sets often identifying only one specific element that mapped to the identity of the codomain. In general, this type of abstraction reduction captures object-process duality with varying degrees of sophistication.

## Complexity of the concept of thought

Hazzan (1999) provided one conception of abstraction within this category: using elements rather than a general set. We found Hazzan's (1999) classification useful and observed students engaging in similar reductions of abstraction. For example, consistent with Asiala et al. (1997), many students conflated the equivalence of left and right cosets with the commutativity of their individual elements. However, we also identified other ways students reduced abstraction by reducing complexity. We expand this category to include: *Moving from cohesive concept to disjoint parts* and *Moving from connected concepts to isolated concepts*. An example of the former category can be found in Melhuish and Fagan (2017). Students' engaging with tasks around binary operations reduced abstraction via attending to only one property. When asked if a given function (such as $x^3$) is a binary operation, majority of students focused on one property: closure. Reducing abstraction to this property is productive in traditional tasks where there are two inputs, however unproductive in a setting where not all functions are binary. This example illustrates that a student may reduce abstraction by attending to one aspect or property of a concept rather than the totality. The consequences of the reduction may be unintentional, especially if the students' concept image does not contain all relevant properties.

Alternately, a reduction of abstraction can occur when students lose relationships between other concepts that connect to the meaning of a concept at-hand. For example, when students were asked to find the inverse of $c$ in the Cayley table below (table 2), many students identified $c$, treating $a$ as the implicit identity element. Note that the identity element is not in the first row and column. When asked to explain their thinking on this task, such students did not attend to the role identity played in the concept of inverses. Rather, students explained inverse as, "[i]t's the opposite element of an element." In this way, their abstraction level is lowered via loss of an important connection to another concept: identity. We see these additional complexity theories as related to abstraction theories of properties and theories of connectedness. Students may reduce abstraction via attending to only a subset of properties or alternately losing important connections to additional concepts.

*Table 2. Cayley Table defined on set {a,b,c}*

| | a | b | c |
|---|---|---|---|
| a | c | a | b |
| b | a | b | c |
| c | b | c | a |

**Degree of precision/formality**

This category was not from Hazzan's (1999) framework; rather this additional category emerged to reflect theories of abstraction such as Davydov (1990) where abstraction level reflects the transition of mathematical object from informal/imprecise and to delineated and concrete. This process is sometimes equated to formalization in advanced mathematics. We identified three literature-based ways abstraction can be reduced from this theoretical lens.

Students may replace a formal definition with an informal definition. Lajoie and Mura (2000) identified this type of abstraction reduction when students engaged in tasks related to cyclic groups. We similarly found students leveraged an informal definition of cyclic when tasked with determining if particular groups were cyclic. Their definitions often relied on a generating action: "Start with the unit element and keep piling that onto itself" until a group is created. Reducing abstraction to this informal definition will be successful for finite group, but become problematic for the infinite cyclic group. See Melhuish (2018) for a discussion of how such an informal definition may be supportive for understanding the convention of powers in group theory.

Students may also use metaphors as a way to reduce abreaction. Rather than deal with a concept mathematically, they may leverage a metaphor such as an input-output machine for function (e.g. Zandieh, Ellis, & Rasmussen 2017). When identifying a specific homomorphism's kernel, a student used such a metaphor: "All the elements in the integers that would fit through the function and go to the identity on the codomain. Then start plugging things into that function." The function is treated as something that elements are fitted through. This particular reduction in abstraction was productive when attempting to identify the kernel of a given map.

A third variant is through creation of visual representations. As noted by Sfard (1991), abstraction can be reduced through returning to a visual image which is "more tangible, and encourag[ing] treating them almost as if they were material entities" (p. 6). In our work, we found this type of reduction of abstraction to be infrequent, but often productive when it occurred. One example can be seen in Figure 1. This student was asked to determine if cosets can always be formed from a given subgroup *H* and if so, what their size would be. The student drew a generic group partitioned into cosets to reason that this can always occur and the size would be the same as *H*. The group *G* was represented as a visual that can be reasoned from. This type of abstraction reduction aligns with Pinto and Tall's (2002) generic abstraction.

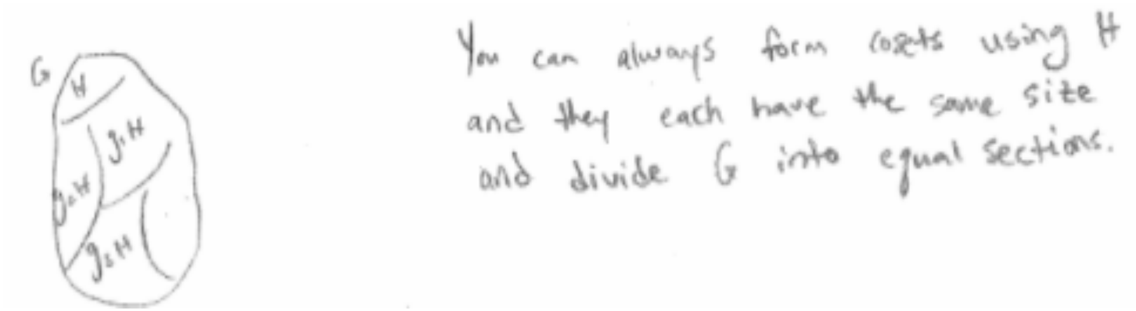


*Figure 1. A student's visual representation of group G with cosets built using subgroup H*

## Discussion

In this paper, we sought to expand Hazzan's (1999) reducing abstraction framework by leveraging a number of abstraction theories. As Hazzan acknowledged, this type of framework cannot be exhaustive, nor mutually exclusive. In fact, the theories of abstraction which inform such an analysis often have overlap themselves. In this sense, we see this framework as a productive lens for analyzing student activity, but not a lens meant to categorize students. As Hazzan did originally, we made our theoretical expansions based on data from abstract algebra tasks. This subject area is populated with concepts that are abstract across many characterizations of abstraction (decontextualized, objects, complex, and stipulated.)

In addition to expanding the framework, we also wished to further highlight that reducing abstraction can be productive. Hazzan (1999) cautioned, "The term 'reducing abstraction' should *not* be conceived as a mental process which necessarily results in misconceptions or mathematical errors" (p. 75). However, Hazzan illustrated student activity that was either erroneous or neutral in problem-solving situations. While this if often the case, we also shared a number of examples of students working productively via reducing abstraction. In fact, we argue the ability to *appropriately* lower abstraction reflects a high level understanding. For example, to move from a formal representation to an accurate generic model reflects an advanced re-construction of a formal idea (von Glasersfeld, 1991). Similarly, Dubinsky and McDonald (2001) identified the ability to de-encapsulate from object to process as an essential feature of object-level conceptions. In this sense, we see parsing reduction of abstraction as more than just a tool for analyzing the cause of inaccurate student responses.

Such a framework can also provide insight into how we meet students where they are at in order to promote *productive* reduction of abstraction. There is power in being able to reduce abstraction in problem-solving (or proving) situations. Weber and Alcock (2004) presented contrasting cases where students (and graduate students) may produce proofs via working in an entirely formal system or through semantic explorations. In some sense, moving out of the formal system reduces abstraction level. It is this reduction that allowed successful provers in their study to gain insight into proofs. In Larsen and Lockwood (2013), students moved between decontextualized and contextualized situations to productively explore conjectures and ultimately reinvent mathematics. The question is not, how do we prevent students from reducing abstraction, but rather how do we promote students in reducing abstraction in productive ways? Through better understanding of reducing abstraction, we may ultimately aid in supporting students as they navigate abstract concepts in advanced mathematics.

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# The Potential Virtues of Wicked Problems for Education

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*We offer a contribution to a theory of transdisciplinary curriculum based in empirical research of an undergraduate mathematics course in quantitative literacy. By organizing around contexts and developing open, semester-long projects, this course blurred disciplinary boundaries. Fortunately, ignoring debates about where mathematics ends and these contexts begin is well-suited for the goals of general undergraduate courses. We found that the language of* transdisciplinary *and* wicked problems *fitted our experiences designing, teaching, and studying the course. We share selected empirical findings, then develop a transdisciplinary curriculum theory for wicked problems.*

*Keywords*: wicked problems, transdisciplinary education, curriculum, quantitative literacy

> This is the first human generation in which the majority will live in crowded cities, whose actions will flood low-lying islands and whose rate of resource use exceeds 2.5 times the production capacity of the planet (Melkert and Vox, 2008). Well-founded projections suggest that future supplies of the air we need to breathe, the water to drink and the food to eat are in doubt. (Schneider et al., 2007 as cited in Brown, Deane, Harris, & Russell, 2010, p. 3).

> It seems as though some problems are tame, such as factoring a quadratic equation, traversing a maze, and solving the tower of Hanoi puzzle. But problems of importance… are invariably 'wicked.' (Coyne, 2005, pp. 5-6)

The first excerpt above is the opening paragraph of a book, *Tackling Wicked Problems* (Brown et al., 2010), which focuses on uniting people across and outside of disciplines to confront the global problems that affect us all. In the book, scholars argued that this particular class of problems requires *transdisciplinary inquiry*--fusion of knowledges across and outside of disciplinary boundaries. Transdisciplinary approaches create knowledge that "is more than the sum of its disciplinary components" (Lawrence, 2010, p. 19).

The *wicked problems* at the center of this transdisciplinary inquiry require imaginative approaches because they cannot be (or at least, have not been) fully resolved through disciplinary techniques. In the second quotation above, Coyne (2005) illustrated wicked problems by contrasting them with tame problems. Coyne used these three examples--factoring, navigating a maze, and the tower of Hanoi puzzle--which inadvertently criticized the prevalence of tame problems in mathematics education. The tame problem examples he chose subtly posed a challenge to mathematics education to consider different problems, called "wicked."

*How might we, mathematics educators, react?* In this paper, we produce a theoretical report grounded in an empirical study of a quantitative literacy (QL) course which involved a focus on wicked problems. The theoretical perspective is a transdisciplinary curricular theory focused on education about wicked problems. We found that such an approach can involve at least four things: a context-based curricular organization, a blurring or erasure of disciplinary boundaries to classroom activity, opportunities to engage in open problematizing of the world, and a repositioning of the teacher relative to students.

## Transdisciplinary Education for Wicked Problems

Rittel and Webber (1973) first conceptualized wicked problems in design and planning. They argued that there are ten distinguishing characteristics of wicked problems:

1. There is no definitive formulation of a wicked problem.
2. Wicked problems have no stopping rules.
3. Solutions to wicked problems are not true or false, but good or bad.
4. There is no immediate and no ultimate test of a solution to a wicked problem.
5. Every solution to a wicked problem is a 'one-shot operation'; because there is no opportunity to learn by trial-and-error, every attempt counts significantly.
6. Wicked problems do not have an enumerable (or an exhaustively describable) set of possible solutions, nor is there a well-described set of permissible operations that may be incorporated into the plan.
7. Every wicked problem is essentially unique.
8. Every wicked problem can be considered to be a symptom of another problem.
9. The existence of a discrepancy representing a wicked problem can be explained in numerous ways. The choice of explanation determines the nature of the problem's resolution.
10. The planner has no right to be wrong.

Wicked problems have been conceptualized within many fields, including: environmental studies (e.g., Kreuter, De Rosa, Howze, & Baldwin, 2004), political science and public policy (e.g., Head, 2008), public health (e.g., Blackman et al., 2006), public risk and defense (e.g., Ritchey, 2001), and economics (e.g., Batie, 2008). The move to consider work in their fields as wicked problems generally emerged alongside recognition of limitations of attempts to quantify complexity. Ritchey (2001) claimed, "if you work with long-term social, commercial, or organizational planning – or any type of policy planning that impacts *people* – then you've got *wicked problems* (p. 1). The presence of wicked problems can be signified by a sense of reactivity, where after attempting resolution, the problem transforms and "fight[s] back when you try to do something" (Ritchey, 2001, p. 1).

Studying complex wicked problems, therefore, poses a challenge to disciplinary approaches to knowledge; in response, transdisciplinary approaches "step outside the limiting frames and methods of phenomenon-specific disciplines" (Davis, 2008, p. 55). The transdisciplinary approach mirrors the collective nature of wicked problems and values not only the multiplicity of knowledges from different disciplines, but also their tapestry (Lawrence, 2008). Transdisciplinary approaches involve redrawing the boundaries of inquiry (to the extent possible) *around the problems themselves*, to ask what disciplines and their unifications can contribute to addressing and resolving a problem, rather than whether a problem belongs inside a discipline.

### Transdisciplinary Wicked Problems and Quantitative Literacy (QL)

In mathematics education, Vacher (2011; 2017) has begun to argue that QL is transdisciplinary. His claim ultimately hinges on the fact that people from different disciplines have used QL or numeracy as terminology to connect the quantitative to their fields. Our approach to transdisciplinary is different in that we are decentering the disciplines, and instead focusing on wicked problems. As a result, the early empirical underpinnings of this transdisciplinary curriculum theory emerge from an undergraduate QL course not centered solely on learning mathematics and statistics; instead, the course is centered on exploring what mathematics and statistics might offer us while we learn about *contexts and wicked problems within them*. Of course, the course certainly did involve students learning significant mathematics and statistics, but that was not the primary organizational element.

This QL classroom was located at a large Midwestern university--a predominantly white institution located in what is often described as a “college-town.” The studied course emerged from institutional efforts to provide multiple routes to fulfilling the university’s general mathematics degree requirement. This course was organized around three different context-based modules: *The World and Its Peoples*, organized around the choices and power involved in counting people and quantifying the world; *Numbers and Media*, designed around the flexibility of numbers as socially constructed, rhetorical, subjected, and powerful; and *Health and Risk*, centered on considering the quantification in health and risk and its implications on fear and safety narratives. In the next section, we used examples from students’ work to further illuminate characteristics of wicked problems. We pulled the examples from a larger study of students’ course projects (Craig, 2017). All names are pseudonyms.

## Examples of Wicked Problems Characteristics

The students chose a wide variety of issues to study, but formulating the particular problem or set of problems was challenging. Many students asked about the suitability of a particular topic for this project. Upon first impression, many students’ topic choices had developed ideas around problems of massive scope (e.g., racism, climate change). Despite students completing the same project phases, their work was unique content because they focused on different problems and formulated similar problems differently.

### (Characteristic #7) Every wicked problem is essentially unique.

We begin our exploration of wicked problem characteristics with Characteristic #7 to share some students’ project topics, found in Table 1. The range project topics illustrates how wicked problems are not confined to any disciplinary boundary, including mathematics. Further, the list also suggests how all disciplines, including mathematics, are relevant to considering wicked problems.

*Table 1. Selected Student Projects Titles and Descriptions*

| Title | Description |
|---|---|
| Censorship | Internet censorship policies and histories across countries |
| Climate Change | Looking at the impacts of polar ice shifts on climate from a religious perspective |
| Do Schools Kill Creativity? | Budget cuts to arts and humanities programs in K-12 public schools in the United States |
| Domestic Violence | The perpetual cycle of domestic violence across generations |
| Drug Abuses and Overdoses | The prevalence of heroin abuse in specific cities across the United States |
| Gun Violence | Advocating ways to avoid gun violence, specifically preventative learning about guns |

| | |
|---|---|
| Impact of Big Money on Politics | The results of the Citizens United court decision on money entering politics |
| Overpopulation | The effects of a one-child policy on China and the persistence of problems of resource use |
| People of Color in Media | The erasure of entertainment and media achievements of actors of color |
| Recycling Practices | Cross-country analysis of recycling practices and constraints on recycling |
| Representation of Women in Media | The disparities in gender representation in political news reporting |
| Stigmas around Mental Illness | How different cultures respond to depression and stigmatize the illness |

**(Characteristic #1) There is No Definitive Formulation of a Wicked Problem**

> Another issue that internet censorship is tied to is that sometimes internet censorship can be a good thing. For example, I don't think it is okay for people to be posting pro-terrorist webpages, or terrorist recruitment forms online. If it really is a threat to national security, then I believe that the government has a right to restrict that. (Matt, Phase 2, Media Synthesis)

Matt confronted the challenge of formulating what the problem of censorship entirely involved. Rittel and Weber (1973) stated that "the formulation of a wicked problem *is* the problem!" (p. 137). To formulate a wicked problem involves establishing a discrepancy between what *is* and what *should be*. But there are plural perspectives on both what is and what should be, and therefore multiple formulations of the problems we face.

**(Characteristic #2) Wicked Problems Have No Stopping Rules**

> The one child policy was published to limit people to have only one child. This stopped the growth of population. And people's life changed a lot from this. Then, population aging became another issue for China. Population aging hurts the economy and the government decides to end the one child policy. The new policy is "One Couple, Two Children. (Leilei, Phase 2, Media Synthesis)

Because wicked problems have no stopping rules, that implies they have a history. Students engaged with the histories of these complex wicked problems through media analyses and syntheses, which usually included their own formulation of the problem and explanation for the cause. The reintroduction of the initial problems underscored the complexity and fluidity of these projects. The challenges involved in testing solutions to wicked problems are underscored by the idea that those solutions cannot be evaluated in the same way that tame problems can.

**(Characteristic #3) Solutions to Wicked Problems are Not True or False, but Good or Bad**

> I think we can measure improvement with the issue of the lack of diversity by continuing to take data on how diverse film characters are in general. But, by how many women are directing mainstream? People of color? Who are running these networks? Who are writing these shows? Orange is the New Black has one of the most diverse casts on

> television right now. The writing room is mostly women, but no people of color. Do we chalk this up to a win anyway? (Aisha, Phase 4, Written Reflection)

Aisha's reflection further complexified the relationship between mathematics and statistics as disciplines and the resolution of wicked problems. She recognized the limitations of mathematics in determining the quality of a solution. The transdisciplinary reasoning emerged from how open problematizing, as Smith (1997) predicted, burst the boundaries of mathematics curriculum.

**(Characteristic #4) There is No Immediate and No Ultimate Test of a Solution to a Wicked Problem**

> I think it's interesting that the world measures depression by suicide count. I don't know how accurate I feel that is but it's interesting and I wish that we could change it. *But how do you measure depression? Through chronic, manic, and other forms of breakdown or do we not measure it by severity and simply mush it all together? It's difficult to measure something that can't be seen*. (Beth, Phase 4, Written Reflection, emphasis added)

Beth summarized how wicked problems change what it means to do problem solving. The choices of what to measure, how, and when, are political and aligned with particular formulations and particular explanations of a wicked problem discrepancy (Best, 2008). Quantitative methods and information cannot provide evidence of improvement on a wicked problem, unless situated within a particular formulation of the problem. Within the boundaries of mathematical problem solving, this deeper consideration of how to engage quantitative methods involves traversing disciplinary boundaries for other information.

**(Characteristic #5) Every Solution to a Wicked Problem is a 'One-Shot Operation'; Because There is no Opportunity to Learn by Trial-and-Error, Every Attempt Counts Significantly**

> He brings up the fact that there are so many murders in Central America and Mexico and other parts of the world, prisons packed, the global black market is estimated at 3 hundred billion a year, *all due to the war on drugs. Yet more people are using drugs than there ever was before*. (Diana, Phase 2, Media Analysis, emphasis added)

Diana's analysis of a piece of media reflects something critical about the urgency of these wicked problems. Despite this course being labeled mathematics, many forms of reasoning were salient. Specifically, quantitative methodologists would have very particular and technical strategies for determining the effects of the war on drugs (not the least of which would be formulating what that means). At the same time, there is very serious moral, social, historical, psychological, and intuitive reasoning involved in my students' projects.

**(Characteristic #6) Wicked Problems Do Not Have an Enumerable (Or an Exhaustively Describable) Set of Possible Solutions, Nor is There a Well-Described Set of Permissible Operations that May Be Incorporated into the Plan**

> I'm not entirely sure what the plot of the story that I am hoping to tell is but I know *that I want to talk about the collective solution to the problem from multiple sources…* community awareness, education on certain matters revolving around mental health, and acceptance are the ideal ways to combat more easily preventable mental health problems

> or at least to lessen the effects of the problem. (Beth, Phase 3, Infographic Check-in, emphasis added)

This aspect of wicked problems came out the most in a class check-in where I asked students to report the plot of the infographic they were creating for the third phase of the project. There are no limits on what can be included in resolving a wicked problem, and the acceptance of multiple forms of data and engagement of multiple forms of reasoning across and outside of disciplinary boundaries is central.

**(Characteristic #8) Every Wicked Problem Can Be Considered to Be a Symptom of Another Problem**

> After doing the research, I found the issue is not only too many people, but also pollution, land, resources and other issues. (Leilei, Phase 4, Written Reflection)

Leilei produced a project on China's One-Child Policy and concluded that overpopulation itself is an amalgam of other interlocking problems. Although the first phase of the project was the explicit time when students formulated their topic, the challenges to problematizing wicked problems persisted through complexity. I had five students make significant changes to their topics during the second phase of the project, as they clarified their own interests, but all students reformulated their problems at some point during the course.

**(Characteristic #9) The Existence of a Discrepancy Representing a Wicked Problem Can Be Explained in Numerous Ways. The Choice of Explanation Determines the Nature of the Problem's Resolution**

> There are a few different issues revolving around the stigma that exists around mental illness; the first being that people sometimes avoid or bully those suffering from mental illness just due to their differences, the next is that mental illness is often misrepresented in the media, those suffering from mental illness in television shows or movies are almost always depicted as some sort of antagonist, murderer, or criminal. (Beth, Phase 2, Media Synthesis)

Beth's project on mental illness evolved into an investigation of cultural differences in identifying and treating mental illness. She focused on a discrepancy between perceptions of mental illness taking the form of stigma and the realities of people suffering from mental illness. She had different explanations for that discrepancy which she explored during her project.

**(Characteristic #10) The Planner Has No Right To Be Wrong**

> …In 2014 it is estimated that at least 6,800 overdose deaths occurred in the European Union. In Oceania, which includes Australia and New Zealand, there were 1,700 and 2,100 drug related deaths in 2013. In Scotland there were 613 drug related deaths in 2014. In South America, the Caribbean and Central America reported between 4,900 and 10,900 drug related deaths in 2013. In the United States, overdose deaths from opioids, including prescription opioids and heroin, have nearly quadrupled since 1999. Overdoses involving opioids killed more than 28,000 people in 2014. During 2014, a total of 47,055 drug overdose deaths occurred in the United States. *These statistics make it known that*

> *the use of drugs is a very serious issue that needs to be fixed somehow.* (Diana, Phase 2, Media Synthesis, emphasis added)

Diana's project on the opioid epidemic developed out of her hometown struggling with drug addiction. Although Diana herself formulated the problem in a particular way with a particular resolution – "our drug policies care more about criminalization over health and this has to be changed" (Diana, Phase 2, Media Analysis) – she explicitly noted the seriousness, urgency, and responsibility for policy planners to act.

## Discussion

Our purpose for sharing these examples of how wicked problem characteristics manifested in students' course projects was to give life to the characteristics, rather than leave them as strictly theoretical. Craig (2017) explored three themes that emerged from a deeper analysis of the course projects, transdisciplinary, complexity, and democratic openness. Here, we focused on transdisciplinary to begin developing a curriculum theory. A *transdisciplinary curriculum theory for wicked problems* both connects with and diverges from disciplinary education in mathematics or statistics. On one hand, the openness of transdisciplinary inquiry is fully inclusive to all disciplines, therefore, mathematics and statistics play indispensable roles. On the other hand, exploring wicked problems within disciplinary boundaries necessarily excludes important considerations and leaves resolutions more fragile, incomplete, and possibly counterproductive.

For wicked problems where boundaries are elusive (or impossible) to draw, the inquiry process should be inclusive. All knowledges are relevant and applicable to resolving wicked problems, and transdisciplinarity "is created by including the personal, the local and the strategic, as well as specialized contributions to knowledge" (Brown et al., 2010, p. 4). This lack of boundaries is conducive to a *transdisciplinary imagination* (Brown et al., 2010). In the transdisciplinary imagination are attempts "to generate fundamentally new conceptual frameworks, hypotheses, theories, models, and methodological applications that *transcend* their disciplinary origins" (Hall et al., 2012, p. 416, emphasis in original).

Wicked problems facing the world regarding sustainability are staggering in scope, and elusive. Further, evidence increasingly suggests that addressing sustainability questions necessarily involves addressing myriad social injustices and complex economic relationships (Peterson, 2016). Nearly two decades ago, education for wicked problems was "taken seriously by no one, even if they are included with some regularity in official curriculum documents" (Parker, Ninomiya, & Cogan, 1999, p. 119). Serious consideration of these wicked problems as justifiable school curriculum remains uncommon. Although some argue we prepare students to be wicked problem resolvers by becoming expert tame problem solvers, we still generally avoid these problems during schooling in favor of self-contained and sanitized word problems, particularly in mathematics education.

That wicked problems theory emerged from design theory is fitting for considering curriculum theory. Designing curriculum involves reactive, complex, and transdisciplinary students, situated among reactive, complex, and transdisciplinary social institutions and wicked problems. Perhaps curriculum design is a wicked problem, without final resolution. But, what right do we have to be wrong about education? We share responsibility for the social problems caused by and embedded in how schools, curricula, assessments, and teaching are designed and planned (Butin, 2002).

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# Generating Equations for Proportional Relationships Using Magnitude and Substance Conceptions

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*We discuss a magnitude conception and a substance conception of fractions and variables that future middle-grades and secondary teachers used when developing and explaining equations for proportional relationships by reasoning about quantities. We conjecture that both conceptions are important for developing equations. The substance conception is useful when a fraction or variable functions as a multiplicand, but not when it functions as a multiplier. The magnitude conception is useful when a fraction or variable functions as a multiplier, but may not be essential when it functions as a multiplicand. Expertise may involve recognizing that the conceptions are distinct and developing a sense of when each conception is useful.*

*Keywords:* Equations, proportional relationships, variables, fractions

The domain of ratio and proportional relationships is a gateway to algebra, other topics in K-12 and undergraduate mathematics, and science (National Center on Education and the Economy, 2013). Yet this crucial domain is also one of the most challenging to learn (e.g., Lamon, 2007). Our research group has been studying how future middle grades and secondary teachers reason about ratios and proportional relationships as they take our mathematics content courses, which focus on multiplicative ideas. In this paper, we are interested in reasoning that takes a *variable-parts* perspective on proportional relationships (Beckmann & Izsák, 2015), a perspective that had been largely overlooked in the research literature, but provides a pathway to developing equations and solving proportions. In these reasoning situations, we are interested in what ideas are useful and generative, and what ideas are especially hard. We discuss a conjecture about two conceptions of fractions and variables—a magnitude conception and a substance conception. Based on preliminary analysis of data, we conjecture that both conceptions play an important role in generating and explaining equations for quantities in a proportional relationship, and that knowing when to use which conception is an aspect of expertise.

## Background and Theoretical Perspectives

We view ratios and proportional relationships as part of the multiplicative conceptual field (Vergnaud, 1988)—a web of interrelated ideas that also includes multiplication, division, fractions, and linear relationships. According to Beckmann and Izsák (2015), a quantitative definition of multiplication can organize and connect multiplication, division, and proportional and inversely proportional relationships. We therefore use quantitative definitions of multiplication and fractions as central organizing ideas in our mathematics content courses for future middle grades and secondary teachers.

### Quantities and Magnitudes

Measurement includes describing the size of entities (objects or stuff) as some number of a chosen measurement unit, which can be a standard unit, such as a liter, or a non-standard unit, such as a strip drawn on a piece of paper. Although quantities are often described as numbers with units (e.g., CCSS; Common Core State Standards Initiative, 2010), we agree with Thompson (1994) that one need not have selected a specific measurement unit to conceive of an entity as a quantity. In this paper, we define “quantity” to mean an entity that either serves as a measurement unit or could be expressed as some number of another measurement unit, where “some number” means any positive whole, rational, or irrational real number. For example, if a student views one strip drawn on a piece of paper as 2/5 of another drawn strip, then we consider the student to be treating both strips as quantities.

The language of linear algebra may be helpful for thinking about quantities. For each measureable attribute, such as length, weight, or volume, we can associate with that attribute a one-dimensional vector space over the real numbers. Given such a vector space, there is no automatic choice for a basis, and we can work with the vector space without having chosen a basis. Therefore, when we view an entity as a quantity, we essentially consider it as an element of one of these one-dimensional vector spaces, but we need not think of the quantity in terms of a basis for the vector space. When we choose a measurement unit for a given attribute, this measurement unit forms a basis for the one-dimensional vector space, and a quantity can be expressed as a scalar multiple of the basis vector, i.e., the quantity can be expressed as so and so many of the chosen measurement unit. We call this scalar (real number) the magnitude of the quantity with respect to the chosen measurement unit (see also Thompson, Carlson, Byerly, & Hatfield, 2014).

**A Quantitative Definition of Multiplication**

Although people can use intuitive models to recognize some multiplication situations (e.g., Fischbein, Deri, Nello, & Marino, 1985), if we want students and teachers to be able to make principled arguments for why multiplication applies in a situation, then we need a definition of multiplication. If multiplication is to be understood as a single coherent operation that applies across many different types of situations and across whole numbers, fractions, and decimals, then we need a definition of multiplication that applies to all these cases. One version of a definition we use in our courses for future teachers is as follows. In a situation involving quantities, we say that $M \bullet N = P$ if $M$ is the number of groups in the product amount, $N$ is the number of base units in 1 group, and $P$ is the number of base units in $M$ groups for a suitable base unit, group, and product amount in the situation. We call $M$ the multiplier, $N$ the multiplicand, and $P$ the product; $M$, $N$, and $P$ can be non-negative whole numbers, fractions, or decimals. This definition is similar to the one given by Beckmann and Izsák (2015). In some of our courses we have reversed the order of multiplier and multiplicand and written the multiplicand first and the multiplier second. Within a course, we use a consistent order to facilitate clear communication.

This definition of multiplication connects multiplication with measurement (e.g., Davydov, 1992). In the definition, $N$, $M$, and $P$ are magnitudes of the quantities “the group” and “the product amount” with respect to the measurement units “the base unit” and “the group.” In particular, the multiplier and the product are the results of measuring the product amount in two ways. In some versions of our definition, we clarify the measurement language by defining the multiplicand as the number of base units it takes to make 1 group exactly, the multiplier as the number of groups it takes to make the product amount exactly, and the product as the number of base units it takes to make the product amount exactly.

Reasoning with the definition of multiplication requires organizing and structuring quantities by unitizing, iterating, and partitioning—ideas that have been identified as foundational to multiplicative reasoning in the literature (e.g., Hackenberg & Tillema, 2009). It requires unitizing because $N$ base units form 1 group, so those $N$ base units function as a unit; it requires iterating because if $M$ is 5, one must consider 5 copies or iterates of that group; it requires partitioning because if $M$ is 1/5, one must consider 1/5 of that group, so one must partition the group into 5 equal-sized parts.

### A Quantitative Definition of Fraction and Fraction Subconstructs

In our courses for future teachers, we use essentially the same definition of fraction as in the *Common Core State Standards for Mathematics* (CCSS, 2010). We define a unit fraction $1/B$ to be the amount formed by 1 part when a unit amount (or whole) is partitioned into $B$ equal-sized parts. A fraction $A/B$ is defined to be the amount in $A$ parts, each of size $1/B$ of the unit amount (or whole). Therefore, this definition relies on partitioning to form unit fractions and on iterating unit fractions to form both proper and improper fractions. Viewing fractions as obtained by iterating unit fractions can be valuable for students (e.g., Behr, Lesh, Post, & Silver, 1983), and we have found that our future middle grades and secondary teachers reason effectively with this definition.

Various fraction subconstructs or interpretations have been identified in the literature, including the measurement and operator subconstructs (e.g., Behr, Lesh, Post, & Silver, 1983; Kieren, 1976). With the measurement interpretation, fractions can be viewed as plotted on number lines via measurement. To plot the fraction A/B we measure A parts, each of size 1/B of the unit (the interval from 0 to 1). With the operator interpretation, the fraction A/B is seen as a transformation that takes one quantity to another, for example by stretching or shrinking.

Later in this paper we identify substance and magnitude *conceptions* of fractions, which are different from the fraction *subconstructs* in the literature. The magnitude and substance conceptions are essentially orthogonal to the measurement subconstruct, whereas the magnitude conception may be a prerequisite for some instances of the operator subconstruct.

### Equations for Proportional Relationships

Proportional relationships in which two unknown quantities are in a fixed ratio can be modeled by equations in two variables, including equations of the form $y = m \cdot x$ or $y = x \cdot m$, where $m$ is a constant of proportionality. By "variable" we mean a letter or symbol that stands for any number from some set (which might not be explicitly specified). Multiplication is numerically commutative, but the multiplier and multiplicand play different roles in quantitative situations. Depending on how the quantities in a situation are structured and organized, one of $y = m \cdot x$ or $y = x \cdot m$ might be better for modeling the situation.

In this paper, we are interested in cases where the constant $m$ is a fraction $a/b$ (so $a$ and $b$ are positive integers). Thus, our quantitative definitions for multiplication and fractions are potentially useful for explaining and generating equations for quantities in a proportional relationship. We are interested in ideas needed to generate and explain equations that relate quantities, especially when the quantities are viewed from the variable-parts perspective (Beckmann & Izsák, 2015), as in the paint task in Figure 1. The 2 parts of blue paint and the 5 parts of yellow paint in that task are all the same size as each other, but that size is unspecified and could vary. The equations $Y = 5/2 \cdot X$ and $X = 2/5 \cdot Y$ (among many others) model the situation in the paint task and fit with the definition of multiplication by taking 1 base unit to be 1 gallon and 1 group to be either all the blue paint or all the yellow paint.

**Paint Task**: The figure below shows volumes of blue paint and yellow paint. The diagram does not show specific numbers of gallons of paint, but each small part of both strips represents the same number of gallons.

Let's say there are X gallons of blue paint (in all) and Y gallons of yellow paint (in all). How are X and Y related?


*Figure 1. A proportional relationship task about paint, from a variable-parts perspective.*

Generating algebraic equations is known to be difficult in part because understanding how algebraic notation symbolizes quantitative situations is difficult (see Kieran, 2007). Even advanced students produce equations with a "reversal error," such as $6S = P$ for a situation in which there are 6 students for every professor (e.g., Clement, 1982). Hackenberg and Lee (2015) explained students' difficulties with generating equations in terms of students' multiplicative concepts, which involve capacities to coordinate multiple levels of nested units and to anticipate, hold in mind, and reorganize such structures. Other authors have pointed to students' conceptions of variables as a source of difficulty, such as treating a variable as a shorthand label for an object or unit (e.g., Küchemann, 1981; Lucariello, Tine, & Ganley, 2014, McNeil et al., 2010). These authors described such a conception of variables as low level or as a misconception. According to Küchemann, using a letter as an object amounts to reducing the letter's meaning from something abstract to something more concrete. He noted that such a reduction often occurs when it is not appropriate, especially in cases where one must distinguish between objects themselves and the number of objects. Yet Beckmann & Kulow (2018) found that future middle grades teachers often used variables as labels when they generated valid equations and produced viable arguments using fractions and multiplication.

### A Knowledge-in-Pieces Stance Toward Cognition

We take Knowledge-in-Pieces as our theoretical frame for studying cognition (e.g., diSessa, 1993). In particular, we assume students' knowledge in a mathematical domain is an ecology consisting of many elements, some of which are primitive and intuitive, and simply taken as given, and some of which are more scientific in nature. Some knowledge elements may be closely coordinated, whereas others may be seen as unrelated. Knowledge elements are highly sensitive to context. A knowledge element might be cued in one context but not in another where an expert might view it as relevant. We view learning as a process that involves refinement and coordination of knowledge elements, not a process of repealing and replacing ideas (e.g., Smith, diSessa, & Roschelle, 1993). In particular, this refinement and coordination consists of separating ideas as well as connecting them, and it consists of discerning features of new contexts that make an idea applicable or not applicable, or that make using one idea preferable over another idea (Wagner, 2006). Thus, becoming proficient in generating and explaining equations could involve distinguishing different ways of thinking about a variable or a fraction and a sense of when each way of thinking is more useful or less useful.

## Methods, Data Sources, and Research Question

As part of a larger ongoing investigation into future middle grades and secondary teachers' reasoning in the multiplicative conceptual field, we are interested in generating and testing conjectures about ways of thinking about fractions and variables that may be important when developing, explaining, or interpreting equations and expressions involving multiplication. This paper is primarily theoretical because it discusses conjectures we have generated based on initial passes through our data. Our research question for this paper is therefore: Based on our project's data, what ways of thinking about fractions and variables, beyond those already identified in the literature, can we conjecture to be important for generating and explaining equations to relate two unknown quantities that are in a proportional relationship, viewed from a variable-parts perspective?

Data come from 104 semi-structured 75-minute interviews conducted individually with 22 participants, 10 from 2 cohorts of future middle grades mathematics teachers (5 interviews each) and 12 from 2 cohorts of future secondary mathematics teachers (6 with 5 interviews each and 6 with 4 interviews each). All participants were taking mathematics content courses focusing on ideas in the multiplicative conceptual field between the fall of 2014 and the spring of 2017. Interview questions were related to course topics, although some interview questions preceded instruction in a relevant topic. The participants were selected to be mathematically diverse based on their performance on a fractions survey (Bradshaw, Izsák, Templin, & Jacobson, 2014). The data included transcribed video-recording of each interview and scanned copies of the written work each participant generated. To analyze the data, members of the research team watched interviews multiple times, attending to words, gestures, and inscriptions, and wrote cognitive memos discussing and summarizing participants' reasoning.

## Conjectures about Conceptions of Fractions and Variables

Based on our initial analysis, we identify two conceptions about fractions and variables—a substance conception and a magnitude conception—that we conjecture play important roles in developing and explaining equations for proportional relationships. To illustrate these conceptions, we use examples that are glosses of reasoning we found across multiple participants, interviews, and interview tasks.

### A Substance Conception of Fractions and Variables

A person uses a *substance conception* of a fraction or variable if the person explicitly views the fraction or variable as a label, name, or descriptor of an entity, or as the entity itself. In the case of variables, the substance conception is essentially the same as the label or object conception of variables that has been described in the literature (e.g., McNeil et al., 2010). For example, if a student describes the second strip in Figure 1 as $Y$ and means it as a label or name for the strip, then at that moment, the student is using a label conception of the variable $Y$. We do not use the term "label conception" because in the case of numbers, we do not want the conception to be confused with cases where a number serves as a non-quantitative label or name, such as a house number or telephone number.

In the case of fractions, if a student describes one of the 5 parts in the second strip in Figure 1 as "a one-fifth-part," or says that the part "is one-fifth," and means that 1/5 is a descriptor or name for the part, or stands for the part itself, then at that moment, the student is using a substance conception of fraction.

We note that the substance conception can also apply to phrases. For example, a student might describe the 5-part yellow paint strip and 2-part blue paint strip in Figure 1 as "the yellow paint" and "the blue paint" respectively, write the equation "the blue paint = 2/5 of the yellow paint," and then write the equation $X = 2/5Y$. In this case, the student uses a substance conception of the phrases "the blue paint" and "the yellow paint," and they might continue to use this substance conception with the variables $X$ and $Y$. In any case, the student treats the blue paint and the yellow paint as quantities, but they might not be thinking of those quantities as some number of a specified measurement unit, and therefore might not be thinking of $X$ and $Y$ as magnitudes. In essence, the student's equations would be like saying that one vector is equal to a scalar multiple of another vector. In fact, if we interpret $X$ and $Y$ as elements of a vector space, then the equation $X = 2/5Y$ makes perfect sense even if no basis has been chosen for the vector space. So even though we expect the equation $X = 2/5Y$ to be about numbers and to fit with the definition of multiplication, this might not fit readily with a student's interpretation.

**A Substance Conception of a Multiplicand may Be Productive.** In the example just presented, which led to the equation $X = 2/5Y$, the variable $Y$ functions as a *multiplicand*: it represents 1 group, and 2/5 of that group is the amount of blue paint, $X$. We conjecture that more generally, when a fraction or variable functions as a multiplicand, a substance conception of a fraction, variable, or related phrase may help the student (1) view the situation in terms of quantities and (2) formulate a correct equation by reasoning about quantities in the situation.

This conjecture is consistent with productive reasoning we have seen with improper fractions. In fact, our definition of fraction almost invites a substance conception. For example, the fraction 5/2 is defined as the amount formed by 5 parts, each of size ½ of the unit amount. According to this definition, 5/2 is essentially the product 5•1/2, where ½ is the multiplicand. Working with the strips in Figure 1, a student might view ½ as a label for each of the 2 parts in the first strip, and also for each of the 5 parts in the second strip. The student might then describe the second strip as 5 parts, each ½, and therefore as 5/2. Even though the student views ½ as a label, the ½ also functions as a quantity for the student because the student considers 5 of the halves. This seems to be a productive way to make sense of improper fractions. What could still be missing, however, is the idea that ½ and 5/2 are magnitudes—the numerical outcome of measurement by the 2-part strip.

**A Substance Conception of a Multiplier may Be Unproductive.** In contrast, when a fraction or variable functions as a *multiplier*, we conjecture that a substance conception can lead to unproductive interpretations of multiplication. For example, if a student is asked to make a drawing to help explain the meaning of 1/6•X according to our definition of multiplication, the student might draw a 6-part strip, call each part a 1/6-group, and write X in each part, explaining that each 1/6-group has X in it. The student sees each part as 1 group, and sees 1/6 as describing the type of the part, thereby taking a substance conception of 1/6. The substance conception doesn't help the student view 1/6 as *how many* groups are being considered.

This conjecture is consistent with Küchemann's (1981) finding that students were especially challenged to formulate correct algebraic expressions in situations where variables stood for (whole) numbers of objects. The students may have interpreted the variables as the names or types of the objects rather than as their number.

## A Magnitude Conception of Fractions and Variables

A person uses a *magnitude conception* of a fraction or variable if the person explicitly views the fraction or variable as a magnitude, i.e., as the result of measuring one quantity by another quantity (which need not be separate from the first quantity). For example, if a student understands that it takes 2/5 of the second strip in Figure 1 to make the first strip, then at that moment, the student is using a magnitude conception of 2/5. Similarly, if a student views $Y$ as the number of gallons of yellow paint in the situation of Figure 1, then at that moment, the student is using a magnitude conception of $Y$.

**A Magnitude Conception of a Multiplicand may not Be Necessary.** To use the definition of multiplication as intended does require understanding the multiplicand as a magnitude. However, some students might be able to formulate and explain valid multiplication equations by reasoning about quantities while using only a substance conception of the multiplicand. They might even be able to use the equations by substituting numbers for variables even though they don't think of the variables as magnitudes.

**A Magnitude Conception of a Multiplier may Be Necessary.** In contrast, when a fraction or variable functions as a *multiplier*, we conjecture that a measurement conception is necessary for a productive interpretation of multiplication. We also conjecture that a measurement conception can be cued by asking a measurement question such as "How many of the second strip in Figure 1 does it take to make the first strip exactly?" A student who answers this question as 2/5 may then see that it takes 2/5 of $Y$ to make $X$ and may therefore formulate the equation $2/5 \cdot Y = X$ even if they have a substance conception of $Y$ and $X$ at the moment.

### The Two Conceptions and Moment-by-Moment Reasoning

Finally, we conjecture that the substance and magnitude conceptions are not mutually exclusive. In particular, we conjecture that (1) students can hold the two conceptions simultaneously or that they may switch between the two from one moment to the next, (2) students may not recognize that are using two distinct conceptions when they are reasoning about fractions or variables, and (3) developing expertise with equations involves developing a sense of the difference in the two conceptions and knowing when to use which one.

## Conclusion and Future Directions

The future teachers in our mathematics content courses on multiplicative reasoning come to us with various ideas about developing equations, including intuitive or rote approaches, such as setting up an equation of the form $a/b = c/d$ from "$a$ is to $b$ as $c$ is to $d$." We teach our students to refine their ideas and develop mathematically sound explanations for equations and solution methods by reasoning about how to structure, organize, and relate quantities. To structure, organize, and relate quantities, students must engage with ideas about unitizing, iterating, and partitioning. In addition to these ideas, we conjecture that students also need to refine how they think about quantities, the measurement of quantities, and the mathematical notation we use to describe quantities and their size.

The conjectures we have formulated for this paper come from an initial analysis of a large amount of data. The next step is to find a principled way to select a circumscribed portion of the data for closer examination, so that the conjectures can be put to a rigorous test. We are especially interested in discussions with the audience about this next phase of analysis.

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# Networking Theories to Design Dynamic Covariation Techtivities for College Algebra Students

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*Covariational reasoning is a challenging form of reasoning for undergraduate students to develop and employ. Yet, students' lack of opportunities to use covariational reasoning may account, in part, for some of their difficulties. Building from the work of mathematics education researchers (e.g., Kaput, Thompson, Moore), we developed a suite of Techtivities—free, accessible, digital media activities linking dynamic animations and graphs. Using a Cannon Man Techtivity to illustrate, we provide four key design components and three theoretically based design principles underlying the Techtivities. To inform design both within and across the Techtivities, we network theories of different grain sizes: Thompson's theory of quantitative reasoning and Marton's variation theory. Developing Techtivities for students in the gatekeeping course, College Algebra, we intend to expand students' opportunities to employ covariational reasoning. We discuss implications stemming from students' opportunities to use free, accessible digital media activites, such as Techtivities, to promote their covariational reasoning.*

*Keywords:* Covariational reasoning, Task Design, Quantitative Reasoning, Variation Theory

Covariational reasoning is a critical form of mathematical reasoning imperative for students' understanding of key concepts of introductory college level mathematics such as functions, rates, and graphs (Thompson & Carlson, 2017). At its core, covariational reasoning entails a twofold conception: conceiving of attributes as capable of varying and possible to measure, then conceiving of a relationship between those attributes (Carlson, Jacobs, Coe, Larson, & Hsu, 2002; Thompson & Carlson, 2017). By investigating situations involving multiple changing attributes, students can have opportunities to employ covariational reasoning (e.g., Moore, Stevens, Paoletti, & Hobson, 2016; Johnson, McClintock, & Hornbein, 2017; Thompson & Carlson, 2017). For example, students might investigate a Cannon Man situation, in which a person is ejected into the air, then falls down to the ground with the help of a parachute. In this situation, students employing covariational reasoning could conceive of two possible attributes as capable of varying and possible to measure: Cannon Man's height from the ground and Cannon Man's total distance traveled while in the air. Students could then conceive of a relationship between Cannon Man's height from the ground and total distance traveled.

Building from the work of mathematics education researchers (e.g., Kaput, Thompson, Moore), we developed a suite of Techtivities—free, accessible, digital media activities linking dynamic animations and graphs. We designed the Techtivities for students in College Algebra, an introductory course that can serve as a gatekeeper for many students (e.g., Gordon, 2008; Herriot & Dunbar, 2009). Using a Cannon Man Techtivity to illustrate, we provide four key design components and three theoretically based design principles underlying the Techtivities. Networking theories of different grain sizes—Thompson's theory of quantitative reasoning (1993, 1994, 2002, 2011) and Marton's variation theory (2015)—we designed both within and across the Techtivities. By designing the Techtivities in Desmos (www.desmos.com), we increase accessibility, and thereby expand students' opportunities to employ covariational reasoning. We conclude with implications for students' use of Techtivities to promote their covariational reasoning and for the networking of theories to design digital media activities.

## Background

Despite the importance of covariational reasoning, researchers have documented challenges that undergraduate university students enrolled in calculus and trigonometry courses face when encountering situations calling for covariational reasoning (e.g., Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008; Moore, 2014; Moore & Carlson, 2012; Moore, Paoletti, & Musgrave, 2013). Broadly, undergraduate students have difficulty using covariational reasoning to make sense of situations involving variation in change that occurs in a single direction, such as a temperature increasing at a decreasing rate (e.g., Carlson et al., 2002; Oehrtman et al., 2008). In addition, students' impoverished conceptions of the "things" that are changing may decrease their likelihood for covariational reasoning (Moore, 2014; Moore & Carlson, 2012). Furthermore, students' lack of covariational reasoning can impact their ability to view graphs as representing relationships between quantities (Moore & Thompson, 2015; Moore, Stevens, Paoletti, & Hobson, 2016).

Through their programs of research, Thompson and Carlson, together with colleagues, have developed and implemented innovative learning materials to provide opportunities for university students in Calculus and Precalculus to employ covariational reasoning (e.g., Carlson et al., 2002; Carlson, Oehrtman, & Engelke, 2010; Carlson, Oehrtman, & Moore, 2010; Thompson & Ashbrook, 2016a; Thompson & Carlson, 2017). In a PreCalculus course designed to foster university students' covariational reasoning, students encountered instructional tasks designed to provide students opportunities to conceive of change in attributes prior to determining numerical amounts of change (Thompson & Carlson, 2017). In their online Conceptual Calculus textbook, Thompson & Ashbrook (2016a) included a task situation involving a droplet of water landing into a bowl of water and creating circular ripples that increase in size (Thompson & Ashbrook, 2016b). We view this situation as having potential to serve as background for a task requiring students to conceive of and represent change in the area and radius of the ripples. Overall, the research programs of Thompson and Carlson have resulted in opportunities for university students to use innovative learning materials designed to promote covariational reasoning. Yet, we argue that there is room for the development of more accessible and multimodal learning materials, so as to provide digital media that broadens access and learning opportunities to an even wider range of students.

## An Approach to Technology Development and Use for Greater Access and Participation

By developing a suite of Techtivities in Desmos, we increase accessibility and opportunities for participation in multiple ways: across operating system platforms (Apple OS, Microsoft Windows), across various browsers (i.e., Google Chrome, Mozilla Firefox, Microsoft Edge), via mobile devices (Desmos is compatible with iOS and Android), and as an app extension via Google's Chrome browser (Desmos has 2.8 million app installations within Chrome). Furthermore, Desmos has low barriers to entry and initial use, which afford more expansive opportunities for student participation. Specifically, learner use of Desmos begins in just a few clicks via a web browser or mobile platform; supports learning in over two dozen languages; complies with WCAG 2.0 accessibility standards for learners who may be blind or visually impaired, with screen reader capability on both web-based and mobile platforms; includes authenticated sign in with Google credentials; and incorporates a robust set of web tutorials on Youtube (over a quarter million views). We have intentionally partnered with Desmos because the development and use of each Techtivity will maintain these technical features for greater access and participation, and also align all Techtivities with the broader "ecosystem" of Desmos users, social media networks, technical supports, and complementary resources.

## Networking Theories to Design Techtivities

Rasmussen and Wawro (2017) called for researchers investigating research problems in undergraduate mathematics education to network theories, thereby providing new lenses and tools to study the complexities of learning and teaching mathematics. By networking theories of different grain sizes to design the Techtivities, we respond to the call put forth by Rasmussen and Wawro (2017). Watson (2016) articulated three different grain sizes of theories: grand theories (e.g., Piaget's constructivist theory), intermediate theories (e.g., Marton's variation theory), and domain specific/local theories (e.g., Thompson's theory of quantitative reasoning). Following Johnson and colleagues (Johnson, McClintock, Hornbein, Gardner, and Grieser, 2017; Johnson & McClintock, in press), we networked Thompson's theory of quantitative reasoning and Marton's variation theory to design both within and across the Techtivities.

### Thompson's Theory of Quantitative Reasoning

In explicating a theory of quantitative reasoning (e.g., Thompson 1993; 1994; 2002; 2011), Thompson employed a constructivist perspective. Thompson's theory of quantitative reasoning focuses on students' mental operations, which individuals can enact in thought as well as action (e.g., Piaget, 1970, 1985). Drawing on Thompson's theory of quantitative reasoning, by quantity we mean how students conceive of the possibility of measuring some attribute. For example, a student might conceive of using a fixed distance between her thumb and forefinger to measure Cannon Man's height from the ground. Thompson's theory of quantitative reasoning undergirds our perspective on covariational reasoning.

Following Thompson and Carlson (2017), we argue that covariational reasoning entails at least four different kinds of mental operations: students' conceptions of attributes as being possible to measure (quantitative reasoning), students' conceptions of attributes as being capable of varying, students' conceptions of a relationship between attributes capable of varying and possible to measure, and students' images of change. Thompson, Hatfield, Yoon, Joshua, and Byerly (2017) built on Saldanha & Thompson's (1998) term, *multiplicative object*, to specify a conception of a relationship between attributes capable of varying and possible to measure. A student conceiving of a relationship between attributes as a multiplicative object can conceptualize a new attribute, which coordinates the constituent attributes (Saldanha & Thompson, 1998; Thompson et al., 2017). For example, a student could conceive of a new attribute, coordinating Cannon Man's height from the ground and total distance traveled at every value of height and distance. By images of change, we mean more than a mental picture, we mean students' mental operations (see also Thompson, 1996). Castillo-Garsow, Johnson, & Moore (2013) posited two contrasting images of change: chunky and smooth. A smooth image of change refers to a conception of change as occurring in progress. A chunky image of change refers to a conception of change as having occurred in particular increments. For example, a student might conceive of Cannon Man's height as changing continually (smooth image of change) or as having changed to reach a certain amount (chunky image of change). Students' use of smooth images of change correlates to more advanced levels of covariational reasoning (Thompson & Carlson, 2017). Researchers have argued for the utility of students' smooth images of change (e.g., Castillo-Garsow et al., 2013), reporting case studies to demonstrate that utility for both undergraduate and high school students (e.g., Johnson, 2012; Moore, 2014).

### Marton's Variation Theory

We used Marton's (2015) variation theory to guide design across the Techtivities. Broadly, Marton (2015) argued that instructional designers should develop task sequences that provide students opportunities to discern critical aspects (Marton, 2015). When interacting with the Techtivities, we view covariation to be a critical aspect for students to discern. Furthermore, covariation is a critical aspect comprised of interrelated aspects. For critical aspects comprised of interrelated aspects, Marton (2015) recommended that task sequences first include variation and invariance in each interrelated aspect, then variation in both aspects. To discern covariation, students need to conceive of two constituent attributes as capable of varying and possible to measure, as well as a relationship between those attributes. Consequently, in designing the Techtivities, we first included variation and invariance in each constituent attribute, then variation in both attributes.

**Networking Theories to Move Beyond Existing Theoretical Perspectives**

Networking theories can take different forms. We network Thompson's theory of quantitative reasoning and Marton's variation theory to design both within and across the Techtivities. To design within each Techtivity, we drew on Thompson's theory of quantitative reasoning to inform our selection of different attributes to use and to inform our design to promote students' use of smooth images of change. To design across the Techtivities, we drew on Marton's variation theory to include variation and invariance in the type and representation of constituent attributes, then variation in both attributes.

For the purposes of designing the Techtivities, we view Thompson's theory of quantitative reasoning and Marton's variation theory to complement, rather than to compete, with each other. From a constructivist perspective, we do not assume that covariation is something that is "out there" for students to notice (see also Johnson, McClintock, Hornbein, et al., 2017). From a variation theory perspective, Marton (2015) asserted that researchers should not assume that students already attend to the critical aspect prior to encountering a task sequence. We concur with Marton (2015), as we do not assume that students already attend to covariation prior to encountering the task sequence. Furthermore, in the design of the Techtivities, the critical aspect for students to discern—covariation—is a conception (see also Johnson, McClintock, Hornbein, et al., 2017). By discernment, we mean students' engagement in mental operations entailed in covariational reasoning. In the next section, we articulate four key design components, encompassing design decisions both within and across the Techtivities.

## Four Key Design Components of Each Techtivity

Building from the work of mathematics education researchers (e.g., Kaput & Roschelle, 1999; Moore et al., 2013; Moore et al., 2016; Saldanha & Thompson, 1998; Thompson, 2002; Thompson, Byerly, & Hatfield, 2013) we provide four key design components of each Techtivity. In explicating these components, we expand on Johnson's previous task design research (2013, 2015). Furthermore, we find our design components to be complementary to the task sequence reported by Moore et al. (2016). In their task sequence, Moore et al. (2016) began first by providing students with a video or animation depicting changing attributes; second, they prompted students to sketch a graph showing a relationship, and third, they prompted students to sketch a second graph, containing either the same or similar attributes. Furthermore, Moore et al. (2016) recommended that tasks not include numerical amounts, concurrent with Johnson's (2013, 2015) recommendations. In our design components, we adapt and expand on the task sequence reported by Moore et al. (2016). We include opportunities for students to vary individual attributes, and we constrain the attributes in the second graph, such that those attributes are the same as the attributes in the first graph.

## Dynamic Animations of Situations Involving Changing Attributes

Johnson, McClintock, and Hornbein (2017) articulated a need for task designers to take into account the types of attributes included in tasks. In the suite of Techtivities, we intended to select attributes that we thought students may more readily conceive of as measurable. Furthermore, alongside the animation, we identify attributes which will serve as the focus of the Techtivity (Figure 1). We use an animation in part to provide students opportunities to conceive of attributes in the process of changing, or put another way, to use smooth images of change.

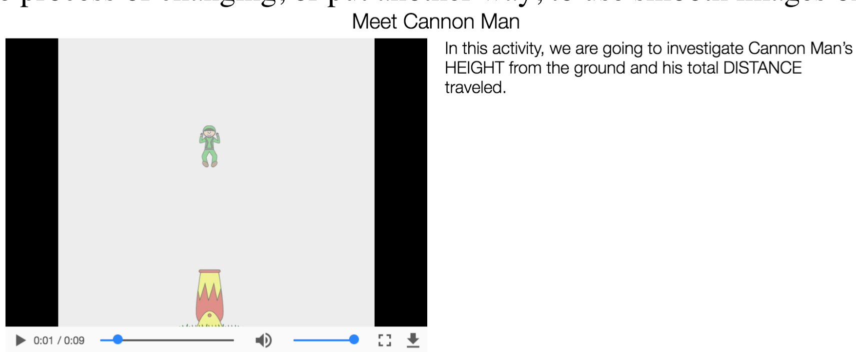


*Figure 1. Cannon man animation*

## Cartesian Graphs Containing Dynamic Segments on the Axes

It is useful for students to use their fingers as tools to represent variation in individual attributes (Thompson, 2002). Through the dynamic segments on each axis (Figure 2, left), we provide students opportunities to use digital media to represent variation in individual attributes. We include freely stretching segments and avoid using numerical amounts to foster students' use of smooth images of change.

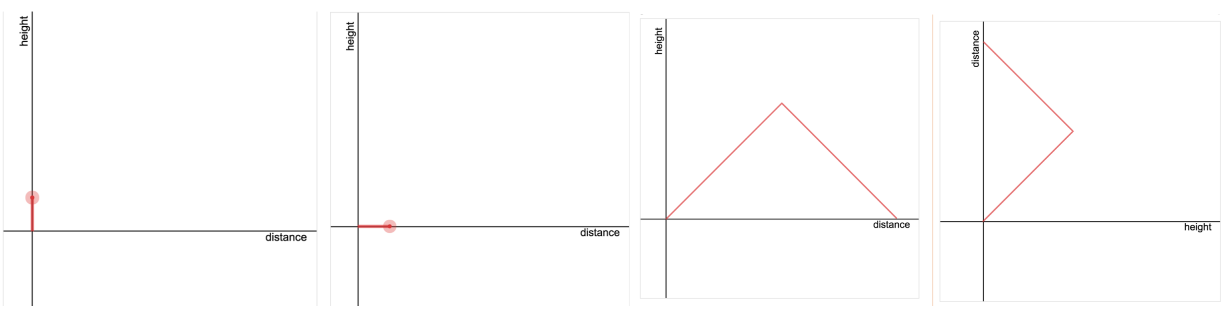


*Figure 2. Dynamic segments (left). Graphs varying representation of the same attributes (right).*

## Opportunities to Sketch a Cartesian Graph after Varying Individual Attributes

Johnson (2015) showed that students' opportunities to conceive of variation in individual attributes impacted their conceptions of covariation. In each Techtivity, after varying individual attributes, students have the opportunity to sketch a Cartesian graph. When working to sketch the graph, students may replay the animation. To sketch a graph, students may select between two digital tools: a free-form pencil or a line segment.

## Variation in Representation of Attributes

Students may find it challenging to conceive of graphs as representing relationships between attributes (Moore & Thompson, 2015; Moore et al., 2016). We incorporated Cartesian graphs that represented the same attributes in different ways (Figure 2, right). In so doing, we intended to provide students opportunities to conceive of graphs as representing relationships, rather than forming a particular type of shape (See also Moore & Thompson, 2015).

## A Blueprint for a Techtivity

Each Techtivity consists of a series of screens, which students move through in a particular order. Table 1 provides a blueprint for a Techtivity. First, students watch an animation of a situation involving changing attributes (Table 1, Item 1). Second, students move dynamic segments to represent change in each attribute. After moving segments, students view the dynamic segments changing together, appearing in conjunction with the animation (Table 1, Items 2-4). Third, students sketch a Cartesian graph representing how both attributes are changing together. After sketching a graph, students view a computer generated graph, appearing in conjunction with an animation (Table 1, Items 5-6). Fourth, students answer a reflection question (Table 1, Item 7). Fifth, students repeat the process for a new Cartesian graph representing the same situation, with attributes on different axes (Table 1, Item 8).

*Table 1. A Blueprint for a Techtivity*

| A Blueprint for a Techtivity |
|---|
| 1. View animation of a situation involving changing attributes. Identify the changing attributes on which to focus in this situation.<br>2. Move a dynamic segment to show how one attribute is changing.<br>3. Move a second dynamic segment to show how the other attribute is changing.<br>4. View both dynamic segments changing together, appearing in conjunction with an animation. (In 2-4, dynamic segments are located on horizontal or vertical axes on a Cartesian Plane.)<br>5. Sketch a Cartesian graph representing how both attributes are changing together.<br>6. View a computer-generated Cartesian graph, appearing in conjunction with an animation.<br>7. Reflect on an aspect of the Cartesian graph. For example, is the graph what you expected? Is there anything about the graph that surprises you? Why might it make sense for a graph to look that way? Is it possible for two different looking graphs to represent the same situation?<br>8. Repeat 2-7 for a new Cartesian graph representing the same situation, with attributes on different axes. |

## Design Principles Emerging from the Development of the Techtivities

### Increase Accessibility to Expand Students' Opportunities to Employ Covariational Reasoning

Kaput (1994) argued that technology could provide students opportunities to investigate areas of mathematics once reserved only for students at more advanced levels. Covariational reasoning is a critical form of reasoning that cannot be reserved only for students at the upper levels of undergraduate mathematics. At CU Denver, the student population is becoming increasingly diverse. In 2016, 57% of new freshman, and overall 43% of undergraduate students identified as students of color (Williams, 2016). Across Spring and Fall 2016, 70% of students enrolled in College Algebra at CU Denver self-identified as students of color. By designing our Techtivities in Desmos, we increase access for undergraduate students, as well as their instructors. We designed the Techtivities so that students could work in ways that are self-paced, or with direction from their instructors. Furthermore, students and educators have free online access to the suite of Techtivities, to use as a just-in-time curricular resource or as an embedded component of a course, allowing for entire cohorts of students to have opportunities to employ covariational reasoning.

### Leverage Domain Specific Theories in Mathematics Education to Design Task Components

By drawing on Thompson's theory of quantitative reasoning, we augment the design of the Techtivities by infusing what we have learned from researchers focusing on students' conceptions. Specific to our focus on covariational reasoning, we leveraged Thompson's theory of quantitative reasoning in three ways. First, provide opportunities for students to conceive of attributes as capable of varying and possible to measure (Table 1, Items 1-3). Second, provide opportunities for students to discern a relationship between attributes, or put another way, to discern covariation (Table 1, Items 4-7). Third, by representing attributes on different axes, provide opportunities for students to conceive of a graph as representing a relationship between attributes capable of varying and possible to measure (Table 1, Item 8).

### Network Theories of Different Grain Sizes to Design Both Within and Across Tasks

Networking theories of different grain sizes, we were able to design both within and across the Techtivities (see also Johnson, McClintock, Hornbein, et al., 2017; Johnson & McClintock, in press). Thompson's theory of quantitative reasoning informed our selection of attributes within each Techtivity and across the suite of techtivities. In the Cannon Man Techtivity, total distance traveled is monotonically increasing. In another Techtivity, we include attributes such that neither is monotonically increasing or decreasing (see also Moore et al., 2016). Marton's variation theory informed the sequencing of design across sections of individual Techtivities as well as across the suite of Techtivities. Individual Techtivities include variation in each attribute, then variation in both attributes. The suite of Techtivities provide different backgrounds.

## Discussion

Despite the existence of some innovative learning materials for Calculus and Precalculus students (e.g., Carlson, Oehrtman, & Moore, 2010; Thompson & Ashbrook, 2016a), we argue that a broader range of university students need access to such materials. We contend there is an *opportunity* gap for university students to develop and employ covariational reasoning. We view this opportunity gap to be particularly problematic for students enrolled in College Algebra. Furthermore, increasing numbers of College algebra students identify as students of color, and university College Algebra courses have had low success rates (e.g., Gordon, 2008; Herriot & Dunbar, 2009). By developing a suite of Techtivities designed to promote College Algebra students' covariational reasoning, we intend to address this opportunity gap.

Broadly, a dual commitment has motivated our design decisions when developing the Techtivities. We intend to increase students' access to opportunities to employ covariational reasoning, and expand learning opportunities through the development of free, accessible, digital media activities that link dynamic animations with graphs. By attending simultaneously to disciplinary and technical barriers, while foregrounding the expansion of learning opportunities for nondominant students at CU Denver, we make explicit our "researcher positionality" (Aguirre et al., 2017), acknowledging that mathematics education research is both a political and equity-oriented endeavor.

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# Mathematics Cognition Reconsidered: On Ascribing Meaning

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*In contrast to the common assumption that mathematics cognition involves the attempt to recognize a previously unnoticed meaning of a concept, here mathematics cognition is reconsidered as a process of ascribing meaning to the objects of one's thinking. In this paper, the attention is focused on three processes that are convoluted in the complex dynamics involved when individuals ascribe meaning to higher mathematical objects: contextualizing, complementizing, and complexifying. The aim is to discuss emerging perspectives of these three processes in more detail that speak to the complex dynamics in mathematics cognition.*

*Keywords:* complexifying, complementizing, contextualizing, mathematics cognition, sense-making

## Introduction

Mathematics cognition is a complex phenomenon that has been addressed and discussed in the literature in different ways and with various emphases. The work presented here arose from a primary cognitive tradition, focusing on critical processes in mathematical concept formation and their complex dynamics. In search for more dialogical possibilities in thinking about mathematics cognition, a new understanding of mathematics cognition emerged (see Scheiner & Pinto, 2017): mathematics cognition does not merely involve the attempt to recognize a previously unnoticed meaning of a concept but the attempt to ascribe meaning to the objects of one's thinking. The purpose of this paper is to provide deeper meaning to the complex processes involved when individuals ascribe meaning. In this paper, three processes are foregrounded: contextualizing, complementizing, and complexifying. Over the past few years, theoretical perspectives and insights emerged (in reanalyzing students' knowing and learning of the limit concept of a sequence) that advance our understanding of these processes. These new perspectives and insights inform research on mathematics cognition and enable one to see not only new phenomena in mathematical concept formation but to think about them differently. In this presentation, emerging interpretative possibilities in thinking about contextualizing, complementizing, and complexifying are discussed that speak to the complex dynamics in mathematics cognition.

## Theoretical Orientations

The work presented here relies on and projects theoretical assertions made by Frege (1892a, 1892b). First, the meaning of a mathematical concept is not directly accessible through the concept itself but only through objects that fall under the concept (Frege, 1892a). Second, mathematical objects (different to objects of natural sciences) cannot be apprehended by human senses (we cannot, for instance, 'see' the objects), but only via some 'mode of presentation.' That we only have access to mathematical objects in using signs and representations, however, leads to what Duval (2006) called a 'cognitive paradox':

"how can they [individuals] distinguish the represented object from the semiotic representation used if they cannot get access to the mathematical object apart from the semiotic representation?" (Duval, 2006, p. 107)

It seems to be an epistemological requirement to distinguish the 'mode of presentation' (or 'way of presentation') of an object from the object that is represented. Frege (1892b) revealed this critical insight, by proposing that an expression has a *sense$_F$* ('Sinn') in addition to its *reference$_F$* ('Bedeutung') (the subscript F indicates that these terms refer to Frege, 1892b). The reference$_F$ of an expression is the object it refers to, whereas the sense$_F$ describes a particular state of affairs in the world, the way that some object is presented. Thus, it seems to follow that we may understand Frege's notion of an idea$_F$ the manner in which we make sense of the world. Ideas$_F$ can interact with each other and form more compressed knowledge structures, called conceptions. A general outline of this view is provided in Figure 1.

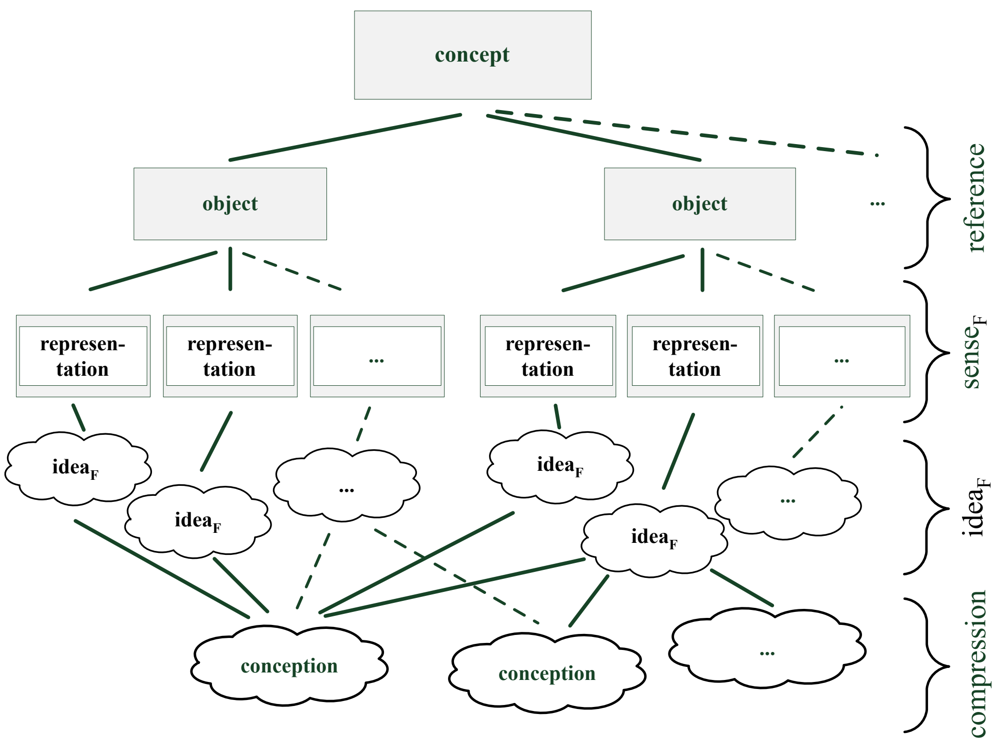


*Figure 1: On reference$_F$, sense$_F$, idea$_F$, and compression (reproduced from Scheiner, 2016, p. 179)*

Duval (2006) argued that via systematic variation of representation registers that is, "investigating representation variations in the source register and representation variations in a target register" (p. 125), one can detach a sense$_F$ from the represented object. This resonates a critical function of reflective abstraction that is, reflecting on the coordination of actions on mental objects (see Piaget, 1977/2001). The special function of reflective abstraction is extracting meaning of an individual's action coordination. Underlying these approaches is the assumption that meaning is inherent in objects and is to be extracted via manipulating objects (or representations of those objects).

Over the past few years, a new understanding of mathematics cognition emerged from reanalyzing students' knowing and learning the limit concept of a sequence (see Scheiner &

Pinto, 2014, 2017; Pinto & Scheiner, 2016): mathematics cognition does not so much involve the attempt to recognize a previously unnoticed meaning of a concept (or the structure common to various objects), but rather a process of ascribing meaning to the objects of an individual's thinking. That is, meaning is not so much an inherent quality of objects that is to be extracted, but something that is given to objects of one's thinking. Three processes are considered as critical in the complex dynamics involved when individuals ascribe meaning to higher mathematical objects: contextualizing, complementizing, and complexifying (see Scheiner & Pinto, 2017).

### Contextualizing: Particularizing Senses$_F$

In Frege's view, a sense$_F$ can be construed as a certain state of affairs in the world and an idea$_F$ in which we make sense$_F$ of the world. In the work presented here, we started from an understanding of sense$_F$ as not primarily dependent on a mathematical object, but as emerging from the interaction of an individual with an object in the immediate context. That is, a sense$_F$ of an object at one moment in time can only be established in a more or less definite way when the process of sense$_F$-making is supported by what van Oers (1998) called *contextualizing*. Van Oers (1998) argued for a dynamic approach to context that provides for the "particularization of meaning" (p. 475), or more precisely, the particularization of a sense$_F$ that comes into being in a context in which an object actualizes.

Recent research suggests that individuals seem to reason and make sense from a specific perspective (see Scheiner & Pinto, 2017). It might be suggested that individuals take a specific perspective that orients their sense$_F$-making, or more accurately: in taking a certain perspective, individuals direct their attention to particular senses$_F$. Contextualizing, in this view, means taking a certain perspective that calls attention to particular senses$_F$. Attention in such cases, however, may not involve an attempt to 'sense' or 'see' anything, but it seems to be attentive thinking: attention as the direction of thinking (see Mole, 2011). As such, calling attention to particular senses$_F$, then, means directing mind to sense$_F$. In this respect, contextualizing is intentional: it directs one's thinking to particular senses$_F$.

### Complementizing: Creating Conceptual Unity

Frege (1892b) underlined that a particular sense$_F$ "illuminates the reference$_F$ […] in a very one-sided fashion. A complete knowledge of the reference$_F$ would require that we could say immediately whether any given sense$_F$ belongs to the reference$_F$. To such knowledge we never attain" (p. 27). (Translated from Frege (1892b): "[mit dem Sinn] ist die Bedeutung aber […] immer nur einseitig beleuchtet. Zu einer allseitigen Erkenntniss der Bedeutung würde gehören, dass wir von jedem gegebenen Sinne sogleich angeben könnten, ob er zu ihr gehöre. Dahin gelangen wir nie"). This is to say, that just from sense$_F$-making of one representation that refers to an object, we are typically not in a position to know what the object is (see Duval, 2006). As contextualizing serves to particularize only single senses$_F$ of a represented object, the same object can be 're-contextualized' (see van Oers, 1998) in other ways that support the particularization of different senses$_F$ of the same object. Notice that senses$_F$ can differ despite sameness of reference$_F$, and it is this difference of senses$_F$ that accounts for the 'epistemological value' of different representations. It is the diversity of senses$_F$ that has 'epistemological significance' and forms conceptual unity (see structuralist approach, Scheiner, 2016), not the similarity (or sameness) of senses$_F$ (as might be advocated in an empiricist approach). This means, what matters is to coordinate diverse senses$_F$ to form a unity, a process called complementizing. However, the notion of 'complementizing' might be misunderstood as accumulating various senses$_F$ (until an individual has all of them); this is not the case. Complementizing means to coordinate different senses$_F$ to create conceptual unity.

As each idea$_F$ is partial in the sense of being restricted (in space and time) and biased (from a particular perspective), it needs to be put in dialogue with other ideas$_F$ that offers an epistemological extension. The function of complementizing, then, is extending the epistemological space of possible ideas$_F$. Complementizing as extending the epistemological space of possible ideas$_F$ brings a positive stance, indicating that seemingly conflicting ideas$_F$ can be productively coordinated in a way such that these ideas$_F$ are cooperative rather than conflicting. Hence complementizing is the ongoing expansion of one's epistemological space, the ever-unfolding process of becoming capable of new, perhaps as-yet unimaginable possibilities.

### Complexifying: Creating a Complex Knowledge System

It is not only creating a unity of diverse senses$_F$, but creating an entity in its own right that forms a 'whole' from which emerges new qualities of the entity. That is, rather than treating the unity as a collection of different senses$_F$ that can be assigned to objects that actualize in the immediate context, it is the forming of the unity that emerges new senses$_F$ that might be assigned to potential objects. In forming a unity, senses$_F$ are not merely considered as the parts of the unity, but "they are viewed as forming a whole with distinct properties and relations" (Dörfler, 2002, p. 342). It is, therefore, not an unachievable totality of senses$_F$ (or ideas$_F$) that matters, but how senses$_F$ (or ideas$_F$) are coordinated that develop emergent structure. This brings to foreground a critical function of complexifying that has not been attested yet: blending previously unrelated ideas$_F$ that emerge new dynamics and structure (for a detailed account of conceptual blending, see Fauconnier & Turner, 2002). The essence of conceptual blending is to construct a partial match, called a cross-space mapping, between frames from established domains (known as inputs), in order to project selectively from those inputs into a novel hybrid frame (a blend), comprised of a structure from each of its inputs, as well as a unique structure of its own (emergent structure). This strengthens Tall's (2013) assertion that the "whole development of mathematical thinking is presented as a combination of compression and blending of knowledge structures to produce crystalline concepts that can lead to imaginative new ways of thinking mathematically in new contexts" (p. 28).

### Discussion

Mathematics cognition, as asserted here, evolves in the dialogue of contextualizing, complementizing, and complexifying. As such, mathematics cognition is ongoing and cannot be pre-stated. That is, mathematics cognition does not follow a determinable developmental trajectory, but the evolution of mathematics cognition is directional: it seems to move toward higher levels of internal diversity, interactions, and decentralization of ideas$_F$.

Scheiner and Pinto (2017) suggested that individuals take a specific perspective in ascribing meaning to the limit concept of a sequence. For instance, individuals, who take the perspective of a limit sequence as approaching, may activate dynamic ideas$_F$ (such as the idea$_F$ of a sequence of points 'getting closer' to a limit point). Consider Figure 2: one might activate the idea$_F$ of a limit that can be approached monotonically (see idea$_F$ A) or the idea$_F$ of a limit that can be approached from above and below (see idea$_F$ B) in making sense$_F$ of the respective representation (see Cornu, 1991; Davis & Vinner, 1986; Tall & Vinner, 1981). On the other hand, individuals, who take the perspective of closeness in thinking about the limit concept of a sequence, might activate rather static ideas$_F$ (such as the idea$_F$ of points of a sequence 'gathering around' the limit point). One might activate the idea$_F$ that infinite many points of a sequence can lie within a given epsilon strip (see idea$_F$ U or V) in making sense$_F$ of the representations (see Przenioslo, 2004; Roh, 2008; Williams, 1991).

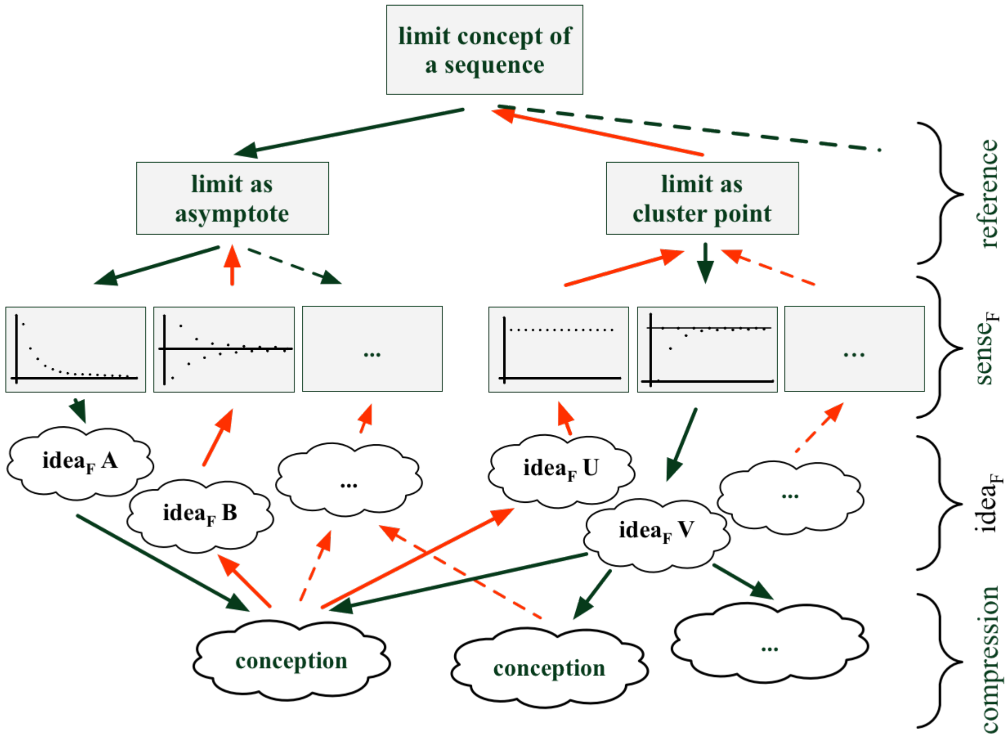


*Figure 2: On the complex dynamics in mathematics cognition*

The critical point here is that it is not one single idea$_F$ around one's thinking is to be organized (such as the idea$_F$ that only finite many points lie outside a given epsilon strip), but a variety of diverse ideas$_F$ that provides a resource of activating productive ideas$_F$ and of making sense$_F$ in the immediate context. Decentralization and internal diversity of ideas$_F$, however, are not only critical for making sense$_F$ in the immediate context, but also for creating novel ideas$_F$. Whereas analogy theory typically focuses on compatibilities between ideas$_F$ simultaneously connected, blending is equally driven by incompatibilities (see Fauconnier & Turner, 2002).

Creating novel ideas$_F$, however, only occurs if there is a certain level of interaction between existing ideas$_F$. That is, only if ideas$_F$ can compensate for each other's restrictions and limitations, one is able to extend the space of possibilities in thinking about a mathematical concept. In this view, novel ideas$_F$ can ascribe *new* meaning to the objects of one's thinking (see Figure 2). This substantiates the assertion that mathematics cognition is as much concerned with creating a meaning of a concept as it is with comprehending it (see Scheiner, 2017).

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# A Model of Task-Based Learning for Research on Instructor Professional Development


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| --- | --- | --- |
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*We present a theoretical framework that synthesizes and increases the descriptive power of existing models of task-based learning. Grounded in social constructivism and activity theory, the framework supports collegiate mathematics education researchers in identifying, investigating, and reporting on task-based learning in instructor professional development contexts. Relevant definitions and connections to the larger realm of inquiry-based, problem-based, and other general inquiry-oriented instruction are addressed. We conclude with a discussion and illustration of how the framework may be used in design, materials development, and evaluation research related to instructor professional learning.*



*Key Words:* Activity, Task-based learning, Professional development, Teaching for robust understanding


Experts in the social and behavioral sciences, such as mathematics education, often deal with the challenge of specialized language. Technical terms can have shades of meaning that differ significantly from everyday language-in-use counterparts. For example, Cook, Murphy, and Fukawa-Connelly (2016) point out that the absence of a concise and consistently applied definition of *inquiry-based learning* (IBL) in science, technology, engineering, and mathematics education has meant researchers use the term to mean fundamentally different things. This divergence has created confusion. Kirschner, Sweller and Clark (2006) claimed that inquiry-based learning does not work, while Hmelo-Silver, Duncan and Chinn (2007) responded that the claim was based on a fundamental flaw: the authors had oversimplified and treated IBL as if it were unguided discovery (which, for Hmelo-Silver and colleagues was a very different thing).

The importance of concise and shared definitions is amplified in research on instructor professional development. Hayward, Kogan and Laursen (2016) note that presenting IBL as a broad and inclusive set of pedagogical practices appeared to be critical in the willingness of college mathematics faculty to adopt it. Instructors viewed questions (inquiry) and learning as existing aspects of their own practice. Faculty saw this new "inquiry-based learning" as an extension of something they already knew, as professionally relevant and useful. As faculty learned more, read more, spoke more about IBL, they practiced using a specialized language, an IBL lexicon, for describing and re-defining their goals, resources, and orientations about teaching, about what learning was, and about what constituted evidence of it.

Here we operationalize a theory of *task-based learning* (TBL). Our focus is in the context of faculty professional development. The goal is to create a sufficiently detailed framework that has descriptive power and is useful for evaluating professional learning and for doing design-based research. In particular, there is a need for a model of TBL for research and development work on professional growth among mathematics faculty new to teaching future school teachers (Masingila, Olanoff, & Kwaka, 2012).

Some might argue with the feasibility of singular definitions in mathematics education. At the same time, the attempt to negotiate a definition, to create a useful model of meaning, can have valuable descriptive power (Schoenfeld, 2000). It is this aspect of research and design in professional development, and the knowledge that there are linguistic and cultural norms related to particular views of teaching and learning, that influences our framework effort. Consider the case of college mathematics faculty in the U.S., most of whom are fluent in one or more natural languages (e.g., English and Chinese) and one or more dialects of research mathematics. These are people who also know the Western academic cultural norms of the transmission and product models for college instruction (Davis, Hauk, & Latiolais, 2009). Place a person with these multiple fluencies and areas of expertise in a room with 20 undergraduates whose life goal is to become a primary school teacher and tell the instructor: Teach them math. Three words: Teach. Them. Math. Each word has a cacophony of meaning. The layers of meaning are large in number and the likelihood of shared definitions for "teach," "them," and "math" are small. What does it mean to teach? What distinguishes "them" from "me" or "us" (if anything)? And *which* math does "math" mean? Indeed, many American teachers perceive mathematics as a static body of knowledge where knowing mathematics is equivalent to efficiently manipulating symbols without necessarily understanding what they represent (Thompson, 1992).

## Mathematical Knowledge for Teaching (MKT) for Grades K-8

Several decades of research rooted in Shulman's (1986) work have indicated that there are particular understandings and skills associated with effective instruction, a sociological synergy of mathematics and mathematics education called *mathematical knowledge for teaching* (MKT; Ball, Thames, & Phelps, 2008). MKT for elementary grades as modeled by Ball and colleagues is made up of six kinds of knowledge. Three are types of subject matter knowledge: *horizon content knowledge,* about how topics are related across the span of curriculum; *specialized content knowledge* which is specialized in the sense that it is specific to the task of *teaching*, and is complementary to *common content knowledge*. In particular, specialized content knowledge includes ways to represent mathematical ideas, provide mathematical explanations for rules and procedures, and examine and understand innovative solution strategies from the student's perspective. This specialized knowledge for teaching K-8 is sparse or absent for many with advanced mathematics expertise but little teaching experience (e.g., mathematics professors; Bass, 2005). As an example, consider fraction division. Most novice instructors can readily use the invert-and-multiply algorithm to divide fractions. Thus, this piece of knowledge is *common content*. Yet, few can explain to someone *why* the algorithm is justified in some problem situations and not in others, thereby making knowing the "whys" *specialized*.

The other three categories in MKT are types of *pedagogical content knowledge* (PCK) and are neither purely pedagogical nor exclusively mathematical. *Knowledge of curriculum* includes awareness of the content and connections across standards and texts (i.e., of the intended curriculum; Herbel-Eisenmann, 2007). *Knowledge of content and students* (KCS) is "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (Hill, Ball, & Schilling, 2008, p. 375). *Knowledge of content and teaching* (KCT) is about teaching actions or moves (i.e., productive ways to respond in-the-moment to students to support learning). So, in our fraction example, teachers who are aware that students often invert the dividend instead of the divisor are demonstrating KCS and might use fraction diagrams to scaffold understanding if they have the appropriate KCT. Both KCS and KCT are associated with improved student learning (Hill et al., 2008; Hill, Rowan, & Ball, 2005).

A related idea at the college level is *mathematical knowledge for teaching future teachers* (MKT-FT) held by college instructors who teach pre-service teachers (Hauk, Jackson, & Tsay, 2017). A rich and textured MKT-FT is especially vital in the inquiry-oriented or activity-based approaches to teaching shown to improve student learning, increase persistence, and reduce inequities (Bressoud, Mesa & Rasmussen, 2015; Freeman et al., 2014; Holdren & Lander, 2012; Laursen, Hassi, Kogan, & Weston, 2014). College instructors acquire MKT-FT in many ways: grading, examining their own learning, observing and interacting with students or colleagues, reflecting on and discussing practice (Kung, 2010; Speer & Hald, 2009; Speer & Wagner, 2009).

## Defining and Illustrating Tasks

With a focus on MKT and MKT-FT in mind, we examined *task-based learning* (TBL) and the task and activity framework of Christiansen and Walther (1986). Growing from social constructivist roots, their view of TBL is as adaptation, human behavior in response to the conditions between the individual and the social, physical, and cognitive environments perceived by the individual. In other words, human behavior is a result of goal-directed seeking for a regulation of mutual relationships between the individual and environment(s). Within this framework, the terms *task, activity, action,* and *plan* each play distinctive roles. Christiansen and Walther explicitly characterize *task* as *interplay* among teacher, students, curriculum and objectified mathematics (see Figure 2). They implicitly express activity as inherent in the relations among the various components indicated by the unlabeled arrows in Figure 2.

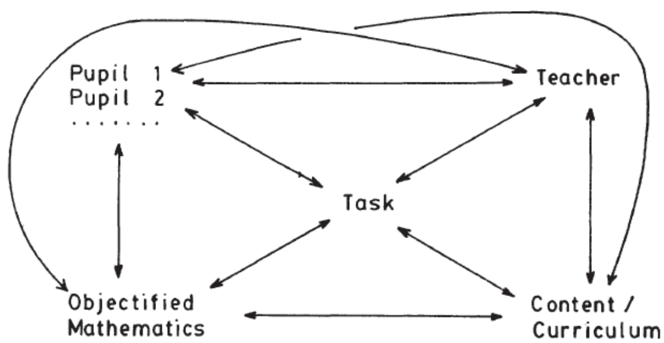



*Figure 2. The relational character of task and activity (Christiansen & Walther, 1986)*

A *task* is the "goal of an action, with the goal being framed by distinct conditions" (p. 256). Specifically, a task is the assignment set by the teacher, which is the object for students' activity. A mathematical task generally includes one or more problems whose solving is expected (by the task designer) to involve mathematics. The task also includes a set of instructions, directives, and/or extensions to which learners are expected to respond. Two caveats here: (1) how explicitly the goals and conditions of the task are communicated varies widely, and (2) replacing "mathematics" with "MKT" or "MKT-FT" in the paragraph above provides parallel definitions for tasks in the context of college instructor professional development for teaching.

*Activity* is a process that includes reactions and adaptations by the student that are in response to the changes in task conditions that arise during the students' work on the task (these are theorized to be based upon student-specific needs and motives). Activity is realized through a collection of *actions*, goal-directed processes arising from the students' motives:

> Activity exists only in actions, but *activity* and *actions* are different entities. Thus, a specific action may serve to realize different activities, and the same activity may give rise to different goals and accordingly initiate different actions. (p. 255).

Each action in activity serves to attain a goal of the task: the collection of actions is goal directed and together forms a *plan*. For Christiansen and Walther, the teacher is the central agent of authority. We argue that in contexts where students are adults, the locus of control may well lie with the learner (e.g., future elementary teachers, faculty who are learning about teaching). And, social, mathematical, and socio-mathematical mediation occur among students and between students and instructor. That is, how an activity induces action depends on the agents and their relationships. Moreover, moving between actions and from actions to related plan (and back again) involves many decisions. Figure 3a summarizes our interpretation of the framework, overall, and Figure 3b illustrates one possible decision process across actions and planning.

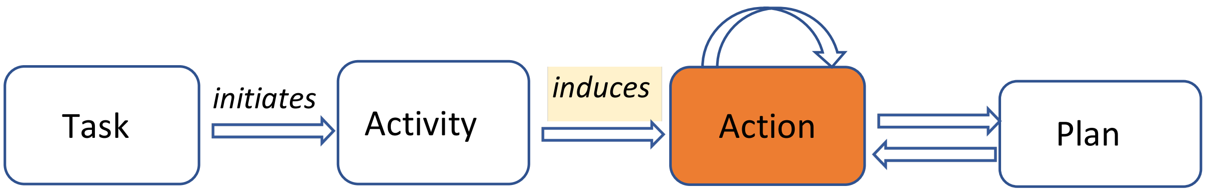



*Figure 3a. Detailed task and activity framework.*

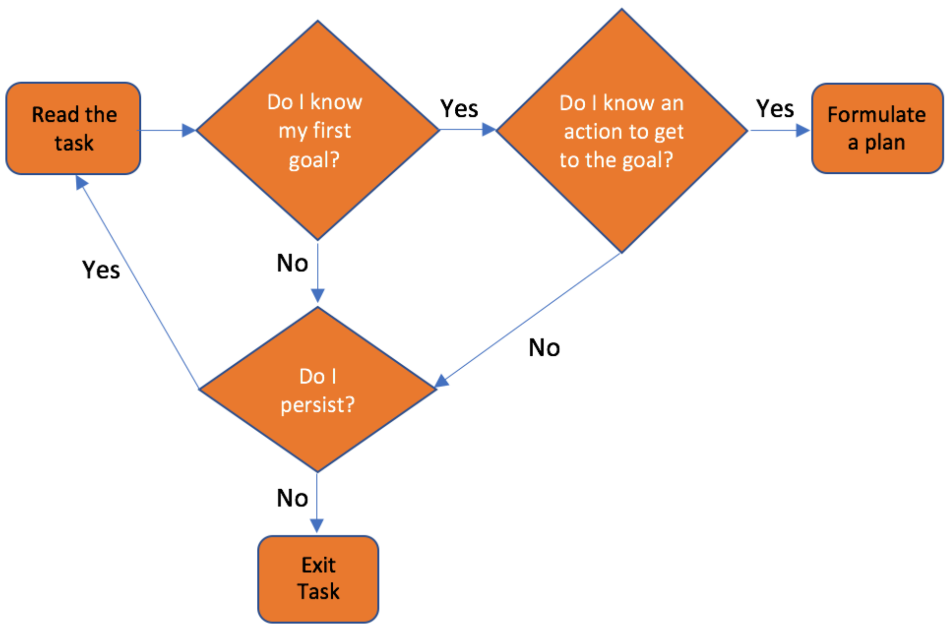



*Figure 3b. Example Action – Plan feedback tree.*

Christiansen and Walther (1986) offer different non-exhaustive types for each element in the framework. For instance, they distinguish different tasks by the type of mathematical activity in which students will engage: exploratory, constructive, or problem-solving. Smith and Stein (2011) offer a further delineation of problem-solving tasks as those that (a) call on memorization, (b) use procedural knowledge but require limited connections to other knowledge, (c) require procedural and connected knowledge, and (d) engage students in actually doing novel (to the learner) mathematics by calling for conjecturing, reasoning, and justification.

For Christiansen and Walther, *educational activity* is what leads to work in response to a task-driven behavioral goal (e.g., produce a graph), while *learning activity* is activity that results in someone achieving the intended learning outcomes. When engaged in an activity, learner actions may be *preparatory, observational/reflective, control-focused, safeguarding,* or *corrective*. Preparatory actions are those that establish conditions for success or which facilitate another action (e.g., *formulating a plan* is a preparatory action). Observational and reflective actions develop or identify information needed to complete or plan other actions. Safeguarding actions ensure that information and results obtained along the way in the task are readily available to the learner later in the task. Control actions are calibrations: learners compare the intended goals/actions with those that were actually achieved/performed. Corrective actions refer to acts by learners to anticipate or remove possible errors.

As an illustrative example of the framework, suppose an instructor of pre-service elementary teachers gives students a collection of questions similar to the one in Figure 4.

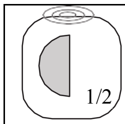
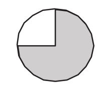
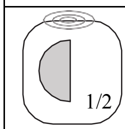
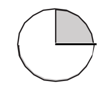
| | | |
|---|---|---|
|  | Suppose a whole serving is 1/2 of a cookie. How many servings (whole or fractional) can I make from 3/4 of a cookie? |  |
|  | Suppose a whole serving is 1/2 of a cookie. How many servings (whole or fractional) can I make from 1/4 of a cookie? |  |

*Figure 4. Fraction word problems (adapted from Gregg and Gregg, 2007).*

Notice that based upon definitions given here, a collection of a dozen such word problems, in and of itself, does **not** constitute a task because there are no instructions or extensions asking for response/resolution. However, the collection might be transformed into a task with an MKT development goal by inducing two different activities:

1. Suppose you are a 6th grader who is completing this activity for the first time. You have never been exposed to an algorithm for fraction division, and so you do not have that knowledge going into the activity. Do the activity accordingly.
2. What algorithm for fraction division does the activity suggest would be appropriate? HINT: To answer this, think about HOW you got your answers to each of the questions.

Now students are being asked to do more than to solve problems. They are required to engage in several activities. One activity introduces planning and actions for imagining (and then thinking like) a 6th grader, and as an extension, deducing the common denominator algorithm from the task by thinking about another (imaginary) person's solution process. In the course of completing the task, common learner actions tend to include reflection on how what they have done in solving each of a parallel set of problems about serving size as it relates to an algorithm, safeguarding as they search for patterns to determine the algorithm, and control actions as they begin to think about what a typical 6th grader might know.

## Characteristics of Task-based Learning (TBL)

Having now established definitions and a framework relating task and activity, we are in a position to elucidate the defining characteristics of task-based learning:

- Learners work on a task collaboratively (usually in groups of 2 to 4 members). Often tasks will include activity with manipulatives, video, and/or other technology.
- As learners work to complete the task, they consistently engage in activity that is mathematical and/or pedagogical in nature. The task is designed to elicit actions such as sense-making, conjecturing, reasoning, justifying, problem posing, questioning, challenging, role playing, reflecting, and anticipating.
- The task makes explicit queries about the nature of learners' thinking, reasons for steps they take, and what they produce as they work to complete the task. Teacher utterances include challenges to student productions, questions that extend activity or call for re-planning, and brokering guidance for struggling students.

Note that the first element is collaboration – working together towards a group goal or outcome. This is different from cooperation – working together for mutual benefit towards individual goals/outcomes. Both can be powerful supports for building community (Banilower et al., 2013).

**Juxtaposition with Other *-Based Learning**

Task-based learning certainly shares characteristics with most uses of "inquiry-based learning" we have encountered. It requires inquiry-oriented instruction (Rasmussen and Kwon, 2007) in that teacher and student play important roles in the process. Within the science education community, inquiry-based learning is often categorized as *structured*, *guided*, or *open* (Biggers and Forbes, 2012; Chinn and Malhorta, 2002; Kuhn, Black, Kesselman, & Kaplan, 2000). In structured inquiry, the instructor provides the materials and procedures necessary to complete the task, with the expectation that students will discover the intended learning outcomes in the process. In guided inquiry, the instructor poses a problem and provides necessary materials, leaving students to devise their own solution methods. In open inquiry, students pose their own problems and seek their own solutions. By design, TBL is either guided or structured, depending on how the task is presented to the learner. This is in contrast to *problem-based learning* which is an open model starting with something problematic for the learner rather than *problems,* which are the starting point for TBL. Likewise, TBL is different from *project-based learning* because tasks as defined here are not generally projects that require synthesizing significant amounts of information over time.

Every task starts with a novel (to the learner) problem (i.e., not an exercise involving a single stream of well-rehearsed actions). The activity and actions of students required in TBL ensure that they are doing mathematics. Actions that occur during task activity form the basis for self-regulation, a critical component of metacognition which is crucial for effective and efficient problem solving. Self-regulation is a behavior that can be acquired over time as learners engage in authentic problem solving regularly (Schoenfeld, 1992). That is, repeated exposure to tasks that scaffold agency and self-regulation can support the taking up of agency and self-regulation. The teacher's role in TBL mirrors that in teaching problem solving: as a cultural broker of mathematically rigorous meaning and facilitator of self-aware use of mathematical language.

Enacting or assigning tasks does **not** guarantee learning. In much the same vein, presenting students with problems to solve does not constitute teaching of problem solving. Other criteria must be met by instructor and students. For example, having future teachers use base ten blocks to demonstrate operations does not mean they can explain common algorithms for the operations. Concretism does not always ensure that intended learning activity will follow. To achieve the desired activity and, ultimately the goal learning outcome, it takes focused effort by the expert (teacher, instructor, facilitator of professional development) during activity in the task to direct attention as needed. Thus, task-based learning for faculty, where the goal is to build MKT-FT must do more than tell participants to watch some mathematics classroom video and reflect on it (Seago, 2004). Specific prompts before video viewing might direct people to prepare themselves to notice and identify evidence of student thinking about the meaning of slope. Twice. That is, the task includes purposeful repetition of activity. The prompt for two viewings makes explicit the goal and sets expectations that participants will do a particular kind of intellectual work (notice, identify) about particular aspects of the video (student utterances and actions that can be considered evidence, slope). These prompts are intentional in preparing the participant for possible extensions like: Create at least two potential responses to the noticed thinking.

**Why Promote TBL Among Mathematics Instructors and in Professional Learning?**

First, TBL is a form of active learning and active learning has been shown to significantly improve undergraduates' performance in science, technology, engineering, and mathematics courses by half a letter grade (Freeman et al., 2014). Second, the recent *Standards for Preparing*

*Teachers of Mathematics* by the Association of Mathematics Teacher Educators (2017) calls for the use of task-based learning in courses for future teachers:

> In such settings, learners are typically provided challenging tasks that promote mathematical problem solving and … discuss their thinking in small and full-group discourse, thus promoting important mathematical practices (Webb, 2016) (p. 31).

As does the *Mathematical Education of Teachers II* (MET II, 2012):

> Courses should also use the flexible, interactive styles of teaching that will enable teachers to develop [mathematical] habits of mind in their students (p. 19).

Indeed, a task-based approach empowers the skilled teacher to meet many (if not all) of the criteria in the Teaching for Robust Understanding (TRU) framework for high quality instruction (Schoenfeld, 2014, 2017). Moreover, Connolly and Millar (2006) noted that faculty in teaching workshops wanted professional development that used the TBL methods being advocated in the workshop. In a current project by the authors, we are offering faculty a task-based approach to professional learning about task design and task use in their own classrooms.

## Conclusion and Avenues for Further Investigation and Research

We end by giving some examples of tasks for faculty professional learning. Our focus is ways to teach mathematics courses for future K-8 teachers. Note that the main goal of these professional learning tasks is not to build mathematical knowledge, but to foster development of MKT-FT. A task requires a problem. Rich problems of instructional practice might center on pedagogical content or specialized content knowledge for teaching future teachers or building understanding of the MKT that future teachers need.

In this spirit, consider the cookies task discussed earlier. In a task for faculty, participants are asked to "put on your student hat" and do the task. Then the nature of the task and activity are discussed. Then comes a meta-aware extension to the task, "Imagine you are a pre-service teacher and have been given this task, what is challenging? Why?" Faculty work involves *knowledge of content and (pre-service teacher) students,* a component of MKT-FT parallel to knowledge of content and students in MKT. Faculty then read a transcript of pre-service teachers completing the original task. They identify things that the pre-service teachers struggled with and compare that with their anticipations. The task has two intended learning outcomes: (1) faculty build knowledge of (pre-service teacher) student thinking and (2) faculty unpack the demands and consequences of designing/revising tasks for achieving particular learning goals.

Areas for use of the TBL framework in research and development include addressing questions such as: How do designers and facilitators know that they are effectively implementing task-based learning in faculty professional development? What constitutes evidence of this? Also, what are indicators of success of a task-based professional learning experience? The productive use of tasks by participants in their own practice is one important factor, but are there others? Finally, the scant research literature on professional development for teaching in higher education has yet to delineate the conditions that promote (or hinder) faculty success. For example, what experiences and supports may be needed for faculty to use, as an instructor, a TBL model they have experienced as learners in professional development?

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Theorizing Silence

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*Different norms govern the use of silence in mathematical collaboration and in every-day Anglo-conversation. Research is therefore needed into the ways students are enculturated into the distinctive uses of silences employed in mathematics collaboration. This project will require a new theoretical perspective that facilitates the study of silence. Drawing off studies of silence and embodiment from multiple disciplines, this paper advances a view of silence and the body, and so lays the groundwork for a rigorous study of silence in mathematics education.*

*Keywords:* Silence, Embodiment, Sociomathematical Norms, Interaction

Two recent conference papers (Petersen, in press; Lim, 2017) have raised the prospect that the mathematics education community may benefit by attending to silence. First, Petersen argued that while engaged in intense collaborative mathematical activity, mathematicians remain silent for lengthy periods of time; a practice at odds with the every-day Anglo conversational norm that exactly one person speak at a time (Erickson, 2004; Liddicoat, 2011). This disconnect between the norms of every-day Anglo conversation and mathematical practice makes mathematicians' collaborative silences pedagogically interesting. Second, Lim, argued that though the reform movement has done a good job giving students voice in the classroom, introverted students who value silence and careful thought, may have a difficult time in reform classrooms.

As I have thought about these issues over the last year, two anecdotes have helped convince me that silence is in fact an important, though understudied aspect of mathematical activity and of learning to be a mathematician. First, while discussing silence with a mathematician, he told me that he had the distinct impression of being apprenticed into silence while working on both his Master's degree and Ph.D. The second episode occurred while tutoring students in a 300-level proof class. On one occasion, as I attempted to answer student questions, I realized that the solution I had worked out in advance was incorrect and, with the mathematics I then knew, I was unable to address the students' questions—though I could tell that my error was small, and relatively easy to fix. My natural strategy was not to speak, but to perform mathematical activity by attending to the problem closely and carefully, in silence. The students, however, responded to my silence as a lapse in conversation, and repeatedly attempted to engage me in further conversation. To my surprise, I quickly realized that though, on my own, I would be able to fix the bug in my proof quickly and easily, the interactional requirements of the tutoring situation, and the norms governing the conversation, made me incapable of performing the mathematical activity necessary to adequately address my students' questions.

Though these are just anecdotes, both stories provide corroborating evidence that in learning to collaborate as mathematicians, students need to learn to employ silence in ways that violate the norms of every-day Anglo-conversation. The second, in particular, points to potential difficulties students may face as they attempt to collaborate on mathematics. If research is to be conducted into silence, strong new theoretical and methodological papers are needed. This paper attempts to make a beginning in providing a theoretical foundation for the study of students and mathematicians uses of silence.

## Communities of Practice

Several important strands of research in mathematics education attend to the way students learn mathematical ideas and concepts in communities of practice. Several are particularly relevant to research on silence.

First, research in the emergent perspective (Cobb and Yackel, 1996; Yackel and Cobb, 1996; Voigt, 1985, 1989, 1995) attends to sociomathematical norms, that is, ways students and teachers negotiate what sorts of answers are normatively treated as expressing mathematical concepts and practices (such as justification) (Cobb and Yackel, 1996; Yackel and Cobb, 1996; Voigt, 1985, 1989, 1995). If, as previous research suggests (Petersen, in press), there are distinct norms governing silence in face-to-face mathematics collaboration that differ from the norms governing silence in every-day conversation, there are peculiar sociomathematical norms governing silence, and, as students become mathematicians, they are enculturated to those norms. On the other hand, Yackel and Cobb attempt to "account for how students develop specific mathematical *beliefs and values*" (p. 458, emphasis mine), whereas though silence may be used peculiarly in mathematical collaboration, it does not itself signify any mathematical reality.

This focus on overtly mathematical aspects of the classroom, however, does not preclude attention to what Voigt (1985) calls "patterns of interaction", e.g. questions of who is authorized to speak when, or how much wait-time teachers give their students, behavioral patterns that are not overtly mathematical in nature, but which give the classroom a particular order in which explicitly mathematical practices can be learned. These patterns of interaction which undergird mathematical activity, however, are not strictly mathematical, and so the norms governing them are not sociomathematical norms; whereas, if the norms governing silence are discipline specific, because they are an aspect of mathematical activity, they are as sociomathematical norms.

On the other hand, a very different line of research in mathematics education attends not to mathematical concepts and beliefs, but to mathematical *activity* (Rasmussen, Zandieh, King, & Teppo, 2005). In learning mathematics, students and teachers engage in activities endemic to the mathematics profession, like justification, algorighmatization, and defining.

Like that from Cobb and Yackel's emergent perspective, research from this perspective focuses on overtly mathematical aspects of learning. This line of research, however, opens up the possibility that actions that are not overtly mathematical nevertheless play an important role in allowing people to perform mathematical actions, and so are an important aspect of mathematical activity and an important line of research in mathematics education. Though not from the same perspective, Savic (2015), can be read as an existence proof for this sort activity. He found that when mathematicians reach a proving impasse, they will sometimes resolve the difficulty by stepping away from the problem and doing something else, e.g. taking a walk, going to lunch with their family. Savic's research does not address the potential for aspects of mathematical activity that are *socially* interesting, and that require students to learn new sociomathematical norms, but together with Cobb and Yackel's research into sociomathematical norms, it raises the prospect that, as part of their collaborative mathematical activity, mathematicians follow norms contrary to the norms used in every-day interaction. If the results in Petersen (in press) hold up, silence falls into this category. This claim, however, needs unpacking unpacking.

Philips' (1792; 1983) ethnography of education on the Warm Springs Indian Reservation in Oregon provides helpful information regarding the ways different norms for silence can structure classroom interaction. According to Philips, the Native community she studied placed a high value on the difficult skill of effective, brief, speech; and therefore, lengthy pauses often preceded responses. On the other hand, in Anglo-conversation, pauses longer than a second carry meaning, often indicating a dispreferred response (Liddicoat, 2011). This disconnect between the norms governing silence for Anglo teachers and their native children meant that student silences, directed at both teacher and peer as signs of respect, were read by the teacher as signs of incompetence. Furthermore, the Anglo teachers would often cut native student's silences short, thus depriving them of the chance to speak. It goes without saying that, in the university mathematics classroom, the power relations are very different than in the elementary classrooms Philips studied, however, this example illustrates the possibility for deep miscommunication caused by different norms for silence. Furthermore, if those norms are specific to mathematical activity, they are sociomathematical norms.

## Silence

Silence seems difficult to study scientifically for two reasons: First, silence seems to be the lack of speech or of sound and not a phenomenon in its own right. This issue has theoretical and methodological aspects: What is silence, and how can we attend to it? Second, silence does not regularly signify anything mathematical, and if it does, it only does so accidentally. It seems therefore, silence should be addressed when it happens to come up, but should not be a topic of research in its own right. This section will address the theoretical aspects of the first question, whereas the subsequent section will address the second question. A separate paper will be required to address the methodological aspect of the first question, though the final section of this paper provides a sketch of a methodology.

Silence is not a mere absence or a lack (Acheson, 2008; Ephratt, 2011), but a phenomenon, actively heard with our ears, that both frames sounds and words, and is in turn framed by sounds and words (Acheson, 2008; Chrétien, 2004). So, for instance, as Dauenhauer (1980) notes, a performance of music is only heard as a unity because of the silences that bracket it. On the other hand, Handel often underscores dramatic moments in his music with lengthy silences (Harris, 2005), which are only heard as dramatic parts of the music because they are surrounded by sound (cf. Kim, 2013). Nor is silence not one-dimensional: The sorts of sounds that bracket a particular silence, and the posture and gestures employed during a silence, give a particular color and meaning to silences (Margulis, 2007a, 2007b; Acheson, 2008). Finally, silence is not a default state, but is actively produced. Thus, for instance, silence can be a design feature of buildings (Kanngieser, 2011; Ergin, 2015; Meyer, 2015; Bonde & Maines, 2015); and we are all familiar with how difficult holding our tongue can be.

Because silences are actively produced and heard, they can bear particular meaning (Acheson, 2008; Ephratt, 2011). They are therefore perhaps best understood as a particular sort of gesture; a gesture which we can perform in concert with others, or alone while others are speaking (Acheson, 2008). For instance, Quaker worship is structured by lengthy collaborative silences (Lippard, 1988), and the bond of a nursing mother with her infant can be strengthened through mutual eye-contact and silence (Maitland, 2008); while, on the other hand, the children Philips (1983) studied on the Warm Springs Indian Reservation communicated that they were actively listening, not through eye-contact or back-channeling (e.g. "mhm"), but through silence.

## Embodiment

Two recent papers (Abrahamson & Sánchez-García, 2016; Abrahamson, Sánchez-García, & Trninic, 2016) have called for attention to the ways students develop their bodily capacities and so, open up new avenues for action in the world. In this call, they have opened up a new avenue for research into the body in mathematics education, the ways skilled uses of the body are a prerequisite for mathematical learning, and how, by training our bodies to be capable of new actions, new affordances for action are opened. Following their lead, mathematics education research can attend to the various ways we train our bodies to perform otherwise difficult actions which can subsequently give rise to mathematical meaning. Since acting according to new norms is difficult, this perspective is a helpful starting point for theorizing silence.

There are, however, two aspects of their theoretical perspective that make it inadequate for theorizing research on silence. First, they do not attend to the ways bodies are used in social interaction. But silence is an interactional accomplishment and challenge. It is relatively easy, however, to modify their perspective to incorporate social interaction. As McDermott (1978) notes, in interaction, we are the environment in which our peers act. Thus, developing new ways of acting in the world means developing new ways of acting on our peers, and of being acted on by our peers. Abrahamson Sánchez-García's (2016) perspective can be modified to say that as we acquire new skilled uses of the body, new affordances are opened up not only for learning mathematics, but for orienting ourselves and our peers collaboratively toward mathematics.

Second, their focus is still on actions that signify mathematical realities—the actions just do not yet have mathematical significance when learned. But there is another way actions, in our bodies both natural and social, can be connected to doing mathematics: They can order the parts of the body in a way that gives the capacity to do mathematics. Morgan and Abrahamson (2016) take something like this tack in their preliminary investigation of the ways meditative practices like tai chi and yoga could be utilized to enable students to engage with difficult mathematics, and Savic (2015) showed that not doing mathematics is an aspect of doing high-level mathematics. But otherwise, I have not encountered research that examines ways the body is used to give agents the capacity to do mathematics; and none that examine interaction. However, ordering the parts of our bodies, natural and social, in a way that facilitates the doing of mathematics is a necessary condition for doing mathematics, and so is a valid topic for mathematics education research. Furthermore, as noted above, if in doing mathematics, the body social is ordered in a novel way that relies on social norms different from those used in every-day interaction, this order, and the way it is learned, is educationally *relevant*.

While little mathematics education research that attends to the ways bodies are utilized to give an agent, or a group of agents, the capacity to perform mathematics, this perspective on the body is akin to some perspectives employed in anthropology. In particular, Marcel Mauss' (1935/1968) concept of a *habitus*, a pre-reflective, bodily know-how, that gives a subject the capacity to engage in an activity, has proven fruitful in examining a number of different phenomena, e.g. the transmission of oral literature (Saussy, 2016), and to the mosque movement (Asad, 2003; Mahmood, 2005). Mauss's concept also has a deep resonance with the theorization of the body Targoff (2001) employed in her investigation of poetry and prayer in early modern England (Mahmood, 2005). Finally, though not related to Mauss, Esaki (2016) argues that Japanese-American gardeners employ silence to give them the capacity to tell what sorts of cuts they should perform on their trees.

## Interaction

The issues surrounding silence, however, are not individual, but arise in interaction. Petersen (in press) argues that, while engaged in intense mathematics, mathematicians collaboratively engage in lengthy silences, in violation every-day conversational norms. How do students learn these norms? And how does conflicting interpretation of silences, and conflicting norms governing its use, influence students capacities to engage in mathematical activities? In order to address these questions, we need to theorize not only the body, but interaction. In this section I will argue that the claim that interacting participants form a complex dynamic system, or what some researchers call a synergy (e.g. Chemero, 2016), is a plausible hypothesis.

A pair of recurring question in behavioral sciences concern the mechanisms involved in the bodily coordination presupposed by the pursuit of a behavioral goal, either by an individual or by a group of individuals (Takei, Confais, Tomatsu, Oha, & Seki, 2017; Ashraf et al., August 24, 2017). Though the addition of multiple agents makes the second question more complex, there are reasons to believe that similar dynamics underlie both. As Marsh (2015) claims "in both cases, some kind of information…leads to entrainment; each involves the creation of a coordinative structure or synergy" (p. 321).

Researchers studying the material aspects of interaction have found that participants mutually entrain multiple aspects of each other's movements, including posture, limb-movement, speaking rate, vocal intensity, and, critically, length of silences (Marsh, Richardson, & Schmidt, 2008; Shockley, Santana, & Fowler, 2003; Sebanz, Bekkering, & Knoblich, 2006; Shockley, Richardson, & Dale, 2009; Noy, Dekel, & Alon, 2011; Fowler, Richardson, Marsh, & Shockley, 2008; Schmidt & Richardson, 2008; Capella & Planlap, 1981). What functional goal has this entrainment evolved to serve? The answer seems to be that it allows people to join together in a common activity, in pursuit of a common good (Richardson, Dale, & Marsh, 2014). This hypothesis is, partially, confirmed, by a recent paper on professional string quartet performance (Chang, Livingstone, Bosnyak, & Trainor, 2017). They demonstrated the body-sway of the musicians is auditorily and visually coupled, is a tool musicians employ to shape performance, and more coupling is correlated to the musicians' perception of successful performance.

These results in mutual entrainment allows the tentative conclusion that the body functions, in part, to knit people together into a body social with a common end, either through the mediation of shared representations, or immediately by allowing them to engage in joint activity (Marsh, Richardson, & Schmidt, 2008). Because research in silence is not attending to mental constructions, the second option seems to better fit for research on silence: In interaction, we utilize our bodies not only to signify the world, but to order each other and ourselves toward a common good, e.g. discovering and proving a new theorem, symbolizing and defining a mathematical object, etc. (cf. Rasmussen, Zandieh, King, & Teppo, 2005), and so to give a body social, and its individual members, the capacity to pursue that good.

This theoretical perspective on interaction requires that the activity of the body social—the linked dynamic system (Richardson, Dale, & Marsh, 2014), synergy (Chemero, 2016), or teleodynamic system (Walton, Richardson, & Chemero, 2014)—be the unit of analysis, not the isolated actions of the particular persons in the interaction. However, because the individual mathematicians are material parts of the body social, the analysis cannot be carried out in abstraction from the bodily actions of the individual mathematicians. Rather, the unit of analysis is the body social precisely because the actions of the individual mathematicians are treated as constraining, and constrained by, the activity of their peers. As certain activities—say, in piano playing—are not difficult for each hand individually, but in the coordination between hands; so some activities that are not difficult for individuals, when working alone, may be difficult to achieve *in common* (Marsh, Richardson, & Schmidt, 2009). On the other hand, because of the mutual entrainment, social order belongs to the body social—that is, to the dynamic system—and cannot be understood merely as the work of the individual participants, considered in isolation. Two points are key here: First, if peers engaged in joint interaction act according to different interactional norms, joint action may be particularly difficult. Second, when participants in face-to-face collaborations engage in high-level activity, and act according to the same norms, the fact that each member of the body social perceives the others are engaged in the same activity should serve to strengthen their own engagement (Walton, et al. 2014).

This theoretical position is heavily influenced by, and very similar to, the position McDermott (1978; McDermott, Gospodinoff, & Aron, 1978) employed in his ethnographic descriptions of a classroom, and to Erickson's (1996, 2004) microethnography. McDermott (1978), attended ways two teacher and student reading groups established an order through the postural positions of each member of the groups and the way the orders facilitated, or did not facilitate, learning to read. Similarly, employing the concept of the *habitus* mentioned above, Erickson (2004) argues that when there is a disconnect between the *habitus* actors attempt to employ in joint activity, "seemingly automatic workings of the players' *habitus* are no longer effective for engagement in the collective activity…If the player is to be able to stay in the new game, that player's *habitus* must change" (p. 12).

On the other hand, it shares similarities with several influential perspectives in mathematics education, while differing from them in key respects, namely Cobb and Yackel's (Cobb & Yackel, 1996; Yackel & Cobb, 1996) emergent perspective on classroom activity, and a Realistic Mathematics Education (RME) (Rasmussen et al., 2005) discussion of mathematical activity.

My research shares with Cobb and Yackel an emphasis on the ways participants in concerted activity co-create the activity, mutually conditioning the activity of all the others, and so forming the group into a single "dynamic system" (Yackel and Cobb, 1996, p. 460), and with a concern with the sociomathematical norms that govern this activity. It differs from them in two key, interconnected, respects. First, though Cobb and Yackel are concerned with one aspect of the way students and teachers mutually position each other around mathematics; the bodily aspects of that activity are not relevant to their investigations. But research into silence attends to one aspect of the way individual students and mathematicians engaged in mathematical activity hold themselves, physically, and so mutually orient themselves and peers toward doing mathematics. Second, their fundamental goal is to determine how mathematical *beliefs* are learned in learned in concert; whereas my focus is more like Rasmussen's research, in that it is focused on joint mathematical *activity*. This second difference shapes a methodological divergence: Whereas they envisage zooming in to a psychological investigation of student beliefs and understandings; I envisage zooming in from an investigation of the materiality, including the silences, of interaction, to an investigation of the bodily activity and gestures, including silence, of each individual mathematician or student.

Second, this perspective is closely related to a RME understanding of mathematics not merely as individual belief, but as particular sociocultural activity (Rasmussen et al., 2005). The key difference is that their work is focused on a different aspect of mathematical activity than research into silence is. Though they sometimes attend to gestures (e.g. Rasmussen, Stephan, & Allen, 2004), these gestures are relevant because of their ability to symbolize and communicate mathematical realities, whereas whether a student working to mathematize is, at that instant, seated or standing, motionless or pacing, etc. is irrelevant. Cooperative student mathematizing is, however, supported by shared bodily orientations that order group participants toward the mathematics at hand, norms regarding what bodily actions are appropriate, etc. It is to these norms that facilitate mathematical activity that research into silence should attend.

## Sketch of a Methodology

The first methodological challenge a study of silence faces is that traditional transcripts make silences invisible, rather than highlighting them (Ochs, 1979). A new form of transcription is therefore required that highlights silences, both collective and individual, and the postural nuances that give silences distinct characteristics. Figure 1 contains a sample transcript from three calculus students' attempt to identify which of three functions represent the position, velocity, and acceleration of a car. Not all the transcription conventions are relevant, but the following are most salient: Individual students' verbal utterances are placed in columns on the left, and non-verbal gestures, on the right. Footnotes show when, relative to the speaking and silences, individual gestures occurred. Mutual silences are highlighted in dark grey, and their length notated in the center column. Silences that do not include all students are highlighted in light grey, and their duration indicated to the left of the column. In the gesture columns, "eg" abbreviates "eye-gaze". These conventions highlight silence and allow its investigation.

In this episode, how do the three students respond to the lengthy mutual silences at the beginning? Andy, responds to the silence by speaking, approaching the board and seeking eye-contact with his peers (bottom row; 2, 3, 5, 7, 8, 10). Jason and Katherine, however, remain mostly still, do not move in response to Andy, and avoid eye-contact with him. A much longer analysis of this episode is possible, but these facts suggest Andy responds to the silence as an awkward pause, and attempts to resume the lapsed conversation; whereas Jason and Katherine, treat the silence as a part of their mathematical activity, and seek to continue it. This leads to the tentative conclusion that Jason and Katherine hear Andy's talk as an interruption of their silent mathematical activity; whereas Andy hears Jason and Katherine's silence as silencing him.

## Conclusions

Silence, though a peculiar aspect of mathematicians collaboration, does not fit well with existing theoretical perspectives in mathematics education research. Novel theorizations are therefore required for the study of silence. This paper provides the beginnings of a new theorization of silence, and the body, and provides a brief analysis of silence.

| Verbal | | | | Non-Verbal | | |
|---|---|---|---|---|---|---|
| Andy | Jason | Katherine | | Andy | Jason | Katherine |
| [1] | of [1]**b** //but] **c** is still °just the derivative of **a** [a]ri:ght°[z] | \\of **b**] | | [1]steps toward graph | [a]turns head slightly to right (still at board) then motionless | [z]starts slowly rocking back with rigid body |
| 35 Seconds | | 2 Sec | 2s | | | |
| [1] | | [z1]>right< | | [1]raises paper | | [z]pulls arm back (hand still pointing) nods |
| [1] | 10 Seconds a | z | 3s | [1]steps back ↓eg to paper | [a]tilts head to right ^squares toward board | [z]lowers hand, tilts head to left. |
| <so> [a](.) anyway[1] I'm just gonna go back to the basics here[2]- | a | | | [1]hands apart [2]steps forward pointing toward graph | [a]^squares hips toward board, crosses arms, tilts head slightly left. | |
| | Actually (.) ↑well,[z] | 1 minute, 44 seconds z | | | | [z]slight pulse of left hand (otherwise motionless) |
| So we do know we have a [1]velocity [2](.) [3]'cause this graph and we have a position[4] from that based[5] on all of our data from[6] all these[7] little tangles and everything based[8] on the position[a] well[9] we [b]can say that (.) and then[c] we[z] have >an acceleration< of **c**[d10] //so[11] | 30 Seconds a b c d | z | | [1]points to where they have written "velocity" [2]→eg peers, ↑back to board [3]traces along graph a [4]beats near word "position" then waves at graph [5]→eg to peers [6]↑eg to graph touches graph **a** with pen [7]draws lines down from **a** to **b** [8]traces hand along **b** [9]pulls hand back then gestures forward & turns hand over [10]traces hand along **c** [11]scratches nose | [a]head right [b]weight to right foot ↓eg toward floor [c]↑weight on both feet eg and pelvis toward board [d]turns head very slightly to right | [z]steps back, head straight (eg still board), crosses arms. |

*Figure 1: A transcript of three calculus students attempt to determine which of three sketched functions represent the position, velocity, and acceleration of a car.*

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# Didactical Disciplinary Literacy

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*Undergraduate mathematics students are routinely asked to learn from various "texts" such as textbooks, videos, and lectures. In order to understand how students read and learn from discipline-specific texts, literacy researchers in recent years (e.g. Shanahan & Shanahan, 2012) have begun to direct their attention to disciplinary literacy: the ways that disciplinary experts themselves interpret, create, and critique materials. In this vein, we set out to investigate the disciplinary literacy practices of calculus students and non-mathematics STEM faculty, but found that focusing on this form of literacy alone was insufficient to explain the differences between the students' and faculty members' practices. To address this, we propose a new construct of didactical literacy, provide examples, and discuss its details and ramifications.*

*Keywords:* Disciplinary Literacy, Textbooks, Expert-Novice Studies

## Background

Over the past several decades, numerous groups have advocated for learning mathematics by participating in mathematical practices, which includes mathematical communication (e.g., National Council of Teachers of Mathematics, 2000; National Governors Association, 2010; Schumacher, Siegel, & Zorn, 2015). Students have long been asked to use and learn from a variety of mathematical texts, such as textbooks, lectures, and videos; and the proliferation of new didactical formats such as "flipped" classrooms have led to increased interest in using these formats in mathematics classrooms (e.g., Maxson & Szaniszlo, 2015a, 2015b).

There has been relatively little research that has focused on literacy in the context of teaching, learning, and doing mathematics. Although there is a body of research focused on undergraduates' reading and comprehension of mathematical proof (e.g., Inglis & Alcock, 2012; Mejia-Ramos & Weber, 2014; Weber, 2015), there is scarce other research that describes how students interpret and learn from printed texts (e.g., Borasi, Siegel, Fonzi, & Smith, 1998; Draper & Siebert, 2004; Shepherd, Selden, & Selden, 2012; Shepherd & van de Sande, 2014) or lectures (e.g., Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016; Weinberg, Wiesner, & Fukawa-Connelly, 2014). As Doerr and Temple (2016) noted, the paucity of research combined with the increased need for students to interact productively with mathematical texts underscores the "need for greater attention to reading in mathematics instruction" (p. 6).

As Moje (2008) noted, as educators have come to view participation in discipline-specific discourses as an essential aspect of disciplinary learning, educators and researchers have begun to focus on literacy as an aspect of disciplinary practice. In this paper, we begin by discussing the concept of disciplinary literacy. Then, we describe selected results from a study we undertook to investigate the literacy practices of students and non-mathematics STEM faculty members. Based on these data, we propose a new form of literacy called *didactical disciplinary literacy*, which, we hypothesize, may play an important role in learning from didactical mathematics texts.

## Literacy in Teaching and Learning Mathematics

The term "literacy" is used in a variety of contexts. In the mathematics education community, it appears most often as part of "quantitative literacy" (e.g., Steen, 2001), which focuses on the capacity of an individual to identify and work within quantitative situations and to use mathematical skills for citizenship. In contrast, the types of literacy that we focus on in this paper involve creating and learning from texts. We use the term “text” broadly, as defined by Draper and Siebert (2010): “a *text* is any representational resource or object that people intentionally imbue with meaning, in the way they either create or attend to the object, to achieve a particular purpose” (p. 28).

Literacy experts have emphasized the pervasive and complicated role that texts play in teaching and learning, and students’ roles as active learners—and their interactions with new forms of media—have led researchers to propose a broad definition of literacy, such as: “the ability to negotiate (e.g., read, view, listen, taste, smell, critique) and create (e.g., write, produce, sing, act, speak) texts in discipline-appropriate ways or in ways that other members of the discipline (e.g., mathematicians, historians, artists) would recognize as ‘correct’ or ‘viable’” (Draper & Siebert, 2010, p. 30).

Each discipline produces its own types of texts (Bass, 2011) and uses distinct, grammatical resources and agency (e.g., Coffin, 1997; Martin, 1993; Veel, 1997, Wingnell, 1994). For example, a history text might take the form of a historical argument (advocating for a particular interpretation of events) while a science text might take the form of a science explanation (describing how and why phenomena occur).

In addition to the disciplinary-specific aspects of texts, recent research has shown that experts in different disciplines engage in reading processes in different ways. In one of the few studies to examine aspects of mathematics disciplinary literacy, Shanahan, Shanahan and Misischia (2011) compared the reading practices of historians, chemists, and mathematicians and identified distinctive features of the ways they interacted with research texts from their respective discipline. For example:

- Historians used the time period in which a text was written and other contextual factors as interpretive tools; chemists attended to time period to determine the value of the text in the context of rapidly-evolving scientific theories; and mathematicians did not contextualize the text.
- Historians used intertextual connections to identify the effects of the author’s perspective; chemists corroborated connections to explain the outcomes of various experiments; and mathematicians used corroboration to limit interpretive differences.

This focus on the “particular norms for everyday practice, conventions for communicating and representing knowledge and ideas, and ways of interacting, defending ideas, and challenging the deeply held ideas of others in the discipline” (Moje, 2008, p. 100) has led some literacy researchers to move towards the construct of disciplinary literacy as a way to identify discipline-specific practices and to structure students’ engagement with these practices. We define *disciplinary literacy* as the capacity to create, interpret, and critique texts using the practices, skills, tools, and ways of constructing, representing, using, and communicating knowledge that are specific to a particular discipline.

## Investigating Literacy Practices for Mathematics Textbooks

The initial goal of our research study was to describe the literacy practices that are important for learning from reading mathematics textbooks. We expected that asking mathematicians to read a mathematics textbook wouldn’t reveal such practices because they would likely already know most of the ideas presented in the textbook. Thus, we interviewed two groups of people. The first group consisted of five student volunteers from the third author’s second-semester calculus class. The students regularly read their textbook outside of class and engaged in various online and in-class discussions of the reading.

Our goal for the second group was to find readers who might have similar disciplinary literacy practices as mathematicians but wouldn't have the same extensive background knowledge. Thus, we recruited five faculty members from our institution, one each from the physics, chemistry, biology, computer science, and economics departments. Although each of these faculty members had taken calculus courses as part of their coursework, none had actively engaged with ideas from introductory calculus for (at least) the past ten years.

For our text, we used two excerpts from a section entitled "Applications [of the integral] to Geometry" in the students' textbook (Hughes-Hallett et al., 2012): the section introduction and an excerpt on arc length. The students had already read about other applications of integration in their textbook, so we posited that they were likely to have constructed the necessary background knowledge for understanding the concepts in the chapter.

To collect data, each person participated in an hour-long interview, which was video-recorded to capture the interviewee's speech, gestures and writing. We asked each participant to read the excerpts for the purpose of learning the content and to stop at any places where they had questions or were confused to engage in discussion. After completing the reading, the participants were asked to describe the main ideas of the sections, and to explain the meaning of each graph and/or formula, how the terms in the graphs and formulas had been derived, and why the text's explanation of the connections and derivations made sense.

## Results

Both the students and the faculty interviewees engaged in numerous interactions with the textbook that mirrored aspects of disciplinary literacy in mathematics—for example, they all engaged in close reading and rereading, understanding terminology and defining variables, interpreting graphics as part of the text, and not attempting to contextualize the text. The faculty members were not familiar with the mathematical concepts that were used in the sections—namely, Riemann Sums and limits—but the students were. However, the faculty members all were able to make sense of and construct mathematically-accurate descriptions of the concepts and, in many cases, to do so more successfully than the students.

In the process of identifying aspects of the students' and faculty members literacy skills, we noticed that the faculty members appeared to be identifying and using what we called "didactical aspects" of the textbook in order to make sense of the reading. We identified patterns in the faculty members' interactions and present them here with several examples from the faculty members' interviews. The italicized portions of the interview excerpts highlight particular language we identified as indicating a didactical focus. Although we also describe aspects of the students' work, space limitations prevent us from including excerpts from the students' interviews.

### Framing the Text in Terms of the Authors' Intentions

All of the faculty members routinely framed their interpretation of the text in terms of the authors' intentions. For example, Professor E, when asked to describe "what's going on" at the beginning of the section on arclength, described the authors' motivation for using the hypotenuses of right triangles to approximate a curve:

> What's going on here is *they want to show you one of these really small lengths* and so along the horizontal axis, the distance is this amount right here, the change in x, but as x

goes from this point to this point, we want to know how long that length is right there, and it's going to be longer because it's not a straight line.

Similarly, Professor K, when asked to describe why the textbook used a Riemann Sum before presenting the corresponding integral, described the authors' didactical intentions:

> I think *they're trying to provide steps for the reader* so there's no great leap in logic. That it makes sense that I have these small lengths. And then what am I gonna do with these small lengths? I'm going to add them up. So *I'm reminded at this step* that I'm doing a summation. That's essentially what an integral is.

Although most of the students also sometimes framed the text in terms of the authors' intentions and choices, their statements tended to focus more on mathematical rather than didactical aspects and ramifications of these choices.

**Thinking about Didactical Motivation**

The faculty members identified didactical motivation as the basis for the ways the authors presented the concepts. For example, when Professor K was asked to explain why the book's method for developing arclength made sense, she stated a didactical motivation and sequence, rather than a mathematical/logical description of the concepts:

> *They're starting with things that are easier to understand*. So delta x and delta y are easy to understand. We can really see those. They're physical lengths in my picture. So I can see them, and I can see how if I [gestures with fingers as if squishing lengths] made these lengths smaller and smaller, I could approximate a curve. *So they've shown me how to get* delta x and delta y, and then *using tools I've learned before*, we can rewrite these in terms of these [indicating a small quantity with her thumb and forefinger] very small steps, these derivatives. And then turning that into an integral, so *they're taking me through their process and explaining why they're doing it. Which is really important*. *So I'm not just facing an equation with no understanding of where it came from*. But with the understanding, I can apply this now with greater confidence.

When other faculty members were asked why the book first used a Riemann Sum before introducing the integral, they framed their description in terms of a didactical motivation that the Riemann Sum would be easier to understand than the integral. For example, Professor D noted that the Riemann Sum was based on intuition:

> I think *they're just reestablishing that intuition* that like, really an integral is just a sum of just an infinitely small pieces. And it's just a natural—and they're saying also the arc length is approximately equal to—that's what the squiggly lines are saying—but in the limit, it's actually equal to.

The students were less likely than the faculty members to frame aspects of the text in terms of didactical motivation. When they did so, they often framed their explanations partly in terms of mathematical necessity rather than solely in terms of pedagogy.

**Thinking about Didactical Structure**

In addition to identifying didactical motivation for various aspects of the text, several of the instructors also identified the didactical *structure* of the text and used this to inform their learning and to wrestle with uncertainty. Professor E summarized an example of such a structure:

> I've taken a lot of math courses and I think that this is the standard pedagogy that I see in math textbooks. So they kind of try to say in words what they're doing, those words get translated into notation and then there's examples.

Another didactical structure is to outline a general procedure and then use that procedure to develop specific applications. In our study, several faculty members described the structure of the chapter introduction, identified how this structure was mirrored in the presentation of the arclength derivation, and used this knowledge to develop a conception of arc length. For example, when asked to describe the book's method for arc length, Professor K framed the entire derivation in terms of the outline in the chapter introduction:

> I'm not quite sure how to describe it, except that *they're following the same steps that they suggested in their box* [in the chapter outline]. So the first step that they have here is that they show how you can find the length by breaking this up into small pieces. So in delta x it's a two-dimensional function, and so delta x and delta y. So they show how you could approximate it. And then they take that into a summation over very small pieces. So you go from these delta y's to derivatives. So that gives me very small pieces. And then they take that sum, and they turn it into an integral. *So they're following their own steps and laying out their procedure.*

There were several faculty members who, at the start of the interview, couldn't describe what Riemann Sums, limits, or integrals were, but identified and used this didactical structure to assist them in constructing a new understanding of what each of these terms meant. In contrast, only one student identified the structure of the chapter introduction and related it to the arc length derivation, but she didn't appear to actively draw parallels between the two structures

**Didactical Authority**

The faculty members also engaged with the text from a position of didactical authority by offering suggestions to change the presentation of ideas to make concepts clearer. For example, Professor M. suggested adding several diagrams to the arclength derivation to illustrate the steps of the derivation:

> The order in which things are presented, I think they could've, you know—it would have taken more paper, but I think they could've done a little bit better job of starting with this curve [points to diagram], just to illustrate what they're talking about, you know. 'Cause like I go straight from here [points to arc length introduction] and I look at this [points to diagram], and I'm like, "What are these?" [points to algebraic expressions in diagram] You know? And they start introducing stuff I haven't read about yet [points to "Length" calculation]. So you know, if I were presenting this in a class, I would sort of say these words [arc length introduction], show this picture [diagram] without these things [blocks off the text in the diagram with his hands]. And then I'd say these words [text following diagram], and I'd start to pop in the new things.

**Recognizing Didactical Conventions/Necessity**

The faculty members also recognized didactical convention of assuming that the readers possess all of the background knowledge that has been previously addressed. For example, when asked whether there were any parts of the textbook that might be difficult for students, Professor I said:

> Hopefully by the time you've got to this point in your calculus book, you know what the relationship is between delta-x and dx, even though that's a little bit fuzzy to me. Or

was—still is a little fuzzy. *So you can't explain what a derivative is every time in the book uses a derivative. You have to remember that's the slope, blah blah blah blah*.

In contrast, several students commented that they wanted the book to re-explain concepts as they were (re-)used.

### Didactical Literacy

As Shepherd and van de Sande (2014) noted, students—particularly at the undergraduate level—are asked to use their mathematics textbooks as a learning tool. Thus, it is important to understand how students read and learn from textbooks. At the same time, we argue that textbooks are a special type of mathematical text. We propose that the literacy most relevant to engaging with mathematics textbooks may also be specialized and merits focused attention.

Richards (2002) noted that "school mathematics" is a different domain of discourse than the domains used by research mathematicians, mathematically literate adults, and academic journals. Love and Pimm (1996) argued that the mathematical writing in textbooks is not just "a special version of mathematics written for a learner" (p. 375), but rather its own type of mathematical text, written specifically for mathematics students; it "provide[s] an authoritative pedagogic version" of mathematics (Stray, 1994, p. 2), is written from the position of a teacher (Kang & Kilpatrick, 1992), employs its own type of rhetorical forms (Fauvel, 1998), and attempts to provoke the development of specific cognitive structures in the reader (Van Dormolen, 1986). Moreover, textbooks employ structural literary devices, including exposition, explanation, questioning, exercises, examples, and formatting, and each of these structures typically has a specific didactical function (Love & Pimm, 1996).

Members of a discipline interact with disciplinary texts in ways that are guided both by disciplinary practices and discipline-specific features of the texts. Moreover, among disciplinary texts, there are *didactical texts*, created for the purpose of teaching. Didactical texts are usually created using particular norms of teaching within the discipline, and these norms are instantiated as conventions in the structure and discourse of the textbook, such as "introduce big ideas to frame later examples" or "connect theory to [what is viewed as] ideas that will be intuitive to students."

To capture the specialized nature of students' and teachers' interactions with didactical texts, we introduce the notion of *didactical literacy*: the capacity to create, interpret, and critique didactical texts using the practices, skills, tools, and ways of constructing, representing, using, and communicating knowledge that are specific to the didactical practices of a particular discipline. Didactical literacy encompasses the knowledge and skills to recognize, interpret, and use these conventions to effectively read the didactical texts—that is, to use, and critique, them as learning tools.

We hypothesize that, like disciplinary literacy, didactical literacy might be specific to particular disciplines. For example, didactical conventions in the teaching of history might be quite different than the conventions in mathematics. In the same way that disciplinary literacy in other STEM fields is similar to mathematics disciplinary literacy, the didactical conventions in other STEM fields might be close enough to those in mathematics to have enabled the faculty participants in our study to recognize and use the didactical structures in the mathematics textbook. However, this is an open question that could be empirically answered in future studies.

Neither disciplinary nor didactical literacy is subordinate to the other. For example, both students and faculty members in this study engaged in various disciplinary literacy practices associated with reading mathematics. However, simply having the capacity to engage in mathematics disciplinary literacy practices was not enough for the students to engage in the didactical literacy practices in the same ways as the faculty members.

Although a certain amount of disciplinary content knowledge is likely necessary to be didactically literate within a discipline, one may engage in didactically literate practices when missing background knowledge relevant to the text. For example, several faculty participants in this study lacked (what we viewed as) essential knowledge for understanding the concepts presented in the textbook, but were able to use the didactical structure in the textbook to construct mathematically accurate meanings for the related terms.

We hypothesize that didactical literacy is distinct from didactical knowledge (i.e., general knowledge about how students learn) and from specialized knowledge about how students learn within the discipline (e.g., mathematical knowledge for teaching (Hill, Ball, & Schilling, 2008)). Didactical literacy is identified with the didactical conventions within a discipline, which may or may not be based on knowledge of how students learn.

## Connections

We believe that the construct of didactical literacy has connections to other research areas in undergraduate mathematics education. It could be used as a way to understand the types of decisions teachers make when preparing materials for their classes. For example, Lai, Mejia-Ramos and Weber (2012) found that mathematicians have various conventions for creating and modifying proofs for the purpose of teaching students; specifically, they advocated for adding introductory and concluding sentences, using formatting to emphasize main ideas, and eliminating redundant information. Although such conventions for constructing or modifying proofs may be accepted by the mathematics didactical community, students may not be in a position to utilize this structure to interpret, construct, and critique mathematical proofs. That is, the ability to engage with these conventions productively may be part of mathematics didactical literacy that students do not generally possess. Didactical (and disciplinary) literacy also has the potential to be used as a theoretical lens to enable researchers to construct alternative explanations for phenomena. For example, Lew, Fukawa-Connelly, Mejia-Ramos and Weber (2016) analyzed students' interpretation of a lecture-based proof and identified both content- and communication-based barriers for the students in comprehending the lecturer's intended main points. Recasting the results in terms of literacy could enable researchers to understand other dimensions of both the structure of the lecture and the students' interpretation, and enable educators to find alternative methods for creating and helping students interpret these texts.

The idea of didactical literacy likely also has implications for teaching. As Fang and Coatoam (2013) noted, teaching disciplinary literacy involves engaging students in the content, literate practices, discourse patterns, and ways of reasoning within the discipline. Enacting disciplinary and didactical literacy practices involves both intellectual and social engagement with the discourse community in the discipline. Thus, helping students use didactical texts effectively may involve making didactical practices explicit and helping students develop the related literacy practices by supporting their participation in the practices of teaching mathematics.

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Drawing on Three Fields of Education Research to Frame the Development of Digital Games for Inquiry-Oriented Linear Algebra


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*Abstract: Demands in undergraduate education are shifting to reach larger student populations - especially learners beyond the brick-and-mortar classroom - which has led to more pressing demands to incorporate technologies that afford such learners access to high-quality, research-based, digital instructional materials. In this article, we explore three theoretical perspectives that inform the development of such instructional materials. In our team's efforts to develop a game-based learning applet for an existing inquiry-oriented curriculum, we have sought to theoretically frame our approach so that we can draw on the corpus of researcher knowledge from multiple disciplines. Accordingly, we will discuss three bodies of literature – realistic mathematics education's [RME's] approach to curriculum development, inquiry-oriented instruction and inquiry-based learning [IO/IBL], and game-based learning [GBL] - and draw on parallels across the three in order to form a coherent approach to developing digital games that draw on expertise in each field.*



*Keywords:* Realistic Mathematics Education, Inquiry-Oriented Teaching, Inquiry-Based Learning, Game Based Learning, Linear Algebra


## Introduction

A number of researchers in undergraduate mathematics education have developed curricula that draw on the curriculum design principles of Realistic Mathematics Education (RME) and are intended to be implemented using an inquiry-oriented (IO) approach (e.g., Larson, Johnson, & Bartlo, 2013 (abstract algebra); Rasmussen et al., 2006 (differential equations); Wawro et al., 2012 (linear algebra)). IO curricula fall within the broader spectrum of Inquiry-Based Learning (IBL) approaches that focus on student centered learning through exploration and engagement (Ernst, Hodge, & Yoshinobu, 2017) facilitated by an instructor's interest in and use of student thinking (Rasmussen, Marrongelle, Kwon, & Hodge, in press). For the purpose of this paper we will give examples from an IO curriculum, but also use quotes and references from the more general IBL literature.

In our current project we are exploring the extent to which technology can help mathematics educators extend inquiry-oriented (IO) curricula into learning contexts that are less conducive to inquiry-oriented approaches. Game Based Learning (GBL) provides a reasonable approach to addressing the constraints that large class sizes or non-co-located learning place on instructors' implementation of IO curricula. GBL studies show a clear relation between games and learning as games provide a meaningful platform for large numbers of students to engage, participate, and guide their learning with proper and timely feedback (Barab, Gresalfi, & Ingram-Goble, 2010; Gee, 2003; Hamari et al., 2016; Rosenheck, Gordon-Messer, Clarke-Midura, & Klopfer, 2016). However, despite advances in technology and policy initiatives that support development of active learning and the incorporation of technology in classrooms, few digital games exist at the undergraduate level that explicitly incorporate a research-based curriculum. In this paper, we explore the three theoretical perspectives of RME, IO/IBL instruction, and GBL in order to identify the ways in which the three perspectives align and might contribute to the development of digital media that incorporate knowledge and practices gained from each perspective.

We begin with a discussion of each of the three theoretical framings illustrated with specific examples. For the first two framings we describe a task sequence and strategies for implementing that task sequence that come out of the Inquiry Oriented Linear Algebra (IOLA) curriculum. For the third framing, we provide a brief outline of a mathematics game, *Rolly's Adventure*, developed by the third author, who drew on GBL principles in her game design. We then draw on each of these examples to demonstrate how aspects of RME, IO/IBL instruction and GBL align with each other and to point out a few ways that RME and IO/IBL might be used to inform design of future games, especially as we, the authors, move towards the development of a new digital game rooted in the existing IOLA curricular materials.

## Realistic Mathematics Education and Inquiry-Oriented Linear Algebra (IOLA)

Realistic Mathematics Education is a curriculum design theory rooted in the perspective that mathematics is a human activity. Accordingly, RME-based curricula focus on engaging students in activities that lend themselves to the development of more formal mathematics. Researchers rely on several design heuristics to guide the development of RME-based curricula (Gravemeijer, 1999; Rasmussen & Blumenfeld, 2007; Zandieh & Rasmussen, 2010). For instance, researchers often focus on the historical development of the concept intended to be taught so that the curriculum supports students' guided reinvention of the mathematics. In this paper, we focus on Gravemeijer's (1999) four levels of activity to show how curricula might reflect the design theory. **Situational activity** involves students' work on mathematical goals in experientially real settings. **Referential activity** involves models-of that refer to physical and mental activity in the original setting. **General activity** involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting. Finally, **formal activity** involves students reasoning in ways that reflect the emergence of a new mathematical reality and no longer require prior models-for activity.

The IOLA curriculum (http://iola.math.vt.edu) draws on RME instructional design heuristics to guide students through various levels of activity and reflection on that activity to leverage their informal, intuitive knowledge into more general and formal mathematics (Wawro, Rasmussen, Zandieh, & Larson, 2013). The first unit of the curriculum, referred to as the Magic Carpet Ride (MCR) sequence, serves as our example of RME instructional design (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). As stated, **situational activity** involves students working toward mathematical goals in an experientially real setting. The first task of the MCR sequence serves to engage students in **situational activity** by asking them to investigate whether it is possible to reach a specific location with two modes of transportation: a magic carpet that, when ridden forward for a single hour, results in a displacement of 1 mile East and 2 miles North of its starting location (along the vector $<1, 2>$) and a hoverboard, defined similarly along the vector $<3, 1>$. As students work through this task and share solutions with classmates, they develop notation for linear combinations of vectors and connections between vector equations and systems of equations, providing support for representing the notion of linear combinations geometrically and algebraically.

The second task in the MCR sequence supports students' **referential activity** – activity in which students refer to and draw generalizations about physical and mental activity, often from the **situational activity** in the original task setting. In the second task, students are asked to determine whether there is any location where Old Man Gauss can hide from them if they were to use the same two modes of transportation from the previous problem. As students work on this task, they begin to develop the ability to conceptualize movement in the plane using combinations of vectors and also reason about the consequences of travel without actually calculating the results of linear combinations. This allows students to form conceptions of how vectors interact in linear combination without having to know the specific values comprising the vectors. The goal of the problem is to help students develop the notion of span in a two-dimensional setting before formalizing the concept with a definition. As with the first task, students are able to build arguments about the span of the given vectors and rely on both algebraic and geometric representations to support their arguments.

As students transition from the second task of MCR to the third, they have experience reasoning about linear combinations of vectors and systems of equations in terms of modes of transportation in two dimensions. In the third problem, students are asked to determine if, using three given vectors that represent modes of transportation in a three-dimensional world, they can take a journey that starts and ends at home (i.e., the origin). They are also given the restriction that the modes of transportation could only be used once for a fixed amount of time (represented by the scalars $c_1$, $c_2$, and $c_3$). The purpose of the problem is to provide an opportunity for students to develop geometric imagery for linear dependence and linear independence that can be leveraged through students' continued **referential activity** toward the development of the formal definitions of these concepts.

In the fourth task, students have the opportunity to engage in **general activity**, which involves students reasoning in ways that are independent of the original setting. In this task, students are asked to create their own sets of vectors for ten different conditions – two sets (one linearly independent and one linearly dependent) meeting each of the five criteria: two vectors in $\mathbb{R}^2$, three vectors in $\mathbb{R}^2$, two vectors in $\mathbb{R}^3$, three vectors in $\mathbb{R}^3$, and four vectors in $\mathbb{R}^3$. From their example generation, students create conjectures about properties of sets of vectors with respect to linear independence and linear dependence. This is **general activity** because students work with vectors without referring back explicitly to the MCR scenario as they explore properties of the linear in/dependence of sets of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$; furthermore, students often extend their conjectures to $\mathbb{R}^n$. Finally, students engage in **formal activity** as they use the definitions of span and linear independence in service of other arguments without having to re-unpack the definitions' meanings. This does not tend to occur during the MCR sequence but rather during the remainder of the semester as students work on tasks unrelated to the MCR sequence.

## Effectiveness and Challenges of Inquiry-Oriented Instruction

Effectively implemented inquiry-oriented instructional approaches have been related to improved levels of conceptual understanding and equivalent levels of computational performance in areas ranging from K-12 mathematics, to undergraduate mathematics, physics, and chemistry (e.g., Cai, Wang N., Moyer, Wang, C., & Nie 2011; Deslauriers, Schelew, & Wieman, 2011; Kwon, Rasmussen, & Allen, 2005; Lewis & Lewis, 2005). To enact an RME curriculum, a classroom must engage students in inquiry into the mathematics of the problems posed. These classrooms are problem-based and student-centered, characteristics that overlap with other Inquiry Based Learning (IBL) and active learning classrooms (Laursen, Hassi, Kogan, & Weston, 2014). Consistent with others in the field (e.g., Kuster et al, 2017), in this work, we consider inquiry-oriented instruction to fall under the broader category of inquiry-based instruction. Research has shown that students who engage in cognitively demanding mathematical tasks have shown greater learning gains than those who do not (Stein & Lane, 1996). Furthermore, Stein and Lane (1996) found that those gains were greater in classrooms where students were encouraged to use multiple representations, multiple solution paths, and where multiple explanations were considered; in contrast, gains were lower in classrooms where the teacher demonstrated a process students could use to solve the task.

Implementation of the MCR task sequence described above is dependent on an inquiry-oriented classroom environment. Rasmussen and Kwon (2007) describe inquiry both as student inquiry into the mathematics through engagement in novel and challenging problems and instructor inquiry into students' mathematics to provide feedback to advance the mathematical agenda of the classroom. The MCR sequence is comprised of tasks that allow for multiple strategies and representations. Since the tasks are non-trivial, students are challenged with debating their answers and explaining their arguments. In addition, Tasks 2 and 3 each allow students to engage in mathematical activity that can be leveraged by the instructor to introduce formal definitions (span in Task 2, linear independence in Task 3). In both cases the instructor serves the role of broker between the classroom community and the mathematical community (Rasmussen, Zandieh & Wawro, 2009; Wenger, 1998) by taking student ideas and connecting them with the formal mathematical definitions. This brokering move of "interpreting between communities facilitates the students' sense of ownership of ideas and belief that mathematics is something that can be reinvented and figured out" (Zandieh, Wawro, & Rasmussen, 2017).

## Game-Based Learning

Game Based Learning (GBL) is the use of digital games with educational objectives to significantly improve learning outcomes. Games are designed to be enjoyable and fun where students overcome challenges and goals (including educational goals) by gaining mastery of the rules within a constrained environment or setting (Dickey, 2005). Research in game-based learning has emphasized the importance of incorporating thoughtful learning theories into the design of games (Williams-Pierce, 2016; Gee, 2005; Gresalfi, 2015; Gresalfi & Barnes, 2016). Recently, there have been several GBL approaches that have been implemented in secondary and post-secondary classrooms (Sung & Gwo-Jen Hwang, 2013; Lester et al., 2014), most successfully when projects have used GBL in conjunction with an existing pedagogical approach (Salen, 2011; Shute, & Torres, 2012). Several learning and pedagogical approaches have been identified that align well with GBL (e.g., Barab, et al., 2012; Hamari, et al., 2016), and many projects approach learning from a constructivist perspective (e.g., Wilson, 1996; Kiili, 2005; Wu, et al., 2012). Curricula developed from constructivist perspectives typically engage students in activities in a problem-solving scenario so that students have opportunities to build on their understanding through reflective abstraction on their prior activity towards more advanced ways of thinking. We illustrate GBL with examples from *Rolly's Adventure* (RA), a videogame developed by the third author to support student learning about fractions.

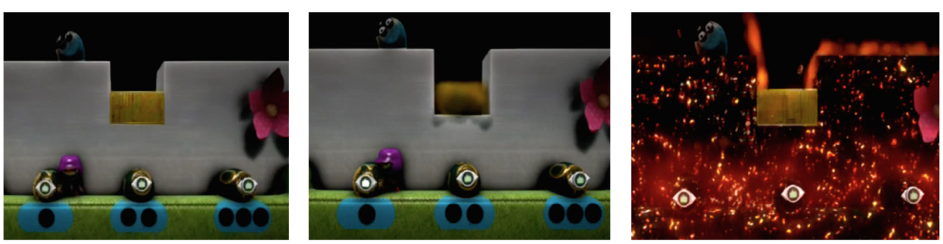

*Figure 1: (a) The player (shown here in a purple helmet) enters the puzzle; (b) the player activates the first button; (c) the puzzle catches on fire.*

RA begins with Rolly in the top left of the screen (see Figure 1). Rolly needs to roll past the obstacle (the gap) in the middle of the screen. The player's avatar is below Rolly in the purple hat. The player can choose from three options to press at the bottom of the screen. If the player chooses incorrectly the area explodes in fire and the golden bricks in the center show the result of the choice (see Figure 1c.)

In Figure 1, the player chose the single black circle and this did not change the size or shape of the golden brick. They then received feedback that their answer was incorrect (the fire that sends their avatar back to start over), and what the direct result of their action was (one black circle results in a single golden brick). Such instantaneous feedback and failure are considered crucial aspects of supporting learning during gameplay (e.g., Gee, 2005; Juul, 2009). If the player chooses the two black circles, the size of the bricks doubles to fill the space and Rolly (and thus the player) is able to move past the obstacle (see Figure 2), thus receiving positive feedback as to the accuracy of their choice.

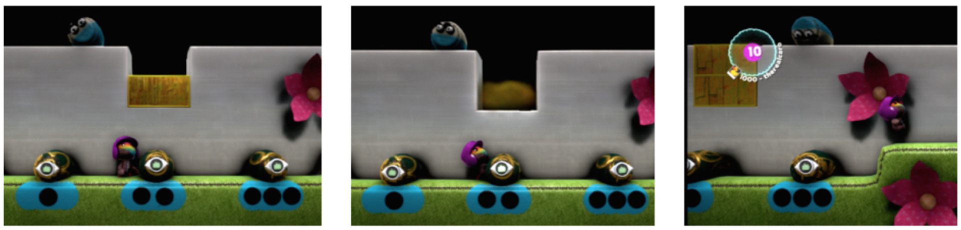

*Figure 2: The player (a) activates the second button, (b) the bricks appear from a haze, and (c) successfully travels over the space now filled with bricks*

As the player progresses through the challenges the brick or bricks in the obstacle will change in relationship to the space, and the way that the choices are indicated will also change. For example, the golden brick in Figures 1 and 2 represent one-half of the hole (the obstacle), and the next puzzle (not shown) has a block that represents one-fourth of the hole, following recommendations that halving a half is a natural next step in the learning of fractions (e.g., Empsen, 2002; Kieren, 1995; Smith, 2002).

*Figure 3: (a) is the fourth puzzle, where the brick is two-thirds of the hole; (b) is the fourteenth and final puzzle in RA, where the brick is one and two-fifths of the hole.*

RA was designed specifically to begin with simpler puzzles and become more complex as players move through the trajectory, such that as players develop generalizations about the game, new puzzles emerge that continue to challenge and nuance these generalizations, so the player has a "pleasantly frustrating" experience (Gee, 2003). Accordingly, mathematical notation becomes introduced that supports the player in being more precise and accurate just as they begin to struggle, as a way of developing a sense of "intellectual need" (Harel, 2013) so that players find the notation immediately useful (following Gee, 2005). Figure 3 shows some examples of how the game becomes more complex. Note that the fourth puzzle (Figure 3a) has bricks that are not an integer multiple of the size of the hole. Correspondingly the player's options include whole and half circles. In the fourteenth puzzle (Figure 3b) the initial bricks are larger than the size of the hole and fraction notation is used to both label the relationship of the brick to the hole (one and two-fifths) and the different choices.

RA was designed specifically with GBL principles to support players in mathematizing their own gaming experience, and engaging in mathematical play (Williams-Pierce, 2017). In this fashion, RA served as a proxy for the role of the instructor in the brokering process (Rasmussen, Zandieh & Wawro, 2009; Wenger, 1998), in that the game required players to act as producers (Gee, 2003) in reinventing the mathematics underlying RA. In other words, an intentionally designed mathematics game can serve as a responsive digital context that mediates interactions between the player, the game, and the mathematical community. Ideally, a well-designed mathematics game uses the principles of failure and feedback to support players in experiencing a pleasantly frustrating and authentically mathematized world. In the following section, we focus more explicitly on how GBL, RME, and IO Instruction can be carefully blended in designs that evoke the best of each world.

## Connecting GBL, RME, and IO Instruction - Blending Theoretical Worlds

The game design principles outlined above and illustrated with Rolly's Adventure align well with the nature of inquiry-oriented instruction using an RME-based curriculum. In Figure 4, we draw heavily on Gee's (2003) notion that good game design is good learning design to show parallels between principles of game design, RME curriculum design, and inquiry instruction and learning. Statements in the boxes of Figure 4 are all quotes or close paraphrases of various authors as indicated.

Looking across the rows in Figure 4 we see that both digital games and RME curricula place importance on the **structure of the task sequence**. The sequence should start with an activity in which students can immediately engage, but that has the potential to be generalized to a more sophisticated understanding that will help in solving more complex problems. We see this both in the increasing complexity of the tasks in Rolly's Adventure (RA) and in the magic carpet ride (MCR) tasks. In particular, student experiences graphically and imaginatively exploring the MCR scenario can be generalized to more formal notions of span and linear independence. As our project progresses, we can envision students being immersed in the MCR scenario through a digital game environment that allows for numerous episodes of growing complexity, from which student generalizations could emerge.

In considering the **nature of the tasks** we see that GBL, RME and IO/IBL all place emphasis on tasks that are novel and ill-structured allowing for a challenging but do-able problem-solving experience. The RA game (Williams-Pierce, in press) and the MCR tasks (Wawro et al., 2012) have both been empirically shown to be challenging, but manageable for students. A digital game based on the MCR sequence would share this novel approach. Through an iterative design process, tasks in the digital game can be created to be challenging but approachable for linear algebra students.

The **teacher's role** in inquiry classrooms is particularly important (Rasmussen & Kwon, 2007; Rasmussen et al., in press). Games can take on some of these roles. A well-designed game can intervene at desired junctures and provide real-time guidance or feedback based on the situation that the player is facing. A game can take on the role of the broker between the player (student) and the larger mathematics community. This brokering occurs both (1) through game play being consistent with the mathematical principles that the students are learning and (2) through students being gradually introduced to accepted mathematical notation and terminology.

| | Theoretical Framing | | |
|---|---|---|---|
| | **GBL** | **RME** | **IO/IBL** |
| Structure of task sequence | Good games confront players in the initial game levels with problems that are specifically designed to allow players to form good generalizations about what will work well later when they face more complex problems. [1] | Lessons should have experientially real starting points and engage in situational, referential, and general activity. [4] | ---- |
| Nature of the tasks | Good games operate at the outer and growing edge of a player's competence, remaining challenging, but do-able **...** **[**therefore**]** they are often also pleasantly frustrating, which is a very motivating state for human beings [1] | Challenging tasks, often situated in realistic situations, serve as the starting point for students' mathematical inquiry. [3] | IBL methods invite students to work out ill-structured but meaningful problems. [2] Students should solve novel problems [3] |
| Teachers' role | Good games give information "on demand" and "just in time," not out of the contexts of actual use or apart from people's purposes and goals… [1] | Teachers to build on students' thinking by posing new questions and tasks. [3] | Students present and discuss solutions; instructors guide and monitor this process. [2] Empower learners to see mathematics as a human activity. [3] |
| Students' role | Games allow players to be producers and not just consumers. [1] | Empower learners to see themselves as capable of reinventing mathematics [3] | Students construct, analyze, and critique mathematical arguments. Their ideas and explanations define and drive progress through the curriculum. [2] |

*Figure 4: Aligning three areas of our team's expertise that inform game design.*
*[1]Gee, 2003; [2]Laursen et al, 2014; [3]Rasmussen & Kwon, 2007; [4]Gravemeijer, 1999*

Ultimately the first three categories are aimed at creating an optimal environment for student learning. The **students' roles** include producing ideas and explanations that allow for their guided reinvention of the mathematics. In RA players create increasingly nuanced generalizations as more complex situations are presented. Student creation of generalizations also occurs in the MCR sequence (Rasmussen, Wawro, & Zandieh, 2015). Our goals as we work toward creating a digital game based on the MCR sequence will be for players of this game to construct, analyze and critique mathematical arguments in the game scenario. For this to happen students need to both (1) experience the mathematical principles/structures through the feedback from gameplay and (2) reflect on their experiences and codify them in some way. In addition to having aspects of the game serve in the teacher role, the game may also need to have aspects that serve in the role of other students in the classroom with whom a student would collaborate in an IO or IBL setting (Ernst et al., 2017).

In conclusion, we believe that these overlapping aspects of GBL, RME and IO/IBL provide a solid starting point for creating a digital game based on the existing IOLA curriculum. As development progresses we will be able to explore affordances and constraints of the digital environment in comparison with the in-person IO classroom.

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Modeling the Spread of Ideas in an Inquiry-Oriented Classroom

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*In this study, we model the spread of student understanding of linear combinations in an Inquiry-Oriented Linear Algebra (IOLA) class based on video analysis. Methods adapted from modeling biological systems were used to estimate the rate of spread of Process-level understanding of linear combinations, measured according to Action-Process-Object-Schema (APOS) theory. The amount of time required for all students to achieve Process-level understanding was also estimated.*

*Keywords:* Inquiry-Oriented, Mathematical Modeling, APOS, Linear Algebra

## Introduction

Over the past thirty years, there has been a movement to use active learning in mathematics instruction or "instructional activities involving students in doing [mathematics] and thinking about what they are doing" (Bonwell & Eison, 1991, p. iii). One instructional design theory is Realistic Mathematics Education (Freudenthal, 1991), which focuses on having students discover mathematical concepts through guided reinvention (Gravemeijer, 2004). Here we examine the spread of ideas in an Inquiry-Oriented Linear Algebra (IOLA) classroom. The tasks used in this study were developed from a larger instructional design project (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). We chose to focus on the spread of the idea of linear combination during the first two class periods of the course, roughly 120 minutes of instructional time.

We used Action-Project-Object-Schema (APOS) theory as a framework to determine whether or not a student "understood" the idea of linear combination (Arnon, et al., 2014). At an Action conception, students are concerned with an external transformation of objects. At the Process level, this activity is interiorized so the student can run through it mentally. At the Object level, that Process is encapsulated into a static entity. At the Schema level, a student is able to coordinate Processes and Objects and thereby act on them. We used Arnon et al.'s (2014) genetic decomposition for spanning set, which included a partial decomposition of linear combination, to determine if a student "understood" linear combinations. In particular, students demonstrating at least a Process-level conception of linear combination were classified as "understanding", specifically:

> Interiorization of [vector addition and scalar multiplication] yields a Process for constructing a new vector which is an element of the vector space, that is, the Process of constructing a linear combination. The reversal of this Process allows the student to verify if a given vector can be written as a linear combination of a given set of vectors. Students who show they have constructed these processes are considered to have a Process conception of a linear combination (p. 36-37).

We operationalized "understanding" linear combinations as verbally articulating a solution method indicating a Process-level conception of linear combination. This includes the ability to add two vectors multiplied by scalar weights and being able to interpret the procedure in at least two contexts (e.g., system of equations, vector equations, graphically).

To follow the spread of the idea of linear combination through an IOLA classroom, we generated two research questions modeled on the language of mathematical biology: (1) what is the infectivity rate for students discussing linear combinations in an IOLA classroom and (2) how long should one expect it to take for all students in the course to reach a Process-level conception of linear combinations?

## Methods

### Data Collection

We watched videos of the first two days of class that recorded three tables of students and captured whole class discussion. We defined a contact as a verbal communication of mathematically relevant content related to linear combinations. We did not consider written work on paper without any verbal explanation to be a contact. Each of the tables' discussions were analyzed for contact rates between members of the table as well as for contacts coming from outside the table. Outside contacts were considered to come from the teacher or the "infected" (understanding) members of the classroom who addressed the whole class. We recorded both contacts from "infected" to "non-infected" (not yet understanding) persons and when we had evidence that a student understood according to our definition.

The coding of the videos for contacts and indications of student understanding was done iteratively. Two researchers independently watched each video and then compared conclusions to check for consistency. After the contacts were counted, we calculated the mean for each student across each researcher's numbers. We only considered contacts for the fifteen individuals at the three tables that we closely observed and assumed their interactions were representative of those in the 35 person class. Due to the variation in duration and types of interactions (e.g. teacher-to-table, student-to-whole class, group member-to-group member), contacts were weighted differently. Contacts between group members at tables were given weight 1 and contacts from the teacher or students from other groups in the class were given weight 0.5 because the group setting allowed for more opportunities for students to engage in each other's reasoning, leading to higher quality interactions. The weighted contact values and times when students understood linear combinations are found in Table 1. Students' names have been replaced with pseudonyms.

### Model Development

We developed an Ordinary Differential Equation (ODE) model and a Continuous Time Markov Chain (CTMC) model for following the spread of understanding. The CTMC was chosen because it better models systems with lower population size than ODE models.

**ODE model.** We used the Susceptible-Infected (S-I) ODE model because we considered all students entering the classroom as being "susceptible" to understanding the concept of linear combination, and we assumed that students who understood the concept did not forget it. Thus there is no "recovery" from understanding, and individuals do not become susceptible again. We assumed that the teacher was the initial infected person, such that $I(0) = 1$ and $S(0) = N - 1$. This yields the following system of equations:

$$\frac{dS}{dt} = -\frac{\beta SI}{N}, \qquad \frac{dI}{dt} = \frac{\beta SI}{N}, \qquad N = S + I$$

The susceptible individuals are the students in the classroom that do not understand yet, and the infected individuals are the individuals that do. The total population, $N = 35$, is the number of individuals participating in the classroom (including the teacher). We used the average number of people from both days as the total population and assumed the size of the population was constant. The parameter $\beta > 0$ is the transmission rate, calculated from the product of the average number of contacts between susceptible and

*Table 1. Students' contacts with understanding persons by researcher and the number of minutes until students displayed Process level conceptions of linear combinations.*

| Table | Student | Weighted R1 | Weighted R2 | Ave Weighted | Minutes to "Infection" |
|---|---|---|---|---|---|
| 1 | Philip | 10 | 9.5 | 9.75 | |
| 1 | Sue | 10 | 9.5 | 9.75 | |
| 1 | Houston | 10 | 9.5 | 9.75 | |
| 1 | Shane | 2.5 | 2 | 2.25 | |
| 1 | Alice | 2.5 | 2 | 2.25 | |
| 2 | Gavin | 1.5 | 0.5 | 1 | 12 |
| 2 | Karl | 4.5 | 8.5 | 6.5 | 23 |
| 2 | Carly | 13 | 21 | 17 | 69 |
| 2 | Mark | 13 | 32 | 22.5 | 76 |
| 2 | Colin | 37 | 67.5 | 52.25 | |
| 3 | Devin | 2 | 2 | 2 | 19 |
| 3 | Ahsan | 4.5 | 12 | 8.25 | 70 |
| 3 | Ernesto | 27 | 23 | 25 | |
| 3 | Kurt | 27 | 23 | 25 | |
| 3 | George | 27 | 23 | 25 | |

infected individuals per time and the probability of infection per contact. Larger $\beta$ values indicate a higher probability of "catching" understanding than smaller $\beta$ values. We assumed that everyone in the classroom had the capacity to understand linear combinations but students did not enter with that understanding, so everyone except the teacher started in the susceptible class.

**CTMC Model.** We also considered a stochastic model using CTMC (Allen, 2008). We define our CTMC on $t \in [0, \infty)$ where $t = 0$ is the time when the class started working with mathematical content on the first day. The states $S(t)$ and $I(t)$ are discrete random variables; that is $S(t), I(t) \in \{0, 1, 2, ..., N\}$ where $N = 35$. Here we are largely concerned with the dynamics for $t \in [0, 120]$, the classroom time in which the students engaged in the tasks. Each random variable depends on the probability functions $p_i(t) = \text{Prob}\{I(t) = i\}$. We assume the Markov property holds: for any sequence of real numbers $t_i$ for $i = 0, 1, ..., n+1$, where $0 \leq t_0 < ... < t_{n+1}$, $\text{Prob}\{I(t_{n+1}) : I(t_0), I(t_1), ..., I(t_n)\} = \text{Prob}\{I(t_{n+1}) : I(t_n)\}$. Thus the transition probability at time $t_{n+1}$ only depends on the most recent time step, $t_n$.

We consider the transition probabilities to be defined for time intervals of length $\Delta t = 1$ minute. We assume this is sufficiently small so at most one person is infected in a single time step. Under this assumption, the transition probabilities are as follows:

$$p_{ji}(\Delta t) = \begin{cases} \frac{\beta i(N-i)}{N}\Delta t + o(\Delta t) & j = 1 \\ 1 - \frac{\beta i(N-i)}{N}\Delta t + o(\Delta t) & j = 0 \\ o(\Delta t) & j \neq 0, 1 \end{cases}$$

Case $j = 1$ gives the probability that one individual transitioned from not understanding to understanding in a time step, $j = 0$ is the probability of no change, and $o(\Delta t)$ is the probability that more than one individual is infected at one time (assumed to be 0). To simplify the probability expression, let $\beta i(N - i) = b(i)$. Then $b(i)$ represents the infection rate of individuals at time $\Delta t$. The simplified version of the model is as follows:

$$p_{ji}(\Delta t) = \begin{cases} b(i)\Delta t + o(\Delta t) & j = 1 \\ 1 - b(i)\Delta t + o(\Delta t) & j = 0 \\ o(\Delta t) & j \neq 0, 1 \end{cases}$$

**Simulation Model**

We used the values in Table 1 to determine the average number of contacts for each of the students at the three tables. We then divided this by 118, the total number of minutes of instruction across the two days. This produced $\beta = 0.1233$. We also found the "best fit" $\beta$ value according to the deterministic ODE model, which was $\beta = 0.0426$ according to nlinfit in Matlab. The code used to generate figures was adapted from a Mathematical Modeling course (L. Childs, personal communication, March 23, 2017) and Allen (2008).

## Results

### Analytical Results

There are two steady states for the ODE system: "disease-free" $(N, 0)$ and "endemic" $(0, N)$. We want understanding to be endemic; that is, we want the entire population to understand. To study the steady states, we used the Jacobian method which yielded eigenvalues of $J(0, N)$ of $\lambda_1 = 0$ and $\lambda_2 = -\beta$. We note that $\beta > 0$; thus $(0, N)$ is always stable, provided there is one infected person initially. Using the same method it is easy to see $(N, 0)$ is always unstable when there is an initial infected person.

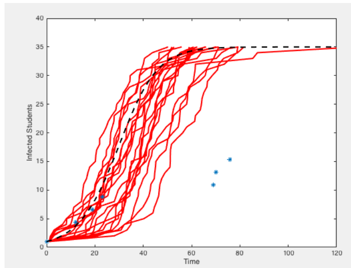
For the stochastic model, we consider the inter-event time, $T_i$, which is the expected time until everyone is infected. $T_i = -\frac{\ln(1-U)}{b(i)}$ is an exponentially distributed random variable with parameter $b(i)$ (defined above) and uniform random variable $U$ on $[0, 1]$. We use this to calculate the expected time until everyone understands, which is equivalent to reaching the endemic steady state. Recall that $I(0) = 1$. Thus the first state is one infected person. We know the expected time to reach the next state (two infected people) is $T_1$. Let $W$ be the time it takes to get from state 1 to state 35. Then $W = \sum_{i=1}^{34} T_i$. Because $T_i$ is exponentially distributed with parameter $b(i)$, the expected value is given by $\mathbb{E}(T_i) = \frac{1}{b(i)}$. Furthermore, because $\mathbb{E}$ is a linear operator, $\mathbb{E}(W) = \sum_{i=1}^{34} \frac{1}{b(i)}$.

### Simulation Results

In Figure 1, we see a comparison of the scaled data points extrapolated from the observed fifteen students to the whole class, the deterministic solution to the ODE, and 20 stochastic paths. In Figure 1A the first three data points are close to the deterministic solution and in the midst of the stochastic paths. However the three later data points are far below the deterministic and stochastic paths. Although $\beta = 0.1233$ is based in our classroom observations, it does not appear to model when students obtain Process conceptions of linear combination, though it did perform better than the non-weighted version. The non-weighted version, based on counting contacts from the teacher, students from other tables, and students from the same table equally (Figure 1B) used $\beta = 0.1870$, which fit even worse.

Unlike the curves in Figure 1, which matched the early data points well but missed later points badly, the curve in Figure 2, which is based on the "best fit" $\beta = 0.0426$, misses most of the early data points but fits the later points well. Additionally, at 120 minutes, not all students are "infected". This fits the data better because a number of our students did not appear to obtain a Process conception of linear combination by the end of the 118 minutes of class time we observed.

Furthermore, using $\beta = 0.1233$ produced an expected wait time for all students to obtain a Process conception of linear combination of 67 minutes, roughly half the time we observed the class. When using $\beta = 0.0426$, which represents a reduced probability of infection, the expected time for all students to understand was 193 minutes. This amount of time seems more plausible because only six of the fifteen students we observed in detail appeared to have obtained Process level conceptions by the end of the second class period.

A 

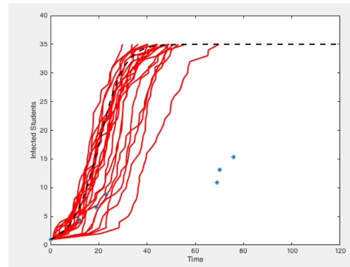

B 


*Figure 1. These graphs use $N = 35$, $I(0) = 1$, and a time step of 1 minute. The black dashed lines represent the deterministic solution to the ODE. The 20 red lines on each graph are stochastic simulations. The blue stars represent the data points obtained by scaling the fifteen students' data to the full class of 35 people. The left simulations (A) ran with $\beta = 0.1233$ and the right simulations (B) used $\beta = 0.1870$.*

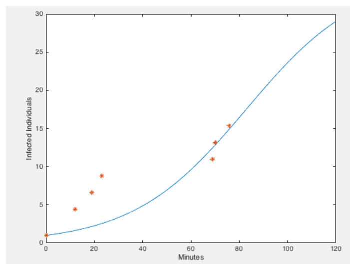



*Figure 2. This graph uses $N = 35$, $I(0) = 1$, initial guess $\beta = 0.1$, and a time step of 1 minute. The blue line represents the number of infected individuals at a given time. The orange stars represent the data points obtained by scaling the fifteen students' data to the full class. The resulting curve results when $\beta = 0.0426$.*

## Discussion

Our goals were to determine the infectivity rate for students discussing linear combinations in an IOLA classroom and to determine the expected time for all students in the course to reach a Process-level conception of linear combinations. We determined that the infectivity rate $\beta = 0.1233$, based on creating weighted contact values, estimated more effectively than the non-weighted contact values, but still did not estimate very well. However, when we used maximum least squares to estimate the infectivity rate from the infection data, we obtained $\beta = 0.0426$, which produced more reasonable long term results, including requiring slightly more than another full period for full class understanding.

We acknowledge that this study and model have a number of limitations, including incomplete classroom data, discontinuous data collection, and a non-constant population of students. Specifically, while we have direct information about 15 students and the teacher, we have limited information about the other 19 students. Even in the data on the 15 students, we were, if anything, a bit conservative on saying a student understood the concept of linear combination. It is possible that students who did not speak as often in the groups also understood; we simply did not have enough evidence to be sure that they did.

Future work could include further refining $\beta$ through more extensive data collection. This could involve observing subsequent class periods to see if 193 minutes was sufficient time for everyone to understand linear combinations. Alternatively, future data collection could involve setting up cameras at more students' desks. We could also refine the weighting of contacts. Perhaps group contacts should be weighted even more heavily or perhaps specific "infected" individuals, like the teacher, have a greater impact than other individuals. Finally, we could consider implementing an age structure or risk structure model instead of basing the model on Markov Chains.

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Curricular Presentation of Static and Process-Oriented Views of Proof to Pre-service Elementary Teachers

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*Engaging students in proof-related reasoning is an important but often challenging task for pre-service elementary teachers. Given that limited mathematics content courses and their associated textbooks offer some of the only opportunities for preservice elementary teachers to engage with proof, it is vital to understand what opportunities they offer to understand proof. I conducted an analysis of two textbooks used for elementary mathematics content courses to investigate the view(s) of proof promoted within and the opportunities to learn about proof-related reasoning. My findings suggest a mixed emphasis on static and dynamic views of proof and proving, but also many opportunities for instructors of mathematics content courses to promote an explanatory, process-oriented view of proof.*

*Keywords:* Preservice Elementary Teachers, Proof, Curriculum Analysis

## Introduction

Increasingly, calls for mathematical reform in the U.S. specify that mathematics instruction should enable students to "recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proof, and select and use various types of reasoning and methods of proof" (NCTM, 2000, p. 56). These calls reflect a larger effort to align classroom work in mathematics more closely with the discipline of mathematics, especially at elementary levels where proof has not historically been emphasized (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). However, existing research indicates that those preparing to teach elementary mathematics have difficulty recognizing an argument as proof (Martin & Harel, 1989) and interpreting and evaluating students' proofs and arguments (Morris, 2007). Likely due to the historical disconnect between proof and other aspects of K-12 mathematics curriculum (Herbst, 2002), pre-service elementary teachers (EPSTs) may not have had meaningful experiences with proving in their own education. Further, the curriculum materials from which they are to teach do not offer adequate support in the area of reasoning and proving (Stylianides, 2007). For these reasons, it is vital that we understand how EPSTs' mathematics content courses provide to develop understandings of proof and proof-related reasoning. These content courses represent some of the first, and potentially only, opportunities for pre-service teachers to engage with proof-related reasoning prior to being expected to teach it to their students, so these courses have a potentially powerful influence on how EPSTs will eventually teach.

One way to understand the opportunities EPSTs have to learn about proof and proving in their preparation coursework is to unpack such opportunities as written in the textbooks used for elementary mathematics content courses. Though the written opportunities may not align with the opportunities as enacted in a course, textbooks influence an instructors' course design and shape EPSTs' independent learning opportunities (McCrory & Stylianides, 2014). This study, broadly, aimed to understand the view(s) of proof promoted in these texts and the opportunities provided for EPSTs to learn about the nature of proof, proving and proof-related reasoning.

## A Vision of Proof Communicated to Pre-Service Elementary Teachers

Numerous scholars have drawn attention to how our conceptions of what proof and argument are, as well as what occurs when we engage in proving and argumentation, shape not only the conclusions researchers make about students' thinking and instructional practice related to proof and argumentation (Balacheff, 1988; Stylianides, Morselli & Bieda, 2016) but also how teachers shape how students think about proof and proof-related activities (Conner et al., 2014). Thus, what kind of conceptions should EPSTs have about proof and argument to support the development of student thinking and mathematical practice as proposed in calls for reform?

Schoenfeld (1991) states, "In real mathematical thinking, formal and informal reasoning are deeply intertwined" (p. 311). Proof can serve several important functions: verification, explanation (that is, offering insight into why a statement is true), systematization of axioms, discovery of new results, communication of mathematical knowledge, construction of an empirical theory, exploration of mathematical statements' meaning or consequences, and incorporation of mathematical truths into alternative frameworks (Hanna, 2000). Thus, if classroom practice should align with disciplinary practice, teachers must be familiar with many of these aspects and be able to link informal, exploratory activities with reading or writing formal proofs in order to help students learn mathematics as they prove. Especially in the early grades, explanation, communication, and exploration will have great importance (Hanna, 2000) and engaging with proof in this way can contribute to a process-oriented view of proof (Schoenfeld, 1991). For EPSTs to support their students to engage in proving in this way, it is important that they experience and understand proof as a process.

Existing literature suggests that it is more common for EPSTs to hold more ritualistic and empirical understandings about proving and argumentation that may stem from experiences with a more static view of proof (Harel & Sowder, 1998; Morris, 2002). Classroom practices in content and methods courses for EPSTs should be structured to promote a dynamic view of proof. In a dynamic view of proof, the process of constructing a proof *is* the vehicle for mathematical learning (Schoenfeld, 1991). In this way, a dynamic view of proof relates integrally with the purpose of explanation promoted by Hanna (2000).

From these assumptions about the aspects of proof that EPSTs must have experience with and promote in their teaching practice, this study investigated the nature of opportunities provided in textbooks for mathematics content courses, guided by the questions: *Within the mathematics content textbooks considered here, which functions of proof are promoted, and by extension, which functions of proof will pre-service elementary teachers become familiar with?*

## Methods

The textbooks analyzed were *Mathematics for Elementary Teachers* by Beckmann (2003) and the series *Elementary Mathematics for Teachers* and *Elementary Geometry for Teachers* by Parker and Baldridge (2004, 2008). Both texts are written for use in a year-long mathematics content courses for pre-service elementary teachers, who are assumed to receive no other specialized mathematical training. Both texts were identified by McCrory and Stylianides (2014)'s analysis of tables of contents for elementary mathematics content course texts to provide explicit treatment of proving and argumentation.

Focusing solely on student materials, rather than teacher's guides, I analyzed each textbook page-by-page to identify instances of proof and proof-related reasoning. Analyzing student materials can provide insight into the opportunities to learn about proof regardless of what occurs in class, which is important because textbooks can be utilized in a variety of circumstances

(McCrory & Stylianides, 2014). An analysis of student materials also informs instructor planning around material that might need to be supplemented or particularly emphasized outside of the textbook. I generated categories of proof-related reasoning present grounded from the textbooks themselves, and narrowed to a list of three categories that could be used to code each of the instances I analyzed:

1. Complete general proofs presented somewhat in isolation;
2. Exploration of a topic which the textbook author links explicitly to proof, for example by stating that the mathematical statement can be proved or by assigning a proof to the student as homework; and
3. Exploration of a topic which leads to a complete deductive proof within the text.

For each instance of proof related reasoning, I recorded the forms of reasoning present in exploration and/or final proof as either: written reasoning, symbolic/algebraic reasoning, diagrams/graphs, specific worked examples, and imagined motion (e.g. sliding, turning). I also noted the mode of argumentation (e.g. direct logic, construction of counterexamples, proof by contradiction), to gain insight into how proof is presented overall in these texts. Finally, for any instances of proof-related reasoning that included exploration I noted whether this led to the emergence of a key idea.

To demonstrate this coding process, consider the proof that the base angles of an isosceles triangle are congruent from Parker and Baldridge (2008). Initial exploration for this is presented in discussions of symmetry, where the text invites readers to fold a paper isosceles triangle along its line of symmetry to discover matching parts (Parker & Baldridge, 2008, p. 44). Since at this point this is not a general argument, but pertains to one particular paper triangle, this represents exploration. The forms of reasoning present in this exploration are a diagram (to describe the construction of paper isosceles triangle), a specific example of one isosceles triangle, and a description of imagined motion (folding the triangle). Further along in the text, a general proof that the base angles of an isosceles triangle are congruent is presented to the reader in the form of a "teacher's solution", which closely mirrors a two-column proof (Parker & Baldridge, 2008, p. 92). The forms of representation in the proof were coded as minimal text, a symbolic representation to describe congruent parts of the triangle, and a supporting diagram. The mode of argument was coded as direct because the argument begins with given information and follows a direct chain of logic to the conclusion.

In addition to looking for instances of proof-related reasoning, I noted any statements by the author that included reference to proof or mathematical reasoning, but were not connected to specific instances of proof-related reasoning. These include definitions and descriptions of proof, descriptions of unfamiliar modes of argumentation, advice on how to write proofs, and statements about the nature of mathematical knowledge more generally. I examined these statements and the instances of proof-related reasoning identified in order to learn how a dynamic view of proof is (or is not) promoted in the textbooks.

## Findings

### Conceptions of What Proof Is and Who Does Proof

My analysis revealed that both textbooks included written descriptions, or definitions, of proof. In Parker and Baldridge (2004), proof is defined as "a detailed explanation of why that fact follows logically from statements that are already accepted as true" (p. 110). This definition highlights the purposes of verification, communication, and construction of an empirical theory. Even though the word "explanation" is included in this definition, the authors do not seem to indicate that this refers to explanation of the underlying mathematics in order to provide insight into the statement's truth. By this definition then, it is enough simply that the statement *is* true. Moreover, this definition highlights proof as a finished object, inconsistent with the process-oriented view of proof that pre-service elementary teachers will need to effectively engage their students in learning through proof.

Similarly, Beckmann (2003) chooses to use the term "explanation" in defining proof: , "Proofs are one of the important aspects of this book too, even if we don't usually call our explanations proofs. A proof is a thorough, precise, logical explanation for why something is true, based on assumptions or facts that are already known or assumed to be true. So a proof is what establishes that a theorem is true" (p. 213). Like Parker and Baldridge (2004), Beckmann's definition highlights the purposes of verification, communication, and construction of an empirical theory. Additionally, though, Beckmann (2003) links proof to other types of mathematical explanation, suggesting proof is useful for gaining insight into the mathematics and connecting between informal and formal reasoning. While this definition of proof discusses its value as a product of verification, it hints at the process involved in its creation.

Beyond exact definitions, these texts include written descriptions of the nature of proof. Beckmann (2003) describes a process-oriented view of proof, stating "…mathematics is about starting with some assumptions and some definitions of objects and concepts, discovering additional properties that these objects or concepts must have, and then reasoning logically to deduce that the objects or concepts do indeed have those properties" (p. 65 of volume 2). However several times throughout the text, pre-service teachers are asked to do exercises that warrant a proof but the text indicates that they are not expected to generate such a proof.

Parker and Baldridge (2004) also indicate that there are many forms of mathematical reasoning involved in and related to proof: "In the classroom the reasoning occurs in explanations and guided investigations, while in mathematics textbooks the reasoning often occurs in formal and informal 'proofs'" (p. 109). In this statement the authors do not specifically present proof as a process, but they do present proof as the domain of mathematics textbooks and the mathematicians who write them, and not an area where teachers are expected to be involved. This suggests that pre-service elementary teachers are expected to read and understand proofs, but not write proofs themselves. However, in the context of geometry, Parker and Baldridge (2008) occasionally present "hints" for how pre-service elementary teachers, and eventually their students, might approach specific proving tasks.

**Instances of Proof-Related Reasoning**

Within Parker and Baldridge (2004, 2008) I identified 68 total instances of proof-related reasoning and within Beckmann (2003) I identified 25 total instances of proof-related reasoning. This amounts to one instance of proof-related reasoning approximately every seven pages in Parker and Baldridge, and approximately every 32 pages in Beckmann. The number of instances of proof-related reasoning in each category (proof only, exploration only, and exploration and proof) for each text is listed in Table 1.

*Table 1. Categorization of instances of proof-related reasoning within each text*

| | Parker and Baldridge (2004, 2008) | Beckmann (2003) |
|---|---|---|
| Proof Only | 25 | 6 |
| Exploration Only | 8 | 7 |

| | | |
|---|---|---|
| Exploration and Proof | 35 | 12 |
| Total | 68 | 25 |

The mode of argumentation for almost all instances of proof and proof-related reasoning from Table 1 is direct, where an initial assumption begins a chain of deductive logic (i.e. modus ponens). The only exceptions to this mode of argumentation are two instances in Parker and Baldridge (2004), where proof by contradiction is utilized. The prevalence of direct proofs within both texts indicates a high value on the transparency and understandability of mathematics. Indeed, Beckmann (2003) describes that a good mathematical explanation, which she uses in lieu of the term proof, is clear and logical, without "[requiring] the reader to make a leap of faith" (p. 11). In addition, Beckmann specifies that these should be convincing and usable for teaching, which highlights the possibility that proofs might give insight into the mathematics, and further, might be useful for learning mathematics.

Within Parker and Baldridge (2004, 2008), 27 of the instances of proof take the form of a "Teacher's Solution" or "Elementary Proof" that are similar to two-column proofs from standard geometry textbooks. These instances provide only the minimum number of highly-abbreviated steps. Further directions on the formatting of these proofs specified, "Do not label the two columns 'statement' and 'reason' (everyone already knows this!)" (Parker & Baldridge, 2008, p. 79). Given the similarity of this form to two-column proofs that highlight verification of facts (Herbst, 2002), these elementary proofs do not readily serve the function of explanation. In addition, their emphasis on abbreviation and standard form may distance the final proof from the process that was conducted to create it.

## Conclusion

Through my analysis of both texts, including written statements regarding the nature of proof, the form and modes of argumentation within opportunities to prove, and the emergence of key ideas, I found evidence of both dynamic and static conceptions of proof. The strong presence of direct proving methods that aid in mathematical understanding, as well as inclusion of exploratory processes that lead to general proofs, often with consistent key ideas linking these stages, offer pre-service teachers opportunities to develop a dynamic view of proof. Many written statements and the prevalence of rote modes of argumentation and purely symbolic proving methods, however, offer pre-service teachers opportunities to develop a static view of proof. Given that the two texts analyzed represent strong inclusion of proof-related reasoning compared to other elementary mathematics content texts (McCrory & Stylianides, 2014), it is likely that other texts present potentially a more static emphasis on proof.

These findings suggest the possible value of supplementing these texts with additional opportunities from instructor materials or other sources to emphasize a dynamic view of proof. Classroom enactment, also, should be structured by instructors to build on existing opportunities to see proof as dynamic and directly confront messages that promote a static view of proof. Considering that roughly half of the instances of proof-related reasoning include an entire reasoning process that consists informal exploration and a complete general proof, there is much within these texts that offers opportunities to engage in the process of proof-related reasoning. It is on this that instructors for content courses for pre-service elementary teachers must build.

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Mathematical Competencies and E-Learning: A Case Study of Engineering Students' Use of Digital Resources

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*This paper explores how an e-learning environment affords the execution of mathematical competencies in an undergraduate engineering context. Considering the students' mathematical practice as action mediated by the digital resources in a sociocultural sense, we employ the competence framework by (Niss & Højgaard, 2011) to make sense of students' learning. Case-study research design has been implemented to thoroughly observe the mathematical practices of a small group of participants. Observing students' group work and following their mathematical discussions elucidated the way this environment afforded the execution of competencies. Closer analysis revealed that the availability of online tools in this environment has the twofold effects on mathematical thinking, mathematical reasoning and problem-tackling competencies.*

*Keywords*: Calculus, Engineering mathematics, E-learning, Mathematical competencies.

## Introduction

The use of digital resources in mathematics education has started since the development of such tools and is still being researched to study its impact on mathematical learning. Increased dependence on digital tools for practicing mathematics is transforming the mathematics education, and to learn mathematics is not the same as it was before the introduction of digital technology. The use of digital resources is of particular relevance in engineering mathematics in the sense that modern-day engineers during their professional activities rely on technology for mathematical tasks (van der Wal, Bakker, & Drijvers, 2017). The framework for mathematics curricula in engineering (Alpers et al., 2013) also recommends how technology should contribute towards fostering the engineering students' mathematical competencies (Alpers et al., 2013). The notion of mathematical competence from the Danish KOM project (Niss, 2003; Niss & Højgaard, 2011) has been adopted to make sense of the engineering students' mathematical learning.

Previous research studies have also employed this competence framework, either to make sense of students' learning in mathematics or to analyse how these competencies are developed in particular situations or through certain activities. For instance, Jaworski (2012) used Niss's idea of mathematical competencies to design and analyse the tasks and to recognise the engineering students' mathematical learning. Jaworski pointed out that a potential use of the competence framework may be to create opportunities for students to achieve certain competencies (Jaworski, 2013). Furthermore, Albano and Pierri (2014) used a role play activity and identified the first-year engineering students' mathematical competencies through the questions students asked. Albano and Pierri concluded that students seemed to possess all the competencies by Niss (2003) which were evident through the words they used in their questions. García, García, Del Rey, Rodríguez, and De La Villa (2014) presented a model for the integrated use of CAS which they implemented and analysed in engineering classrooms. They suggested that the use of CAS in all learning and assessment activities has the potential to positively influence the development of mathematical competencies. Recently, Queiruga-Dios et al. (2016) analysed the development of mathematical competencies among industrial engineering students through their teamwork which included the use of CAS for solving mathematical problems as an integral part. While their main aim was to integrate these mathematical competencies with the required engineering competencies in Spain, they claimed that the students acquired all the mathematical competencies during this task.

Our study focuses particularly on nature of mathematical competence afforded by an e-learning environment. Realising the contemporary and the future state of mathematics education, we attempt to add to the research literature within the context of engineering mathematics education. In this paper, we analyse engineering students' engagement within a calculus course to report on how their mathematical competencies are supported within an e-learning situation. We attempt to answer the following research questions: What traces of mathematical competencies are observed in students' work when they practice mathematics digitally? How does this environment afford the execution of these mathematical competencies?

## Theoretical perspective

We consider students' mathematical practice in the present situation as mediated action in sociocultural terms (Vygotsky, 1978). The provided resources which support the learning of mathematics serve as mediating artefacts between students and the mathematical concepts. The mediating artefacts used in the present situation are MyMathLab, tutorial videos, textbook, Maxima for programming, and other internet-based resources. The students' homework and eventually the students' assessments are done digitally. There were no regular face-to face lectures thus the situation is considered as e-learning in which students remotely work with the resources. A brief introduction of these resources follows.

MyMathLab is an online interactive learning environment for practicing mathematics digitally. While the main aim of this resource is to provide a platform for digital homework and assessments, it also facilitates in solving the tasks by providing illustrated worked examples and personalised feedback. The tutorial videos replace traditional university lectures and are linked topic-wise with the textbook sections. The videos are recorded by the mathematics teacher using a document camera, and they consist introduction to each mathematical topic along with worked examples. The tutorial videos and the homework in MyMathLab were clearly linked with the chapters in the textbook.

We employ the competence framework by Niss and Højgaard (2011) to make sense of engineering students' mathematical learning (Jaworski, 2012, 2013). The framework is complemented by sociocultural notion of resource mediation. The Danish KOM project (Niss & Højgaard, 2011) enlisted eight mathematical competencies, divided into two groups as follows (Figure 1):

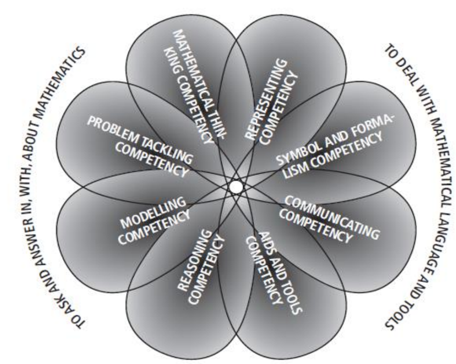


*Figure 1: A visual representation of eight mathematical competencies (Niss & Højgaard, 2011, p. 51).*

### The Ability to Ask and Answer Questions in and with Mathematics

The first group comprises the competencies of mathematical thinking, mathematical reasoning, problem tackling, and mathematical modelling. Mathematical thinking competency involves "awareness of the types of questions which characterise mathematics" (Niss & Højgaard, 2011, p. 52) and "being able to recognise, understand and deal with scope of given mathematical concepts" (Niss & Højgaard, 2011, p. 53). Mathematical reasoning includes following and assessing chains of arguments, comprehending a mathematical proof, and devising formal and informal mathematical arguments (Niss, 2003). In the present study, the proofs were not a part of the mathematics curriculum. Thus, the reasoning competency is only observed within the context of problem solving. Mathematical modelling is neither a part of the curriculum in the present situation.

### The Ability to Deal with Mathematical Language and Tools

The second group includes the competencies of representing mathematical entities, handling mathematical symbols and formalism, communicating in, with and about mathematics, and making use of aids and tools.

## Research Design and Methods

This study is carried out following a case study design (Yin, 2013) and the data has been collected in a Norwegian public university. A small group of three male students, enrolled in the first year of an electronics engineering program, has been observed over the whole semester. The methods used to generate data include group observations, group interviews, individual weekly journals and field notes by the researcher.

For the participant observations, video recordings of their group work, and screen recordings to follow the activity on computer screens have been collected. Additionally, participants provided screen recordings of their individual work, and weekly journals containing self-reports about the use of resources for practicing mathematics. In this paper, we analyse three episodes of the students' group work in order to look for how these competencies are supported in an e-learning environment.

## Analysis

The two sets of competencies are not mutually disjoint, in general, and are intertwined which is evident from the so-called competency flower. Although each competency has a well-defined identity in theory, execution of each competency in practical will draw on some other competencies. This makes it empirically challenging to disentangle one competency from the others (Niss, Bruder, Planas, Turner, & Villa-Ochoa, 2016). We adhere to these considerations and our purpose here is to rather we look for possibilities in which e-learning influences each sets of competencies.

In the quest for finding correct answers to the given tasks in present situation, participants needed to go through certain procedures where they could demonstrate these competencies. Geogebra (https://www.geogebra.org/) and WolframAlpha (https://www.wolframalpha.com/) were main tools used by the students to make sense of various mathematical functions, checking for the functions' behaviour and to look at the solutions of the tasks. Textbook served as a main written help material in terms of consulting for mathematical formulas, explanations or illustrations, and for checking whether their solutions were correct by comparing these with the answers to tasks provided in the end of textbook. At several occasions, the textbook served as an aid to get acquainted with the mathematical topics, as the students read the textbook to understand the mathematics. The introduction of Maxima was done in a project in this course, and the purpose was to make engineering students capable of using this programming language to solve mathematical problems thus it also served as a resource.

The exposure to Google and different online calculators, in this case, for finding solutions of the given tasks, has shared the role for computing and calculating the solutions. We noticed that in participants' arguments, the element of tool dependence was evident.

In this regard, WolframAlpha and GeoGebra have a central role, since it in the present situation supported students in making sense of the functions, expressions and mathematical concepts in different ways. For example, when the students were not able to solve an integral $\int_0^1 \frac{\sin(x)}{x} dx$ by programming with Maxima, they started wondering whether it was solvable at all, and they used WolframAlpha to make sense of the scope of the task or to know the answer:

Per: (…) Maybe it… (we) can't solve it? Have you tried Wolfram? [Per is addressing Jan and visits WolframAlpha website himself. Per has looked up $\int_0^1 \frac{\sin(x)}{x} dx$ on WolframAlpha (Figure 2)]

Per: No, you're supposed to get an answer.

In this example, when asked by Jan, Per was trying to handle the scope of this integral. He used WolframAlpha to see what this integral is all about, and based on the output, he decided that it could be solved. This example illuminates how the mathematical thinking and problem-tackling competencies are being executed along with the obviously observed aids and tools competency.

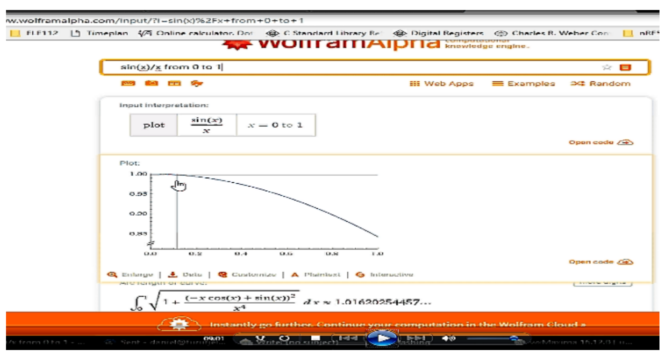


*Figure 2: Screenshot of a participant's work on WolframAlpha.*

The online tools mediated in the students' abilities to think and reason mathematically either by providing the complete calculations or the opportunities to explore the tasks at hand. By using paper and pencil techniques, both of these functions require a different kind of knowledge and skills as it says in the competence framework.

The following excerpt indicates how this environment is supporting the competencies of dealing with mathematical language and tools. While trying to solve a definite integral $\int_{-1}^{1} e^{-\sqrt{-1}wt}\, dt$ using Maxima, they got apparently a different outcome than what it said in the book.

Per: It is the same? It is the same thing, just written in a different way.
Jan: Yeah
Per: Simplify [Per tries to use the "simplify" command on the expression in Maxima]
Jan: Yeah, it just looks that much nicer when you do it in…
Per: In Wolfram.
Jan: Yeah. Yeah, or at that. Did you get…You got the same in Wolfram?

Per: Nnn… I haven't checked it. I assume I get what it says in the book.

[Per looks up $\int_{-1}^{1} e^{-\sqrt{-1}wt}\, dt$ in WolframAlpha.]

Per: Then I get sine w to…2 sine w divided by w, and that's exactly the same as it says in the book.

Jan: There, it… If you go back. Wolfram has moved -1 outside.

[Jan is trying to make Per aware how WolframAlpha has changed the representation.]

Per: Where?

Jan: Put the square root outside the parentheses.

Per: Yeah, but that's just if… I don't think it matters if...

[Meanwhile Per writes the original expression slightly differently in Maxima and gets the same output]

Per: It is exactly the same. I think it is correct.

Here, Per and Jan were trying to make sense of the different representations of the expression when both resources offered the result in a slightly different manner. The second set of competencies concerning representing mathematical entities, handling mathematical symbols and formalism, communicating in, with and about mathematics, and making use of aids and tools are *in action*.

An interplay of different resources had also been helping to approach a given task from different perspectives and to gain more information about the task in hand. Also, the use of Maxima apparently seemed as a short cut for getting ready-made answers. However, it has been observed that it required some effort from the students to decode the mathematical language into programming language.

## Discussion

We intended to look for the execution of mathematical competencies in an e-learning environment in our case, and the findings of this study differ from the previous findings by (García et al., 2014). We found that while this environment supports some competencies, it does not ensure enhancing all of these in all learning environments. The way in which this online learning environment provides possibilities for practicing mathematics makes it different from the traditional way of doing mathematics in a paper and pencil environment.

For instance, from the first set, when the competencies of thinking and reasoning mathematically have to be executed in an online environment. We conjecture that the effects are twofold. On one hand, the resources are facilitating in computing, calculating and providing answers requiring less effort from the students thus limiting the possibilities for exploration. However, on the other hand, when used for comprehension of the tasks at hand they have potential to enhance the possibilities of exploration. We further observed that e-learning is certainly not on the same lines as it means to think and reason mathematically in a traditional way. In a traditional paper and pencil environment, students use their own knowledge and skills for performing the tasks at the hand.

The second set of competencies has more scope in the present context owing to the use of different tools and aids for practicing mathematics. When students used different tools for practicing mathematics, and each one of those tools uses different symbolism which provides some opportunities for the students to experience and handle varied mathematical formalism in a way.

Question for discussion: How to devise a better systematic scheme for analysing mathematical competencies in this environment?

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## Shape Thinking: Covariational Reasoning in Chemical Kinetics

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*This work addresses the following research question: In what ways do students use mathematics in combination with their knowledge of chemistry and chemical kinetics to interpret concentration versus time graphs? The study was designed and implemented using a resource-based model of cognition as the theoretical framework. Data was collected through the use of an assessment involving short-answer test items administered to 109 students in a first-year, non-majors chemistry course at a Swedish university. The student responses were translated from Swedish to English and subsequently coded. Data analysis involved using the shape thinking perspective of graphical reasoning as a methodological framework, which was adapted to analyze the covariational reasoning used by students in the context of chemical kinetics. Open-coding and considerations of shape reasoning have provided insight into student understanding of mathematical models of chemical processes.*

*Keywords*: Graphing, Covariational Reasoning, Shape Thinking, Rates, Chemistry

### Introduction and Rationale

Chemical kinetics is concerned with the rate of change of concentration of compounds in a chemical reaction, which readily lends itself to be described using differential calculus. Mathematical operations and graphical reasoning centered around the derivative provide useful tools for modeling systems that are changing over time. However, a review of the literature indicates students lack a clear understanding of rate and rate-related ideas, with ample evidence supporting the claim that students struggle with a conceptual understanding of functions, covariational reasoning, and assigning meaning to variables (Aydin, 2014; Bain & Towns, 2016; Castillo-Garsow, Johnson, & Moore, 2013; Moore, 2014; Moore, Paoletti, & Musgrave, 2013; Rasmussen, Marrongelle, & Borba, 2014; White & Mitchelmore, 1996).

Given this backdrop, it is not surprising students have difficulty using and applying calculus in other contexts, such as modeling physical systems (Becker & Towns, 2012). The act of modeling, in which processes are translated into mathematical formalism, is a common practice in the sciences, and it has been identified as a foundational scientific practice that students should engage in at all levels of education (Bruce, 2013; National Research Council, 2012; Edwards & Head, 2016; Posthuma-Adams, 2014). However, problem-solving, reasoning, and modeling in the physical sciences is particularly challenging because it introduces an additional domain of (scientific) knowledge that must be integrated with a student's mathematical knowledge, a problem that is further compounded when considering that chemistry requires students to think abstractly at the particulate-level, which is not readily observable or accessible (Becker & Towns, 2012). Nevertheless, researchers agree that making connections across different domains of knowledge through modeling is necessary to promote a deeper understanding of chemistry (Becker, Rupp, & Brandriet, 2017; Sjostrom & Talanquer, 2014; Taber, 2013; Talanquer, 2011).

Based on this rationale, research studies have investigated student understanding of mathematical expressions and their relationship to chemical phenomena, and published literature reviews indicate there have also been a number of other studies that focus specifically on chemical kinetics (Bain & Towns, 2016; Becker et al., 2017; Becker & Towns, 2012; Greenbowe & Meltzer, 2003; Hadfield & Wieman, 2010; Jasien & Oberem, 2002; Justi, 2002). In their review paper, Bain and Towns (2016) echo the call of the National Research Council for more discipline-based education research (DBER) that focuses on studies at the undergraduate level and emphasizes interdisciplinary work, such as collaborations between chemistry and mathematics communities. They also comment specifically on the need for more studies that incorporate prompts aimed at investigating graphical reasoning in a chemical context such as kinetics (Bain & Towns, 2016).

Among the reviewed literature, few studies focus on the overlap of chemical and graphical reasoning, and among the chemical kinetics studies reviewed, none focus exclusively on reasoning related to graphical representations. However, student difficulties with graphs are discussed briefly as part of larger studies and the general consensus among the literature is that students are often unable to make conclusions about the chemical mechanism that is implied in graphical representations of chemical processes (Cakmakci, 2010; Cakmakci & Aydogdu, 2011; Cakmakci, Leach, & Donnelly, 2006; Kolomuç & Tekin, 2011; Tastan, Yalçinkaya, & Boz, 2010). This study seeks to fill the gap in the literature and contribute to the body of knowledge related to graphical reasoning in the physical sciences. To this end, our guiding research question is the following: In what ways do students use mathematics in combination with their knowledge of chemistry and chemical kinetics to interpret concentration versus time graphs?

## Theoretical Underpinnings

This study was developed using the resource-based model of cognition as a theoretical framework (Hammer & Elby, 2002, 2003). The resources perspective describes student knowledge as being defined by resources that are activated in specific contexts. These resources are broadly defined as pieces of knowledge or ideas about the nature of knowledge. Hammer and Elby (2002) emphasize that resources may be productive or unproductive, and instruction should focus on understanding what resources students have and how to encourage students to use resources that are useful for a given context. As mentioned by Becker and colleagues (2017), using the resources perspective to frame data analysis and dissemination of results provides the opportunity for researchers and practitioners to consider what instructional support would be useful to help students productively use their knowledge. By considering the nature of the student responses and the reasoning elicited from the prompt, appropriate scaffolding can be developed to encourage scientific reasoning and promote scientific practices such as modeling.

To aid in the analysis of our data, we used the shape thinking perspective as a methodological framework (Moore & Thompson, 2015). Within the shape thinking framework, reasoning related to graphical understanding and problem-solving is characterized as static or emergent: static thinking is reasoning that describes graphs as objects ("a wire") that have associated properties; emergent thinking is reasoning about graphs as a mapping of all of the possible inputs and outputs, a trace in progress (process) involving covarying quantities.

## Methods

### Data Collection

The primary source of data was an assessment administered to 109 students following the chemical kinetics unit in a first-year non-majors chemistry course at a Swedish university. The prompt given to the students provided a concentration vs. time graph along with three short-answer questions related to the graph (see Figure 1). One of the learning objectives for the kinetics unit involved getting students to extract information about what is happening at the molecular level from a graphical representation of a reaction. This is reflected in the design of the prompt, which focuses on conceptual understanding and requires students to integrate chemical and mathematical knowledge.

**Kim got the results of an experiment that a friend performed, where a particular substance X was involved in a reaction. The graph shows how the concentration of X changed with time during the experiment. Given this graph:**

**(a) Is X better described as a reactant or product in the reaction? Explain your answer.**

**(b) At which of the time points t = 1 min, 5 min, 10 min was the rate of reaction the highest? At which time point was it the lowest? Explain carefully how you arrived at your answer.**

**(c) Kim is trying now to figure out what could have led to the concentration of X behaving the way it did and is asking you for some help and ideas. Suggest to Kim possible things that could have happened in the reaction that could explain the difference in reaction rates at t = 1 min and t = 10 min. Be sure to justify your response with explanations at the molecular level for why your suggestions can reasonably explain the difference.**

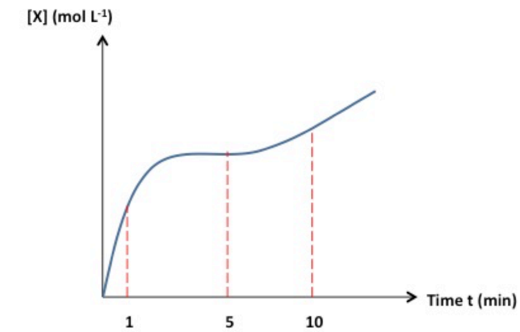


*Figure 1. Prompt used for assessment.*

This emphasis on conceptual understanding led the researchers to consider ideas of transfer. Transfer involves applying knowledge to unfamiliar situations, and the ability to transfer knowledge has been identified as a key component of conceptual understanding (Holme, Luxford, & Brandriet, 2015). Within the resources framework, transfer is conceptualized as the *activation of resources*, and in order for students to be able to use knowledge in novel situations, resources related to the task need to be coherently organized in such a way that they are not dependent on a single context (Hammer, Elby, Scherr, & Redish, 2005).

The prompt was designed, in part, to evaluate the extent in which students are able to use the appropriate knowledge in a different context. The graph in the prompt did not reflect the concentration vs. time graphs normally depicted in textbooks, and although chemically possible, it exhibited deviations from empirical results one would observe in typical laboratory work done in a general chemistry course (see Figure 2). In addition to representing a somewhat unfamiliar problem-solving scenario, item (c) in the prompt reflects what is described as an “ill-defined” problem, in which the question is more open-ended and there is not just one correct answer (Singer, Nielson, & Schweingruber, 2012). For this problem students are prompted to suggest a plausible explanation for the observed graph shape, which could encompass a myriad of possible justifications. Content validity of the assessment items was achieved by discussing and co-developing this prompt among a group of four researchers, and the wording in the prompt was refined after initially being piloted (in both English and Swedish) with a group of participants that included three professors, a postdoctoral researcher, and two Ph.D. students.

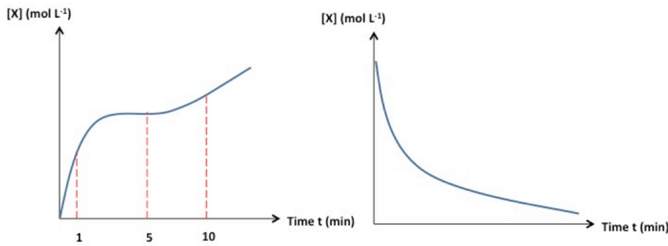

*Figure 2. Graph used in the prompt (left) and a graph used in a typical chemistry textbook (right).*

After translating the student responses from Swedish to English, they were analyzed using open coding and the shape reasoning framework. Through the process of constant-comparison a list of codes was created and refined, with a graduate student and a postdoctoral researcher coding in tandem and requiring 100% agreement for assignment of codes (Patton, 2002). The coding scheme developed into a multi-tier categorization system that was used to characterize student reasoning. The scheme characterizes the student responses, first based on whether the student answered the question correctly, then on the discipline-specific (chemistry vs. mathematics) resources, such as the content and reasoning the students used. This was developed through a combination of inductive and deductive analysis, in which the chemistry categories developed as a result of the observed student responses and the mathematical reasoning categories were modeled after the previously mentioned delineation of emergent vs. static reasoning in the shape thinking framework (Moore & Thompson, 2015). For the chemistry categories, a "Less Productive" sub-category was created to encompasses responses that involve ideas that are not useful for problem-solving in this context and/or reflect incorrect reasoning about relevant ideas, and a "Kinetics Concepts" sub-category was created to encompasses ideas and reasoning that more appropriately address the prompt. It is also important to note that chemical and mathematical reasoning categories are not mutually exclusive (e.g. a student response can employ both chemical and mathematical reasoning), and here the authors describe the development of a new construct called *process thinking*, which encompasses chemically plausible explanations and emergent reasoning, illustrating higher-level modeling that involves the productive use of cognitive resources (example to follow).

## Preliminary Results

Analysis of the data reveals that students employ multiple different types and combinations of chemical and mathematical reasoning when interpreting concentration vs. time graphs.

When considering student responses to the first item on the assessment, (a), which asks the students to decide if the graph depicts changes in the amount of product or reactant, it can be seen that there was little variation in student reasoning; most students responded with the same chemically plausible idea that since reactants are consumed (decrease) and products are formed (increase) over the course of the reaction, the graph represents products increasing over time. In responding to this prompt, students also tended to consider how both variables change over time, displaying emergent reasoning. For the second assessment item, (b), which tests students' abilities in making connections between a graphical understanding of the derivative and ideas related to rate, the students tended to respond in purely mathematical terms without bringing in chemical knowledge, with most students reasoning statically, only thinking about the general shape and the steepness of slopes, rather than considering more formal definitions of the derivative. In the case of the final prompt, (c), which asks students to essentially trace the function and discuss the chemical phenomena that could explain the observed graph, most students responded in general terms, providing lists of factors that affect rate, rather than specifically considering the chemistry occurring at each point. However, a few students expressed a deeper level of understanding. Consider Eleanor's response to (c):

> We can see from the graph that from t = 0 to about t = 3, the rate of reaction increases, it means the concentration of reactants is greater than the concentration of products. Such a difference in concentration leads to the increase in the products' concentration. But when the reaction reaches t = 5 we can see that the product's concentration has stopped increasing, this means that the reaction has reached an equilibrium. That is why we do not get an increase in the concentration of X. But we see how at t = 7 the reaction will keep forming products. This is because we no longer have an equilibrium. And one way to change the equilibrium can for example be through changing the temperature in the reaction or through adding more reactants to the reaction so that they can continue to form products.

In her response, Eleanor considers multiple points on the graph and provides chemically plausible explanations that could justify the observed shape of the graph. Responding to item (c) requires mechanistic thinking about the process the graph models, and as instructors, we would like to move students toward a more sophisticated understanding that encompasses practices such as modeling, a level of reasoning (exemplified by Eleanor) that defines the construct we call *process thinking*. Process thinking combines chemically plausible explanations with emergent reasoning (mathematical reasoning related to functions and covariation), and preliminary results indicate process thinking was not common among the student responses. This does not necessarily imply students are unable to engage in this level of reasoning, because this prompt may not have been effective in activating or eliciting this type of reasoning.

Preliminary analysis also yielded some interesting considerations regarding language and culture. In Swedish, the standard mathematical term for the gradient of a line in two-dimensional space is *lutning*, which is the noun form of the verb *luta*, "to lean". Both words are of general, everyday usage, but are nevertheless used in Swedish to describe the characteristics of a line in the more specialized, mathematical context. Furthermore, this description of the derivative is dynamic in the way that it connotes action. For instance, from its grammatical construction, the word *lutning* is literally "leaning-ness" or the act of leaning. Also, a line can *luta skarpt*, "lean strongly", as opposed the more static description of a line *having* a steep gradient/slope, as is the norm in English. This suggests some level of cultural and colloquial familiarity or association for the students.

## Conclusion and Questions

When viewing the student responses as variations in reasoning that reflect cognitive resources available to the students, it is worth evaluating which resources are more productive for the context, and investigating the extent in which each item in the assessment elicited the desired reasoning. This will provide a better understanding of how to scaffold student reasoning, promote modeling, and develop exams that can assess deeper levels of understanding. Further analysis is warranted and the following questions reflect potential future avenues for inquiry:

(1) How are the types of chemical and mathematical reasoning related for each response?
(2) How do we promote modeling and activate productive resources in unfamiliar situations?
(3) What role do cultural influences such as language have on student mathematical reasoning and our coding scheme?

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Building Lasting Relationships: Inquiry-Oriented Instructional Measure Practices

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*This study examines the relationships between instructional practices in the Inquiry-Oriented Instructional Measure (IOIM). The IOIM consists of seven practices developed from four guiding principles of Inquiry-Oriented (IO) instruction: generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation. A 2-tailed correlation test was applied to IOIM scores from 36 instructors and found six of the practices had strong positive correlations to each other and the seventh had a moderate positive correlation. Cronbach alpha was calculated indicating the IOIM is an internally consistent measure.*

*Keywords:* Inquiry-Oriented, Instructional Measure, Quantitative

Inquiry based learning (IBL) encompasses a broad range of teaching approaches focused on engaging students in mathematical argumentation while performing a sequence of tasks (Yoshinobu & Jones, 2013; Laursen, Hassi, Kogan, & Weston, 2014). Studies have shown better student outcomes from self-reported IBL instructors than from non-IBL instructors (Laursen, et al., 2014; Kogan & Laursen, 2013). However, IBL is a "big tent" with different meanings to different researchers (Kuster, Johnson, Keene, & Andrews-Larson, 2017). Here we focus on the more narrow Inquiry-Oriented (IO) instruction, which generally adheres to the tenets of IBL.

Measures have been developed in other branches of math education, with purposes such as teacher noticing (Jacobs, 2017) or determining the mathematical quality of instruction (Learning Mathematics for Teaching Project, 2011). These measures can help clarify the degree to which a standard is met and can clarify how researchers are conceptualizing phenomena (Jacobs, 2017). For IO instruction, this conceptualization is particularly important because IO curricular materials have presented a number of challenges for implementation. These challenges include developing mathematical knowledge for teaching, anticipating how to build on students' ideas, and facilitating whole-class discussions (Johnson & Larsen, 2011; Rasmussen & Marrongelle, 2006; Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007). Therefore, it is essential to define what IO instruction looks like, to develop a clear measure to better understand the nature of improved outcomes observed in IBL courses, and to see to what extent they were observed in IO intending classes. This measure can also help address implementation challenges by highlighting specific aspects of high-quality IO instruction.

Researchers have created the Inquiry-Oriented Instructional Measure (IOIM), a rubric that quantifies the degree to which a class can be characterized as IO. For more background information on this measure as well as the measure itself, refer to Kuster, Rupnow, & Johnson (2018) in this volume. We used the IOIM to score 36 Abstract Algebra, Linear Algebra, and Differential Equations instructors. Based on those scores, the purpose of this paper is to explore the relationships between different practices in the IOIM to determine the value of using the IOIM to measure IO instruction.

## Theoretical Perspective

The IOIM is based on four guiding principles from Kuster et al. (2017): *generating* student ways of reasoning, *building* on student contributions, developing a *shared understanding*, and *connecting* to standard mathematical language and notation. Generating student ways of reasoning includes engaging students in mathematical tasks so their thinking is shared and explored with the class. Building on student contributions involves taking students' ideas and using them to direct class discussion, potentially in unforeseen ways. Developing a shared understanding describes helping individual students understand one another's thinking, reasoning, and notation so that a common experience can be "taken-as-shared" in the classroom (Stephan & Rasmussen, 2002). Connecting to standard mathematical language and notation involves transitioning students from the idiosyncratic mathematical notation and terms used in class to standard descriptions and notation, such as "groups" or phase planes. These four principles are enacted though seven instructional practice. The four principles and the seven practices supporting them are listed in Figure 1.

| Principles | Practices Supporting Each Principle |
|---|---|
| **Generating** student ways of reasoning | 1. Teachers **facilitate** student engagement in meaningful tasks and mathematical activity related to an important mathematical point. |
| **Generating** student ways of reasoning<br>**Building** on student contributions | 2. Teachers **elicit** student reasoning and contributions. |
| **Generating** student ways of reasoning<br>**Building** on student contributions | 3. Teachers **actively inquire** into student thinking. |
| **Building** on student contributions<br>Developing a **shared understanding** | 4. Teachers are **responsive** to student contributions, using student contributions to inform the lesson. |
| Developing a **shared understanding** | 5. The teacher engages students in **one another's reasoning**. |
| **Building** on student contributions | 6. The teacher guides and manages the development of the **mathematical agenda**. |
| Developing a **shared understanding**<br>**Connecting** to standard mathematical language and notation | 7. Teachers support **formalizing** of student ideas/contributions and introduce language and notation when appropriate. |

*Figure 1: Principles and their supporting practices*

Practice one reflects the extent to which the teacher engages students in "doing mathematics," or the extent to which students engaged in cognitively demanding tasks and used mathematical argumentation to support or refute any claims (Stein, Engle, Smith, & Hughes, 2008). Practice two reveals the degree to which the teacher elicits rich mathematical reasoning from students, as opposed to simple recitation of procedures. Practice three signals the level to which the teacher further probes students' statements and reasoning in order to improve their own understanding of what students meant and in order to help students reflect on their own thinking. Practice four indicates how much the teacher uses students' questions and ideas as a springboard for further discussion in class that enriches the mathematical development for the class as a whole. Practice five examines the extent to which the teacher prompts students to directly compare and contrast each other's reasoning without the teacher needing to act as a filter that interprets statements for the students. Practice six exhibits the level to which the teacher guides and manages the development of a lesson in a coherent way that reaches a mathematical goal while using student reasoning and contributions to reach that mathematical goal. Practice seven displays the degree to which the teacher transitions from students' own language and notation, which have been developed to address tasks, to standard mathematical language and notation and the extent to which the teacher allows students to take ownership of this transition (i.e., at a high level, the teacher provides the standard name but the students translate their notation into standard notation once given a template for the standard form). Based on this perspective, we explore the following question: To what extent are the practices related?

## Methods

This quantitative study uses a relational research design to look at the relationships among the seven IOIM practices by investigating data collected from a project designed to support instructors interested in implementing IO instructional materials. Five volunteers trained for five days to understand how to score videos with the IOIM. Coders then scored videos of professors teaching Abstract Algebra, Linear Algebra, and Differential Equations that had been collected during the IO project. Mean scores for each video were calculated and examined using correlation and linear regression analysis to determine the relationships among the practices.

### Coders

Classroom videos were coded by one expert coder and five graduate students recruited by researchers involved in a large project designed to support instructors as they implemented IO curricular materials. The expert coder was a graduate student involved in the development of the IOIM, who had been trained by an IO project researcher on coding each practice. The other five coders were recruited from three different universities associated with the IO project. These five coders completed a week of training conducted by the expert coder to learn about scoring the IOIM practices from 1 to 5, with 1 being low and 5 being high (Kuster, et al., 2018). The first three days were spent in online meetings watching and discussing different teaching scenarios representing the five levels of IO teaching described in the IOIM. Special emphasis was placed on characterizing low, medium, and high levels of IO teaching to aid interpretation of the IOIM. During this time, the expert coder explained each IOIM practice and the associated score for each of the videos. The expert coder also answered the coders' questions and facilitated debates about scores to ensure all coders gained an understanding of the IOIM practices and scores. The last two days involved coding practice videos. Each coder individually scored a video, discussed their scores with another coder, and then met as a group online with the expert coder. Once the coders and expert coder reached agreement on a score for each IOIM practice, they scored the next video. This repetitive process continued throughout the last two days. Coders had to be within one score from the expert coder for each IOIM practice before coding another video. This benchmark helped ensure coders understood the IOIM practices and scoring.

### Data Collection

The five coders individually watched eight to twenty-one classroom videos from the Abstract Algebra, Linear Algebra, and Differential Equations IO project professors. The videos were from TIMES fellows, who had engaged in professional development while using IO materials. After watching each video, coders used the IOIM to score each practice and wrote a justification of the score. Individual coders met online with the expert coder after every fifth video to discuss scores. If all of the coder's IOIM scores were at most one away from the expert's scores, the coder proceeded to the next set of videos. However, if the coder's IOIM scores were off by more than one score, the coder was asked to re-watch and recode the video. This benchmark ensured consistency in coding. Final IOIM scores were compiled in a spreadsheet for each video. The goal was to have at least two coders score each video.

### Data Analysis

To determine the relationships among the IOIM practices, correlations and linear regression analysis were conducted using the mean scores of each IOIM practice for each video. The goal was to determine the strength of the relationships between practices and if the score of one practice predicted the score of other practices from the IOIM rubric. A total of 36 scored videos were used, each containing one mean score for each of the seven IOIM practices. Simple linear regressions were conducted by defining one practice as the independent variable with all other practices defined as the dependent variables for all 36 videos. To assess this measure's internal consistency, Cronbach's alpha analysis was conducted using all seven practices.

## Results

The preliminary results indicate each practice is positively correlated with every other practice, which provides justification for the cohesion of the measure (Table 1). Cronbach's alpha was calculated to assess internal consistency for the seven practices ($\alpha = .969$). This indicates the IOIM has high internal consistency and is a reliable measure for assessing IO instruction. As a video receives high scores for one practice, it receives high scores for the other practices, and likewise if the scores are low. We found practices one through six had very strong correlations to each other, and practice seven had a moderate correlation with the other practices (Table 1). This means video scores for practices one though six strongly depended on each other, whereas video scores for practice seven were only moderately dependent on the scores from practices one though six.

*Table 1. Correlations between IOIM Practices*

| Correlations | | | | | | | |
|---|---|---|---|---|---|---|---|
| Practices | Practice 1 | Practice 2 | Practice 3 | Practice 4 | Practice 5 | Practice 6 | Practice 7 |
| Practice 1 | 1 | .892** | .932** | .883** | .817** | .932** | .716** |
| Practice 2 | .892** | 1 | .910** | .893** | .889** | .917** | .726** |
| Practice 3 | .932** | .910** | 1 | .834** | .798** | .911** | .659** |
| Practice 4 | .883** | .893** | .834** | 1 | .883** | .871** | .695** |
| Practice 5 | .817** | .889** | .798** | .883** | 1 | .846** | .642** |
| Practice 6 | .932** | .917** | .911** | .871** | .846** | 1 | .660** |
| Practice 7 | .716** | .726** | .659** | .695** | .642** | .660** | 1 |

## Discussion

The preliminary results indicate practices one through six have strong, positive correlations between each other, but practice seven is only moderately correlated with the other practices. According to our theoretical perspective, the first six practices map to the generating student ways of reasoning, building on student contributions, and developing a shared understanding IO principles. This explains the strong correlation between them since they rely primarily on student thinking and how the instructor responds to such thinking. However, practice seven is the only practice mapped to the connecting to standard mathematical language and notation principle. Practice seven focuses on formalizing student contributions to standard mathematical language and notation, which does not appear to strongly depend on student thinking stemming from an IO task. Due to the difference in mapping, this could explain the difference in correlations between practices one through six with practice seven. For practices 1-6, the high correlations suggest, for example, that a teacher who can probe student thinking also has students engaged in mathematics and vice versa.

Because only TIMES fellows who were trained in doing IO instruction were scored with this rubric, a future area of research would be to use the measure with professors who lecture, who use other forms of IBL, or who use a mixture of IO and lecture to see if the measure can distinguish among teaching styles. Professors who are excellent lecturers could also be a potential subject pool to investigate whether the correlations would be similar with professors who excel with a different instructional method. It is also worth studying whether there are differences between practice scores when the data is broken down by course or coder. Additional research could investigate the interaction between practice, course, and coder.

**Questions for Audience**

1. We analyzed the data with correlations. What other data analysis methods would be appropriate and for what purposes?
2. What might we learn by using this rubric on other data sets (e.g., IBL or lecture based)?
3. Do you think the rubric would be applicable for K-12 instruction? If so, how?
4. Do you think the rubric would be applicable for mathematics preservice teacher evaluation? If so, how?

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Productive Failures: From Class Requirement to Fostering a Support Group


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*Mistakes occur frequently in mathematics. In two classes (Abstract Algebra and Calculus II), mistakes were brought to the forefront in the form of a "productive failure." Through five interviews with students, we initially looked for affectual responses to the pedagogical allowance and student-led demonstration. Many of the responses, both benefits and drawbacks of the productive failure, were interpreted by the research group to resemble peer-led support groups such as Alcoholics Anonymous. Descriptions of both productive failure and support groups, as well as quotes from the students, aim to shed light on psychological benefits of valuing mistakes.*

*Keywords:* productive failure, affect, inquiry-based learning

## Introduction

At one point in their life every student will reach a mathematical impasse when attempting to solve a problem. What students do after such an impasse might define how they view mathematics as a process. Additionally, what instructors do to cultivate such a process may further (and perhaps ultimately) influence students' thoughts about mathematics. The present investigation focused on the pedagogical action of allowing students to demonstrate their problem-solving impasses and explain their struggle positively. We call this struggle a "productive failure." At first, we investigated affect in students' interview responses to productive failures. However, we conjecture that many of the affectual responses may also be found as benefits and drawbacks in peer-led support groups such as Alcoholics Anonymous. While the two do not equate on a societal level, the characteristics and effects seemed to align. This proposal describes what a productive failure is, gives background on affect and support groups, and argues the resemblance of a support group to the demonstrations of a productive failure.

## Background Literature

### Productive Failures

The notion of using mistakes, difficulties and impasses as productive has been discussed in many capacities, often with success. However, both what constitutes "productive" and what kind of difficulty arises, varies in the literature. For example, Granberg (2016) defined productive failure as a "result in the restructuring of mental connections in more powerful, useful ways through which the problem at hand would make sense and new information, ideas and facts would become assimilated" (p. 34). Granberg, again, stated that errors play a large part: "It appears that making, discovering and correcting errors may generate effort that can engage students in productive struggle" (p. 34). However, productive to Granberg meant to obtain a correct solution, whereas the authors mean productivity in how students learned about their own problem-solving methods. What must occur for a student to be productive in their failure is a recognition of the failure or mistake (the "checking" phase of Carlson and Bloom's (2005) problem-solving process), subsequent recovery or additional approach (the cycle back to "planning" and "executing" phase of Carlson and Bloom (2005)), and the metacognitive awareness of modifying their approach for future problem solving. Research has suggested that during productive struggle, students activate prior knowledge and intuitive ideas (Kapur & Bielaczyc, 2012; Kapur, 2014). Furthermore, the more problem-solving methods that students construct during their struggles, the more prior knowledge is to be activated (Kapur, 2014).

As a pedagogical tool, there is an indication in previous literature that an environment structured for utilizing failures or mistakes can be successful in refining students' problem-solving skills. For example, an explicit incentive to correct their mistakes can be an effective formative assessment tool (Black & Wiliam, 2009). This incentive could be points or other credit in the course: "Offering grade incentives to diagnose and correct mistakes can go a long way to close the performance gap between struggling and high-performing students" (Brown, Singh, Mason, 2015, p. 4). A by-product of this pedagogical action is that it can create "failure tolerance" (e.g., Clifford, 1984; 1988), turning potentially negative occurrences into positive outcomes. Tulis (2013) stated that research into pedagogical actions on failure and mistakes is scarce: "little is known about adaptive classroom practices for dealing with errors and the reciprocal effects of students' and teachers' attitudes towards learning from mistakes" (p. 56). These effects on attitude led us to search for affect in our project, which will be described next.

**Affect**

McLeod (1992) stated that the definition of affect "refers to a wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition" (p. 576). He goes on to state that there are three general categories to the affective domain: beliefs, attitudes, and emotions. While others have added categories to the domain (namely, values, motivation and engagement (Attard, 2014)), for the purposes of this project, the focus will be on these three categories, and on affect as a whole. Beliefs are "psychologically held understandings, premises, or propositions about the world that are thought to be true" (Philipp, 2007, p. 259). For example, an instrumentalist view of mathematics may state that mathematics is all about rules and procedures. Attitudes are "develop[ed] from several similar and repeated emotive responses to an event or object" (Grootenboer & Marshman, 2016, p. 19). Emotions are more visceral and momentous. Positive emotions include AHA! moments (Liljedahl, 2013), while negative emotions involve frustration. Negative emotions can largely contribute to how students approach problem solving tasks: "Furner (2000) suggested that two-thirds of Americans either hate or loathe mathematics" (Grootenboer & Marshman, 2016, p. 21).

The difficult part about affect is that it can be influenced by a variety of factors, some that can be controlled by pedagogical actions. For instance, Grootenboer and Marshman (2016), citing Pajares (1992), stated that "because central beliefs have been developed through experience, new activities giving rise to positive experiences and reflection upon those experiences is critical to belief change" (p. 17). Therefore, while affect is personal, and can influence cognition and learning, it is difficult and lengthy to foster or change in students. Nevertheless, a demonstration on the productivity of failure may be an influence students' affect.

## Research Question

What are the effects of demonstrations of a productive failure on a student and the classroom? In particular, what changes in affect occurred during and after a productive failure demonstration?

## Methods

This investigation focused on two classes: an undergraduate/graduate abstract algebra course using TAAFU (Teaching Abstract Algebra For Understanding) materials (Larsen, Johnson, and

Bartlo, 2013); and calculus II, covering from the definition of definite integral to integration techniques. The algebra class was in Fall 2015 with 32 students, and the calculus class was in Fall 2016 with 137 students. Demonstrating a productive failure in front of the class accounted for 5% of the final grade, with 2% extra credit in the calculus course if the student demonstrated in front of the large lecture instead of the discussion sections.

Productive failures generally occurred in the same manner. The instructor asked if any students had a productive failure, and if one did, the instructor would ask them to come to the document camera and demonstrate it. Students would describe their mistakes and were encouraged to reflect on them. Unless already mentioned, they were typically asked why it was productive for them. Often the problem or theorem in question was an entry point to discuss the topic for that day. These demonstrations lasted for an average of five minutes. After questions from other students and the instructor, the presenting student would walk back to their seat while their peers applauded.

The first author was the instructor for both courses and taught using inquiry-based learning (IBL) (Cook, Murphy, & Fukawa-Connelly, 2016). The second author researched the calculus course, taking observation notes of daily classes, interviewing four students (including the fourth author), and conducting an online survey (different than the end-of-course evaluations). All interviews were conducted and transcribed by the second author. One question in the interview focused specifically on productive failures and their presentations. Due to space, both the full interview questions and survey questions are omitted.

The third author was a student in the abstract algebra course and presented a productive failure after the second test, which occurred on week 10. The fourth author was a student in the calculus course, and presented her productive failure to the large class before the first test. Both were asked to participate in a reflection session with the first author about 7 months after their demonstrations, where they discussed the demonstration of the failure, the reactions that they had during the time, and future effects. The first and third authors watched the video of the third author's productive failure presentation (collected for another project), and discussed instances together in an unstructured group reflection. The third author then transcribed that discussion. The first and second author analyzed and coded utterances using affect, and then discussed the importance and significance of those codes. While coding for affect, the authors then found resemblances between the responses given and characteristics of support groups.

## Results

These math classrooms, when incorporating the presentation of productive failure, can be viewed as analogous to a support group. It is prudent to reiterate that this is an analogous relation only and that it is not the intent of the authors to imply that the support that these students are receiving is of the same magnitude to other formal support groups. By the theory of Schopler and Galinsky (1995), support groups have certain characteristics that include:

- "organizational sponsorship or be the creation of an innovative practitioner" (p. 4)
- being member-centric, with members providing experiences, information, advice, and occasionally leadership within the group.
- leaders sharing authority with the members, having their legitimacy often being based on training
- providing a supportive environment and a means for developing coping abilities

The instructor implemented the productive failure requirement in his courses beginning Spring 2016, but was influenced by the IBL community (e.g., Yoshinobu, 2014) and previous literature about impasses (e.g., Savic, 2015). This wasn't necessarily the "creation of an innovative practitioner," but a practitioner that created the productive failure requirement influenced by an innovative community. All productive failures were done by the students, and frequently ended with a round of applause from the majority of the students, hence were "member-centric." The third author stated in the follow-up interview, "We have to clap! This person did such a good job! I was so excited for anyone to get up there and do it that even if it was horrible." The instructor shared the class time (and the power) with the students, and was trained to teach IBL, therefore satisfying the third requirement. As for "providing a supporting environment and a means for developing coping abilities," both may be apparent when discussing the positive and negative effects of the productive failure.

**Positive Effects**

Positive effects of social groups can include "greater social resources, increased knowledge about the focal concern, a sense of relief and reassurance, and enhanced skills for coping" (Schopler & Galinsky, 2014, pp. 6-7). In the interviews, each benefit seemed to align with multiple affectual quotes from the students, which are portrayed in Table 1. The affect code in the quotes is interspersed as normal font.

*Table 1: Benefits with Student Responses*

| Benefit | Student Response |
| --- | --- |
| Greater social resources | "I just remember a lot of people having the same questions that I did and following down the same path that I did [Belief], so I didn't feel all that bad about having a failure and a lot of people [Emotion], I feel, benefited from me going up, because a lot of people were making the same mistake as I was. And we all got to work together to figure out the right way to do it as a class, which felt awesome. It made the class a lot more interactive and I felt that I learned a lot from my first presentation. [Belief]" – Calculus Student 4, Follow-up Interview |
| Increased knowledge about focal concern | "For example, in BC Calc, I really struggled with integration by parts; it never really made sense, I didn't know where it came from, but this year, integration by parts, now that I actually understand all of the background to it, makes so much more sense and it comes so much easier now. And it's because I had that opportunity to try and then fail and then see where it came from [Belief]." – Calculus Student 4, Initial Interview |
| Sense of relief and reassurance | "So, I went to the board and presented my productive failure and, I didn't feel bad [Emotion], which was odd because, you know, most classes when you make a mistake, people just look at you like 'wow, she's so dumb' and not in this class. They value when you make a mistake and then you realize why you made the mistake [Attitude] and you can fix it because then you're not gonna forget it, you're not gonna make the mistake again [Belief]." – Fourth Author, Initial Interview |
| Enhanced skills for coping | "[W]hat I learned from that was I try, really try not to fail [Attitude], but I'm not afraid of it anymore [Emotion]. So, now whenever I'm doing homework or whatever, I'm not thinking about 'I'm not going to get this right.' I think about, 'What can I do to not fail and get it right?' [Attitude] … Like, if I fail, well I fail. I just restart again [Attitude]." – Fourth Author, Follow-up |

**Negative Effects**

There were students that stated negative effects of productive failures. This is also reflective of the support group research literature; Schopler and Galinsky (1994) found that participants felt "pressure to conform, stress related to group obligations, feeling overwhelmed and less adequate, learning ineffective and inappropriate responses, embarrassment, and overconfidence" (Schopler & Galinsky, 2014, p. 7). Calculus Student 2, in her initial interview, stated that productive failures are "terrifying," and preferred a large class because she could "hide with all those people," both are affectual responses that can be categorized as *pressure to conform* and *feeling overwhelmed*. Calculus Student 4, in his follow-up interview, stated that he enjoyed productive failures but did not find that it would transfer to his major in medicine, where he hoped to specialize in cardiovascular surgery. A student evaluation of the course stated that "I don't feel like the productive failures are effective cause it's a hit or miss whether they'll explain it well," which can be categorized as *learning ineffective and inappropriate responses*.

## Discussion

The pedagogical action of a productive failure demonstration seemed to create a support-group environment. Therefore, we expect similar benefits to support groups. Although we have not found any evidence of this yet, Brown, Tang, and Hollman (2014), citing Brown (2009), stated that "Part of [support groups'] strength lies in their empowering nature, where participants help each other as equals rather than taking on dependency roles where they rely on the advice of professionals" (p. 84). Therefore, in addition to inquiry-based learning, demonstrations of productive failures may help shift power to create a more equitable classroom (Tang et al., in press).

The socio-mathematical norm (Yackel & Cobb, 1996) of learning from mistakes has effects on students' approaches to future problems. For example, the fourth author stated in her interview that she is "not afraid of failing," thus her self-efficacy may have increased for subsequent courses. Finally, this study was first conducted in order to figure out affective as well as cognitive and metacognitive shifts due to productive failures. Thus, there may be many metacognitive gains for students when demonstrating a productive failure. The third author stated in her demonstration interview that "I really do think [the productive failure] impacted me. I don't know if it impacted other people but I think that specific instance has changed how I perceive problems when I see them. I had a lot more success in Abstract Algebra 2 I think because of it."

## Conclusion

Productive failure demonstrations allow mistakes to be open and psychologically constructive instead of damaging, give a platform and power that otherwise may not be available, and may influence both the presenter and their peers affectively. The intention is to investigate and collect further data, especially for the gains in problem solving. A conjecture is that as failures tend to be recast, more students will persist in their problem solving. Time and effort may improve their mathematical skills, and allow them to grow to be more content with their abilities. Encouraging productive failures in a classroom can give students the affectual support to grow as practicing mathematicians.

Questions for the readers:

1. What other pedagogical actions can create environments where mistakes are valued?
2. What other pedagogical actions can create support groups?

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## Surveying Professors' Perceptions of Incorporating History into Calculus I Instruction

*The goal of this study is to document undergraduate mathematics professors' perceptions of incorporating the history of mathematics into their Calculus I instruction. Although research has been documented on benefits of incorporating history into mathematics teaching and learning, little has been documented on undergraduate professors' beliefs and how they may incorporate history into Calculus I. To address this question, we created a survey based on Schoenfeld's (1999) theoretical framework of knowledge, goals, and orientations to capture perceptions about instructional decisions related to history incorporation. Calculus I professors in a southeastern state were surveyed to gain an understanding of perceptions on the importance of history and how they incorporate history. The majority of professors (80%) view history as important for Calculus I learning for a variety of reasons and incorporate it in different ways. Implications for supporting undergraduate Calculus I teaching and learning are shared along with questions for further research.*

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*Keywords:* Calculus, History, Surveying, Instruction

Numerous educational theorists agree that learning the history of a discipline is essential to learning the content of that discipline (e.g. Frederick & Katz, 1997; Munakata, 2005). Regarding mathematics, Berlinghoff and Gouvea (2002) explain, "Learning about math is like getting to know another person. The more you know of someone's past, the better able you are to understand and interact with him or her now and in the future" (p. 1). While a number of researchers have investigated the usefulness of incorporating history into mathematics instruction, we found little research on undergraduate mathematics professors' views on this topic. Understanding professors' perceptions is important to the field of undergraduate mathematics education. Smestad (2011) points out that teachers' views on incorporating history into mathematics instruction must be taken into account before attempting to influence their teaching.

To better understand the views of Calculus I professors, the study documents aspects of professors' perceptions on usefulness of mathematics history and how they may employ history. The research begins to fill the particular gap in the literature regarding the perceptions of Calculus I professors. The results of this work may better inform the field of undergraduate mathematics education on how to best support professors seeking to enhance Calculus I teaching and learning through incorporating history.

### Background and Literature

The National Science Foundation (NSF) has supported a number of scholars in their attempts to disseminate research and resources on the benefits of incorporating history into undergraduate mathematics. Frederick and Katz (1997) established the NSF supported endeavor, Institute in the History of Mathematics and its Use in Teaching. The institute was associated with the Mathematical Association of America, and one of its many aspirations was "to increase the presence of history in the undergraduate mathematics curriculum" (p. 1). Another NSF supported project, TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS), narrows the focus employing history in undergraduate mathematics through developing curricular materials based on primary historical sources. Barnett, Clark, Klyve, Lodder, Otero, Scoville, and White (2017) explain:

> That faculty with an interest in primary sources can use them in their teaching is well and good, but we are convinced that there are so many benefits derived from their use that we would like to see them available to all instructors of university mathematics (p. 1).

A relevant question emerges of what type of historical resources professors use, primary or otherwise, and how they perceive them as useful.

Ferreira and Rich (2001) reviewed literature on incorporating history to conclude that using history improves perceptions of mathematics and increases enthusiasm for learning. While these benefits have been established, Calculus I professors' perceptions of incorporating history and how they use history in their instruction are largely unknown. Furthermore, while most undergraduate mathematics textbooks include historical anecdotes throughout the text (e.g. Stewart, 2015), it is also unknown to what degree such anecdotes influence instruction. Research on undergraduate mathematics education must account for Calculus I professors' perceptions in order to adequately guide curricular resource development and instructional practice.

**Theoretical Framework**

While there is general consensus that the history of mathematics is important to mathematics teaching and learning (e.g. Frederick & Katz, 1997; Fauvel & van Maanen, 2000), we found little research on undergraduate faculty perceptions of incorporating history into their Calculus I curriculum. A gap exists in the field of undergraduate mathematics regarding the beliefs, knowledge, and goals professors exhibit on the inclusion of history to teach Calculus I. Research questions were formed and analyzed with the lens of Schoenfeld's (1999) framework of knowledge, goals, and orientations as these three dimensions capture professors' perceptions about both short and long term instructional decisions. The following research questions were addressed: 1) How important do Calculus I professors view the incorporation of history into their instruction and why? 2) What kinds of pedagogical practices do professors use to incorporate history into Calculus I and why?

**Methods and Data Analysis**

Surveying methodology provided an overall picture of professors' perceptions of using the history of mathematics in their Calculus I instruction. Rea and Parker explain, "[t]he ultimate goal of survey research is to allow researchers to generalize about a large population by studying only a small portion of that population" (1997, p. 2). After identifying all undergraduate mathematics professors in the state of North Carolina that reported Calculus I in their professional teaching portfolio, we sent each a mailed letter and emailed invitation to participate in an online survey. Of the 599 survey invitations, we received 96 completed surveys. This provided us a 16 percent response rate.

Survey questions were designed to gain an understanding of the demographic of the responding professors (survey items 1-3), their beliefs about incorporating history (survey items 4-6), how and why they use mathematics history in teaching Calculus I (survey items 7 & 8), and interest in learning more about history incorporation (survey item 9). Survey items six and eight were open ended, and the others were multiple choice. Figure one below summarizes survey data for these categories.

| Survey Item | Responses with Percentage |
|---|---|
| 1) Years teaching at a four year | 0-2 years: 6% 3-6 years: 11% |

| | |
|---|---|
| undergraduate institution | 7-10 years: 16% 11-15 years: 10%<br>15+ years: 57% |
| 2) Number of undergraduate Calculus I courses taught | 0-5 courses: 33% 6-10 courses: 15%<br>11-15 courses: 12% 16-24 courses: 14%<br>25+ courses: 26% |
| 3) Ever incorporated history into Calculus I instruction | Yes: 71%<br>No: 29% |
| 4) Believe incorporating history into Calculus I instruction may benefit students | Yes: 80%<br>No: 20% |
| 5) Importance of incorporating history into Calculus I instruction for teaching and learning (Scale from 0 to 5 with 5 being highest) | 0: 6% 1: 18%<br>2: 20% 3: 24%<br>4: 19% 5: 13% |
| 6) If believe history is beneficial, how will incorporating history into Calculus I instruction benefit students? | See math as *human endeavor*: 28%<br>Make *connections* to history/other disciplines/real world: 27%<br>Increase *motivation*: 27%<br>Increase *understanding*: 14%<br>Broadens *perspective*: 4% |
| 7) If you use history, how often do you incorporate history into Calculus I instruction during a one semester course? | Never: 12% Rarely: 19%<br>Once: 11% 2-4 times: 35%<br>5-9 times: 14% 10+ times: 9% |
| 8) If you include history, describe way(s) you use history and why? | Mentioned during *lecture*: 36%<br>Assigned outside historical *readings*: 30%<br>*Projects* with historical component: 11%<br>*Discussion* on mathematics history: 10%<br>*Videos* about mathematics history: 8%<br>*Papers* with historical component: 5% |
| 9) Interest in learning more about incorporating history into Calculus I instruction (Scale from 0 to 5 with 5 being highest) | 0: 6% 1: 11%<br>2: 14% 3: 36%<br>4: 20% 5: 13% |

*Figure 1. Summary Survey Responses*

To analyze the data, summary statistics were calculated for all numeric data along with thematic analysis of open response data. The thematic analysis work was guided by Creswell's (2012) recommendations for analyzing qualitative data to document emergent themes in responses. We applied thematic analysis to items six and eight with the key word of the theme italicized for clarity in Figure 1 above. A subset of survey participants responded to items six and eight with 75 responding to six and 67 responding to item eight. To address the first research question, we examined summary statistics on professors' beliefs as related to their reporting of how often they incorporate history and the degree to which they are interested in learning more about incorporation. To address the second research question, we examined summary statistics on how history was used and why they reported incorporating it into their instruction. While claims will be generalized to professors in North Carolina, study findings can reasonably be used to inform knowledge of professors' perceptions outside of this state.

## Results

**Beliefs on Incorporating History**

Recall the first research question. How important do Calculus I professors view the incorporation of history into their instruction and why? The majority of professors (80%) indicated that history is useful for students learning in Calculus I however only 71% reported using history in their teaching. On the importance scale from zero to five (see item five), the majority of respondents were clumped towards the middle with 63% providing ranks from 2 to 4. Professors reported multiple reasons for viewing history as beneficial to the Calculus I learner. The top three reasons were the following: 1) See math as *human endeavor* (28%); 2) Make *connections* to history/other disciplines/real world (27%); and 3) Increase *motivation* (27%).

Figure 2 below provides a small sample of the explanations professors offered for viewing the incorporation of history into Calculus I instruction as beneficial to the student.

| **Categories of Benefits of Incorporating History** | **Sample Response in Each Category** |
|---|---|
| See math as human endeavor (28%) | Students benefit greatly from being helped to see that mathematics is, like any other human invention, the product of human activity, often social activity. |
| Make connections to history/other disciplines/real world (27%) | At face value, this can be a dry subject for students who do not see the bigger picture. Incorporating history gives students another connection to the material, and helps to position mathematics. |
| Increase Motivation (27%) | I have found that by placing results within context students tend to be more engaged in the topic. |
| Increase Understanding (14%) | Giving the context of the discovery of l'Hospital's rule, I believe, helped students retain information.<br><br>Gives an understanding of why there are multiple notations for the same things. |
| Broadens Perspective (4%) | It broadens student perspectives, and gives students a break from the "normal" routine. |

*Figure 2. Sample Responses on Beliefs*

In summary, the majority of professors saw incorporating history into Calculus I instruction as beneficial for reasons that were categorized into five groups. The reasons given largely align with research on incorporating history. For instance, increased motivation has been documented as a result of incorporating history (e.g. Ferreira & Rich, 2001). One interesting finding is that the reason, increase understanding, did not make it into the top three reasons. Finally, the majority of professors indicated a substantial interest in learning more about using history in Calculus I instruction.

**How and Why History is Used**

Professors reported teaching Calculus I with history in a variety of ways with varying frequencies. Regarding frequency of use in a one semester course, the bottom three categories (never, rarely, and once) account for 42% of respondents. About a third (35%) reported using history two to four times during the semester, and the remaining 23% use history five or more times. Some professors reported using history in multiple ways while others reported only one. The most common ways reported were in lecture and reading representing 66% of responses. See Figure 3 for a sample of reasons professors gave for incorporating history corresponding to a particular instructional method.

| How History is Used | Description/Why |
|---|---|
| Mentioned during *lecture*: 36% | To compare the methods of either Fermat or Barrow to modern methods of finding tangent lines |
| Assigned outside historical *readings*: 30% | To compare the past with the present |
| *Projects* with historical component: 11% | [T]he biggest project, which I include EVERY time I teach Calc[ulus] I, is a re-enactment of the Newton-Leibniz controversy, in the form of a civil trial that the students plan for about a month and then stage in class.The project, often initially received skeptically, is almost always UNIVERSALLY acclaimed by students by the end. |
| *Discussion* on mathematics history: 10% | In discussions I give short explanations on the history and showcase where/why/how the math in their textbook was developed. |
| *Videos* about mathematics history: 8% | So students can see the evolution of concepts |
| *Papers* with historical component: 5% | To show students the history lets them know that mathematics is a dynamic subject and reinforces the idea that the concepts are most essential so they can apply Calculus to the applications that will arise in the future |

*Figure 3. Sample Responses on How History is Used*

Professors reported augmenting their instruction with history in multiple ways. One dominant theme was that of professors finding that their students were more engaged in the class and found historical ideas interesting. Similar to responses on benefits of incorporating history (item six), few professors mentioned increased understanding in response to item eight (see Figure 3). In categorizing reasons why professors use a particular method of incorporating history, we found no mention of primary historical source use.

## Findings and Implications for Future Research

Two findings stood out. First, while the majority of professors have used history in their Calculus I teaching, even more professors believe that their students would benefit from its use. This study identifies a need to assist the segment of professors who feel that incorporating history into Calculus I is important, even though they reported incorporating such history no more than once per semester. Furthermore, the majority of professors indicated a substantial interest in learning more about incorporating history. Study findings on how history is currently incorporated may assist those concerned with developing curricular supports that enhance Calculus I curriculum with history.

The second primary finding is that only 14% of professors reported increased understanding as a perceived benefit of incorporating history. It is possible that many professors viewed increased understanding as a secondary benefit of an initial benefit they reported. For instance, those that listed increased motivation (28%) may believe this increase leads to increased understanding. Another possible explanation for the low percentage of professors listing increased understanding is their lack of reporting using primary sources to teach Calculus I content. More refined studies are needed to parse out perceptions in these regards.

As further understanding of professors' perceptions on incorporating history into Calculus I instruction is established, the field of undergraduate mathematics education will be better prepared to assist those seeking to harness the potential of using history to teach mathematics. We see at least three relevant research questions for the RUME audience. How can RUME researchers develop adequate resources to assist Calculus I professors to incorporate history in meaningful ways? What are professors' perceptions of using primary historical sources to teach Calculus I? How might RUME researchers add to our knowledge of using primary sources?

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Prerequisite Knowledge of Mathematics and Success in Calculus I

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*This preliminary report describes how prerequisite content knowledge is related to success in a first semester calculus course. Data collected included adaptive assessments administered in both Pre-Calculus and Calculus I, standardized test scores, prior enrollment in Pre-Calculus, prior enrollment in Calculus I, and final grades in Calculus I. Analysis revealed that (1) standardized metrics such as ACT, SAT, and placement test scores did not reliably predict students' success in Calculus I, (2) passing Pre-Calculus directly impacted students' prerequisite content knowledge which in turn led to a stronger performance in Calculus I, and (3) students lost a significant amount of knowledge between the end of Pre-Calculus and the beginning of Calculus I. Lastly, in an effort to identify how deficits in specific knowledge domains impact student performance in Calculus I, additional analysis revealed that students' ability to graph trigonometric functions was most predictive of their performance in Calculus I.*

*Keywords:* Prerequisite Content Knowledge, Pre-Calculus, Calculus I

Despite national reports calling for additional Science, Technology, Engineering, and Mathematics (STEM) degrees over the next decade, students are choosing to leave STEM programs of study, in part because of their inability to pass Calculus I (Bressoud, Camp, & Teague, 2012). Although research teams have explored reasons why students struggle with college level mathematics and some have even pinpointed specific topics for which students lack sufficient prerequisite knowledge (e.g., concept of function, composition of functions, quantitative reasoning) failure rates in Calculus I remain problematic nationally (Breidenback, Dubinsky, Hawks, & Nichols, 1992; Carlson, Madison, &West, 2015).

Mathematics instructors in higher education have been regularly contending with students who are unprepared to take college level courses. Porter & Polickof (2011) have found that as many as 20% of students at PhD granting institutions and 60% of community college students are required to take remedial courses before they are permitted to take college level courses. Although, many students have high school credit for precaclulus and calculus courses, Bressoud et al. (2012) added that students who pass high school calculus courses are not necessarily better prepared for success in college level calculus courses (Bressoud et al., 2012).

Most colleges and universities utilize some type of placement procedure with their first year mathematics students. The purpose of a placement procedure is to assess students' prerequisite knowledge and subsequently place them in a course that is commensurate with that knowledge. A land-grant university in the Mid-Atlantic Region of the United States implemented the following placement procedure for all mathematics students taking Calculus 1 during the Spring 2017 Semester. New students were placed in Calculus 1 via their ACT Math score, SAT Math score, scores on a math placement exam, or successful completion of a pre-calculus course. Regardless of this placement process, failure rates (students earning a D, F or withdrawing from the class) in Calculus 1 have remained high. The failure rates from Fall 2015, Spring 2016, Fall 2016, and Spring 2017 were 44%, 55%, 34%, and 50% respectively.

In an effort to better understand the relationship between students' prerequisite knowledge and their performance in Calculus I, this preliminary report will specifically address the following research questions:

1) How does students' prerequisite knowledge influence their success (earning an A, B, or C) in Calculus I?
2) Are students who take Pre-Calculus more likely to be successful (earning a grade of A, B, or C) in Calculus I than those who does not?
3) How do deficits in specific knowledge domains impact students' success (earning an A, B, or C) in Calculus I?

## Method

### Data Collection

Data were collected from 118 students who were enrolled in Calculus I at a land-grant university in the Mid-Atlantic Region of the United States during the Spring 2017 Semester. Forty-eight of the 118 students successfully completed the institution's Pre-Calculus Course during the Fall 2016 Semester. All 118 students took an Initial Assessment during the first week of their Calculus I course. The Initial Assessment was part of a commercial software package that uses artificial intelligence to assess the student's current course knowledge by asking him 20-30 questions open-ended questions. Students who took Pre-Calculus prior to taking Calculus I took a Final Assessment similar to the Initial Assessment in Calculus I at the conclusion of their Pre-Calculus course. Both of these assessments measured the students' level of mastery with respect to 21 knowledge domains including: Equations and Inequalities, Quadratic Equations, Rational Equations, Radical Equations, Lines, Polynomial and Rational Functions, Graphs and Transformations, Logarithmic and Exponential Functions, Trigonometric Functions and Equations. For the 48 students who completed both Pre-Calculus and Calculus I, a change score was determined. This change score was calculated by subtracting the Initial Assessment Score in Calculus I from the Final Assessment Score in Pre-Calculus. This was used to help identify which topics students did not retain between the end of the fall semester and the beginning of the spring semester. In addition to the scores from these assessments and the change score between the two assessments, the following data were also considered: standardized test scores (ACT Math, SAT Math, and Math Placement Exam) used for placement into Calculus I, prior enrollment in Pre-Calculus, prior enrollment in Calculus I (number of students repeating the course), and final grades in Calculus I.

### Data Analysis

A hierarchal regression analysis was used to explore predictors of students final scores in Calculus I, which included a final sample of $n = 83$ (removing students who withdrew from the course, or who did not have standardized test scores to report). Step 1 of the analysis included students' standardized test scores (converted to ACT units, $M = 25.92$, $SD = 2.73$), prior enrollment in Pre-Calculus (about 49% of the sample), and past enrollment in the Calculus I course (about 25% of the sample). Step 2 included students' overall performance on the Initial Assessment in Calculus I ($M = 51.49$, $SD = 20.62$). The only significant predictor of Calculus I performance from these variables was students' Initial Assessment scores, which explained about

20% of the unique variance in their Calculus I scores (see Table 1). No other predictors contributed any significant or meaningful direct impact at any step in the regression model.

*Table 1. Hierarchal regression analysis examining predictors of students' performance in Calculus I.*

| | $B$ | $SE\ B$ | β | $t$ | $p$-value | unique $R^2$ |
|---|---|---|---|---|---|---|
| **STEP 1** | | | | | | |
| Standardized Test Scores | 1.10 | .943 | .140 | 1.17 | .246 | .02 |
| Pre-Calculus enrollment | 2.32 | 5.99 | .054 | .356 | .700 | ~.00 |
| Repeating Calculus I | 1.15 | 6.58 | .023 | .175 | .892 | ~.00 |
| $F(3,79) = .461$, $p = .710$, $R^2 = .02$ | | | | | | |
| **STEP 2** | | | | | | |
| Standardized Test Scores | .720 | .580 | .092 | .846 | .400 | ~.00 |
| Pre-Calculus enrollment | -2.19 | 5.47 | -.051 | .401 | .690 | ~.00 |
| Repeating Calculus I | 1.59 | 5.90 | .032 | .270 | .788 | ~.00 |
| **Initial Assessment Scores** | **48.06** | **10.70** | **.462** | **4.49** | **< .001** | **.20** |
| $F(4,78) = 5.47$, $p < .001$, Δ, $R^2 = .21$, adj. $R^2 = .18$<br>Durbin-Watson = .902 | | | | | | |

*Note.* Significant predictors bolded for ease of interpretation.

A follow-up analysis considered the potential for prior enrollment in Pre-Calculus to indirectly impact Calculus I performance through a direct impact on Initial Assessment scores. This analysis showed that prior Pre-Calculus enrollment was a significant direct positive predictor of Initial Assessment scores (β = .25, $p < .001$, explaining about 6% of the variance in those scores), which in turn had a significant positive impact on Calculus I Final Grades (β = .48, $p < .001$explaining about 24% of the variance); there was no direct impact of Pre-Calculus enrollment on Calculus I. As a robustness check (and to provide mean group comparisons), students who had previously enrolled in the Pre-Calculus course scored significantly higher, nearly 10% more ($n = 36$, $M = 56.34$, $SD = 20.18$) than those who had not ($n = 47$, $M = 47.76$, $SD = 20.37$), $F(1,80) = 4.42$, partial η2 = .05(and controlling for standardized placement scores, which has no impact, $p < .334$, partial η2 = .01).

Additionally, to determine a cut-off score on the Initial Assessment score for determining the odds of students passing Calculus I, a receiver operating characteristic (ROC) curve was used to determine the test's predictive utility, with area under curve $C = .751$ ($SE = .53$, 95% CI from .647 to .855, $p < .001$). A minimum score of 40% on the initial exam (sensitivity = .825, specificity = .539) was the lowest score predictive of passing Calculus I. Notably, standardized tests scores (such as ACT scores) had no predictive value in a student's odds of passing Calculus I, area under curve $C = .557$ ($SE = .072$, 95% CI from .416 to .697, $p = .389$).

Finally, as more discrete performance data was available on the Pre-Calculus students who eventually enrolled in Calculus I ($n$ = 48), we did additional analysis on their performance on specific dimensions of the Initial Assessment, as well as their Calculus I performance. First, given that these students would have taken the initial assessment twice (once at the conclusion of Pre-Calculus in December (as the final assessment in that class) and once at the start of Calculus I the following January), we compared the change scores on these two tests between students who passed Calculus I and those who did not—higher changes scores are indicative of a decline in performance on the Assessment. Overall, students who eventually passed Calculus I forgot less information on the Initial Assessment after taking Pre-Calculus ($n$ = 26, $M$ = 21.86, $SD$ = 24.18) than students who failed Calculus I ($n$ = 22, $M$ = 44.43, $SD$ = 22.45), $t$(46) = 3.33, $p$ = .002, $R^2$ = .19—students failing Calculus I had a nearly 20% higher discrepancy score between the first and second iteration of the initial assessment that those who passed the course.

For the 21 specific knowledge domains, three tests were conducted. First, we compared tests scores on each domain directly. There was a universal and significant drop in knowledge retention on all domains. The lowest drop was observed with Equations and Inequalities at 10% ($p$ = .003, $R^2$10= .05); all other domains experienced significant ($p$'s < .001) and substantial ($R^2$ ranging from .09 to .75) drops of at least 20% (Slopes) to as much as 67% (Polynomial Functions). Second, we compared change scores for students who passed Calculus I to those who failed the course. In nearly every case, students who passed Calculus I retained more information—the smallest significant discrepancy being Unit Circle Trigonometry with a 20% discrepancy, $t$(46) = 2.02, $p$ = .049, $R^2$ = .07) and the largest being Right Triangle Trigonometry with a nearly 36% discrepancy, $t$(46) = 4.14, $p$ < .001, $R^2$ = 26. Domains that did not observe significant differences in knowledge retention were Composite, Polynomial, and Rational Functions (three separate domains), and the domains of Graphing Trigonometric Functions, Inverse Trigonometric Functions, Trigonometric Identities, and Trigonometric Equations (four different domains)—the average knowledge loss for these domains was $M$ = 50.37, $SD$ = 24.38).

Finally, we used a hierarchal regression to determine which knowledge domains seemed to be most predictive of performance in Calculus I, controlling for standardized test scores. As with the earlier analysis, standardized scores had no impact on Pre-Calculus students' performance in Calculus ($R^2$ ~ .00). The collective addition of the 21 knowledge domains increased $R^2$ by about 16%—of which, nearly half was explained by Graphing Trigonometric Functions ($R^2$ = .08). Notably, significance levels were not interpreted due to the small sample size for this post-hoc analysis.

## Discussion

The analysis showed that although the institution utilized standardized metrics such as ACT, SAT, and placement test scores, these metrics did not reliably predict if a student will pass his or her first calculus course. Interestingly, the Initial Assessment that was given to all students as they entered Calculus I was a better predictor of course performance than any other predictive variable utilized. The score on the Initial Assessment explained 20% of the overall variance in the final course grade in Calculus I. In other words, students who demonstrated weak prerequisite skills began the class two letter grades behind those who exhibited a strong prerequisite knowledge base.

Since the Initial Assessment was a strong predictor of student success in Calculus I, additional analysis was used to determine a cut-score capable of predicting the odds of students passing Calculus I. The analysis revealed that students who obtained a score of at least 40% on the Initial Assessment had an 80% chance of passing Calculus I. Although 40% appears to be a low score, the overall average on the Initial Assessment was only 51.49%. The Initial Assessment has the potential to be quite valuable when assessing students' prerequisite knowledge along with students' ability to be successful in Calculus I.

Students who completed Pre-Calculus successfully were not more likely than their counterparts to be successful in Calculus I. While the analysis revealed that no connection between the two courses existed directly, it did reveal a connection indirectly. Students who passed Pre-Calculus scored higher on the Initial Assessment in Calculus I than their counterparts and subsequently students who did well on the Initial Assessment were more likely to pass Calculus I. This was a significant finding as the Initial Assessment was the only predictive variable for students' final grade in Calculus I. Thus, passing Pre-Calculus directly impacted students' prerequisite content knowledge which in in turn led to a stronger performance in Calculus I.

Furthermore, it was found that students who enrolled in Pre-Calculus had a significant decrease in content knowledge in all 21 knowledge domains between the Final Assessment in December of 2016 in Pre-Calculus and the Initial Assessment in January of 2017. Despite this loss, Pre-Calculus students still outperformed their counterparts by 10 percentage points. Furthermore, students who lost the least amount of knowledge between semesters performed significantly better in Calculus I. On average, students who failed Calculus I had a nearly 20% higher discrepancy between the two assessments than those who passed Calculus I.

Finally, in an effort to identify which prerequisite topics had the most significant impact on students' performance in Calculus I, change scores between the Final Assessment in Pre-Calculus and Initial Assessment in Calculus I were analyzed. Although, the analysis revealed that students' ability to graph trigonometric functions was most predictive of students' success in Calculus I, other interesting findings emerged. First, students retained the most prerequisite knowledge in the domain:  Equations and Inequalities (10% drop). In fact, it was the only knowledge domain with no significant knowledge drop. Second, students lost on average 67% of the content knowledge related to polynomial functions. Last, students who failed Calculus I, had a significant knowledge drop in seven out of 21 knowledge domains.

## Conclusion

It is troubling that some of the predictive metrics utilized in the institution's placement process did not accurately predict who would be successful in Calculus I. Many institutions solely use standardized tests to place students into mathematics courses. If these metrics are not providing an accurate snapshot of students' prerequisite knowledge, then perhaps colleges and universities should consider adjusting their placement procedure to include adaptive assessments such as the Initial Assessment discussed in this report.

Another concern raised in this report is the significant decrease in content knowledge between the end of Pre-Calculus and the beginning of Calculus I, only one month later. Researchers and instructors alike must find ways to mitigate this knowledge loss. These results should inform teaching decisions in pre-calculus courses especially, as students are clearly not retaining topics critical to their understanding of calculus. Lastly, it is important to acknowledge the small sample size used in this preliminary research project. Further research should be conducted to see if these outcomes are replicated in future semesters.

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# Validating Proofs in Parallel Mathematical and Pedagogical Tasks

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*Educators often use tasks that situate teachers in pedagogical contexts, under the assumptions that such tasks activate knowledge authentic to teaching; and, furthermore, purely mathematical contexts may not activate such knowledge. These assumptions are based on analyses that contrast actual engagement with pedagogical context to hypothetical engagement without pedagogical context. We propose that it is important to conduct a direct comparison of responses, and we report on such a study using a set of tasks with and without pedagogical contexts – featuring the same underlying mathematics. The results revealed differences in how secondary teachers validated proof based on context. Context also influenced the importance participants placed on algebraic notation in validating a proof. This study has implications for how and when secondary teachers attend to validity and the role of algebraic notation, and the messages they may convey to their students about validity and notation.*

*Keywords:* Mathematical Knowledge for Teaching, Proof Validation, Secondary Teachers

Many secondary mathematics teachers find their undergraduate mathematical preparation irrelevant to or disconnected from their teaching (Goulding, Hatch, & Rodd, 2003; Ticknor, 2012; Wasserman, Villanueva, Mejía-Ramos, & Weber, 2015; Zazkis & Leikin, 2010). One possible response to this problem is to embed mathematics into pedagogical contexts (e.g., Stylianides & Stylianides, 2010; Wasserman, Fukawa-Connelly, Villanueva, Mejía-Ramos, & Weber, 2016). The strategy behind this design is that situating teachers in *pedagogical tasks*, as opposed to pure *mathematics tasks*, helps activate "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball, Thames, & Phelps, 2008, p. 395). Because the work on the task resembles work done in teaching, teachers can experience ways in which mathematics applies to teaching, and thus find these experiences useful for their future teaching.

One implicit assumption underlying development efforts is that pedagogical context activates knowledge that is authentic to teaching. Furthermore, this knowledge may not be activated or perceived as relevant in pure mathematics contexts. For instance, all examples of items in Hill, Ball, and Schilling (2008) contain names of students or teachers, and the authors discussed debates of "how much to contextualize [items]" (p. 379) – not whether to contextualize items. Following a description of a task with pedagogical context, Stylianides and Stylianides (2010) concluded, "Presumably, it would be hard for a teacher educator to engage prospective elementary teachers in a discussion of such a subtle but important mathematical issue in the absence of a 'motivating' pedagogical space" (p. 168). Arguments in support of this assumption rely on analyses of teachers engaged in tasks with pedagogical context, contrasted with hypothetical cases where the pedagogical context is absent. Although the arguments are compelling and have advanced the field, there is no empirical evidence for this assumption based on direct comparisons between responses to tasks with and without pedagogical context.

We propose that it is productive to conduct such a comparison. Suppose that tasks set in pedagogical contexts do in fact activate mathematical knowledge differently than mathematical contexts. For example, if different criteria are used to determine whether or not a proof is valid depending on the context, then that may have implications for how mathematical knowledge is used in teaching. Similarly, the explanation of a mathematical idea might have different features when presented with a pedagogical context, which has implications for how that idea would be understood. Differences based on context might reveal unaddressed incoherence in teachers' mathematical knowledge. Differences might also suggest places where connecting undergraduate mathematical content to the work of teaching is particularly difficult.

We hypothesize that if there are differences in responses to tasks set in pedagogical contexts and mathematical contexts, then these differences might be explained by the norms and values teachers hold about mathematics. We base this hypothesis on the observation that different contexts, including orientations toward problem solving, can influence the norms and values brought to bear in solving tasks (Aaron & Herbst, 2012); and that different contexts can prime different knowledge on identical tasks (e.g., Gick & Holyoak, 1980; Ortner & Sieverding, 2008; Yeager & Walton, 2011). In this paper, we report on a study in which we compared 17 high school teachers' responses to parallel mathematics tasks, one situated in a pedagogical context and the other in a university mathematics context. The tasks were exactly the same except for context; see Table 1. We asked: Do teachers validate proofs based on similar norms and values when situated in teaching mathematics compared to when situated in learning mathematics? Our results are highly suggestive that contexts do elicit different orientations to mathematics, in the form of norms and values.

Throughout this paper, we use *pedagogical context* refer to contextual elements of elementary school, middle school, or secondary teaching practice contained in the task text such as student talk or curriculum materials (Phelps & Howell, 2016). In contrast we use *university context* to refer to tasks that are set in the context of an undergraduate mathematics course, and do not have contextual elements related to teaching. Distinguishing these two contexts explicitly highlights the potential differences in teachers' undergraduate mathematical preparation and the mathematical work of their teaching.

## Theoretical Perspective and Frameworks Used

Teaching decisions are shaped by orientations (Schoenfeld, 2010), which encompass norms and values. Norms refer to expectations and understandings; values refer to what is perceived as important or beneficial; both have forms specific to the discipline of mathematics (Kitcher, 1984) as well as its learning and teaching (Yackel & Cobb, 1996). The norms and values for mathematics inform those of teaching mathematics, but they are not the same (Ball et al., 2008), and priming with different contexts can potentially activate different resources (e.g., Gick & Holyoak, 1980). Consequently, mathematics teaching entails negotiating mathematical and pedagogical norms and values (Ball & Bass, 2003a, 2003b).

Since formal proof is part of secondary mathematics (NGACBP & CCSSO, 2010), a practice of mathematics learning that arises in teaching is validating mathematical arguments, including proof. The validity and communication of a proof can be contextual (Weber, 2014, 2016). Additionally, Lai and Weber (2014) found that mathematicians would improve proposed proofs differently depending on whether the proof had come from a student or a mathematician.

## Data & Method

### Rationale

To determine whether the contexts of teaching and learning would elicit different mathematical norms and values, we used parallel tasks. One task featured pedagogical context to situate the participant in teaching secondary mathematics; the other situated the participant as a student in a university mathematics course. We chose to contrast the pedagogical secondary context with a university context because the most recent and intensive context in which teachers experience proofs as learners is university. We determined that these two contexts served as productive contrasts to inform future work in teacher education.

Table 1 shows the set of tasks used to address the first research question. The university context could be considered a pedagogical tertiary context, however, we note that the task situates the participant as a student, not a professor. Moreover, responses from our participants indicate that they were reasoning from the stance of student, not university instructor.

*Table 1. Parallel tasks for validating mathematical proofs. The tasks are based on the TEDS-M released item #MFC709 (TEDS-M, 2010).*

<table>
<tr><th>Pedagogical context</th><th>University mathematics context</th></tr>
<tr><td>In a unit on mathematical justification, you ask your high school students to prove the following statement:</td><td>In a unit on mathematical justification, your mathematics professor asks you to consider proofs of the following statement:</td></tr>
<tr><td colspan="2">When you multiply 3 consecutive natural numbers, the product is a multiple of 6.</td></tr>
<tr><td>Below are three responses. Determine whether each student’s proof is valid.</td><td>Below are three responses. Determine whether each proof is valid.</td></tr>
<tr><td>Kate’s answer:</td><td>1.</td></tr>
<tr><td colspan="2">A multiple of 6 must have factors of 3 and 2. If you have three consecutive numbers, one will be a multiple of 3. Also, at least one number will be even and all even numbers are multiples of 2. If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2.</td></tr>
<tr><td>Leon’s answer:</td><td>2.</td></tr>
<tr><td colspan="2">$1 \times 2 \times 3 = 6$    $2 \times 3 \times 4 = 24 = 6 \times 4$<br>$4 \times 5 \times 6 = 120 = 6 \times 20$    $6 \times 7 \times 8 = 336 = 6 \times 56$</td></tr>
<tr><td>Maria’s answer:</td><td>3.</td></tr>
<tr><td colspan="2">$n$ is any whole number<br>$n\times(n+1)\times(n+2) = (n^2+n)\times(n+2)$<br>$= n^3 + n^2 + 2n^2 + 2n$<br>Cancelling the $n$’s gives $1 + 1 + 2 + 2 = 6$</td></tr>
</table>

## Data source

**Participants.** We interviewed 17 practicing secondary mathematics teachers who had 1 to 14 years of experience teaching, and who had worked with a variety of grade levels and courses.

**Tasks.** To ensure that the pedagogical context was realistic, we used existing tasks that had been extensively reviewed as representing mathematical knowledge for teaching. For the research question reported, we used tasks, shown in Table 1, based on the TEDS-M released item #MFC709 (TEDS-M, 2010), which represents pedagogical content knowledge (Tatto et al., 2008). The full study considers a second set of parallel tasks, focused on explanation.

**Protocol.** All participants answered in teaching context first and learning context second, with a distractor between contexts to prime their identity as university students. We asked participants in each context whether they agreed or disagreed with the statement, “Kate’s proof/Proof 1 is less valid because it does not use algebraic notation". This question targeted teachers' potential belief in the importance of algebraic notation in proof (e.g., Knuth, 2002).

**Analysis**. We first coded the reasons why each proof was judged valid or invalid. In the second coding, we looked for differences across context in the determinations about the proofs, their reasoning, and agreement or disagreement about the role of algebraic notation.

## Results

Clear differences emerged based on context. Table 2 shows how participants validated each of the proofs, in each context. Table 3 illustrates how participants judged the role of symbolic notation in each context. We now highlight two themes of the teachers' reasoning, and will present the remainder of our results in the full paper.

*Table 2. Number of participants determining whether proofs are valid, by context*

| | Pedagogical context | | University context | |
|---|---|---|---|---|
| | Valid | Not valid | Valid | Not valid |
| Kate / 1 | 16 | 1 | 5 | 12 |
| Leon / 2 | 2 | 15 | 0 | 17 |
| Maria / 3 | 2 | 15 | 3 | 14 |

*Table 3. Number of participants who disagree or agree: "Kate's proof is less valid because it does not use algebraic notation"/"Proof 1 is less valid because it does not use algebraic notation"*

| Pedagogical context (Kate's proof) | | | University context (Proof 1) | | |
|---|---|---|---|---|---|
| Disagree | Agree | Other* | Disagree | Agree | Other* |
| 14 | 1 | 2 | 3 | 10 | 3 |

*"Other" denotes equivocation, e.g., "If the teacher wants algebraic proof, then yes, less valid. If that's not the learning target, then it's not more or less valid." In the university context, one participant was unintentionally not asked this question, so $n = 16$ instead of 17.

### Privileged norms of communication

In the university context, teachers valued precision and clarity and privileged algebraic notation: "the algebraic notation is clearer, precise, or better than just words, and it is a skill you should have in university." One teacher's explanation of why Proof 1 was not valid, while Kate's proof was valid, captured the privileging of algebraic notation in a particularly remarkable way: "In university, you have to use mathematical reasoning not logical reasoning."

In the pedagogical context, teachers privileged words and focused on explanation: "If you can get down your idea, that's all that matters" or "Using words is important". Some teachers expressed discomfort, wondering whether it was "okay" to hold differences across context. Several teachers insisted that algebraic notation is absolutely needed at the university level, while at the same time not expecting high school students to use algebraic notation.

### (Not) Attending to the logical structure of proof

In the pedagogical context, some participants praised Maria's "good start" and stated she needed to explain her work more in order to have a valid proof; these same participants in the university context stated that the reason Proof 3 was not valid was because there was an algebraic error. In both contexts, participants implied that the proof approach would work, for instance saying, "This proof is almost correct, however it is not adequate to simply 'cancel the $n$'s'." One common theme in evaluations of this proof was ascribing validity to the *approach*, even when disagreeing with the details to the extent of calling Maria's proof/Proof 3 not valid. (In fact, the approach would only work with a much more complicated structure that considers cases by divisibility.) There was a sense among some participants that the algebraic approach would eventually lead to a valid proof, especially in the university context.

## Significance

Using a novel study design with highly parallel task sets, we contribute a striking example of how the contexts of learning and teaching may activate teachers' norms and values differently. We found that different norms of communication were privileged between the two versions of the tasks. Precision and clarity arose almost entirely only in the university context, and explanation arose almost entirely only in the pedagogical context. The teachers paid explicit attention to algebraic notation, for merits of precision and clarity and because "that's what university professors expect"; teachers at times turned a blind eye to algebraic notation in the pedagogical context, professing that they would be impressed with Kate's work. This contrast raises the issue of how and when secondary students learn to attend to algebraic notation, and what messages teachers send about algebraic notation. Using tasks in varying contexts, especially featuring the same underlying mathematics, can elicit tensions between norms and values about mathematics so that they can be problematized to benefit teachers' use of mathematics in teaching as well as their identities as doers of mathematics.

## Questions for the Audience

This preliminary work has helped us shape several questions that we intend to discuss during our RUME presentation. In the presentation, we plan to share a sample of participant work, and discuss how it might change our thinking about approaches to teacher education. We then ask:

1. How compelling is the framing of the problem?
2. We used references from cognitive science to substantiate our hypothesis (that differences in responses to tasks can be explained by differences in norms and values held by teachers in pedagogical and university contexts). Are there results in mathematics education that make an equivalent point or a related point?
3. What are productive strategies for engaging with these data that attend to differences in reasoning across parallel tasks?

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Self-Regulation in Calculus I

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*Improving STEM retention is a major focus of universities and studies have shown calculus to be a barrier for STEM intending students. Prior to this study, local data indicated students did not pursue STEM fields because they were not passing calculus. In this work, I report on the results of a study on factors that seemingly impacted student success in Calculus I. In particular, I examined the relationship between final grades and self-reported self-regulatory aptitudes after accounting for incoming math aptitude. Results indicate self-regulatory aptitudes predict final grades above and beyond math aptitude. In addition, measures of self-regulation differed amongst high and under achievers as well as low and over achievers. This indicates self-regulation plays a role in student success. Furthermore, gender differences were present in measures of self-regulation which may be of importance for improving retention of women in STEM.*

*Keywords:* Calculus, Motivation, Self-Regulation

Calculus I is known to be a barrier to success for students desiring a career in science, technology, engineering, and mathematics (STEM) fields (National Academies of Sciences, Engineering, and Medicine, 2016). Recent national data shows that little more than half of students in calculus I receive a grade of an A or B and DFW rates are around 22-38% depending on the type of institution in which the course is taken (Bressoud, 2015). Of particular concern is the number of women who do not persist into calculus II with 20.1% of females switching their calculus II intention at the end of calculus I (Ellis, Kelton, & Rasmussen, 2014).

Research has correlated student self-regulation with final grades (Pintrich, Smith, Garcia, & McKeachie, 1991). In particular, recent studies have shown self-regulation measures can predict exam scores in Calculus I (Worthley, 2013) and a calculus based engineering analysis course (Hieb, Lyle, Ralston, & Chariker, 2015). This suggests that addressing self-regulation factors may be important aspects of the curriculum that could potentially improve success for some types of students. However, there is a gap in the literature regarding achievement group differences in self-regulatory aptitudes. Prior regression models indicate self-regulation predicts grades above and beyond incoming math aptitude when considering the sample as one group (Hieb, et al., 2015; Worthley, 2013). However, when classified into four achievement groups based on performance relative to the median incoming math aptitude and median final grade (see Figure 1), it is not known if achievement groups report the same type of self-regulation.

Furthermore, it remains unclear what role gender may play in the relationship between self-regulatory aptitudes and final grades in Calculus I. Prior studies have shown gender differences among self-regulatory aptitude measures (Pintrich & DeGroot, 1990; Zimmerman & Martinez Pons, 1990). In addition, although prior studies have shown aspects of self-regulation impacts success after taking into account incoming math ability, there is a gap in the literature regarding if a model of success for males would differ from a model for females. Better understanding of differences in performance according to gender and achievement groups can aid in designing interventions that cater to specific student populations. To address these gaps in the literature, three main research questions guided data analysis for the current study:

1) Are gender differences present in self-reports of self-regulation among students enrolled in Calculus 1?
2) Is there a relationship between final grades and self-regulation according to gender?
3) How do achievement groups differ in their self-reports of self-regulation?

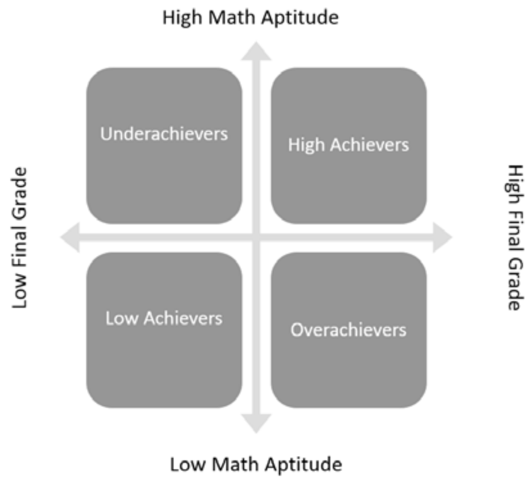


*Figure 1. Achievement groups*

**Theoretical Framework and Literature Review**

Broadly, self-regulation involves setting a standard or goal, monitoring progress toward the goal, controlling oneself to make adjustments if needed, and reflecting on one's performance (Pintrich, 2004). Self-regulation is rooted in social cognitive theory, examining reciprocal interactions between the individual, their behavior, and their environment (Zimmerman, 1989). For example, Pintrich and Zusho (2007) argue classroom contexts such as academic tasks and instructor behavior impact students' self-regulatory processes which in turn impacts student outcomes.

This study draws upon Pintrich and Zusho's (2007) and Pintrich's (2004) frameworks for self-regulation. Pintrich and Zusho's model places self-regulation within the context of the classroom. They argue students' personal characteristics and the classroom context impact students' motivational and self-regulatory processes. While some self-regulation models consider motivation to fall under self-regulation, Pintrich and Zusho distinguish motivational processes apart from self-regulatory processes. They argue motivation only becomes self-regulatory when there are active attempts to monitor and control motivation. In Pintrich and Zusho's model, motivational and self-regulatory processes then affect student outcomes. The outcomes then feed back into the model to impact future classroom context, motivation, and self-regulatory processes. According to Pintrich and Zusho's model, interventions to alter the classroom context could lead to changes in motivational and self-regulatory processes. However, it must first be understood which motivational and self-regulatory processes are impacting outcomes.

Pintrich's (2004) framework provides a means of examining motivational and self-regulatory processes within categories. In his framework, Pintrich (2004) places motivational processes under the umbrella of self-regulation. Pintrich classifies self-regulation as occurring in four areas: cognition, motivation, behavior, and environment. In addition, he considers self-regulation to occur over four phases: forethought and planning, monitoring, control, and reflection. While Pintrich acknowledges that self-regulation does not necessarily occur linearly through the phases and some aspects of self-regulation don't neatly fit into one area, thinking of self-regulation in terms of phases and areas does allow for distinction among self-regulation processes.

Pintrich's (2004) framework stems from his work developing the Motivated Strategies for Learning Questionnaire (MSLQ). The MSLQ is a questionnaire designed to measure students' course specific self-regulatory aptitudes (Duncan & McKeachie, 2005). The MSLQ has 15 subscales which Pintrich (2004) later mapped onto his classification framework.

In recent years researchers have used the MSLQ to consider the role of self-regulation in success among calculus students. In particular, some studies have attempted to utilize models that predict student success in calculus considering variables such as self-regulatory factors. For instance, Worthley (2013) and Hieb, et al. (2015) used subscales of the MSLQ in their models. Worthley found MSLQ subscales of test anxiety and self-efficacy for learning and performance were good predictors of first midterm grades when combined with math placement test results. Hieb, et al. found that of the select MSLQ subscales administered to their subjects, time and study environment management, internal goal orientation, and test anxiety were good predictors of exam scores. These studies indicate self-regulatory factors play a role in student success and should be examined in more detail.

Furthermore, studies have shown males and females differ in their mathematics interest and self-efficacy beliefs as early as middle school (Pajares, 2005) and the trend continues into college (Pajares & Miller, 1994). In addition, females maintain higher test anxiety than males (Hong, O'Neil, & Feldon, 2005; Pajares & Miller, 1994). Considering these results, it seems plausible that different gender groups may need attention on different areas of self-regulation. Thus it is necessary to examine whether the impact of self-regulation aptitudes on grades vary by gender.

## Method

### Participants

All autumn 2016 Calculus I students at a large Midwestern university were invited to participate in the study. Of the 2539 students enrolled in the course on the 15$^{th}$ day of class, 603 consented to have their data be used in research. Among these 603 students, 29 withdrew from the course. Of the 573 remaining students, 36% (*n* = 149) of students had missing data leaving a complete data set for 424 students.

### Measures and Procedure

The Calculus Concept Readiness (CCR) assessment (Carlson, Madison, & West, 2015) was administered to students during the first week of the academic semester. The CCR was used as measure of students' conceptual preparedness for calculus. In addition, the CCR provided an incoming math aptitude measure taken at the same time-point for all students. Students' ACT and SAT Math scores were collected from the university's database system. For students with no ACT Math score their SAT Math scores were converted to ACT Math equivalent scores (Dorans, 1999). In order to create a single composite incoming math aptitude score, ACT/SAT math and CCR scores were combined. The composite math aptitude score was computed by transforming scores on the CCR and ACT/SAT math test into z-scores and then summing the scores.

Students completed 12 of the 15 Motivated Strategies for Learning Questionnaire (MSLQ) (Pintrich, Smith, Garcia, & McKeachie, 1991) subscales during the fifth week of the semester. The motivation subscales that were used were intrinsic motivation, task value, control of learning beliefs, self-efficacy, and test anxiety. The learning strategy and resources management subscales used were elaboration, organization, critical thinking, metacognitive self-regulation, time and study environment, effort regulation, and peer learning.

Final grades as a decimal percentage were collected from the university's learning management system gradebook after the semester was complete and final grades for courses had been submitted.

## Results

### Gender Differences

A multivariate analysis of variance (MANOVA) was performed to determine gender differences in MSLQ subscale scores. Using a Wilks's Lambda, there was a significant effect of gender on MSLQ subscales, $\Lambda = .800$, $F(12,411) = 8.548$, $p < .001$. The MANOVA was followed up with one-way ANOVAs. Adjusting for Bonferroni's correction, significant differences in gender were found on intrinsic motivation, self-efficacy, test anxiety, critical thinking, organization, intrinsic motivation, and time and study environment (*ps* < .004). Females reported significantly lower intrinsic motivation, self-efficacy, and critical thinking than males. Females reported significantly higher test anxiety, organization, and time and study environment structuring than males.

A hierarchical regression was performed in order to determine predictability of final course grade. Math aptitude was entered in the first step. Then all ten MSLQ subscale scores were entered in the second step via forced entry. Finally, gender was entered as the third step. In the first step, math aptitude was a significant predictor of final grades, $R^2 = .352$, $F(1,422) = 228.82$, $p < .001$. In the second step, the MSLQ subscales were added to the model and contributed a significant change in $\Delta R^2 = .136$, $F(12,410) = 9.076$, $p < .001$, for a total model $R^2 = .488$. The third step, entering gender, did not result in a significant change in $R^2$ ($\Delta R^2 = .002$, $F(1,409) = 1.534$, $p = .216$). This final step indicates that after accounting for math ability and MSLQ scores, gender does not significantly predict final grade.

In addition, a secondary hierarchical linear model was applied to determine if the same hierarchical linear model of math aptitude and MSLQ subscales to predict final grades could be used for both men and women. Comparing the fit of the models using Fisher's Z-test ($z = 1.45$, $p = .147$) and the structure of the models using Steiger's Z, ($Z_H = -2.11$, $p = .034$) the same hierarchical linear model can be used for both men and women ($R^2 = .488$, $p < .001$).

### Achievement Level Differences

In order to determine how self-regulation may differ amongst achievement groups, students were categorized into four clusters. Students were ranked according to both their math aptitude and final grade scores. Students below the median in math aptitude and final grade were categorized as low achievers. Overachievers were those students below the median in math aptitude but above the median in final grade. Students above the median in math aptitude but below the median in final grade were categorized as underachievers. Finally, students above the median in math aptitude and above the median in final grade were categorized as high achievers (Figure 1). There were 167 low achievers, 73 overachievers, 67 underachievers, and 176 high achievers in the sample.

## Results

A multivariate analysis of variance (MANOVA) was performed to determine achievement group differences on MSLQ subscale scores. Using a Wilks's Lambda, there was a significant effect of achievement group on MSLQ subscales ($\Lambda = .699$, $F(36,1209) = 4.336$, $p < .001$). The MANOVA was followed up by post hoc Hochberg's GT2 tests and confirmed with Games-Howell tests. Group differences at a $p < .05$ level are indicated in Figure 2. When comparing to low achievers the post hoc tests indicate both high achievers and overachievers have greater intrinsic motivation, task value, and self-efficacy but lower test anxiety. Only high achievers have greater metacognitive self-regulation and control of learning beliefs than low achievers. When comparing underachievers, both high achievers and over achievers have greater task value, self-efficacy, time and study management, and effort regulation. Only high achievers have greater intrinsic motivation and metacognitive regulation but lower test anxiety than underachievers. At a $p < .05$ level, no statistically significant differences were found between high achievers and over achievers or under achievers and low.

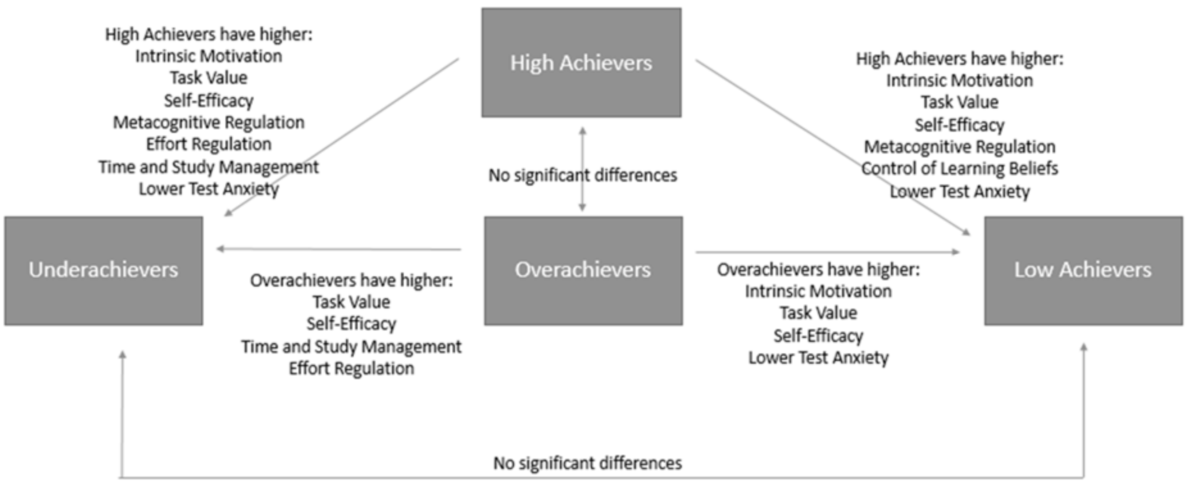


*Figure 2. Differences in MSLQ subscales by achievement group*

## Discussion

Results indicate that the CCR adds significant predictive power when used in combination with ACT/SAT math scores. Combined, these scores can account for 32% of variance in final grades. In addition, adding MSLQ measures of self-regulation, the model accounts for 48% of variance in final grades. This indicates that self-regulation attributes are important for success in calculus I. Incoming math aptitude and pre-requisite knowledge is not enough to ensure success.

Results show self-regulation predicts final grades the same in males and females. However, females reported lower intrinsic motivation, self-efficacy, and time and study environment management, as well as higher test anxiety than males. This indicates addressing motivation and self-regulation for females may be important to retaining females in STEM.

In addition, differences in MSLQ scores amongst achievement groups indicate different populations have different self-regulatory needs. While both high achievers and underachievers came in with above median math aptitude, underachievers ended the course with a grade below the median. Differences in self-regulation may account for the underperformance of underachievers as these students differed significantly on several MSLQ subscales compared to high achievers. Furthermore, low incoming math aptitude does not necessarily doom a student to failure. Self-regulation may again play a role as overachievers and low achievers scored significantly differently on several MSLQ subscales. Data indicates addressing self-regulation in low and under-achievers may promote success in calculus.

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Cognitive Resources in Student Reasoning about Mean Tendency

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*The ability to conceptualize the sample mean as having a distribution is essential to the development of statistical reasoning. Considerable research on student thinking exists on this topic, but this literature largely assumes a misconception model. This study takes a grounded theory approach to investigate the cognitive resources incoming students possess to reason about sampling distributions and mean tendency. This preliminary report includes data from a pilot study with one student enrolled in an introductory statistics course. She completed both a pre- and post-instruction interview that involved prompts about the distribution of the ages of pennies in circulation and related questions about average ages of groups of pennies. We identify several cognitive resources elicited by the pre- and post-interviews, consider the influence of instruction on the activation of these resources, and briefly discuss implications to statistics teaching. Finally, we outline next steps for data collection with 8-10 students.*

*Keywords:* Statistics, Sampling Distributions, Resources, Constructivism

The heart of statistical reasoning in the introductory statistics course is linking the core concepts of sampling, variability, and distribution to a unified conceptualization of sampling distributions and inference (Garfield, 2002). Unfortunately, superficial knowledge of this and other central ideas results in students resorting to procedural, cookbook approaches to tasks involving statistical inference (Garfield & Ben-Zvi, 2008; Garfield & Zieffler, 2012). Sampling distributions and mean tendency have been identified as extremely difficult concepts (Chance, delMas, & Garfield, 2004; Lunsford, Rowell, & Goodson-Epsy, 2006). Through their research on student thinking involving sampling distributions, Chance and colleagues offered specifics regarding what students should understand about sampling distributions, what they should do with that knowledge, and common misconceptions they may hold. Many pieces in the literature on this topic have aligned with a misconception framing (e.g., Posner, Strike, Hewson, & Gerzog, 1982) and view learning as adoption of expert thinking and replacement of novice thinking (e.g., delMas, Garfield, & Chance, 1999; Garfield, Le, Zieffler, and Ben-Zvi, 2015; Sotos, Vanhoof, Van den Noortgate, and Onghena, 2007).

This study takes a different perspective by seeking to identify cognitive resources students apply to reason about mean tendency. Smith, diSessa, and Roschelle (1993) define *resources* as designating "any feature of the learner's present cognitive state that can serve as significant input to the process of conceptual growth" (p. 124). Resources may be visualized as "knowledge-in-pieces," representing fine-grained intuitions drawn from experiences and activated in multiple contexts where the learner identifies potential connection (diSessa, 1988). We seek to identify resources students use to reason about mean tendency before formal instruction and how resource activation is influenced by formal instruction. Our research questions are as follows:

1) What cognitive resources do students activate when reasoning about mean tendency?
2) How does formal statistics instruction influence student activation of resources on these topics?

## Conceptual Framework

The statistics education literature has traditionally framed novice reasoning about probability and sampling distribution in terms of misconceptions (e.g., ASA, 2016; Cohen, Smith, Chechile, Burns, & Tsai, 1996; Garfield & Ahlgren, 1988; Kahneman, Slovic, & Tversky, 1982; Sotos et al., 2007). *Misconceptions*, as defined by Sotos and colleagues, may represent "any sort of fallacies, misunderstandings, misuses, or misinterpretations of concepts, provided that they result in a documented systematic pattern of error" (p. 99). Misconceptions may be defined to represent any number of ideas, but, a misconception framing has implications on how we view learning, and thus how we define effective teaching. This framing suggests that students have incorrect, stable conceptions of statistical ideas under what Hammer (2004) would term a "unitary" model. In contrast to a unitary model, Hammer suggests we view learners as possessing a "manifold" model that is more fine-grained. Within a manifold model, we will see students applying various resources to arrive at conceptions and conclusions. We view these two alternative perspectives not necessarily as looking at different mountains, but often as looking at the same mountain from different sides. A unitary model tends to look at the outcomes and conclusions the learner makes by synthesizing cognitive resources, while the manifold model searches for starting places and seeks to isolate the learner's cognitive resources.

Some research on student misconceptions suggests that effective instruction involves "confronting" or "eradicating" these misconceptions (e.g., delMas et al., 1999; Eaton, Anderson, & Smith, 1984). For example, delMas and colleagues found it was beneficial to present students with an anomaly to help them understand the Central Limit Theorem (CLT). The researchers modified their simulation program to encourage students to compare predictions with the actual shape of the sampling distribution. While such strategies may have resulted in students accepting the CLT as a fact, it is difficult to ascertain whether students in the study restructured their knowledge cognitively to develop deep and lasting understanding of the CLT.

In his study on student reasoning about probability with dice, Pratt (2000) took a resource perspective. He noted that resources may be contextually appropriate (e.g., the more data we collect, the more stable the results will be) or inappropriate (e.g., the next observation will be 'random' because I cannot steer or control the result). He noted the contextually inappropriate resources were based on short-term, "local" observations that emphasized randomness and unpredictability. The appropriate resources were founded in long-term "global" observations and recognition of probabilistic patterns. Pratt's work contributed to research on children's understanding of probability, but has remained relatively undeveloped in understanding college students' reasoning regarding statistical inference. We intend to work towards that goal.

## Methods

### Setting

This study is ongoing and taking place at a large public university in the southeastern United States. The target population is students enrolled in introductory applied statistics courses for non-majors. This preliminary report discusses findings from a pilot study with a sophomore Biology major at the university. Karen (pseudonym) was enrolled in a small introductory statistics course for Biology majors and reported having limited high school exposure to statistics content. The instructor of her college course was a Chinese teaching assistant enrolled in the Ph.D. program in statistics, in his third semester of teaching this course.

**Data Collection**

Data include a 20-minute pre-instruction and 30-minute post-instruction interview with Karen, field notes from the first author's observations of the class periods, and two interviews with the instructor. All interviews were video recorded and transcribed, and hand-written work from Karen's interviews were also kept as data.

In each of the interviews, Karen was provided with an x and y axis template and asked:

> Think about the age of pennies in circulation, like pennies in cash registers or people's money wallets and purses. What is the range of penny years that we would see in circulation? Draw a line to represent how many pennies you would expect there to be across the range of penny years.

Karen was also asked:

> Now think about if we were to take 5 pennies randomly from circulation and find their average age. If we repeatedly did this and collected a list of 5-penny averages, then what would be the range of averages we would see? Draw a line to represent how many of each average would we see.

The third prompt was a variation of the second prompt with 25 pennies instead of 5. The interviewer (first author) asked clarification questions and probed Karen's reasoning during the interviews. Karen answered the same prompts in both interviews.

The instructional period of interest spanned from the introduction to distributions through the end of instruction on the Central Limit Theorem. Field notes from observations focused on the kinds of tasks students completed and the knowledge that was privileged. Interviews were completed with the instructor to capture both reflections about his instruction and his perceived goals for students. These interviews served to triangulate observations and limit bias in the first author's account of the events taking place in class.

**Methods of Analysis**

We take a grounded-theory approach to identify resources and search for links between formal instruction and student reasoning after instruction. To answer our first research question, we examined the pre-instruction interview for resources by identifying specific statements Karen made and relating them to more general kinds of reasoning (e.g., "I think a larger sample will buffer out the line" was interpreted as "more data means more accuracy"). We tagged spots that were ambiguous, often representing points when Karen's conceptual structure of distributions and mean tendency was potentially underdeveloped or self-contradicting. Using NVivo software, we applied general codes to interview statements and identified three resources.

To answer the second research question, we open-coded Karen's pre-instruction interview, focusing first on the elements of distribution to which Karen attended and the order to which she attended to those things (e.g., the range, the center/middle, the height, the extremes, etc.). We consulted the characteristics from Arnold and Pfannkuch (2015) to guide our organization of these elements. The post-instruction interview was coded in a similar fashion. We compared and contrasted Karen's reasoning before and after instruction to understand how the instruction influenced her reasoning. We also attempted to link elements of the instruction to reasoning approaches and explanations Karen provided in the second interview. We created a causal network diagram to relate Karen's reasoning before and after instruction through elements of the instruction and to synthesize these connections.

Throughout the process, we consulted tactics for generating meaning from Miles, Huberman, and Saldaña (2014). For example, we began the analysis by coding to look for patterns and themes, we clustered and partitioned codes, made comparisons and contrasts across Karen's pre and post interview, subsumed particular statements from the interviews into potential resources, and worked to make conceptual coherence of Karen's reasoning.

## Findings

Figure 1 shows Karen's drawings for the population distribution (on the left) and the sampling distribution for n=5 (on the right).

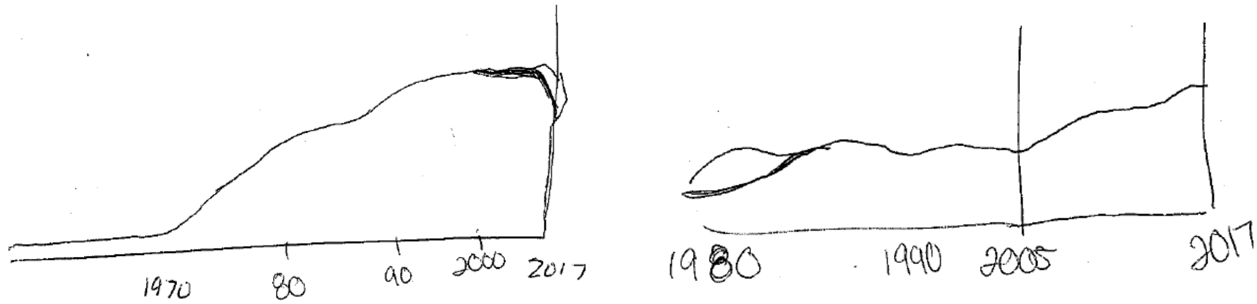


Figure 1. Karen's pre-interview drawings for the population and for sampling distribution with n=5.

### Resources

**Average is middle.** This was the first resource Karen used as she reasoned about the second prompt. She fabricated a likely sample she could imagine pulling from her change purse and thought about the average of that sample: "Let's say you get 5 that are like 1980, 1987, 1992, 1997, and then 2005, then it would be like, the average between that, so, like somewhere around 1993." This resource served as a starting point for reasoning about individual sample means, but was no longer activated when reasoning about the shape of the sampling distribution.

**The sample will resemble the population.** Karen said the following while drawing the sampling distribution for n=5: "Shapewise, [the population and sampling distribution are] not too different, like I have the lower down here and then it increases over time, and then it's kind of like a dip." In this description, Karen recognizes a relationship between these two distributions. What she is noting is a similarity in shape, with the sampling distribution (in her view) being a smoother, slightly flatter version of the population.

**Larger sample will produce more consistency.** This resource was activated as she continued to reason about the second prompt, and again as she reasoned about the third prompt with n=25. "I feel like this [the population distribution] would be less…like consistent than this [sampling distribution for n=5] would be, I feel like this [sampling distribution for n=5] would be not straight but straighter of a line." When reasoning about the distribution of sample means from even larger samples, she said, "if you take 1000 pennies, if there's not that many outliers, or if it's like evenly, dispersed number of years I guess, it would kind of make it a flatter line." She also mentioned penny averages being "more likely to change" when coming from small samples, thus viewing sample size as a sort of weighting component to variability in the mean.

### Two Resource Perspectives

In trying to make sense of the path of reasoning that Karen took, we identified two perspectives of reasoning. The first perspective focuses on the individual pennies and the influence of the population on the samples. This perspective was used frequently in early statements as Karen reasoned about individual pennies that could plausibly be in a sample she took from her change purse. The resources associated with this perspective were "the sample will resemble the population" and "average is middle."

The second perspective focuses more on abstract patterns. When Karen was asked to reason about larger samples, she shifted to thinking about "larger sample means more consistency." As a result, Karen believed there was a correct answer that the sampling distribution shape was approaching. Because she could no longer reason about specific plausible samples, she resorted to this more abstract resource.

**Instructional Influence on Karen's Activation of Resources**

Consistent with a constructivist perspective on learning, we view Karen's pre-instruction reasoning as representing an existing conceptualization of the relevant statistics content. Even though this knowledge structure was inconsistent and flexible, we do not believe Karen's conceptualization was a "blank slate" ready to receive and adopt correct knowledge structures. We view the formal instruction as a filter on Karen's reasoning and resource activation. Such a framing is neutral: the filter can be beneficial by refining, challenging, or introducing new resources, but this filter can also be detrimental by discouraging resource activation and promoting rote memorization with no deeper conceptual connection.

As she began to reason about the sampling distribution in the post-instruction interview, Karen frequently activated the resource that "average is middle." She pulled heavily on the Central Limit Theorem as evidence for why averages cluster in the middle, but as a definition absent of cognitive conviction. When probed to compare her initial interview drawings to her current ones, Karen recalled the previous resource of "larger samples produce more consistency" and her belief that larger samples will "buffer out" the sampling distribution to a flat line. At this point, Karen struggled to reconcile these seemingly contradicting resources. At the close of the interview, she questioned whether there might be a distinction between a frequency distribution and a probability distribution. Overall, Karen had not radically changed her thinking; she instead appeared to be suppressing a key resource, "larger samples produce more consistency," because it did not align with what she learned in class. "Average is middle," however, could still be aligned with the instruction.

## Implications and Future Work

On the surface, it is easy to miss that Karen did not cognitively adopt the Central Limit Theorem. Instead, it was a fact that she could articulate. While she attempted to justify it with one resource she had, she suppressed other resources with the potential to do so. If this pattern can be generalized, it is possible Karen might no longer attempt to reconcile future statistical content with her other resources when such a conflict exists and, instead, resort to a "cookbook" approach to statistical inference. Therefore, arming instructors with a list of relevant resources rather than a list of common misconceptions might lead to more cognitively robust reasoning and avoid leading students to a "cookbook" approach to statistical inference.

In the fall, we will conduct interviews with 8-10 students as we attempt to test and refine the resources we identified, search for others, look for negative cases, and make if-then tests about student reasoning in the instruction. Students will be pooled from a large-lecture introductory course, and students will again be interviewed before and after relevant instruction.

## Audience Questions

- Are there papers we are not aware of in the statistics education literature that take a resource framing of learning?
- In looking at our evidence, do you agree with the resources that we have identified?
- Are there benefits to still identifying misconceptions in student thinking on this topic?

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# Exploring the Pedagogical Empathy of Mathematics Graduate Teaching Assistants

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*Mathematics graduate teaching assistants (GTAs) are an important part of the mathematics education community. Recently, there has been a concentrated effort to better understand GTAs' pedagogical beliefs and teaching practices. The purpose of this study is to explore how GTAs would respond to student feelings and if their feedback to student questions can be characterized as attending to emotion. Data was collected through interviews of current GTAs in which participants were shown samples of student work and asked to respond to questions about that work. Preliminary analysis has revealed varying abilities of GTAs to express student feelings.*

*Keywords:* Graduate teaching assistants, students, empathy, emotion

Learning mathematics is an emotional experience for students (Hannula, 2002). Many studies have focused on the role of student affect in learning mathematics. However, little attention has been given to the relationship between teachers and student affect (Philipp, 2007). At the collegiate level, both emotional reactions and interpersonal relationships between teachers and students have been shown to influence what is learned in the classroom (Lowman, 1994).

Often, graduate teaching assistants (GTAs) play a large role in the instruction of lower level undergraduate mathematics courses (Speer, Gutmann, & Murphy, 2005). As a result, GTAs have opportunities to interact with a variety of students on a day-to-day basis and develop interpersonal relationships with them. These interactions likely influence GTAs' identities as teachers and shape their teaching philosophies (Kung, 2010). However, little is known about GTAs' teaching experiences and only recently has the mathematics education community begun to study their development as teachers and potential future faculty members (Kung, 2010; Speer et al., 2005).

This study seeks to add to the growing body of information about mathematics GTAs' pedagogical beliefs and teaching practices by investigating the awareness of GTAs to student feelings using a qualitative research design. The purpose of this study is to explore how GTAs would respond to student feelings and if their responses to questions can be characterized as attending to student emotions. With this in mind, our central research question is: *What are the characteristics of the responses that GTAs have to student questions on a typical pre-calculus problem?* To help refine our focus, we also pose the following two sub-questions:

1. Given sample written work on a typical pre-calculus problem, what feelings might GTAs attribute to students?
2. How might GTAs take student feelings into account when answering student questions?

After providing a brief summary of relevant literature, we give a detailed description of the methods that were used for this study and the data that was collected. Findings from preliminary analysis are also included, followed by some discussion and areas of future work.

## Literature

The basis for this study is found in three key areas of literature: the role of emotions in learning, the relationship between teacher affect and student affect in mathematics education, and the importance of empathy and caring in undergraduate mathematics education.

The process of learning is complex and involves both cognitive and affective factors. In particular, emotions have an effect on student learning and "the teacher has a significant role to play in shaping those emotions" (Mortiboys, 2012, p. 2). Many educational studies have discussed the role of two different types of knowledge needed for teaching: content knowledge and pedagogical knowledge (Grossman, Wilson, & Shulman, 1989; Ball, Thames, & Phelps, 2008; Shulman, 1987). However, Mortiboys (2012) contends that teachers should develop and employ a third type of knowledge, which he terms "emotional intelligence," in order to enhance teaching and address the needs of their learners. Thinking of learning as only a cognitive process deemphasizes the central role of emotions in decision making and learning. Teachers must be able to use emotional intelligence to acknowledge and address the emotions that their students feel while learning (Mortiboys, 2012).

In recent years, neuroscience researchers have found that interconnected neural processes support both emotion and cognition. In fact, it is "impossible to build memories, engage complex thoughts, or make meaningful decisions without emotion" (Immordino-Yang, 2015, p. 18). To better understand the relationship between emotions and learning, Hannula (2002) developed a framework to analyze a student's attitude towards mathematics using the psychology of emotions as a foundation. This framework separates attitudes into four evaluative processes:

> 1) the emotions the student experiences during mathematics related activities;
> 2) the emotions that the student automatically associates with the concept 'mathematics';
> 3) evaluations of situations that the student expects to follow as a consequence of doing mathematics; and 4) the value of mathematics-related goals in the student's global goal structure (Hannula, 2002, p. 26).

With respect to this study, we will focus on the first part of this framework, which attends to the emotions that students experience while working on math problems. Whereas the framework was analyzed from the perspective of a student, we aim to explore how the framework might be viewed from a GTA's perspective and how the responses of a GTA might take into account the initial process of the framework when interacting with students. In addition, we also explicate the relationship between feelings and emotion. Hansen (2005) defines feelings as conscious perceptions used to describe emotions. Because feelings are perceivable and can be articulated by the individuals who experience them, we use this term for discussing student displays of emotion. We also define pedagogical empathy as "the ability to express concern and take the perspective of a student" in accordance with the definition of teacher empathy given by Tettegah and Anderson (2007, p. 50).

In the math education literature, few studies have specifically addressed the intersection between teachers and affect (Philipp, 2007). However, it has been noted that, "all research in mathematics education can be strengthened if researchers will integrate affective issues into studies of cognition and instruction" (McLeod, 1992, p. 575). With respect to math education, the affective domain has been described as encompassing the beliefs, attitudes, and emotions of both students and teachers (McLeod, 1992; Philipps, 2007). In a summary of studies focusing on teacher affect and student affect in mathematics education, Philipp (2007) acknowledges that he knows of no research linking teachers' affect to their instructional decisions. Furthermore, he does not mention any research that connects teachers' responses to student affect.

Although there is limited research connecting teachers' responses to student emotions, previous studies have been conducted which highlight the importance of caring and empathy in higher mathematics education. Weston and McAlpine (1998) present a study where six math professors' characterized as outstanding teachers were interviewed to explore their views on teaching and learning. The most prominent teaching theme that emerged from the interviews was the importance that the professors placed on caring and concern for students. In their paper, the authors include suggestions of how to help teachers become more aware of having an "intentional caring perspective" which in turn, relates to developing pedagogical empathy. One recommendation they provide is to have professors engage in reflection upon their own experiences as learners in order to "recognize the importance of caring as part of the learning process" (Weston & McAlpine, 1998, p. 154).

In another study, Duffin and Simpson (2005) examine the link between cognitive styles and higher levels of cognitive empathy in graduate teaching assistants. As part of their study, the authors interviewed thirteen mathematics PhD students to explore their cognitive style of responding to learning new mathematics. During the interviews, many of the participants unexpectedly brought up experiences with teaching undergraduate students, which prompted the authors to consider the relationship between cognitive style and cognitive empathy. From the data, three levels of cognitive empathy emerged showing increasing levels of understanding how students might struggle with mathematics. These levels of cognitive empathy were then compared with the cognitive styles of the graduate students (Duffin & Simpson, 2005).

While it is apparent from the literature that there is a natural connection between student and teacher affect and that addressing emotions and feelings in the classroom is essential to student learning, this area has been understudied. Our study aims to help fill this gap and provide a qualitative way to capture pedagogical empathy.

## Data and Methods

The participants in this study were 14 mathematics GTAs at a large Midwestern university. Each GTA had at least one semester of experience as an instructor of record for a pre-calculus class. Data was collected during the 2016-2017 school year through structured interviews with the GTAs. Participants selected their own pseudonyms and are used below. During the interview, participants were asked to solve a typical pre-calculus problem in order to familiarize them with the problem. They were then shown five different samples of student work for the problem and asked to respond to questions about the work. The samples of student work were fictitious examples of actual student work based on the author's experiences as a pre-calculus instructor. Student questions about the work were presented through audio recordings intended to simulate an actual student asking the question. At the end of the interview the participants were asked to reflect on how they thought each student might have felt when working on the problem. Participants were also provided with a list of feeling words to use a reference during this part of the interview. After data was collected, select interviews were transcribed and analyzed using open coding.

## Preliminary Findings

By conducting this study, we aimed to find overarching themes related to the nature of responses that GTAs have while helping students in a pre-calculus class. Many of the responses to interview questions focused on helping students develop either procedural or conceptual math knowledge. However, several comments about student feelings surfaced during the interviews, even before being prompted by the interviewer to think about what feelings students might be experiencing in certain situations. In a few notable cases, GTAs were unable to articulate possible student feelings using descriptive feeling words. These responses were categorized under the code "Non-feeling" and were common among only a few participants. Table 1 shows the primary codes that have emerged from the interviews along with representative excerpts from the interviews. These codes help to categorize the characteristics of GTA responses.

*Table 1. Characteristics of GTA Responses*

| Code | Description | Interview Excerpt |
|---|---|---|
| Procedural Math Knowledge | Responses involving procedures, algorithms, rules, or formulas that do not directly attend to conceptual ideas | "If I plug in 25,000 students, do I get out the 13,000?" |
| Conceptual Math Knowledge | Responses connected to underlying concepts of math content including discussion of abstract ideas or relationships | "What sort of function are we trying to come up with here?...Think of the function as a machine." |
| Student-centered Reflection | Evidence of reflection centered around student thinking or past experiences with students, but not directly related to student feelings or emotions | "I think it's important to teach them how to identify their own mistakes." |
| Instructor-centered Reflection | Evidence of self-reflection that is centered around the participant, rather than students, including personal beliefs | "I would look through all of it…so that I'm prepared when something goes wrong." |
| Student feelings or emotions | Anything about what a student might be feeling or anything related to emotions that students might experience | "It's less that they don't know the math and more sort of fear or being uncomfortable with story problems." |
| Non-feelings | Use of words that are unrelated to emotions to describe what a student might be feeling when specifically prompted | "They probably feel medium." |

## Discussion and Future Work

The characteristics listed above help answer the central research question by classifying the types of responses that were common among GTAs. Both procedural and conceptual knowledge was a central focus for most GTAs. However, it is also important to note that GTAs also used both student-centered and instructor-centered reflection during their interviews when responding to student questions. In addition, the presence of potential student feelings or emotions was also brought up by many of the GTAs, although only some of them were able to express those feelings clearly.

Further analysis is ongoing to identify common feelings that GTAs might attribute to students. A few of the feelings that were mentioned in interviews included fear, confusion, uncertainty, and confidence. From the preliminary analysis, it is evident that the GTAs who participated in this study were aware of the potential for student feelings to arise when working on a math problem, but varied in their ability to express those feelings. For example, one GTA, James, found it difficult to attribute emotions to students:

*James*: I guess I have a hard time ascribing emotion to people as they're working on math problems. That's not something I really consider too much.

However, there were other GTAs who were able to articulate student feelings. In addition, a couple of these GTAs mentioned how taking account of student feelings was something that they already did when answering student questions:

*Nicole*: I think that's something that I do think about when a student asks me a question, like where they are not only mathematically but also emotionally.

*Aubrey*: I try to think on the spot about how they're feeling and look at people's faces…I try to pay attention to how they're feeling.

This preliminary analysis of the data has revealed differences between the abilities of the GTAs to describe and account for possible student feelings. These differences provide a rich area for further analysis of the data that has already been gathered, the results of which we hope to present at the conference. In addition to presenting further analysis of the data, we also look to ask the audience a few central questions to help us better answer the research questions.

**Questions for the Audience:**

Within the field of GTA professional development (PD) there is a great deal of anecdotal experience. We wish to check our data to see if it is representative of the experience of others who provide GTA PD. To this end, we intend to ask the audience:

- In your experience, working with both instructors and students, are there characteristics of responses to student questions that are missing?

These interviews provide a rich set of data on GTA responses to students. However, the data did not provide an answer to our second research sub-question: How might GTAs take student feelings into account when answering student questions? To help direct future avenues of research, we also intend to ask the audience:

- What data, or analysis, would you recommend we collect in order to better answer research sub-question 2?

Finally, neither the research questions nor the data collected directly address the content of professional development programs for GTAs. Nevertheless, we are interested in how PD activities can better incorporate aspects of pedagogical empathy. To this end, we intend to ask the audience:

- How can the findings from this research study be incorporated into professional development activities for GTAs?

Existing literature shows the value of empathy in the classroom. However, the existing literature does not address how that empathy is developed or expressed in the collegiate mathematics classroom. This study has begun to outline some characteristics of empathetic interactions that might exist in the classroom. Further research should expand upon these characteristics and help connect experiencing pedagogical empathy with communicating that empathy to students.

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# Physics students' construction of differential length vectors for a spiral path

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*As part of an effort to examine student understanding of non-Cartesian coordinate systems and differential elements related to vector calculus, we interviewed students using tasks similar to typical electricity and magnetism problems. In one task, students were asked to calculate the change in electric potential along a spiral path, involving a common line integral. Analysis focused on conceptual understanding and symbolic expression of differential length vectors. Students were heavily drawn to the angular motion of the path through the radial electric field, often only expressing the angular component of the length vector. This contrasts with earlier work, suggesting context may distract from correct mathematical expression.*

*Key words:* Physics, Differential Elements, Vector Calculus, Multivariable

## Introduction and Relevant Literature

Students' use of vector calculus in upper-division physics is fundamental to developing an understanding of various principles in electricity and magnetism (E&M). However, the choice of differential elements in vector calculus, specifically line integration, depends on physical symmetries of electric and magnetic fields created by charged objects and current-carrying wires, respectively. As these radial and curling fields are often most simply expressed using non-Cartesian coordinates, the appropriate differential lengths differ from the Cartesian $dx\hat{x} + dy\hat{y} + dz\hat{z}$. The curved paths resulting from angular movement lead to differential length components that are arc lengths and that include scaling factors in cylindrical and spherical coordinates (*i.e.*, $dl_\theta = rd\theta\hat{\theta}$ for spherical coordinates in physics).

Additionally, line integration in physics takes on a different form than in typical mathematics courses. Line integrals having physical application are rarely solved using parametric curves (Dray & Manogue, 2003), and vector calculus in physics is typically non-Cartesian (Dray & Manogue, 1999). The high incidence of symmetry in physical situations allows students to choose a particular component of a differential length vector in a given coordinate system, rather than compute the specific dot product explicitly.

Given the different expressions of differential length elements across coordinate systems and the importance of vector calculus in E&M, we seek to investigate the following questions:

- To what extent do students understand the symbolic expressions and conceptual aspects of differential length vector elements [in non-Cartesian coordinates]?
- How do students use their understanding to construct differential length elements?

Earlier work explored students' understanding of the differential length vector absent of physics context by asking students to construct a length vector in an unconventional spherical coordinate system that we named schmerical coordinates (Schermerhorn & Thompson, 2016a; Schermerhorn & Thompson, 2016b). Current work seeks to build on this by investigating students' solving of line integrals closely related to E&M tasks. This allows insight into how aspects of E&M tasks do or do not influence students' determinations of differential elements, which informs the instruction of differential elements and coordinate systems in physics courses.

Previous work related to mathematics in E&M has sought to address students understanding of integration and differentials where the differential is a scalar element of charge or resistance

(Doughty *et al*., 2014; Hu & Rebello, 2013; Nguyen & Rebello, 2011). Research attending to student understanding of vector calculus in E&M has primarily addressed student application and understanding of symmetries associated with Gauss's and Ampère's Laws, two common aspects of an E&M course that involve a surface integral and line integral, respectively (Guisasola *et al*., 2008; Manogue *et al*., 2006; Pepper *et al*., 2012). Researchers turning to vector differential operators have explored student understanding and calculation of gradient, divergence, and curl in both mathematics and physics settings, finding student difficulty in interpreting vector fields despite excelling at calculation (Astolfi & Baily, 2014; Bollen *et al*., 2015; Bollen *et al*., 2016). Little of this work has specifically explored student understanding of the differential vector element as it appears in the non-Cartesian systems used commonly in physics.

## Theoretical Perspective

To explore students' construction of differential length vectors in more typical E&M contexts, we extend theoretical perspectives from previous work on differential length construction in the unconventional coordinate system to allow comparison between tasks. The *symbolic forms* framework (Sherin, 2001) provides insight into students' development of the structure of differential vector elements and determination of how each component is represented in the final equation, while a *concept image* analysis (Tall & Vinner, 1981) gives insight into the particular ideas and aspects to which students attend during construction.

Based on the knowledge-in-pieces model (diSessa, 1993), symbolic forms was developed to explain students construction of expressions when modeling physical situations common to introductory physics (Sherin, 2001). A symbolic form represents the combination of a symbol template and a conceptual schema. The symbol template, an externalized structure such as $\square + \square + \square$, represents the skeleton of an expression containing variables and/or numbers. A student's conceptual schema is the requisite internalized (mathematical) understanding of the role of the template. For example, if students recognized the need to sum multiple quantities that added to a larger whole, they would invoke the $\square + \square + \square$ template. The resulting template-schema pair used here is known as *parts-of-a-whole* (Sherin, 2001).

Meredith and Marrongelle (2008) adapted the conceptual aspects of symbolic forms to describe students being cued to integrate by recognizing reliance on a particular variable (*dependence* symbolic form), or the need to sum up pieces (*parts-of-a-whole)*. The ideas of symbolic forms were expanded to address calculus students' understanding of integrals, often mediated by graphical representations (Jones, 2015). Work exploring physical chemistry students' use of partial derivatives in thermodynamics found that recall mediated students' use of symbolic forms (Becker & Towns, 2012).

A constraint of a strict symbolic forms analysis is that it only yields procedurally based mathematical justifications for the symbolic arrangements and expression structure, neglecting how content understanding plays a role in why the structures or terms are needed. Importing the concept image framework (Tall & Vinner, 1981) from mathematics education rounds out the investigation of conceptual schemata. A student's concept image is a multifaceted understanding including any properties, processes, etc., a student may have about a given topic. A concept image for integration may contain area under the curve or Riemann sums (Doughty *et al*., 2014). It may also contain a specific rule such as that the indefinite integral of $nx^{n-1}dx$ is $x^n + C$, with or without a specific understanding of why that is the result. By incorporating the concept image framework, the symbolic forms analysis gains a contextual meaning associated with students elicited content understanding, which is not explicitly addressed by the conceptual schemata.

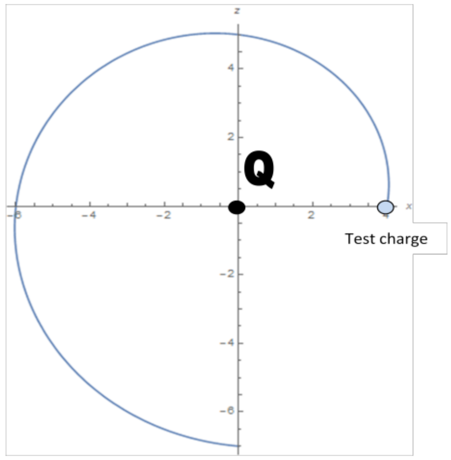

Figure 1: Image given to students, depicting the charges and spiral path of the test charge.

## Methodology

In order to investigate students' performance on typical E&M problems students were given a point charge, Q, centered at the origin (Fig. 1). Students were asked to find the differential length vector for a spiral path given by $r = 2\theta/\pi$ in the *xz*-plane and to find the change in potential experienced by a test charge as it moved along the path from the point (4,0,0) to (0,0,-7). The spiral path complicates the task since it requires two differential length components to describe it completely: $d\vec{l} = dr\hat{r} + rd\theta\hat{\theta}$. The electric field due to a point charge is a highly symmetric case where electric potential depends only on changes in the radial direction. For a typical task students only need $dr\hat{r}$ when computing this line integral. This report focuses mainly on students' construction of $d\vec{l}$ to make comparison to generic $d\vec{l}$ construction.

The task was administered in a clinical think-aloud setting with two pairs of students (B&H, D&V) and six individual students (J, K, L, M, N, O) at one university and one individual (T) at a second university. All students were enrolled in the second semester of a two-semester, junior-level E&M sequence. Pseudonyms are provided for students corresponding to their identifying letter (*i.e.,* Jake for J). This particular question took students about 10-20 minutes in interviews.

Video interview data were transcribed and analyzed using a modified grounded theory approach, with the goal of identifying student attention to *symbolic forms* and the associated aspects of students' *concept images* in line with previous findings, while additionally looking for new aspects now appearing because of the applied context. Previously identified symbolic forms include those consistent with Sherin (2001): *parts-of-a-whole*, *coefficient*, and *no dependence*; and new forms to account for the increased mathematical sophistication: *differential* and *magnitude-direction* (Schermerhorn & Thompson, 2016b). The concept images often spurring the need for these templates or necessary terms included *component and direction*, *dimensionality, differential*, and *projection*, as well as specific associated actions, such as *recall, grouping*, and *transliteration* to other coordinate systems (Schermerhorn & Thompson, 2016a).

## Results

Data analysis showed attention to many of the relevant symbolic forms and concept images identified in the schmerical differential length task, but among fewer students.

In particular, *parts-of-a-whole* (PW) and *magnitude-direction* (MD), both prominent in the acontextual task, were generally absent for students' construction in the spiral task. Five students invoked PW, described earlier as students' recognition of parts summing up to a whole with the template $\square+\square+\square$. However, only one applied a polar coordinate system and initially included MD. MD accounts for the magnitude and unit vector parts of a quantity and is associated with the template $\square\hat{\square}$. Both these symbolic forms are associated with the *component and direction* concept image, where students would recognize that differential length vectors need multiple components, and that each component corresponds to motion in a specific direction. The following transcript illustrates a correct response and highlights the *component and direction* aspect needed for differential length vector construction:

> Molly: Yeah, and then you go a little bit…I'm picturing you go from this point to this point …So first I travel in the $r$ direction so I go dr in the $\hat{r}$, and then I travel in the $\hat{\theta}$ direction and the arc length of a circle is the radius times the angle that you move so that is $rd\theta$, here in the $\hat{\theta}$. (Fig. 2a)

Molly appropriately separates each component as two distinct motions ("I travel"), then encodes each length as the magnitude and the corresponding direction as the unit vector, resulting in a correct $d\vec{l}$.

Two other students invoked PW without encoding components with a MD template. Neither student specifically attended to the directions each component traced out, resulting in differential length components absent of unit vectors (Figure 2b, 2c). Kyle's transcript demonstrates this:

> Kyle: We stay in the one plane… so we're only changing by $\theta$ and $r$, so it we have some $d\theta$ or let's say $\Delta\theta$, then $\Delta r$ is going to be $2\Delta\theta/\pi$, so the actual length is the change in the radius and the change in the angle times the radius so that we stay in units of length.

Upon recognizing a need to account for a dot product during the later integration, both students added unit vectors to each of their terms.

Both of the above transcripts also highlight students' multiple concept images of the *differential,* accounting for "a little bit" of or "changes" in variables, consistent with students' ideas of differentials identified in the literature (Artigue *et al*., 1990; Hu & Rebello, 2013; Roundy *et al*., 2015; Von Korff & Rebello, 2012). These ideas cue students' invocation of the differential symbolic form: representing a differential quantity with template d$\square$.

The last two students to invoke the PW template used Cartesian coordinates. They both mentioned needing small changes in $x$ and $y$, rather than starting in a more appropriate polar coordinate system. Oliver attempted to differentiate coordinate transformations for $x$ and $y$ with respect to $\theta$ in order to express $dx$ and $dy$. Tyler began similarly but then suggested that a spherical transformation would produce $dl = r^2 \sin\theta \, dr d\theta$. He reduces his $dl$ down to one component without addressing a need to maintain a sum of two components, or directionality.

The remaining interview subjects only attend to one component, neglecting both the PW and MD symbolic forms. Dan and Victor addressed just the change in the $r$ direction, ignoring the change in $\theta$ as irrelevant to calculation (Fig. 3a). While this does lead to the correct solution for the potential difference, the length element for the path is incomplete without the $\theta$ component.

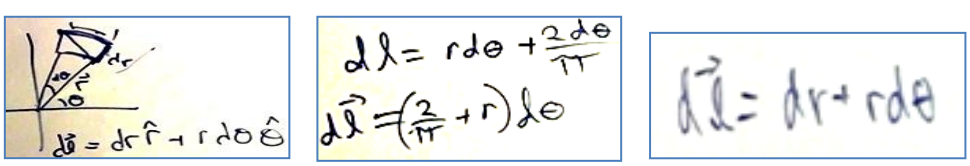


Figure 2. Left to right: (a) Molly's correct differential length elements. (b) Kyle's and (c) Jake's differential length elements absent of unit vectors.

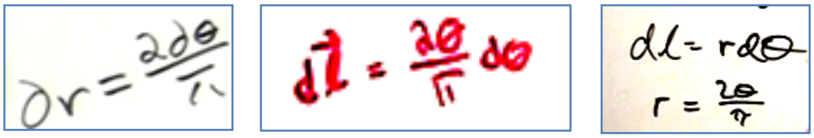


Figure 3. Left to right: (a) Dan and Victor's accounting for only change in $r$-direction and converting to terms of $\theta$. (b) Nate's $dl$, with function replacing $r$ in $rd\theta$. (c) Bart and Harold's $dl$, where the function for $r$ is written with the term to account for changes in $r$ along the path.

The three remaining students only account for the $\theta$ component (Figs. 3b, 3c), correctly including the $r$ in the arc length and including the functional relationship to write the length component in terms of $\theta$:

> Nate: I think I'm going to move just a tiny bit. This point changes, and so $r$ is going to change and $[\theta]$ is going to change… $r$ is going to be obvious because I think it's going to be $[2\theta/\pi]$ and then $[\theta]$ would just change some $d[\theta]$… To me it makes sense, because you're moving some infinitesimal amount in $\theta$ and then you have that $r$ change.

This reasoning appeared across multiple interviews. Students still recognize the need for change in particular variables, an evoked concept image that results in the differential symbolic form. Here students use the functionality of $r$ on $\theta$ and the inclusion of $r$ in arc length to account for $r$ changing. This appears to supersede their need to include change in $r$ as a separate component of the differential length. The need to include a $dr$ is entirely absent from their constructions.

## Conclusions

Analysis of student interviews on differential length construction on a more typical E&M task reveal that students are not as attentive to the vector nature of differential elements compared to similar construction in the unconventional spherical coordinates. This may be due to familiarity with the high symmetry of many tasks in E&M that allow students to select one component of a length or area vector and disregard others. Typically for a task involving a spherically symmetric electric field, students would usually select the $\hat{r}$ component. However, students interviewed on the spiral task are largely only selecting the $\theta$ component. For these students the change in $r$ is activated, but where in schmerical coordinates this would result in an expression of $dr$, it appears the salience of $\theta$ changing and a functionally dependent $r$ being a variable in the arc length, allows students a justification for their choice of one component.

Student use of $\theta$ was prominent. Almost all students, even those expressing multiple components, worked to express their final differential length vector in terms of $\theta$, despite the simplicity of integrating a radially dependent field in terms of $r$. This focus on $\theta$ is most likely due to the salience of the circular nature of the path and/or the functional form of $r$ given to students. It is additionally possible the recent familiarity with circular symmetry in E&M II and Ampère's Law, played a role in students' emphasis on the $\theta$ component.

Whereas students easily recognize the need for multiple components for the general expression of the differential length vector, in this more typical task embedded in a physics context, students have difficulty recognizing the need to separate out directions. We seek to investigate students' work on these tasks without a function for the path, to see if this leads to inclusion of the $\hat{r}$ term. Current instructional implications speak to more emphasis on connecting whole differential length vector construction to the determination of terms based on symmetry arguments However, more work is needed to make specific claims regarding students' choices.

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# Engaged Learning Through Creativity in Mathematics


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## Abstract

*This study examined the impact of a deliberate attempt to present creativity as a mathematical endeavor on students' conceptions about mathematics, attitudes toward learning mathematics, and approaches to mathematics. Course modules were developed as part of a grant-funded project on creativity in STEM education and implemented in a course for non-mathematics majors. Throughout the course, students engaged in hands-on, active learning of mathematics through discovery. The mathematical topics for the course were chosen based on their relevance to students' everyday lives and their suitability for encouraging creativity. Data were collected through surveys, focus group interviews, and written artifacts. In this paper, we describe the preliminary results of our study and offer implications for both research and teaching.*



*Keywords:* Creativity, General Education Mathematics Courses, Instruction


## Introduction

Creativity is often associated solely with the arts. An online search gives the following definition: "The use of the imagination or original ideas, especially in the production of an *artistic* work." (Oxford Dictionary Online, 2017; italics added). Although creativity is valued among mathematicians (Sriraman, 2004), undergraduate mathematics students do not consider creativity to be important to mathematics (Munakata & Vaidya, 2015). Our Creativity in Mathematics and Science Project, funded by the National Science Foundation, challenges these traditional notions of both creativity and mathematics. Through our project, we developed and implemented course modules that embraced creativity in mathematics for a general education mathematics course and assessed the impact of some innovative teaching methods on students' conceptions of STEM (science, technology engineering, and mathematics), their approaches to mathematics, and their understanding of what it means to "do math".

## Problem statement

Our research explored the impact of a course that deliberately drew upon theories related to creativity in its development and implementation. Namely, we sought to uncover whether this course influenced students' conceptions about mathematics and their approaches to mathematics. The following were our research questions:

1.) Does a deliberate attempt to infuse the undergraduate curriculum with a focus on the creative process lead to changes to students' perceptions of mathematics and attitudes toward mathematics learning?

2.) What is the impact of this instructional strategy on students' creativity, especially as it concerns approaches to mathematics?

## Relation of this work to the research literature

Recent discussions in undergraduate education have proposed active (Wieman, 2014), inquiry-based (Singer, Hilton, & Schweingruber, 2006), and problem-based learning (Freeman et al., 2014). In the STEM fields, especially, there has been a call to make instruction more relevant to students' needs (DeHaan, 2009), and to have it exemplify the work of scientists and mathematicians (NRC, 2000). All too often, undergraduate STEM education is relegated to learning through traditional, lecture-style instruction, with problems and laboratory experiences dictated by questions and exercises posed without regard to context (DeHaan, 2009). There is a need to shift learning away from the acquisition of facts and procedural knowledge and to environments that encourage innovation (Southwick, 2012). This runs parallel to the need to cultivate adaptive expertise in our students whereby they are exposed to opportunities to be flexible and adaptable in problem-solving situations (Cropley, 2015).

The discussion about effective mathematics instruction is particularly important when considering students in general education courses. The fact that these courses for non-majors are often students' terminal courses in mathematics--and that they are composed of students representing various disciplines and interests--adds to the complexity of the issue. Many universities offer courses in the application of mathematics, especially related to societal issues (such as voting) or everyday interests (such as sports and arts), while others focus on practical uses such as finite mathematics for finances. We decided to put aside the list of mathematical topics usually covered, and have the topics emerge naturally from the processes we sought to encourage in our students. That is, we first identified creative processes and learning objectives for the course, then developed content-based modules we believed encouraged these processes.

Our work set out to consider the impact of such a course on students and to use the results to inform further revisions to the course.

## Conceptual framework

Our project draws from different works in creativity--from both psychology and education. As creativity is notoriously difficult to define, we chose to focus on the various traits of creativity identified by researchers (e.g., Amabile, 1996; Hadamard, 1954; Sternberg and Lubart, 1996). These traits include the ability to connect ideas, see similarities and differences, be flexible, have aesthetic taste, be unorthodox, be motivated, be inquisitive, and question norms. We also considered what others have noted as being essential to the work of mathematicians: divergent thinking, and the ability to identify new problems and contribute new knowledge (Nadjafikhah, Yaftian, & Shahrnaz, 2012). In the mathematics classroom, some have suggested problem posing as a way to encourage creativity (e.g, Silver, 1994). With the understanding that the difference between the nature of creativity of mathematicians and of students is chiefly that of degree and level (Hadamard, 1954), we sought to develop a mathematics course that centered around creativity.

Our course introduced students to topics in mathematics of relevance to their daily lives. The course aimed to expose students to the wonders of mathematics and covered various topics in discrete and continuous data modeling, fundamental aspects of Euclidean and non-Euclidean geometry, fractal geometry and probability theory. We took a hands-on approach to learning since we felt that mathematics is best understood by active learning methods such as doing problems, discussions and debates and even performing experiments rather than passively listening to a lecture. No formal textbook was prescribed; all necessarily materials were handed out to the students in class or through our online system, as needed. The only prerequisite was a very elementary knowledge of mathematics which may be required for any college course.

## Research methodology

This study was conducted at a state university in the Northeast US. The institution was recently designated as a Research III doctoral institution and enrolls a little over 20,000 students, including undergraduate, master's, and doctoral students. The university has historically enrolled large numbers of students who are first-time college attendees in their families, and prides itself on its diverse population of students, having the distinction of being a Hispanic Serving Institution.

The course, Contemporary Applied Mathematics, is one of three courses for students who are not STEM or education majors: it fulfills their mathematics general education requirement. The course was taught in Spring 2017 and enrolled 36 students. Three of the four authors developed the course and co-taught the course during the semester: one instructor was the lead, and the other two led certain classes and otherwise assisted or took notes on the course discussions. We met weekly to debrief about the most recent class meeting and to plan for future meetings.

Since this was a general education course for non-science and math majors, the class represented diverse majors from outside science and mathematics including the arts and humanities. Few of these students, if any, had experience with the kinds of mathematical topics that were being discussed and few had seen mathematics presented quite in the hands-on and open-ended form that we adopted.

We employed both quantitative and qualitative research methodologies. Data were collected from the 35 students (21 female, 14 male) who consented to participating in the study. We collected data through semi-structured focus group interviews, surveys, journals, class assignments and two well-known measures of creativity (Guilford, 1958; Torrance, 1965). For the quantitative measures, we compared pre- and post-test gains against a those of a comparison group. This paper will focus on the results of the qualitative data—namely, the focus group interviews, reflective journals, and classroom artifacts. We have completed analysis of the interviews (of 12 students) and are expecting to complete analysis of the journals and written artifacts (from all 35 consenting students) in the next month.

Seven semi-structured group interviews were conducted with 12 students during the last two weeks of the course. The interviews were audio taped and transcribed. The purpose of the interviews was to collect information on the students' attitudes, beliefs, and opinions about the creativity course. The interviews were coded into seven initial categories and later broken down into subcategories. (Please see Table 1.)

During the interviews, students most often discussed the instructional and teaching strategies ($n$=106 times) employed by the instructors of the course. Overwhelmingly, these excerpts distinguished differences between traditional instruction of mathematics and science and the instruction in the creativity course. Students often described specific instructional activities they completed during the course ($n$=72) and discussed how the course influenced their thinking and learning of mathematics ($n$=67).

*Table 1. Subcategories of the Main Themes*

| Theme | Subcategories |
|---|---|
| Mentions Specific Activities | Alignment Among Curricular Materials |

| | |
|---|---|
| | Connections to Personal Life and Knowledge<br>Non-typical Problems<br>Thinking Differently<br>Working as a Community |
| Discusses Instructional/Teaching Strategies | Teaching How to Think<br>Atypical Answers<br>Instructor Qualities<br>Active Learning<br>Discussing Ideas<br>Unfinished Problems<br>Flexibility<br>Vision for Education |
| Discusses Their Ways of Thinking or Learning | Depth of Understanding<br>Discussing Ideas<br>Confidence and Stress<br>Thinking Differently |
| Discusses Creativity | Surprise Math Can Be Creative<br>Value of Creative Approach |
| Mentions Future Aspirations | Helpful for Their Career Choice<br>Applicability to Multiple Majors<br>Transfer to Their Everyday Lives |
| Conceptions About Mathematics | Confidence<br>Broadens Thinking<br>Challenges Past Beliefs |
| Other | Connection to Other Disciplines<br>Frustration: not given final answer/unfinished<br>Procedural vs Conceptual Knowledge<br>Standardized Tests<br>Enjoyable<br>Grading |

The following offers descriptions with exemplar excerpts for some of the subcategories:

**Teaching How to Think**

Students pointed out that the instructors encouraged students to think mathematically. Therefore, they created a culture where students were not taught what to think, but were active participants in their learning. An excerpt from Mia highlights the contrast between the creativity course and her experiences in the past:

> I think it's really different from like a lot of the courses I've taken um, just because they don't teach in like a conventional way. You know, like, in every other math course I've ever taken it's been like you learn the theory and then like you learn um, they give you like examples, and then you do the homework, and then you take a test. And like it's just like one week doing that whole thing, and this is kind of teaching you the theories but not like I don't, it's hard to explain, with the numbers and stuff in it. So, it's teaching you the way of thinking, I guess, without making it seem as difficult. If that makes sense.

**Discussing Creativity**
There were two subcategories for students discussing creativity: surprise that math can be creative and value of the creative approach. For example, Ann stated, "I think it's different because it's the first class I've taken that tells you that math can be creative instead of just logical. You needed it for practical things, not creativity." Many students had initial thoughts of creativity as art, photography, film, or other endeavors not generally associated with science or mathematics. In general, students believed mathematics to be rule-based, logical, and formula-driven, so they were surprised when they were encouraged to be creative.
Students found that an emphasis on creative approaches stimulated new ideas, allowing them to look at things differently and be less afraid to try something different. They claimed that the creative activities helped them remember what they had learned. Many remarked that younger learners tend to get more creative opportunities in math, but that diminishes in middle and high school. Corey further explained:

> They are more pushing creativity than like getting a solid right answer using the right formula...I've taken Calc so always have to memorize things always have to get the right answer in order for me to get credit. So it is definitely different. Because as long as you are being creative, like supporting how are you getting an answer, it is acceptable...But once you get going and like you said, be creative and thought out of the box helps me become more creative. Because I am very like, I have been learned to go straight forward, use these problems, get the right answer and you're done. Whereas now it's pushing me to like think outside the box and that's not something I am used to.

## Implications for Teaching Practice and Further Research

This study was based on a first attempt at implementing our newly developed course modules. Our plan is to revise our modules based on our results and implement them once more in several sections of the general education course. The results of our pilot study thus far, however, have indicated changes to students' conceptions about mathematics. The study has potential to inform the curriculum of other mathematics courses for non-majors. Namely, the preliminary results indicate that a deliberate attempt to encourage creative thinking among students can influence their confidence, broaden their thinking about mathematics, and even guide their career choices. These traits are especially relevant for non-mathematics majors, who will most likely not take additional mathematics courses.

We are currently implementing the course modules in a similar course (for non-mathematics majors) at our local community college and also in a first-year seminar course for mathematics majors at our institution. The aim of the implementation in the latter course is to expose students to a new way of thinking about mathematics as they begin undertaking the mathematics course sequence. Our continuing research is expected to further elucidate the impact of our instructional strategy on various populations of undergraduate students.

## Discussion Questions

1. What is the place of mathematics content in a course that promotes creative approaches to mathematics (and other disciplines)?
2. How can mathematics courses for majors embrace creativity?
3. How do you assess creative approaches to mathematics?
4. What research would help practitioners consider this teaching innovation?

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# Student Understanding of Linear Combinations of Eigenvectors


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*Student understanding of eigenspace seems to be a particularly understudied aspect of research on eigentheory. To further detail student understanding of eigenspace relationships, we present preliminary results regarding students' reasoning on problems involving linear combinations of eigenvectors in which the resultant vector is or is not an eigenvector of the matrix. We detail three preliminary themes gleaned from our analysis: (a) using the phrase "is a linear combination of" to support both correct and incorrect answers; (b) conflating scalars in a linear combination with eigenvalues, and (c) reasoning about the dimension of eigenspaces versus a number of eigenvectors.*



*Keywords:* Linear Algebra, Student Reasoning, Eigenspace, Linear Combination


## Purpose and Background

Linear algebra is particularly useful to science, technology, engineering and mathematics (STEM) fields and has received increased attention by undergraduate mathematics education researchers in the past few decades (Dorier, 2000; Artigue, Batanero, & Kent, 2007; Rasmussen & Wawro, in press). A useful group of concepts in linear algebra is eigentheory, or the study of eigenvectors, eigenvalues, eigenspaces, and other related concepts. Eigentheory is important for many applications in STEM, such as studying Markov chains and modeling quantum mechanical systems. Despite this importance, research specifically focused on the teaching and learning of eigentheory is a fairly recent endeavor and is far from exhausted.

One aspect of eigentheory that seems to be particularly understudied is eigenspace, including how students understand linear combinations of eigenvectors. Some research on eigentheory has included eigenspaces but not as the main focus. For instance, Salgado and Trigueros (2015) found that students struggled to construct the concept of eigenspace as well as to coordinate the number of eigenvectors corresponding to a given eigenvalue with the dimension of the space spanned by the eigenvectors of that eigenvalue. Gol Tabaghi and Sinclair (2013), on the other hand, found that exploration of a two-dimensional "eigen-sketch" in Geometer's Sketchpad helped students understand the existence of multiple eigenvectors for a single eigenvalue as they dragged the vector $\boldsymbol{x}$ along the line of the eigenspace. Lastly, Beltrán-Meneu, Murillo-Arcila, and Albarracín (2016) gave students a test question asking if various linear combinations of eigenvectors in $\mathbb{R}^2$ would also be eigenvectors; they found students either reasoned symbolically by explicitly verifying the eigen-equations for the numerically given matrix and vectors, or formally by reasoning about the resultant vectors belonging or not belonging to an eigenspace.

In order to more explicitly explore students' understanding of eigenspaces and extend research beyond 2×2 matrices, the research question for this study is: How do students make sense of and reason about linear combinations of eigenvectors?

## Theory and Literature Review

This report is part of our ongoing effort to analyze students' understanding of eigentheory. In doing so, we ground our work in the Emergent Perspective (Cobb & Yackel, 1996), which is based on the assumption that mathematical development is a process of active individual construction and mathematical enculturation. In this report we focus on the mathematical conceptions that individual students bring to bear in their mathematical work (Rasmussen, Wawro, & Zandieh, 2015). The literature on the teaching and learning of eigenvectors and eigenvalues points to several aspects of eigentheory that are important as students build their understanding. Here we summarize that literature by highlighting what we have found to be the most important aspects for building a theoretical framework for eigentheory.

Thomas and Stewart (2011) found that students struggle to coordinate the two different mathematical processes (*matrix* multiplication versus *scalar* multiplication) captured in the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense of equality as "yielding the same result" between mathematical entities (i.e., two equivalent vectors), an interpretation that is nontrivial or even novel to students (Henderson, Rasmussen, Sweeney, Wawro, & Zandieh, 2010). Furthermore, students have to keep track of multiple mathematical entities (matrices, vectors, and scalars) when working on eigentheory problems, all of which can be symbolized similarly. For instance, the zero in $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ refers to the zero vector, whereas the zero in $\det(A - \lambda I) = 0$ is the number zero. This complexity of coordinating mathematical entities and their symbolization is something students have to grapple with when studying eigentheory.

Thomas and Stewart (2011) also posit that this complexity may prevent students from making the symbolic progression from $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ through the introduction of the identity matrix, which is often an important step in solving for the eigenvalues and eigenvectors of a matrix $A$. In their genetic decomposition of eigentheory concepts, Salgado and Trigueros (2015) also point out the importance of understanding the equivalence of the two equations through a coordination of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and solutions to homogeneous systems of equations. Harel (2000) posits that the interpretation of "solution" in this setting, the set of all vectors $\boldsymbol{x}$ that make the equation true, entails a new level of complexity beyond solving equations such as $cx = d$, where $c, x,$ and $d$ are real numbers. Our own work indicates student reasoning when solving eigentheory problems may be influenced by their reliance on or preference for one of the two eigen-equations (Watson, Wawro, Zandieh, & Kerrigan, 2017).

Hillel (2000) found that instructors often move between geometric, algebraic, and abstract modes of description without explicitly alerting students; although the various ways to think about and symbolize linear algebra ideas are second nature to experts, they often are not within the cognitive reach of students. In fact, Thomas and Stewart (2011) mentioned that students in their study primarily thought of eigenvectors and eigenvalues symbolically and were confident in matrix-oriented algebraic procedures, but "the vast majority had no geometric, embodied world view of eigenvectors or eigenvalues … losing out on the geometric notion of invariance of direction" (p. 294). In contrast, other researchers have shown how exploration of eigentheory through dynamic geometry software (Çağlayan, 2015; Gol Tabaghi & Sinclair, 2013; Nyman, Lapp, St John, & Berry, 2010), stretching geometric figures by a linear transformation (Zandieh, Wawro, & Rasmussen, 2017), gesture, time, and space (Sinclair & Gol Tabaghi, 2010), or real-world contexts (Beltrán-Meneu et al., 2016; Salgado & Trigueros, 2015) can be beneficial to developing conceptual understanding of eigentheory. We similarly agree on the importance of understanding eigentheory concepts in multiple ways and successfully navigating between these various modes of description.

## Methods

The data for this study come from student written responses to the 6-question Eigentheory Multiple-Choice Extended (MCE) Assessment Instrument (Watson et al., 2017). This MCE aims to capture nuances of students' conceptual understanding of eigentheory and to inform our working framework of what it might mean to have a deep understanding of eigentheory. This work is part of a larger study of student understanding of eigentheory in mathematics and physics. However, this paper focuses on data from one sophomore-level introductory linear algebra class, at a university in the eastern United States. For this paper we focus on student responses to Questions 3 and 5 (Q3 and Q5), which are about linear combinations of eigenvectors (Figure 1). Of the 28 students in this class, 27 answered Q3 and 23 answered Q5. For each, students selected an answer to the multiple-choice stem and then were to respond to the open-ended prompt: "Because…(Please write a thorough justification for your choice)."

3. Suppose $A$ is a $n{\times}n$ matrix, and $\boldsymbol{y}$ and $\boldsymbol{z}$ are linearly independent eigenvectors of $A$ with corresponding eigenvalue 2. Let $\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$. Is $\boldsymbol{v}$ an eigenvector of $A$?
   (a) Yes, $\boldsymbol{v}$ is an eigenvector of $A$ with eigenvalue 2.
   (b) Yes, $\boldsymbol{v}$ is an eigenvector of $A$ with eigenvalue 5.
   (c) No, $\boldsymbol{v}$ is not an eigenvector of $A$.

5. Suppose a 3x3 matrix $B$ has two real eigenvalues: for eigenvalue 2 its eigenspace $E_2$ is one-dimensional, and for eigenvalue 4 its eigenspace $E_4$ is two-dimensional. Also suppose that vector $\boldsymbol{x} \in \mathbb{R}^3$ lies on the plane created by the eigenspace $E_4$ and $\boldsymbol{y} \in \mathbb{R}^3$ lies on the line created by the eigenspace $E_2$, as illustrated in the graph below. If $\boldsymbol{z} = \boldsymbol{y} + 0.5\boldsymbol{x}$, which of the following is true?
   (a) The vector $\boldsymbol{z}$ is an eigenvector of $B$ with an eigenvalue of _____ [fill in the blank]
   (b) The vector $\boldsymbol{z}$ is not an eigenvector of $B$.

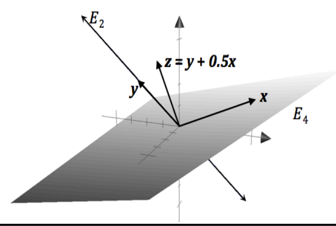


*Figure 1.* Questions 3 and 5 of the Eigentheory MCE Assessment Instrument.

Using Grounded Theory (Glaser & Strauss, 1967), each author of this paper open coded the students' open-ended responses to Q3 independently and discussed our results as a team to find interesting emerging themes. We repeated this process for Q5. In addition, we began comparing a student's open-ended response to Q3 with their response to Q5 to see if the pair of responses provided further insight into each student's understanding. Some of the themes that have emerged from our initial analysis are reported in the following section.

## Results

We detail three preliminary themes from our analysis of justifications that students provided to support their conclusions on Q3 and Q5: (a) using the phrase "is a linear combination of" to support *both* correct and incorrect answers; (b) conflating scalars in a linear combination with eigenvalues, and (c) reasoning about *dimension* of eigenspaces versus *number* of eigenvectors.

### Reasoning about linear combinations

In Q3, we noticed that 13 students wrote "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$" in their open-ended justification; however, 3 used it to support (a), 3 used it to support (b), and 7 used it to support (c). We note that the phrase "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$" was not written anywhere in the Q3 prompt; rather, the symbolic expression "$\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$" was given. We find it notable that so many students expressed this algebraic relationship in words and that this correct phrase was used to support all three solution options. Below we provide a few examples of responses supporting each solution option. Students are identified using labels of the form B#.

Examples of justifications given to support the correct solution (a), that $\boldsymbol{v}$ is an eigenvector with eigenvalue 2, are: "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ which have the same eigenvalue"

[B72], and "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$. Since the value 2 already causes $\boldsymbol{y}$ and $\boldsymbol{z}$ to equal zero, adding a multiple to it will not change that" [B66]. We note that B72's response includes the critical information that $\boldsymbol{y}$ and $\boldsymbol{z}$ have the same eigenvalue – if this were not true, $\boldsymbol{v}$ would not be an eigenvector of $A$. It is not clear to us what B66 meant by his/her response, but we conjecture that it involved reasoning about solutions to the equations $(A - 2I)\boldsymbol{y} = \boldsymbol{0}$ and $(A - 2I)\boldsymbol{z} = \boldsymbol{0}$. In fact, in Watson et al. (2017) we highlighted B66, using data from work on other Eigentheory MCE questions, as an example of a student who showed some reliance on or preference for the homogeneous equation $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ rather than $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Examples of justifications given to support (b), that $\boldsymbol{v}$ is an eigenvector with eigenvalue 5, are: "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$. Both $5\boldsymbol{y}$ and $5\boldsymbol{z}$ are scalar multiples of their previous form so the resultant vector will be an eigenvector as well" [B71], and "Since it is a linear combination of the other eigenvectors, it would also be an eigenvector" [B69]. Note that 5 is the scalar associated with both $\boldsymbol{y}$ and $\boldsymbol{z}$ in the linear combination $\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$ given in the problem. However, 2 is the eigenvalue for both $\boldsymbol{y}$ and $\boldsymbol{z}$, and thus also for $5\boldsymbol{y}$, $5\boldsymbol{z}$ and $5\boldsymbol{y} + 5\boldsymbol{z}$. The explanations given by each of these two students would be correct if they had circled the correct eigenvalue in the multiple-choice portion of the question. It may be that both B71 and B69 made a simple error in choosing 5 as the eigenvalue for $v$ rather than the correct eigenvalue of 2; however, as we detail in the next subsection, it may be that these students conflated the scalar in the linear combination with the eigenvalue in a way more rooted in their thinking about what it means to be a linear combination of eigenvectors.

Examples of justifications given to support (c), that $v$ is not an eigenvector, are: "Eigenvectors must be linearly independent from each other so if $\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ then it cannot be an eigenvector" [B58], and "Because they all correspond to the same eigenvalue they all must have unique eigenvectors and $\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ and therefore not unique and not an eigenvector of $A$." [B79]. One can understand how aspects of B58's reasoning were sensible to him/her, given that eigenvectors from *distinct* eigenvalues of a matrix are linearly independent. In addition it is common in textbooks to list a basis for the eigenspace as the solution to an eigenvector problem; this might lead students to believe that these linearly independent basis vectors are the only eigenvectors.

The phrase "is a linear combination of" was not as common in student responses to Q5. One notable exception is Student B69. This student answered Q3 correctly but gave the vague justification of "since it is a linear combination of the other eigenvectors, it would also be an eigenvector." On Q5, however, B69 explained that the vector would only be an eigenvector if the two vectors in the linear combination had the same eigenvalue (which is true). When considering B69's Q3 response in light of his/her Q5 response, we hypothesize that B69's vague response to Q3 was most likely based in a correct understanding of linear combinations of eigenvectors.

**Conflating scalars in the linear combination with eigenvalues**

We noticed that some student struggles with Q3 could possibly be explained by a conflation of the scalar 5 in the linear combination $\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$ with the scalar 2, which is stated as the eigenvalue for both $\boldsymbol{y}$ and $\boldsymbol{z}$. In addition to B71's justification that $\boldsymbol{v}$ is an eigenvector with eigenvalue 5 (seen in the previous subsection), consider B81's justification given to support (c):

> "No, because an eigenvector is defined as some linear combination defined by the eigenvalue so that $A\boldsymbol{x} = \lambda\boldsymbol{x}$, where $\boldsymbol{x}$ is the eigenvector and $\lambda$ is the eigenvalue. The vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ are being scaled by a factor of 5 and $\lambda = 2$ so they cannot be corresponding eigenvectors."

This student seems to conflate the scaling by 5 of the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ in the linear combination with the scaling by 2 of the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ when acted upon by the matrix $A$. In the former, $\boldsymbol{y}$ and $\boldsymbol{z}$ have not been acted upon by a transformation – the 5 is used to define the amount of each vector that is needed to create the vector $\boldsymbol{v}$. In the latter, the 2 is used to define that the result of multiplying each vector by $A$ is twice the input vector. B81's reasoning seems to explain the role of the 5 in ways that would be more compatible with the role of the 2 and, because the scalars are different, concluded that "they" could not be eigenvectors. It is unclear what vectors are implied in the student's use of "they" – it could be some combination of $\boldsymbol{v}$, $5\boldsymbol{y}$ and/or $5\boldsymbol{z}$.

This preliminary result reminded us of another data set from our research group. In written data from final exams from an introductory linear algebra courses at a large public university in the southwestern United States, the instructor asked a question specifically targeting this potential conflation. The question first gave students a 3x3 matrix $A$ and the eigenspace $k\begin{bmatrix}-1\\2\\2\end{bmatrix}$ and asked them to find the associated eigenvalue (the correct answer was $-1$). The question then asked students to complete the following and fill in the blank if appropriate: "The vector $\begin{bmatrix}-2\\4\\4\end{bmatrix}$ is … [an eigenvector of $A$ with eigenvalue = ___ ] or [not an eigenvalue of $A$]." 13 of the 32 students correctly found that $-1$ was the eigenvalue for both vectors; however, 11 put that 2 was the eigenvalue for the second vector. Because it is two times the vector representative of the eigenspace, we hypothesize that these students conflated the scalar in the scalar multiple with the eigenvalue. These data need further investigation.

**Reasoning about the dimension of eigenspaces versus the number of eigenvectors**

Our third preliminary theme concerns students' reasoning about the possible number of eigenvectors in contrast to the dimension of the eigenspaces. On both Q3 and Q5, some student justifications referred to a finite number of eigenvectors; this is a potentially problematic view because each eigenspace has an infinite number of eigenvectors. For instance, on Q3, B78 reasoned that the linear combination of eigenvectors could not be another eigenvector because "Technically, you could multiply the eigenvectors by any number and if you did so and another eigenvector was achieved there would be a possibility for infinite eigenvectors which doesn't make sense." On Q5, reasons given by some students to support the correct choice (b) similarly focused on finite numbers of eigenvectors: "Matrix $B$ already has 3 eigenvectors so there's no room for a 4$^{\text{th}}$" [B59], and "$\boldsymbol{z}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{x}$, and there are already 3 eigenvectors for 3 dimensions, so $\boldsymbol{z}$ cannot be an eigenvector of $B$" [B66]. We conjecture these students may have been conflating the total number of possible eigenvectors (infinite) for a 3x3 matrix with the number of linearly independent vectors (three) needed to create the bases for the one- and two-dimensional eigenspaces. Alternatively, B58's justification for Q5 focuses on dimension: "In a 3x3 matrix there can only be 3 dimensions to the eigenspace. $E_2$ and $E_4$ together span the entire space of $\mathbb{R}^3$ so there cannot be another eigenvector of $B$ besides $E_2$ and $E_4$" [B58]. We conjecture grasping the difference between finiteness of dimensions and infiniteness of eigenvectors may be particularly important for understanding eigenspaces.

## Discussion Questions

During our presentation we would like to discuss: how can we further investigate our three preliminary research themes, and what additional analyses might help us uncover students' creative and productive ways of reasoning about eigenspaces?

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## First-generation Low-income College Student Perceptions about First Year Calculus

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### Abstract

*The purpose of this study was to explore first-generation low-income students' experiences with first-year calculus, including their self-belief in being successful in math. As part of the Progress though Calculus project, one STEM-focused institution was studied with survey results from students enrolled in first year calculus, and interviews and a focus group of three first-generation low-income students who completed first year calculus. Qualitative findings emphasized the value of creating connections with other students and faculty, and faculty's impact on students' sense of belief in being successful in calculus. Quantitative results illustrated statistically significant higher rates of faculty interaction outside of class and increased confidence in math while taking first year calculus for first-generation low-income students, in comparison to their continued generation, higher income peers. Promoting non-cognitive factors such as student support and self-belief in math success may influence math completion of first-generation low-income students.*

**Key Words**: First-generation low-income students, self-belief, first year calculus

Along with innovative pedagogies and curriculum enhancements to improve math education, it is also important to consider as part of the formula for student success in mathematics, the increasingly diverse student population, gaps of math completion with marginalized students, and non-cognitive factors such as self belief and support networks. The population of diverse students for this study was first-generation students as the first in their immediate family to be working towards a bachelor's degree, and low-income students.

In this study I explore the experiences of first-generation low-income students in first year mathematics from an asset approach, meaning I focus on what these students bring from their identities to support their success, as well as attending to barriers these students face. For decades college administrators and researchers have viewed the first-generation and low-income identities as "at risk", which is reinforced by the well documented national graduation gaps of these students. For instance, among 4.5 million college students from 1995-2002, six-year graduation rates for first-generation low-income students were 44% lower than continuing-generation higher-income students (Engle & Tinto, 2008).

To address these graduation gaps and move beyond the focus on the disparities of underserved students as a disadvantage to being successful in college, a paradigm shift is needed to to support students that attend college rather than require students to adapt to college. One way to provide support is focusing on first-year mathematics completion since it is highly correlated to graduation rates. By understanding what assets students bring to first-year mathematics success, we can better understand how to support a higher graduation rate among this population.

### First-generation Low-income Students

Most prominent research on first-generation low-income students has focused on deficits as a disadvantage to being successful in college. Academic deficiencies of these students include: higher need for remedial courses (Chen & Carroll, 2005), undeveloped student success skill sets (Collier & Morgan, 2008), less academic and co-curricular engagement (Pascarella, Pierson, Wolniak, & Terenzini, 2004; Warpole, 2003), and lower educational aspirations (Pike & Kuh, 2005). Non-cognitive disparities include a lack of parental support (Ward, 2012), not as much social capital (Lin, 2011), lower levels of a sense of belonging (Aires & Seider, 2005; Ward, 2012), and a cultural mismatch with the university (Roberts & Rosenwald, 2001; Stephens, Fryberg, Markus, Johnson, & Covarrubias, 2012).

A different approach to defining deficits and expecting students to compensate for deficits is research done within the perspective of promoting the strengths and assets of students as an advantage for collegiate success. Although not as prevalent, research within an asset framework focused on self-authorship (Pizzolato, 2003), high motivation to attend college (Martin, 2012) , and a desire to contribute to society (Olive, 2009). Along with research on the strengths of these students, is the importance of meaningful individual connections to support students. For instance, a strong network of faculty who care and have high expectations, and peers who offer encouragement has been found to help first-generation college students transition to college (Coffman, 2011), and obtain a college degree (Lourdes, 2015).

## Math Completion of First-generation and Low-income Students

In considering the success in math of first-generation low-income students, lower levels of math completion have been documented. An analysis of first-generation student college transcripts from 1992 to 2000 shared that 55% of first-generation students took at least one math course in college compared to 81% of students whose parents had a bachelor's degree (Chen & Carroll, 2005). Additionally, at Colorado State University (2016) after controlling for prior academic preparation, first-generation, students of color, and Pell eligible students were significantly less likely to place into college algebra and to complete three credits of math in the first year compared to their peers.

## Self-belief Theoretical Framework

To begin to reflect on ways to enhance math completion with first-generation low-income students, one non-cognitive factor to consider is self-belief based on the power of positive psychology, which is the study of conditions that influence the optimal functioning of people (Gable & Haidt, 2005). Theories to inform this perspective of developing student assets are stereotype threat (Steele, 1997), which challenges college success, and self-belief (Bandura, 1977; Dweck, 2006), which can potentially mediate challenges and promote academic success. Stereotype threat theory asserts that negative stereotypes of one's performance based on his or her social group can put individuals at risk of lower performance (Steele, 1997).

In response to the negative influences of stereotype threat, positive psychology theories of self-belief are used with Bandara's theory of self-efficacy and Dweck's theory of a growth mindset. Bandara's theory of self-efficacy is a social cognitive theory based on the belief that one can achieve his or her goals (Bandura, 1977). Expanding upon self-efficacy is growth mindset, is the belief that one may improve through engagement with the learning process (Dweck, 2006).

## Self-belief and Math Achievement

Research on the relationship of self-efficacy and math achievement is evident both with students who have not performed well in math along with engineering students with high levels of math performance. Investigating students who were repeating a developmental math course, they identified high self-efficacy as the essence of their persistence despite a low self-concept in mathematics (Canfield, 2013). For engineering students who usually excel in math, self-efficacy was correlated with mathematics achievement scores and cumulative grade point averages (Loo & Choy, 2013).

Reinforcing a growth mindset, research has demonstrated greater course completion rates in challenging math courses (Yeager & Dweck, 2012). Many studies have also focused on the growth mindset as a mediating factor to stereotype type threat of marginalized populations in math performance. Dar-Nimrod and Heine (2006) studied math achievement and gender, and illustrated that females with a growth mindset performed better than females with a fixed mindset on math assessments similar to the Graduate Record Examination.

Lower math completion rates of first-generation low-income students, along with the positive impact of self-belief and math achievement, warrant further investigation into ways that self-belief can enhance success in mathematics. The purpose of this study is to explore first-generation low-income students' experiences with first-year calculus, with particular focus on their self-belief in being successful in math. Specifically, the following research questions guide this work: (1) How do first-generation low-income college students experience first year calculus at a STEM focused institution? (2) How does first year calculus influence the self-belief of first-generation low-income college students to be successful in math?

## Methodology

To provide context of this research, a broad overview of the Project through Calculus research that studied ways to enhance student calculus completion rates is summarized. A part of this research project was a pilot study at one STEM institution, which is the focus of this paper.

### Progress through Calculus Research

The Progress through Calculus study is sponsored by the Mathematical Association of America and funded by the National Science Foundation (NSF) to research student success in calculus. Twelve higher education institutions were identified by the research project team as institutions using structural, procedural, curricular, and pedagogical approaches to the pre-calculus and calculus program that has been successful in higher math completion rates, especially with underrepresented students. Prior to researching the twelve institutions, three pilot studies were held at institutions based on geography, convenience, and access; to refine data collection content and procedures.

### Research Design, Participants, and Data Collection

One of the pilot studies for the Progress through Calculus research was done at a private institution that is focused on STEM degrees. This study was implemented with five researchers including a two-day site visit with interviews, class observations, focus groups, and surveys.

The qualitative subset of the pilot study included three first-generation low-income students who were interviewed mid-semester and participated in focus group at the end of the semester. The students that participated were a first year white female, a junior African American female, and a third year Asian male. The student survey was developed by the Progress through Calculus Research team, and was distributed by the instructors in one of the first year calculus course sessions during the middle of the spring semester.

The mixed methods design was a convergent parallel design with both the interviews/focus group and the survey gathered and analyzed independently, and then the results interpreted together (Creswell & Plano Clark, 2011). For the interviews and focus group holistic data analysis was accomplished with an inductive process to identify relevant emerging themes (Yin, 2003), making sense out of the data collection (Miles, Huberman, & Saldana, 1994). To begin, the interviews and focus groups were transcribed and then coded with MaxQDA, a qualitative coding software program. The researcher began with a first-cycle coding process and then reviewed each code and coded segment to illuminate connections between the categories in the second-cycle coding process, and used a second coder to refine the codes (Miles et al., 1994). Survey questions on student self-belief and interactions with faculty and peers as support resources were studied with chi-square analysis.

## Results

Studying this institution may offer insights to math success since they have decreased the first year calculus DFW rates from 22% in fall 2006 to 10% in spring 2015. The qualitative results from interviews and a focus group illustrated that a common theme of their math experiences was the major significance of working with faculty and other students outside of class.

In exploring faculty connections, key factors that emerged were the importance of how faculty responded to questions, and the value of small individual interactions. An example of how faculty reacted positively to questions is illustrated by the following statement by one student, “When you ask a question and a faculty member is really supportive and they don’t look down on what you ask, they just answer this is what it is.” A less supportive response is illustrated by the statement by another student, “if we ask a question that is dumb he looks down on us, so it’s really intimidating.”

In addition to responding to questions, short interactions with faculty had a big impact on the student’s experiences in math courses. One student illustrated the impact of a faculty connection as being the most positive experience in calculus.

> “My math teacher was sitting outside on one of the picnic tables and I didn’t want to sit with him and talk about math….so I sat on a bench …. He was going back into his office and he stopped by and was talking to me….How are your classes going? Then he said he didn’t care about the other classes just mine, it kinda made me laugh… It was kinda of like your cool and we joke around now. I feel like I know him a lot better. Listening to him lecture I have that connection: you know what you’re talking about I will believe what you are saying. I mean I guess it showed because I did a lot better on my last test. That was the best positive experience that I have had in calculus. It was getting that connection.”

Along with faculty interactions, the importance of peer support was highlighted, describing how students worked with other students in math courses. Two students shared that they looked for students that were doing better than them, and then would ask them to be in a study group. Another example was a network of students beginning with two students working together, each branching out to other friends, and then coming back together to complete the homework. The value of peer support was further highlighted when a student shared that she would rather work with other students than a faculty member, even if it took longer to get the correct answer.

The other prominent research finding was the tremendous impact that faculty had on student’s self-belief in being successful illustrated by the quote below.

> “I went in [to her office] and said I can’t do Calc II, I’m a fraud, and she said yes you can. She said we are going to sit down and go through this exam and she went question by question and she said what did you do wrong? It’s not like you don’t understand what’s going on sometimes you are reading the question incorrectly. You know the material you just need to interpret the question and answer it correctly. Okay that clicks. She didn’t give up she didn’t brush me aside as one of twenty students. She remembered my name which was important.”

These results illustrate the power of faculty and student connections integral to first-generation low-income student experiences in math, and especially the impact of faculty believing in student success. Another student experience was how they were able to improve their low grades with new strategies and continued effort which relates to having a growth mindset.

Concerning the survey results, there was 322 respondents, with a 67% response rate. The questions analyzed in this study focused on faculty and student interactions, and self-belief in mathematics. Focusing on faculty interaction, survey frequencies found a higher percentage of first-generation students (21%) compared to continuing generation students (16.5%) saw their instructor outside of class. Chi-square results indicated a statistically significant association between first-generation status and faculty interaction, $\chi 2$ (5) = 11.879, $p < .05$. The effect size was small (Cohen, 1988), Cramer's V = .172. Focusing on interactions with peers for all students, there were higher percentages of working with peers than instructors, and higher percentages with first-generation and low-income students compared to their continuing generation and higher income peers. Aspects of self-belief studied were confidence, ability to do math, and growth mindset. The survey results indicated that most first- generation (66%) and continuing generation (53%) significantly or moderately increased their confidence in math by taking calculus. More importantly, chi-square results indicated a statistically significant association between first-generation students and increased confidence, $\chi 2$ (5) = 14.477, $p < .01$. The effect size was small (Cohen, 1988), Cramer's V = .19.

Along with confidence, findings about a student's ability to learn mathematics revealed that most first-generation students (68%) and low-income students (76%) said that their math ability "moderately or significantly increased" with taking calculus. Additionally, 73% first- generation students and 80% low-income students shared that their growth mindset "significantly or while taking first year calculus. There were no statistically significant differences of the ability to do math and growth mindset between first-generation and low-income students compared to their peers.

## Discussion

A major highlight of this research was the importance of faculty and staff connections and the positive impact that calculus had on students' increased confidence in math which was higher for first-generation and low-income in comparison to their continuing generation and higher income peers. Although strong faculty and student connections reinforce well established high quality teaching practices, it is an important reminder to keep these qualities at the forefront especially in college courses. Additionally, in light of the research findings that first-generation low-income students are working with peers outside of class at higher rates than instructors, perhaps more intentional integration of student study groups would be impactful.

Although the survey findings are based on small sample sizes, results may suggest a possibility that historically marginalized first-generation and low-income students are gaining self-belief as part of their experience in calculus courses at this institution. This is an important finding considering the stereotype threat that is well documented with students having marginalized identities. Learning even more about how faculty can provide an environment for enhancing student self-belief is recommended. Additionally repeating this same study at other institutions as part of the Progress through Calculus research project, will provide cross institutional results and additional insights. These findings will hopefully suggest ways to create an environment that promotes self-belief in developing the talent of first-generation low-income students, thereby increasing success in math.

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# Gauging College Mathematics Instructors' Knowledge of Student Thinking About Limits

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*A theme in the literature surrounding instructional practices and knowledge for teaching is that knowledge of how students think about mathematical ideas plays important roles in supporting effective instruction. However, the undergraduate mathematics education community lacks tools for assessing this kind of knowledge. As an initial step toward the development of such assessments, we documented instructors as they examined students' work on calculus tasks during individual interviews. Transcripts were coded as exhibiting robust, limited, or no evidence of knowledge of student thinking using Jacobs, Lamb, and Philipp's (2010) framework. The coding process highlighted the varying depth and breadth of instructors' knowledge. Once refined, this coding process can be used to develop instruments for gauging knowledge of student thinking through means other than interviews. Such instruments will be of use to researchers, to those who design professional development for experienced and novice instructors, and for evaluation of professional development efforts.*

*Keywords:* mathematical knowledge for teaching, limits, instructor professional development

## Introduction

There have been many calls for increased attention to the teaching of undergraduate mathematics and professional development for those who do such teaching as part of efforts to improve enrollment and retention rates in STEM disciplines (Bok, 2013; Holdren & Lander, 2012). From extensive research at K-12 levels, we know multiple factors shape teachers' instructional practices (see, e.g., Borko & Putnam, 1996) and developing practices consistent with findings from research on teaching and learning can be challenging (see, e.g., Fennema & Scott Nelson, 1997). In this body of literature, a recurring theme is that knowledge of how students think about particular mathematical ideas plays important roles in supporting effective instruction (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Hill, Rowan, & Ball, 2005). Moreover, recent findings encourage increased active learning approaches in undergraduate instruction (Freeman, et al., 2014; Laursen, Hassi, Kogan, & Weston, 2014), and a necessary ingredient in enacting such instruction is a rich understanding of and ability to interpret student meanings. In particular, to respond effectively and support further student learning, instructors need to infer what thinking (correct or incorrect) might underlay what students write and say.

There are many ways in which teachers can and do develop such knowledge, including from experiences with their own students. Researchers of teacher professional development (PD) have touted the value of designing opportunities for teachers to examine and analyze students' written work (see, e.g., Little, Gearhart, Curry, & Kafka, 2003). College mathematics instructors frequently engage in this practice while grading homework or tests and/or while interacting with students who are working on problems as part of classroom activities. However, often the goal is to assess students' understanding rather than to unpack and make sense of that understanding.

Although knowledge of student thinking supports effective instruction and examining student work appears to be a context in which teachers learn how students think, the undergraduate mathematics education community lacks tools for assessing this kind of knowledge. Being able to gauge the depth and breadth of this knowledge for particular topics will aid both researchers (seeking to evaluate and/or study the development of such knowledge) and those who provide PD (who wish to determine the need for attention to particular topics). For example, such an assessment could inform PD activities so that they are accessible to the instructors based on their current knowledge of student thinking. This, in turn, could enhance the PD we provide to graduate students and other novice instructors of college mathematics.

The value of these kinds of assessment instruments is readily apparent from the body of work that was made possible at K-12 levels because of the existence of such tools (e.g., Hill et al., 2005). Such studies have been extremely powerful in establishing the importance of this form of knowledge in the K-12 mathematics community. Because of the content-specific nature of this knowledge, it is not possible to merely adopt existing instruments for use at the undergraduate level. As an initial step in this development process for topics in the undergraduate curriculum, we documented instructors as they examined students' written work on calculus tasks with the aim of developing a rubric for gauging the extent of their knowledge of student thinking. Our analysis was focused on answering two questions: (1) What meanings and/or ways of thinking do instructors attribute to the student work? (2) How extensive (or not) is each instructor's catalog of such meanings? Answers to these questions are needed to inform future development of short answer, multiple choice and/or case-based assessment items.

## Research on Knowledge for Teaching

Findings from decades of research point to the important roles components of mathematical knowledge for teaching (MKT) play in teachers' practices and the learning opportunities they create for students. Of particular relevance to this study are the Pedagogical Content Knowledge (PCK) components of MKT that represent knowledge teachers use when hypothesizing what a student meant when they show their thinking when speaking, writing, and interacting with others. Studies have illuminated links between this kind of teaching-specific knowledge and both teachers' instructional practices and their students' learning (Carpenter et al., 1989; Hill et al., 2005). As part of these efforts, instruments have been developed to assess teachers' MKT. These instruments were based on findings from the substantial body of research on student thinking in the K-12 literature and on a robust set of classroom- and interview-based studies of teachers (Hill, Schilling, & Ball, 2004; Krauss, Neubrand, Blum, & Baumert, 2008).

As part of an effort to weave these findings from the K-12 MKT body of literature with findings from research on undergraduate student thinking, we are focused on categorizing college mathematics instructors' knowledge of student thinking. Our approach shares features with that of Jacobs, Lamb and Philipp (2010), who investigated the extent to which teachers paid "…attention to children's strategies but also interpretation of the mathematical understandings reflected in those strategies" (p. 184). By examining responses to assessments from teachers with varying levels of experience, these researchers were able to shed light on and characterize expertise in knowledge of student thinking, and document that this expertise can be developed.

## Research Design and Methodology

We conducted our work from a cognitive theoretical perspective because of the prevalence of this perspective in the research on knowledge and knowledge development as well as the primarily individual nature of the out-of-classroom teaching work that is the focus of our investigation. This perspective, with the premise that human cognitive activity is accessible via written and spoken communication, has been used productively to examine teachers' knowledge and its roles in teaching practices (Borko & Putnam, 1996; Escudero & Sanchez, 2007;

Schoenfeld, 2007; Sherin, 2002). One specific way to access what teachers know about their students' thinking is by attending to what they notice when looking at student work (Jacobs, Lamb & Philipp, 2010). In this paper, we leverage Jacob, Lamb, and Phillip's noticing framework as a way to unpack what mathematicians know about their undergraduate students' mathematical thinking related to limit.

Interview data came from task-based individual interviews with seven research mathematicians at three institutions who had been recognized for their excellence in teaching, through being nominated for or winning a teaching award. The interviews were audio-recorded and transcribed to aid in the coding and data analysis. Tasks were taken from or modeled after tasks used in research on student thinking about limit, function (as it appears in calculus), and derivative. Interview design was adapted from one used previously to examine college instructor MKT (Speer & Frank, 2013). This consists of three parts per task for each interviewee: (1) Solve the task and describe the solution, (2) describe how students would solve the task, including difficulties they may encounter and/or mistakes they might make, and (3) examine and discuss student work, noting productive and unproductive ways of thinking demonstrated in each response. Here our focus is on the participants' insight and understanding of student thinking based on their examination of sample written work from the limit tasks.

Data analysis was guided by grounded theory (Corbin & Strauss, 2008) but also made use of findings from research on student thinking about limit, particularly those that provide insights into common productive and unproductive ways of thinking demonstrated by calculus students (e.g., Oehrtman, 2008, 2009). Following the approach of Jacobs, Lamb and Philipp (2010), chunks of interview data were labeled as demonstrating robust evidence, limited evidence, or absent of evidence of insight and understanding of student thinking. Descriptions of these levels were then developed considering similarities and differences among coded interview excerpts and among various levels of evidence. Our unit of analysis was each participant's discussion of the set of student responses to an individual task.

## Findings

We found that Jacobs, Lamb, and Philipp's (2010) framework for classification of knowledge of student thinking was easily adapted to our data set. We demonstrate the applicability of this approach for use with instructors at the post-secondary level with interview excerpts and descriptions of how we operationalized the robust-limited-absent evidence rubric for use with our data. Although mathematicians examined several samples of student work, we illustrate our findings with data from their discussions of one student response to the prompt: "Describe what it means when we say 'the limit of $f(x)$ as $x$ approaches 3 is 12' ($\lim_{x\to 3} f(x)=12$)." The sample written student response said, "It looks like *f(x)* is 12 although *x* never actually reaches 3." Descriptions and transcript excerpts are shown in Table 1.

In interview excerpts coded as being *absent* in demonstrating insight into student thinking, the instructor does not explain what the student might be thinking and only notes that the response is not correct, remarking that she would assign it partial credit on a test. In the interview excerpt coded as *limited*, the instructor recognizes that this is a typical student response, suggesting one possible reason why a student might come up with this answer. However, this response is somewhat vague and does not demonstrate extensive or rich knowledge of student thinking. Responses coded as *robust* demonstrate an understanding of and validity in the student work, despite the not completely correct response. To demonstrate robust interpretation of student work and knowledge on this scale, one must provide examples of why students might answer in such a way, recognizing common productive/unproductive ways of thinking and describe possible origins for such ways of thinking.

*Table 1. Examples demonstrating three levels of knowledge of student thinking*

| Level | Description | Example from Interview |
|---|---|---|
| **Absent** | Does not articulate understanding about student thinking when interpreting the student work. Further, does not consider the student perspective and see validity in the portions of students' incorrect responses. | **Interviewer:** If that was an answer you saw on a test, or someone said to you in office hours, how do you feel about that one?<br>**Instructor 1:** (pause) I don't know.<br>**Interviewer:** It has that approaches 3 thing that you mentioned before.<br>**Instructor 1:** Yeah, there are some parts that I would give some partial credit, but I certainly would not get full credit because you know it is like "it *looks* like f(x) is 12", that is not quite correct, and "although *x* never reaches 3", so I would probably give something like partial credit, but no full credit. |
| **Limited** | Some articulation of student thinking; able to explain thinking demonstrated in common student responses, but either some pieces are left unexplained, or the interpretation of student work is vague. | **Instructor 2:** I mean this is – the very apt response, arcitypical [archetypal].<br>**Interviewer:** arcitypical?<br>**Instructor 2:** No I think it is fine; … I think this is a reasonably good approach. A good answer, if imprecise. It looks like f(x) is 12, so that's – I mean, it is just lacking the language to say that f(x) is within epsilon-delta of 12, right?<br>**Interviewer:** There is a tolerance –<br>**Instructor 2:** One way to imagine is that this is accompanied by a graph. And maybe they are thinking of the graph picture. |
| **Robust** | Able to take on the student's perspective in all responses, including less common errors, finding validity in aspects and recognizing where the student is still lacking in complete understanding. | **Instructor 3:** That's fine, this is exactly Newton's way of approaching limits.<br>**Interviewer:** What is the student thinking?<br>**Instructor 3:** So that's exactly this picture of as x *approaches* 3, but never touches…the limit they learned that you don't necessarily hit, and Newton had exactly this issue of what happened when you were trying to calculate the derivative…<br>**Interviewer:** Alright, and what do you make of the "it looks like f(x) is 12"?<br>**Instructor 3:** That's a response to this stuff, limit of f(x)=12, that's a reinterpretation of what they would think there. |

## Discussion and Implications

The goal of this work was to categorize what instructors do when interpreting student meanings/thinking in order to gain a better understanding of what it means to have knowledge of student thinking of undergraduate mathematics topics. This is the first step to developing an assessment grounded in practice adapting Jacobs, Lamb, and Philipp's (2010) framework of classifying the professional noticing of teachers.

The coding process revealed that mathematicians demonstrated multiple levels in their interpretation of student work across various tasks. This suggests that even experienced instructors may have had different opportunities to hear students' reasons for answers or may engage differently with student work on different tasks or topics. Although the central goal of the analysis was to inform assessment development, we offer preliminary thoughts on this apparent variation. Differences could be related to their dispositions towards student thinking and willingness to engage with the student work (although we note that all participants appeared to engage with the tasks). Alternatively, they may have had different opportunities to engage with student thinking due to differences in their instructional approach (i.e., there may be fewer opportunities to hear student thinking in a lecture-based class than one utilizing collaborative groupwork). Further analysis is needed to test and refine this tool to gauge instructor knowledge. This will entail examining data from additional instructors (including more novice instructors) as they examined written student work on limit tasks and also an expansion to data we have from the same instructors as they examined student work from function, derivative and integral tasks.

By operationalizing the Jacobs, Lamb, and Philipp's (2010) framework for our limit tasks we begin the work of identifying characteristics of varying depths and breadths of knowledge of student thinking. We see two features as varying across the levels. One relates to the richness of the participants' descriptions of the student thinking. These range from non-existent to including multiple diagnoses for what a student might have been thinking. Participants also varied in the extent to which they articulate hypotheses for *why* responses might have seemed reasonable to students. We view this empathetic disposition as aligned with Smith III, diSessa, & Roschelle's (1994) perspective on student misconceptions and as an important avenue for further study.

Once refined, these characteristics can then be used to develop instruments for gauging this knowledge through means other than interviews. Such instruments with items in open response, multiple-choice or case-based formats would make other approaches to this work feasible, including assessment in larger instructor populations and/or coupled with observations of teaching. Such a tool could also be used to track growth in instructors' abilities to interpret student thinking and depth of knowledge over time and/or after participating in PD.

Instances of robust understanding provide us with evidence that mathematicians can develop such knowledge. We note that development of this knowledge most likely occurred from their on-the-job experiences of examining student work and interacting with students given the scarcity of teaching-specific professional development opportunities typically available for college instructors (Holdren & Lander, 2012). The varied levels demonstrated by participants suggest that examining student work could be productively used in college instructor PD to further enhance their practice-based learning opportunities. In addition, such an approach could provide an opportunity for exposure to not only student work/student thinking, but also help develop instructors' understanding that knowledge of student thinking is indeed a set of knowledge that is (1) desirable for instructors to possess to support effective instruction, and (2) something that can be attended to and enriched over time. Moreover, this leads to the possibility for more targeted PD when armed with insight into types of responses one might expect from experienced and novice instructors. It also has the potential to illuminate and document growth in knowledge of student thinking over time, pointing to the influence of PD or experience in this development. Utilizing such approaches in the professional development of undergraduate instructors can help our community improve the learning opportunities we create for students.

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# A Student's Use of Definitions in the Derivation of the Taxicab Circle Equation

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*Research shows that by observing properties and making conjectures in other geometries, students can better develop their understanding of concepts in Euclidean geometry. It is also known that definitions in mathematics are an integral part of understanding concepts, and are often not used correctly in proof or logic courses by students. APOS Theory is used as the framework in this preliminary data analysis to determine one students' understanding of certain definitions in Euclidean and Taxicab geometry, and her use of these definitions in deriving an equation for a circle in Taxicab geometry.*

*Key Words:* Definitions, Geometry, Taxicab, Geometrical Reasoning

## Introduction

It has been found that commonalities exist in higher-level math courses regarding students' inability to properly complete tasks involving definitions (Edwards & Ward, 2004), despite the expectations held for students enrolled in such courses. Edwards and Ward (2004) state that there were misconceptions in students' understanding of "the very nature of mathematical definitions, not just from the content of the definitions," (p. 411). In the context of geometry, since the properties of geometric figures are derived from definitions within an axiomatic system, it is important to note that a figure is "controlled by its definition," (Fischbein, 1993, p. 141).

In college geometry courses, Euclidean geometry and its axiomatic system is deeply studied, but other axiomatic systems receive little consideration (Byrkit, 1971; Hollebrands, Conner, & Smith, 2010), although research shows that by exploring concepts in non-Euclidean geometry, students can better understand Euclidean geometry (Dreiling, 2012; Hollebrands, Conner, & Smith, 2010; Jenkins, 1968). For example, Dreiling (2012) found that "through the exploration of these 'constructions' in taxicab geometry…[students] gained a deeper understanding of constructions in Euclidean Geometry." (Dreiling, 2012, p. 478).

For this preliminary report from the larger research study, we present results and discussion on the following research question: What is the learning trajectory one student followed to accommodate her understanding of *distance* and *circle* in Taxicab geometry?

## Theoretical Framework

As a constructivist framework, APOS Theory is based on Jean Piaget's theory of reflective abstraction, or the process of constructing mental notions of mathematical knowledge and objects by an individual during cognitive development (Dubinksy, 2002). In APOS Theory, there are four different levels of cognitive development: Action, Process, Object, and Schema. In addition, there are mechanisms to move between these levels of cognitive development, such as interiorization and encapsulation. An Action in APOS Theory is when a student is able to transform objects by external stimuli, performing steps to complete this transformation. As a student reflects on an Action and has the ability to perform the Action in his or her head without external stimuli, we refer to that as an interiorized Action and call it a Process. Once a student is able to think of this Process as a whole, viewing it as a totality to which Actions or other

Processes could be applied, we say that an Object is constructed through the encapsulation of the Process. Finally, the entire collection of Actions, Processes, Objects, and other Schemas that are relevant to the original concept that form a coherent understanding is called a Schema (Dubinsky, 2002).

## Methodology

This research study was conducted in a College Geometry course during Fall 2016, which has an introduction to proof course as a prerequisite. Since it is a cross listed course, there were seven undergraduate and 11 graduate students enrolled in the course, many of whom were pre-service teachers. The study is defined as a teaching experiment, as described by Cobb and Steffe (2010) and Steffe & Thompson (2000), which consisted of sessions of instruction, followed by individual interviews. The textbook used in the course was College Geometry Using the Geometer's Sketchpad (Reynolds & Fenton, 2011), which is written on the basis of APOS Theory. The material of the course covered concepts and theorems often seen in a College Geometry course, with Taxicab geometry included at the end of the semester. Videos from in-class group work and discussion, as well as written work from the semester were collected. After the semester, semi-structured interviews were conducted with the 15 of the 18 students enrolled in the course who volunteered. We focus our attention in this paper to one student and some of her answers to the interview questions, as they provided good insight as to how students transfer definitions to a new context. The following questions are relevant to this paper, and are a subset of the questions asked during the interview:

1. Define and draw an image (or images) that represents each of the following terms, however you see fit: Circle, Distance.
2. For any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$
   (i) Euclidean distance is given by $d_E(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
   (ii) Taxi distance is given by $d_T(P,Q) = |x_2 - x_1| + |y_2 - y_1|$
   Using the grids below, illustrate each of these two distances. Be as detailed as possible in labeling them.
3. Using the grids below, sketch the following circle in both geometries: Circle with center at $C(3,3)$ and radius $r = 2$.

These questions were used in the analysis for this preliminary report, since they help to identify student understanding of *distance* and *circle*, and the possible pathway a student takes to transfer and possibly modify her definitions to new situations. Specifically, for this paper, we focus on how a student used her definition of *distance* and *circle* in her attempt to derive the algebraic representation of a Taxicab circle.

A genetic decomposition is defined as a "description of how the concept may be constructed in an individual's mind," (Arnon et al., 2014, p. 17). A preliminary genetic decomposition was developed for this study to identify the development pathway students may follow to derive (or understand the derivation of) this equation (see Figure 1). To specify, *geometric representation* includes any sketch or drawing in addition to the students' verbal description of the definition, unless they specifically state the definition is an equation.

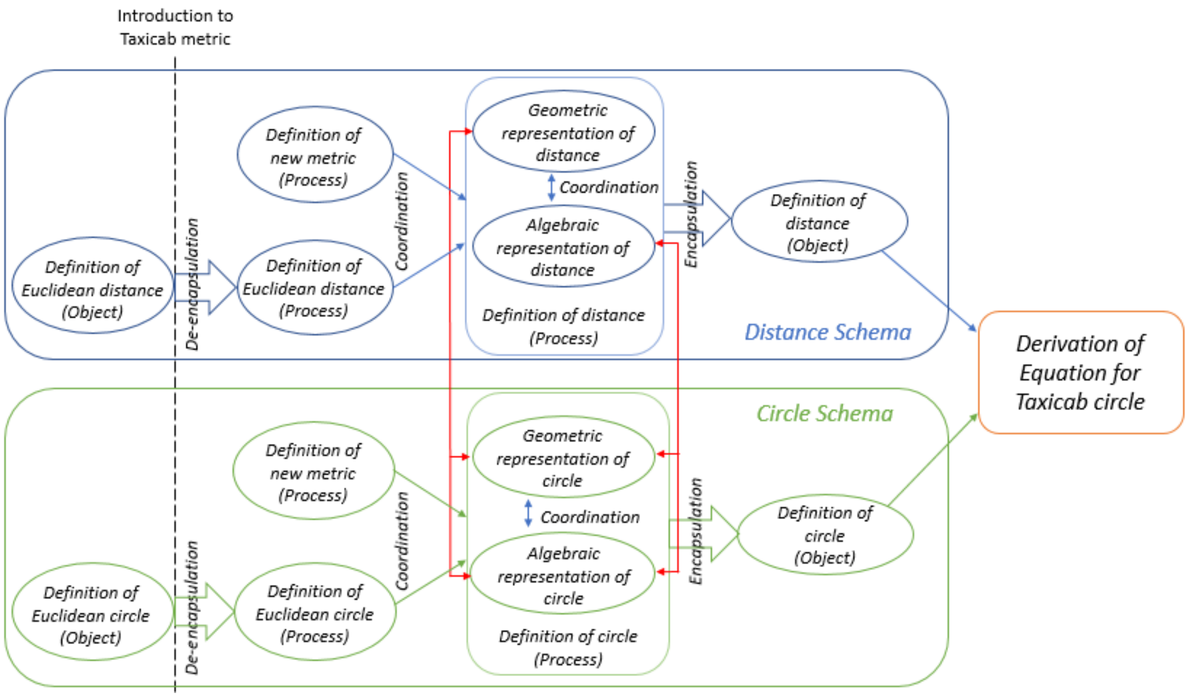


Figure 1: Preliminary genetic decomposition.

Based on our own understanding of historical development of these concepts, along with existent research results, we partitioned the concepts of *distance* and *circle* in terms of their schemas, and illustrated how students develop this equation in Taxicab geometry. As illustrated in Figure 1, we propose that in dealing with concepts of *distance* and *circle* in these two geometries for this task, students exhibit an interplay between two schemas. For the sake of length, we omit a full explanation of Figure 1.

## Preliminary Results

We provide the APOS Theory based analysis of one student's answers, Nicole, as it corresponds to this preliminary genetic decomposition.

### Geometric Representation of Distance

Nicole stated, "I know with Taxicab I can't just look at the distance as this from A to B, I have to go…up and around I mean up, like I'm using the road and not going through," and "there are certain steps I have to take or a certain route…" Illustrations of her conception of these metrics can be found in Figure 2. Later, in responding to a question about her Taxicab circle, Nicole was able to use a definition of Taxicab distance to justify that every point on her diamond shaped Taxicab circle was equidistant from the center. Therefore, she clearly exhibited an object conception of *geometric representation of distance* in Taxicab geometry since she was able to think of it as a totality and apply an action (comparison) to it.

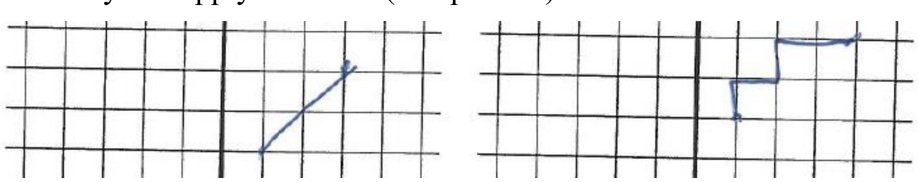

Figure 2: Nicole's illustrations of Euclidean and Taxicab distance, respectively.

## Algebraic Representation of Distance

Nicole defined *distance* as "the measurement in between…two or more points that someone would ask me." Further, Nicole clearly explained that with the two geometries, "the definition [of distance] would be the same, but how to find it with the equation won't be the same," indicating that she distinguishes between these metrics. This is evidence of a process conception of *algebraic representation of distance* since she seemed to have a general definition for distance and would be able to find the distance between any two points in either geometry.

## Geometric Representation of Circle

Nicole demonstrated and stated, that she is able to construct a Euclidean circle easily without plotting specific points on the circle (see Figure 3 - red ink indicates her drawing during the interview discussion). Developmentally, with the addition of Taxicab distance, Nicole reorganized her Euclidean Circle Schema to accommodate this new metric in order to describe and draw the Taxicab circle. When describing this circle in comparison to a Euclidean circle, she says "when I think of in Taxicab geometry …visually it won't be the same, but I do think the definition [of it] would be the same, because it has to be equidistant to be a circle." She explained that when she drew her Taxicab circle, she had to "follow strict routes, making my radius." She attempted to apply her written definition of circle by using the property of equidistance, and constructed her Taxicab circle incorrectly (see blue square in Figure 3, sketched during her individual work). This may be due to her inability to coordinate her *geometric representation of distance* in Taxicab geometry and her definition of circle as a set of equidistant points from a fixed point. More specifically, she was not able to imagine traveling a given distance in all directions from the center following 'strict routes.'

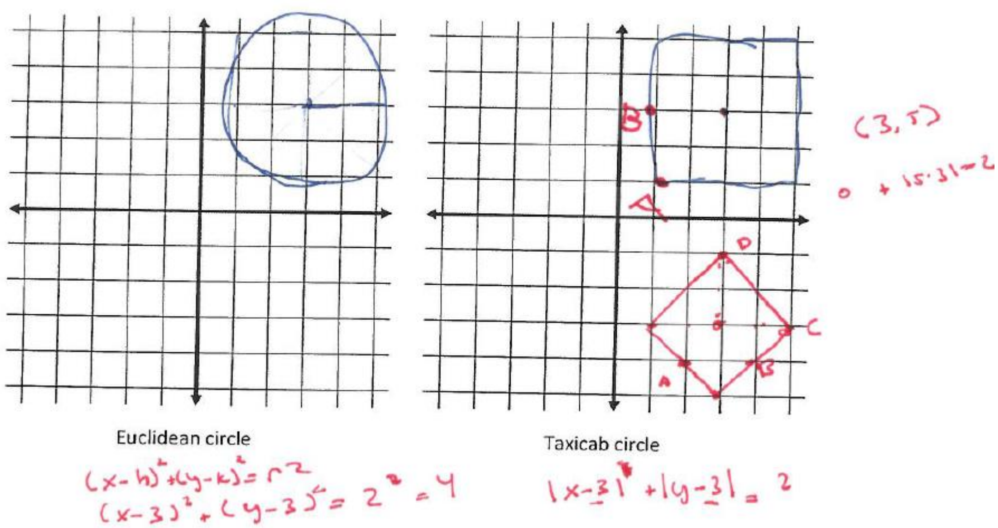


Figure 3: Nicole's illustrations of Euclidean and Taxicab circles, respectively.

During the interview, with prompting, Nicole recalled the shape of a circle in Taxicab geometry is a "diamond." She constructed this by finding four points (vertices) on the diamond and connected them (see red diamond in Figure 3), saying "oh, but it was like this". Thus, Nicole exhibited an action conception of *geometric representation of circle* in Taxicab geometry, since she relied on her memory for the shape of a Taxicab circle and needed external cues to draw it.

### Deriving the Algebraic Representation of Circle

Nicole, with prompting, could recall the equation for the Euclidean circle, and arrived at the correct equation (see Figure 3). She seemed to be completing a sequence of steps, each provoked by the previous, since she first needed to write the general equation for a Euclidean circle, then identified the variables that would be replaced by the given center and radius, and finally plugged them in. By working off memory, Nicole exhibited an action conception of *algebraic representation of circle* in Euclidean geometry. Later, Nicole recognized some relationship between the formula for distance and the equation for a circle when prompted to derive the equation for a circle in Taxicab geometry. She stated, "I'm wanting to use…absolute values simply because we use absolute values for the distance? But that could be wrong." These statements imply Nicole saw the Euclidean distance formula is used in the Euclidean circle equation, and thus inferred the same must be true in Taxicab geometry. It appears she relied on this pattern to create her equation for the Taxicab circle. Thus, we believe that Nicole has an action conception of *algebraic representation of circle*.

## Discussion and Concluding Remarks

Fischbein (1993) explains that in geometrical reasoning, a major obstacle is the tendency to "neglect the definition under the pressure of figural constraints," (p. 155). The results presented in this paper support this notion, with Nicole exhibiting a slightly different path to derive her Taxicab circle equation other than what our preliminary genetic decomposition illustrated. Our data indicates that Nicole had an action conception of *geometric representation of circle* and an object conception of *geometric representation of distance*, which allowed her to eventually draw the given Taxicab circle.

We expected that to derive the equation for a Taxicab circle, Nicole would need to have an object conception of *algebraic representation of distance.* Although she eventually arrived at the correct equation for her Taxicab circle with a process conception of *algebraic representation of distance*, her success was reliant on reproducing patterns instead of logic. Further, we claimed that she must have an object conception of a definition in Euclidean geometry to consider applying it in a new geometry. Nicole demonstrated that out of the concepts considered, she had an object conception of *geometric representation of distance* only, which could be why she struggled to derive the Taxicab circle equation from logic. Vinner (1991) and many others support this, since knowing a definition does not imply a real understanding of the concept.

As evidenced in this study, Nicole knew the definitions of *circle* and *distance*, but was unable to apply them effectively to derive an equation, and instead relied on patterns. We still believe an object conception of definitions is necessary prior to operating in another geometry to logically derive this equation. We plan, for future analysis, to identify the cognitive paths for each student and further investigate misconceptions these students illustrate in their discussion about deriving the equation for a Taxicab circle.

## Questions for the Audience

1. What obstacles have your students faced when transferring their knowledge of definitions to a new context?
2. What are some good activities that can assist students in encapsulating their definitions?

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# Assessing Group Learning Opportunities in a First Semester Calculus Course

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*The purpose of this study was to examine how undergraduate calculus students positioned themselves within group work and how that positionality influenced their own and others' learning opportunities. Using qualitative methods, this study examines the specific group social interactions and positionalities that led to productive and unproductive group problem solving. The study used a sociocultural lens to identify productive group work and learning. The findings of this paper suggest the roles students assume are very fluid throughout the problem solving process. In addition, the roles that the students assume influenced the learning opportunities. Furthermore, groups that utilized individualistic group practices were not able to build opportunities for conceptual understanding nor have productive group learning.*

*Keywords:* Calculus I, Collaborative Learning, Positionality

This study adds to the body of research on group work and collaborative learning in the mathematics classroom. Prior research suggests that collaborative learning aids in positive learning outcomes. For example, research suggests that in the mathematics classroom small group learning can increase academic achievement and promote positive attitudes to learning mathematics (Draskovic, Holdrinet, Bulte, Bolhuis, &Van Leeuwe, 2004; Smith, McKenna, & Hines, 2014; Springer, Stanne, & Donovan, 1999). Forbes, Duke and Prosser (2001) found that students perceive group-based instruction as effective as traditional lecture based learning models. A related body of literature examines group compositions and how to promote productive group learning (Cohen, 1994; Dolmans, Wolfhagen, Scherpbier, & Vleuten, 2001; Engle & Conant, 2002; Esmonde, 2009; Haller, Gallgher, Weldon, & Felder, 2000; Johnson & Johnson, 1999; Webb, 1991). Cohen (1994) recommends moving away from the standard academic achievement measures and examining group interactions and group engagement as a means of defining successful group work. Additionally, this body of research suggests that status within groups can cause inequitable interactions and learning discrepancies (Haller et al., 2000). This research is mainly situated in primary and secondary mathematics classrooms with very little research at the undergraduate level and more specifically within the context of a first semester Calculus course. The purpose of this study is to add to this collection of literature a meaningful analysis of how group interactions and positionality impact the learning opportunities and problem solving process.

## Theoretical Framework

Two bodies of literature are utilized to build the theoretical framework for this study. The first, sociocultural theory, provides a means for understanding and defining conditions in which learning occurs. The second body of literature is the work on positionality and roles within groups as defined by Cohen (1994), Draskovic et al. (2004), and Esmonde (2009).

### Sociocultural Theory and Defining Learning Opportunities

Sociocultural frameworks are rooted in the Vygotskian school of thought where all learning is socially constructed (Goos, 2004). Within this framework all knowledge is constructed through the lens of social interaction institutions (Nasir et al., 2008). Saxe (1999) identifies cultural activities as integral to understanding cultural changes but also cognitive change. It is in the cognitive change where knowledge is developed through a communicative process engaging and reacting to others. Lastly, learning is facilitated through the use of tools and language available to the student. It is the social interactions in which the individual participates that generates the Zone of Proximal Development (ZPD) (Steele, 2001). Within the ZPD there is an intersection of individual meaning making and social constructs that allows for active engagement in the learning process. ZPD is the space between the individual's current understanding and potential for new understanding. Within this space is where learning and growth occur within the sociocultural framework (Lerman, 2001).

**Defining Roles and Positionality**

Another body of literature that informs this study examines how group interactions develop, nurture, or impede upon the learning and growth of the individual. How individuals choose to participate and are positioned influences learning (Draskovic et al. 2009; Esmonde, 2009; Webb, 1991). Results have shown some roles aid in productive group work and others hinder the process. One significant role within groups that has been studied is the role of *facilitator* (Cohen, 1994, Esmonde, 2009, Draskovic et al. 2004). Facilitators or tutors direct the path of knowledge and problem solving by providing explanations and rationales to the group. This role is significant for two reasons. First, this role provides a valuable resource to the other group members, aiding in the extension of knowledge and potential for learning of others within the group. Secondly, the individual assuming this role is able to build conceptual understanding through articulating his or her ideas. Esmonde (2009) identified the complementary roles of expert and novice. Experts have significant influence over the direction of the group and provide a source of information and resources. However, contrary to the facilitator role, experts go unquestioned with their authority. This role can cause a problem in instances when experts provide incorrect knowledge to the group. The counterpart to this role is the novice. The novice is the receiver of the knowledge from the expert. This role is counterproductive to aiding in building conceptual knowledge or the problem solving abilities of either role. In Vyogotskian thought, learning is developed through conversation that supports understanding and meaning making. In the roles of expert and novice, the unidirectional communication is directive rather than conversational. Thus, meaning making and conceptual learning is severely restricted when students assume these two roles.

Although many studies have found positive gains through collaborative learning, there are positionalities that impede upon learning in a group context. Dolmans, Wolfhagen, Scherpbier, and van der Vleuten (2001) recognize the impact of *sponging* on the learning of the individual and overall group. Sponging occurs when an individual sits idle with little to no input yet expects to profit from the work of others in the group. Additionally, a dominant personally as described by De Grave, Dolmans, and van der Vleuten (2001) can also impede the overall group interactions and learning. Draskovic et al. (2004) hypothesize that collaborative group problem solving aids in positive knowledge gains when the occurrences of undesired positionalities are mitigated.

The goal of this study is to examine students' positionality as a means of understanding productive group problem solving and learning opportunities. With a dual perspective that learning is constructed through social interactions and developed through internalization of the individual's new constructs with existing constructs, the primary research questions are (1) How does an individuals' positionality in the group influence the group problem solving process? (2) How do the roles that students take influence their learning opportunities?

## Methods

The participants of this study were enrolled in a first semester calculus course (Calculus I) at land-grant institution in the Mid-Atlantic region of the United States. Participants in this study were placed in groups based upon student major. This particular grouping method was significant in order to control for the individuals that may be automatically placed in the role of expert by the group due to field-specific knowledge relevant to solving the problem. Group A consisted of four female biology majors. Group B consisted of three males and one female bio-chemistry majors. There was no preference given to gender or prior academic performance in the course in the group construction.

## Data Collection and Analyses

This study examined the group interactions in a clinical setting where students were presented the group work within the context of their normal recitation portion of the course. Students were first presented with standard pedagogical methods for introducing optimization problems to first semester calculus students as part of their standard lecture-based instruction. These instructional methods are out of the scope of this study. However, as noted by Crooks and Alibali (2013), the way students encode prior knowledge significantly influences how students perceive and ultimately solve problems. The findings of Llinares and Roig (2008) suggest that students use particular cases in the development of the modeling process. Given a particular case, students will base mathematical decisions from the model constructed. This encoding was addressed by presenting each group with a novel optimization problem related to their intended field of study. Both groups were required to minimize the resistance of blood flow in veins. Minimizing of resistance problems required the participants to have some understanding of the biological concepts to aid in the orientation phase in the problem.

The group problem solving cognitive process was documented through video recording. This structure builds upon the work of De Grave, Boshuizen, and Schmidt (1996) as a way to investigate cognitive and metacognitive processes in a group problem solving setting. Recording the problem solving process without interfering provides an authentic representation of students' conceptualization and problem solving. The recordings were transcribed for analysis.

To understand the group interactions and positionalities the transcripts were coded using the work of Esmonde (2009). The first round of coding identified the group work as collaborative, individualistic, or helping. Collaborative group work was identified as interactions where the group members asked questions, debated ideas, and worked together toward a common goal. Individualistic group work was identified as situations in which the individual group members worked separately and then used each other as a resource for checking and verifying purposes only. Lastly, helping group work involved one or more individuals who instructed other group members. In this case there was no back and forth conversation or questioning of ideas. Rather, helping was clearly a unidirectional flow of information from one individual to another.

The second tier of coding examined the roles each group member held. First, phases of the problem solving process were identified for each group, including the orienting, planning, executing, and checking phases as described by Carlson and Bloom (2005). Within each phase the work of Esmonde (2009) was used to develop provisional categories of facilitator, expert, and novice to describe the positionality of each individual. Through reiterative coding, additional categories were defined of associate and by-stander. These two positionalities were not identified in the work of Esmonde (2009). Associates were fully engaged in the group discussion, however were not considered facilitators as they were not the gatekeepers of information. Associates were individuals that were perceived as equals in knowledge acquisition. By-standers were still engaged in absorbing information, however these individuals did not provide feedback, questions, or suggestions. Unlike a novice, by-standers were not directly engaged with an expert and appeared when other group members were engaged as associates. By-standers did not provide any meaningful direction to the problem solving.

## Preliminary Results and Discussion

The analyses revealed there were two key findings. The first finding suggests that the roles that students hold significantly impact the individual's and the group's learning opportunities. These roles were very fluid throughout the problem solving process. Secondly, the group's work practices influenced the learning opportunities. For example, the group that utilized individualistic group practices did not have the same learning opportunities as the group that utilized more collaborative group practices nor successful completion of the problem.
Phases of the problem solving process become more fluid in a collaborative setting. This allowed the individuals the opportunity to take on various roles throughout the entire process. Figures 1 and 2 show the fluidity of the roles individual group members held. Several of Group B member's phases were individualistic and therefore positionality could not be determined. Thus, a sixth category, individualist, was introduced to describe the positionality of the group.

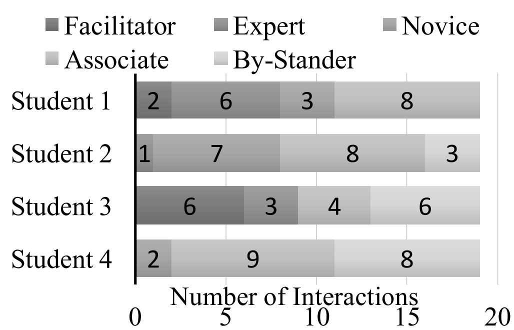

*Figure 1. Group A Positionality*

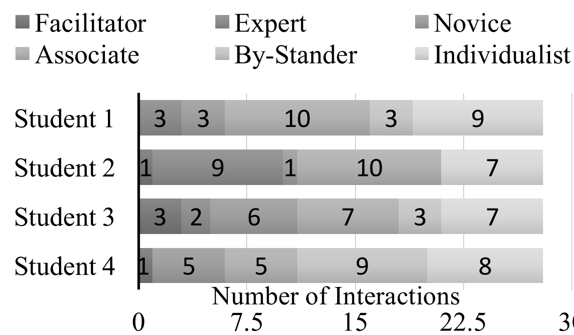

*Figure 2. Group B Positionality*

By examining the positionality of each group member, the expectation is to understand how the positionality of the individual influenced learning opportunities. Results of this analysis indicate that, although the roles of facilitator and expert are critical in successful problem solving, students in those roles do not necessarily benefit from the collaborative work. However, students benefit from the role associate as a means of building conceptual knowledge and mathematical skills. The roles of facilitator and expert can be interpreted as knowledge disseminators. These key roles are distinctive in a way that these students are the knowledge holders. In Vygotskian thought, students in the expert role are not in a position to learn. The only case in which learning would occur would be if there was a contradiction between the meaning held of the student and the interpretation of the other group members. This was not observed due to unquestioned authority of experts. Therefore, only a facilitator who is engaged in conversation with others would have the opportunity for learning. Interestingly, in both groups, even if the content knowledge from the expert was incorrect or the explanation by the facilitator was inappropriate, the other group members failed to recognize the inconsistencies. This may be due to the fact that students in both of these roles where viewed as having stronger mathematical knowledge, leading to these students not being questioned thoroughly or at all. It was only in the role of associate that group members questioned each other and expected full explanations. In these instances, students used the approaches cautiously until either the approach was validated by the instructor or a consensus among the group. Within the sociocultural framework, students engaged in the associate role found themselves in the ZPD by positioning themselves with the highest potential for learning and growth to occur. Thus, in terms of productive group work, it is suggested that students be encouraged to engage in this equal playing field of questioning ideas and approaches.

The role distribution played a significant function in whether a group could successfully complete the problem. Thirty-eight percent of Group A's interactions were identified as associate role. Contrary to this, only 29 percent of Group B's interactions were identified as associate roles. Furthermore, the problem given restricted the students' ability to model a previous example to solve the problem. There was evidence that both groups used the diagram provided in order to relate to a previous example. However, Group A was able to dismiss the incorrect approach through a continuous back and forth discussion. Group B continuously reverted back to individualistic roles once a strategy was introduced. This led the group to continue down paths that led to incorrect solutions. Group B struggled significantly at orienting themselves to the problem and determining a clear and defined approach to solve the problem. This group relied heavily on the instructor for direction and clarity.

The group work practices also contributed to the groups' ability to complete the problem. Group A never worked in an individualistic manner. All the interactions were either collaborative or helping. This led to more group discussion thereby creating ZPD. In the Vygotskian thought, these dialectic conversations created more learning opportunities for this group. In contrast, Group B spent over a third of the interactions working individually rather than collaboratively. This may be attributed to the group composition. Group A was entirely female whereas the majority of members in Group B were male (three out of four). The findings here support those of Haller et al. (2000) who found women prefer collaborative interactions to competitive interactions. The all female group solely utilized collaborative group practices. The primarily male group not only utilized collaborative work practices less, but the group applied individualistic practices which are more indicative of competitive interactions. Furthermore, this may have significant impacts for the one female in Group B (Student 4). She spent 60 percent of her interactions as either a by-stander or working individually compared to Group A where only 22 percent of interactions were described as by-standers (no individualistic interactions were recorded). This significantly reduced Student 4's group discussion and therefore significantly reducing her learning opportunities.

## Conclusion

This study confirms the findings of Draskovic et al. (2004) and Esmonde (2009) where positive group interactions lead to successful learning opportunities in a group context. However, it was found that within the role of associate, where the students provided meaningful back and forth dialog and questioning, that one can find evidence of increased ZPD and learning opportunities. Moreover, individualistic group work can lead to unproductive group work and this may be more prevalent in male dominated groups opposed to female groups. These results of this study suggest that educators should be fostering a strong back and forth dialog amongst students and help initiate those types of interactions. Furthermore, additional research should focus on the gender gap that was observed in the positionality and group practices of the group along with the potential impact on women in male-dominated groups. Lastly, future research may include the student perception of roles and the impact on positionality.

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# Leveraging the Perceptual Ambiguity of Proof Scripts to Witness Students' Identities

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*Recognizing identity not only as an important educational outcome but also being inter-related to students' knowledge and practice, this paper explores an affordance of proof scripts; the witnessing of students' identities. Drawing on proof scripts from teaching experiments and the construct of perceptual ambiguity, this paper argues that proof scripts afford access not only to students' understandings, problematics, and ways of reasoning but also students' identities.*

*Key words*: Identity, perceptual ambiguity, proof scripts

There exists a host of reasons for why researchers have grown increasingly interested in identity. Yet, this interest has not been accompanied by a growth in methodologies that afford the study of identities. The aim of this paper, therefore, is to demonstrate how the perceptual ambiguity of proof scripts can be leveraged to explore students' identities. To accomplish this goal, three steps are taken. First, I explore why interest in identity has grown in recent decades. Second, I describe the proof script methodology. Third, I present a characterization of perceptual ambiguity and then draw on proof script data to illustrate and discuss how proof scripts afford access not only to students' understandings, problematics, and ways of reasoning but also their identities, if we work to leverage their perceptual ambiguity.

## Identity

Students' identities have become increasingly of interest to researchers. As researchers have grown in their capacity to document student affect and its impact on learning (cf. Bishop, 2012), organizations (e.g., NCTM, 2000) have increasingly called for teachers and researchers to work to better understand the emergence of productive dispositions. Also, researchers interested in equity oriented instruction have increasingly recognized not only that knowledge and practice are interactively constituted but also that students' identities impact students' knowledge and practice. Boaler (2002), for instance, has argued that classroom learning is constituted through interactions between students' knowledge, identities, and practices (see Figure 1); arguing, like Wenger (1998), that learning "is an experience of identity" (p. 215). Last, Bishop (2012), has argued that due to their impact on dispositions, affect, persistence, and achievement, identities are recognized as an important educational outcome.

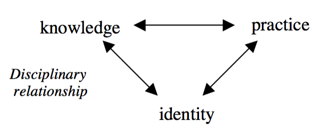


Figure 1. Boaler (2002) Learning Model

Despite agreement on its importance, definitions of identity vary widely. Due to length limitations, the discussion will focus on the definition of identity proposed by Bishop (2012). Bishop (2012) defines *identity* as: "a dynamic view of self, negotiated in a specific social context and informed by past history, events, personal narratives, experiences, routines, and ways of participating…. (it) is both individually and collectively defined" (p. 38). This definition highlights that identities are interactively constituted within environments and, in part, by others.

One of the primary means for interactive constitution is *discourse* – a point emphasized by Gee (2001, 2005) who argued that identities are created through discourses, by Sfard and Prusak (2005) who define one's identity in terms of internalized communications and narratives, and by Bishop (2012) who argues "discourse plays a critical role in enacting identities." Indeed, identities become visible through discourse as interlocutors position themselves and others in relation to their current social context, institutional setting, and history – a point poignantly illustrated by Setati (2005), who studied how class and power are enacted through mathematics classroom discourses. Like, Bishop (2012) and Setati (2005), the position taken in this paper is that discourses can serve as a primary means for exploring identity.

## The Proof Script Methodology

Interest in students' reading strategies, difficulties with, and comprehension of mathematical proofs, has led to a host of studies. These studies have either employed proof comprehension assessments (*cf*. Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012), clinical studies (*e.g*., Weber, 2015), or novel methodologies, such as *proof scripts* (Koichu & R. Zazkis, 2013; D. Zazkis, 2014; R. Zazkis & D. Zazkis, 2016). The latter entails having students produce a written dialog where the interlocutors discuss a given proof, highlighting *problematics* (i.e., difficulties identified by students) and elaborating on key points to promote understanding. The dialogs are then analyzed by researchers to create models of students' understandings of the content and practices attended to, their perceptions of key points and ways of reasoning about *problematics*. This methodology emerged for a variety of reasons. First, as noted by Koichu and Zazkis (2013), past research examined students' difficulties from the researchers point of view. Methods were needed that enabled the identification of problematics, a point also emphasized by Brown (2017). Second, the methodology aligns with theoretical perspectives that see discourse as integral to thinking. Specifically, the methodology builds on Sfard's (2007) commognitive theory, in which Sfard argues thinking can be viewed as individualization of "the activity of communicating" (p. 571) that is derived from the collective patterned activities one experiences.

## Perceptual ambiguity as a means to witness students' identities

At its core the proof script methodology calls on students to produce a written dialog in which the participants discuss a proof, paying special attention to the key ideas and problematics observed. Taking the perspective that the dialogical interactions generated are reflective of students' thinking, recent studies have shed light on students' understandings of important mathematical topics and practices (Koichu & R. Zazkis, 2013; D. Zazkis, 2014; R. Zazkis & D. Zazkis, 2016; D. Zazkis & Cook, *in press*). However, is this all that we can learn? The position taken in this paper is that the methodology affords not only a means to explore students' understandings but also to witness students' identities, if we attend to their perceptual ambiguity.

Figure 2. W. E. Hill's Cartoon

In 1915, W.E. Hill published the drawing shown in Figure 2. Staring at the drawing one of two images will appear, either a young lady with her head turned or an elderly woman looking down pensively. Both images are present. Yet, there is just one drawing. Both reside in the same set of lines. Yet, *we can only see one image at a time*. This is why the drawing has what psychologists call *perceptual ambiguity*. Perceptual ambiguity refers to instances in which one's grouping of certain contours, images, or ideas supports one's perception of a figure, object, or meaning while the grouping of other contours within the same image promotes a different singular perception. Such drawings were of interest to psychologists for they demonstrated that vision is an active rather than passive process; what is seen is constructed by the viewer actively. In this paper, perceptual ambiguity is of interest for it aptly describes an affordance of proof scripts: they afford observation of students' ways of attending to proofs while at the same time enabling us to witness students' identities, if we engage actively in the process of *seeing* students' positioning of themselves in relation to others, to the discipline, its practices and knowledge.

## The Study

To examine students' ways of attending to contradictions, 43 proof scripts were collected from 2nd and 3rd year university students enrolled in an IBL - Introduction to Proof course who were given the proof task shown in Figure 3 during the last week of the term. Data collection occurred at a designated Hispanic-serving institution, where the majority of students are first generation college students, who qualify for need-based financial assistance. The original research question was "Which problematics and key ideas are salient to and noticed by students, when producing scripts for proofs involving contradictions?" However, when analyzing the data it became increasingly apparent that the discursive interactions did more than provide a window into students' reasoning about contradictions, for they also afforded an opportunity to witness students' enacted identities. In other words, the proof scripts embodied a form of perceptual ambiguity. Taking the position that students' identities are an important learning outcome of *all* mathematics courses, this affordance became the focus of the research. The purpose of this preliminary report is to provide an existence proof of proof scripts' perceptual ambiguity and in so doing establish a methodological approach to the study of students' identities that is distinct from but in harmony with the discursive approaches taken by Bishop (2012) and Setati (2005).

***Assignment:***
***Part 1***. Start by reading the proof and identifying what you believe are the "*problematic points*" for a learner when attempting to understand the theorem and its proof. A *problematic point* is anything you think is incorrect, is confusing, or is correct but warrants further discussion. List these "*problematic points*" in a bulleted list.

***Part 2.*** Write a dialogue between you and Gamma in which you and Gamma discuss the theorem and its proof. The dialog should address the *problematic points* you identified (and listed in your bulleted list) through questions posed either by you or Gamma.

***Theorem*:** For any real numbers $x$ and $y$, if $x \leq y$ and $y \leq x$ then $x = y$.
***Proof*:**
1) Assume $x$ and $y$ are real numbers such that $x \leq y$ and $y \leq x$.
2) Then ($x < y$ or $x = y$) and ($y < x$ or $y = x$).
3) We will consider four cases
Case 1. $x < y$ and $y < x$.
Case 2. $x < y$ and $y = x$.
Case 3. $x = y$ and $y < x$.
Case 4. $x = y$ and $y = x$.
4) In Cases 1 through 3 our assumptions contradict the Law of Trichotomy.
5) We are left with Case 4.
6) Case 4. $x = y$ and $y = x$.
7) Therefore, $x = y$.
8) The result follows. ♦

Figure 3. The Proof Script Task

The remainder of the paper, which will constitute the findings of the study, will proceed in two parts. In the first part, we examine excerpts from proof scripts to demonstrate how they afford the opportunity to examine students' understandings of the attended to problematics and content.

In the second, the same excerpts are analyzed to witness students' identities. Here it is important to note that the term *witness* is used intentionally and in reference to the belief that, at best, researchers can only hope for *disciplined subjectivity* (LeCompte et al., 1999) when seeking to understand others' identities and histories.

**Findings, Part I: Proof scripts as a means to examine student understandings**

While it may seem that there is not much to say in relation to the proof, the IBL-Introduction to Proof students had little difficulty elaborating on the given, arguably brief, argument. The majority of students produced discourses in which either Gamma or the student elaborated on why four cases were called for. Furthermore, many explained not only the generalized Law of Trichotomy but also the role of axioms and/or definitions in mathematics. To see this we consider Excerpt A, in which Student A explains the Law of Trichotomy to Gamma.

*Excerpt A*

| | |
|---|---|
| *Gamma:* | *Why can't the first three cases be true?* |
| *Student A:* | *Because of the law of trichotomy.* |
| *Gamma:* | *What's the law of trichotomy?* |
| *Student A:* | *The Law of Trichotomy is an axiom we use. An axiom is a statement that is regarded as being the truth or accepted as true. So this axiom states that only one of the three following cases may happen: either $x < y$, $y < x$, or $x = y$. Applying this knowledge we can see why the first three cases don't work.* |

This brief excerpt demonstrates several key understandings: (1) the status of the Law of Trichotomy within the theory of the real numbers; (2) the status of axioms within the discipline of mathematics; and, (3) a perhaps tentative understanding of contradictions – namely that they indicate an inconsistency has occurred within a mathematical theory, which must be resolved by deference to that which is taken to be true (i.e., given a choice between a result and an axiom we choose the axiom and label the result "*false*" or "*impossible*"). Beyond students' understanding of the components of mathematical theories, the proof scripts also often indicated students' understandings of a generalized proof structure. In particular, several scripts included the proofs for Cases 1 and 2, often citing basic axioms and definitions, and an explanation as to why Case 3 was not needed, if a proof of Case 2 was given.

*Excerpt B*

| | |
|---|---|
| *Student B:* | *What's the word Mockingbird?* |
| *Gamma:* | *No much ese, just working on this pinche proof!* |
| *Student B:* | *Chale, that stuff ain't easy homes.* |
| *Gamma:* | *Que no, check it out and see if I got this mierda right?* |
| *Student B:* | *It looks good Homes except in linea 2 and 3 you got no detail ese. You need to explain cases 1 – 3!* |
| *Gamma:* | *Don't yell at me ese!* |
| *Student B:* | *Stop acting like a chavala!* |
| *Gamma:* | *Whatever homes!* |
| *Student B:* | *Anyways, lleva, with Line 2 you didn't state Axiom 6 which lets you split the inequalities foo.* |
| *Gamma:* | *Chingada madre! I forgot Axiom 10 in Line 3 for the cases 1, 2, 3.* |
| *Student B:* | *And for Line 4 you forgot Definition 6. Another thing for Case 2, if you prove it by contradiction using Axiom 10, because $x \neq y$, Case 3 is found without loss of generality because of Case 2.* |

As is the case with Excerpt A, Excerpt B sheds light on several key understandings that are employed by the student. First, the remarks clearly indicate an understanding of the symbol $\leq$ and the fact that it has a mathematical definition which takes the form of a disjunctive statement; i.e., a form that justifies the partitioning of the proof into cases. Second, the script indicates the student has an observable understanding of a practice that is important to proofs at this level; namely, that sub-proofs which are identical in structure are not replicated within in a proof.

**Findings, Part II: Proof scripts as a means to witness students' identities**

When producing a dialog, a student must decide on how to position the interlocutors, their knowledge and status, goals and relationships. Moreover, the students must share or question specific actions taken within the proof and respond to questions about those actions by either drawing on interlocutors' knowledge and ways of reasoning or their interpretations of the expectations of participants of a discipline. Hence, by attending to students' positioning, use of language, characterizations of practice or articulation of expectations within a dialog, proof scripts afford the opportunity to perceive the inter-relationships between identity, knowledge, and practice experienced by students. It is for these reasons they afford opportunities to witness students' identities. Consider for example, Excerpt A. In the dialog, Student A is not positioned as an unknowledgeable or uncertain peer (a stance taken by authors in several proof scripts) but rather is positioned as a knowledgeable other, who can decidedly determine the status of statements ("*The Law of Trichotomy is an axiom")* while also acknowledging the relative status of "truth" in mathematics ("*regarded as being the truth or accepted as true"*). Likewise consider, Excerpt B, where the student skillfully pinpoints key gaps in the proof and the axioms and definitions necessary to elaborate on those gaps, while at the same time maintaining the dialect common to students in the area – essentially translating sophisticated mathematical ideas into an urban dialect. Here we see not only evidence of a student's content knowledge but also evidence of the *blending of identities*: identities common to the discipline of mathematics, where attention to detail and structure reigns supreme, and identities common to our urban youth, which are expressed through specific temporal and situated dialects that employ terms outside of formal English and Spanish (e.g., *ese* means "that" in Spanish but is slang for "man" or "dude" in parts of Mexico and the southwestern United States). Viewed in this way, the dialect presents an instance of true ownership, for the mathematics has bridged the great divide between the institutional home of the discipline, where the practices of mathematics are both recognized and valorized, and a community that is often structurally excluded from the discipline. Hence, the bridging represents an expression of identity, where the individual has taken ownership of the mathematics through its acculturation (as opposed to the student's).

## Discussion & Concluding Remarks

Recognizing that, as argued by others (Bishop, 2012) identities are not only an important educational outcome but also critical to learning (Boaler, 2002), I have sought to demonstrate how proof scripts' perceptual ambiguity affords an opportunity to witness students' identities. Perceptual ambiguity in this context refers to an affordance of a static artifact, when that artifact can convey particular meanings through one's intentional focus on particular attributes, yet convey a distinct set of meanings should one's intentional focus shift to other, present but not yet attended to, attributes. The artifact – the student's proof script – being static does not change but rather our goal oriented activities do when actively "seeing" the artifact. As such, this work proposes that proof scripts can serve as a productive means for examining students' identities.

**Questions**

1. Is *perceptual ambiguity* the appropriate construct for describing the dualities of proof scripts?
2. What issues are there with describing researchers' inferences of identity as *witnessing*?

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# Emerging Instructional Leadership in a New Course Coordination System

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*This paper reports on the instantiation of a coordination system in a university mathematics department, and in particular the transition of three faculty members into their new roles as course coordinators. Course coordination, characterized by uniform course elements and instructor meetings, is a programmatic feature that supports student success in introductory mathematics courses. When courses are coordinated, the person or people responsible for the coordination play a critical and complex role in ensuring that all students experience comparable, well-designed classes – but building such a system is complex and has not been studied in situ. In this report, I explore one coordinator's transition from a peripheral participant in discussions of teaching to a highly central figure with significant influence on instructors and colleagues in the department. Surveys and interviews with involved parties reveal nuance of this shift in leadership and shed some light on the process.*

*Keywords:* Precalculus/Calculus, Institutional Change, Course Coordination, Leadership

This report is about the implementation of a course coordination system in an undergraduate mathematics department, situated in a broad change initiative, considering in particular how one coordinator in particular rose from relative obscurity to a strong position of instructional leadership – a result which has the potential to support student success in introductory mathematics courses. In recent years, numerous reports and recommendations have been published at the national scale calling for improvements in undergraduate STEM education, and mathematics education in particular (e.g., CBMS, 2016; National Research Council, 2013; PCAST, 2012; Saxe & Braddy, 2015). Critical to STEM majors, and so to improvement efforts, is the Precalculus to Calculus 2 (P2C2) sequence which is required for most upper division STEM courses. In particular, these documents have called for the implementation of evidence-based instructional practices (notably active learning) and resources to support students in social as well as academic aspects of their lives. In the same time period, research has identified particular programs and features that support student success in the calculus courses that are critical for STEM majors (e.g., Bressoud, Mesa, & Rasmussen, 2015; Bressoud & Rasmussen, 2015). However, these practices and programs are not widespread – lecture still dominates the classroom, and even those departments which have such programs do not feel they are as successful with their implementation as they would like (Apkarian & Kirin, 2017; Rasmussen et al., in review). Furthermore, change is difficult and little is known about best practices for initiating and sustaining change in undergraduate departments. The role of social and cultural factors is viewed as an important part of the puzzle, and this has been demonstrated at the K-12 level repeatedly (Borrego & Henderson, 2014; Daly, 2010; Henderson & Dancy, 2007; Penuel, Frank, & Krause, 2010). By investigating the process by which one department implemented a major change initiative, and in particular how one member of that department grew into a new and critical role, this project contributes to our understanding of the social factors that affect the instantiation of evidence-based change in undergraduate mathematics department.

Apkarian and Rasmussen (2017) report on a sample of successful programs with coordination, where the coordinators held both formal power, by way of their titled role, and informal power, by way of social influence and leadership. In light of their findings, this paper considers three faculty members who were assigned the role of course coordinator, their social standing prior to this assignment, and how this shifted during the first two years of the new program. The analysis includes interviews with the coordinators and their colleagues in the mathematics department and social network surveys which capture informal power through interaction patterns and the nomination of expertise.

## Theory and Literature Review

The choice to focus on course coordinators, in this report, is due to their potential to impact multiple elements of the P2C2 course experience. The Characteristics of Successful Programs in College Calculus (CSPCC) study identified course coordination as one of the features of successful Calculus 1 programs (Rasmussen & Ellis, 2015). Through management of uniform course elements (e.g., common textbook, common exams) coordinators affect the basic elements of course curriculum, and through regular meetings with instructors they can affect the culture surrounding teaching. Their position can be leveraged to nudge instructors toward specific practices, particularly powerful during a systematic change effort, and their actions have the potential to engender communities of practice. Apkarian and Rasmussen's (2017) further investigation of successful departments in the CSPCC study revealed that, at those institutions, the course coordinators were primary sources for advice and information about teaching, meaning that they have informal social influence as well as formal, official power from their position. Their work suggests that alignment of informal and formal leadership with regards to teaching is a feature of more successful course coordination programs. The departments in that study, however, have had coordination systems intact for many years. This study investigates the development of such a system, and in particular the shift in coordinators' informal roles as they adopt their new, formal roles.

Wenger (1998) defines the practice of a community of practices as "doing in a historical and social context that gives structure and meaning to what we do. In this sense, practice is always social practice" (p. 47). This report considers both the practice of the coordinators and the social context in which they practice – the interactions and attitudes they carry and those of their colleagues. To do so, I draw on social capital theory and social network analysis (SNA). Social capital refers to the "resources embedded in social relations and social structure, which can be mobilized when an actor wishes to increase the likelihood of success in purposive actions" (Lin, 2002, p. 24), and these resources are considered to be "the potential and actual set of cognitive, social, and material resources made available through direct and indirect relationships" (Bridwell-Mitchell & Cooc, 2016, p. 7). SNA is one productive way to investigate social capital and its distribution among members of a community, because "an actor's network of social ties create opportunities for social capital transactions" (Adler & Kwon, 2002, p. 24). Thus, I leverage the tools of SNA to investigate interaction patterns identify central and peripheral participants, based on their potential access to social capital. Interviews and observations are used to characterize the social capital resources that are accessible through that network.

## Methodology

### Data Collection and Participants

The data for this study comes from a three-year longitudinal mixed methods study of a single mathematics department at a large state university. The P2C2 courses in particular were considered a problem at the university due to low pass rates, low persistence, lack of preparation in future courses, and student dissatisfaction. A newly elected chair set out to improve the situation using evidence-based practices, specifically considering the seven features of successful Calculus 1 programs laid out by the CSPCC project (Bressoud et al., 2015; Bressoud & Rasmussen, 2015). He determined that the P2C2 program had none of these characteristics and so, along with a task force, set out to implement them all. This report focuses on one of these characteristics: coordination systems for P2C2 courses that consist of regular instructor meetings and uniform course elements, organized by course coordinators. This department appointed three coordinators, one for each P2C2 course: Precalculus, Calculus 1, and Calculus 2. Data for this report comes from three major sources: (1) a survey to all people involved in the P2C2 sequences and changes therein; (2) interviews with instructors, coordinators, and the P2C2 committee; and (3) observations of P2C2 committee meetings. Data collection for this project has been completed.

A survey was distributed to all those involved with P2C2 courses at the university at three time points: before any changes occurred and at the end of the first and second academic year of the change initiative. It was distributed to all instructors (regardless of rank) all members of the mathematics and mathematics education divisions of the mathematics department, directors of faculty and student support programs, and selected administrators. Part of the survey consisted of Likert-style questions about the culture and climate of the department and P2C2 program, adapted from similar work in both K-12 and higher education contexts (Antonakis, Avolio, & Sivasubramaniam, 2003; Daly, Der-Martirosian, Moolenaar, & Liou, 2014; Moolenaar, 2012). These were aimed at measuring changes in attitudes over time. Another major part of the survey was a set of social network questions aimed at uncovering interaction patterns surrounding instruction of lower-division mathematics courses. Instructors were asked who they go to for advice, for instructional materials, and who they consider influential on their instructional practice. Everyone was asked with whom they discuss lower-division courses, they discuss their own research, discuss the ongoing changes in the department, and who they consider to be friends. This set of questions allows for an understanding of who opinion leaders are with regards to instruction, who is involved in conversations about what is going on and how things are changing, and to what extent this is or is not the same as who are friends. This selection is in line with standard approaches to social network data collection (Daly, 2010; Kadushin, 2011).

In order to understand the goals, implementation, and evolution of the change initiative in general, and the coordination system in particular, semi-structured interviews were conducted with a subset of the large pool that was surveyed. Particularly relevant to the coordination system are the interviews with instructors of coordinated courses and the P2C2 committee, which included all the new coordinators and the department chair. This group has been involved in the decision-making surrounding the change initiative, and so are primary resources for understanding the how and why of changes being made. Two rounds of interviews were conducted, toward the end of the first and second year of the change initiative. These interviews asked about the main goals of the change initiative, how it came into being, who the key players are, their role in the process, how progress toward goals will be assessed, and how well things seemed to be working in their view. Many of those interviewed in the first year were interviewed again in the second year, in which case the follow-up interview was tailored based on their first interview. The purpose of these interviews was to collect (potentially changing) information about and to assess participants' perceptions of P2C2 program, and the ongoing changes. All interviews were audio-recorded and the interviewer took field notes.

The final source of data comes from observations of P2C2 committee meetings. The committee consisted of the department chair, the Precalculus coordinator who is also director of the new learning center, the Calculus 1 and 2 coordinators, a senior mathematics education faculty member, and the GTA professional development leader who is also a mathematics education researcher. This group met on a regular basis to discuss plans, concerns, and strategies for constant improvement. The intention of attending these meetings was to obtain real-time information about the evolution of the change initiative and any concerns that presented themselves, as well as how each member of the committee spoke about the program and the changes. In some instances, the observer was able to ask clarifying questions of the committee. These meetings were audio-recorded, any artifacts (e.g., agenda, official notes) were collected, and field notes were taken. While data collection is complete, following the end of the second year of the change initiative, the department is not finished with their changes. Therefore, committee meetings are continuing to be observed to monitor and document any further significant changes.

**Data Analysis**

The data for this project is being analyzed in a coordinated fashion, with each piece reflexively informing iterative rounds of analysis. Social network data is first being analyzed using basic graph theory ideas of degree and centrality. These measures allow for the identification of key figures (those with higher degree or in-degree) and distribution of ties: high centrality corresponds to a concentration of ties in a few key players, lower centrality corresponds to more even distribution. These results can then be used to identify important characters, and the interviews and observation notes can be used to better understand their influence or position. Likert data from the survey is analyzed to give each person a score for each scale (e.g., perceptions of students and teaching, individual innovative climate), and the department as a whole. This data is compared across time points to identify shifts in attitudes and/or departmental climate and individual perceptions.

Interview data is being transcribed in full, checked, and coded. A first coding pass identified all interview segments pertaining to the seven features from the CSPCC study, which the department hoped to implement. A separate round of coding identifies segments where participants talk about goals and evaluations of any aspect of the change initiative. In addition to this coding using *a priori* schemes, the data is being open coded for emerging themes in line with the principles of grounded theory (Strauss & Corbin, 1994). These codes will be used to ascertain attitudes, priorities, and goals of individuals and how those shift over time. Of particular importance to this report are the segments which touch on the coordination system and coordinators. Meeting observations are being selectively transcribed, omitting discussions of budget and minutiae. Again, the most relevant segments for this report are those which include discussion of the coordination system and the roles of the coordinators, particularly when the coordinators are commenting.

Analyses of the network, Likert-scale, and interview data will be combined to look for characteristics and attitudes that coincide with network connections (e.g., do conversation partners share attitudes about teaching; do those who discuss the change initiative have similar ideas about goals; what interview language coincides with Likert scale scores).

## Preliminary Results and Discussion

Interview analysis and coordination has not been completed, but the social network analysis is well underway. This has revealed that two of the three new coordinators were quite involved in conversations about teaching, and their advice was sought after, before they were selected as coordinators. They retained and/or increased their prominence throughout the initiative, though increased network activity (more involvement in discussions; higher maximum numbers of ties) altered the relative position of many members of the community. The third coordinator, however, was more peripheral, and nearly absent in those conversations. By the end of the second year of changes, he became highly involved and is now one of the most central members of teaching-related networks. This shift results in an alignment of formal position and informal influence, seen in Apkarian and Rasmussen's (2017) study of successful departments. This university's networks are somewhat more distributed than those in Apkarian and Rasmussen's study, but the shift is a sign that this coordinator has taken up the mantle of coordinator and others respect him as such. Further analysis of the interviews and observations, especially his views of the position and other's perception of him, will shed light on how this transition occurred. An understanding of this transition and coordinator may be able to inform future change agents who choose to implement coordination systems and select coordinators.

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# Using Machine Learning Algorithms to Categorize Free Responses to Calculus Questions[1]

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*Researchers in various science disciplines have begun exploring use of machine learning algorithms to categorize students' answers to constructed-response tasks, achieving inter-rater reliability on par with that between expert raters. We report on a proof-of-concept experiment in which we categorized student responses to conceptually-focused tasks on a calculus final exam. Our results were only modestly successful, but promising. We identify ways in which responses to mathematics tasks are uniquely challenging for these algorithms, and ways in which the algorithms' performance on mathematics tasks can be improved.*

*Keywords:* assessment, calculus, machine learning, constructed response

Advances in machine learning algorithms have introduced the possibility of using computers to categorize written responses to questions based on linguistic patterns. When provided a corpus of hand-scored responses, these programs are able to identify patterns in the responses, and are able to automatically evaluate future responses based on similarities to responses already scored. This has been demonstrated to be successful in several science disciplines and statistics, but these techniques have not yet been applied to mathematics courses more broadly.

Application of these algorithms to education research and to the classroom has many potential benefits. In the classroom, these tools may allow for automated categorization of responses to open questions, so that students in large classes (e.g., large lectures or MOOCs) can receive immediate feedback, as they might in a system such as WeBWorK, but on open-ended, conceptually-focused questions. For research, these algorithms have the potential to identify both students' ways of thinking and how they are connected.

Prior efforts at measuring conceptual understanding have mainly relied on multiple-choice-based concept inventories (see, e.g., Libarkin, 2008); specifically, Epstein (2013) developed a concept inventory for calculus. However, the multiple choice format of a concept inventory is inherently restrictive, and prior research has identified problems with the CCI (Gleason, White, Thomas, Bagley, & Rice, 2015). Machine learning algorithms may allow for the creation of new instructional and assessment tools which capture nuances that concept inventories cannot.

In this study, we consider a proof-of-concept experiment in which we gathered student answers to free-response questions on a calculus final exam and attempted to use these algorithms to analyze the data. In doing so, we identify unique challenges to using these methods in mathematics, and provide ideas for how these challenges can be overcome.

## Background

Our motivation for conducting this study is to improve assessment of conceptual understanding in calculus. Our understanding of assessment is informed by the National Research Council's (NRC, 2001) model: cognition, or models of student understanding;

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observations, or the tasks by which students' understanding is elicited in an observable form; and interpretation, in which the assessor uses their cognitive models to make sense of students' observed behavior. In particular, in order to make valid inferences, the tasks must be appropriate to elicit the desired types of cognition, especially when measuring conceptual understanding.

Most tasks targeting conceptual understanding are either forced-response tasks (e.g., multiple-choice or true-false questions, where students must select the correct answer from a prescribed list of possibilities) or constructed-response tasks (where students must explain concepts in their own words). While both item types can serve to elicit students' conceptual understanding, students' own writing on open-ended questions reveals more about their conceptions and misconceptions than does their performance on a multiple-choice exam (Birenbaum & Tatsuoka, 1987), as constrained-response questions obscure nuances and partial conceptions in student thinking (Hubbard, Potts, & Couch, 2017).

Our work is informed by the Automated Analysis of Constructed Response (AACR) project (https://msu.edu/~aacr/), an ongoing NSF-funded project which seeks to use machine learning to rapidly and efficiently categorize students' responses to open-ended conceptual questions. First, researchers develop and administer constructed response questions to students, and experts categorize the student responses. Then, the coded data are given to a machine-learning algorithm, which builds a model of expert rating using a subset of the coded data. The model's performance is assessed through randomized crossvalidation, calculating a measure of inter-rater reliability (usually Cohen's kappa). Further, several different models built with different machine learning techniques can be combined into an ensemble model, weighting each model by its confidence level; the version of the algorithm we used is an ensemble of 8 separate algorithms.

AACR has achieved impressive performance in several disciplines, including biology (Beggrow, Ha, Nehm, Pearl, & Boone, 2014; Prevost, Smith, & Knight, 2016), chemistry (Haudek, Prevost, Moscarella, Merrill, & Urban-Lurain, 2012), interdisciplinary understandings of energy (Park, Haudek, & Urban-Lurain, 2015), and even statistics (Kaplan, Haudek, Ha, Rogness, & Fisher, 2014). This evidence suggests the AACR approach may be successful in other mathematics classes, though there may be unique obstacles not faced in other disciplines.

## Methods

A total of 67 students in two sections of a coordinated introductory calculus course at a mid-sized university in the Rocky Mountain region of the United States were given the following questions on a common final exam:

1. The limit definition of the derivative of a generic function f(x) is: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ .
   a. What does the numerator mean?
   b. What does the denominator mean?
   c. Why are we taking the limit as h approaches 0?
   d. Explain why the limit definition given above aligns with the overall <u>meaning</u> of the derivative.
2. Suppose the derivative of a function f(x) is negative everywhere on the interval x = 2 to x = 3. Where on this interval (i.e. for what x-value) does the function f have its maximum value? Carefully explain how you know your answer is correct.

We chose to examine student responses to these two questions for several reasons. First, these are conceptually-focused questions, and our ultimate aims are to move toward the development of a library of computer-gradeable items assessing students' conceptual understanding of calculus. Second, student responses to these two questions tended to contain more words than symbols or calculations, which makes the data they produce more amenable to analysis using these machine learning techniques. Third, these questions were designed to elicit a wide variety of ways of interpreting and thinking about the derivative concept.

The two instructors, who are not authors on this report, provided us with anonymized data in the form of digital scans of students' responses to these two questions. Not every student answered every part of both questions; see Table 1 for specific values of N for each question.

We transcribed students' handwritten responses into machine-readable text. In particular, we rendered mathematical symbols into words (for example, we rendered " $\rightarrow$ " as "to") and corrected spelling errors (for example, we corrected various misspellings of "infinitely").

The first two authors independently coded student responses using an emergent open coding process. We independently examined all the student responses to each part of the two questions to identify recurring themes and regularities in student responses, and used these themes and regularities to produce coding schemes. We then met to compare categories, resolve discrepancies, and produce a consensus coding scheme. In the final consensus coding scheme, each part of the two questions had four to six non-exclusive categories (or bins) of student responses, and we coded each student response as either belonging or not belonging to each.

After this coding process was complete, the third author used the coded data as input for the AACR algorithms. The first two authors then performed an error analysis by reviewing the output of the algorithms, looking for bins that performed particularly well or particularly poorly, then seeking to discover possible reasons for the performance of each bin.

## Results

The emergent open coding process resulted in the creation of categories for each question. While space constraints preclude us from giving a full description of every bin of student responses, categories included features of responses such as whether the student identified a "delta" or change in particular values, used common arguments such as "the derivative is negative, so the function is decreasing," or highlighted a geometric or graphical interpretation of a piece of the difference quotient.

Overall, our efforts were only modestly successful, but promising. Our main measure of how well the algorithm performed is inter-rater reliability, as measured by Cohen's kappa. Our kappas ranged from 0.759 to -0.09 (see Table 1). According to Landis and Koch (1977), kappa values between 0.61 and 0.8 represent "substantial" agreement; values between 0.41 and 0.6 represent "moderate" agreement; between 0.21 and 0.4, "fair" agreement; between 0 and 0.2, "slight" agreement; and below 0, "poor" agreement.

| Question | N | Kappas for each category (largest to smallest) | | | | | |
|---|---|---|---|---|---|---|---|
| Question 1a | 67 | 0.616 | 0.576 | 0.290 | 0.000 | | |
| Question 1b | 67 | 0.546 | 0.546 | 0.533 | 0.000 | | |
| Question 1c | 66 | 0.746 | 0.579 | 0.488 | 0.417 | 0.159 | 0.000 |
| Question 1d | 65 | 0.690 | 0.278 | 0.177 | 0.000 | 0.000 | -0.090 |
| Question 2 | 67 | 0.759 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 1: Summary of inter-rater reliability between expert coding and algorithm coding

To gain more insight into the practical performance of the algorithm, we conducted an error analysis, carefully examining all the false positives and false negatives. We looked in particular for commonalities in misclassified responses, as they might reveal the specific difficulties the algorithm faced when attempting to classify responses to math tasks, and thus suggest ways to improve the performance of the algorithm. We illustrate our analysis with a few examples.

**Homogeneity in Responses**

While conducting this investigation, we discovered some elements of these algorithms that worked well in this setting and others which did not. When responses that we coded into a single category all contained the same key phrase (or only small variations), the algorithm tended to be particularly successful in matching our coding. For example, in question 2, one bin included noticing that the derivative was negative. There were very few phrasings students used to capture this idea (for example, "since the derivative of f(x) is negative that means that f(x) is decreasing"), and these phrases showed up repeatedly. In this category, the algorithm achieved a kappa value of 0.759, indicating substantial agreement with our coding. Question 1c included a category about estimating the slope, which performed well for similar reasons; most responses in this bin used the phrase "slope of the tangent line" or the word "secant." Here, the kappa value was 0.579. In categories like these, the algorithm could more easily pick out a pattern that characterizes all the responses in the bin, and thus was able to perform better.

On the other hand, we noted that the algorithm performed poorly on several categories in which the responses were not homogeneous enough. For example, one bin in question 1c was the "cancels out" bin; responses in this category expressed the idea that the *h* in the definition of the derivative should not appear in the correct end result. This idea was expressed in many different ways, such as "cancel out," "get rid of," "plug 0 into *h*," "be removed," "equal zero," "eliminate," and "substitute 0 for *h*." Due to the inability of the computer to recognize these phrases as describing the same ideas, this bin had a kappa value of 0.

**Sample Size**

One clear limitation of our data set was the sample size. Machine learning algorithms require a large enough sample size (often a few hundred) so that patterns in the responses can be identified. In order for the algorithm to differentiate responses, it must detect patterns that exist within the categorized responses *and* do not exist among the remaining responses. Because there will be variation in the phrasing of ideas, enough responses need to exist in order to identify the various ways the same concept may be expressed. This was particularly clear in question 1c's category of "cancels out," as discussed above. Increasing the sample size would increase the likelihood of the algorithm learning that the many variations express the same idea.

**Bag-of-Words Model**

We also consider particular issues which may occur using these techniques in mathematics which may not occur in other subject areas. The ensemble of algorithms used by the AACR project used a bag-of-words (BOW) approach. The raw data is broken into 1-3-word n-grams (that is, words, pairs of words, and three-word phrases). When only 1-word n-grams are used, there is no sense of order or proximity, only the collection of words in a response. When 3-word n-grams are used, there is some sense of proximity, but only within a distance of 3 words.

For many responses to our tasks, the only substantive difference between examples and non-examples of a category was the order of the words they used. One category for question 2 was discussing the derivative being negative. One of the false positives in this category was this response: “If the function is negative on the interval $x = 2$ to $x = 3$ because of the derivative, then the function's maximum value is at $x = 2$. If the function is negative then it means that the slope is decreasing causing the maximum value between 2 and 3 to be at $x = 2$.” Compare this response to a true positive: “The maximum is located on $x = 2$. Since the derivative is negative at every point on the interval, the function's slope is known to be negative for the entirety of the interval. Since $f(x)$ is always decreasing on the interval, the leftmost point is the maximum.” The false positive response contains many of the same words as the true positive response (e.g., “derivative,” “negative,” “slope,” “maximum,” “decreasing”), but uses them in a different order. A strictly-BOW model cannot easily detect the difference between these two responses.

We suspect that the order of words matters more in mathematical writing than it does in writing in other STEM disciplines. Writing a correct description of a mathematical procedure, or rendering mathematical notation into text, likely requires comparatively more precise attention to word order; in contrast, a correct description of the principles underlying evolution likely depends more on the word choice than the word order.

## Future Directions

While we haven’t achieved inter-rater reliability on par with that between human experts, these results provide promise that these machine learning algorithms may be applied to mathematics, while also indicating some ways in which these methods may need to be modified. We aim to begin with a larger sample size to determine whether some of the challenges are solely due to the sample size or are connected with the language of mathematics. We also aim to iteratively refine our coding scheme to allow for a more even split of the responses to aid in the pattern recognition. Additionally, we may need to consider details of how the input is parsed by the software, in particular by considering how to best represent mathematical notation not easily translated into words, such as arrows or functional notation. For instance, our initial exploration with another text classification tool called LightSIDE suggests that we may achieve better performance by replacing the character Δ with the word “delta.”

We also aim to develop more, and better, questions with which to gather data. We initially thought question 1 would be a good choice, because it elicits student understanding of the limit definition of the derivative and tends to produce “wordy” responses. However, our analysis suggests that the question is problematic, since it asks students to make meaning of mathematical notation, leading to responses containing notation or direct translations of notation into words. We hope to create questions whose answers would be more descriptive than notational.

## Audience Questions

- What questions might elicit more descriptive language in student responses?
- The more data we have, the better the algorithms will perform. Are there existing repositories of large numbers of text-based student responses to conceptual questions?
- We want to make our models useful for instructors and researchers. What common understandings exist about the concepts that are most important for students?

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Investigating Student Success in Team-Based Learning Calculus I and in Subsequent Courses

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*With recommendations for active-learning strategies and challenging courses, we applied mixed methods to examine students' success in Calculus I and subsequent courses following instruction using Team-Based Learning (TBL). Overall, TBL students performed better on midterm and final calculus exams, gave more explanations, and completed Calculus I at a higher rate than their peers. These results remained true when students' incoming competencies for calculus were considered. TBL students performed comparably to their peers in Calculus II and Physics.*

*Keywords:* Calculus, Team-Based Learning, Flipped, Active, Large

Research in mathematics education calls for active learning in post-secondary mathematics courses. Freeman et al. (2014) published the findings of a meta-analysis of 225 studies reporting data on exam scores or failure rates for undergraduate STEM students in active learning classrooms versus traditional lecture classrooms. Students in active learning classes consistently performed better. In their nationwide study, Bressoud, Mesa, and Rasmussen (2015), encouraged challenging and engaging courses along with student-centered pedagogies and active-learning strategies as two of seven recommendations leading to successful calculus programs. Active learning also promotes the transfer of knowledge (Billing, 2007). When learning occurs through active engagement while embedding understanding and reflecting upon practices, Billing (2007) says justifications, principles and explanations are socially fostered, generated and contrasted. To identify which active learning strategies benefit learning, Wieman (2014) and Freeman et al. (2014) call for examination of the strategies implemented in classrooms.

Team-Based Learning (TBL) is a specific form of active learning designed to engage students in problem-solving discussions and hold students accountable for their preparation, no matter the size of the class (Sibley & Ostafichuk, 2014). TBL utilizes the flipped classroom model. Many instructors in recent years have investigated the impact of the flipped classroom in Calculus I in both large and small class settings, with the primary benefit to students in flipped sections being higher final exam scores when compared to students in non-flipped sections (Schroeder, McGivney-Burelle, & Xue, 2015, Anderson & Brennan, 2015, Jungic, Kaur, Mulholland, & Xin, 2015, and Maciejewski, 2015). Studies reported mixed performance in students' subsequent performance in Calculus II after having flipped Calculus I, with only one study reporting flipped students performing significantly better (Schroeder, et al., 2015 and Anderson & Brennan, 2015). The study presented here explores the benefits of TBL Calculus I taught to students in large and small class settings both during the semester of engagement and in subsequent courses.

Developed by Michaelsen, Knight, & Fink (2004) to help students learn how to apply concepts, TBL maintains that groups must be formed heterogeneously and remain permanent, students must be made accountable for their individual and group work, group tasks must promote critical thinking and team development, and students must have frequent and timely performance feedback. Aligning to Michaelsen's methods, at the start of the semester instructors assign groups of 5-7 students to teams. A module in TBL covers 2-3 weeks' worth of content and consists of the Readiness Assurance Process (RAP) and application exercises. The RAP holds students accountable for their work completed outside class time and encourages better team functioning. The RAP includes reading and/or viewing videos to gain initial understanding of course information, taking an individual Readiness Assessment Test (iRAT), completing a team Readiness Assessment Test (tRAT), an appeals process, and an instructor mini-lecture. The RATs consist of multiple choice or short answer questions. Students do not receive feedback on the iRATs prior to completing the identical tRATs with their teams. Following the tRATs, instructors give the answers and highlight the main concepts and procedures addressed by the RAP. The RAP positions students to work on the team-based application exercises occurring during subsequent class session(s). The application exercises are rich problem-solving activities designed so each team solves the same significant task, makes a specific choice for an answer, and simultaneously reports an answer. During the tRATs and application exercises, the instructors circulate the classroom to answer questions and nudge students in ways to consider the concepts and solution paths. As done by Nanes (2014) and Prudente (2017) for small linear algebra courses, we modify the TBL process by limiting the length of a module to 2-3 class sessions instead of 8-12 class sessions, and administer the iRATs online outside of class time. At various points throughout the course, students evaluate their team members, further enhancing the team's function and assuring student accountability to the team.

With the emphasis on team communication, interaction, and development of shared understanding, the theoretical framework of social constructivism underlies the foundation and implementation of TBL. Vygotsky highlighted the role of language and social interactions to develop meaning and understanding of a concept (Bigge & Shermis, 1999). The TBL process provides scaffolding for students and situates students in their zones of proximal development. When encountering a new topic, students first engage with material by reading and/or watching instructor-made videos. During the tRATs and application exercises, students engage with other students in interesting, meaningful collaborative problem-solving activities (scaffolding), as they record language and equations and create graphics to communicate their shared understanding (Bigge & Shermis, 1999). The tRATs and application exercises also position students in their zones of proximal development (ZPD) as the designed tasks target skills beyond what students can do independently but can achieve with assistance from both instructors and more knowledgeable peers (Bigge & Shermis, 1999). By design, the heterogeneous teams in TBL offer students different opportunities to serve as the more knowledgeable others to lend assistance in the ZPD (Sibley & Ostafichuk, 2014). Once students in teams complete the application exercises, the students likely will be able to complete future similar tasks individually, thereby raising the ZPD (Shabani, Khatib, & Ebadi, 2010).

## Research Questions

Do students in TBL calculus perform better on departmental midterm and final exams when compared with Non-TBL calculus students? How do TBL students perform on both conceptual and procedural problems on the comprehensive final exam? How well do TBL students transfer their calculus knowledge to subsequent courses, for example, to calculus-related questions on their first physics exam? How do TBL students fare in calculus I and subsequently in calculus II and physics when compared to their peers?

## Methodology

The population at a large (36,000) Midwestern Research I institution includes 43% women, 11% US minority, and 12.6% international students. In Fall 2016, 1845 students enrolled in Calculus I for science and engineering majors. Students registered for a section of Calculus I not knowing the course-delivery system. Three of the authors taught 366 students using TBL (N = 301 students across two large classes and N = 65 students across two small classes). All Calculus I students completed uniformly graded departmental midterm and final exams. TBL students completed pre- and post-tests of the Calculus Concept Inventory (CCI) and an eight-item semantics differential assessing students' confidence in mathematics. The 1507 Non-TBL Calculus students received primarily traditional instruction. For one subset (N=136) of the Non-TBL students, researchers gathered pre- and post-CCI, pre- and post-confidence surveys, midterm exam scores, final exam scores and students' answers to three of the final exam problems. For a second subset (N=108) of Non-TBL students, researchers collected pre- and post-confidence surveys and midterm and final exam scores. During Spring 2017, 232 students consented to the release of information regarding their first exam in calculus-based physics. For 771 students, the Registrar's office provided demographic information and Calculus I, II, and Physics course grades for consenting TBL and Non-TBL Calculus I students and Physics students.

For TBL students, a subset of Non-TBL Calculus students, and students who consented after the drop date to provide physics exam information (i.e., survivors in physics), researchers calculated "incoming competency scores" by averaging students' ALEKs placement score (of 100), students' pretest CCI score (of 22), and the number of high school calculus units (of 6). If students did not take the CCI, the incoming competency score was the average of the number of high school calculus units and the ALEKs placement score. Table 1 displays the averages of the scores for the three groups.

*Table 1. This table shows the incoming calculus competency averages.*

| TBL Calc | | | Non-TBL Calc | | | Non-Calc I Physics | | |
|---|---|---|---|---|---|---|---|---|
| Mean | SD | N | Mean | SD | N | Mean | SD | N |
| .477 | .113 | 298 | .478 | .116 | 354 | .583* | .141 | 116 |

Note. *p < 0.001. The Non-Calc I Physics completed Physics in Spring 2017 but not Calculus I in Fall 2016.

Researchers determined the quintiles of the incoming competency scores for 744 students. For each quintile, researchers analyzed students' performance in Calculus I, II, and Physics. Additionally, for both TBL and Non-TBL Calculus I students for whom researchers possessed pre- and post CCI scores, final exam scores, including scores on three problems of the final exam, researchers partitioned the final exam scores into quintiles. Based on the final exam, researchers selected exams from a subset of each quintile and of each instruction type. For quintiles one through five, researchers randomly selected 14, 24, 24, 24, and 14 exams[1], respectively and performed qualitative analysis on three of the eight final exam questions. Applying a detailed scoring rubric, two of the authors blindly evaluated students' solutions from the 200 final exams. Of the three questions selected, a conceptual question provided the graph of a derivative ($f'$) and asked students to graph the second derivative ($f''$), identify where $f$

[1] Researchers sought to oversample the middle quintiles but the Non-TBL group had exactly 24 exams in quintile 3.

increases, decreases, is concave up and concave down, and where *f* has critical points and inflection points. A procedural question asked students to evaluate a definite integral whose integrand involved the sum of three parts: a simple polynomial, a function for which substitution was required, and a third requiring interpretation of an integral as an area. The third question required optimizing an area given a function containing a parameter. This problem required students to consider multiple techniques including integrating, calculating derivatives, and verifying an optimal value. Each of these three problems was worth 15 points.

Two graders applied the rubric for the conceptual, procedural, and complex procedural problems having inter-rater reliability calculated using percent agreement of 93.79, 86.75, and 96.18, respectively. The two raters and another researcher discussed and resolved all differences.

## Results

When considering the final exam scores for all students in Calculus I, Table 2 shows TBL students outperformed Non-TBL students on the uniformly graded departmental exam. Additionally, Table 3 displays the rate at which students earned D, F, or withdrew (DFW) from the course, demonstrating a significantly lower DWF rate for TBL students when compared to Non-TBL students.

*Table 2. This table gives Fall 2016 departmental final exam scores.*

| TBL | | | Non-TBL | | |
|---|---|---|---|---|---|
| Mean | SD | N | Mean | SD | N |
| 55.52* | 21.88 | 325 | 50.05 | 21.89 | 851 |

Note. *p < 0.001.

*Table 3. This table gives DFW Rates for Calculus I.*

| Fall 2016 | TBL | Non-TBL |
|---|---|---|
| Overall | **19.1** (70/366)* | 32.0 (473/1479) |
| Female only | **24.7** (18/73) | 30.1 (85/282) |
| Ethnic Underrepresented only | **34.8** (8/23) | 45.5 (61/134) |
| International Only | 50.0 (5/10) | **37.1** (65/175) |

Note. *p < 0.01. Boldface type indicates lower DFW rate.

For students categorized based on incoming competency scores, researchers analyzed student success on the departmental midterm and final calculus exams, on the first physics exam, and the DFW or DF rates for Calculus II and calculus-based physics course. As shown in Table 4, for students with comparable incoming competency, for all but the first quintile, students in TBL Calculus outperformed their Non-TBL peers on the midterm exam. For the final exam, TBL students performed better than Non-TBL students significantly so for quintiles 4 and 5.

For students who took Calculus II or Physics during Spring 2017, very few differences occurred in performance on the first Physics exam or when considering the rates at which students completed Calculus II. TBL students in the top four quintiles completed Physics at a higher rate than Non-TBL students, but never significantly different.

Qualitative analysis performed on three of eight of the final exam problems for 100 TBL and 100 Non-TBL students partitioned by quintiles showed TBL students outperformed Non-TBL students on the conceptual problem, with significance for quintiles two and four. Differences in the numbers of explanations and correct explanations arose as shown in Table 5. TBL students sometimes gave three times as many correct explanations as their peers. For the procedural and complex procedural questions, TBL students and Non-TBL students performed comparatively.

*Table 4. This table shows students' success on Calculus I exams depending on an incoming competency score.*

| | Calculus 1 Midterm Exam | | | | | | Calculus 1 Final Exam | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | TBL | | | Non-TBL | | | TBL | | | Non-TBL | | |
| | Mean | SD | N | Mean | SD | N | Mean | SD | N | Mean | SD | N |
| Q1 | 54.8 | 20.8 | 63 | 57.2 | 19.4 | 64 | 37.4 | 21.1 | 57 | 36.9 | 20.3 | 62 |
| Q2 | 72.9* | 13.9 | 52 | 66.11 | 17.5 | 63 | 51.6 | 16.5 | 52 | 44.4 | 20.6 | 58 |
| Q3 | 78.2* | 13.8 | 80 | 72.5 | 17.3 | 49 | 58.15 | 18.3 | 80 | 53.2 | 20.6 | 47 |
| Q4 | 83.1* | 9.57 | 51 | 72.8 | 16.8 | 35 | 62.2* | 18.9 | 50 | 53.2 | 21.7 | 34 |
| Q5 | 89.3* | 8.7 | 41 | 80 | 12.8 | 29 | 74.3* | 17.6 | 40 | 60.3 | 24.4 | 29 |

Note. *p < 0.05.

*Table 5. This table shows the number of explanations given for the conceptual question of the final exam.*

| | TBL | | Non-TBL | |
|---|---|---|---|---|
| | Explanations | Correct Explanations | Explanations | Correct Explanations |
| *f* increases/decreases | 72** | 61 | 56 | 47 |
| *f* critical points | 73** | 59 | 55 | 21 |
| *f* concave up/down | 64* | 44 | 52 | 34 |
| *f* inflection points | 72** | 29 | 55 | 15 |

Note. Numbers are based on 100 TBL exams and 100 Non-TBL exams. *p < 0.05. **p < 0.01.

## Discussion

Similar to the studies examining flipped calculus, (Schroeder, et al., 2015, Anderson & Brennan, 2015, Jungic, et al., 2015, and Maciejewski, 2015), this study demonstrates that when compared to their peers, TBL students performed higher on calculus midterm and final exams, gave more explanations and correct explanations on a conceptual question, and performed comparatively in downstream courses relying on calculus knowledge. This study adds to the literature in that we explored students' performance based on their competencies brought to calculus. The few distinctions in performance of TBL students with Non-TBL students in downstream courses could be due to a mismatch of assessments. TBL students were frequently assessed in their conceptual understanding during Calculus I while assessments in downstream courses likely targeted more procedural understanding.

Noteworthy in this study are the smaller DFW rates in Calculus I for TBL students compared to Non-TBL students. The higher DFW rates for female and ethnic underrepresented students show significant work yet remains by mathematics departments to better serve all students. In addition, the emphasis on communication in TBL likely challenges international students beyond what occurs in Non-TBL courses.

## Questions

What additional aspects of the data should be investigated? Does the incoming competency calculation fairly assess a student's position at the start of Calculus I? What are effective ways to measure students' performance in downstream courses and to capture students' transfer of calculus knowledge?

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# Investigating Students' Meta-Level Object-Reflections and Discourse-Reflections: The Provocative Power of Primary Historical Sources


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*We discuss research based on Sfard's theory of mathematics as a discourse, which we used to investigate the potential of engagement with primary historical sources for motivating undergraduate students to participate in and accept new mathematical discourses. This preliminary report focuses on characterizing the nature of students' participation in mathematical discourse in their written work on primary source projects (PSPs), as well as the question of what constitutes evidence of students' noticing of meta-level rules in that work. We present our analysis of a brief excerpt from one PSP, and provide an analysis of two student work samples to exhibit students' object- and discourse-reflections at the meta-level.*



*Keywords:* Primary Historical Sources, Analysis, Rigor, Metadiscursive Rules


## Introduction

Recently we initiated a study based on Anna Sfard's theory of mathematics as a discourse to investigate the potential of engagement with primary historical sources for motivating students at the undergraduate level to participate in and accept new mathematical discourses. Part of a larger project focused on the use of primary sources in the teaching and learning of undergraduate mathematics, the investigation we report on here seeks in particular to contribute to the growing body of research on the "metadiscursive rules" that govern participation in a mathematical discourse community (Sfard, 2008).

Prior research suggests, for instance, that engagement with primary historical sources may help students learn the metarules that govern mathematicians' discourse (Kjeldsen & Blomhøj, 2012). Given these and related findings, we believe it is important to look more closely at students' interactions with unfamiliar mathematical discourses and investigate their progress in "figuring out" (Sfard, 2014, p. 201) the meta-level rules that govern a new mathematical discourse as a result of those interactions. We are further interested in determining the extent to which students' (verbal/written/other) actions both during and after engagement with the primary source projects provide evidence of their acceptance of a new discourse. In this preliminary report, we focus on the following two questions within the context of an undergraduate analysis course:

- How can we characterize the nature of students' participation in mathematical discourse in their written work related to primary source projects?
- What constitutes evidence of students' noticing of meta-level rules in this written work?

## Theoretical Framing and Literature

In an attempt to resolve certain quandaries related to mathematical thinking and learning, Sfard (2008) operationally defined thinking as a personalized version of communication. Given the collective nature of communication, she introduced the term *commognition* to highlight the communicative nature of activities in our minds, emphasizing that individual cognitive processes (thinking) and interpersonal communication are "but different manifestations of basically the same phenomenon" (Sfard, 2008, p. 83). Using this communicative, or discursive lens, Sfard

(2008) determined that "mathematics begins where the tangible real-life objects end and where reflection on our own discourse about these objects begin" (p. 129). Cobb, Bouf'i, McClain, and Whitenack (1997) also noted the connection between students' mathematical development and *mathematizing* (or *reflective*) *discourse*, which they described as "characterized by repeated shifts such that what the students and teacher do in action subsequently becomes an explicit object of discussion" (p. 258). That is, what identifies the objects of communication in mathematics is their discursive nature: they come to exist as we talk about them. From this viewpoint, mathematics emerges as a highly situated human activity which generates itself. As a result, the learner of mathematics faces a paradoxical situation: How can a person join a discourse for which familiarity with the discourse is a precondition for participation in that discourse?

As a further complication, Sfard (2008) noted that participation in any discourse requires adopting the rules that govern that discourse, in addition to becoming familiar with the objects of the discourse. She referred to the former rules as *meta-level*, or *metadiscursive*, and the latter as *object-level*. For instance, asserting that a particular function is differentiable constitutes an object-level narrative about functions. However, a student's method of justifying this assertion (e.g., sketching a graph versus an ε–δ proof) would be indicative of the metadiscursive rules that govern her discourse about functions. Despite the usual implications of the word *rule* as being invariable and strictly deterministic, metadiscursive rules are subject to change in time and space, and are tacit, contingent, constraining, flexible, and value-laden. Sfard posits that these characteristics render meta-level learning possible only through direct encounters with a new discourse that is governed by meta-level rules different from those governing the learner's current discourse (p. 256). Furthermore, such encounters generally entail a *commognitive conflict* when the discursants unknowingly operate under completely different meta-level rules.

Given their role in governing the actions of the participants in a mathematical discourse, researchers have paid particular attention to factors that affect the learning of metadiscursive rules in mathematics. In a number of these studies, the history of mathematics, and primary source readings in particular, emerged as an instructional approach with strong potential to promote such learning. In their study of university mathematics students, for example, Kjeldsen and Blomhøj (2012) showed that a careful selection of historical sources can help students learn about the metadiscursive rules that govern mathematicians' discourse about functions, and allow them to recognize that these rules changed during the development of that concept. This meta-level learning, they argued, fostered students' learning of mathematics at the object-level as well. In her teaching experiment with pre- and in-service teachers, Güçler (2016) designed an instructional sequence in which the metadiscursive rules implicit in various historical sources were made explicit to students. She showed that by reflecting on their own and mathematicians' definitions of function, students experienced changes in their discourse; within the commognitive framework, it is precisely such changes that constitute evidence of learning.

## Data Sources and Methods of Analysis

For our metadiscursive rules investigation, we collected data in a one-semester *Introduction to Analysis* course for senior mathematics majors. The instructor (the second author) has extensive experience in the development and use of primary source materials for teaching undergraduate mathematics courses. During the semester in question, students completed two Primary Source Projects (PSPs). Analysis PSP #1 (Barnett, 2017a) examines nineteenth century concerns about the foundations of analysis that led to an increase in formal rigor at that time; it was implemented in the second week of class through a combination of individual advance reading/preparation followed by 1.5 days of whole-class discussion. Analysis PSP #2 (Barnett, 2017b) also relates to standards of rigor in analysis, but within the context of counterexamples satisfying certain function properties (e.g., a continuous but nowhere differentiable function). This PSP was implemented over two weeks via a combination of individual advance reading/preparation, whole class discussion, and small group work. A traditional textbook (Abbott, 2015) was also used in the course. Students were guided in their reading and study of the PSPs and the textbook by daily "Reading and Study Guides" (RSGs) prepared by the instructor.

The data collected for this study include video recordings of all class meetings, audio recording of each group during small group work for Analysis PSP #2, students' written work on both PSPs and the related RSGs, instructor class notes, pre-interviews with nine students prior to work on Analysis PSP #2, and post-PSP interviews with two of those nine. We also implemented four student surveys: a pre-course survey, two post-PSP surveys, and a post-course survey.

Since our goal in this report is to share our preliminary findings related to the evidential foundation of this metadiscursive rules investigation, we limit our analysis to just one data source: students' individual written work on Analysis PSP #2 and related RSGs. One reason for this choice is that written work is generally narrower in terms of the variables involved. In particular, students' written work allows us to focus exclusively on the individual's interactions with the material, in contrast to interview or small group work data that also involves students' discourse with each other, the instructor, and/or the interviewer. However, the primary source excerpts and the student tasks contained in Analysis PSP #2 do include considerable breadth and variety of discourse. We thus anticipate that the preliminary analysis and findings we present here will serve as a useful guide to our analysis of the more complex data sources which we will need to complete in order to align our investigation with Sfard's situated-learning framework.

For this preliminary report, we analyzed the PSP itself, the related RSGs, and students' written work on these instructional materials. Given our focus on metadiscursive rules, the main consideration that guided our analysis was whether and how the written narratives of the different discursants did or could provide indications of the implicit rules governing the various discourses. In our analysis of the PSP, for example, our goal was to identify its potential to motivate student noticing of and/or reflection on the various metadiscursive rules, either those of the student or of the discursants in Analysis PSP #2. In that PSP, there are three different discursants: Darboux, Houël, and the project author. Darboux's and Houël's voices are represented through excerpts drawn from letters exchanged during a ten-year correspondence in which they debated issues related to rigor in analysis. The voice of the project's author is present in the background narrative that describes the historical context, in the selection of particular excerpts from the Darboux-Houël correspondence, and in the student tasks based on those excerpts. The instructor-prepared RSGs directed students to read specified portions of the PSP and complete preliminary work on certain PSP tasks for the next class period.

In our analysis of student work on the RSGs, we were interested in aspects of students' written work that could be interpreted as "talking" about the actions of the PSP discursants. We completed this analysis in three stages. First, we examined student responses on specific PSP or RSG items that our document analysis identified as having potential to provoke a meta-level response, and made note of those responses in which students wrote about the actions of the discursants. In the next stage, we examined each of those student responses in detail. As we completed this analysis, we became aware that students' meta-level responses could be further classified as either reflections about specific mathematical objects, or reflections about the discourse itself. Finally, based on this new sub-categorization scheme, we analyzed student written work on all PSP and RSG items to determine if we could document evidence for students' noticing of metadiscursive rules in the form of meta-level reflections of either kind.

## Findings

Based on our analysis, we developed a two-tiered categorization scheme for student discourse in their written work. First, students produced narratives at either the object-level (i.e., they simply "did the math"), or at the meta-level (i.e., they "talked about" – or reflected on – doing mathematics). Second, the focus of students' meta-level reflections was either on the mathematical discourse in the PSP, or on the mathematical objects under discussion in that discourse. We limit our attention in this preliminary report to the second tier of this categorization scheme and consider the focus of students' meta-level reflections, as we believe these findings best characterize the nature of students' noticing of metadiscursive rules. For each subcategory within this tier, we also analyze one student response for evidence of such noticing.

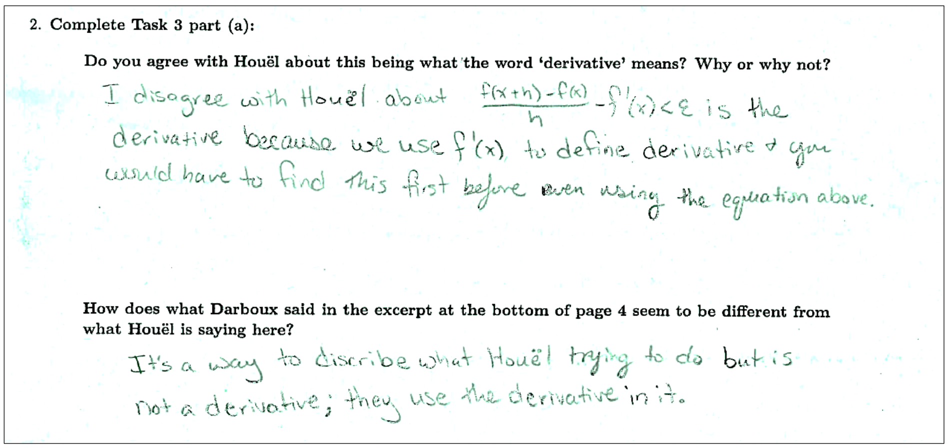
We start with a short sample from the PSP to illustrate how its design could motivate students' noticing of metadiscursive rules. In Figure 1, we read from two of the discursants in the PSP: the author in the narrative before the excerpt, and Darboux in the excerpt itself. As noted by the PSP author, Darboux had been raising his concerns regarding the rigor in Houël's proofs for a fairly long time. We argue that this disagreement results from a difference in the metadiscursive rules that govern their respective discourses regarding rigor. As Darboux noted in his letter, there was a shift occurring in the discourse on rigor among mathematicians of that time, which he felt implied that "no one would find [Houël's reasoning] rigorous." We believe that, through their observations of these shifts in the discourse, students will come to notice the metadiscursive rules that govern the discourse, and begin to experience the commognitive conflict required for a shift in their own metadiscursive rules as a result.

In this same letter, Darboux also returned to a second specific concern that he had about the proofs that Houël's provided for certain theorems involving derivatives. In fact, Darboux raised this other concern as early as January 18, 1875, when he wrote[2]:

> Here is what I reproach in your reasoning which no one would now find rigorous. When we have
>
> $$\frac{f(x+h)-f(x)}{h} - f'(x) = \epsilon,$$
>
> $\epsilon$ is a function of two variables $x$ and $h$ that approaches zero when, $x$ remaining fixed, $h$ approaches zero. But if $x$ and $h$ [both] vary as they do in your proof, or worse yet, if to each new subdivision of the intervals $x_1 - x_0$ there arise new quantities $\epsilon$, then I find it altogether unclear and your proof has nothing but the appearance of rigor. [Darboux, as quoted in (Gispert, 1983, p. 99)]

*Figure 1. Excerpt from Analysis PSP #2 (Barnett, 2017b).*

We now share a representative student response related to this particular excerpt, and a subsequent statement made by Houël about the inequality $\frac{f(x+h)-f(x)}{h} - f'(x) < \varepsilon$ and the meaning of the word "derivative." Figure 2 displays a student response that we characterized as an *object-reflection*: although the RSG prompt invites participation at the meta-level, and the student is engaged with the discourse of the PSP, her response focused on talking about the mathematical object "derivative," rather than about the discourse itself. Analyzing this response for the student's noticing of metadiscursive rules in the discourse, we highlight her response to the second part of the RSG item: "It's a way to discribe *[sic]* what Houël *[sic]* trying to do but is not a derivative; they use the derivative in it." We are aware of a potential objection here, in that the student provided neither an explicit or implicit narrative on the metadiscourse. However, we interpret the commognitive conflict she appears to have experienced regarding definitions in mathematics – that terms should not be explicitly used in the equations/inequalities that define them – to be a result of her noticing of the disagreement between the metadiscursive rules that governed Darboux and Houël's discourses.


2. Complete Task 3 part (a):

**Do you agree with Houël about this being what the word 'derivative' means? Why or why not?**

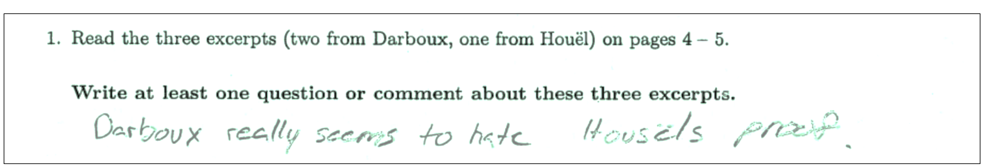
I disagree with Houël about $\frac{f(x+h)-f(x)}{h} - f'(x) < \varepsilon$ is the derivative because we use $f'(x)$ to define derivative & you would have to find this first before even using the equation above.

**How does what Darboux said in the excerpt at the bottom of page 4 seem to be different from what Houël is saying here?**

It's a way to discribe what Houël trying to do but is not a derivative; they use the derivative in it.

*Figure 2. Student object-reflection at the meta-level (Student Response, RSG).*

In the sample of a *discourse-reflection* student response shown in Figure 3, we argue that the student noticed the tension that arose between Darboux and Houël surrounding their lack of communication: by talking about (the nature of) the discourse itself, the student participated in the discourse at the meta-level. We also pay attention here to what the student did not say, as well as what he said. The student did not, for instance, evaluate Darboux's or Houël's letters for mathematical correctness, but provided instead a statement regarding the nature of their communication that has a metadiscursive characteristic. Again, we acknowledge the potential criticism that the student did not explicitly talk about the rules that might be governing Darboux's and Houël's discourse; he did, however, clearly notice the ineffective communication between them, which, in time, created the tension in the letters.


1. Read the three excerpts (two from Darboux, one from Houël) on pages 4 – 5.

**Write at least one question or comment about these three excerpts.**

Darboux really seems to hate Housël's proof.

*Figure 3. Student discourse-reflection at the meta-level (Student Response, RSG).*

Although space considerations allow us to share only very brief examples to suggest the richness of our data set, we anticipate that analysis of further examples will open up discussion of other topics of research interest, including the role of commognitive conflict in promoting metalevel learning and the implications of the discursive framework and our classification scheme for instructional practice.

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## Testing the Stability of Items in a Survey to Measure Relative Instructional Priorities Among Graduate Teaching Assistants


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*The results presented in this paper are part of a larger mixed-methods study examining relative instructional priorities among mathematics graduate teaching assistants (MGTAs). In this paper we share some early results and observations from a limited test-retest analysis of a pilot survey administered to MGTAs in two large public institutions in the Southeast United States. This is not intended to be an exhaustive statistical analysis of the pilot survey results or test-retest analysis. Instead, we focus on specific items to serve as a lens for better understanding the complexity of the choices MGTAs make in instructional settings.*



*Keywords:* Graduate student teacher identity, Instructional choices, Survey validation


### Introduction and Background

The notion of *teacher identity*, the extent to which one identifies as a teacher, informs a large body of literature related to professional preparation and development of secondary mathematics teachers (Ward, Nolen, & Horn, 2011; Beauchamp & Thomas, 2011; Ball & Bass, 2004; Beauchamp & Thomas, 2009; Flores & Day, 2006; Hamman, Gosselin, Romano, & Bunuan, 2010; Heyd-Metzuyanim & Sfard, 2012; Horn, Nolen, Ward, & Campbell, 2008; Lasky, 2005; Sexton, 2008; VanZoest & Bohl, 2005; Hodges & Cady, 2012). Although novice secondary mathematics teachers and new mathematics graduate teaching assistants (MGTAs) share many characteristics such as undergraduate mathematics coursework, stage of life, and assumption of new professional teaching duties, the frameworks developed for understanding secondary teacher identity do not translate easily to work with MGTAs (Gallagher, 2016).

In this paper, we build on previous work that explored the ways in which experienced secondary teachers saw themselves as subject matter, didactical, and/or pedagogical experts, and then assigned those teachers locations within a "personal knowledge triangle" to represent the relative weight given to each of those types of expertise (Beijaard, Verloop, & Vermunt, 2000). A multi-year multiple case study following the development of teacher identity and instructional practice among four MGTAs led the lead author to develop a modified framework to represent the types of expertise and actions valued by MGTAs. For that population, instructional decision-making and value judgments were stratified into three zones: subject-matter concerns, class-level structures, and individual-level needs. For a more complete description of the boundaries of each of those zones, we refer the reader to (Gallagher, 2016).

We refer to the collective framework of these choices as Relative Instructional Priorities (RIP) and visualize each individual MGTA's balance as being situated within a triangle, where the vertices represent the components of the RIP (individual needs, subject-matter knowledge, and class-level considerations). *Individual needs* items are focused on the students as individuals and prompt for instructor prioritization of individualized instruction, supporting diversity, and awareness of campus resources for students. Items in the *subject-matter knowledge* category focus on content and curriculum, including content mastery and preparation for subsequent courses. *Class-level consideration* items include the use of technology in the classroom, physical classroom arrangement, choice of instructional activity, and pacing of a class session.

The relative position of a point along an edge of the triangle represents the weight given to that RIP component. For instance, in Figure 1, the point along the I-S edge is closer to vertex I meaning that this respondent prioritizes the individual needs of students over subject-matter concerns while the point along the C-S edge reveals that the respondent is relatively balanced when choosing between class-level versus subject-matter. Triangulating the three edge points gives the respondent's overall RIP placement, the “star” inside the triangle.

## Survey Development

Our goal with the RIP survey was to have students make choices between pairs of statements that were anchored strongly at one of the three vertices. For example, a descriptor such as “knowing how the course content is used in subsequent courses” would be anchored at S, while “having a range of strategies to encourage group discussion” would be anchored at C and “knowing what resources are available for a student who is upset about a non-academic issue” would be anchored at I. On the other hand, a descriptor such as *‘adapting class activities based on students’ prerequisite knowledge’* contains aspects of subject-matter knowledge (S), individual needs (I), and class-level decision-making (C).

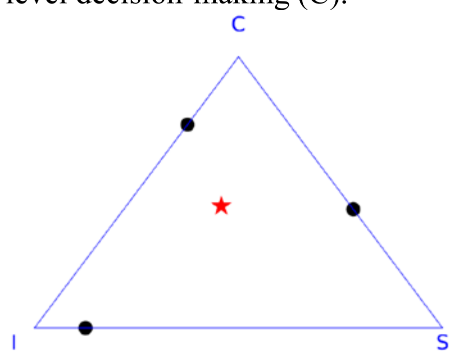


*Figure 1. Visualization of an Example RIP*

For our first implementation, we drew on our prior multiple case study interview and survey data to draft items capturing different aspects of each category, as shown in Table 1. Each item in a category was then paired with every item in the other two categories, generating a total of 48 pairings; all possible cross-category pairings were present and each pairing was prefaced with the prompt, *‘For each item, circle the statement that is more important to you in your teaching.’*

*Table 1. Early Version of RIP Items*

| Individual (I) |
|---|
| **I1** Knowing how to help underrepresented students feel welcome in your class |
| **I2** Knowing how to adapt instruction for individual students |
| **I3** Having a range of strategies for encouraging a struggling student |
| **I4** Being able to support the emotional needs of your students |

Class-level (C)

**C1** Having a range of strategies for evaluating class outcomes
**C2** Knowing how to manage a classroom setting
**C3** Having a range of instructional options for structuring a class session
**C4** Knowing how to plan a class

Subject-matter (S)

**S1** Knowing how to apply the mathematical content in other contexts
**S2** Being able to answer questions about the content
**S3** Being able to identify mathematics mistakes in student work
**S4** Knowing the content of the course

This early paper-based version of the survey was administered to a group of 21 first-year MGTAs in a mathematics professional development course near the beginning of the semester at one large, public, southeastern university. In this version, students circled the item from each pair that was more important to them. Following administration of the paper version, we engaged in a focus group discussion to understand the students' perceptions of the individual prompts. In this discussion, it became apparent that some of these prompts were too broadly phrased, as some students saw an overlap of items. For example, one participant interpreted the item, *'Knowing how to help a struggling student,'* as an S item, as evidenced by her statement, "Knowing your content *is* helping your struggling student, they're the same thing!" Similar statements from other participants allowed us to refine the phrasing of prompts within the C, S, and I categories.

For the second version of the survey, which is the focus of this paper, we used results from the paper-based pilot administration and focus group discussion to generate a bank of 120 items, with each of the four authors generating ten potential items within each category. Each of the potential items was then classified by the other three members of the team as either C, S, or I. Items that did not reach unanimous agreement on classification were discarded or modified until they reached consensus of classification. That winnowing process resulted in 21 C items, 28 I items, and 20 S items. From those, each member of the team selected the eight that he or she felt best captured the range of archetypal prompts for the category. Those votes resulted in the selection of four prompts for each category.

We then created 27 pairings distributed evenly among C-S, C-I, and I-S pairings. Each of the four prompts within a category occurred 2-3 times and was never paired twice against the same prompt from another category. The nine prompts within each pairing were evenly distributed among three groupings to explore whether participants differentiated between preferences, importance, and appeal in choosing between prompts. Each group of nine pairings was prefaced by one of: *'Which is more important to you?' 'Which scenario appeals to you more?'* and *'Select the one you prefer.'* To avoid biasing generation of new archetypal prompts in the discussion portion of the session, we purposefully do not include here a list of all twelve prompts we selected. Some specific prompts are discussed below, and the full set is available from the corresponding author.

## Survey Administration and Retest

We used Qualtrics® to develop, edit, and distribute the survey. At the beginning of the survey, participants were asked to enter a participant ID or code that was generated to protect their anonymity. Additional prompts after the item-selection blocked pairs asked demographic questions about their undergraduate and graduate majors, languages spoken, year of study, teaching background, etc. Participants were given the option of entering into a random drawing for a Visa® gift card incentive, and a separate option to enter their email address to indicate willingness to participate in a follow-up interview.

Thirty-six MGTAs from two large southeastern universities responded to the survey, out of 88 who were invited to participate (40.1% response rate); there was an incentive of two $25 gift card in a random drawing from among respondents. From the responses received, we eliminated those who did not complete the survey and those who completed it in under four minutes, leaving us with 28 responses (31.8% valid response rate). The participants at one university (n=25) could voluntarily enroll in one of several course-specific professional development seminar courses, while participants at the other university (n=3) were required to enroll in a general professional development course. We conducted single-session observations of each semester-long seminar course; each seminar leader indicated that the session we observed was typical of the course. From those observations, the seminars ranged from a focus on instructional planning, to a focus on content mastery, to a focus on planning instruction around common student misconceptions. Participants were a mix of domestic and international, and traditional and non-traditional. All were in their first, second, or third year of graduate school. They held a range of teaching duties including teaching assistant, classroom assistant, and instructor of record.

We sent follow-up emails to all 28 participants six weeks following their original survey completion, inviting each to retake the survey for test-retest comparison. Fifteen participants (53.57%) completed the re-test. It is worth noting that we had 12 retest responses from one institution (48%) and 3 retest responses from the other (100%). Agreement statistics were calculated using Cohen's κ in JMP® Pro 12 (DeVellis, 2016; McHugh, 2012) and we use these results to direct our attention to avenues for improvement of the survey.

## Results and Discussion

We use McHugh's recommendations for Cohen's κ estimates of reliability (McHugh, 2012), and recognize that with only 15 respondents our values of κ are likely underestimates of stability. Under those guidelines, 4 of our 27 items were classified as 'moderate agreement ($0.60 \leq \kappa \leq 0.79$)' and another 9 as 'weak agreement ($0.40 \leq \kappa \leq 0.59$).' Six met the criteria for 'minimal agreement ($0.21 \leq \kappa \leq 0.39$)' and the remaining eight were classified as 'no agreement ($\kappa \leq 0.20$).' We discuss here two specific item pairings: one with weak (nearly moderate) agreement, and one with no agreement. We have selected these items not because they represent the extremes, but because they provide insight into issues with specific phrasings. It is worth noting that although we chose one C-I and one C-S item for this discussion, all three category pairings had pairs that scored well and pairs that scored poorly.

### No Agreement, κ = -0.05. PROMPT: *Which is more important to you? Knowing how to evaluate class activities, OR writing a thorough test that assesses content mastery.*

Here, we have a C item paired against an S item. We suspect that *'evaluate class activities'* may be too broadly worded and could have also elicited elements of subject-matter although it was intended as a class-level item, which includes planning, delivering, and assessing instructional activities. This is because one may perceive the evaluation of class activities as dependent on subject-matter knowledge.

**Weak Agreement, κ = 0.59. PROMPT:** ***Which scenario appeals to you more? You know how and when to adjust class pace and focus, OR you know what resources are available on campus for a student who is upset in class about an issue not related to content.***

This question is a pairing of a C item to an I item. This pairing shows weak agreement likely because each individual phrasing is a strong item in its own category and is not likely to be confused with the other categories. We anticipate that with a larger sample, this item might reach moderate or strong stability.

In general, the items with minimal or no stability are those that have one or both prompts conflated with a second category, rather than being strong anchors to just one category. For example, the prompt, *'Having multiple styles of teaching the same concept,'* was interpreted by some respondents as reflecting full mastery of the content. Although we built this item to exemplify C, it can also be seen as a strong S item. Confounding the stability test, several of the respondents were engaged in teaching and professional development during the six-week period between test and retest. Thus, some lack of stability may reflect an actual shift in instructional priorities. Since we do not have specific professional development enrollment linked to individual survey responses, we cannot clearly account for this effect.

## Questions for the Audience

While most items in this version of the RIP survey failed to meet the criterion for moderate or strong stability, analysis of items that had minimal to no agreement as compared to items that had weak or moderate agreement has provided additional insight into the complexities of the choices MGTAs make as they develop their instructional practice. We continue to struggle with developing prompts that are anchored at the poles of our model in such a way that we can quantitatively capture the nuances that emerged in previous qualitative analysis. We welcome audience input to help guide our next steps. In particular:

- Our attempts to prime for different aspect of decision-making (importance, appeal, preference) do not appear to have produced meaningful differences in response category or test-retest stability. In our next iteration, we plan to prompt for what the MGTA's response has been, or would be, in specific scenarios. Do the additional insights we could potentially gain from pairing an action-based prompt with either an importance- or preference-based prompt outweigh the much higher participation rate we would need in order to analyze the data for differences in responses and stability between those two prompts?
- We have struggled particularly with crafting prompts that isolate aspects associated with the I category: meetings the needs of individual learners. Suggestions from the audience for specific prompts in this category would be most welcome.
- One eventual goal for the RIP survey is to use it to capture change in MGTA's instructional decision-making over time. Do you see an 'ideal' location in the triangle as an outcome for professional development for MGTAs?

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# A Course in Mathematical Modeling for Pre-Service Teachers: Designs and Challenges

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*The increased status of mathematical modeling in the K-12 curriculum requires teacher preparation programs to adapt. This design experiment examines a course in mathematical modeling for pre-service secondary mathematics instructors that was co-developed and co-taught by a mathematics educator and an applied mathematician. The students in the course, all mathematics majors, experienced growth as well as challenges, some rooted in quantitative reasoning.*

*Keywords:* mathematical modeling, teacher preparation, quantitative reasoning

Mathematical Modeling is one of just six conceptual categories for high school in the Common Core State Standards for Mathematics (CCSSM) and is one of the eight CCSSM mathematical practices which span all of K-12 mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). These standards are adopted in 42 of the United States ("Standards in Your State," 2017), however there is reason for concern about teachers' preparation for implementing mathematical modeling tasks. It is rare for teacher preparation programs to even introduce students to mathematical modeling (Doerr, 2007; Lingefjärd, 2007a). Moreover, programs that do currently include or wish to develop instruction in mathematical modeling may be hindered by the lack of a robust research base about best practices, both for teaching mathematical modeling and for preparing teachers to teach modeling.

The elevated status of mathematical modeling in the curriculum, both as a practice and as a conceptual category, requires many secondary teacher preparation programs to adapt. Herein, we report on a design experiment in which the authors, a mathematics educator and an applied mathematician, co-designed and co-taught an undergraduate course in mathematical modeling for mathematics majors intending to be secondary teachers (N=9). In this first iteration of our design experiment, we were guided by a very broad research question about the nature of students' dispositions for engaging in and teaching mathematical modeling both before and after the course.

## Perspective

The Guidelines for Assessment & Instruction in Mathematical Modeling Education (GAIMME) report described mathematical modeling as a process used to answer "big, messy, reality-based questions" (Garfunkel & Montgomery, 2016, p. 7). The process begins with identifying a problem and ends with reporting results. In between, the mathematical modeler makes assumptions; defines variables; refines the original question; develops and implements models; and analyzes the outputs of the model. This all transpires in a non-linear, often cyclic manner. The messiness, openness, and time requirements of authentic mathematical modeling tasks present an array of both pedagogical and conceptual challenges for teachers and for teacher preparation programs.

Some of the challenges for learners of mathematical modeling are documented by Thompson (2011) in his description of mathematical modeling as emerging from quantitative reasoning, which serves as grounding for several nontrivial abilities that are essential to mathematical modeling. Foundationally, the ability for quantitative reasoning allows a student to conceptualize a situation quantitatively. Extending this, covariational reasoning is needed for students to make sense of dynamic situations in which quantities vary in relation to each other. The ability to generalize, in the context of mathematical modeling, allows a student to represent these relationships. Thompson describes a mathematical model as a generalization "of a situation's inner mechanics—of 'how it works'" (p. 51).

Doerr (2007) noted that the pedagogical knowledge for teaching modeling is distinctive and she enumerated some specific pedagogical tasks for teachers of mathematical modeling, among them: choosing and adapting modeling tasks; anticipating and evaluating students' varied strategies; and helping students make rich mathematical connections. This description of a teacher's role in supporting mathematical modeling is largely echoed in the GAIMME report (2016), which also devotes considerable attention to the challenge of assessing mathematical modeling. Unfortunately, there is a dearth of research which investigates the development of the pedagogical knowledge teachers need for teaching modeling. Indeed, Doerr observed that "how teachers acquire this knowledge… remains an open question for researchers" (p. 77).

Doerr (2007) also describes a mathematical modeling course for pre-service teachers (N=8) which she developed and taught. In her course, students read about the modeling cycle and they engaged in and reflected on the modeling processes. She suggested that pre-service teachers engage in a variety of modeling tasks that require explanations, justifications, and reflection. Zbiek (2016) designed and taught a course for a similar audience. She focused on productive beliefs and corresponding unproductive beliefs about teaching and learning mathematical modeling. For example, it is productive to believe that mathematical modeling is a messy process, as opposed to the unproductive belief that problem solving should follow a clearly determined path. Through modeling and pedagogical tasks, students in her course moved toward productive beliefs, though this was accompanied by some persistent confusion about mathematical modeling. She echoed Lingefjärd's (2007b) recommendation that modeling be integrated throughout teacher education programs, not just in a single course.

## The Mathematical Modeling for Teachers Course

We co-designed and co-taught the Mathematical Modeling for Teachers course from the joint perspectives of our disciplines, mathematics education and applied mathematics, and with direction from the GAIMME report. Students in the course engaged in the modeling cycle through in-class team activities, homework/exam questions, and a final team project. From the first day of class, we were explicit about the modeling cycle and, after modeling tasks were completed, students reflected on their work as an expression of the cycle.

We did not organize the course as instruction in a sequence of different modeling techniques. The first half engaged students in a variety of modeling tasks in which they relied primarily on their existing algebraic, geometric, and statistical knowledge. This was followed by three weeks of instruction in linear programming, statistical and mathematical simulations, and some useful features of Microsoft Excel (2013) such as visualizing data, using random numbers to do simulations, making predictions with models, and solving linear programming problems. The next three weeks focused on pedagogical content knowledge such as modifying high school textbook tasks, analyzing curricular materials, and evaluating student work; some of this was foreshadowed by similar tasks in the first half of the course. The rest of the course was devoted to final projects by teams of students in which they identified a question, developed a model, reported on the model, and connected their work to the CCSSM. Throughout, we adjusted instruction according to what we perceived to be difficult parts of the modeling process for students. For instance, students struggled with generalization, as described by Thompson (2011), so we focused on this piece of the modeling process with some matching activities; students linked equations to verbal scenarios and linked the structures of equations to scenarios.

Rather than using a textbook, we developed, adapted, and curated tasks for the students. Students did readings from the GAIMME Report and from teacher-focused articles about mathematical modeling. An often used resource was the set of high school textbooks used by the local school district which claimed to be aligned with the CCSSM. The books labeled questions as “Modeling with Mathematics” within each problem set. However, to borrow phrasing from the GAIMME report (2016), the tasks would more aptly be described as “traditional word problems or textbook applications where all of the necessary information is provided and there is a single, known, correct answer” (p.28). This echoes Meyer’s (2015) analysis of two different supposedly CCSSM-aligned textbooks — tasks labeled as “modeling” rarely required students to model. The local textbooks were valuable both as illustrations of some of the curricular challenges our students would face as teachers and as a source of tasks for students to analyze and modify.

## Methodology

We approached the development and implementation of the Modeling for Teachers course as a design experiment in which course design and theory development are “iterative and interactive” (Schoenfeld, 2006, p. 198). Herein, we report on the first iteration of the course; our research goals were to: 1. identify emergent themes related to the mathematical modeling preparation of teachers, and 2. generate hypotheses to be tested in future iterations. Our data are comprised of student-generated artifacts from the course (e.g., homework, projects, exams), notes from weekly planning meetings between the researchers/instructors, and an end-of-course survey. Seven of the nine students were undergraduate students in a Bachelor of Science (BS) program in Mathematics, Option in Mathematics Education. The remaining two students had already completed the BS program and were taking the course out of interest. All of the students had at least completed Linear Algebra and a first course in proof.

We are in the process of iteratively coding the data to expose patterns in student work (Coffey & Atkinson, 1996). This initial round of coding is guided by the components of the modeling cycle as defined in the GAIMME report, e.g., “make assumptions and define essential variables” (Garfunkle & Montgomery, 2016, p. 13). Within each of these components, subcodes are based largely on the knowledges and dispositions needed for doing and teaching mathematical modeling that Thompson (2011) and Doerr (2007) enumerated. For example, we are identifying challenges and patterns related to generalization and to types of pedagogical content knowledge. Subsequent rounds will lead to a refinement of the codes.

## Preliminary Results

Given the preliminary nature of this report, we will briefly document some emergent themes, some initial results related to the students’ pedagogical and content knowledge of mathematical modeling, and some plans/hypotheses for the next iteration of the course. The end-of-course survey indicated that students found the course to be worthwhile. They reported that their knowledge of mathematical modeling increased and that they were excited to teach mathematical modeling. They also expressed comfort with the openness of the tasks they did in class. They reported that they intend to, as teachers, adapt textbook tasks to engage students in various aspects of the mathematical modeling process though they expressed somewhat less confidence in their ability to do so.

Our initial analysis and reflections have made us rethink our decision to begin the course with a discussion of the modeling cycle. Throughout the course, we asked students to reflect on the modeling process and connect it to their work. These reflections revealed that the process began to make sense only after substantial engagement with mathematical modeling. Moreover, there were instances in which students unproductively looked to the cycle for quasi-procedural guidance in the problem solving process. Even after successful completion of a modeling task students had trouble answering, “What is the model?” As a remedy for this discomfort, many students later communicated that they would have preferred to begin the course by watching the instructors demonstrate the mathematical modeling process. Though we are unlikely to honor that request in the next iteration, it may be a sign that the students have greater comfort with more traditional modes of teaching.

By delaying the explicit naming of the components in the modeling process, we can first begin to address some unproductive problem-solving dispositions of students. In particular, we found that we had to encourage the students to approach modeling tasks by first considering specific examples and exploring a “toy model.” Without intervention, students often became mired in premature attempts to define appropriate variables and develop an abstract model. Furthermore, their work with abstraction often betrayed a lack of comfort connecting verbal and symbolic representations. In an early linear programming task adapted from a local school district’s quarterly Algebra II exam, six of the nine students made errors with units while connecting an inequality to the problem’s context. In general, most students experienced some level of difficulty with quantitative and covariational reasoning; ongoing analysis aims to characterize this with more granularity.

We also observed challenges with more pedagogically focused tasks. For example, students’ attempts to modify textbook problems to create authentic mathematical modeling tasks often resulted in tasks that were not open enough or were imprecisely stated. As instructors, we sympathized with this as we also experienced the challenge of finding or producing appropriate tasks. Other pedagogical tasks posed challenges that were likely not exclusive to the context of modeling. Of note, on an exam we asked students to make sense of and recommunicate a hypothetical student’s linear model for a scenario that our students had already modeled (geometrically) during an in-class activity. In our analysis of their work, we have not yet been able to parse out the sources of difficulty, whether they be related to the nature of their content knowledge or insufficient practice evaluating student work or something else.

We have thus far documented several challenges faced by students. We view these challenges as opportunities to improve the Mathematical Modeling for Teachers course and to frame an examination of students’ experience throughout our teacher preparation program. Certainly, as instructors, we faced several challenges that stemmed from our lack of familiarity with and resources for teaching mathematical modeling to preservice teachers. But that is a subject for another paper.

## Discussion

Though students were satisfied with the course and we generally viewed it as a success, students and instructors encountered significant challenges that extended beyond those reported above. The existence and persistence of these challenges may give credence to the call for greater integration of mathematical modeling throughout teacher preparation programs, not just in a single course. However, if we accept Thompson’s (2011) view that mathematical modeling depends on quantitative and covariational reasoning, our preliminary analyses indicate that increased focus on these foundational abilities is merited and, from a pragmatic perspective, are perhaps more feasible to implement throughout the program. This is not to say that engagement in mathematical modeling cannot be done in service of developing those reasoning abilities (e.g., see Swan, Turner, Yoon, & Muller, 2007), but our experience illuminated that, even with two instructors for nine students, teaching mathematical modeling as envisioned in the GAIMME report and finding (or developing) good modeling tasks requires time and expertise. Furthermore, the challenges students faced related to generalizing (e.g., translating from verbal or specific scenarios to symbolic representations) and quantitative reasoning (e.g., working with units) may not be detected by assessments in more computationally-focused lower-division courses. Building foundational reasoning abilities and mathematical dispositions for mathematical modeling in those courses would yield impactful benefits throughout the students' undergraduate careers and their careers as teachers.

As we continue analysis and interpretation of data from the Mathematical Modeling for Teachers course, we are also planning the content for and research of the next iteration of the course. Input from the RUME community will provide valuable guidance. Audience discussion will be prompted, in part, by the following questions:

1. Given the breadth of our data, we could investigate it with various foci: the teacher educators, the students as mathematics majors, the students as future teachers, the curricular materials, and the teacher preparation program. How can we capture that in our analysis? Or how should we narrow the focus?
2. Widening the scope, what are the broader questions about the nature of mathematics teacher preparation across the curriculum? How can this work contribute to answering those questions?
3. What are particularly salient opportunities for research during the second iteration of the course?
4. What are the broader implications of this study for undergraduates who are not pre-service mathematics teachers or mathematics majors? What attainable goals should we set in designing college-level mathematical modeling courses or experiences at various levels?

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# Assessing Visual Literacy Competency in Undergraduate Mathematics

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*We investigated how linear algebra students acquired mathematical knowledge from visualization objects, and to what extent these students exhibited visual literacy standards in higher education. Seven linear algebra students were the subjects of this research project. The data were collected through questions with high visual content and through semi-structured interviews. We analyzed the data by using descriptive and content analysis techniques. Our study found that linear algebra students were not sufficiently competent in using visualization techniques.*

*Keywords:* visual literacy, visualization, linear algebra, assessment.

## Introduction

Research into the teaching of undergraduate linear algebra confirms the advantages of using the visual approach when introducing mathematical content, and that visual representations of mathematical notions have a positive effect on students' learning (Hannah, Stewart & Thomas, 2013; Dorier & Sierpinska, 2001; Dubinsky, 1997; Harel, 1989). Guided by the Visual Literacy Competency Standards for Higher Education (ACRL, 2011), we have designed a framework for assessing students' visual literacy competency level in undergraduate mathematics and used this framework to indicate students' use of visualization objects in linear algebra.

The earliest attempt to define visual literacy was in Debes (1969, p.27; as cited in Avgerinou & Ericson, 1997, p.281). Following his definition, visual literacy will "…enable a visually literate person to discriminate and interpret the visible actions, objects, symbols … that he encounters in his environment." As Bieman (1984) noted, Debes's definition tells what a visually literate person can do, rather than what visual literacy is. Researchers in distinct fields have offered various definitions of visual literacy (Bristor & Drake, 1994; Braden, 1996; Burns, 2006). For example, Ausburn and Ausburn (1978) defined visual literacy as a group of skills that will enable an individual to understand and use visualization objects to communicate with others. Hortin (1980) defined visual literacy as the ability to understand and use images, and to think and learn in terms of images. We adopt the definition of visual literacy given by Stokes (2002) as the ability to interpret images, and to generate images for communicating ideas and concepts.

In 2011, the Association of College and Research Libraries (ACRL, 2011) published standards providing tools for educators seeking to measure visual literacy competency (VLC) of college and university students in undergraduate education. ACRL emphasized that standards outlining student learning outcomes have not been articulated in the research on visual literacy. They proposed the following standards:

The visually literate student should be able to:

**1.** Determine the nature and extent of the visual materials needed.
**2.** Find and access the needed images and visual media effectively and efficiently.
**3.** Interpret and analyze meanings of images and visual media.
**4.** Evaluate images and their sources.
**5.** Use images and visual media effectively.
**6.** Design and create meaningful images and visual media.
**7.** Understand many of the ethical, legal, social, and economic issues surrounding the creation and use of images and visual media, and access and use visual materials ethically.

In terms of assessing VLC, there are a limited number of instruments in existing literature (Avgerinou, 2007; Arslan & Zeren-Nalinci, 2014). However, we could not find assessment instruments for specific disciplines. We develop a framework based on standards for assessing VLC in undergraduate mathematics, with attention to linear algebra.

**Adjusted Framework for Assessing VLC in Undergraduate Mathematics**

We adjust the Visual Literacy Standards in Higher Education to undergraduate mathematics, to assess students' visual literacy competencies. The adjusted standards are as follows:

The visually literate students in undergraduate mathematics should be able to:

| | |
|---|---|
| – perceive a given visualization object and recall prior knowledge related to a given visualization object. | PERCEPTION |
| – understand a given visualization object and make connections between prior knowledge and the given visualization object. | UNDERSTANDING |
| – analyze the properties of a given visualization object and interpret that given visualization object. | ANALYSIS and INTERPRETATION |
| – use a given visualization object. | USAGE |
| – create a meaningful visualization object. | CREATION |
| – evaluate a given or personally created visualization object. | EVALUATION |

Each standard has sub-categories which can be used to assess students' VLCs. The order in which the standards are given should not indicate their significance. Being focused on the mathematical skills and procedures, we did not attend to ethical and social components addressed in ACRL (2011). For a working definition of a "visualization object" we embraced the definition of a visualization object as a physical object that is viewed and interpreted by a person for the purpose of understanding something other than the object itself. These objects can be drawings, pictures, 3D representations, animations, etc. (Philips, Norris & Macnab, 2010 p.26).

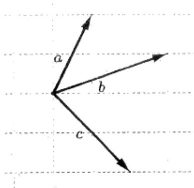
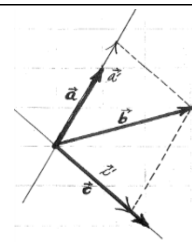
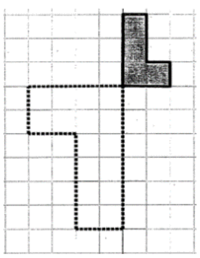
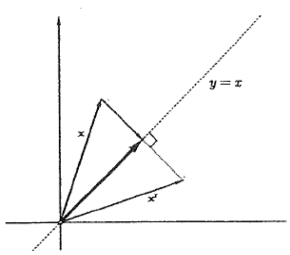
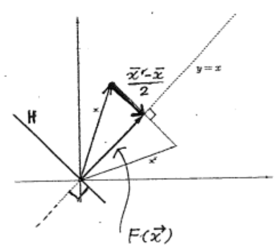
We specifically focused our research on the following adjusted standard: Use a given visualization object (usage standard). To gain insight into the extent linear algebra students' usage of visualization object, our main research questions are as follows:

- How do linear algebra students use a visualization object in the problem-solving process?
- To what extend do linear algebra students exhibit usage standard in the problem solving-process?

## The Method

We adopt the qualitative-interpretative paradigm (Lodico, Spaulding & Voegtle, 2006, p. 264) applied to a holistic single case study (Yin, 2003, p. 39). Seven undergraduate linear algebra students were selected via the purposive sampling technique (Cohen, Manion and Morrison, 2007, p. 114). Collected data consisted of students' responses to three linear algebra questions with high visual content (Table 1). These questions were given to students at different occasions as test questions. Webb's (2009) Depth of Knowledge model was used to identify questions' complexity level; six mathematics professors assisted with classifying questions' complexity level. We also conducted five semi-structured interviews with student volunteers that were recorded and transcribed. Obtained data was analyzed and interpreted using percentage frequency distribution (Shapiro, 2008, p. 292) and content analysis techniques (Cohen, Manion and Morrison, 2007, p. 475).

*Table 1. Questions and their complexity levels*

| Level | Questions | Possible Solutions |
|---|---|---|
| **1- recall and reproduction** |  Determine if the vectors given in the figure below are linearly independent or linearly dependent without computing. Please, use the figure to justify your answer. |  One possible answer is to construct a parallelogram in which $\vec{b}$ is a diagonal and $\vec{a}$ and $\vec{c}$ are on adjacent sides. Then (following the figure) $\overrightarrow{a'} = \alpha\vec{a}$ for some $\alpha > 1$ and $\overrightarrow{c'} = \gamma\vec{c}$ for some $0 < \gamma < 1$. We have $\vec{b} = \alpha\vec{a} + \gamma\vec{c}$. |
| **2- skill and concepts** |  Assume T: $\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that is a composition of two transformations: reflection with respect to the origin ($\boldsymbol{x} \mapsto -\boldsymbol{x}$), followed by scaling by factor 2. Write the standard matrix of the inverse transformation. Please justify! | Note that $T = S \circ R$ where $R(\vec{x}) = -\vec{x}$ and $S(\vec{x}) = 2\vec{x}$. So $R^{-1}(\vec{x}) = -\vec{x} = R(\vec{x})$ and $S^{-1}(\vec{x}) = \frac{1}{2}(\vec{x})$. Therefore $T^{-1}(\vec{x}) = (S \circ R)^{-1}(x) = R^{-1}\left(S^{-1}(\vec{x})\right) = R\left(\frac{1}{2}(\vec{x})\right) = -\frac{1}{2}\vec{x}$ and standard matrix of $T^{-1}$ is $\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ |
| **3- strategic thinking** |   Assume that the mapping $F: \mathbb{R}^2 \mapsto \mathbb{R}^2$ maps each vector $\boldsymbol{x}$ into the vector $\boldsymbol{x} + \frac{\boldsymbol{x^r} - \boldsymbol{x}}{\boldsymbol{2}}$ where $\boldsymbol{x^r}$ is the reflection of vector $\boldsymbol{x}$ through the line $y = x$. Find the subset $H \subseteq \mathbb{R}^2$ such that $F(\boldsymbol{x}) = \boldsymbol{0}$ for every $\boldsymbol{x}$ in $H$. Sketch the subset *H*. |   Note that if $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ then $\overrightarrow{x^r} = \begin{bmatrix} b \\ a \end{bmatrix}$ so $F\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} b \\ a \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \end{bmatrix}$ so F is a linear transformation. All vectors $\begin{bmatrix} t \\ -t \end{bmatrix}$ ( t a real number) will be mapped into the zero vector or $H = \left\{\begin{bmatrix} t \\ -t \end{bmatrix} \middle\| t \in \mathbb{R}\right\}$ |

The sub-categories for standard 'usage' are auxiliary drawing, algebraic interpretation and justification. The scores based on students' responses ranged from zero (0) to three (3) as shown in Table 2. In each of the three problems presented in Table 1, we expected that the student would use the given visualization object as a tool to advance his introspective visualization (Phillips, Norris and Macnab, 2010, p. 10). We also expected the student to produce an auxiliary drawing. We consider that would indicate higher level of VLC of the students. Students had opportunities in class to see how linear transformations transform particular sets of points in the plane, but we could not measure to what extend they have developed their intuition and benefitted from those opportunities in their solutions to Problem 3. Some of the difficulties students had with problem 2 could be result of inexperience with the notion of the inverse of a composition of bijections, which they have encountered in their previous courses.

*Table 2. Scores and explanations*

| Score | Explanation of expected response in the standard of usage |
|---|---|
| 3- excellent | Adequate and effective auxiliary drawing<br>Accurate and relevant algebraic interpretation<br>Valid and relevant justification |
| 2- satisfactory | Adequate but ineffective auxiliary drawing<br>Accurate but irrelevant algebraic interpretation<br>Inappropriate justification |
| 1- fair | Inadequate or ineffective auxiliary drawing<br>Inaccurate or irrelevant algebraic interpretation<br>Invalid justification |
| 0 | No-response |

## Preliminary Findings

In Figure 1, we present the results of three test questions in the percentage distribution bar chart. (Y-axis represents the percentage of students achieving the measured category)


*Figure 1. Using visualization objects*

The bar chart (Figure 1), gives students' performance on these three questions and the three categories: effectiveness of their auxiliary drawing, the extent of their algebraic interpretation of a problem and appropriateness of their justifications. Notice that auxiliary drawing in all three problems is present, but it varies depending on the complexity of the question. One can also see a very mixed show of algebraic interpretation in all three questions. Students' ability to justify mathematical statements was weak, with good results only for question number one.

We will illustrate some of these conclusions with two examples of students' responses to question 3 and a question that was given as a part of the interview process.

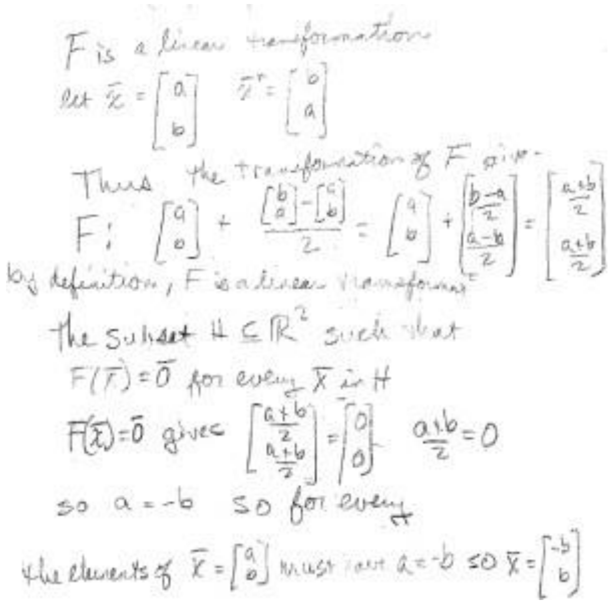



*Figure 2a. An example response to the third visual linear algebra question*

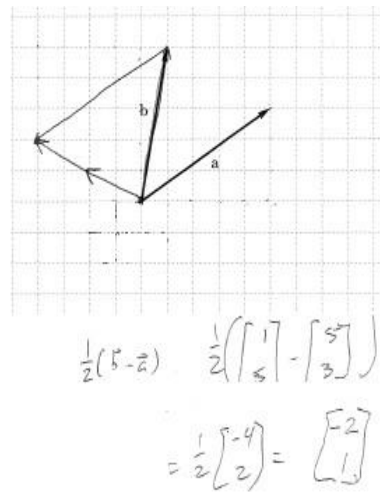



*Figure 2b. An example response to interview question: Sketch (**b-a**)/2*

The above two examples of student responses illustrate their primary learning strategy in solving the problems. Figure 2a illustrates a solution to problem 3 with correct response and conclusion to which student arrives in purely algebraic way, almost ignoring the image showing $F$ as an orthogonal projection on the line $y=x$. The required sketch of $H$ is obtained *after the* analytic solution to the problem was obtained. We observe a similar pattern in Figure 2b, where the student uses the unit grid as a coordinate system and assigns specific coordinates to given vectors. After computing the components of the vector (***a-b***)/2, the student sketched the answer. Again, the analytic reasoning precedes the visual way and illustrates the low 'usage' level of the student.

## Conclusion

In this ongoing research, we proposed to use a new framework for assessing undergraduate mathematics students' visual literacy competency based on ACRL's (2011) standards and presented the findings of this usage standard. We found that students struggle to use given visualization objects in linear algebra. Students did not use auxiliary drawings very much, despite their usefulness, a phenomenon reported in Krajcevski and Keene (2017). We intend to continue developing the framework in our further research.

## Intended Questions for the Audience

1. Are the components of the adjusted framework (perception, understanding, analysis and interpretation, usage, creation, and evaluation) sufficient to characterize visual literacy?
2. What sub-categories might be useful for assessing students' visual literacy?

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**Measuring Self-Regulated Learning:**
**A Tool for Understanding Disengagement in Calculus I**

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*Calculus 1 has been and continues to be a key gateway course to STEM majors, which contributes to a loss of students in the STEM pipeline. Student-learning behaviors impact performance and, in turn, the student experience. By analyzing early online homework activity and help seeking, rich descriptions of students can be used for early prediction for at-risk students, but can be misrepresentative for students who have not yet engaged with these resources. This preliminary report presents self-regulated learning (SRL) theory as a way to understand student behaviors. Using this framework, online tools were designed to collect behavioral data which was used to create a SRL score based on in-course student activity. This preliminary report presents findings on the relationship between student behaviors in Calculus I, a behavioral SRL score, and failure rates, particularly with students disengaged with course content.*

*Keywords:* Calculus I, Self-Regulated Learning, Learning Behaviors

Calculus I is commonly identified as a weed-out course for students majoring in STEM disciplines. This is further supported by data gathered from the Mathematical Association of America national calculus study that reports a 25% DFW rate nationally at research institutions. Additionally, they report that Calculus I students experience lowers confidence in, enjoyment of, and the desire to continue pursuing a degree requiring mathematics (Bressoud & Rasmussen, 2015). Research evidence suggests that how students engage with their studies effects success (Vandamme, Meskens, Superby, 2007). By leveraging the high level of data that can be collected from online engagement and digital interactions, we theorize that it may be possible to identify students early in the semester, based on their behaviors with digital content, who are at-risk of being unsuccessful (defined as a grade of D or F here) in Calculus I (Fonti, 2015; Hu, Lo, & Shih, 2014; Macfadyen & Dawson, 2010). In the past, techniques from learning analytics and data mining have been employed with relative success using early performance data to predict course outcomes. In this study we use data from the Canvas LMS (quiz-log data), the online homework system, and sign-in logs from when students visit the calculus help center (CHC).

While fine-grained interactions with digital resources can provide a rich set of data about individual students, those who do not engage with resources can be easily be misrepresented by their digital footprint, as their temporary disengagement can result for many reasons. Self-regulated learning (SRL) theory provides a way to better understand student behaviors.

This preliminary report presents initial findings toward understanding disengaged students' self-regulation, academic performance, and learning behaviors. We draw on student interactions with online tools developed around SRL theory and then organized into a behavioral SRL score. We aim to address the following research questions:

1. *Can we quantify SRL through student interactions with online tools?*
2. *How does SRL relate to academic performance, particularly for those students that are disengaged in the course content early on?*

## Theoretical Framework: Self-Regulated Learning

Mathematics students often cannot identify mistakes in their work, why the mistakes exist, or how to change their study habits to address their mistakes (Zimmerman, Moylan, Hudesman, White, & Flugman, 2011). SRL - "the self-directed process by which learners transform their mental abilities into academic skills" (Zimmerman, 2002, p. 65) - enables students to develop an understanding of their learning processes so that they can implement strategies and modify their study habits and learning behaviors to address difficulties to become more successful learners. Zimmerman's (2000, 2002) three-phase process model provides the SRL framework for this proposal. The phases focus around a learning task and consist of a planning phase (forethought), performance phase, and self-reflection phase – each occurring before, during, and after the learning task, respectively. The model is cyclic, with each phase informing the next. The cyclic nature of SRL allows students to continually review course material and deepen their understanding of concepts, which can promote learning, transferability, and retention (Bannert, Sonnenberg, Mengelkamp, & Pieger, 2015; Sonnenberg & Bannert, 2015).

### Forethought

The forethought phase consists of task analysis and self-motivational beliefs around the learning task. Task analysis involves setting goals and identifying strategies to employ so that those goals can be achieved (Zimmerman, 2000, 2002). Self-motivational beliefs involve "self-efficacy beliefs, outcome expectations, task interest or value, and goal orientation" (Zimmerman, 2008, p. 178). A student's beliefs regarding their self-efficacy about a task impact the value placed on that task and, in turn, the motivation and expectations of how effort for task will be executed. When a strategy cannot be identified, confidence and motivation play a role in determining if the learner intends to seek help. The forethought process connects directly to beliefs about one's learning (Zimmerman, 2002), influencing how the performance phase will be carried out.

### Performance

The performance phase is where strategies identified in the forethought phase are implemented. Elements of regulation of performance require self-control and self-observation, where one can modify or adapt the strategies identified during forethought to optimize the learning process. Self-control competencies such as time management and attention focusing are key during this phase to be able to make such adjustments (Zimmerman, 2000, 2002). By monitoring and having a record (mental or physical) of event details and duration during the performance phase, one can assess current and future adjustments that may need to occur. Upon completing the task (i.e. finishing the performance phase), learners reflect on their processes.

### Self-Reflection

Self-reflection involves learners looking back on their performance to assess what went well and where improvements could be made. This phase involves judging performance and then reacting to that judgment. Self-judgment means evaluating one's own performance to some personal standard, and self-reactions will differ depending on whether or not that standard was met (Zimmerman, 2000, 2002). For example, a student may receive a C on an exam, when, in fact, they had anticipated an A. The student may then react by changing study habits or strategies. Results from the self-reflection phase then impact subsequent forethought and performance phases of future tasks in this cyclical process as students move forward.

## Measuring Self-Regulated Learning

The most commonly used instrument for measuring SRL is the Motivated Strategies for Learning Questionnaire (MSLQ) - an 81-item self-report questionnaire that assesses “college students’ motivational orientations and their use of different learning strategies for a college course” as well as their “goals and value beliefs” (Pintrich, Smith, Garcia, & McKeachie, 1991, p. 3). The MSLQ has primarily been used to study components of SRL and the relationship of the components to academic performance (Pardo, Han, & Ellis, 2016; Pintrich & De Groot, 1990; Pintrich, Smith, Garcia, & McKeachie, 1993; Zimmerman & Kitsantas, 2014). However, due to the nature of self-reports, student responses on the MSLQ tend to reflect “how they [think] they should study, rather than how they [do] study” (Worthley, Gloeckner, and Kennedy, 2015, p. 137). Further, the validity of the MSLQ has been put into question, as it does not always align with observable behaviors such as strategy usage (Winne & Jamieson-Noel, 2002). While the three-phase cyclic model of SRL describes the learning process, methods to find evidence of SRL outside of self-reports are non-trivial (Winne & Baker, 2013). To address this discrepancy, we designed online tools specifically to coax SRL into observable, measurable events that can be recorded, which led to the formulation of a behavioral SRL metric.

## Methods

For this research, online tools were developed and implemented through the university’s learning management system (LMS). They were based on the three SRL phases: forethought, performance, and self-reflection. Since students often struggle with precalculus content (Agustin & Agustin, 2009), we focused on SRL around the task of assessing and remediating one’s knowledge of precalculus topics at the start of the semester. A self-assessment (forethought), content quiz (performance), and post-quiz reflection (self-reflection) were created, all of which were optional for the students.

Focusing on the forethought phase, we designed a Prerequisite Self-Assessment (SA), an 8-item survey asking students to rate their confidence in correctly answering questions on relevant prerequisite material on a Likert Scale from one (No confidence) to five (Very Confident). By assessing their confidence in precalculus topics, the tool determines if students were engaging in the task analysis component of forethought – how well they think they know the material.

Students’ participation in performance phase of SRL was determined by whether or not the student took the Prerequisites Content Quiz (CQ). The CQ is comprised of 12 multiple-choice and multiple-answer questions about prerequisite material essential for Calculus I. Upon finishing the CQ, students received information on what questions they answered right and wrong. When any question was answered incorrectly, immediate feedback was provided, including the relevant topic to review and available resources. Student responses, time spent per question, and order in which questions were answered can provide insight to better understanding the performance phase of SRL, as these data provide information on self-control and strategy implementation.

Students were given a five-item survey called the prerequisite reflection tool (RT) which was intended to be used after the CQ. The RT asked students questions such as ‘What topics from the prerequisite content quiz do you plan to study?’ and ‘How do you plan to study/practice problems from the prerequisite content quiz material?’. Use of the RT provides evidence that a student is reflecting on his or her performance on the CQ. This behavior indicates that a student may be planning to address possible content weaknesses, but does not provide evidence of subsequent follow through (i.e. additional forethought and performance of the intended task) without further investigation, such as tracking access of resource materials.

## Formulating a Behavioral SRL Metric

Of students that used the SA, they were considered to have either high confidence (mean confidence score of three or greater) or low confidence (mean confidence score less than three). Students that completed the CQ were considered to have either high precalculus ability (score of eight or greater) or low ability (score less than eight). Results from the SA and CQ were combined with use of the RT (used or did not) as well as precalculus resource access (accessed or did not) to formulate a behavioral SRL score. Results in each of these categories produced 36 different possible outcomes, and each outcome was then evaluated using Zimmerman's three-phase model as to whether or not they needed to remediate and if they were self-regulating appropriately. Each behavior was then assigned behavioral SRL scores of 0, 20, 40, 60, 80, or 100, from 0 (no self-regulation) to 100 (highly self-regulating).

## Data and Initial Findings

In addition to data gathered to compute the behavioral SRL score for each student, course performance data, online homework access data, and CHC attendance was collected. Online homework for the entire course was due at the end of the semester. Success in Calculus I was identified by a final letter grade of A, B, or C. Grades of D and F were classified as failure.

Table 1 presents the distribution of behavioral SRL scores across the 376 consenting students with the failure rate for each group. The same data is also shown for students that were considered 'disengaged' in the course with regard to both digital interactions with online homework and in person help-seeking in Calculus I as of week four. For example, 64 students had an SRL score of 60, 31.2% of which failed the course. In addition, 23 of these students were identified as being disengaged with the course, and 52.2% of these disengaged students failed.

*Table 1: Statistics for Behavioral SRL Score Within Two Groups: All Students and All Disengaged Students.*

| Behavioral SRL Score | Number of Students | Failure Rate | Number of Disengaged Students | Failure Rate |
|---|---|---|---|---|
| 0 | 32 | 46.9% | 20 | 60% |
| 20 | 50 | 32% | 16 | 56.2% |
| 40 | 18 | 27.8% | 6 | 16.7% |
| 60 | 64 | 31.2% | 23 | 52.2% |
| 80 | 104 | 24% | 29 | 31% |
| 100 | 108 | 17% | 23 | 21.7% |
| Total | 376 | 26.3% | 117 | 41% |

Within both groups, failure rates tend to decrease as students' behavioral SRL score increase, with the exception of the small group of students who have a behavioral SRL score of 40. Additionally, the subset of disengaged students has a particularly high rate of failure.

To begin validating the behavioral SRL score, we compared mean behavioral SRL scores with students' behavior with online homework and help seeking in the CHC. Of these four groups, the disengaged students, those who had neither been to the CHC nor worked on their Calculus I course online homework as of week four, had the lowest mean behavioral SRL score (56.1), while students who both sought help and used the online homework had the highest mean score (72). Those who only worked on online homework had slightly higher mean behavioral SRL score (70.3) than those who only sought help (mean score=63.2). A Kruskal-Wallis nonparametric test verified that these four behavioral groups differ in mean rank behavioral SRL score, $\chi^2(3)=15.0625$, $p=0.013$. Post Hoc Dunn's test with FDR correction revealed that the mean rank of disengaged students are statistically lower than students who only engage in the calculus course online homework before Exam 1, $z=-3.67$, $p=0.0018$, $r=0.21$.

## Discussion

These preliminary findings show promise for being able to use an SRL framework to develop tools that measure students' SRL behaviors and identify which disengaged students are potentially at-risk of failing Calculus I. Using these tools, we developed a method for generating SRL scores for students by analyzing their behaviors, specifically those around prerequisite remediation and readiness for Calculus I. The relationship between SRL scores and academic performance metrics suggests that more self-regulatory behaviors around prerequisite material promote success in course performance, which aligns with what is seen in the literature (Labuhn, Zimmerman, & Hasselhorn , 2010; Zimmerman et al., 2011; Zimmerman & Schunk, 2001). These statistical relationships grow stronger when looking at only those students who are disengaged with Calculus I before their first exam. Students who are disengaged, but have a higher SRL score tend to have higher success rates in the course than those who are disengaged with lower SRL scores. Similarly, when looking across all students (not just those who are disengaged), we see that students who have higher SRL scores generally fail less on average, showing the benefit of measuring SRL for all students.

## Limitations and Future Direction

In this report, presence of the different phases of SRL was determined by whether or not the students used the designed online SRL tools. This method relies on students' understanding the purpose for each tool and makes the assumption that lack of use is a conscious effort to avoid the tool and the associated SRL phase. We recognize that this has limitations as we have yet to further develop ways to measure SRL for students that do not use online tools. In moving forward, while we recognize there are several limitations, we plan to address the following two: (1) student awareness of online tools and their purpose, and (2) student usage of a selection of tools rather than engaging with all tools, which leads to a lack of evidence of students' SRL behaviors (e.g. despite a student reflecting on CQ performance, they fail to use the RT). As part of our efforts in addressing these limitations, we plan to merge multiple tools into one.

Student self-regulatory behaviors around prerequisites leave breadcrumbs about their self-regulation in Calculus I. This may help frame the temporary disengagement of some students as intentional regulated prolonging of engagement. For instance, a student well positioned in Calculus I may temporarily divert their exam study time to a different class in which they are struggling. Preliminary reports show some success with using high and low behavioral SRL scores as predictors for success and failure in Calculus I. Further, when focusing on disengaged students, our methods for identifying those at-risk of failure become more precise. Based on student interactions with online tools and other external resources, we are deepening our understanding of SRL's role in Calculus I with student learning behaviors. While this data combined with the SRL framework is informing modification, enhancements, and addition of online tools, we plan to conduct student interviews as a way of triangulating our data. We plan to use Zimmerman and Pons' (1986) protocol as another way to measure SRL. Student interviews will provide an opportunity to better understand student engagement in the course as well as validate our quantitative findings. SRL scores and qualitative data can then inform intervention support to improve student success in Calculus I and STEM.

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**The Emergence of a Video Coding Protocol to Assess the Quality of Community College Algebra Instruction**

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*The Evaluating the Quality of Instruction in Post-secondary Mathematics (EQIPM) is a video coding instrument that provides indicators of the quality of instruction in community college algebra lessons. The instrument is based on two existing instruments that assess the quality of instruction in K-12 settings—the Mathematical Quality of Instruction (MQI) instrument (Hill, 2014) and the Quality of Instructional Practices in Algebra (QIPA) instrument (Litke, 2015). EQIPM addresses three dimensions focused on quality of instruction via 17 codes. In this paper, we describe two codes: Instructors Making Sense of Procedures from the Quality of Instructor-Content Interaction dimension, and the Mathematical Errors and Imprecisions in Content or Language, a code spanning all three dimensions. The purpose of the paper is to illustrate what we have learned from these codes and the new instrument to advance our understanding of post-secondary mathematics instruction.*

*Keywords:* Algebra, Instruction, Video Coding, Community Colleges

Various reports have established an indirect connection between students leaving science, technology, mathematics, and engineering (STEM) majors because of their poor experiences in their STEM classrooms (Herzig, 2004; Rasmussen & Ellis, 2013). Interestingly, however, most of these reports are based on participants' descriptions of their experiences in the classroom, rather than on evidence collected from large scale observations of classroom teaching (Seymour & Hewitt, 1997). When such observations have been made, they usually focus on superficial aspects of the interaction in the classroom (e.g., how many questions instructors ask, how many students participate, or who is called to respond, Mesa, 2010) or their organization (e.g., time devoted to problems on the board, or lecturing, Hora & Ferrare, 2013; Mesa, Celis, & Lande, 2014). Undeniably, these are important aspects of instruction, yet these elements are insufficient to provide a characterization of such complex activity as instruction in classrooms.

A key concern in post-secondary mathematics education is the lack of teacher training that mathematics instructors received in their graduate education (Ellis, 2015; Grubb, 1999). We argue that the lack of a reliable and valid method to fully describe how instruction occurs hinders our understanding of the complexity of instructors' work in post-secondary settings and therefore limits the richness of professional development opportunities focused on the faculty-student-content interactions (Bryk, Gomez, Grunow, & LeMahieu, 2015). As part of a larger project that investigates the connection between the quality of instruction and student learning in community college algebra, we have developed an instrument, EQIPM, that seeks to characterize instruction.

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In this paper, we present the current form of the instrument and describe two codes that show promising findings from our pilot data.

## Theoretical Perspective

We assume that teaching and learning are phenomena that occur among people enacting different roles—those of instructor or student—aided by resources of different types (e.g., classroom environment, technology, knowledge) and constrained by specific institutional requirements (e.g., covering preset mathematical content, having periods of 50 minutes, see Chazan, Herbst, & Clark, 2016; Cohen, Raudenbush, & Ball, 2003). We focus on instruction, one of many activities that can be encompassed within teaching (Chazan et al., 2016), and define instruction as the interactions that occur between instructors and students in concert with the mathematical content (Cohen et al., 2003). Such interaction is influenced by the environment in which it happens and it changes over time. Empirical evidence from K-5 classrooms indicates that ambitious instruction is positively correlated with student performance on standardized tests (Hill, Rowan, & Ball, 2005). The definition of instruction requires attention to the discipline and is fundamental in understanding mathematics teaching practice. Therefore, we assume, first, that the experiences of instructors and students while interacting with mathematical content have a significant impact on what students are ultimately able to demonstrate in terms of knowledge and understanding, and second, that it is possible to identify different levels of quality of the instruction that is enacted in mathematics classrooms.

## Methods

In the pilot phase of the larger research study, we video-recorded 15 lessons in introductory, intermediate, and college algebra classrooms from three different community colleges in three different states during the Fall 2016 semester. The lessons ranged in duration between 45 and 120 minutes, and were taught by six different instructors (two part-time and four full-time). The lessons covered one of three topics: linear equations/functions, rational equations/functions, or exponential equations/functions. These topics were chosen because they offer us opportunities to observe instruction on key mathematical concepts (e.g., transformations of functions; algebra of functions) and to attend to key ways of thinking about equations and functions (e.g., preservation of solutions after transformations; covariational reasoning), which are foundational algebraic ideas that support more advanced mathematical understanding. The development of EQIPM was similar to the process used by Hill and colleagues (2008) and by Litke (2015). Their instruments describe and qualify instructional practices from video-recorded lessons deemed representative by rating all individual 7.5-minute segments.

EQIPM evolved through various iterations of segment and lesson coding and discussion with a subset of segments. In the final phase of development, all 151 segments in the data corpus were double-coded using an earlier version of EQIPM. Each code received a score ranging from 1 to 5. The team of 10 researchers, all co-authors on this paper, worked in pairs to independently code three lessons; for each of their lessons, each pair held calibration meetings to discuss codes with a discrepancy in ratings greater than one point.

The instrument consists of three dimensions: (1) Quality of Instructor-Student interaction, (2) Quality of Instructor-Content Interaction, and (3) Quality of Student-Content Interaction; two cross-cutting codes (*Mathematical Explanations and Mathematical Errors and Imprecisions in Content or Language*); and three additional codes that help characterize the type of work done on each segment in a lesson (i.e., Mathematics is a focus of the segment, Procedure taught in the segment, and Modes of instruction, see Figure 1). In this paper, we describe one code from the

Quality of Instructor-Content Interaction dimension (*Instructors Making Sense of Procedures*) and one cross-cutting code (*Mathematical Errors and Imprecisions in Content or Language*) to provide the reader with a sense of how these two codes are useful in characterizing key practices in the community college algebra classrooms that we have observed.

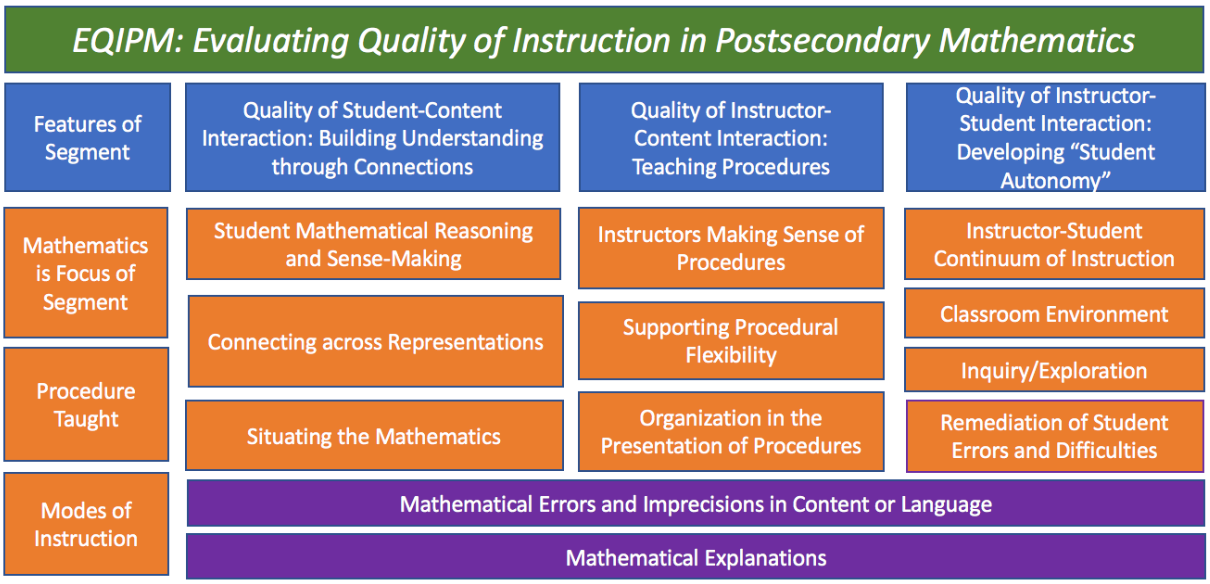


*Figure 1: Dimensions and codes for the EQIPM instrument.*

## Preliminary Findings

*Instructors Making Sense of Procedures* was a code originally from the QIPA instrument, which defined a procedure as "instructions for completing a mathematical algorithm or task" (Litke, 2015, p. 160). With this code, we sought to identify ways in which instructors used mathematical relationships or properties to motivate a particular procedure. Such work includes activities that attend to, for example, the type of solution generated by a procedure and its interpretation or to the conditions of the problem that may suggest what procedure to apply and where in the process to use it. This work also includes activities that attend to the symbols used in mathematical expressions and equations, as well as to the structure of an algebraic expression and how it is transformed by each step in a mathematical procedure. Thus, in general, this code seeks to capture all mathematical work that instructors do to make salient mathematical properties, relationships, and connections embedded in a particular mathematical procedure. Making sense of procedures helps students to understand the underlying logic of the procedure of how to get from one step to the other, not merely reproducing the work from a textbook example. We believe that when the instructors make explicit the sense-making behind procedures, then their students will have an opportunity to make sense of the mathematics as well so that they can engage more substantively with the mathematics.

In order to make an assessment of the evidence found in the videos, each segment was rated on a scale of 1 to 5 depending on whether the instructor did not engage in sense-making while teaching a procedure (a rating of 1) or when the instructor consistently engaged in sense-making throughout the segment (a rating of 5). A rating of 3 is reserved for cases in which sense-making is observed on several occasions in the segment, but they are brief, or for cases in which procedures are not the focus of instruction. Ratings of 2 and 4 were used when the evidence was not sufficient for a 3 or a 5. Out of segments in which a procedure was taught, we only identified one in which no sense-making was present; 59 segments (43%) had a rating of 3, and 55 segments (39%) had a rating of 4 or 5 (30% and 9%, respectively). Thus, in these lessons, we were able to provide evidence for all of the ratings, which suggests that the instrument allows for differentiation of the role of sense-making in the classroom. In most cases, we can say that instructors were making a genuine effort of assisting students in making sense of the procedures taught during the video recorded sessions.

For example, in a lesson on linear functions, instructor 0613 presented a word problem in which students are asked to model the value of a copy machine, $v$, as a function of time, $x$. The instructor asked students to consider how to write a linear function $v$ as a function of $x$. Students contributed three answers: $f(x)$, $f(v)$, and $v(x)$. The instructor reasoned through all three responses using the information in the problem to make sense of the appropriate way to write the function as $v(x)$ (0613_L1, 2016, 26:22). Later in the segment, the instructor asked, "What does the value of $120,000 mean in this problem? What does a slope of negative 12,000 mean in this problem?" (0613_L1, 2016, 29:00). The subsequent conversation detailed the meaning of the values of the $y$-intercept and slope for this specific context. This segment received a rating of 3 because within the segment, the instructor made sense of the procedure more than briefly, but sense-making was not the focus of the instruction on the procedure (how to write an equation to model a situation given in a problem). Instructor 0112 demonstrated sense-making that was rated as a 5, when working with a growth problem modeled by $y = 3(2)^x$. He asked students to think about the meaning of the general equation $y = ab^x$ with a concrete example that used paper folding to demonstrate the meaning of $2^x$, where $x$ was the number of times a piece of paper was folded by half and $y$ the size of the stack of papers generated by the fold: One fold created a stack of 2, two folds created a stack of 4, three folds created a stack of 8, and so on (0112_L1, 2016, 30:00). This segment was rated 5 because, sense-making was the focus of the segment and it saturated the segment.

*Mathematical Errors and Imprecisions in Content or Language* was a code originally from the MQI. The code is intended to capture events in the segment that are mathematically incorrect or that have problematic uses of mathematical ideas, language, or notation. This code applies to the work and utterances of the instructor. Errors made by students are ignored except when the instructor does not correct them. This code also captures cases in which problems are solved incorrectly, when definitions are incorrect, or when the instructors do not use or forget to mention a key condition in a definition. Finally, we apply this code when instructors use imprecise or colloquial mathematical language. Our interest in this code stems from the realization that in some of the lessons we observed, instructors used language that was not mathematically correct to convey ideas, and whereas such uses were appropriate for the lesson—their meaning had been negotiated within the classroom—the continued use of that language could put students at a disadvantage because they would not be gaining proficiency in using correct mathematical language. A rating of 1 indicates that no errors or imprecisions were observed, a desirable situation, whereas a rating of 5 indicates major content, notation, or language errors were made throughout the segment, an undesirable situation. A rating of 3 is reserved for some errors that obscure the mathematical meaning for part of the segment. Out of 138 segments in which a procedure was taught, we only identified one segment with a rating of 5 signaling that major errors were seen, and 45 segments (33%) in which no errors were observed. Fifty-eight segments (42%) had minor imprecisions (rating of 2, e.g., using "bottom" for denominator) and 34 segments (24%) had a rating of 3 or 4 (23% and 1%, respectively). Given that about one fourth of the segments were rated with a 3 or more in this sample, we note that the instrument may suggest areas for professional development that relate to strengthening the rigor in using accurate mathematical language, relationships, and notations. For example, in a lesson on rational equations, instructor 0112 used a graphing calculator to graph $y = \frac{x+1}{x^2-5x+6}$. While identifying the asymptotes using graphical and symbolic representations of the function, the instructor stated that the zeros of the denominator "equal" the vertical asymptotes. We considered this statement as an error in language because that precise statement would require writing a linear equation, as well as an error in content because it does not recognize that there is a removable discontinuity. Later in the segment, the instructor asked the students: "If I let $x$ be equal to 2, with what do I end up at the bottom?" (0112_R1, 2016, 57:12). We considered the replacement of precise mathematical terms (e.g., denominator) with an everyday, colloquial term (e.g., bottom) an imprecision in language. However, the segment was rated as 2 in this code because the imprecise language did not hinder the procedure of identifying the asymptotes.

## Questions for the Audience

The current version of the EQIPM instrument seeks to gather evidence on the quality of the instructor-content, instructor-student, and student-content interactions, thus mirroring the framing on instruction of our work. To advance our work, we have the following questions:

- Are there features of quality instruction that are not being captured in this version of the EQIPM instrument? A preliminary factor analysis with the pilot data points to a three-factor structure (with mathematical errors and imprecision being by itself). During the presentation, we will share the instrument and the current definitions, and we will illustrate how some of the codes fit in this factor analysis.
- The labels for the main categories of codes, mirror our definition of instruction. Are there other possible structures or organizations of the codes? What theoretical framing about quality of instruction could be used for such reorganization?
- Which additional video coding protocols could be leveraged to make the EQIPM instrument more robust?

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The Ways Graduate Teaching Assistants Learn to Teach Through Various Professional Development Interactions

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*Across the nation, there is increased national interest in improving the way mathematics departments prepare their GTAs. However, without an understanding of how GTAs interpret and make sense of various teaching practices, we are working without all of the information. I report preliminary results on the ways in which the understandings of GTAs of various teaching practices changed over a term. With this analysis, we will be able to better understand how to better support GTAs with their teaching in the future. The research presented here represents the start of an increased understanding of how GTAs form their own teaching practices.*

*Keywords:* Graduate Teaching Assistants, professional development, teaching practices

Across the nation, many mathematics departments have begun to change the way they structure the teaching of the Calculus sequence based on the seven recommendations that emerged as a result of the MAA sponsored study of successful Calculus programs (Bressoud, Mesa, & Rasmussen, 2015). One of the recommendations was to improve the professional development (PD) offered to the Graduate Teaching Assistants (GTAs) involved in the teaching of Calculus. GTAs comprise a larger percentage of Calculus I instructors and teach a larger percentage of Calculus I students than tenure-track professors (Ellis, 2014), making their PD all the more important. Though these various mathematics departments have the common goal of improving the teaching practice of GTAs through PD, the structure of the PD programs for GTAs varies greatly among them (Belnap & Allred, 2009). Research on the PD programs across the nation is becoming more common place, as seen by the growth in the number of people in the PD working group at the annual Research in Undergraduate Mathematics Education conference.

Much of the research done on GTA PD programs have focused on the various structures of PD programs, on the outcomes of the programs, or on a small, in-depth case study (e.g., Kung & Speer, 2009). While it is important to know what the outcomes of the program are, it is equally important to understand how those changes occurred so as to improve our PD programs. In a review of the research, Speer, Gutmann, and Murphy (2005) stated the need for studies with longitudinal designs so as to “inform the design of exemplary programs that have a lasting influence on instructional practices” (p. 79).

The mathematics faculty at a large southwestern university made several changes to the calculus program, including to the structure of the PD program for the GTAs. The PD is oriented around supporting GTAs to teach in with a more student centered approach. As part of this program, the GTAs discuss effective teaching practices with each other, their course coordinators, and mathematics education researchers. This paper discusses preliminary results of analysis on the evolving understanding of effective teaching practices as evidenced in their discussions in the various formal meetings attended by the Calculus I and II GTAs. With a better understanding of the ways in which the discourse around various teaching practices evolve over time, we can better support GTAs in their learning to teach in the future.

## Background

In many ways, professional development can feel like a complex game of telephone. The leaders and creators of the professional development have certain ideas of effective teaching practices that they are attempting to convey to the teachers or the facilitators with whom they are working. However, the facilitators and teachers are appropriately going to interpret it in their own way and share and use their transformed version of their ideas of effective practice. Research on the Standards movement reform of the 1980's and 1990's documented only a modest impact of the initiative on teachers' practice and that teachers selectively took up reform ideas and adopted only the surface-level features (Spillane & Zeuli, 1999). Researchers explained the adaptation in terms of teachers' learning processes and suggested that implementation varied because teachers drew on prior knowledge and practices when interpreting the message about the new standards and instructional practices (Coburn, Hill, & Spillane, 2016; Coburn, 2001; Cohen & Ball, 1990).

The similarity with the K-12 context is that when faculty and graduate students undertake reform teaching, all of those involved, including the department chair, course coordinators, faculty who take on the PD of teaching assistants, and the teaching assistants themselves, co-construct the message of the reform. It begins with a small group of faculty with the goal to promote high-quality instruction and its success ultimately, in large part, depends upon the learning of the teaching assistants who interact with the college students most frequently. My particular study focuses on how GTAs make sense of and interpret what they learn about how to lead a student-centered classroom.

There have been only a handful of studies done exclusively on the state of professional development of GTAs across the nation (Belnap & Allred, 2009; Ellis, 2015; Kalish et al., 2011; Palmer, 2011; Robinson, 2011). In addition to national level studies, there are also several case studies of particular programs at specific institutions, with a focus on the structure of or the efficacy of the program (e.g., Griffith, O'Loughlin, Kearns, Braun, & Heacock, 2010; Marbach-Ad, Shields, Kent, Higgins, & Thompson, 2010). So, while there have been studies that describe the various forms of PD or that give an idea of what GTAs have learned from their experiences in PD, little work has been done on the ways in which the GTAs have constructed their understandings of various teaching practices – "what teachers do and think daily, in class and out, as they perform their teaching work" (Speer, Smith, & Horvath, 2010, p. 99). This research contributes to understanding how the GTAs are appropriating and transforming various teaching practices to fit their own needs over time.

## Setting

At the large, public southwestern university in this study, Calculus is taught in large lectures of approximately 160 students. The GTAs lead break-out sessions with approximately 35 students twice a week, with one meeting focused on active learning activities. To support the GTAs in facilitating these active learning activities, the GTAs participate in a three-day teaching seminar the week before classes begin in the Fall. The GTAs continue to meet approximately eight times throughout the term with mathematics education faculty. In addition to the formal PD, the GTAs have weekly meetings with their course coordinator where they talk about the activity for the following week and any additional administrative issues.

The structure of the GTA program has been changed to include a lead TA for each of Calculus I and II. The lead TA is a more experienced GTA who provides support to his or her fellow GTAs with a PD aspect that occurs both before the term begins and throughout the term (Ellis, 2015). Throughout the term, the lead TA visits the activity day sections of his or her fellow GTAs to observe the class and meet with the GTAs afterward to debrief. The lead TA visits all of the other GTAs two or three times a term. A representation of all of these various meetings and observations throughout the term is given below in Figure 1.

| Monday | Tuesday | Wednesday | Thursday | Friday |
|---|---|---|---|---|
| Meeting w/ Calc II Coordinator<br>Calc II Break-out sessions | Calc I Break-out sessions | Calc II Break-out sessions | Calc I Break-out sessions | PD Class w/ math ed researcher<br>Meeting w/ Calc I Coordinator |

*Figure 1: A sample week for GTAs from the Fall 2016 term.*

## Methods and Analysis

Sixteen GTAs agreed to be part of the study, including both of the lead TAs, seven new GTAs, and seven returning GTAs. I either audio or video-recorded each PD meeting, course coordinator meeting, debrief between the lead TA and a fellow GTA, and any break-out sections observed by the lead TAs. I transcribed each of the video and audio recordings and coded each utterance about teaching practices using descriptive coding (Bakhtin, Emerson, & Holquist, 1986; Miles & Huberman, 1994).

A theoretical perspective that includes both a social and a cognitive aspect is useful in making sense of the evolving nature of discussions around teaching practices. The socio-cultural learning theory put forth by Vygotsky posits learning occurs through a reflexive relationship between the individual and the community in which the individual interacts (John-Steiner & Mahn, 1996). In order to understand the ways in which the discussion evolved over time amongst the GTAs, I am using a modified version of a framework within the socio-cultural learning theory known as the Vygotsky space (Harré, 1983). With this framework, the understanding of a teaching practice can be tracked as it is appropriated and transformed by the GTAs throughout the term. A representation of the Vygotsky space can be seen below in Figure 2.

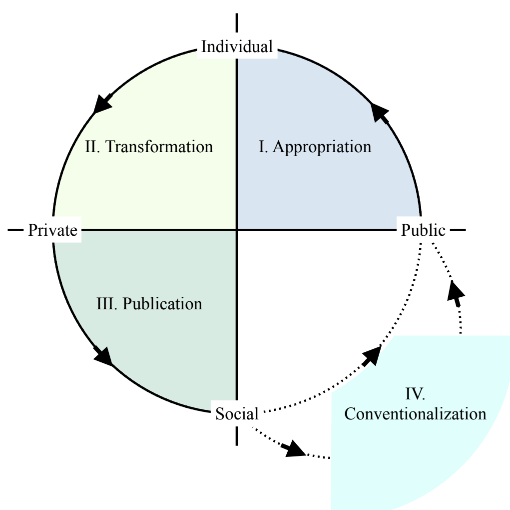


*Figure 2: Modified diagram of the Vygotsky space.*

Within this diagram, there are two axes: Public-Private and Individual-Social. These two axes make up four quadrants, which work to explain the four aspects of the Vygotsky space. For instance, *appropriation* is within the Public-Individual quadrant because it describes how an idea comes to a person from the public. *Transformation* is within the Individual-Private quadrant because it describes the way a person has made the original idea their own. In the third quadrant is *publication*, which is described as how the person makes their own private understanding of the idea known to the social group they are within. At this point, the idea may go through several iterations of these three quadrants before it lands within the fourth quadrant, *conventionalization*, which represents that idea has become normalized within a community (Gallucci, DeVoogt Van Lare, Yoon, & Boatright, 2010).

The modified Vygotsky space framework facilitates our understanding of how changes occur within a community over time. In this study, it sheds light on how GTAs make sense of various teaching practices. Using this framework, a researcher attends to the speaker, the publicized interpretation, and the timing of the utterance, revealing how individuals may affect the community as well as how the community may affect the individual. An example of this sort of analysis is given in the preliminary results below.

**Preliminary Results**

One particular teaching practice that was discussed by the GTAs throughout the term was that of asking students to repeat or rephrase something that had just been said. This may be to repeat or rephrase something another student said, state a given task in their own words, or rephrase something the GTA has just explained. An experienced instructor in mathematics education introduced this teaching practice during the three-day seminar before the semester began:

> "There was something that I wanted to add that I think is really productive [in] engaging your students in a task is to make sure that if someone gives an answer… and they're kind of going in the right direction, you want to make sure the rest of the class understands it as well so you can say 'can somebody revoice what Nick just said or revoice what Joe just said' or basically say what they said but in your own words to make sure that other students do understand…"

After this, several other references to this practice were made public by the professional development leaders, including as a way to reinforce an idea, to get students to interact with one another, to have the students state when they could not hear a response, and to make sure the students understand the task given to them.

Once the term began, a transformation in the way this teaching practice was discussed could be observed in the ways in which various GTAs made public their understanding:

> "One of them is if you see someone who's talking, you say 'hey can you repeat what, repeat what Christian said.' And put them on the spot a little. But if they can't, don't make a big deal. They're already going to make a big deal about it."

Independently of this interaction, the lead TA for Calculus I made a similar suggestion to one of his GTAs:

> "And you had another student re-explain the directions which is always good because that means at least somebody is paying attention. Also, it makes them think oh what if he calls on me."

As the term went on, the understanding of the teaching practice seemed almost become conventionalized around the thought that it was a good way to make sure students are paying attention in class. However, there was one more experienced GTA who continued to push her different version of a more "student-understanding" approach to the teaching practice amongst her peers:

> "I think even asking [inaud] students to like revoice or talk about what just happened is good because it gives different perspectives than you teaching them and you make sure

someone in some group out there understands and maybe when they say it, others will get it better."

When the term began to close, the student-understanding approach became more dominant in the ways GTAs discussed this teaching practice in formal settings, with the lead TA for Calculus I making public this transformed version after watching a fellow GTA's teaching video:

"I think, um, one thing I do, 'cause I do the same thing. I ask them 'do you understand that' and then no one says anything so pick on someone you know, maybe not all the time, but occasionally pick someone you know usually struggles and see if they actually understand. Have them try and explain it. And then at that point they either do and they explain it, or they say 'well, I don't actually get it.' Okay well, take some time, talk about it with your groups and then we'll come back and then tell me what it means."

While there is evidence to suggest the understanding of the teaching practice as useful in determining their students' understandings was becoming conventionalized, the change only took place near the end of the term and so it cannot be said whether or not that understanding continued. What makes this particular example an interesting and important one to consider is the fact that the understanding of this teaching practice as a disciplinary tool may have been inadvertently encouraged by the professor of the professional development course. Approximately one month into the term, the professor was engaging the GTAs in a discussion about what they noticed in a video they had just watched and said the following:

"If you use things like asking them to repeat what somebody else said, asking them to explain what somebody else said, those types of things, those can help get students to listen to each other."

This could be interpreted as the professor suggesting this particular practice as a way to get students to engage with one another but since there was evidence of the GTAs understanding this practice as a disciplinary tool, I believe this could have been interpreted as something that would support such an understanding. So, without a good understanding of how the GTAs are making sense of various teaching practices, we may inadvertently encourage a belief that we ourselves may not believe.

## Conclusion

Analysis for this study is currently on going but the preliminary results are proving to show some interesting conclusions. It is my belief that with the results from this study, we will have a better understanding of the ways in which graduate teaching assistants make sense of various teaching practices and therefore will be able to better support them in the future. Without taking into account the understandings and interpretations of the graduate students we are working to help, we may inadvertently enforce beliefs we do not hold our selves. With this information, the field can begin to understand how GTAs change their practice over time and improve the professional development offered to graduate students who are new to the practice of teaching.

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Integrals, Volumes, and Visualizations

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*Many studies have been done on student understanding of integration and this research aims to add to that knowledge base with the study of student understanding of integration when applied to volume problems and how visualizations and sketches are used in the problem-solving process. Participants were recruited from a large, public, research university and interviews consisted of students working through routine and novel volume problems while discussing their thought processes aloud. Preliminary results show that students rely heavily on memorized formulas and have difficulties explaining the concepts behind the formulas. The idea of the integral as a sum of small pieces is present in most students studied, but they have trouble relating this idea to the formulas in their volume integrals. All students drew sketches of the geometric situation for all the problems, but the extent to which they could use their sketch meaningfully varied greatly.*

*Keywords:* Calculus, definite integral, volume, visualization, Riemann sum

## Introduction and Literature Review

After an introduction to the concept of the definite integral, some of the first applications that students encounter are volume problems. Volume problems found in second-semester calculus classes involve a combination of visualization, geometry, and integration skills. Previous studies have found that when solving definite integral application problems, students often rely on formulas, patterns, and previously encountered methods for setting up integrals (Yeatts & Hundhausen, 1992; Grundmeier, Hansen, & Sousa, 2006; Huang, 2010). In one of the first studies on student understanding of integration, Orton (1983) found that students had very little idea of the dissecting, summing, and limiting processes involved in integration when solving area and volume problems. Several authors (Sealey, 2006, 2014; Jones, 2013, 2015a, 2015b; Meredith & Marrongelle, 2008) have found that students are most successful when they are able to conceptualize the definite integral as the limit of a sum of products. Moreover, Sealey's (2006, 2014) work shows that students may have an idea of the underlying structure of the definite integral, but may not fully understand the layers that comprise the whole. In particular, students can easily conceptualize the summation layer but have the most trouble when working in the product layer of the Riemann sum structure.

One key component of a calculus volume problem that students can use as an aid is a visualization of the situation, generally in the form of a sketch made by the student. Stylianou and Silver (2004) found that, even though the construction of a diagram or picture is helpful, it is the quality of the picture that is most important. Bremigan (2005) had similar results, finding that although diagram production was related to correctly solving the problem, the presence of a constructed or modified diagram was not a sufficient condition for problem-solving success. In their study on expert and novice visualization practices, Stylianou and Silver (2004) observed novices' cognitive disconnect between visualizations and the problem-solving process. They state that, "although novices appear to have aspects of the declarative knowledge associated with visual representation use, they lack the necessary procedural knowledge that would allow them to use visual representations functionally and efficiently" (p. 380).

## Research Aim

As volume problems are one of the first applications of the definite integral that students encounter, the aim of this study is to further explore how students view and use the underlying structure of the definite integral when solving these types of problems. We are also interested in how students use their sketches of the geometric situation to aid in solving volume problems.

## Conceptual Framework

Sealey's (2014) Riemann Integral Framework was used to inform both the data collection and analysis of student understanding of the structure of the definite integral. This framework breaks the constituent parts of the Riemann integral down into pieces – product, summation, limit, and function – and it allows us to pinpoint the parts of the underlying structure of the definite integral that students have the most trouble with when solving volume problems. For the visualization aspect, we will be using Zazkis, Dubinsky, and Dautermann's (1996) Visualization/Analysis Framework to analyze student use of pictures and diagrams in the volume problem-solving process. In this model, there is a first visualization, $V_1$ (for example, a sketch of a 2-dimensional region), which is then acted on by an analysis event, $A_1$. In the following act of visualization, $V_2$, the student is still attending to the same picture used in $V_1$, but its nature has changed (due to $A_1$) and could lead to a reinterpretation of the picture or a new image construction. No matter the form the visualization takes in this step, $V_2$ results in a richer understanding of the original situation. The process goes on like this, from visualization to analysis back to visualization, optimally resulting in a more complete understanding of the physical situation.

## Research Methodology

Interviews with students were conducted during summer 2016 (Study 1) and summer 2017 (Study 2). The participants were recruited from summer classes at a large, public, research university. In summer 2016, the participants were four Calculus 2 students (all male) and three Elementary Differential Equations students (one female and two male). In summer 2017, the participants were two Calculus 2 students (one male and one female). The interviews were one-on-one and videotaped, and the students were asked to write their math work on paper or a white board and discuss their thoughts aloud.

During the interviews, the students were asked to complete three second-semester integral volume problems. In Study 1, the problems were three routine solid of revolution problems (e.g., "Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = 3x$ about the line $x = -1$") and the students drew their sketches and wrote their math work on the same paper. In Study 2, the problems were two routine solid of revolution problems and one geometric solid problem, which we will call the pyramid problem ("Find the volume of the pyramid whose base is a square with side length $L$ and whose height is $h$."). In order to more clearly observe when students were referring to their drawings during the problem-solving process, the method was adjusted in Study 2 as follows. One student in Study 1 drew sketches and performed math work on separate sheets of paper; the second student sketched drawings on a white board and wrote math work on a sheet of paper.

During the interviews for both studies, students were probed about their responses and were asked to explain their work and thought processes. Some typical questions asked during the interview process were: "How do you know this integral gives you a volume?", "How does that particular statement give you the volume of a cylinder/washer/etc?", "What does the *dx* mean?", and "Can you show on your picture the different parts of the volume integral?"

The video data was transcribed and analysis is in the beginning stages. We use thematic analysis (Braun & Clark, 2006) to identify themes and patterns in the data. In particular, we have begun by employing theoretical thematic analysis, which is "driven by the researcher's theoretical or analytic interest in the area, and is thus more explicitly analyst-driven" (p. 84). We feel that this will be an appropriate method for this study, since we have pre-determined codes that we will be looking for in student responses (e.g., working in a specific layer for Sealey's framework or being in one of the visualization/analysis phases of Zazkis' framework).

## Preliminary Results

The students in these studies exhibited a strong attachment to memorized volume integral formulas when solving the routine solid of revolution problems.

*Interviewer*: Do you understand where that [*their volume integral*] came from?
*Student 2*: I treat that just as a formula. Physics is the class where I think about and understand, you know, but, it just could be because they throw a lot of numbers at you fast.
*Student 3*: I know the formula, but sometimes I don't know where to apply them.

Using the formulas is not a detrimental method, but we would prefer that students are also able to unpack the underlying definite integral structure when asked to do so. Few students in this study were able to accurately and consistently discuss the details of their memorized formulas or how they produced a volume measurement.

Another observed occurrence was students linking the line of rotation to the variable of integration without being able to produce meaningful explanations.

*Student 1*: Since the line I'm rotating about is parallel to the $y$-axis, if I use cylindrical shells method, I need to integrate $x$.
*Student 2*: So, if, like it's [*the line of rotation*] parallel to the $y$-axis, I'll integrate $y$.

There were only two students who were given the pyramid problem, but their methods for attacking the problem were very different and highlighted some problems of relying on memorized formulas and mimicking methods seen in class. Student 5 was very successful with the routine solid of revolution problems by relying heavily on the "volume formulas" and was able to produce accurate volume integrals that would receive high marks on an exam, even though her explanation of the details was shaky. When confronted with the pyramid problem, Student 5 continued to try to use a memorized volume formula but with poor results. Student 5 had to be heavily guided in the pyramid problem due to over-reliance on memorized formulas that did not fit with this problem. Even though Student 5 did not succeed in solving the problem completely, there was the presence of "cutting into small pieces and adding them up" in her thought process that seemed to be accessible but not heavily used. Student 5 also had a very hard time visualizing the situation and had to be heavily guided into a 2-dimensional side-view of the pyramid so that she could reduce the cognitive load of the 3-dimensional solid situation.

Student 6 was less successful with the solid of revolution problems; he relied heavily on the memorized formulas but was unsure of how to use them effectively. When probed using the questions stated above, he was very unsure and stated this fact many times. When confronted with the pyramid problem, he had trouble visualizing the situation at first, but then transformed the 3-dimensional solid into a 2-dimensional side view (on his own) and was able to make significant progress. Once the weight of "you must use this formula" was lifted, Student 6 was able to make some strong connections between what he knew of "adding up small pieces" and this novel volume problem. This problem forced Student 6 to give up on his attachment to memorized volume formulas and rely solely on the concept of the definite integral as the sum of smaller pieces.

The students in this study had many misunderstandings but we want to emphasize more what they could do than what they could not do. When faced with volume integral application problems, almost all of the students exhibited some understanding of the dissecting, summing, and limiting structure of the definite integral; they just had trouble applying it to the problem.

> *Student 1*: In that case you're using the, um, areas of [*stacks hands horizontally on top of each other*] … circles. Um, so you're making a series of washers.
>
> *Student 1*: So if you were to take one of those, a slice of the inside of the cylinder, it would be like this sheet with a depth. But when we integrate, we're basically taking that depth to zero. The limit of that depth. Right?
>
> *Student 2*: What I'm looking for then is my radii of my, you know, infinitely concentric circles going on here.
>
> *Student 4*: Basically it's going to be a bunch of different-sized cylinders stacked upon each other.

Students' drawings varied in sophistication and accuracy. All students drew a sketch of the geometric situation for all the problems, but the extent to which they could use their sketch meaningfully varied greatly. The most glaring disconnects came when students produced a correct (or mostly correct) volume integral, but could not relate their integral back to the physical situation of the sketch. One student in particular – Student 2 – was able to produce very detailed sketches of the sum-of-pieces structure (Figure 1), but his description of the underlying mathematics was full of inaccuracies and nonsense. According to Sealey's (2014) framework, this student is appropriately attending to summation layer, but he had great difficulties in the product layer.

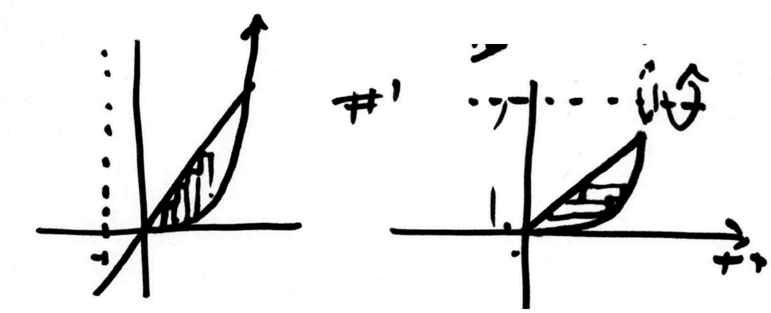


*Figure 1: Drawings produced by Student 2 when describing his volume integrals.*

### Teaching Implications/Future Research

The preliminary results of this study imply that students' heavy reliance on memorized formulas and mimicking of methods observed in class can lead to misunderstandings and brick walls when students are faced with more complicated or non-routine integral application problems. It was also observed that producing an accurate definite integral to a solid-by-revolution problem does not necessarily imply that the student understands integration and how the integral produces a volume. Solid by revolution problems are given a lot of face time in most calculus books, but they can inadvertently tell an incomplete story of how integrals can be used to find volumes in general and how they can be used in other application situations. We believe that more time should be spent on non-revolution and non-routine volume problems, so that students are required to exercise their definite integral muscles and not be tempted to fall into the trap of relying solely on memorized formulas. From this study, it is clear that the idea of “dissecting and summing” is present in many students, and we need to find ways to employ and enrich it.

We plan to continue analyzing our data set, as well as conduct more interviews with the aim of examining student visualization and picture-use when solving volume problems, like the interviews conducted during summer 2017 involving the pyramid problem. In particular, we would like to investigate how they use their sketches to build the pieces of the corresponding volume integral and how they interact with their drawing in the problem-solving process. Furthermore, we would like to develop ways in which students can more meaningfully engage in constructing and understanding the product layer. In the future, we would like to study other aspects of visualization that students use when solving volume problems, like gesture.

## Questions for Audience

1. Aside from eye-tracking software, are there more sophisticated ways to capture when students are going back and forth between picture and math work on camera?
2. How can we determine if separating the drawing space from the math work space bring about any unintended consequences for student problem-solving?
3. Are there any studies on visualization that we have missed that could help us out with this work?

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Examining Students' Problem Posing Through a Creativity Framework

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*Understanding how students pose problems can inform the development of posing activities to further enhance students' understanding of mathematics. Analyzing students' problem posing through the lens of mathematical creativity provides insight into the creative process of posing problems; namely, the cognitive tools students use to formulate questions. Three undergraduate students, enrolled in a developmental mathematics course, participated in a problem-posing intervention to examine the cognitive resources students used as the foundation for their mathematical problem posing. Session transcripts were analyzed using an analytical framework derived from an investment perspective on creativity, and identified resources were organized into two categories: mathematical knowledge and skills, and social interactions and experiences. Preliminary findings from the fifth session suggest that students associated the mathematical content of a posing task with previously encountered problems, as well as appealed to their familiarity with the situational context of the posing task.*

*Keywords:* Problem posing, Undergraduate Students, Developmental Math, Math Creativity

Problem posing is considered as the creation of mathematical problems, often from a given set of information or from previously presented problems (Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman, 2005; English, 1997; Silver, 1994). Problem posing is a naturally-occurring activity in which individuals engage during their daily interactions. Kilpatrick (1987) notes that people encounter and recognize problems frequently, proceeding to solve those problems as they arise. Problem posing further occurs as part of the problem-solving process, acting as the foundation for developing a solution strategy (Brown & Walter, 2005) or as a form of reflection on and verification of solution strategies (Carlson & Bloom, 2005). In this capacity, individuals engage in problem posing to gain an improved understanding of the problem scenario in front of them. Understanding how individuals pose mathematical problems can inform the development of problem-posing activities to further enhance students' understanding of mathematics.

One way to examine individuals' problem posing is through the lens of mathematical creativity. Silver (1997) describes mathematical creativity as "closely related to deep flexible knowledge in content domains" (p. 75), viewing the connection between problem posing and problem solving as venue for mathematical creativity. Silver notes, "It is in the interplay of formulating, attempting to solve, reformulating, and eventually solving a problem that one sees creative activity" (p. 76). As students closely examine a mathematical situation, they can begin to generate hypotheses about the situation, develop flexibility in the ways in which they think about the situations, and begin to develop new ideas that expand upon their understanding of mathematics. In other words, students have an opportunity to use and develop the cognitive tools they have cultivated for doing mathematics. Viewing problem posing as an act of mathematical creativity, the descriptors of the creative process can be used to illustrate the creation of mathematical problems.

In this proposal, a framework based on the investment theory of creativity (Sternberg & Lubart, 1996) is used to describe undergraduate developmental mathematics students' problem posing. Under investment theory, individuals use a confluence of cognitive resources, such as content knowledge, thinking styles, and environmental influences, to "invest" in their ideas and develop them over time. These resources are tools individuals use as the foundation of their creative process. The guiding question to this inquiry is, "*How do students use cognitive resources to pose mathematical problems?*" Undergraduate students in developmental mathematics courses are an interesting population to observe; knowledge of mathematics is a mediating factor while posing problems (Kontorovich, Koichu, Leikin, & Berman, 2012; E. A. Silver, Mamona-Downs, Leung, & Kenney, 1996), and these students have been identified as underprepared for the expectations of college mathematics courses. As knowledge is a resource for creativity, one wonders to what extent these students use their knowledge as a resource for posing problems.

## Methods

### Participants & Study Design

Three undergraduate students enrolled in a developmental mathematics course at a mid-Atlantic public university participated in a five-week problem-posing intervention during the spring semester of 2016. The purpose of the intervention was to examine how students' problem posing evolved after learning about two problem-posing strategies described by Polya (2009): accepting the given, and "what-if-not". When accepting the given, students posed problems using only the numerical information and situational context provided in a posing task. When using "what-if-not", students were asked to pose problems by either changing the information they were given or adding new information to the scenario. The intent for the instruction on the two posing strategies was to encourage students to reflect on the given information in each task, using their understanding of the scenario as a resource for creating math problems. This proposal will focus on students' resource use during the final session of the intervention, to illustrate the variety of resources the students' used.

The final session of the intervention consisted of one posing task called "Payment Plan", shown in Figure 1. In this task, students were presented two payment options: one option where the payment increased by $1,000 each day, and a second option where the payment doubled each day. The posing scenario presents an opportunity for students to examine the two rates of growth and make comparisons between the two options. Students worked together to pose ten problems for the task as a group and were not required to use any specific posing strategy when creating their problems. After posing the ten problems, the primary investigator asked the students to describe their thinking behind the problems they posed. Students were not asked to solve the problems they posed. At the end of the session, students were asked to reflect on their experience and discuss their thoughts about posing problems. Recordings of conversations with the students were transcribed, and students written work was collected.

**Payment Plan**

You are given the choice to be paid in one of the following two ways:

1. You will be paid $1,000 the first day, $2,000 the second day, $3,000 the third day, $4,000 the fourth day, and so on for one month.
2. You will be paid $0.01 the first day, $0.02 the second day, $0.04 the third day, $0.08 the fourth day, and so on for one month.

(1) Work with your partner(s) to write ten mathematical word problems.

*Figure 1. Prompt for the Payment Plan task.*

## Data Analysis

The transcript of the session was partitioned into three episodes based on the session activity, and each episode was partitioned into several smaller events based upon what students were doing within the activity. Events were established around the topic of conversation, typically a student's explanation of a response or continued discussion around an idea. As a result, events were varied in length so that a more complete picture of each event could be achieved (Schoenfeld, 1985). Across the three episodes, there was a total of thirty-one events in the transcript. Using the categories of resources outlined by Sternberg and Lubart (1992) as a basis, students' actions, concepts mentioned by the students, and students' experiences were grouped into three types of resource types: task resources; mathematical knowledge and skills; and social interactions and experiences.

This proposal focuses on mathematical knowledge and skills, and social interactions and experiences. *Mathematical knowledge and skills* refers to students' mathematical thinking during the posing activities. This category relates to what students know about mathematics, such as students' understanding of concepts, association with previously encountered problems, use of mathematical terminology, as well as problem posing strategies. The design of the intervention focused on the use of the accepting the given and what-if-not posing strategies; therefore, it was expected that the students would use these strategies. *Social interactions and experiences* refer to students' personal experiences, non-mathematical knowledge, and interactions with other individuals. This resource category primarily relates to students' use of the situational context provided in a posing task, but also includes interactions between the students during the session, such as building from other students' thinking or seeking verification from other students.

# Students' Resources for Posing

Students exhibited use of both their mathematical knowledge and skills, and social interactions and experiences as resources for creating mathematical problems. To illustrate how students used these resources, an example of a resource type under each category will now be discussed. Under the mathematical knowledge and skills category, students associated the posing task with problems they previously encountered. Under social interactions and experiences, students related the situational context of the task to their personal experiences.

## Mathematical Knowledge – Problem Association

To engage with the mathematical content in the posing tasks, the students would relate the posing task to types problems they had previously encountered in the past. Brianna associated the Payment Plan task to comparison problems, posing the problem, "How long will you have to work for the second plan to equal the first plan?" She noted, "We've done problems like this before, where you have two rates of growth, and you compare them. So I was just curious at what point would they intersect?" Brianna recalled that with previous comparison problems, she would often be asked to identify the moment that two mathematical relationships would have the same value.

Students would also recall specific examples of problems they had encountered. Jason posed the problem, "Which would make you more money, a minimum wage job, or the second option?" (Jason later clarified that minimum wage stood for $7.50 an hour with an eight-hour work day.) When asked what motivated him to pose this problem, Jason recalled a previous experience with a teacher he had in high school:

> I looked at the problem, and I remembered that I had seen a similar math problem posed by one of my old teachers, as a…example to show what exponential growth was. Would you rather have a minimum wage job, or the one that starts off paying one cent and then doubles every day? I made it [the second option] because it's pretty obvious that the minimum wage is never going to beat the [first] job in terms of pay.

In recalling this past experience, Jason focused on the exponential relationship presented in the second option, explicitly naming the pattern as an example of exponential growth. It was a combination of his recognition of the exponential relationship and the similarity of the posing task to his prior experience that led him to associate the task with the previously encountered example.

**Social Experience – Familiarity with Context**

Appealing to the situational context of the posing task, the students framed their posing around their familiarity with the context. Brianna interpreted the two options as two jobs offering different pay. Brianna posed the problem, "If there was a 5% tax that was taken every day, how much would you have at the end of the month?", relating the scenario to a recent experience: "I was thinking of real life. Over spring break, I just worked, so that's what I was thinking about. A percentage is taken out every time. It's just a real example." Brianna took her recent employment experience as a resource for the problem she posed, introducing a type of income tax to the payment options.

Students could be familiar with situational context yet not have personal experience with that context. Kelsey had posed the problem, "If you take a sick day on the seventh day, what will your pay be at the end of the month for the first job?" Trying to describe how she created the problem, Kelsey questioned whether her response was realistic:

> *Kelsey:* Well, I just did a real-life problem… I just thought of another problem that could come from this. I don't know if the pattern would just continue and you would just lose that $7,000, or is it going to continue? I don't know. Do you get paid during the sick day?
>
> *Interviewer*: Possibly. I guess it would depend upon the job. What do you think?
>
> *Brianna*: It depends upon the job.
>
> *Jason*: I think that for this problem, just the way it's worded, you're supposed to imply that you aren't paid for it.
>
> *Kelsey*: That's what I thought, but I didn't know if you would get paid or not, because I only volunteer. I don't work. When I was thinking about this [problem], I didn't think of the fact that you might not get paid.

Kelsey began to pose a problem by introducing the idea of sick days and how payment would be impacted by a sick day. Kelsey ultimately revealed that she had not had employment experience, which led her to be uncertain about how her problem would fit within the situational context. Although she did not have employment experience, she was familiar enough with the situational context to pose a problem with a context related to employment.

## Concluding Remarks

The preliminary findings suggest that students can use both their knowledge of mathematics and their social experiences as foundations for engaging in mathematical activity. Students could use this foundation to spring into further discussion around mathematical ideas. This could be especially valuable for students in developmental math courses, as it provides another venue for students to gain access to mathematical activities. A notable limitation of this study is that students were not asked to solve the problems they posed. Because students were only asked to pose problems, it is difficult to determine the extent to which students' resource use shaped their mathematical thinking while posing problems. Due to this limitation, feedback sought from the audience will focus on identifying the depth of students' engagement with the mathematical content while posing problems in the absence of students working towards a solution.

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# The Instructor's Role in Promoting Student Argumentation in an Inquiry-Oriented Classroom

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Abstract

*Four class sessions in inquiry-oriented differential equations were analyzed to understand the role of the instructor in supporting student argumentation. Three coding schemes were developed to identify arguments, characterize instructor utterances, and connect instructor talk to argumentation goals in inquiry-oriented instruction. Results show that students generated the majority of arguments tendered in the four class sessions. The instructor used questions to generate student arguments more than other types of instructional utterances (e.g., revoicing, telling). Nearly half of the instructor's utterances were aligned with argumentation goals. More detailed examples of student-generated arguments in the class sessions are being constructed to illustrate the flow and function of different goal alignment routines to understand what it is that the instructor did during class to promote student argumentation.*

*Keywords*: Inquiry, Teaching, Active learning, Case study

Mounting research evidence points to the benefits of active learning in improving student outcomes in undergraduate STEM courses compared to more traditionally taught courses (e.g., Freeman et al., 2014; Kogan & Laursen 2014; Larsen, Johnson, & Bartlo, 2013; Rasmussen & Kwon, 2007). Recently, professional societies explicitly recognize the need for faculty to increase their use of active learning to improve student success (Saxe & Braddy, 2015). But to what extent do mathematics departments across the country value active learning and actually use various active learning strategies? In a recently completed census survey of all mathematics departments that offer a graduate degree in mathematics, researchers found that 44% of departments report that active learning is very important in their Precalculus through Calculus 2 courses but only 15% say their program is very successful in implementing active learning (Apkarian & Kirin, 2017). Moreover, recent comprehensive literature reviews (Larsen, Marrongelle, Bressoud, & Graham, 2017; Rasmussen & Wawro, 2017) reveal that very few studies provide detailed analyses of what instructors actually do to create and sustain active learning classrooms. All of this points to the need for in depth case studies of how instructors successfully implement active learning in undergraduate mathematics classes. The research reported here begins to address this pressing, national need.

The analysis presented here focuses on a specific active learning classroom, in particular an inquiry-oriented differential equations class. We define inquiry in terms of three principles: student deep engagement in mathematics, peer to peer interaction, and instructor interest in and use of student thinking (Rasmussen, Marrongelle, Kwon, & Hodge, in press; Rasmussen & Wawro, 2017). This definition of inquiry follows from over a decade of work in creating and investigating active learning classrooms in undergraduate classrooms (Rasmussen & Kwon, 2007; Rasmussen & Marrongelle, 2006) and parallel framing of the inquiry based learning movement (Laursen, Hassi, Kogan, Hunter, & Weston, 2015). At the intersection of deep engagement in mathematics and peer to peer interaction is argumentation, where argumentation refers to classroom discussion featuring significant mathematics, conjectures, reasoning to support conjectures, and students making sense of others' reasoning. In keeping with this focus, we chose for our analysis four days in an inquiry-oriented differential equations class because on these four days students made considerable progress on debating what graphs of solutions to a system of two linear homogeneous differential equations in the phase plane look like and on justifying their conclusions. In other words, argumentation was a distinguishing feature of this class and hence provides an opportunity to unpack the role of the instructor in initiating and supporting student debate. In particular, we address the following research questions:

1. To what extent did the instructor and students contribute to arguments and what was the nature of their respective contributions?
2. What did the instructor do to promote student argumentation?

## Background

The role of student argumentation in mathematics classrooms has a long history in mathematics education reform, both at the K-12 and post-secondary levels. For example, in the late 1980's Cobb and colleagues investigated how classroom argumentation supported student learning and intellectual autonomy in elementary school classrooms. They argued that classroom argumentation provides "opportunities for children to articulate and reflect on their own and others' mathematical activities" (Cobb, Yackel, & Wood, 1989, p. 126). These researchers also examined the role of the teacher in supporting student argumentation, leading in part to the articulation of social and sociomathematical norms (Yackel & Cobb, 1996) and how teachers initiate and sustain productive discursive norms that support argumentation (Wood, Cobb, & Yackel, 1990). This work was later extended to the university setting with further articulation of the role of the instructor (Yackel, Rasmussen, & King, 2000; Rasmussen, Yackel, & King, 2003). Part of this extensive literature has also included the elaboration and extension of particular social norms for argumentation in terms of four broader goals for inquiry-oriented instruction and specific teacher prompts that can function to realize these goals. The four argumentation goals are: (1) getting students to share their thinking, (2) helping students to orient to and engage in others' thinking, (3) helping students deepen their thinking, and (4) building on and extending student ideas (Rasmussen, 2015; Rasmussen, Marrongelle, Kwon, & Hodge, in press).

At the university level, research is just beginning to provide in depth portraits of what inquiry-oriented instruction actually looks like and what instructors do on a daily basis to promote argumentation. For example, one of the studies that examined effective instructional practices in differential equations focused on the instructor-student interaction patterns that facilitated students' reinvention of a bifurcation diagram (Rasmussen, Zandieh, & Wawro, 2009). These researchers identified three instructor "brokering" moves that forged connections between the different small groups, the classroom community as a whole, and conventional terminology and notations of the broader mathematical community. This work resonates with the fourth argumentation goal, building on and extending student ideas. In other work examining instructor-student interactions that contributed to significant student progress in creating, interpreting, and using phase portraits, Kwon et al. (2008) identified and illustrated four functions of instructor revoicing (O'Connor & Michaels, 1993). In this analysis, revoicing was shown to function in multiple ways in support of argumentation - as a binder of ideas among students, as a springboard for new ideas, for ownership of ideas, and as a means for socialization into the discipline of mathematics. These functions resonate with argumentation goals two and three.

In inquiry-oriented classrooms instructors need to decide when and how to insert information, formalize students' informal ideas, and make connections to related mathematics in the midst of students exploring ideas and doing mathematics. Doing so requires a blend of mathematical expertise, pedagogical knowledge, and pedagogical content knowledge. In a case study of two mathematicians implementing an inquiry-oriented differential equations curriculum, Rasmussen and Marrongelle (2006) identified two different ways that inquiry-oriented instructors connected to student thinking while moving the mathematical agenda forward – transformational records and generative alternatives. Transformational records are defined as notations, diagrams, or other graphical representations that are initially used to record student thinking and that are later used by students to solve new problems. Generative alternatives are defined to be alternate symbolic expressions or graphical representations that a teacher uses to foster particular social norms for explanation and that generate student justifications for the validity of these alternatives. Johnson (2013) illustrates other ways that instructors can build on and extend student thinking. In particular, Johnson identified a variety of ways in which two abstract algebra instructors interpreted student ideas, analyzed and evaluated these ideas, and made connections between students' ideas and conventional mathematics.

In more recent work, Kuster, Johnson, Keene, and Andrews-Larson (in press) specify four components of inquiry-oriented instruction: generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation. Each of these components, which connects well with the four goals for argumentation, are further refined by specifying practices that support each component. These components and practices are culled from the K-16 literature and their own work supporting and studying inquiry-oriented teaching in abstract algebra, linear algebra, and differential equations. As this emerging body of research focused on the work of inquiry-oriented instruction suggests, the work of instructors in inquiry-oriented classrooms goes well beyond the typical teaching preparation that mathematicians receive. In depth case studies of such work, as one in this report, can offer useful practical and theoretical accounts of practice.

## Methodology

Data for this analysis comes from four class sessions in an inquiry oriented differential equations class. These sessions were part of an eight-week classroom teaching experiment. Data sources consisted of video recordings of whole class and small group discussions, researcher field notes, and copies of student work. We began the data analysis by making complete transcripts of all whole class discussions from the four classroom sessions.We engaged in three passes of coding. In the first pass, we conducted a Toulmin analysis of all whole class discussion. In the second and third passes, all instructor utterances were coded for the nature of the instructor utterance. The second pass focused on the type of utterance (referred to as talk move) and the third pass focused on the alignment of the utterance with the four argumentation goals. Details on each of these passes follows. We are currently coordinating these three passes to develop a detailed, empirically grounded portrait of how this instructor promoted student argumentation. This preliminary report therefore focuses on results related to the first research question.

**Toulmin coding**: In his seminal work, Toulmin (1969) created a model to describe the structure and function of an argument. The core of an argument consists of three parts: data, claim, and warrant. In any argumentation, the speaker makes a claim and presents evidence or data to support that claim. Typically, the data consist of facts that lead to the conclusion that is made. In order to clarify what the data has to do with the conclusion, a person might also present a warrant that serves as a kind of bridge between the data and the conclusion. Often, warrants remain implied by the speaker and are elaborations that connect or show the implications of the data to the conclusion (Rasmussen & Stephan, 2008). Backings, when offered, provide legitimacy for the core of the argument (that is, the data-claim-warrant).

Videotaped data of the four class sessions and transcripts were reviewed by a group of eight mathematics education researchers and initial arguments identified according to Toulmin's model of argumentation. In particular, elements of claim, data, and warrant needed to be identified and present to be considered an argument. Identification of the arguments was done for a portion of the video data as a whole group. The remainder of the video data was coded in smaller groups. The groups would reconvene as a whole in order to review problematic data or interesting episodes. The collaborative coding process enabled shared interpretations of the codes and decreased instances of interpretations not grounded in the video data. Authorship of an argument was determined by who offered the justification (i.e., data, warrant or backing). Authorship could be attributed to the instructor, a student, or jointly constructed by the instructor and students.

**Talk move coding:** A coding scheme was then developed as we observed video and simultaneously highlighted the teacher's discourse in the transcripts. We broke what the instructor said into identifiable utterances that served a different function. An utterance is not a conventional unit, like the sentence, but a unit nonetheless in the sense that it is marked out in the boundaries of speech (Bakhtin, 1986). We refined and revised our coding scheme of teacher utterances based on review of the literature (e.g., Forman et al, 1998; Krussel, Edwards, & Springer, 2004; Lobato et al, 2005; Mehan, 1979) and multiple passes through our data. The final coding scheme consisted of four main codes: revoicing, questioning/requesting, telling, and managing. Each of these main codes had four subcodes. Full descriptions of the codes can be found in (Rasmussen, Marrongelle, & Kwon, 2009). We used problematic or especially interesting episodes to sharpen and refine the coding scheme and a collaborative, iterative process to share and defend interpretations of the video and corresponding transcripts. In addition, we explained our coding scheme to a mathematics education graduate student who then independently coded all transcripts, resulting in over 80% agreement.

**Goal alignment coding.** Recall that the four argumentation goals for inquiry-oriented instruction are: (1) getting students to share their thinking, (2) helping students to orient to and engage in others' thinking, (3) helping students deepen their thinking, and (4) building on and extending student ideas. In this pass through the data, we aligned the instructor's discourse with these four goals for inquiry-oriented instruction. We did not attempt to make judgements about the thinking or rationale of the instructor; rather, we attempted to align the instructor's speech with how the speech functioned in furthering argumentation in the classroom. Three mathematics education researchers coded the data independently, and discussed differences to reach 100% coding agreement.

## Sample Results

We begin with a top level view of who was responsible for arguments and the nature of the instructor's contributions. As shown in Table 2, the students (Student) gave 35 of the 52 of the arguments (67%), with another 7 of the 52 arguments (13%) being co-constructed between students and the instructor (S&I). Clearly the instructor was not the primary source of justifications for argumentations tendered.

| Day | Student | Instructor | S&I | Total |
|---|---|---|---|---|
| 4/18 | 6 | 3 | 0 | 9 |
| 4/20 | 5 | 3 | 4 | 12 |
| 4/22 | 2 | 2 | 0 | 4 |
| 4/25 | 22 | 2 | 3 | 27 |
| **Total** | **35** | **10** | **7** | **52** |

*Table 2*. Number of arguments given in whole class discussions

To gain insight into what else he was doing we also examined the instructor's utterances for frequency at which he used revoicing (R), telling (T), Questioning/requesting (Q), and Managing (M). As shown in Table 3, the instructor used questioning more than the other types of utterances (approximately 37% of his utterances) over the four days. The other utterances types – revoicing, telling, and managing – were about equally represented (approximately 20% each). The full analysis will look more closely at the role these utterances in the production of arguments, but preliminary analysis points to the key role of questioning/requesting and revoicing (which comprise nearly 60% of the utterances.

| Day | R | T | Q | M | Total |
|---|---|---|---|---|---|
| 4/18 | 16 (25.4) | 12 (19) | 23 (36.5) | 12 (19) | 63 |
| 4/20 | 8 (25) | 8 (25) | 8 (25) | 8 (25) | 32 |
| 4/22 | 6 (20) | 8 (26.7) | 12 (40) | 4 (13.3) | 30 |
| 4/25 | 20 (20) | 20 (20) | 44 (44) | 16 (16) | 100 |
| **Total** | **50 (22.2)** | **48 (21.3)** | **84 (37.3)** | **40 (17.8)** | **225** |

*Table 3*: Total number of instructor utterances per day

Further insight into the function of the instructor utterances is revealed by examining how each utterance relates to the four argumentation goals. Recall that the four argumentation goals are: (1) getting students to share their thinking, (2) helping students to orient to and engage in others' thinking, (3) helping students deepen their thinking, and (4) building on and extending student ideas. Table 4 shows the frequency of utterances that aligned with each of the four goals, as well as the number of utterances that were not aligned with any of the four argumentation goals.

| Day | G1 | G2 | G3 | G4 | No G | Total | G/Tot |
|---|---|---|---|---|---|---|---|
| 4/18 | 13 | 6 | 7 | 7 | 33 | 66 | 50% |
| 4/20 | 1 | 5 | 2 | 4 | 17 | 29 | 41% |
| 4/22 | 4 | 1 | 3 | 1 | 20 | 29 | 31% |
| 4/25 | 21 | 12 | 7 | 8 | 56 | 104 | 46% |
| **Total** | **39** | **24** | **19** | **20** | **126** | **228** | **45%** |

Table 4: Alignment of instructor utterances with argumentation goals

There were a total of 102 total utterances that were argumentation goal-aligned. Approximately 38% of these were aligned with goal 1, getting students to share their thinking. This makes sense because this is the first step for students to explicate their reasoning. The percentage of goals 2-4 were fairly equally distributed, ranging from 19% to 23.5%.

By providing detailed examples of student-generated arguments from initiation to conclusion, we will illustrate the flow and function of different goal alignment routines to understand what it is that the instructor did during class to promote student argumentation, moving beyond simply coding for questions or revoicing. In one such detailed example we will discuss how the teacher, making use of a generative alternative (Rasmussen & Marrongelle, 2008), supports three student arguments about solutions to a system of differential equations. This rich example provides a prototype for how instructors can enact this particular global strategy for promoting student argumentation. We anticipate identifying other prototypes in the full report.

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# Teachers' Use of Informal Conceptions of Variability to Make Sense of Representativeness of Samples

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*A key factor in statistical thinking is reasoning about variability. This paper contains data on how in-service middle school teachers and a community college faculty member reasoned through two statistical tasks. The researcher presents his analysis of the data through the lens of how teachers reasoned about variability as they worked through the two statistical tasks.*

*Keywords*: Statistics, Professional Development, Teacher Education

As data and statistical thinking have become more important in the information age, middle school mathematics teachers have been tasked with placing greater emphasis on statistical concepts than may have been previously required before the introduction of the Common Core State Standards for Mathematics [CCSSM] (National Governors Association, 2010; Tran, Reys, Teuscher, Dingman, & Kasmer, 2016). To reason about and teach statistical concepts in a productive manner, it is critical for teachers to attend to the notion of variability in their personal reasoning and in teaching statistics (Franklin et al., 2015; Garfield & Ben-Zvi, 2005; Moore, 1990; Shaughnessy, Watson, Moritz, & Reading, 1999; Wild & Pfannkuch, 1999). Specifically, Garfield and Ben-Zvi present a framework from which teachers can develop tasks to support and assess strong conceptions of variability in their students. Developing intuitive ideas of variability, using variability to make comparisons, and using variability to predict random samples or outcomes are key ideas in this framework.

In this paper, we compare the following: (1) how middle school math teachers used intuitive notions of variability to make comparisons between sets of outcomes of random processes for two statistical tasks and; (2) how a community college instructor answered the same tasks using more formal notions of variability. We attempt to answer the following question: *How do teachers' informal ways of reasoning using variability compare to an expert's more formal ways of reasoning about variability while working through statistical tasks?*

## Methods

The data collected for this study were gathered in a large-scale professional development and research program, focusing on middle school teachers in a Southwestern state in the United States. Each teacher in the program was asked to participate in professional development activities for two years. The project focused on increasing teachers' mathematical and pedagogical content knowledge.

In the second year of the project, participants were involved in nine full-day workshops focused on both functions and statistics content. Upon the teachers' completion of their second year in the program, the researcher conducted four videotaped, individual, task-based, clinical interviews (Clement, 2000). The subjects for this study were three middle school teachers (Joy[1], Leia, and Nina) and one community college faculty member (Kory) responsible for co-leading the statistics content for the workshops.

[1] All subject names are pseudonyms.

Prior to their second year on the project, the three teachers responded to a free-response survey question that asked them to describe their own personal background in statistics. Joy responded that she was "very limited" in statistics with her background being one college course in educational statistics, the material that she taught to her students, and the statistics-related material learned in her collaborative community of learners[2] (CCOL) facilitated by the other co-leader of the statistics content at the workshop. Leia described her background as limited in the variety of statistics that she used and uncomfortable in justifying her methods for doing statistics, but comfortable in analyzing data and displays. Leia also participated in the same CCOL as Joy. Nina described herself as having "very little background" in statistics other than working with spreadsheets. Kory taught statistics at the secondary and post-secondary levels for over 20 years.

The researcher analyzed each subject's raw and transcribed video data with careful attention to habits or inclinations that the subjects may have shown while reasoning through the tasks. The researcher then created themes based on these habits or inclinations before refining said themes through subsequent passes through the data using open coding principles developed by Strauss and Corbin (1998). Once the researcher felt these themes were sufficiently well-defined, his attention shifted to themes that related to how the subjects utilized concepts pertaining to variability to reason through the statistical tasks. The researcher then analyzed these variability-related themes for each subject individually before coordinating common themes across subjects.

**Task Description**

Participants engaged in two tasks: The Coin Flip Task[3] (Figure 1) and The Orange Bin Task (Figure 2). The Coin Flip Task was chosen to determine how participants would reason about the probabilities of two distinct events when the proportion of outcomes for each event was the same. From prior data collection efforts, the researcher suspected that the teachers would have limited mathematical background, thus making this The Coin Flip Task non-trivial. Thus, the teachers would need to reason about the similarities or differences between the two events in order to provide an answer they deemed to be reasonable.

Event A: A machine flips a fair coin 10 times with the outcome of 7 heads.
Event B: A machine flips a fair coin 1000 times with the outcome of 700 heads.

Which one of the following is true?

1) Event A and Event B are equally probable.
2) Event A is more probable than Event B.
3) Event A is less probable than Event B.
4) Unable to determine given the information.

*Figure 1: The Coin Flip Task*

The Orange Bin Task was chosen to determine if subjects would reason about the role of sample size in the variability of outcomes and use this reasoning to support the grocer's choice of whose method to choose. As an explicit verification of how subjects made a connection between sample sizes and the variability in the two sets of potential mean weights, the following question was asked to any teacher who did not give an example of two mean weight lists[4]: *Suppose that both*

[2] The project's version of a Professional Learning Community (PLC).
[3] Modified task from Schrage (1983) p. 353
[4] Each mean weight was determined by oranges randomly generated from a normal distribution with a mean of 131 grams and a standard deviation of 15 grams.

*of these people had repeated their method only five times. One person yielded the following five mean weights. 133 grams, 124 grams, 129 grams, 129 grams, 140 grams. The other person yielded the following five mean weights. 126 grams, 130 grams, 128 grams, 135 grams, 133 grams. Which set of mean weights belong to which person, and why?*

A grocer wanted to determine the typical weight for oranges in his store. One employee, Jeff, suggested finding the mean weight of 5 randomly-selected oranges, placing the oranges back into the bin, and repeating this process several times. Another employee, Krystal suggested finding the mean weight of 15 randomly-selected oranges, placing the oranges back into the bin, and repeating this process several times.

1) Generate what you believe could be the outcomes for Jeff's process.
2) Generate what you believe could be the outcomes for Krystal's process.
3) The grocer decided to go with Krystal's suggestion of using 15 randomly-selected oranges instead of Jeff's 5 randomly-selected oranges. Give specific reasons for why you believe the grocer decided to go with Krystal's suggestion?

*Figure 2: The Orange Bin Task*

## Preliminary Results and Discussion

### The Coin Flip Task

Kory read the problem and immediately determined that Event A was more probable than Event B due to the difference in the number of trials for each event. To support his reasoning, Kory drew upon his prior statistical knowledge and saw the context of the coin flip problem as a binomial situation. Kory calculated the expected value and standard deviation for each event to determine how far each outcome deviated from the expectation. Kory explained for Event A that "seven is a little bit more than a standard deviation away from the center, it's fairly likely to occur. It's not incredibly likely, but it's still within the realm of possibilities." However, Kory voiced the opposite stance when considering the probability of Event B: "If you think about where 500 and 700 is, that's over 10 standard deviations away, that more like 12 or 13 standard deviations….Yeah not gonna [sic] happen."

$$\mu = 10(.5) = 5$$
$$\sigma = \sqrt{10(.5)(.5)} = 1.58$$
$$\mu = 1000(.5) = 500$$
$$\sigma = \sqrt{1000(.5)(.5)} = \underline{15.8}$$
$$500 \rightarrow 700$$

*Figure 3: Kory's Calculations of Mean and Standard Deviation*

Leia and Nina anticipated that the outcomes of the coin flips should be equally represented. For both teachers, the number of trials in Event A made it seem possible that seven heads could occur in 10 coin flips because this event didn't deviate too much from their initial anticipation. However, when reasoning about Event B, both teachers saw this as a major aberration from what they had anticipated. In fact, when first engaging with Event B, Nina immediately said "Wow! … That seems crazy. Which one of the following is true? The coin is not fair." Nina elaborated on her reasoning about why Event A was more probable than Event B by stating the following:

"It's a fair coin, … the more times I do it (trials), it should be, it (distribution of outcomes) should approach tails and heads should be appearing in an equal frequency, 50% 50%." Leia provided similar explication for her stance on why Event A was more probable than Event B by invoking her image of the law of large numbers "The higher the number of trials is, the less likely it is that these numbers are going to be far away from that theoretical probability….When you only do ten, there's a lot of chance for it to be different." In both instances, the Leia and Nina reasoned that a larger number of trials would decrease the amount of variation that between what they would anticipate for the outcomes of flipping a fair coin, and what the observed outcomes would be.

The only subject to answer The Coin Flip Task incorrectly was Joy. Joy gave the response that Event A and Event B were equally probable because "seven tenths is the same as $\frac{700}{1000}$." When asked by the researcher to create a new event in the same context that would have the same probability of Event A and Event B, Joy created Event C and Event D where the outcomes were 100 coin flips with 70 heads and 50 coin flips with 35 heads, respectively. As Joy reasoned about the differences between Event A and Event B, the only difference that she verbalized was the fact that Event B had 100 times as many trials.

When Joy was presented The Coin Flip Task, she immediately made mention to a prior teaching experience where she presented her students with a probability lesson related to coin flipping. She jokingly lamented about a troublesome student who had challenged her claims to a coin flip outcome because "the coin wasn't fair because one of the sides was heavier." She stated that "he likes to get nitpicky about stuff like that." Initially, the researcher considered this to be a throwaway comment. However, shortly after the Joy recounted her classroom experience, reading the phrase *fair coin* seemed to trigger the response "It's a fair coin, that's what I should have said, one of the sides is not heavier than the other," as if she had just realized a quelling to her troublesome student's refutation. These utterances seem to provide evidence for why Joy had not analyzed the situation in an analogous way to the other subjects. By not considering fairness, it is possible Joy was not anticipating that the outcomes of the coin flips should be equally represented given a large enough size of observations. By not anticipating this equal share of outcomes, she was not perturbed by 70% of the outcomes being heads as deviating far from 50% of the outcomes being heads in either the 10 or 1000 flip case.

The three subjects who were able to correctly respond to The Coin Flip Task shared a common focus that seemed to be paramount in their thinking. Establishing a reference for what they had *anticipated* would happen allowed the subjects to compare the degrees of variability for each event. Kory compared expectations with observations using a measurement tool of standard deviations. Leia and Nina showed more informal reasoning that a small number of trials would result in a greater chance for aberration from expected outcomes than would a greater number of trials. In reasoning through this task, all three successful subjects reasoned using some aspect of variability to fuel their thought processes.

**The Orange Bin Task**

Using some aspect of variability to reason about a statistical situation was present again for participants for The Orange Bin Task. Knowing that there would be variability in the collection of sample means, Kory set off to make the variability explicit. Kory calculated the standard deviation for each sampling mean distribution under the assumption that orange weights would be normally[5] distributed. Comparing the spread of the distribution of sampling means for both Jeff and Krystal's methods allowed Kory to reason about how each sample would be distributed around the true mean weight for the population of oranges (Figure 2). Kory reasoned that: "… she's going to have much less variation…. The grocer will probably go with, he goes with Krystal's suggestion because she's probably closer to the truth than Jeff is. She has got much less variability in her distribution of sample means. Larger samples give better results typically."

Leia, Nina, and Joy also utilized the fact that sample size was the factor in determining why the grocer chose Krystal's method. A common theme in their approach was how extreme values would potentially influence the mean weights for each method. Joy provided two lists of mean weights (Figure 3) where the underlying reasoning was that larger samples would produce more consistent means.

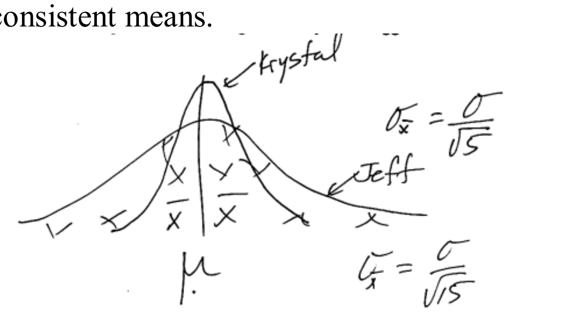


*Figure 4: Kory's Representation of Distributions of Sample Means*

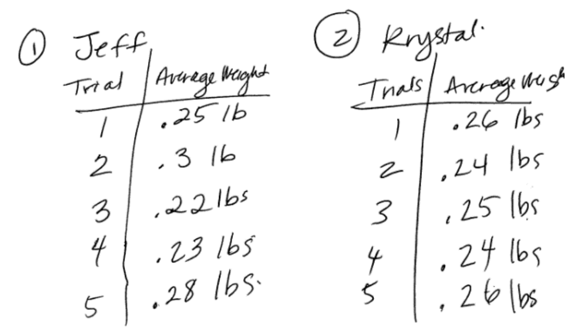


*Figure 5: Joy's Constructed Lists of Mean Orange Weights*

Thus, Joy felt the variation for Jeff's means would be larger than the variation for Krystal's means. While, Leia and Nina did not explicitly produce lists of mean weights, when given the follow-up question, both Leia and Nina reasoned that since the second list of numbers showed less variation, the list had to belong to Krystal.

The teachers were able to use informal ways of reasoning about variability in mean weights to support their arguments about how larger sample sizes would produce more representative sampling results. The teachers were able to reason that a larger sample size would allow for fewer relative aberrations, which in turn would make the collection of means for this larger sample size more consistent with each other. Though not synonymous with Kory's way of reasoning, the three teachers definitely displayed reasoning that can potentially precede thinking about the standard deviation of the sample mean.

## Conclusion

While the ways of statistical thinking presented by the community college faculty member are beyond what most middle school mathematics teachers would teach, these underlying ways can be preceded by the ways of thinking that the teachers displayed. Developing these intuitive ways to use variability to reason about statistical situations can lead teachers to develop more normative, robust ways of reasoning. This in turn may allow middle school teachers to better understand where their students are headed in terms of concepts which will enable them to prepare a better statistical foundation for their students.

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[5] Fruit weights are typically lognormal distributed.

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Classroom Experiences of Students in a
Community College Intermediate Algebra Course

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*There is little understanding of the ways in which students experience developmental mathematics courses at community colleges (Crisp, Reyes, & Doran, 2015). This study investigates the instructional experiences of students in an Intermediate Algebra course using qualitative methods that rely on interviews, surveys, classroom observations and classroom artifacts. I aim to understand (1) what are the experiences of students in a developmental mathematics class at a community college and (2) how students make sense of particular experiences. The findings from this study will support college mathematics departments by providing evidence of the classroom instructional experiences of students.*

*Keywords:* Developmental Mathematics, Student Success, Classroom Experience, Postsecondary Instruction, Community Colleges

Developmental mathematics is an important area of postsecondary mathematics education. Nearly 60% of first-year students at public two-year colleges take developmental mathematics (Radford, Pearson, Ho, Chambers, & Ferlazzo, 2012). Community colleges offer more developmental courses than any other type of postsecondary institution (Nora & Crisp, 2012; Parsad, Lewis & Greene, 2003; Radford et al., 2012). Developmental courses are often viewed as a gatekeeper for students to make progress to degree completion (Attewell, Lavin, Domina, & Levey, 2006; Bettinger & Long, 2005), but are also a pre-requisite for many students aiming to enter technical fields (e.g., science, technology, engineering, and mathematics [STEM], business). Historically, Intermediate Algebra has been the bridge between developmental courses and college courses (Lutzer et al., 2007). To be able to increase persistence, to provide a valuable learning experience and also to keep students in the STEM track, it is essential to know how parts of the "black box" work and student classroom experience is one that is often overlooked.

The most common way to measure student success in higher education is through academic achievement, such as GPA, course completion, the need to repeat courses, student persistence, and degree attainment (Howard, 2010; Valencia, 2015). This scholarship also seeks to predict such "success" using students' previous academic preparation, socio-economic status, mathematics placement test scores, and SAT scores (e.g., Bahr, 2010; Crisp & Delgado, 2014; Crisp & Nora, 2010; Nora & Garcia, 2001). Although such studies attempt to understand what contributes to the low success rates for developmental mathematics students, they, however, do not help us to understand what happens *while* students are enrolled in their math class, undermining the mathematical influence we as instructors can have on student success. The interactions and experiences that students have in a classroom can impact student learning (Cohen & Lotan, 1997) and those experiences in turn, can shape students in ways that can affect the quality of other subsequent educational experiences they may have (Dewey, 1938). Unfortunately, to this date, we know very little about how classroom experiences contribute to the success of developmental mathematics students.

The purpose of this study is to better understand the mathematical experiences of students enrolled in a developmental mathematics. This paper will address two research questions: 1) What are the instructional experiences of students in a developmental mathematics class at a community college? and 2) How do students make sense of these particular experiences? This qualitative case study focuses on how students experience the instruction in a developmental mathematics course, attending to their perception of these experiences, and to the ways in which these experiences influence their mathematical understanding.

## Theoretical Framework

I define instruction as the interactions that occur between instructors and students with the mathematical content (Cohen, Raudenbush, & Ball, 2003). Teaching and learning are essential aspects of instruction that occur within a specific environment, in this case, a developmental mathematics classroom at a community college. The roles of both student and teacher are supported by different resources (e.g., previous educational experiences, classroom environment, technology) and constrained by specific institutional requirements (e.g., classroom assignment/layout, covering preset mathematical content, having periods of 50 minutes) (Chazan, Herbst, & Clark, 2016; Cohen et al., 2003). In order to characterize students' experiences, I choose to focus on instruction. The classroom is an important space within a community college campus because it is the space in which the most interaction occurs for many students and is also where many students draw from when reflecting on their educational experiences (Wood & Harris III, 2015). Given the high attrition rates for STEM fields (69 % of associate's degree students who entered STEM fields between 2003 and 2009 had left these fields by spring 2009; see Chen, 2013), it is particularly important to understand classroom experiences. For example, in a literature review of reasons for dropping out of engineering programs, more than half of studies identified the classroom as a factor for why students leave (Geisinger & Raj Raman, 2013). I believe that the experiences of students, while interacting with their instructor and the mathematical content, significantly impact the success a student has in mathematics.

## Methods

The study takes place at Clear Water Community College[1], a Hispanic serving institution in California, during Fall 2016. Around 90% of the student population at this college enroll in developmental mathematics. I observed one section of Intermediate Algebra taught by a part-time faculty member, following nine focal students throughout the semester. The class met three times a week for 95 minutes. Table 1 describes information about the nine focal students. All students identified as being of Latinx decent. I chose a part-time faculty member because oftentimes developmental mathematics courses are taught mostly by part-time instructors (Blair, Kirkman, & Maxwell, 2013). The instructor was a Black female, who graduated with a Bachelor's degree in Engineering and a Masters in Applied Mathematics. She has 2 and a half years of teaching experience, two of which were while she was a graduate student. During this semester, she is also teaching 3 courses at two other community colleges.

*Table 1. Student Demographics*

| Student | Gender | Age | First Generation | First Year |
|---|---|---|---|---|
| Adriana | F | 19 | Yes | Yes |
| Chris | M | 21 | No | No |
| Guillermo | M | 18 | Yes | Yes |

[1] All institutions and names in this study are pseudonyms

| | | | | |
|---|---|---|---|---|
| Layana | F | 20 | No | No |
| Marisa | F | 18 | No | Yes |
| Nancy | F | 18 | No | Yes |
| Raquel | F | 18 | Yes | No |
| Santiago | M | 18 | Yes | Yes |
| Teresa | F | 18 | Yes | Yes |

The data sources that I used for this paper include interviews, classroom observations, and diary entries (surveys). I interviewed each student three times at the beginning, middle and end of the semester. I also observed the course 12 times, which included video-recording the lesson as well as taking fieldnotes. After each formal observation, the focal students filled out a survey, detailing their experience of that specific class meeting which included reflections on the math content, moments that went well/did not go well, interactions with peers, and ways that they participated in the course.

This study is a case study analysis of one intermediate algebra class. I engaged in open coding of the student interviews and surveys as well as the observation fieldnotes. I captured comments that related to the ways in which the students described the classroom instruction as well as how issues such as race/ethnicity, gender, culture, class, or language affected their classroom experience. To code the video-recordings, I used the *Evaluating the Quality of Instruction in Postsecondary Mathematics* (EQIPM) instrument (Author and colleagues, 2017), which assesses the quality of mathematical instruction at community colleges by providing ratings from 1 to 5 on various codes. In this paper, I will talk about two codes, specifically *Organization in the Presentation of Procedures* and *Mathematical Errors and Imprecisions in Content or Language*. *Organization in the Presentation of Procedures* captures how complete, detailed, and organized the instructor's (or students') presentation (either verbal or written) of content is when outlining or describing procedures, or describing the steps of a procedure used to solve problems. A rating of 1 on this code implies that the instructor's (or students') presentation of the procedure is disorganized, incomplete, illegible, or unclear. A rating of 5 indicates that the teaching is not only clear, but it is also exceptionally organized and/or detailed. *Mathematical Errors and Imprecisions in Content or Language* capture events in the segment that are mathematically incorrect or that have problematic uses of mathematical ideas, language, or notation. A rating of 1 on this code implies that there were no errors or imprecisions in the segment while a rating of 5 implies that content errors and/or imprecisions occur in most or all of the segment or muddle the opportunity for students to make sense of the procedure. I rated the video in 7.5-minute segments using the EQIPM instrument. Given that there are both good and bad moments within our teaching, I averaged the ratings among the segments.

## Preliminary Results

I will describe preliminary findings from Observation 2, which occurred during week 3 of the semester. There was a total of 12 7.5-minute segments in this observation. In particular, I will first describe ratings and evidence given for *Organization in the Presentation of Procedures* and then for *Mathematical Errors and Imprecisions in Content or Language*. I will also describe how students specifically made sense of their experiences in relation to those particular codes.

There were 38 students in the class meeting and all of the focal students were in attendance. The students sat at long table rows, each seating about eight students, with an aisle down the center. Some students sat in chairs at the back of the room as they walked in late. The lesson covered topics such as solving one-variable linear inequalities, absolute values, and solving absolute value equations and inequalities.

**Instruction.** The modes of instruction during this meeting were lecture, individual student work, and student presentations. Out of the 95 minutes of class, the instructor lectured for 60 minutes (63.2% of class time), students worked individually at their desks for 20 minutes (21.1%), three students presented at the board for 13 minutes (13.7%), and two minutes were devoted to classroom business (2.1%). The classroom was considerably quiet with very little interaction, and the instructor sat at the document camera for the entire class session.

***Organization in the Presentation of Procedures.*** The mean rating for *Organization in the Presentation of Procedures* was a 2.5. A rating of 3 indicates that the teaching of the procedure is acceptable, complete, and mostly clear, but not exceptionally organized or detailed. I noted two general areas where organization greatly affected the instruction. First, the instructor used a set of guided notes to lecture from.

The instructor created online note packets for every chapter in the textbook. She lectured from these packets every class meeting. Some say that there are affordances of using such guided note packets in that problems are pre-selected providing scaffolding or that by providing guided notes, students can spend more time focusing on the lesson while also having a set of coherent notes (Montis, 2007). Upon review of the video, it was difficult to follow the notes throughout the lesson: the instructor jumped around from page to page, often left directions or entire problems out of view of the document camera while working on a problem, and also appeared to run out of space when working on a problem. From fieldnotes, I noted that only a handful of the 38 students had the notes printed out. Therefore, the organizational affordances of providing lecture notes were not capitalized during the lesson.

The second area of organization that was evident during instruction is related to the way that the instructor selected the problems in the lesson. Specifically, the instructor did not scaffold the problems so as to set up student success, leaving the students to work individually on very difficult problems. There were three moments when students were asked to work on problems individually at their desks. In each of these instances, the instructor first worked on one to three examples, and then selected a problem within the same section of notes for students to work on.

At one point in the lecture, the instructor assigned the students to work individually on an absolute value problem where the directions said to "Solve the equations". Prior to this moment, the instructor worked through two examples at the document camera, $|y| = 8$ and also $|4x + 1| = 9$. She asked the students to solve the following: $3\left|\frac{3}{2}a + 1\right| + 2 = 14$. The increase in the level of complexity in this problem jumps quite quickly. Students were given three and a half minutes to find and check the solutions to this problem. Later in the class period, the instructor gave students a set of three absolute value equations to solve and indicated that she would ask students to volunteer to present their work at the board. One of the problems in this section was particularly challenging: $\left|\frac{4w-1}{6}\right| = \left|\frac{2w}{3} + \frac{1}{4}\right|$.

Throughout the lesson the instructor selected challenging problems for students to work on, when the problems she selected to use during the note-taking were uncomplicated. In particular, the instructor did not work on any problems in the notes that involved fractions. In the class surveys, five students mentioned these individual practice problems as extremely challenging. Layana said that she felt uncomfortable, "when [the instructor] involved fractions and didn't give examples and kind of let us do it on our own." Teresa said, "when we began to deal with fractions I started to get confused and compared my notes with my partner but turns out we were both confused."

***Mathematical Errors and Imprecisions in Content or Language.*** The mean score for *Mathematical Errors and Imprecisions in Content or Language* was a 3. This implies that there were on average content errors or imprecisions in every segment. This is extremely problematic given the already possible set of misconceptions that students in developmental mathematics classes may already have (Author, 2014). One particular segment scored a rating of 5. In this segment, the instructor says phrases that overly simplify a complex idea. For example, when solving an equation, she tells students to "remember, no matter what, in Algebra the goal is to isolate the term or the variable." This phrase appears to simplify all work done in Algebra down to one notion: solving. By framing mathematics in such a way, students can begin to lose sight of the purpose and utility of mathematics beyond simply solving.

At another point in the segment, the instructor solved an equation and was left with a solution of $x = \frac{-10}{4}$. She asks the students, "What's the LCD between 10 and 4?" A few students say "2". The instructor continues on talking about how she can simplify to $\frac{-5}{2}$, however, catches herself and says, "Sorry, Greatest Common Factor between 10 and 4. The answer is still 2…Remember factors break down, multiples multiply out." It has been documented that developmental math students tend to struggle with the differences between least common multiples and greatest common factors (Stigler, Givvin, & Thompson, 2010). In this exchange, the instructor asks for the least common multiple, which would be 20, but students gave the greatest common factor. This could have been because students assumed what the instructor was asking for. However, later when the instructor catches her mistake, she tells the students that the answer would still be two, not correcting that the LCD between 10 and 4 is in fact 20. Later, when she checks that the solution $x = -\frac{5}{2}$ works for the absolute value equation, $|4x + 1| = 9$, uses inaccurate mathematical language. When simplifying the term $4\left(-\frac{5}{2}\right)$, she indicates to students that they can reduce the fraction by "cross cancel[ing] a little" such that the 4 in the numerator "cross cancels" with the 2 in the denominator to make the mathematics simpler.

Most students did not seem to catch the different mistakes during the lesson, and when they do they usually catch copy mistakes (e.g., missing a negative, not writing the correct number). In student interviews, some focal students indicated that they have overheard others correcting the instructor's mistakes and that they do not mind because it is useful to see your instructor make a mistake. For example, Chris mentions that he sees her making mistakes pretty regularly. Instead of outright correcting her, he tries to ask questions in order to help her catch them with the intention of not embarrassing her. Raquel and Adriana indicate how it really confuses them when she makes mistakes in her teaching. Raquel said,

> Her teaching methods, I'm just not feeling it. I mean, she tries but it's like, she makes too many mistakes. And it's like, you're a professor. You're supposed to know what you're doing. And I, everybody makes mistakes, but not like constantly, when we're trying to really learn and pass this class…I notice 'em, but I don't want to like, say it. Because what if I'm wrong too? So I don't want to look dumb. But I do notice.

Adriana says that that every time the instructor makes a mistake, she feels like they are doing all of the work for nothing, which both confuses and frustrates her.

**Questions.** At the talk, I aim would like to ask participants: 1) What are your thoughts on the mathematical errors that the instructor demonstrates? 2) The participants in my study are all Latinx students. In the larger study, I intend to use Critical Race Theory, specifically Latino Critical Theory to investigate these experiences further. In what ways have audience members used LatCrit theory in their work in undergraduate mathematics education?

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# How may Fostering Creativity Impact Student Self-efficacy for Proving?

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*Mathematical creativity has been emphasized by mathematicians as an essential piece of doing mathematics, yet little research has been done to study the effects of fostering creativity in the undergraduate classroom. In this paper, we seek to understand creativity in the classroom using Sriraman's (2005) five principles for fostering mathematical creativity by studying how these principles impact student self-efficacy for proving. Using online student surveys, interviews, and classroom observation, we demonstrate how Sriraman's five principles and Bandura's four sources of self-efficacy may be used to explore the impact of fostering mathematical creativity on student's self-efficacy for proving. Then from the interviews, we highlight how the use of the free market and scholarly principles may influence student self-efficacy via vicarious role-modeling, and explain why the use of these two principles may be of particular importance in fostering student self-efficacy for proving.*

*Keywords*: mathematical creativity, discrete math, proving, self-efficacy

A large body of research has underscored the importance of *mathematical creativity* (1) in learning and doing mathematics (Mann, 2006), (2) in retaining students in STEM fields (Atkinson & Mayo, 2017), (3) in advancing the field of mathematics (Sriraman, 2005), (4) as a platform for creativity in general (Nadjafikhah et al., 2012), and (5) as one of the aims of mathematics education (Levenson, 2013). But how and to what extent should creativity be incorporated in an introductory proofs course? How does creativity in the classroom impact students' long-term mathematical development? In this paper, we endeavor to better understand these questions by studying the research question: how does instructor use of creative "principles" in the classroom impact students' development of self-efficacy for proving?

## Literature Review

### Importance of self-efficacy in mathematical learning and performance

Perceived self-efficacy (often referred to simply as *self-efficacy*) is one's belief in their own ability to accomplish something, and is highly predictive of general academic performance (Bouffard-Bouchard, Parent, and Larivée, 1991). Self-efficacy for mathematical problem solving is a better predictor of mathematical performance than mathematical ability or prior experience with mathematics (Siegel, Galassi, & Ware, 1985; Pajares & Miller, 1994). Why is this?

Bandura (1997) explains that "academic performances are the products of cognitive capabilities implemented through motivational and other self-regulatory skills." (p. 216). A wide range of studies in cognitive psychology have confirmed that beliefs of self-efficacy mediate the skills that determine how consistently and effectively students apply what they know (Pajares & Kranzler, 1995; Randhawa, Beamer, & Lundberg, 1993). Students with high self-efficacy show increased motivation and use of strategic thinking (Boffard-Bouchard, 1990), are more successful in solving conceptual problems, manage their time better, are more persistent, and are less likely to reject correct solutions (Bouffard-Bouchard et al., 1991). Thus, we suggest that self-efficacy for mathematical problem solving and proving mediates learning in a way that results in long-term benefit of educational principles and practices correlated with increased self-efficacy.

How can self-efficacy be influenced or changed? Social cognitive theory identifies four primary sources of self-efficacy: enactive attainments, vicarious influence, verbal persuasion, and physiological reactions (Bandura, 1997). *Enactive attainments* are experiences of mastery in a given task and are often the most reliable indicators of future capability. *Vicarious influence,* through observation of someone else's competencies in comparison with one's own, provide the next most reliable indicator of ability. *Verbal persuasion*, or direct verbal appraisal of one's ability by someone else, is a somewhat less reliable source of self-efficacy depending on the credibility of the persuader. Lastly, *physiological reactions,* are feelings such as strength, stamina, stress, fatigue, or pain that can provide indicators of ability or inability. Here we note that these sources of information themselves do not influence self-efficacy, but through cognitive processing of this information and reflective thought, they are selected, weighted, and integrated into self-efficacy judgements: "a host of personal, social, and situational factors affect and direct how socially mediated experiences are cognitively interpreted" (Bandura, 1997, p. 79).

### Definition and Principles of Mathematical Creativity

There is considerable variation when seeking definitions of mathematical creativity (Mann, 2006). In this project, while considering mathematical creativity relative to the student, we chose a view influenced by the perspectives of Liljedahl and Sriraman (2006): "a process of offering new solutions of insights that are unexpected for the student, with respect to his/her mathematical background or the problems s/he has seen before" (Savic et al., in press, p, 2). This is based on relative (Beghetto & Kaufman, 2007), process-oriented (Pelczer & Rodriguez, 2011), domain-specific (Baer, 1998) mathematical creativity.

In seeking to study students' experience of creativity in the classroom, we searched for observable characteristics of creativity that instructors and students engaged in the classroom. According to Sriraman's (2005) extensive review of the literature on mathematical creativity, there are five principles that "maximize potential for mathematical creativity" (p. 26):

- the *Gestalt principle* conveys the importance of students engaging in "suitably challenging problems over a protracted time-period, thereby creating the opportunities for discovery and to experience the euphoria of the 'Aha!' moment of illumination" (p. 27);
- the *aesthetic principle* is concerned with appreciating the beauty in discovering and connecting new ideas;
- the *free market principle* encourages risk-taking and atypical thinking;
- the *scholarly principle* encourages students debating and challenging the validity of teachers' and peers' approaches to problems; and
- the *uncertainty principle* embraces ambiguity in mathematics, emphasizing the importance of giving open-ended problems and providing "affective support to students who experience frustration over being unable to solve a difficult problem" (p. 28).

How does instruction seen through the lens of these principles directly or indirectly impact student development of self-efficacy for proving?

## Methods

Pilot data collection was conducted in an 8-week summer session of a discrete mathematics course serving as an introduction to proofs. Five classes were videotaped, one of which was transcribed, then coded by both authors for evidence of the five principles. Differences in codes were discussed until arriving at an agreement for each coded action; otherwise, both would continue for the remainder of the class period. An online survey was given to the students to measure student *experience*[1] of the five principles consisting of ten questions, two per principle. The survey was given twice, once for student experience in *prior* math classes, and once at the end of the semester for experience in *this* class. Then, in both instances, students rated their confidence in their ability to do five tasks related to proving with respect to three specific problems classified as moderately routine, moderately non-routine, and very non-routine (Selden & Selden, 2013). Selection of task questions was based off the EP-spectrum (Hsieh, Horng, & Shy, 2012) deconstruction of the proving process and followed Bandura's guide for constructing self-efficacy (2006, p. 307-337).

At the end of the course, two randomly selected students (Abe and Ben) and the instructor (Dr. One) were interviewed. Each interview was transcribed and coded once by both authors (again compared for inter-rater reliability); once for explicit and implicit examples of instructor use of each of Sriraman's (2005) five principles in the classroom, and again for examples of the student's experience of each of Bandura's (1997) four sources of self-efficacy.

## Results

For the purposes validating the test questions (Bandura, 2006), the surveys were given two weeks apart. The mean scores of the self-efficacy surveys 79 compared to 80 only two weeks later, showing that both surveys were of similar and appropriate difficulty for distinguishing levels of efficacy. Also, the correlation between the two questions for each of the five principles ranged from 0.45 to 0.92, showing convergent validity.

Only nine students participated in both surveys, so with this data we cannot yet say which of the five principles are quantitatively correlated with changes in student self-efficacy. Also, from classroom observation, we were not able to observe enough instances of instructor actions demonstrating the use of the five principles of creativity. Thus, the remainder of this paper focuses on evidence from individual student surveys and interviews.

In the interviews, both students reported teacher actions for the free market and scholarly principles. Abe explained two actions coded as free market and three coded as scholarly. For example, his comment that "quite a few people went up to the board," and that "if we saw, or said what was on our mind," were both coded as an implicit use of the free market principle, since it showed that students were comfortable enough to take risks: going up to the board and saying what was on "our mind." Abe saying "[The instructor] would break questions down to the point where everyone could possibly have an input on why, or on the steps building up to the proof," was coded as scholarly principle, as well as part of the following:

*Interviewer*: Is there any way that the classroom environment did help you learn some of those skills?

*Abe*: It helped me learn a lot faster because if I didn't know the solution right away in most cases someone else did, and once someone else was called up to the board and started writing their proof**, I would follow along and then at a certain point I'd be able to figure out this is where they were going**. I'd finish out the proof and try to continue on. But having people around me that were like-minded and also enjoyed doing these types of proofs... It helped my learning because I wasn't just having to rely on what I gained from the instructor. I could rely on what other people brought to the class as well.

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[1] In future research, we realize it will be more directly answer our research question to measure to student observation of *instructor use* of the five principles.

Ben described that the instructor would "engage the class like earlier on in the semester and I felt comfortable about like speaking up and answering occasionally." This was coded as implicit use of the free market principle since it showed Ben was able to take the risk of speaking up at that point in the semester. He also stated, "a lot of times [the instructor] would introduce a new problem and tell us to work on it... it'd be like a completely brand-new problem, which I guess is good to try to be able to think of how you'd approach like a brand-new concept." This was coded as uncertainty principle because the instructor did not answer the problem immediately, thus perhaps implicitly allowing the students to tolerate ambiguity.

In coding for sources of self-efficacy, instances of enactive experiences and vicarious role-modeling were identified in both interviews. Abe described seven instances coded as enactive experiences, for which we provide one.

*Abe*: I'm very confident now. I feel like these eight weeks have really given me enough time to work on the formats for everything, to be able to look at a problem rather than just as a solution and more as a problem within a problem and I think I'm pretty confident with solving it now."

However, he did not indicate whether they were due to any actions from the instructor. Ben cited four instances coded as enactive experiences, all on his own, outside of class. He cited *not* having the confidence to engage in class without the "right answer."

Abe describe three instances of vicarious role-modeling. Two were from the instructor: "[the instructor] was very excited when he was talking about proofs and I feel like I feed off the energy of my professors… I think that really helps me learn," and "as he broke it down, he would slowly work it out with us, as we were talking to him." Abe also experienced vicarious role-modeling from other students in the class: "I would follow along and then at a certain point I'd be able to figure out this is where they were going" (quoted above).

Ben described four instances of vicarious role-modeling: two as positive sources of self-efficacy from the instructor: "seeing a teacher like do the proofs repetitively" and "seeing it how you're supposed to approach a proof… seeing like where to start" (in response to the question "what made you confident?"), and two as negative sources from the students:

*Ben:* In this class setting I felt like there were people in this class that already knew, like there's like two people in particular that would always answer all the questions and they seemed… I just kind of like deferred, if it… like the questions to them, so if they didn't… like if the teacher posed a question to the class and they didn't answer it, then I felt it like well, I definitely can't answer it if they can't.

*Ben:* There are other times when he would engage the class like earlier on in the semester and I felt comfortable about like speaking up and answering occasionally, but [later on] I didn't feel comfortable and like around my peers to like answer questions, because I didn't have the confidence.

Lastly, we found that although Ben expressed some confidence in his proving ability in his interview, his self-efficacy score (from one survey) was 8 points lower than average. Abe's self-efficacy score was 6 points (average of two surveys taken) above average.

## Discussion

In Abe's interview, we might infer a potential association between the free market and scholarly principles and increased self-efficacy via vicarious role-modeling. In the same quote, he believed that students were comfortable enough to say, "what was on [their] mind," (free market) then the instructor "would slowly work it out with us, as we were talking to him" (vicarious role-modeling). But, without more data, it is difficult to understand the degree which the free market principle promotes, or may be necessary, for the positive influence of vicarious role-modeling in the classroom. Abe's experience also highlights the importance of the scholarly principle in gaining self-efficacy via vicarious influences. Engaging and considering other's solutions appears to have improved his self-efficacy because he "wasn't just having to rely on what I gained from the instructor." According to Bandura (2007), these kinds of influences (i.e. from peers), provides a stronger source of self-efficacy (than that from the instructor), since the attainments of those who are more like oneself gives better indication of one's own ability.

Although Ben's interview contained evidence of instructor use of scholarly and uncertainty principles, the use of the scholarly principle may not have promoted self-efficacy in this student's case. Firstly, his self-efficacy scores were lower than average. Secondly, when asked, "did you become more confident by the end [of the course]?" he responded, "I still don't think I'd be confident… like if the teacher posed a question to the class and [the two confident students] didn't answer it, then I felt it like well, I definitely can't answer it if they can't." This is an example of vicarious role-modeling giving a negative source of self-efficacy information for the reason described by Bandura: "observing others perceived to be similarly competent fail lowers observers' judgment of their own capabilities and undermines their effort" (1997, p. 87). Interestingly, Ben did not cite any evidence of the free market principle.

After comparing both interviews, we believe that the scholarly with the free market principles *together* may better promote positive student self-efficacy: creating an environment where students can take risks (where mistakes are ok) levels the playing field, helping students see others' experience of both success and failure as part of the proving process. Thus, the way an instructor handles the free market principle may mitigate potential negative effects on self-efficacy associated with the threat of being wrong (inherent with the scholarly principle).

## Conclusion

Although not directly identified thus far in our research, there are some potential connections between Sriraman's principles and self-efficacy suggested by other research. For example, the Gestalt principle may foster self-efficacy via enactive attainment since students' experience of creating proofs though sustained time and effort gives evidence of their future proving ability. The aesthetic principle may serve as verbally influencing student self-efficacy, by convincing students of the joy and beauty inherent in mathematics.

Additionally, as we found above, other principles may have a stronger combined influence on self-efficacy, which may be seen via other mediatory mechanisms such as intrinsic motivation. For example, giving students opportunities to state and defend their solutions (free market) while promoting the understanding of problem design (scholarly) may give students greater ownership, intrinsic motivation, and promote the development of evaluation. As a result, when doing tasks with high intrinsic motivation (i.e. proving), task feedback can have a stronger effect on self-efficacy (Arnold, 1976).

Furthermore, student self-efficacy may influence how students engage or experience these principles. Students with high self-efficacy tend to pursue higher challenges (Pintrick & DeGroot, 1990), suggesting that students with high self-efficacy are more likely to engage the free market principle in and out of class. Also, student experience of the gestalt principle can be influenced by their persistence, which may be supported by self-efficacy beliefs (Selden & Selden, 2010). More research is needed to understand the connections between mathematical creativity and self-efficacy for proving and problem solving. How might this study be modified or applied in a larger context? How might the results inform our use of creative principles in the classroom? What other connections might one find with these theoretical framings?

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Teacher Learning About Mathematical Reasoning: An Instructional Model

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*First, I describe an instructional model for Teacher Learning about Mathematical Reasoning (TLMR), designed for pre-service (PSTs) and in-service teachers (ISTs) to: (a) build knowledge of the various forms of mathematical reasoning that students naturally make use of in their justifying solutions to problems, (b) attend to the development of students' mathematical reasoning from studying videos and student written work, and (c) learn about the conditions and teacher moves that facilitate student justifications of problem solutions. Second, I provide a detailed description of activities from a representative cycle of the TLMR model. Finally, I report briefly on preliminary results indicating teacher growth in identifying and recognizing student reasoning for PSTs and ISTs who underwent the TLMR model compared to a comparison group.*

*Keywords:* mathematical reasoning, pre-service teachers, video-based intervention

As the knowledge required for effective mathematics teaching has become more clearly defined, knowledge needed for PSTs and ISTs to attend to student reasoning in justifying solutions to problems has been recognized as essential (e.g., Francisco & Maher, 2011). In fact, earlier work on student reasoning has been influential in shaping US national policy by developing a set of Standards for Practice in identifying behaviors that are desirable for students as they engage in doing mathematics. (NCTM, 2014). It is essential that PSTs and ISTs become knowledgeable of these practices so that they can attend to and encourage student mathematical reasoning in their classrooms. In addition, while models on how they can do this have been theorized, there is little research testing the effectiveness of these models. In this paper, I will describe an intervention model that is designed to promote teachers' attention to student reasoning through: (a) open-ended problem solving of mathematical tasks, (b) studying videos of children building solutions to the same tasks, and (c) in-class and online discussions about their own and students' problem solving, as well as relevant readings and the results of a quantitative study to test the effectiveness of the model in having PSTs and ISTs attend to student mathematical reasoning.

## Children's Mathematical Reasoning and Justification

Research on the knowledge for teaching mathematics identifies that knowledge of students' mathematical reasoning is essential (Ball, 2003) and that there is a relationship between following how students builds their knowledge and their performance in mathematics (Rowan et. al, 1997). An important early finding in research into mathematical reasoning is that in a natural way, children – even young children – build proof-like justifications, providing convincing arguments that take the form of reasoning by cases, induction, contradiction, and upper and lower bounds (Maher & Martino, 1996). Students' justifications are driven by an effort to make sense of the problem situation, notice patterns, and pose theories (Mueller, Yankelewitz, & Maher, 2012). Three key components surface in promoting the development of mathematical reasoning. These are: (a) reflecting on and revisiting earlier mathematical concepts, (b) collaborating and discussing strategies and modes of solution; and (c) engagement in open-ended tasks that elicit justification (Maher, Powell, & Uptegrove, 2010). Research results across all ages and contexts, formal and informal, indicate that certain tasks tend to elicit particular forms of reasoning (e.g., upper and lower bound arguments, inductive arguments) when students are encouraged to provide a justification for their solutions (Yankelewitz, Mueller, & Maher, 2010).

## Teacher Learning About Mathematical Reasoning: A Model

TLMR was implemented as a design-research study in a graduate mathematics education course over 5 years, with each implementation co-taught by the author. The problems used during class were from the domain of counting and combinatorics. Each iteration was over a 14-week period. Overall, 86 teachers participated. The model has six cycles of interventions with each cycle containing five components. These are: (a) teacher collaborative problem solving (b) teacher study of videos of children working on the same problems, (c) teacher analyses of samples of student written justifications of the same problems, (d) teacher small-group online discussions (with guiding questions) designed for synergistic reflection on their own and students' problem solving, in light of the assigned readings, video study, personal experience, and collaborative problem solving. Throughout the course, teachers engaged in each of the six cycles by first working collaboratively on the related mathematical tasks. I offer an example for each of the five components of a cycle.

## A Sample Cycle

**The Tasks**: The TLMR intervention utilized tasks from earlier research studies that were shown to elicit student reasoning in justifying solutions to problems. All tasks were introduced prior to the mandated school curriculum. During this cycle, teachers worked on two variations of pizza problems: Pizza with Halves, selecting from two toppings; Whole Pizzas, selecting from four toppings. They were asked to discuss the strategies used in solving both problems, attend to similarities or differences, compare their strategies, reflect on strategies for previous problems, and report findings to the entire class.

**The Videos:** Teachers were then assigned to study two video clips online, read and react to two articles, and examine four samples of student work from fifth graders working on the same Pizza with Halves Problem. The first video, http://dx.doi.org/doi:10.7282/T3HM57PQ, followed twelve, fifth-grade students across two class periods as they worked on the Pizza Problem with Halves selecting from two toppings. The clip showed students constructing various representations to justify their solution of the ten pizzas. The second clip, http://dx.doi.org/doi:10.7282/T3VX0FRD, was a task-based interview with fourth grader Brandon. It shows Brandon, explaining his solution to the Pizza Problem selecting from four toppings (without halves). After explaining his solution, Brandon is asked whether this problem reminds him of any other problem he had worked on earlier and he said it reminded him of the Towers Problem, a problem where he had to find the unique number of towers he could build four high selecting from two colors of Unifix cubes. After resolving the Towers problem, he makes a connection between the similarity in structure of the two problems, recognizing that the two choices for a pizza topping (represented by 1 or 0) for being on or off the pizza is similar to the two choices for the color of a particular block of the tower, e.g. red or yellow (PUP Math, 1999).

**The Readings:** The first readings discussed details of the Brandon Video (Maher and Martino, 1998). The second reading dealt with the topic of isomorphisms in mathematics education (Greer and Harel, 1998). The Maher and Martino, paper situated the Brandon video as a part of a longer study and included details that preceded the interview as well as an analysis of Brandon's problem solving. The Greer and Harel paper referred to Brandon as an example of a nine-year old student having an insight in recognizing an isomorphism, similar to the mathematician, Poincare.

**Student Work:** The student work module contained four pieces of student solutions from the Pizza with Halves problem, selecting from two toppings. These were chosen to illustrate the variety of representations and arguments produced by the students. The teachers were asked to review the students' representations and work and specifically address: (1) the correctness of the solution, (2) description of the strategy, (3) the validity of the reasoning, and (4) whether or not they find the solution convincing and, if so, why. If they did not find the solution convincing, they were asked to indicate from studying the student work what pedagogical moves they might take to help the student develop a convincing argument.

**Online Discussion:** For this module, the guiding questions focused on the notation that Brandon used in his problem solving. Teachers were asked to discuss how, if at all, Brandon's choice of notation was helpful to him in recognizing the relationship between the Pizza and Tower problems. They were also asked to discuss the forms of reasoning displayed by Brandon in the video, and the role of isomorphisms in mathematical cognition. Finally, they were asked to compare their own problem solving with that of the students in the Pizza with Halves and in the Brandon video.

## The Study

Limitations in time and space allow only a brief description of preliminary results. A reasoning assessment (RA) was administered as a pre-test before the course and as a post-test after the course. The group that underwent the intervention above was 86 teachers over the five iterations. Additionally, pre and post data was collected from 48 teachers from the same course taught by a different instructor. The comparison group's course had the same emphasis on attending to mathematical reasoning, but did not use the TLMR model. The comparison group is used to check if the students just got more on the post-test because they watched the video twice, if that is true, we would expect to not see a significant difference in the post-test scores between the comparison and experimental group. The RA consisted of a ten-minute video of fourth-graders sharing their arguments for an open-ended problem-solving task. Teachers were asked to: (a) identify the arguments presented by the children, (b) determine the validity of the arguments, (c) provide evidence to support their claims, and (d) explain whether or not the arguments were complete. The clip contained arguments by induction, cases, an alternate cases argument, and contradiction. The responses were scored by a group of three people using an established rubric to determine if the participant noted no features, partial features (the alternate cases and contradiction responses did not contain a partial feature, the other two did), or complete features of the argument. Initially, the group worked with responses from a pilot study for training and to establish reliability. For the analysis, growth was defined as recognizing more features on the post-test than the pre-test. Each argument was analyzed using a Wilcoxon rank-sum test, a nonparametric alternative to the t-test to determine if the growth from pre-to-post was significant. The effect sizes were calculated using an estimator suggested by Grissom and Kim (2012) which takes the U statistic generated by the test and then divides it by the product of the two sample sizes that will estimate that a score randomly draw from one population will be greater than the other. This methodology was chosen over using Cohen's d due to the smaller sample size and non-normality of the data.

## Results

The pre-test results for the comparison and experimental group were not significantly different and the differences in each iteration for the experimental group was not significant, so the experimental group was put together into one group, giving a size of 86 teachers. For the first cases argument, on the pre-test the experimental group had 50% missing the argument, 2.3% had a partial argument, and 47.7% had the complete argument compared to 61.2% missing, 6.1% partial, and 32.7% complete for the comparison group. On the post-test, in the experimental group 21.6% was missing the argument, 1.1% partial, and 77.3% complete compared to 57.1% missing, 8.2% partial, and 34.7% complete. The growth from pre-to-post for the experimental group was significant with a moderate effect size ($p < 0.01$, effect size = 0.351) and for comparing the experimental to comparison post with a moderate effect size ($p < 0.01$, effect size = 0.248). For the alternate cases argument, the experimental group went from 69.3% missing on the pre-test to 33% on the posttest and 30.7% complete on the pre-test to 67% complete on the posttest. This growth was significant ($p < 0.01$, effect size = 0.277) and the post score compared to the comparison group was significant ($p < 0.01$, effect size = 0.32). The inductive argument for the experimental group had 55.7% missing, 37.5% partial, and 6.8% complete on the pre-test and 23.9% missing, 53.4% partial, and 22.7% complete on the post-test for a significant growth ($p < 0.01$, effect size = 0.317), and a significant difference in post compared to the comparison group ($p < 0.01$, effect size = 0.324). Finally, the argument by contradiction was missing from 95.5% of the experimental teachers' pre-test and complete in 4.5% compared to missing in 68.2% of the post-test with 31.8% complete for a significant growth ($p < 0.01$, effect size = 0.364) and significant difference than the comparison post results ($p = 0.011$, effect size = 0.402).

## Discussion and Implications

The design TLMR research study produced an extensive and valuable database about teachers learning to attend to student reasoning. Preliminary analyses suggest significant changes in teacher beliefs in recognizing the potential for student reasoning. In addition, there is evidence that teacher recognition of the forms of arguments used by children in the video as they expressed their justifications of solutions improved over the course of the intervention (Maher et. al, 2014). Analysis of teacher online discussions of one cycle of intervention also indicated some interesting findings. As teachers compared their own problem solving with that of (a) other teachers, (b) students from the videos, and (c) student work samples, they pointed out similarities and differences, especially when the representations differed from their own and were in their view, "more elegant". These comparisons prompted further reflection about what constitutes a convincing argument in posing a solution to a challenging mathematical task and raised expectations about students' creativity in representing their solutions. It is interesting that teachers focused heavily on attending to details in the videos, relating their observations to their own personal experience. Reference to the readings was also made by teachers who tried to situate their learning within a particular theoretical perspective. Implementation of the TLMR holds promise for teacher growth in attending to the development of mathematical reasoning in students. Further studies building off this work can focus on the application of teacher learning through the TLMR model to their practice.

## Questions

(1) I collected some qualitative data as well (online discussions, in-class problem solving) – would those help strengthen the argument that the model is successful in promoting attending to reasoning?

(2) Is it worthwhile to follow PSTs into their practicum and initial classroom experience to see how it affects them or would it be too messy?

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Student's Semantic Understanding of Surjective Functions

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*Reasoning and proof are essential to mathematics, and surjective functions play important roles in every mathematical domain. In this study, students in a transition to proof course completed tasks involving composition and surjective functions. This paper explores students' semantic understandings of surjective functions, both individually and in the context of composition of functions. Most students demonstrated productive semantic understandings of surjective functions that allowed them to produce counterexamples and arguments for the truth of statements. Furthermore, in the struggle of using the syntactic definition of surjective in a proof, some students used their semantic understanding to try to make sense of the definition. This demonstrates the potential of students' ability to reason semantically to build understanding of the syntactic definition and structure of proofs of surjective functions.*

*Keywords:* Proof and Proving, Semantic and Syntactic Reasoning, Surjective Functions

Reasoning and proof are fundamental aspects of mathematics on which mathematical teaching and learning should focus. Weber and Alcock (2004, 2009), describe two distinct reasoning styles and approaches to proof production that they call *semantic* and *syntactic*. Semantic reasoners produce proofs through a focus on general understanding guided by examples, diagrams, or other informal explanations, and syntactic reasoners produce proofs mainly through deductive reasoning based on axioms, definitions, theorems, and standard proof frameworks (Weber & Alcock, 2004). Although a mathematical proof is a syntactic product, understanding the proving process involves both types of reasoning. Thus, "neither of these approaches should be used exclusively by students and both syntactic and referential [semantic] approaches to proving are necessary for proving competence" (Alcock & Weber, 2010, p. 96).

Students typically encounter surjective functions for the first time in precalculus. Although they are not necessarily emphasized at this level, surjective functions are important in upper-division courses as bijections and isomorphisms permeate nearly every mathematical domain. My students consistently struggle with proofs of statements involving surjective functions, so as a step toward understanding why, this paper addresses the following research questions: In what ways do students approach proofs of statements involving surjective functions? What are students' semantic understandings of surjective functions?

## Literature Review

Both semantic and syntactic reasoning present students with opportunities and difficulties in proof production. Semantic reasoning can provide a basis for and support the development of a syntactic proof or counterexample by suggesting a main idea or underlying structure (de Villiers, 2010, Moore, 1994; Raman, 2003; Weber & Alcock, 2004). However, students often do not make these connections due to inaccurate or incomplete semantic understanding (Moore, 1994; Tall & Vinner, 1981) or difficulty relating their semantic understanding to a syntactic definition or proof (Raman, 2003; Weber & Alcock, 2009). Additionally, students may use semantic reasoning as a substitute for syntactic proof (Harel & Sowder, 1998, 2007).

When students have such difficulties with semantic reasoning, syntactic reasoning can help them produce proofs even if they do not fully understand them. Understanding may then develop through students' reflection on how syntactic proofs relate to their semantic understanding of the concepts involved (Weber & Alcock, 2009). On the other hand, students' struggles with syntactic reasoning, such as use of imprecise or incomplete definitions (Harel & Sowder, 2009; Vinner, 1983) or failure to use definitions to structure proofs (Harel & Sowder, 2009; Moore, 1994) may limit their ability to construct and understand mathematical proofs.

Both semantic and syntactic reasoning are important in proving, and the affordances above suggest that students may come to understand proving and proof in one of two ways: by using semantic reasoning to build syntactic proofs, or by making sense of syntactic proofs through reflections on their semantic understanding (Weber & Alcock, 2009).

## Method of Inquiry

The data in this paper come from a larger study that investigates students' proofs of statements involving relations and functions.

### Participants

The participants were ten undergraduate students at a public university in Ohio enrolled in a transition to proof course. Six students were secondary mathematics education majors, and one each was a computer science, meteorology, mathematical statistics, and applied mathematics major. Although the course was intended for sophomore level students who had not taken a proof-based mathematics course, only one participant met these criteria. The other students were juniors and seniors with varied levels of experience with proof-based mathematics.

### Course Structure

The transition to proof course was an inquiry-based learning course taught by the author of this paper. The topics in the course were: problem solving, logic, set theory, proof techniques, counting, induction, relations, orderings, functions, and cardinality. Students read about and completed ungraded pre-work on new topics before class. In class, they discussed the pre-work in small groups, followed by whole class discussions and ungraded student presentations. Students had graded post-work due weekly, which could be discussed with others, but write-ups were to be individual. In addition, there were four quizzes, a midterm, and a final exam in class.

### Data

The data come from the assigned coursework in the transition to proof course. Although the students completed a variety of tasks involving surjective functions, this paper focuses specifically on the three tasks below involving composition and surjective functions. Overall, three weeks of class were spent on functions, with four days including study of surjective functions. Surjective functions were introduced the first day on functions with the following definition from Schumacher (2001):

> A function $f: A \to B$ is said to be onto if for each $b \in B$, there is at least one $a \in A$ for which $b = f(a)$. In other words, $f$ is onto if the codomain and the range of $f$ are the same set.

In this definition, I consider the first sentence the syntactic definition and the second sentence a semantic understanding of the definition. The next three days of class focused on injective and surjective functions, composition of functions, and their interactions. Students constructed examples and explored conjectures on the composition of functions with both finite and infinite domains satisfying varied combinations of injective and surjective. The tasks examined in this paper were explored as pre-work and in class before being assigned as post-work and on in-class assessments, but complete solutions were not provided.

**Task 1.** Task 1 was on a post-work assignment due on the fifth day of study of functions.

True or False? If true, prove it; if false, provide a counterexample.

Let $f: A \to B$ and $g: B \to C$ be functions. If the composite function $g \circ f: A \to C$ is onto, then $g$ is onto $C$.

**Task 2.** Task 2 was on an in-class quiz, four class days after the due date for the post-work containing Task 1.

Let $A, B,$ and $C$ be nonempty sets and $f: A \to B$ and $g: B \to C$ be onto functions. State the domain and codomain of $g \circ f$. Prove that $g \circ f$ is onto its codomain.

**Task 3.** Task 3 was on the final exam, four class days after the quiz containing Task 2.

True or False? If true, prove it; if false, provide a counterexample.

Let $f: A \to B$ and $g: B \to C$ be functions. If the composite function $g \circ f: A \to C$ is onto, then $f$ is onto $B$.

## Results

### Task 1

Every student correctly identified the statement in task 1 as true. Seven of the ten students used an indirect proof, but it was often unclear whether they were using proof by contradiction or contrapositive. Not a single student used the word "contradiction," and each indirect argument concluded $g \circ f$ was not onto, many starting similarly to "if $g$ is not onto, then $g \circ f$ is not onto because…." Most subsequent arguments were based on semantic understandings of surjective functions instead of the syntactic definition. It is unclear what provoked the use of an indirect proof strategy, but it aligned almost naturally with their semantic reasoning in this context.

The students demonstrated five different semantic understandings of surjective functions, some specifically in the context of composition, and some which overlapped. Two students' arguments included diagrams such as those discussed in Task 3 below. Non-surjective functions in the diagrams are represented with an element in the codomain that is not in the range. Additionally, two students' arguments expressed this idea in words, specifically speaking of mapping elements:

> Assume $g$ is not onto. If $f$ is onto $B$, then all the elements in $B$ can be mapped back to $A$. When we map $B$ to $C$, not all of the elements of $C$ can be mapped back to $B$. Since $B$ is not onto $C$, $A$ cannot be onto $C$. Therefore $g$ must be onto $C$.

Another student argued similarly to the students above in the language of codomain and range:

> If $g \circ f$ is onto, is $g$ onto? True. This is true because if $g: B \to C$ had an element show up in its codomain that was not in the range, then the mapping from $A \to C$ would contain that same element in its codomain and not its range.

Two students argued on the consequences of $g$ be the last function applied in the composition:

> Suppose not, that $g$ is not onto $C$. Therefore $g \circ f$ would also <u>not</u> be onto $C$. This is because $g$ is the highest level function that provides the final range of the entire composite, and if $g$ can't reach all of $C$, then the composite $g(f(x))$ certainly won't either.

Finally, three students used the idea that $g$ and $g \circ f$ have the same range when $g \circ f$ is onto:

> True because the range of $g$ will also be the range of $g \circ f$. So, if $g \circ f$ is onto, then that means $f$ has its domain and range, the domain of $g$ that has the same elements as range

of $f$ will have a range also, and that range of $g$ of those elements will be the same range of $g \circ f$.

Only two students used the syntactic definition of surjective on task 1. Each student gave a correct proof, with one being a direct proof and the other a proof by contrapositive.

**Task 2**

Nine of the ten students used a direct proof strategy on Task 2, and six students attempted to use the syntactic definition of surjective function. This approach was in stark contrast to students' approach to Task 1. However, in attempting to follow the forward structure of a prototypical direct proof – start with the assumptions and use definitions to work to the conclusion – students missed the backward structure of the definition of surjective and were unable to use it appropriately to structure their proofs. Each proof attempt started in the domain of $g \circ f$ and moved toward the codomain as in this example:

> Let $a \in A, b \in B,$ and $c \in C$. Note $(g \circ f)(x) = g(f(x))$. Since $f$ is onto, $\forall b \in B$, there is an $a \in A$ such that $f(a) = b$. Furthermore, since $g$ is onto, $\forall c \in C, \exists b \in B$ such that $c = g(b)$. Suppose $(g \circ f)(a)$. $(g \circ f)(a) = g\big(f(a)\big) = g(b) = c$. Hence, $g \circ f$ is onto.

For an analysis of the difficulties that lead to this type of proof, see Epp (2009).

Four of the students who used a version of the syntactic definition also used semantic reasoning in their proof attempt, as illustrated in the following example:

> Let $a \in A$. Since $f$ is onto, $\exists b \in B$ such that $\forall a \in A, f(a) = b$. Every value in $B$ is mapped to. Similarly with $g$, $\forall c \in C, \exists b$ such that $g(b) = c$. And since all values in $B$ are mapped to, and $g$ is also onto, all values in $C$ get mapped to. $g \circ f$ is onto $C$.

Finally, four students used semantic arguments only – three based on all elements in the codomain of surjective functions getting mapped to, and one cardinality argument presumably based incorrectly on surjective functions having the same codomain and range.

**Task 3**

Every student correctly identified the statement in Task 3 as false and attempted to construct a counterexample using a diagram to represent the sets and functions as in Figure 1:

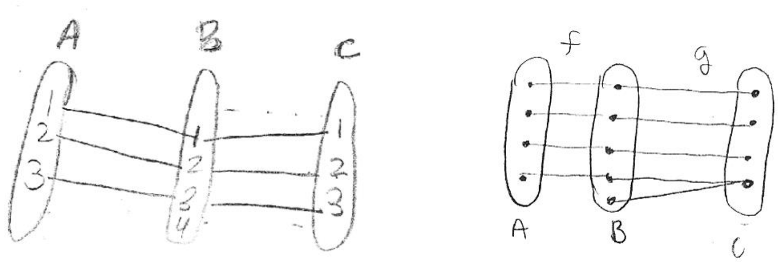


*Figure 1. Sample counterexamples for Task 3*

Three students provided a diagram only, although two of these students circled the element in B that was in the codomain of $f$ but not the range. Four students accompanied their diagram with some version of the statement "$g \circ f$ is onto, but $f$ is not onto." The other three students included explanations with their diagrams. One student used the syntactic definition of surjective and reasoned semantically about mapping elements in their explanation:

Assume $g \circ f$ is onto, this means for each $c \in C$, there is at least one $a \in A$ for which $c = g \circ f(a)$. This means each $c$ must map to a $b \in B$ so $c$ can map to $a$. But there does not need to be an $a \in A$ for which $b = f(a)$ for every $b$ as long as there is a path from $c \in C$ to $a \in A$.

This student's diagram was similar to the diagram on the right in Figure 1. The other two students used only semantic reasoning with their diagrams, with one student arguing that there was an element in $B$ that was "unmapped by any element in $A$" and the other student using an argument based on the cardinality of $B$ being greater than the cardinality of both $A$ and $C$. Although each student's counterexample correctly showed that $g \circ f$ was onto and that $f$ was not onto, $f$ and/or $g$ were not functions in half of the students' counterexamples, as is shown in the example on the right in Figure 1.

## Discussion

The discussion will focus on two promising results: (1) Most students exhibited valid and useful semantic understandings of surjective functions (2) Some students tried to use their semantic understanding to make sense of the syntactic definition of surjective.

### Semantic Understanding of Surjective Functions

Every student in this study demonstrated at least one semantic understanding of surjective functions, notably, some form of the diagram in Figure 1. Eight students exhibited at least one other semantic understanding. For the other two students, one used diagrams on every task, displaying no other semantic or syntactic understandings, and the other used the syntactic definition on Tasks 1 and 2. Overall, the students' semantic reasoning about surjective functions was correct and useful in arguing for the truth of statements and constructing counterexamples.

In addition to the diagram the students exhibited the following semantic understandings of surjective functions: having the same codomain and range, and all elements in the codomain being mapped to. Additionally, students easily negated these concepts for semantic understandings of non-surjective functions: diagrams with unmapped elements, elements in the codomain not being mapped to, and unequal codomains and ranges. Although these are simply different representations of the same concept, only one student demonstrated all three semantic understandings. It would be interesting to see if students could recognize and articulate the connections between these semantic understandings.

Finally, the students reasoned semantically about surjective functions specifically in the context of composition, including: considering the impact of which function was applied last in the composition; using the fact that $g$ and $g \circ f$ have the same codomain; using diagrams representing both surjective functions and composition; and arguing using cardinality as mentioned above in Task 3.

### Connecting Semantic and Syntactic Reasoning

On Tasks 1 and 3, only one student used both semantic and syntactic reasoning about surjective functions, but most students demonstrated useful semantic understandings on which they could build. However, on Task 2, as six students struggled to use the syntactic definition of surjective, four of them included semantic reasoning in their proofs to try to make sense of and connect to the syntactic definition. With more time and practice, these students' semantic understandings have the potential to be valuable in helping them understand the syntactic definition and structure of proofs of surjective functions.

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A Framework for Analyzing Written Curriculum from a Shape-Thinking and (Co)variational Reasoning Perspective

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*This preliminary study provides a framework to analyze the extent and nature of (co)variational and quantitative reasoning in written curriculum. In order to test and refine our framework, we examined both the narratives and worked examples in calculus textbooks on lessons dealing with the topic of functions. We present examples from those textbooks to illustrate the categories of our framework. We conclude with questions concerning potential areas to improve our framework.*

*Keywords:* Textbook Analysis, Quantitative Reasoning, Covariational Reasoning, Calculus

Over the past couple decades, researchers have studied students' quantitative and covariational reasoning – the cognitive activities in which students conceive of measurable attributes varying in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) – and have categorized specific forms of reasoning. They argue its importance to understanding numerous K-12 topics (Carlson et al., 2002), and yet "many popular U.S. textbooks do not emphasize or support students in conceptualizing quantities and viewing function formulas and graphs as representing how two varying quantities change together" (Thompson & Carlson, 2017, p. 457). Paoletti, Rahman, Vishnubhotla, Seventko, and Basu (in press) have started analyzing graphs used in STEM textbooks and journal articles. Thompson and Carlson (2017) and Mesa and Goldstein (2014) reported how secondary level precalculus textbooks addressed the conception of function and inverse trigonometric function, respectively, in their textbook reviews. However, we were unable to find any textbook analysis frameworks that attend to the degree to which textbook narratives and worked examples provide students with the opportunity to conceptualize quantities or reason (co)variationally.

In this report, we describe our attempt to create such a framework. To do so, we adapt two cognitive-focused categorizations – Moore and Thompson's (2015) shape thinking constructs for graphs and Thompson & Carlson's (2017) variational and covariational reasoning framework into categorizations appropriate for analyzing static curricular materials. In order to test and refine our framework, we analyzed calculus textbooks sections readily available to us. We specifically analyzed the introductory material to calculus textbooks (i.e., the pre-calculus topics the authors included) because it provides insights into the conceptions of graphs, functions, etc. the textbook authors believe are foundational for students to have before entering calculus. In this paper, we introduce the framework with specific examples from our analysis.

## Background and Rationale

Two main sources – shape thinking constructs (Moore & Thompson, 2015) and the framework for variational and covariational reasoning (Thompson & Carlson, 2017) – informed our construction of a framework that enables users to analyze the narratives and worked examples when textbook authors explain, define, or use terms, expressions, formulas, and graphs. Firstly, as we describe in more detail when introducing the framework, we adapt the shape thinking construct to analyze the extent to which the narratives and worked examples provide students with opportunities to develop quantitative and covariational reasoning.

Secondly, Thompson and Carlson's (2017) frameworks for variational reasoning and covariational reasoning enabled us to distinguish between various levels of covariation in the narratives and examples provided in the curriculum. We also benefit from other research (i.e., Carlson & Oehrtman, 2005; Cooney & Wilson, 1993; Confrey & Smith, 1994) to construct our framework. The structure in which those researchers provided various levels of understanding functions (i.e., correspondence vs. process/covariation view of functions) was useful in providing a way to analyze algebraically and geometrically defined functions in the narratives to determine how they promote opportunities for students to understand and use functions as values of two variables or quantities covarying.

Researchers have demonstrated that textbooks have significant influence on student learning and teacher practice (Begle, 1973; Schmidt, McKnight, & Raizen, 1997; Kilpatrick, Swafford, & Findell, 2001; Stein et al., 2007; Valverde, Bianchi, & Wolfe, 2002). For example, Kilpatrick et al. (2001) stated that "what is actually taught in classrooms is strongly influenced by the available textbooks" (p. 36). In particular, Carlson, Oehrtman, and Engelke (2010) showed a positive influence of a curriculum (i.e., *Precalculus: Pathways to Calculus*) on students' productive understanding of functions. They reported that students who completed a curriculum focused on quantities and their covariation scored significantly higher on the *Precalculus Concept Assessment* at the end of the course than at the beginning. However, given the important role of textbooks in students' learning and classroom instruction, there is limited investigation regarding how (co)variational reasoning is promoted in textbooks. Hence, we decided to develop a framework to analyze curriculum materials in order to determine the extent and nature of (co)variational reasoning provided students in the narrative and worked examples.

## Framework

We had two main categories in our framework: static and emergent. We illustrate each of these categories along with examples from five calculus textbooks we investigated.

### Static

We use the term *static* to refer to any instances of narratives or worked examples that do not reference quantities and relationships among them in ways that entail those quantities[1] varying. For example, we code things as *static* when they entail instances that provide students images of variables and formulas based on perceptual associations among visual shape, analytic form, and perceptual features. *Static* instances encountered in the narratives and worked examples during our initial work fell into several categories (see Table 1).

**Perceptual Associations**. One category is what we call *perceptual associations*. This category has subcategories (i.e., form-name, form-shape, shape-name, and property-shape associations). *Form-name associations* involve perceptual associations between an analytic form and a function class terminology (e.g., linear, quadratic, or exponential). We adapted this particular category from Moore and Thompson's (2015) shape thinking construct to account for representations that were not graphs but still seemed associated with a particular form of an equation. For example, the following description of linear function was provided based on its analytic form without giving attention to an invariant relationship between the variables $x$ and $f(x)$ that change together: A function of the form $f(x)=ax+b$ is called a linear function (Larson & Edwards, 2010, p. 24; Rogawski, 2012, p. 13). *Form-shape associations* involved perceptual

[1] We use quantities here to refer to both a quantity's magnitude and a quantity's value. We return to this idea in the discussion section.

associations between an analytic form and shape of graph. For example, Johnston & Matthews (2002, p. 20) describe "a nonvertical line in the Cartesian plane, or ($x$, $y$) plane, can be described by an equation of the form $y$=$mx$+b" with little or no attention to the coordinate system or axes' scales and no attention to the invariant relationship between variables $x$ and $y$ as they vary. We also recorded instances as *form-shape associations* when the narratives provide a perceptual association between a change in a parameter in the analytic form and a change in the shape of graph. For example, Edwards and Penney (2014) stated "[the] *size* of the coefficient $a$ in Eq. (9) [$y=ax^2$] determines the 'width' of the parabola; its *sign* determines the direction in which the parabola opens" (p. 18). *Shape-name associations* involve perceptual associations between the shapes of graphs (e.g., "line" or "curve up") and a specific function class terminology or name of a mathematical object. For example, Steward (2008) stated, "When we say that $y$ is a linear function of $x$, we mean that the graph of the function is a line" (p. 24). *Property-shape associations* involve perceptual associations between the shape of graph and a feature of the graph (e.g., slope). For example, Rogawski (2012) provided a pair of parallel lines on a Cartesian coordinate axis that were not scaled and labeled, and stated that "[l]ines of slopes $m_1$ and $m_2$ are parallel if and only if $m_1 = m_2$" (p. 14–15)—with no attention to changes in one variable with respect to changes in another variable by considering the axes' scales or orientations.

*Table 1. Framework for coding the extent and nature (co)variation provided in the narrative of a lesson*

| **Static** | **Emergent** | |
|---|---|---|
| | Variation | Covariation |
| Perceptual associations | Continuous | Continuous |
| • Form-Name Associations | Gross | Gross Coordination of Values |
| • Form-Shape Associations | Discrete | Coordination of Values |
| • Name-Shape Associations | | |
| • Property-Shape Associations | | |
| Variable as unknown | | |
| Correspondence | | |

**Variables as Unknown.** A second category of *static*, named *variables as unknown*, involves presenting a variable as having a fixed unknown value of a quantity or being only a visual symbol that is not varying in the way Thompson and Carlson (2017) categorized as "no variation" and "variable as symbol." One example in Stewart (2008) is to "express the cost [$C$] of materials as a function of the width of the base" of a rectangular storage container (p. 15). In the solution, they note $w$ and $2w$ as "the width and length of the base, respectively, and $h$ be the height" and "the area of the base is $(2w)(w)=2w^2$" without considering $h$ or $w$ as varying. Furthermore, they use these variables to write an equation for $C$, also indicating a treatment of $C$ as an unknown variable whose unique value needs to be computed.

**Correspondence.** We used the code *correspondence,* the third category under static, when the narratives provide instances in which there exists an established static link among numbers in sets, but there is no consideration of either the covariation of variables or the dynamic relationship between number of sets (Cooney & Wilson, 1993; Vinner & Dreyfus, 1989). We also coded instances as correspondence when they simply provide a rule for students to calculate a unique value of a variable or quantity by using any given value of another variable or quantity (Confrey & Smith, 1994). For example, Edwards and Penney (2014) provide the following definition of a function: "A real-valued function [*originally bolded*] $f$ defined on a set $D$ of real numbers is a rule that assigns to each number in $x$ in $D$ exactly one real number, denoted by $f(x)$" (p. 2). This definition is common across all the textbooks we investigated. These definitions do not provide a *process* view of function of how input values covary with output values, emphasizing the change over a continuum of values (Carlson & Oehrtman, 2005).

**Emergent**

The other main category in our framework, named *emergent,* identifies the narratives and worked examples representing various levels of varying and covarying quantities or variables based on Thompson and Carlson's (2017) outline of levels of reasoning variationally and covariationally. We adjusted those levels to fit a textbook analysis and used them as our sub codes under *emergent* to determine the level of opportunities provided in a written curriculum for students to develop quantitative and covariational reasoning. We acknowledge that Thompson and Carlson included smooth and chunky distinction for both variational and covariational reasoning. For the purposes of this framework, however, we chose not distinguish between chunky and smooth continuous (co)variation. We made this decision because of the research (Castillo-Garsow, Johnson, & Moore, 2013) done to indicate that it is the *student* who conceives of a situation as either chunky or smooth, and in our preliminary analysis, we did not find a narrative or example that attempted to distinguish between the two.

**Covariational Reasoning (Thompson & Carlson, 2017).** The other part of our framework under *emergent* outlines the level of opportunities to develop covariational reasoning. *Gross coordination of values* involves representing two variables or quantities whose values increase or decrease together without mentioning the individual values of variables as varying together in the narratives. For example, "As the independent variable $x$ changes, or varies, then so does the dependent variable $y$" (Edwards & Penny, 2014, p. 3). *Coordination of values* involves instances of coordinating the values of one variable or quantity with values of another by providing specific and discrete pairs of values without providing the opportunity for students to conceive two variables or quantities whose value varies together in between those pairs of values. For example, a narrative in Edwards & Penny's (2014) box problem wants students to "[s]tart by expressing the box's volume $V = f(x)$ as a function of its height $x$, and then use the method of repeated tabulation to find the maximum value $V_{max}$" (p. 12). Here, the textbook offers various values for $x$ and asks for students to find their corresponding values in order to determine when the value of $V$ will be maximum. *Continuous covariation* involves instances providing a simultaneous and continuous change in the values of two variables or quantities. We do not have an example of this category; however, the narrative presented as an example for coordination of values would have been an example of continuous covariation if each of the functional representations was linked to a motion in a dynamic geometry software with a slider for students to change the value of $x$ and simultaneously see the corresponding changes in the values of $V$ continuously.

**Variational Reasoning (Thompson & Carlson, 2017)**. One part of our framework under *emergent* outlines the level of opportunities to develop variational reasoning (i.e., discrete, gross, continuous variation). *Discrete variation* involves presenting a variable or quantity in the narratives as taking specific values, but without providing the opportunity for students to conceive the variable or quantity whose value varies in between those specific values. *Gross variation* involves presenting a variable or quantity whose values increase or decrease without mentioning the specific values of the variable or quantity while increasing or decreasing in the narratives. C*ontinuous variation* involves presenting a variable or quantity whose values increase or decrease continuously. We do not provide examples from textbooks for each category of variational reasoning because they can be seen in the examples for covariational reasoning. For example, we can see discrete variation in Edwards and Penny's (2014) box problem when they ask students to change the values of $x$ to find the maximum values of volume. Here, the textbook provides the variable $x$ as having specific values but without considering how its value varies in between those specific values.

## Discussion

The aforementioned research upon which we base our framework describes how students think about quantities. We recognize that the frameworks we chose to adapt were cognitive in nature, and we contend that students conceptualize the written materials differently. For example, as Thompson and Carlson (2017) pointed out, "A variable's variation comes from *a person thinking* [emphasis added], either concretely or abstractly, that the quantity whose value the *letter* [emphasis added] represents has a value that varies" (p. 425). In other words, we cannot know whether a student will interpret a variable provided in a written curriculum as varying, a letter that has a fixed value, or as a symbol. Nevertheless, the curriculum (including textbooks) students receive will influence how students think and learn. Thus, although there will invariably be discrepancies in the intended curriculum, the written curriculum, and what students interpret from the written curriculum, we argue certain narratives promote ways of reasoning that scaffold students in a way that supports reasoning covariationally. Hence, we are constructing a framework to determine which conceptualization students are likely to have based on what we see as evidence from written curricula.

In our framework, we did not include the attention to the distinction between quantities' values and quantities' magnitudes because we have not seen any instances from textbooks representing this distinction. We note that the most sophisticated version of quantitative and covariational reasoning includes explicit attention to such difference (Ellis, 2007; Thompson & Carlson, 2017). Even though textbooks provide opportunities for students to develop productive ways of thinking about quantities that were identified in this study, we found it unfortunate how little evidence we found of curriculum materials intentionally supporting student development of sophisticated quantitative and covariational reasoning schemas. This missing emphasis in written materials may be a partial explanation for why researchers have identified students having difficulties with ideas such as rate of change (Carlson et al., 2002)

To conclude this report, we identify some of the challenges we experienced in developing the framework. For instance, when coding the narratives and worked examples, we determined any instance the textbook seemed to promote (co)variational reasoning or to develop static meanings for quantities and variables. The units of analysis varied from phrases, to sentences, to whole paragraphs, to specific representations of mathematical objects (e.g., graphs, tables, etc.), but we would like to define parameters for our unit of analysis. We will provide examples of how our current unit of analysis influences how we code specific textbook examples and narratives. Lastly, we have been considering various methods of reporting and further analyzing the data (e.g., by textbook vs. by topic, international vs. national) and discussing affordances each offers.

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**Impacts of Peer Mentorship in a Calculus Workshop on the Mentors' Identities and Academic Experiences in Undergraduate STEM**

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*Research has shown the positive impact of peer mentorship on the educational experiences of mentored students from underrepresented backgrounds. National survey data of peer leaders indicate that peer mentors also benefit from the mentoring experience. This report unpacks this survey finding related to peer mentors' increase in a sense of belonging in college and academic persistence as a result of participating in the mentorship. Our data draws from interviews with six historically marginalized students of color after their participation as mentors for a group of first-year calculus students during a summer bridge calculus workshop. The mentors' main responsibility was facilitating critical conversations about racial and gender in Science, Technology, Engineering, and Mathematics (STEM). Preliminary analyses found that mentoring contributed to their confidence in succeeding in a STEM field and their ability to make sense of gendered and racialized educational experiences in STEM.*

*Keywords:* equity, identity, peer mentors, STEM

Research has documented the benefits of mentorship in facilitating persistence of students of color in Science, Technology, Engineering, and Mathematics (STEM; Griffin et al., 2010). Some research has shown that, second to faculty mentoring, peer involvement is the strongest predictor of African American and Hispanic students' academic performance and educational satisfaction (Cole, 2008). Some qualitative studies have also documented the desire for many successful students of color in STEM to give back and be role models in their communities (e.g., Ellington & Frederick, 2010; McGee & Martin, 2011). Peer mentors can be defined as, "students who have been selected and trained to offer educational services to their peers [that] are intentionally designed to assist in the adjustment, satisfaction, and persistence of students toward attainment of their educational goals" (Ender & Kay, 2001, p.1 as cited in Shook & Keup, 2012). Benefits of peer mentorship on students being mentored documented in the literature include: the development of communities and relationships with students of similar identities, and the sharing of resources among students (Shook & Keup, 2012). Higher education research has also explored the benefits of peer mentorship on the institution (e.g., cost-effective student support). For this paper, we focus on the benefits of peer mentorship on the mentors themselves.

The National Peer Leadership Survey by the National Resource Center for The First-Year Experience and Students in Transition in 2009 found that, among the nearly two thousand students in peer leadership programs in different institutions, 81 percent of them said that their experience increased feeling of belonging at the institution. Seventy-one percent indicated an increase in their desire to persist academically (Shook & Keup, 2012). This paper aims to unpack those findings from the survey. In particular, we want to explore possible mechanisms behind the increase in sense of belonging and desire to persist in the discipline. Thus, in this report, we focus on the following research question: How does participating in a peer mentorship program about race and gender in STEM impact the mentors' STEM identities and participation?

## Theoretical Frameworks

This paper employs a sociopolitical perspective, which focuses on investigating some of the accepted norms and practices within the field of mathematics that privilege some people while excluding others (Gutiérrez, 2013; Valero, 2004). Gutiérrez (2013) explains that adopting a sociopolitical perspective involves considering the interrelatedness of knowledge, power, identity, and social discourse. This is to say that power and positioning as a result of existing narratives about groups of students impact the way they see themselves as learners and how they learn. Past research in mathematics education has illustrated how students' identity constructions shape participation and vice versa in mathematics learning specifically (Boaler & Greeno, 20002; Esmonde, Brodie, Dookie, & Takeuchi, 2009; Cobb, Gresalfi, & Hodge, 2006; Langer-Osuna, 2011; Martin, 2000; Oppland-Cordell, 2014). We build on these insights to explore the connections between identities and participation in STEM and the more informal learning context of peer mentoring.

Martin (2000) presented a multi-level framework on sociohistorical, community, school, and intrapersonal influences on African American students' racialized opportunities for mathematical participation and co-constructions of mathematics and social identities. Martin (2000) defined *mathematics identity* as individuals' beliefs about "their ability to perform in mathematical contexts, the instrumental importance of mathematical knowledge, constraints and opportunities in mathematical contexts, and the resulting motivations and strategies used to obtain mathematics knowledge" (p. 21). Our analysis is framed by an extension of Martin's (2000) framework and definition to allow for consideration of how other social identities (e.g., gender) intersect with race to shape peer mentors' identities and participation in STEM.

## Methods

### Context

The Calculus summer workshop was originally designed to increase the representation of underrepresented minority students in the mathematics major. The five-day summer calculus workshop prepares students for their first calculus course in college, highlights productive study skills, and builds a peer support network. Students also receive individual academic advising. As part of the workshop, students engaged in five critical conversations about race, gender, and STEM. The curriculum for the conversations was designed in collaboration with the Dean of Students on Diversity and Inclusion. Two conversations were held during the five-day workshop, two in the fall semester, and the final session in the early spring semester. The conversations served two goals. First, they aimed to empower students with language and tools to make sense of and navigate racialized and gendered experiences they might encounter in being a STEM major (McGee & Stovall, 2015). Second, the conversations were opportunities for students to check in about their first semester in college and their calculus course. Topics for the conversations included students' hopes and fears about their first semester, the importance of a STEM network, and stereotype threat and management.

### Data Source

This study is part of a larger study investigating the impact of the summer workshop and critical conversations on participants' personal and academic success. Eight peer mentors received training in facilitating conversations about race and gender in STEM. We recruited these students from different cultural centers on campus. Their training mainly involved engaging students in activities that their mentees would complete. They also received some training on opening and facilitating discussions. Six of the eight peer mentors for the workshop participated in an individual 60-90 minute exit interview at the end of their participation. Their background information is provided in Table 1. In addition, as part of their training for the 2017 workshop, they briefly reflected as a group on their experiences as a peer mentor in the 2016 workshop.

*Table 1. Peer Mentors Personal and Academic Background*

| Student | Racial/Ethnic Background | Academic Major | Grade Level |
|---|---|---|---|
| Fernando | Hispanic/Latinx | Physiology[a] | Sophomore |
| Hamza | African American/ Black | Physiology | Junior |
| Hugo | Hispanic/Latinx | Biochemistry & Molecular and Cell Biology | Senior |
| Lorena | Hispanic/Latinx | Physiology | Sophomore |
| Sarah | Hispanic/Latinx | Biochemistry | Sophomore |
| Sabrina | Hispanic/Latinx & White (Non-Hispanic) | Physiology | Junior |

[a]Most students planning to attend medical school major in Physiology

We were able to observe impacts of their participation as students reflected on their own STEM learning experiences and related them to their social identities. We looked for statements where there was a clear attribution of a change in behavior to participation in the peer mentorship program. We transcribed all of the interviews. The first author of this report is the interviewer in the study.

## Analysis

We specifically focus on the impacts of the peer mentorship on the way students perceived their experiences and participation in the STEM field. Some themes emerged from our analysis. The critical conversations that these students facilitated focused on empowering students with language and tools to make sense of and navigate any racialized and gendered experiences. Most of the themes that emerged related to students' identities and sense-making of their own racialized and gendered STEM experiences.

One of the documented benefits of peer mentorship for mentored students was that it provided them opportunities to connect with mentors who shared similar social identities. We found that the peer mentors experienced similar benefit. Hamza spoke about the impact of the peer mentorship and critical conversations on him as a Somali-American:

> It was impactful because it was the first type of conversation I was able to have on campus here at the University. It was special for me. Myself, I am going through social identities and finding out who I am /…/ I don't get to have conversations with other immigrants because of distinct language barrier. I didn't grow up knowing Somali. Me and like all the other refugees that came, there is that distinct separation. When my parents came in the 1970s and other families came in 2002 and 2001. So I, there is a very big disconnect between that point and I was able to have the discussion, yeah this is what it's like. I really liked having our conversation.

There happened to be two Somali students in the workshop that summer, and it provided Hamza with a particularly powerful experience. Lorena shared a similar experience of resonating with a

Hispanic mentee's feelings of isolation on September 16th, Mexican Independence Day. The mentee shared with Lorena that he noticed that no one in class knew the significance of that day.

Five of the six students mentioned at least one specific racialized and gendered experience that they identified after participating in the program. Sabrina shared her awareness of being the only woman of color in her research lab. In her reflection, she spoke about how the training she received as part of this program helped her make sense of this experience and allowed her to learn from and relate to another peer mentor.

*Sabrina:* Especially in a lab or something and I think I have thought about it more because of this peer mentor stuff because the first time I walked in there I was like oh this is what doctor Adi and everyone was talking about. This is weird [*laughs*]. So I have never had an experience like that. So that was just so weird on the very first day.

*Interviewer:* So we had a conversation like this? Was before the training or was it during the training?

*Sabrina:* Yeah. It was during the training. I don't remember who it is, but someone was saying, it might have been Hamza, how he was saying how he was the only Black person in a sea of White people. I think it was one of his classes. I never actually had to deal with being the only person of some sort in a sea of different people. That was the first instance where I was the only person. So I understood how he felt on the gender spectrum of it. I feel like that's something he has to deal with all the time especially here at [the university]. So, I was thinking about that couple days ago. I hope he doesn't always feel like this and coping with it and dealing with it one way or another because this is a terrible feeling. It's not a good one.

*Interviewer:* It really isn't. How has this experience helped you with your experience? The first time you recognized it but do you do anything about it. Do you do anything different?

*Sabrina:* I almost do it where we're in the lab meetings and almost intently make sure that everyone knows I'm listening like I'm nodding my head like, oh yeah. So it's almost like I want them know that I'm engaged in what they're saying and not just some undergrad who is there to not just get credit or just working in a lab, because I don't want that perception to go back on me.

McGee and Martin (2011) have shared similar accounts from other students of color in managing stereotypes by staying on top of things, sitting in front of the class, and appearing engaged. What is powerful about Sabrina's learning is that her awareness led to this change in behavior, but it also allowed her to empathize with other students' experiences at the university.

Lorena provides another example of a mentor making sense of a racialized STEM experience. Initially in her reflection she could not articulate what bothered her about seeing students who did not put in as much effort into their education continuing to be in her program. The interview was the first time she made sense of it. After recalling a conversation she had while walking to class with some students, she came to this conclusion:

> You know, my parents did not go to college. I'm totally here because of me. It's 100% me because I'm very smart and all these things. It is true and I don't know if it's a bad thing or it is, I'm saying true things. These people, it's not a bad thing that their parents are educated. I hope to be a very educated parent living financially stable and comfortable. But does the way that they [pause]. See, I'm talking about they. You know certain types of groups [pause]. You know like I'm just so thankful and that I can do all of this. To them it's like, "Oh. All I need to get is like a D so that my dad can keep paying

for college." It's like you're paying 12,000 dollars so that you can go to fun parties. And that's kind of my stereotype to that group of people. It's like you're doing all of that and I'm doing all this and we're still on the same track.

In this quote, Lorena not only positioned herself as a smart and resourceful student, but she also challenged the lack of consequence to the more privileged students who did not work as hard as her. This brings us to the next pattern we observed in our data: a change in self-perception and perspective as STEM students.

During their group reflection, Fernando spoke about an increase in confidence in being a STEM student as a result of his ability to give advice on the spot to the students.

*Fernando:* You've been through enough and you've done well enough. You can provide that information to help other students succeed. It gives you confidence.

*Interviewer:* Others feel free to chime in. Do you realize you're successful *after* you give the advice, or before you give the advice? Or the act of giving advice, "Oh I didn't know I could do that!"

*Fernando:* I'd say during.

*Lorena*: Yeah.

*Fernando*: They asked you a question. And it's not like you prepared for these questions. And they asked you. And you're like oh [*everyone laughs*].

*Interviewer:* Lorena is patting herself on the back.

Fernando and Lorena shared this increase in confidence as a STEM student. Hugo provided a different impact on his participation. He shared that, leading up to the peer mentorship, he had felt worn out by his class work. Interacting with students in the mentorship program gave him "more positive energy" to finish his studies.

## Discussion and Implications

The aim of this report is to unpack some of the findings from the National Peer Leadership Survey (2009). In particular, we were interested in understanding the impact of a peer mentorship program on the peer mentors' identities and participation in STEM. We found that the program impacted the peer mentors by: 1) providing them opportunities to connect with other students who shared their social identities; 2) helping them recognize and make sense of their own racialized and gendered experiences as STEM students; 3) giving them a new perspective and confidence as STEM students. We note that Fernando, Hamza, and Hugo were inspired and acquired other leadership positions on campus as a result of their participation in this program.

This report has implications on the learning and teaching of undergraduate mathematics. The aim of these conversations is to provide both the peer mentors and the calculus students a space to process racialized and gendered experiences in their STEM education. The calculus workshop and the critical conversations serve as academic and social forms of support in students' STEM educational experiences. How can we intentionally and systematically link these initiatives with calculus courses across mathematics programs? How might we extend this work to the training of graduate students as future instructors? We hope to discuss these questions during the session.

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## Appendix

### Peer Mentor Exit Interview Protocol

1. Tell us about your experiences as a peer mentor?

   a. Describe an impactful experience
   b. Describe a challenging experience
2. How would you describe the STEM atmosphere at [the university]?
3. What successes have you had in your major?
4. What challenges have you had in your major?
5. Is there anything that set you apart from other students or students of color/female students in your program?
6. How do you think your different identities impact your experiences in STEM? Be sure to ask about positives.
7. Re-answer the application questions
   a. Have you ever had an experience in a STEM class where you were made aware of your race and/or gender? If so, how did you respond?
   b. Imagine you come across a 1st year student of color who is interested in becoming a STEM major. What types of advice would you give them to be successful at [the university] and in the major. Feel free to assume that they are interested in your major.
   c. Imagine you come across a 1st year student who is interested in becoming a STEM major. What types of advice would you give them to be successful at [the university] and in the major. Feel free to assume that they are interested in your major.

Validation of an Assessment for Introductory Linear Algebra Courses

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*Research-based and validated open-ended assessments are useful tools to explore students' reasoning and understanding of a subject. The primary goal of this study is to validate an assessment which can accurately measure students' conceptual understanding of four focal topics, typically covered in an introductory linear algebra course; span and linear independence, systems of linear equations, linear transformations, and eigenvalues and eigenvectors. I used the assessment data of 255 students, from nine linear algebra classes at eight different institutes across the country to validate the assessment. By administering the assessment in their classes, linear algebra instructors can gauge their students' conceptual understanding of linear algebra concepts and can identify the concepts which are generally vexatious for students.*

*Key Words:* Assessment Validation, Linear Algebra, Conceptual Understanding

I draw on the research that has developed and validated assessments of student understanding at the undergraduate level in the areas of physics, calculus, and abstract algebra to inform the process of validation for the linear algebra assessment. A review of the literature on assessment development and validation has revealed that there are few research-based instruments available for the assessment of students' reasoning and conceptual understanding in undergraduate mathematics courses. Additionally, there is no reliable instrument available for large-scale usage which can measure students' conceptual understanding of linear algebra.

Math education literature privileges conceptual understanding of mathematics and identifies a disconnect between students' conceptual understanding and their ability to follow a procedure to produce correct answers. Students should learn mathematics with conceptual understanding and they should actively build new knowledge from their prior experience and knowledge (NCTM Principles and Standards for School Mathematics, 2000). Understanding mathematical concepts are critical in advanced mathematics but not trivial (Melhuish, 2015).

> "Conceptual knowledge is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information…Procedural knowledge consists of rules or procedures for solving mathematical problems. It is also a familiarity with the individual symbols system and with the syntactic conventions for acceptable configurations of symbols (Hiebert & Lefevre, 1986, pp. 3,7)."

However, attending to procedures is not only about memorizing the list of steps to solve problems (Star, 2005; Hassenbrank & Hodgson, 2007). Researchers have also provided evidence that both methodologies of teaching concepts and teaching procedures can be integrated. Keene & Fortune (2016) have advocated the importance of the connection between teaching concepts and teaching procedures and proposed a framework for Relational Understanding of Procedures, the framework can help instructors to merge both types of teaching to help students better learn the subject matter.

The goal here is not to discuss the artificial dichotomy between procedural and conceptual knowledge but to support the argument that the connection between both is important for better learning. However, sometimes instructors can neglect the testing of conceptual understanding in regular class assessments and focus only on procedural questions (Tallman & Carlson, 2012). Therefore, assessments must be designed carefully to ensure that students have attended to the concepts in the course along with procedural efficiency.

Validated and reliable assessments are not only to assign grades, but many other important goals can be achieved through the assessment results. Teachers, principals, researchers, and organizations can use validated assessments for different purposes; for example, to provide feedback to students on their learning, to use as a diagnostic tool, to inform a placement criteria, to motivate students, to design and adjust student instructional activities, to distinguish between high and low performing students, to evaluate instructional innovations, and to sense overall performance of students, teachers, and organizations (e.g., Brown & Knight, 1994; Gibbs, 2003; Hestenes, Wells, & Swackhamer, 1992). Sainsbury and Walker (2007) argued that tests can also help students in focusing their attention and drive their learning process in the right direction. Additionally, a validated instrument can also evaluate efforts to improve learning and can help researchers to measure the quality and achievements of instructional innovations (Melhuish, 2015).

Research-based validated instruments are required to measure students' learning accurately. The central goal of this work is to validate an assessment which is sensitive to students’ ways of reasoning and understanding of linear algebra concepts.

## Literature Review

In this section, I organize my summary of the literature into two main categories: studies that focus on describing different phases of assessment development and studies that focus on processes for assessment validation. Most assessment development studies also discuss validation, but the focus of the studies remain to elaborate different phases of assessment development (e.g., Carlson, Oehrtman, & Engelke, 2010; Melhuish, 2015; Sadaghiani, Miller, Pollock, & Rehn, 2013). Similarly, the validation studies also briefly describe the process of their assessment development (Barniol & Zavala, 2014; Wilcox & Pollock, 2014). Since the goal of this study is to validate the linear algebra assessment, consulting the literature on assessment validation is critically important to driving my work.

Concept inventory is another assessment development and validation approach which is gaining popularity in undergraduate STEM areas. While there is no universally accepted definition of a concept inventory, Epstein (2013) defined concept inventory as a test of a student’s most basic conceptual comprehension of a subject’s foundations, not the computational skills involved. Concept inventories measure only conceptual understanding and usually concentrate on specific topics within the course curriculum (e.g., Halloun & Hestenes, 1985a, 1985b; Hestenes & Wells, 1992; Hestenes, Wells, & Swackhammer, 1992). Although all initial work on concept inventory was in the field of physics, recently researchers have developed some concept inventories for Pre-Calculus, Calculus, and Abstract Algebra (Carlson, Oehrtman, & Engelke, 2010; Epstein, 2013; Melhuish, 2015).

It is worthwhile to do the challenging work of assessment development and validation because validated assessments produce more accurate results of students' learning than usual classroom assessments. Typically, classroom assessments are loosely structured and have several limitations including a) the instructor's expertise in the subject, b) amount of time the teacher can invest to administer, grade, and provide feedback to students, and c) performance of students in one section of a course cannot be compared with the performance of students in another section of the same course (Thissen-Roe, Hunt, and Minstrell, 2004).

Assessment validation studies usually focus on validity, reliability, and discriminatory power of the entire test and individual items on the test (Barniol & Zavala, 2014). Validity is the extent to which an instrument can measure what it is supposed to measure. To establish the validity of a test, researchers use a variety of validation techniques. Content validity is a measure of how accurately test items covered the content domain the test planned to cover, and reliability of a test is the likelihood that the test will produce consistent results repeatedly (Crocker & Algina, 2008). Cronbach's alpha is a well-known statistical method to determine how closely related a set of items are as a group. A reliability index of 0.7 and higher indicates that the test is reliable for group measures. Discriminatory power is the characteristic of a test to differentiate among high and low achievers. Some statistical analysis can also determine the quality of individual items on the test. Researchers typically determine item difficulty index and item discrimination power of individual items of assessment to validate the assessment (e.g., Barniol & Zavala, 2014; Gleason, White, Thomas, Bagley, & Rice, 2015; Wilcox & Pollock, 2014). Item discrimination is the ability of an item to differentiate between high achieving and low achieving students by establishing a relationship between how well students performed on the item and their total score on the exam (Crocker & Algina, 2008).

My review of the literature on assessment development and validation revealed that there is no valid instrument available to assess conceptual understanding of undergraduate linear algebra topics. Therefore, the purpose of this work is to validate a linear algebra assessment tool which will focus on specific concepts of linear algebra and instructors can use it in their classes.

## Data Sources

Our research group has collected the assessment data as part of a broader NSF project which aimed to develop and assess a system of support for undergraduate mathematics instructors interested in teaching in inquiry-oriented ways. The project supported three subject areas: abstract algebra, differential equations, and linear algebra. The project provided participating instructors three types of support: a summer workshop, inquiry-oriented linear algebra (Inquiry-Oriented Linear Algebra IOLA; http://iola.math.vt.edu) teaching material, and a weekly online instructors' work groups (Bouhjar, Andrews-Larson, Haider, & Zandieh 2017). Previously, a team of mathematician and math educators has developed the IOLA instructional materials. The IOLA covers four major topics span, linear dependence and independence, transformations, and eigenvalues eigenvectors (Wawro, Rasmussen, Zandieh, & Larson, 2013). Linear algebra instructors usually cover these topics in introductory linear algebra classes, and the linear algebra assessment was developed to cover these four topics.

The goal of the NSF project is to improve students' learning experience in undergraduate mathematics courses; this creates a need to develop a validated assessment to measure the difference of understanding of students who attended inquiry-oriented classes. In a previous work, members of our linear algebra research group and I developed the linear algebra assessment and collected data (Haider et al., 2015). In this study, I have used the assessment data of 255 students, which were collected from linear algebra classes of nine different instructors at eight different institutions across the country to validate the assessment.

## Methods of Analysis

For this study, we first need to score the assessment data. Therefore, our research group worked together to develop a reliable scoring rubric to score the assessment copies. Statistical analysis was branched into the analysis of individual items on the test and the analysis of the entire test. More details on the development of scoring rubric, scoring process, and the analysis of data are provided in the section below.

Naturally, scoring the assessment data is the first step towards the analysis of the assessment data. To maintain the reliability of the assessment, it needs a well-defined scoring rubric so different iterations of the assessment produce comparable results. I worked with three other members of our research group to develop a scoring rubric for the assessment. Initially, a senior math education researcher developed a solution key by using a variety of student approaches from pilot data. The solution key was discussed, adjusted, and explained to other members of the group to make sure that every team member completely understood every question and a potential solution of each item. I randomly selected ten copies of the assessment from the entire data set and four researchers independently identified if the given response is correct, incorrect, or partially correct according to the solution key. If we noticed other correct approaches apart from initial solution key, I added those approaches to the potential solutions.

Later, the researchers discussed and resolved any disagreements that appeared and made four categories of student responses: fully correct (awarded 3 points), partially correct (awarded 2 points), some relevant information provided (awarded 1point), and completely incorrect and irrelevant answers (awarded 0 points). For the pilot testing of the scoring rubric, I again randomly selected six different copies of the assessment from the data, made four copies of the six assessments, and every researcher in our research group scored the first question of the assessments following the scoring scheme independently, and then we compared the scores among the team members. We repeated this processes for all the questions on the assessment, and on average, there was more than 85% agreement among the researchers. We also discussed all the disagreements and came to a consensus and fine-tuned the scoring rubric accordingly.

After finalizing the scoring rubric, the next step was to score the assessment data. I randomly selected one-third of the assessment copies (i.e., 85 copies out of 255) with the help of random number generator tool. To ensure the accuracy of my scoring, I will randomly select 20% of the scores copies, and members of my research group will double code those assessment copies. We will be looking for more than 80% of intercoder reliability. The process will be repeated for the rest of the data.

## Analysis of the Assessment and Validation Results

The linear algebra assessment was designed to align with the four main topics of IOLA material, which were mentioned earlier, and the goal of this study is to validate the assessment. Initial analysis shows that all the items on the test have discriminatory power and item are reasonably correlated with each other. Overall, the assessment is reliable enough to use for large groups. For the initial findings, I have scored and used 51 assessment copies in the statistical analysis. Next analysis with larger data set will support the current findings.

During the development of the test, the content validity was established through expert validation. The content areas of the assessment and selected questions for each area were consulted with three mathematics faculty members at three different institutions. The field experts helped us to identify the items which focused on the four focal topics and had potential to measure students' conceptual understanding of those topics.

I used Cronbach's alpha to measure the overall test reliability and found $\alpha = .74$ for all questions (including multiple-choice and open-ended parts), which shows that the assessment is acceptably reliable. However, when I checked the Cronbach's alpha for multiple-choice and open-ended questions separately, the values of $\alpha$ were dropped to .49 and .66 respectively. Item-total statistics showed that deleting any item from the test will decrease the reliability of the assessment. These statistics show that separating MCQs and open-ended items or deleting an entire item will adversely affect the assessment reliability. Statistical analysis of the assessment also revealed none of the items on the assessment have a negative correlation with other items and the corrected item-total correlation for all items is between 34% and 68%. This shows that items are not completely disconnected, but they also do not measure the same construct redundantly.

At item level analysis, the average score of all items was between 54% and 83%, which indicates that some items on the assessment were easier than others. Overall, the average score of students on multiple-choice items was 69%, and slightly lower, 65% of the open-ended items. A separate analysis of the performance of students in four focal areas showed that students performed better on the questions related to span, linear independence, and system of linear equations where the average score was above 75%. However, students were struggling with transformation and eigenvalues & eigenvectors questions where the average score was less than 65%. These results indicate that the linear algebra assessment can help to differentiate among high and low achievers and to identify the linear algebra concepts which are typically challenging for students.

## Questions for Audience

- What are the methodological issues and disadvantages for having different types of questions (variety of MCQs, true/false, fill in the blanks, and open-ended) in one assessment?
- What are other appropriate validation techniques for an assessment with mixed format items?
- How can I gradually shift this work towards concept inventories? What could be possible methodological difficulties in the shift?

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Guiding *Whose* Reinventions?
A Gendered Analysis of Discussions in Inquiry-Oriented Mathematics

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*The under-representation of women in STEM fields is well-documented and undisputed. Evidence suggests that students' experiences in undergraduate mathematics courses contributes to this disparity, and that student-centered approaches to instruction may be more equitable than lecture-based approaches. However, the generalizability of this finding has not been established. In this study, we explore how female students are positioned in whole class discussions in two inquiry-oriented mathematics classes selected to reflect differences in how female students reported experiencing whole class discussions.*

*Key words:* inquiry, equity, linear algebra, argumentation

Gender-based disparities in various forms of mathematics participation are well documented. For instance, women make up 50% of the workforce but only 25% of the STEM workforce (Beede, Julian, Langdon, McKittrick, Khan, and Doms, 2011). Two possible explanations for this disparity are that biological differences give rise to different abilities or preferences, or that these differences are socially constructed. In a critical analysis of literature related to gender and mathematics learning, Leyva (2017) argues that studies of both achievement and participation in mathematics suggest these differences are socially constructed. The rates at which women choose to discontinue study in math-intensive fields following first semester college calculus suggest that inequities in the way students experience collegiate mathematical learning environments likely contribute to these gender disparities (Ellis, Fosdick, and Rasmussen, 2016).

Recent research has suggested that student-centered approaches to instruction in undergraduate mathematics are related to improved and more equitable outcomes for students, particularly when considering gender differences (Laursen, Hassi, Kogan, & Weston, 2014). However, the mechanisms by which such instructional approaches relate to more equitable outcomes for women are not well understood. Some have raised questions about whether Laursen and colleagues' (2014) findings in the context of Inquiry Based Learning (IBL) classes apply to their own efforts to teach in student-centered ways (e.g. Hagman, 2017). This begs the question: Are emerging research-based, student-centered approaches to instruction contributing to or disrupting the pattern of underrepresentation of women in mathematics? More broadly, when and under what conditions are student-centered approaches to instruction more equitable? These broader questions are beyond the scope of this preliminary report, but we aim to move toward answering them by taking on the following, more modest set of questions:

- How do instructors distribute opportunities to contribute to whole class discussions in inquiry-oriented mathematics classes, and how does this relate to the gender composition of the class? (Who is invited to / does contribute and how are contributions framed?)
- How do whole class mathematics discussions vary in relation to how male and female students experience them?

## Literature & Theoretical Framing

We broadly adopt a socio-political perspective, taking the view that knowledge is constructed through social discourses, and that power and identity play important roles in the construction of that knowledge (Adiredja & Andrews-Larson, 2017). Inquiry-oriented instructional approaches aim to support students' reinvention of important mathematical ideas through sequences of carefully designed tasks; they are instructionally complex in that instructors inquire into students' thinking as students are inquiring into mathematics (Kwon & Rasmussen, 2007). Such approaches reposition students to take mathematical authority in a way that may reorganize traditional norms of knowledge construction associated with lecture-based classes. We aim to relate this instructional approach to literature on gender equity in mathematics, as well as settings of cooperative learning and decision making.

Some literature suggests that learning environments that require students to develop their own problem-solving strategies may favor male students in that development of invented approaches aligns with traits traditionally valued as masculine (e.g. independence and confidence), whereas use of standard algorithms aligns with traits like compliance, which are traditionally valued as feminine (Fennema, Carpenter, Jacobs, Franke, & Levi's, 1998; Hyde & Jaffe, 1998). Other literature suggests that female students acclimate better than their male peers to learning environments that emphasize collaboration, work on open-ended problems, and conceptual understanding (Boaler 1997; 2002). Laursen et al.'s (2014) work suggests that reform-based approaches that involve collaborative problem solving may 'even the playing field' for male and female students.

Research from other fields on group decision-making suggests female students are likely to experience marginalization in instructional settings that involve collaborative group work. Karpowitz, Mendelberg, and Shaker (2012) found that when a group was charged with arriving at a decision, women spoke significantly less and were interrupted more frequently (undermining their ability to influence the group's decisions) when they were in the minority. When a group was required to come to consensus, women did not experience this. In mathematical classrooms where discussions are facilitated, students from non-dominant groups (including women) are often marginalized (Becker, 1981; Black, 2004; Walshaw & Anthony, 2008). If emergent research-based instructional approaches are to broaden participation in mathematics by constructing more equitable learning environments, it is important that we consider the gendered and racialized experiences of students in these courses. In this preliminary report, we contribute to this goal by examining gender dynamics in two inquiry-oriented classrooms.

## Data Sources, Case Selection, and Methods of Analysis

We draw on data taken from a broader project interested in the teaching and learning of undergraduate mathematics through inquiry-oriented pedagogy. Instructors in this project received three forms of instructional support: access to research-based instructional sequences with implementation notes, a 16 hour summer workshop, and facilitated weekly online workgroups in the semester when they implemented the instructional materials. In this analysis, we use student surveys from seven classrooms at different institutions to select cases in which there was evidence of gender-based differences in how students were experiencing whole class discussions. We then analyzed video recordings of the selected cases to examine the relationship between mathematical discussions and gendered interactions in these classess.

Students' views of their experience in the course were captured using the Student Assessment of their Learning Gains in Mathematics (SALG-M) instrument (Seymour, Wiese, Hunter, & Daffinrud, 2000; accessible at http://salgsite.org/); surveys were conducted at the end of the semester via Qualtrics, an online survey platform. Classroom videos were recorded from two different instructional units, each of which included 2-3 days of classroom instruction. We selected video from the second instructional unit to analyze, as it took place later in the semester when instructors were more likely to be familiar with the instructional approach and classroom norms were more likely to be well-established. The instructional unit is described in Zandieh, Wawro, and Rasmussen (2017).

To select cases, we first eliminated classess with survey response rates lower than 40%. We then disaggregated students' survey responses by their self-reported gender for each class, and identified classes in which female students reported learning more from whole class discussions than male students and vice versa. We selected two classes, one from each of these categories, with similar class size and gender composition (15-20 students, approximately 25% female). On the survey, no students in either of the classes selected for analysis identified as a gender other than male or female. In video analysis we relied on visual and audio cues (e.g. hair length and style, clothing, vocal pitch, names and pronouns used) to make inferences about the gender of participants when analyzing video data. As such, all claims about participants are based on the researchers' interpretation of gender expression.

Following the selection of two cases, we created summaries of the video recordings of the two classrooms, where we first attempted to characterize instruction in each class broadly. We paid particular attention to the framing of student contributions by the instructors, how students and the instructor attributed mathematical authority, and distribution of opportunities for students' participation to the mathematical discourse in the classroom – with an eye toward gender throughout. We then transcribed the whole class discussions to analyze the mathematical argumentation, paying particular attention to how opportunities and expectations for female students to participate were framed. To this end, we selected mathematically similar focal episodes of similar length (~9 minute long whole class discussion) in the two classrooms. Both discussions addressed the image of $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ under a linear transformation from $R^2 \rightarrow R^2$ that fixed points along the line $y = x$ and stretched points in the direction $y = -3x$ by a factor of 2.

## Initial Findings

The number and nature of contributions made during whole class discussions by students of each gender in each class appear in Table 1. We note that female students contributed 50% of the student ideas in instructor B's class, which is a greater portion than instructor A's class. Table 2 reorganizes student contributions according the ways in which those contributions were solicited (also sorted by the gender of student who made the contribution). Similar portions of female students offered unsolicited contributions in both classes, but instructor A called only on male students by name, and instructor B called on only female students by name. In instructor A's class, female students volunteered to speak at slightly lower rates than in instructor B's class.

Table 1: Number of contributions by nature of student contribution

| | Word or phrase | | Question | | Idea | |
|---|---|---|---|---|---|---|
| Instructor | Female | Male | Female | Male | Female | Male |
| A | 1 | 5 | 0 | 2 | 1 | 7 |
| B | 6 | 17 | 0 | 2 | 3 | 3 |

Taken together, this suggests instructor B made deliberate efforts to include female students in whole class discussion. Based on this information, the reader might think that female students reported getting more out of whole class discussions in instructor B's class than instructor A's class. Interestingly, this is not the case. To better understand the nature of differences in whole class discussions, we describe for each instructor, the task set-up, group formation and composition, and mathematical content of whole class discussion of the focal episode.

Table 2: Number of contributions by nature of instructor solicitation

| | Unsolicited | | Called by name | | Volunteer requested | |
|---|---|---|---|---|---|---|
| Instructor | Female | Male | Female | Male | Female | Male |
| A | 1 | 4 | 0 | 3 | 1 | 7 |
| B | 1 | 4 | 3 | 0 | 4 | 19 |

**Development of Mathematics: Instructor A**

*Task setup and grouping:* Before students began working on the task in groups, the instructor framed the task as difficult to understand, noting that she did the problem incorrectly in her first attempt. The instructor asked students to read the task to themselves, then to explain their understanding of what is happening to make sure they are all interpreting the task the same before the students began working on the task in small groups. Students were in groups before the camera was turned on. In interviews, the instructor had indicated an explicit effort to avoid isolating female students in predominantly male groups.

*Whole class discussion:* After students had worked in the groups for some time, the instructor stops students, telling them she has asked one particular student to share his idea. The instructor noted that this student didn't have it all figured out but that his group's ideas might be helpful for everyone to consider.

MS1: So, basically what we did is we started by sketching $y={}^{-}3x$ and $y=x$. We decided to draw parallel lines next to it so we could get a better visual understanding to see how to sketch. We understand from the above these parts are going to stay and these are going to stretch like this. So, we tried to fix points on corners of the box to see how it goes. What we understood is, the farther it gets from the $y=x$ axis, you could say, the points will stretch farther. So it'll have this sort of diagonal look, if that makes sense.

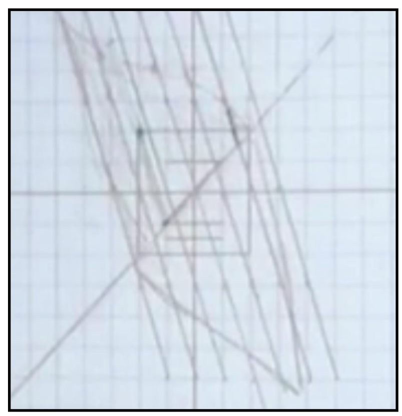

Figure 1. MS1's drawing

After the student had argued that points farther from the "$y=x$ axis" will stretch further than points close to that axis, the instructor asked students where points were stretching "from" – eliciting responses that revealed disgreement on this point. One student (incorrectly) suggested they stretched from the origin, so the instructor drew a line from the origin to the point and noted that wouldn't be in the direction of stretch stated in the problem. The instructor clarified that points stretch from the y=x axis, as the presenter had indicated, before extending this argument to geometrically show how $\begin{bmatrix}-2\\2\end{bmatrix}$ maps to $\begin{bmatrix}-3\\5\end{bmatrix}$ under this transformation by doubling its distance from the $y = x$ line in the direction of the $y = -3x$ line.

**Development of Mathematics: Instructor B**

*Task setup and grouping:* Students were expected to complete the first part of the task before class. Students were asked to "share with everybody in your table what you think this image looks like." Then, students spent about three minutes in their groups to talk about their drawings. Although students were in groups before the camera was turned on, the instructor changed that formation, saying, "I wanna form a did-the-work-[assigned at home] group over here. The rest of you can work for two minutes without the benefit of […] the people who did their work." In contrast with instructor A's class, there were no female majority groups.

*Whole class discussion initiation:* The instructor requested a group to volunteer to share their solutions for the second part of the task following their work in small groups. There were no volunteers, and the instructor called a female student by name to ask if her group would share. The discussion began:

FMS1: We use that matrix to transform the two vectors $\begin{bmatrix}2\\0\end{bmatrix}$ and $\begin{bmatrix}-2\\2\end{bmatrix}$ to see what their transformed values would be. So, we did matrix multiplication with matrix A times $\begin{bmatrix}2\\0\end{bmatrix}$ and then times $\begin{bmatrix}-2\\2\end{bmatrix}$ to find this. Any questions?

I: I got a question. How did you go about finding the matrix with those two you knew?

FMS2: Well, so, we say that T is $x_1$, $x_2$, and then $x_3$, $x_4$, and then you multiply that out again then using the rules of matrix multiplication. You get these four… equations and then you can use equations to solve for the four unknowns and hence you get that. [pointing to the matrix on the board.]

I: Did anybody do it a different way?

MS: I used linear combinations… first of all $\begin{bmatrix}2\\0\end{bmatrix}$… I got the linear combination of 1.5 times $\begin{bmatrix}1\\1\end{bmatrix}$ and 0.5 times $\begin{bmatrix}1\\-3\end{bmatrix}$. So, then when you multiply the $\begin{bmatrix}1\\-3\end{bmatrix}$ times 2 you got $\begin{bmatrix}2.5\\-1.5\end{bmatrix}$, which I was glad to see because that is a, that's what it would look like in my graph. And, then I used the same procedure to get a vector transformation of $\begin{bmatrix}-2\\2\end{bmatrix}$ to equal to $\begin{bmatrix}-3\\5\end{bmatrix}$.

The instructor then agreed that both methods were sensible and correct, referring to the first group's approach using matrix muliplication as "Method 1" and the second group's approach using the linear combinations as "Method 2." The instructor recapped the two methods and offered an explanation for how they related to one another.

## Discussion

Our analysis suggests that identifying female students with correct solutions and asking them to share does not ensure a more equitable learning environment for female students. Taken together with the literature, our findings suggest that female students report getting more out of whole class discussions in the class where we observed explicit discussion of the ambiguity of mathematics and underlying meanings, intuition, and interpretation. It is both plausible and likely that factors beyond what we were able to observe in whole class discussion contributed to different student experiences, and we are eager for feedback to inform our ongoing analysis.

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Modus Tollens in Modeling

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*The purpose of this paper is to present a case study of a mathematics major exhibiting logical reasoning to validate her mathematical model. The case study demonstrates how constructing a mathematical model can be construed as making an argument for its validity.*

*Keywords:* mathematical modeling, mathematical argumentation, mathematics majors

There is a plurality of views and foci on teaching and learning mathematical modeling (Cai et al., 2014). The cognitive view on modeling has focused on how the modeler transforms the nonmathematical problem into a mathematical one (Kaiser & Sriraman, 2006). Several frameworks have been introduced to capture this transformation and allow it to be finely analyzed according to modeling competencies (Blum & Leiß, 2007), prior mathematical knowledge (e.g., Stillman, 2000), prior real-world knowledge (e.g., Czocher, under review), and theories of metacognition arising from problem solving (e.g., Galbraith & Stillman, 2006; Panaoura, Gagatsis, & Demetriou, 2009). While prior analyses have explained a great deal of how productive and unproductive moves within the modeling process may be characterized, they are limited to examining only specific modeler moves *within* the modeling process. With respect to mathematical reasoning, these frameworks are limited to examining only the mathematics the modeler uses to set up, analyze, compute, or solve the resulting model which can usually be explained in terms of the mathematics content intended by the task writer. That is, these frameworks do not allow documentation of validating the model if the means to do so fall outside of the expected mathematics or modeling context. This paper presents a case study of how an individual might use logic to guide her use of mathematical content knowledge. We follow with a discussion of why students' logic might have been overlooked in other frameworks and then discuss why an alternative lens for examining modeling behavior, especially of more advanced students, is promising for shedding light on similarities among modeling, problem solving, and proving.

## Background

From a cognitive perspective, studying mathematical modeling means attending to the mathematical thinking that produces the model (Borromeo Ferri, 2007). Mathematical modeling is viewed as a process that transforms a question about the real world into a mathematical problem to solve (Frejd, 2013). The answer to the mathematical problem is then interpreted as a solution to the real world problem. This process is often represented as a cycle (e.g., Blum & Leiß, 2007), which is summarized in Table 1. Much of the research on modeling from the cognitive perspective focuses on the simplifying/structuring phase (identifying variables, making assumptions) and on the mathematizing phase (introducing conventional representational systems). Comparatively less research has focused on validating, which involves checking that the mathematical model is representative of the situation and that it is correctly analyzed (solved) mathematically. Validating is challenging to study because of how the modeler perceives and resolves cognitive conflict between their expectations of their model (e.g., predictions) and outcomes (e.g., empirical observations) (Czocher, 2014, 2015). Students may respond to cognitive conflicts in less-than-ideal ways (Goos, 2002). They may fail to notice that something is amiss, perceive difficulties that do not exist, provide an inadequate response, or even change the problem to suit their readily-available knowledge (Goos, 1998, 2002; Stillman, 2011). Indeed, some have observed that validating is a "uniform shortcoming" of students' mathematical modeling because they do not *reflect* to improve their models at all (e.g., Blum & Leiß, 2007). However, some small amount of work has revealed that engineering undergraduates do engage in validating their models, typically through techniques like dimensional analysis, checking special and limiting cases, making comparisons to empirical results, and relying on number sense (Czocher, 2013). On the other hand, mathematics majors' conditional reasoning has been documented, particularly as it relates to comprehending an argument (Alcock, Bailey, Inglis, & Docherty, 2014). The following analysis is an effort to begin to document and understand the reasoning mathematics majors use to validate their mathematical modeling work.

*Table 1 Indicators from the observational rubric to identify subprocesses in the MMC* (Czocher, 2016)

| Modeling Subprocess | Definition | Examples of Observed Student Activity |
|---|---|---|
| Understanding | Forming an initial idea about what the problem is asking | Reading the task<br>Clarifying what needs to be accomplished |
| Simplifying & structuring | Identify critical components of the mathematical model (i.e., create an idealized view of the problem) | Listing assumptions or specifying conditions<br>Identifying variables, parameters, or constants<br>Operationalizing quantities or relationships |
| Mathematizing | Represent the idealized model mathematically | Writing or speaking mathematical representations of ideas (e.g., symbols, equations, graphs, tables, .) |
| Working mathematically | Mathematical analysis | Explicit algebraic or arithmetic manipulations<br>Making inferences and deductions without reference to nonmathematical knowledge<br>Changing mathematical reprsentation |
| Interpreting | Recontextualizing the mathematical result | Speaking about the result in context of the problem or referring to units<br>Considering if the result answers the question posed |
| Validating | Verifying results against constraints | Implicit or explicit statements about the reasonableness of the answer/representation<br>Checking extreme or special cases of variables, parameters, relationships, etc.<br>Dimensional analysis of units |

## Methods

Qualitative data were generated via an individual task-based interview (Clement, 2000; Goldin, 2000). The tasks were a variety of modeling and application problems drawn from previous research (e.g., Ärlebäck, 2009; Czocher, 2016; Schoenfeld, 1982; Swetz & Hartzler, 1991). The 10 tasks were sufficiently open to allow participants to select their own variables, assumptions, and solution techniques. The purpose of the interviews was to elicit participants' mathematical thinking as they engaged in mathematical modeling; the interviewer did not guide participants to a solution, but intervened only to request clarification or to extend the task. In this paper, we focus on a single case to illustrate a mathematics major's reasoning on a conventional word problem. The case is illustrative of a mathematics major using clearly outlined logic despite arriving at a wrong answer. The data are presented and analyzed as a narrative, a "spoken or written text giving an account of an event/action or series of events/actions, chronologically connected" (Czarniawska, 2004, p. 17). To do so, we view the interview participant, Safi, as presenting an account her series of decisions during mathematical modeling.

Safi was a senior mathematics major at a large southwestern university. She was enrolled in a vector calculus course and stated that her favorite subjects thus far were "linear algebra, hands down, and differential equations." She was nearing completion of her mathematics requirements and was seeking secondary teacher certification. Safi had completed her first classroom internship in geometry at a local high school, but stated a preference for teaching algebra.  The following semester, before graduation, she was scheduled to do her student teaching in an algebra 2 classroom. Safi did not describe herself as good at mathematics. She said, "since being here [at university] I have struggled with like my math classes and everything but I've worked really hard to get even like the C's I have gotten." Safi valued the hard work she put into her classes, which fueled her drive to be a teacher, despite the fact that the higher level mathematics courses she didn't "really see being useful, like the proof classes." She elaborated that the content of the proof classes would not be something she used in her high school classes but that "maybe the different way of thinking" would be useful.

Below, we present Safi's work on the Turkeys & Goats problem (Czocher & Maldonado, 2015) and analyze it in terms of the correctness of her response, her engagement in mathematical modeling, and the reasoning she used to arrive at her conclusions. The problem was: *A nearby farm raises turkeys and goats. In the morning, the farmer counts 48 heads and 134 legs among the animals on the farm. How many goats and how many turkeys does he have?* The problem is a word problem (see Gerofsky, 1996) that is ubiquitous in secondary school algebra textbooks and on standardized tests. The answer, 19 goats and 29 turkeys, can be obtained in a variety of ways including setting up a system of two equations in two unknowns. Because of Safi's mathematical training and recent experiences in mathematics pedagogy, the task was well within her capabilities. In order to analyze Safi's engagement in modeling, the observational rubric from Table 1 was applied. When Safi was observed, in speech or writing, to be carrying out one of the activities in the right-most column, her activity was coded with the corresponding modeling subprocess from the left-most column.

## Presentation of Safi's Reasoning

Safi began by reading the Turkeys & Goats problem aloud [understanding]. She then emphasized some information, "48 heads and 134 legs" which she repeated aloud and wrote down [simplifying/structuring]. She then explicitly identified what needed to be accomplished, "and then they're asking how many of each animal" [understanding]. She narrated her reasoning, "48 heads means he has 48 animals in total because he wouldn't have more heads than animals because that wouldn't make sense." In this statement, Safi engaged in both simplifying/structuring because she established the condition that 48 heads means 48 animals in total and validating because she was evaluating its sensibility. To carry out her validating, she used counterfactual reasoning (reasoning from a situation that doesn't or can't exist) to set up and evaluate a brief propositional logic argument positing a one-to-one correspondence between heads and animals. She assigned the variable $x$ to the number of turkeys and the variable $y$ to the number of goats [mathematizing]. She then wrote the two equations $x + y = 48$ and $2x + 4y = 134$ [mathematizing], checking that "two legs per turkey will give you the amount of turkey" legs [validating]. Safi used elimination method to solve the system [working mathematically]. She obtained $y = 19$ which she interpreted to mean "there should be 19 goats" [interpreting]. Then to obtain the number of turkeys, she computed $48 - 19 = 27$ using the standard algorithm [working mathematically]. She wrote 27 turkeys [interpreting]. To check her work, she used standard algorithms to compute $2 \times 27 + 4 \times 19 = 134$ [validating]. She obtained 130 for the left hand side. She asked "Am I allowed to ask you the amount of turkey legs?"

Safi had arrived at a contradiction: her solution 19 goats and 27 turkeys did not yield the same number of legs set by the conditions in the problem statement. Her first recourse was not to doubt her computation but to doubt whether turkeys had 2 legs. The interviewer followed up by exploring whether 2 legs per turkey was a logical antecedent or logical consequence of 19 goats.

*Safi*: I solved it with turkeys having two legs, but I am short 4 legs.

*Interviewer*: You're short 4 legs. And you are certain that they are turkey legs?

*Safi*: No. But if turkeys have 2, then I am not sure. Well, 'cause I solved it to where goats have the 19, there were 19 goats.

*Interviewer*: Okay, so given that turkeys have 2 legs, there must be 19 goats. Is that what you're saying?

*Safi*: Yeas. Oh wait, wait wait. But okay wait. The goats here…they have 38 legs, and then [[talks quietly then laughs]]. Yeah, so given that turkeys have 2 legs, there should be 19 goats.

Safi continued this chain of logical reasoning to argue that given that turkeys have 2 legs, there must be 19 goats, so there have to be 27 turkeys. There can't be 27 turkeys because $27 \times 2 = 54$, meaning just turkeys alone would have 54 legs. Given that there are 19 goats and goats have 4 legs, they would have 76 legs. Altogether there would be 130 legs, which is too few legs. Safi "called into question" the assumption that turkeys have 2 legs.

After a brief discussion about why Safi had chosen to use the operations + and × where and how she did to set up her system of equations, the interviewer extended the problem. Instead of Turkeys and Goats, the interviewer posed a problem in which the farm had pigs and goats, with 48 heads and 134 legs. The resulting system of equations was inconsistent. Safi set up the equations, solved them via elimination and obtained the result "0 equals negative." She interpreted it to mean that there could be no goats and therefore there were 48 pigs.

*Safi*: But then if you have 48 pigs, each pig should have 4 legs, which would mean 192 legs. But there is 134. So that's, that's not accurate.

*Interviewer*: Which isn't accurate? The 192 or the 134 or something else?

*Safi*: Well, if you're paying attention to the heads, like it depends on what you're looking for, if you're looking at the heads. Then the legs, the 134 doesn't make sense because you have more, you realistically have 192 legs here with 48 pigs. And it says you only have 134. So that's not enough to complete your farm [[laughs]].

*Interviewer*: So, when you, I noticed you like put in this adjective there, you "realistically" how would you have 192 legs? What did you mean by that?

*Safi*: So, we mean, you could have pigs missing legs. Um, 'cause they don't need four legs to be able to live so if you take out some, we mean we guess you could get to 134. But realistically, if they all have 4 legs then that's how many you would have, you have 192.

In follow-up questioning, Safi revealed that she noticed that both versions of the problem were similar to those she had seen in "algebra and algebra 2 and linear algebra" and so she was readily able to set up the system of equations and "in order to solve for each variable you usually just do any process that you can," though she did not use the vocabulary of linear algebra to seek solutions or explain the lack of solutions to each system.

## Discussion and Conclusions

Safi did not arrive at the correct solution for either the Turkeys & Goats problem, due to the arithmetic error $48 - 19 = 27$. She also did not realize that there was no solution to the system of equations she derived for the extension problem although she recognized a contradiction for the number of legs required. However, in both versions of the task she did engage in the cognitive activities underlying mathematical modeling (as suggested by the observational rubric) and she did reach conclusions that were logically consistent with the information she gleaned from the task statement. On the surface, it seems unreasonable that Safi would doubt a basic fact like *turkeys have two legs*. Closer inspection reveals that it is a logical consequence of an argument she constructed to validate her model (the system of two equations in two unknowns) and its prediction (the number of turkeys and goats on the farm). Table 2 shows her argument's structure mapped to propositional logic:

*Table 2. Safi's logical argument*

| | | | |
|---|---|---|---|
| 1. | There are 134 legs on the farm (premise) | 5. | There are 27 turkeys (3, 4) |
| 2. | Turkeys have two legs (premise) | 6. | There are 130 legs on the farm (3, 4, 5) |
| 3. | The system $x + y = 48$, $2x + 4y = 134$ describes the number of animals on the farm (1, 2) | 7. | Contradiction (1, 6). |
| 4. | There are 19 goats (3) | 8. | Reject (2). |

Safi checked her by-hand computations twice to be sure that (4) and (5) turned out correct (committing the same mental arithmetic error each time). Her only course of action, logically, is to reject one of the two premises upon which (3) stands. Since (1) is given in the problem, she must reject (2). Her spoken argument can be reduced to the form of modus tollens: If turkeys have two legs, then there are 130 legs on the farm. There are not 130 legs on the farm. Therefore, turkeys do not have two legs (she expressed an equivalent summary verbally). She displayed similar reasoning patterns on the extension to the pigs and goats problem.

What is interesting about Safi's response is not that she is a math major who is a preservice teacher who got a routine word problem incorrect (which is the sort of result documented in the past); rather, the novelty of Safi's work is how she used logical reasoning from her advanced mathematics courses to make sense of and support her conclusions about the validity of the mathematical model she constructed. Students' untrained reasoning may be incompatible with mathematical logic and students' application of logical structure largely depends on the semantic context (Dawkins & Cook, 2017). Safi was a student trained in logic and mathematical reasoning with knowledge of the semantic context. Deconstructing Safi's response in terms of a first-order propositional logic revealed how it supported her interpretation and validation of her model, and opens questions about whether students' mathematical thinking during modeling may be productively analyzed according to argumentation models (e.g., Toulmin schemes). Her responses also suggest that such lenses might reveal insights into the interaction between content knowledge and mathematical modeling. Further, Safi's use of logic shows that mathematics majors may not all have the same validating techniques at their disposal as engineering or science majors, implying that caution must be exercised when generalizing conclusions about modeling behavior among any of these populations (Czocher, 2013). In particular, if mathematics majors are using the skills and patterns of reasoning that they learn in advanced proof-based courses in other domains it raises new questions about the natures of mathematical modeling, problem solving, and proving and what characteristics they may share. Scholars in either area must be cautious of overlooking kinds of reasoning not typically linked to the domain of inquiry. For these reasons, further work needs to be done to document what validation processes students are likely to bring from various backgrounds and how they contribute to the students' mathematical modeling processes.

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# Identifying Subtleties in Preservice Secondary Mathematics Teachers' Distinctions Between Functions and Equations

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*For more than thirty years, the secondary school mathematics curriculum has seen a shift to functions-based approaches to algebra. Advancing comprehension of the equals sign as an equivalence relation is critical for beginning algebra students studying equations, and developing understanding of functions is foundational as a gateway to courses required of science, technology, engineering, and mathematics majors. This study explores the ways in which mathematics majors seeking secondary mathematics teaching certification distinguish between the concepts of function and equation. Participants (n=24) completed a ten-item pre- and post-assessment on functions and equations. Open coding techniques were used to identify emerging categories that describe participants' distinctions between the concepts. After a mathematics course experience with an eight-week unit on functions, the participants' concept image for functions focused primarily on input and output whereas their concept image for equations centered broadly on the equivalence of two quantities.*

*Keywords:* preservice secondary mathematics teacher preparation, function, equation

The topic of *functions* has been well-documented in the research literature as "difficult for students to learn, challenging to teach, and critical for students' success as learners and in their future lives and careers" (Cooney, Beckmann, & Lloyd, 2010, p. v). Students in the United States are commonly introduced to functions in secondary school (National Governors Association Center for Best Practices and Council of Chief State School Officers [CCSSM], 2010). Given the importance and difficulty of functions, it is essential that secondary mathematics teachers have the depth and breadth of understanding necessary to teach this critical topic (e.g., Stacey, 2008), and undergraduate studies offer an opportunity for teachers to build a profound understanding of functions.

Part of a profound understanding of function includes a clear articulation of the differences between functions and equations. Although there are ways to relate the topics of *function* and *equation*, they are sometimes inappropriately conflated by students and teachers alike. In this study, we investigate the following research questions:

1) How do preservice secondary mathematics teachers distinguish between functions and equations?
2) What subtleties exist in preservice teachers' distinctions?

## Theoretical Framework

To frame this study, we draw on Tall and Vinner's (1981) theory of *concept image* and *concept definition*. Throughout their school studies, preservice secondary math teachers develop a concept image of the topic of functions, which includes "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). A teacher's concept image of function (for example) may be well-developed and align with the formal definition of function, or their concept image may be fragmented, incomplete, or misaligned with the formal definition. Teachers may also have a personal concept definition for function—that is, the words the teacher uses to define function. A teacher's personal concept definition may reflect their concept image, or it may be misaligned from their concept image. At the same time, one's personal concept definition may align with (or be a memorized recitation of) the formal concept definition in the mathematical community, or it may be inconsistent with the formal definition.

Alignment between concept image and the formal concept definition is important because conflicts between these two may cause difficulties in students' learning (Tall & Vinner, 1981). In addition, concept images or personal concept definitions that are misaligned with the formal definition may cause students to think that the formal definition is "inoperative and superfluous" (Tall & Vinner, 1981, p. 184). Alignment between concept image and concept definition is especially important for teachers who are guiding students' learning of the concept. In this study, we investigate preservice secondary teachers' concept images and personal concept definitions of function and equation.

## Research Literature

Throughout high school and undergraduate mathematics, students are accustomed to working with functions which can be defined by algebraic formulas, and students often use formulas to identify the functions they discuss (Cooney et al., 2010). Formulas for functions are especially useful in calculus, and undergraduate courses such as calculus can reinforce students' concept image of functions being defined by formula. In fact, students' conceptions of functions can be limited by thinking of them as defined by formulas. For example, Even (1993) surveyed 152 preservice secondary mathematics teachers about functions, and ten additional preservice teachers were interviewed. Many of these preservice teachers thought that functions could always be represented by an algebraic formula. Similarly, using questionnaires with 30 secondary teachers, Hitt (1998) reported that many teachers believed that functions could always be represented by a single algebraic expression, and Carlson (1998) reported the same finding for students who earned A's in College Algebra.

Secondary school curriculum emphasizes that zeros of a function *f* are the solutions to the equation $f(x) = 0$ (CCSSM, 2010). Although this connection is valuable, students sometimes muddy this relationship. For example, in a study with students earning A's in College Algebra, Carlson (1998) found that these top-performing students "do not make a distinction between the zeros of functions and solutions to equations" (p. 141). In a 1999 study, Carlson also reported that second-semester calculus students had similar confusions between solutions to equations and zeros of functions.

To further complicate matters, in high school as well as undergraduate mathematics, a formula such as $f(x) = 3x + 2$ is sometimes referred to as *the equation for the function* $f$ or *the defining equation for the function* $f$. Perhaps perpetuated by this terminology, many preservice teachers have some incorrect conceptions about the relationships between functions and equations. For example, in Even's (1993) study, some preservice teachers provided definitions of function in which they claimed a function *was* an equation or expression. Breidenbach, Dubinsky, Hawks, and Nichols (1992) found that some preservice mathematics teachers described a function as "a mathematical equation with variables" (p. 252). Not surprisingly, Chazan & Yerulshamy (2003) documented that learners also have difficulty in distinguishing between functions and equations.

## Methodology

This study was conducted at a large, urban university in the southwestern United States with an on-campus student enrollment larger than 37,000 students. Due to the large enrollment (greater than 25% of the student body) of Hispanic students, the university carries a US Department of Education Hispanic Serving Institution designation. In addition, the university is described as one of the most diverse national universities in the United States.

The population for this study was preservice secondary mathematics students who were enrolled in a second-year mathematics course in the fall semester of 2016. The course, Functions and Modeling, is a required course for mathematics majors seeking secondary mathematics teaching certification. The intent of the course, which carries a second-semester calculus prerequisite, is to deepen preservice secondary mathematics teachers' experiences with the mathematics that they will teach, immerse them in an inquiry-based learning environment, and develop a profound understanding of important concepts for secondary school mathematics. In the 15-week fall 2016 semester, approximately eight weeks of the course focused on functions and patterns, four weeks on regression and modeling, and three weeks on various topics such as parametric equations, polar coordinates, vectors, and the geometry of the complex numbers.

Thirty students (17 females and 13 males) were enrolled in the course and 24 students participated in the research study. The overall student population enrolled in the university's science and mathematics secondary teacher certification program is 41% Hispanic, 38% White, 14% Asian, and 7% Black.

A written instrument consisting of ten items (and corresponding sub-items) targeting the students' understanding of function and equation was used as a pre- and post-assessment. The items on the assessment required the preservice teachers to explain their reasoning and, where appropriate, provide multiple representations. The assessment took the students approximately one hour to complete. This study examines student responses to two of the assessment questions.

- "Can the terms *function* and *equation* ever be used interchangeably? Why or Why not?"
- "If a student in Algebra I asked you to explain the difference(s) between a function and an equation, what would be your response?"

The pre-assessment was administered during the first week of the course. The post-assessment was completed after the course final exam. Qualitative methods were used to analyze the written responses from the assessments. Participant responses were systematically coded by elements in their explanations and by themes that emerged in the data relevant to their descriptions comparing the concepts of function and equation.

In the analysis of the pre-and post-assessments, we used principles of the grounded theory method (Strauss and Corbin 1990), allowing the data to be coded through the lens of emerging themes. The data were then grouped into similar conceptual themes characterize the preservice teachers' descriptions of contrasting function and equation.

## Results

Participant responses to "Can the terms *function* and *equation* ever be used interchangeably? Why or Why not?" on the pretest were coded as ambiguous (AMB), relationship/both (RLB), non-answer (NAN), some equations are not functions (ENF), some or all equations are functions (SEF), and some or all functions are equations (SFE) (see Table 1).

On the posttest, the new codes relationship vs. equivalence (RVE) and definition (DEF) arose from the posttest data. Responses that rejected interchangeability by mentioning the difference in the way the terms are defined were coded DEF. For example, "no, their definitions are not the same" was coded DEF. Responses that claim equations assert equivalence between two quantities but functions depict an input-output relationship were coded RVE. The codes RLB and ENF did not appear while there were 2 AMB, 6 NAN, 3 SEF, 8 SFE, 3 RVE, and 2 DEF.

*Table 1. Codes arising from the Pretest interchangeability question and their frequency.*

| Code | Description | Selected Response | Freq. (n=24) |
|---|---|---|---|
| AMB | Ambiguous response that does not offer reasons. | "not always interchangeable; depends on how it is written" | 3 |
| RLB | Claim that both express a relationship. | "Yes, because they both describe a relationship between variables…" | 4 |
| NAN | Non-sensical or non-mathematical response. | "No! hmm maybe…wow, you've got me stumped…" | 6 |
| ENF | Asserts that not all equations are functions. | "no, because not every equation is a function" | 5 |
| SEF | Asserts that some or all equations are functions. | "they can be interchanged sometimes there are equation that describes functions, but not always" | 2 |
| SFE | Asserts that some or all functions are equations. | "they can be, for example the function of x (f(x)) can be displayed as y" | 4 |

Three of the six participants who provided an NAN-coded response on the pretest also provided an NAN-coded response on the posttest. Three other participants' response codes remained the same from pretest to posttest—two SFE responses and one SEF response. The five ENF-coded responses on the pretest provided two NAN-coded responses, two SFE, and one SEF-coded response on the posttest.

Participant responses to "If a student in Algebra I asked you to explain the difference(s) between a function and an equation, what would be your response?" on the pretest were coded as NAN, SEF, input-output (IO) with sub codes equation equivalence (EE) or equation number specific (NS), and relationship (RL) with sub codes equation equivalence (EE) and equation number-specific (NS) (see Table 2).

*Table 2. Codes arising from the Pretest Algebra I student question and their frequency.*

| Code | Description | Selected Response(s) | Freq. (n=24) |
|---|---|---|---|
| IO<br><br><br>IOEE<br><br>IONS | Refers to input-output or independent-dependent variables for functions and<br>-equations asserting equivalence of two quantities, or<br>-equations as specific situations when numbers are used. | -IOEE: "Function: independent variable dictates the value of the dependent variable. Equation: something equals something else."<br>-IONS: "…equation may just involve solving for one variables [sic] given a number…" | 6<br><br><br><br><br><br>2 |

| | | | |
|---|---|---|---|
| RL<br><br><br>RLEE<br><br>RLNS | Refers to mapping or relationship between variables and<br>-equations asserting equivalence of two quantities, or<br>-equations as specific situations when numbers are used. | -RLEE: "An [sic] function assigns all the elements in set x to set y simultaneously. While an equation does not assign it simply equates.<br>-RLNS: "…equation takes that relationship and puts numbers in it…" | 2<br><br><br><br><br><br>5 |
| NAN | Non-sensical or non-mathematical response. | "I'm not sure I'd have the best response right now." | 7 |
| SEF | Asserts that some or all equations are functions. | "An equation can be a type of function…" | 2 |

On the posttest, the new code representation vs. equation equivalence (RPEE) was needed to code answers that referred to a representation to distinguish between function and equation; for example, "…a function has to pass the vertical line test." The sub codes IONS and RLNS disappeared while there were 2 RPEE, 7 IOEE, 4 NAN, 7 RLEE, and 4 SEF. The six of the seven participants with NAN responses on the pretest coded for IOEE or RLEE on the posttest with one receiving a SEF code.

## Discussion

Although participants completed several inquiry-based lessons that focused on precise definitions of functions and equations as well as several lessons using functions to model data, only one participant, on the "Algebra I student question," used the terms domain and codomain when attempting to make an equation-function distinction. Somewhat akin to Carlson's (1998) findings, no participants attempted to contrast equations and functions by referring to solution sets or domain and range, respectively. As in Even (1993), the use of the equal sign when defining a function with an algebraic expression may explain why 8 of 48 responses—aggregating the responses to both questions—still assert that some functions are equations.

The prevailing concept image for function entailed input-output or the idea that a function establishes a relationship between inputs and outputs, regardless if their description of an equation also used the idea of a relationship between quantities. Possibly a result of a lesson specifically focusing on the role of the equal sign in defining, equivalence, and computation may have influenced a shift from number-specific responses about equations to 14 of 24 responses that gave a mostly-correct equation concept definition.

The purpose of this study is to further investigate the subtleties in preservice secondary mathematics teachers' conceptual distinctions between function and equations. Further input from researchers is needed regarding developing interview protocols, alternative assessment questions, and ways to interpret the data that inform curriculum development and instruction.

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Supporting Prospective Teachers' Understanding of Triangle Congruence Criteria

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*This poster describes an instructional sequence for supporting college geometry students' justifying Euclidean triangle congruence criteria using properties of isometries. We hypothesized that investigating transformations in the taxi-cab metric would perturb students' understandings of the relationships between triangle congruence criteria and isometries, so they would more explicitly identify the properties of transformations as a necessary part of their justifications of triangle congruence criteria. We report on the results of pre-post written assessments of our college geometry students' justifying SAS to a hypothetical 10th grade student.*

*Keywords:* Geometry, Congruence, Mathematical Knowledge for Teaching

Research has demonstrated that supporting college geometry students' understandings of transformational geometry remains a challenge (e.g., Hegg & Fukawa-Connelly, 2017). Because many of our college geometry students are prospective secondary teachers, supporting their mathematical knowledge for teaching secondary geometry is an important course goal. The United States' Common Core State Standards state that high school geometry students should be able to justify triangle congruence criteria (ASA, SAS, SSS) as a consequence of properties of rigid motions (NGACBP/CCSSO, 2010). Hegg and Fukawa-Connelly (2017) found that college geometry students struggle with explicitly using relevant properties of transformations in such justifications, and they suggest that "asking for the kinds of explanations of ideas that [college geometry students] would give [secondary geometry] students has value in both giving researchers insight into their understanding of the content and giving policy-makers a better understanding of what additional supports will be needed going forward" (p. 8).

We designed a written task prompting our college geometry students to justify SAS to a hypothetical 10th grade student, and we administered the task before and after an instructional sequence investigating transformations in taxi-cab geometry in our Spring 2017 courses. With consideration of Harel's (2013) notion of intellectual need, we theorized that after experiencing perturbation that some Euclidean isometries are *not* isometries in a different metric (the taxi-cab metric, $d_t((x_1, y_1),\ (x_2,\ y_2)) = |x_1 - x_2|\ + |y_1 - y_2|$), our students would understand triangle congruence criteria in Euclidean geometry as depending on the *properties* of the isometries and thus be better-prepared to support future secondary students' in constructing that way of thinking about triangle congruence criteria. Using a constant comparative method (Glaser, 1965), we each independently coded students' written justifications of the SAS congruence criteria, considering aspects such as their (a) understanding of the premises and conclusions of SAS, (b) use of other Euclidean axioms or theorems to justify SAS, and (c) use of transformations to justify SAS. We then rated the written justifications on validity, substantiveness, and appropriate use of transformations. Our analysis suggests that students' responses were complicated by their beliefs about (and conceptions of) what it means to justify in secondary school mathematics. In our poster we will present the task, explicate findings, and future directions.

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Adjunct Instructor Learning Through Implementing Research Based Curriculum

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*Due to high demand, part-time adjunct instructors play an increasingly important role in introductory classes at many higher education institutions. As part of a project to support adjunct instructors teaching Precalculus, we are exploring the impact of course coordination and support on content knowledge and instructional practice of instructors.*

*Keywords:* Teacher learning, adjunct instructors, Precalculus teaching

There is a need for research on retaining students in STEM disciplines (Carnevale, Smith & Melton, 2011; Hurtado, Eagan, & Chang, 2010). Students' classroom experiences influence their decisions to pursue STEM degrees, especially initial experiences in introductory math courses (Hutcheson, Pampaka, & Williams 2011; Pampaka, Williams, Hutcheson, Davis & Wake, 2012). Thus, improving instruction quality may influence their decision to stay in STEM fields (Ellis, Kelton & Rasmussen, 2014). At our institution, Precalculus sections are taught by adjunct instructors. This proposal presents ongoing data analysis from a project focusing on promoting adjunct instructors' learning through course coordination and job supports. Specifically, this analysis aims to answer the following research question: In what ways does implementing a research based Precalculus curriculum impact adjunct instructors' mathematical content knowledge and instructional practice?

To explore the impact on instructor content knowledge, we use Shulman's (1987) conception of content knowledge. According to Shulman, there are three facets of content knowledge: structures of subject matter (rules, procedures, definitions, and axioms), principles of conceptual organization (conceptual web of content), and principles of inquiry (mathematical habits of mind (Cuoco, Goldenberg, & Mark, 1996)). These three facets of content knowledge allow teachers to develop a broad understanding of their field by grasping the main concepts, gaining expertise, and learning modes of analysis that take the field forward (Shulman, 1987). We are analyzing transcripts from pre and post semester interviews with adjunct instructors teaching our newly implemented research-based Precalculus curriculum. Each member of our research group reads the transcripts, identifying segments of talk focused on mathematical content, and categorizing each segment as one of the three facets of content knowledge, while allowing for segments to be placed in the intersection of one or more category. As we continue this work, we are thinking about how to incorporate other measures of instructor content knowledge, including pre and post knowledge assessments as well as interviews from more recent semesters.

To explore the impact on instructional practice, we use Teucher, Moore, and Carlson's (2015) construct of decentering introduced by Piaget (1955) as a way "to characterize the actions of an observer attempting to understand how an individual's perspective differs from her or his own" (Teucher, Moore, & Carlson, 2015, p. 5). We are analyzing transcripts from classroom observations from the first semester of implementation to explore how our instructors may, or may not, be employing decentering practices. As we continue this work, we will explore how the decentering practices of our instructors relate to course coordination and supports.

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Students' Engagement with a Function Vending Machine Applet

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*The purpose of this study is to examine the ways students engaged with a Vending Machine applet designed to problematize common misconceptions associated with the function concept. Findings indicated a need to redesign the applet to further disrupt students' misconceptions of the concept of function. Design decisions for the redesigned applet and the new version will be shared.*

*Keywords:* Functions, Calculus, Teaching with Technology

Research has revealed common misconceptions that persist among undergraduate students with respect to the definition of function (Vinner & Dreyfus, 1989), use of function notation (e.g., Oehrtman, Carlson, & Thompson, 2008) and connections between function representations (e.g., Dreher & Kuntze, 2015; Stylianou, 2011). Hence, we designed and studied the ways that undergraduate students, all who have completed Calculus I, from six universities engaged with an applet designed to test and improve their understanding of the function concept.

The Vending Machine applet (https://ggbm.at/qxQQQ7GP) is a four-page GeoGebra book. When the user presses a button (input), one or more cans appear in the bottom of the machine (output). Students are asked make conjectures about why the machines are or are not functions. These machines were designed to provoke dilemmas (Merizow, 2009) with the students' common function misconceptions to lead them toward a robust understanding of the function concept such as students' use of the term "unique" when describing outputs of functions, misunderstanding of what represents an element in the range, and misidentifying horizontal lines as non-functions.

## Method & Results Summary

To answer our research question, How do undergraduate students engage with a vending machine applet designed to provoke dilemmas with their understanding of the function concept?, we analyzed screencasts from 123 students that completed the vending machine assignment. Results showed that even after engaging with the applet, many students applied their previous understandings of the function concept to each of the machines and continued to demonstrate two misconceptions that we had intended to disrupt: 1) a horizontal line (each button returning the same can of soda) as not representing a function and 2) what represents elements in the range (a button consistently producing two identical cans).

## Conclusion

Despite our Vending Machine applet's intended design to provoke dilemmas related to students' understanding of function which we hoped would promote reflection and ideally deepen students' understandings related to the function concept, we found that many students continued to apply their common misconceptions when engaging with the machines. Based on these results, we have redesigned the applet and will share how the new design arose from our analysis of the students' engagement with the applet.

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Pre-Service Teachers' Mathematical Understanding of the Area of a Rectangle

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*This poster will share contrasting responses of two pre-service teachers (PTs) to problems that were part of an ongoing study of PTs' conceptions of area of a rectangle. They were asked to a) find the area of a rectangle in terms of a non-square rectangular unit and relate that to multiplication, and b) interpret a fictional child's attempt to connect the area formula with counting square units. These cases showed that an ability to explain a systematic covering of a 2D space with an area unit does not imply an ability to respond to a student who might think "the corner square gets counted twice". Further, the ability to describe this structure, L rows with W area units per row, does not imply readiness to understand the area formula for a rectangle.*

*Keywords*: area, rectangle, geometry, pre-service teachers

Numerous studies indicate that when teachers obtain appropriate knowledge of mathematics, their instructional practices change in ways that improve their students' mathematics learning (e.g., Cobb et al., 1991; Fennema et al., 1996). As mathematics educators who design and teach mathematics courses for future teachers, we are constantly reevaluating our course curriculum and goals in light of our current understanding of what mathematics will be needed by our PTs in their careers. In our courses, explanations of all mathematical ideas are emphasized, and our role is to orchestrate sharing of different answers, ideas, questions, and solution methods. In our daily observations of their work in class, we developed theories of the PTs' conceptions of area and its measurement that are explored in our research.

This study took place at a public university in a mid-western state with ten PTs who were enrolled in a geometry course for future elementary teachers. We conducted clinical interviews (Clement, 2000) with these PTs outside of class. These focused on (a) the idea of measuring the area of a rectangle with non-square units and explaining the meaning of the covering process used, and (b) the interpretation of a fictional child's thinking about the connection between the area formula and counting square units. Each interview was videotaped and transcribed. Videos and transcripts were analyzed to capture the progression of PTs' thinking from task to task (Auerbach & Silverstein, 2003). Specifically, each of the researchers read the transcripts multiple times, and documented the changes over the interview sessions. Key excerpts were flagged, reviewed, and examined closely for insights about PTs' reasoning.

The poster will present data from interviews with two of the PTs. These were selected because of contrasts across their own answers as well as contrasts with each other's answers. One PT covered a rectangle with rows of area units and used this to explain why multiplication will yield the total number of units. The same PT then had difficulty assisting a fictional student who, looking at an array of squares, was worried that the corner square was counted twice. The second PT also iterated the unit to cover the rectangle. She got confused when trying to explain this process in terms of "L x W", but then clearly explained to the imaginary student how to use multiplication to count the squares. These PTs' responses prompted questions and ideas that we wish to consider with the other researchers and the RUME audience.

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# How Experts Conceptualize Differentials: The Results of Two Studies

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*The mathematical symbol "dx" is a symbol for which there can exist different views about its characteristics, purposes, and roles. We conducted two studies to see how experts viewed the dx in a variety of contexts. For our first study, we interviewed four mathematicians in order to understand their various concept images of the dx, and for our second study, we interviewed two mathematicians and one physicist about both their own concept images and the concept images they would like for their students to have. Overall, we found little agreement among all of the experts' responses, and we believe that that further study of experts' concept images of the differential is warranted.*

Keywords: Calculus, Differentials, Concept Image

The differential is a symbol that is found in various, commonly-used mathematical notations, such as the derivative $\left(\frac{dy}{dx}\right)$ and integrals $\left(\int f(x)dx, \int_a^b g(t)\,dt\right)$. However, while these notations are standard, the meaning behind the differentials in these notations is not necessarily so: do those differentials represent small amounts of a quantity (Hu & Rebello, 2013; Von Korff & Rebello, 2012), do they only exist to indicate important variables (Artigue, 1991; Jones, 2015), are they merely notation with no intrinsic meaning (Artigue, 1991; Hu & Rebello, 2013), or can they possess some combination of all three of these meanings (Tall, 1993)? We wished to interview experts about differentials not only in an attempt to understand their concept images (Tall & Viner, 1981) of differentials, but also to see how much agreement existed among all interview subjects.

Our first study (McCarty & Sealey, 2017), conducted during the summer of 2016, involved interviewing four mathematics professors about how they perceived differentials in various contexts. In no context did all four subjects view the differential similarly, and while every context had agreement between some subjects, no two successive contexts had agreement between the same subjects. Three of the four subjects exhibited strong, personal images throughout all contexts, but these individual images were dissimilar, suggesting that no formal, unifying concept image can be found. Our second study, conducted during the summer of 2017, involved interviewing two mathematicians and one physicist about differentials in various contexts, as well as giving them potential concept images of differentials and asking if they would accept these potential images from their students. Again, there were no contexts in which all subjects agreed. Moreover, the responses from the physicist differed markedly from those of the mathematicians, and we found instances where the subjects might hold a concept image that they would not want their students to possess and vice versa.

In both studies, there were many and varied rich concept images suggested by some interview subjects; nevertheless, we conclude that there is no formal concept definition for the differential. We feel that these studies can be used to stimulate additional research, including, but not limited to, deeper study of differential concept images, further explorations into either how mathematician and physicist concept images might differ, or the differences between expert concept images and acceptable concept images from their students. We wish to present our results to encourage feedback and suggestions, as we move forward with this work.

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Transitional Conceptions of the Orientation of the Cross Product in CalcPlot3D

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*Students struggle with computing the direction of the cross product in relation to the two vectors that form it, but very little research has involved a non-contextual geometric cross product activity, especially in an online context. This study uses grounded theory to analyze student work completed for a dynamic, online visualization activity. Our preliminary research aims to develop categories that could outline a conceptual model of student understanding of the cross product.*

*Keywords:* vectors, cross products, visualization, transitional conceptions

The topic of determining cross products is prevalent in multivariable calculus, engineering, and physics curricula. Yet, research indicates that students struggle with problems involving the cross product (Knight, 1995; Barniol & Zavala, 2014). In particular students have difficulty determining the direction of the cross product and may not comprehend the non-commutative nature of the cross product (Kustusch, 2016; Scaife & Heckler, 2010). Research on understanding of the cross product has focused on symbolic manipulations. When graphical manipulations have been examined, it has been in a static environment on paper (e.g., Van Deventer, 2006; Zavala & Barniol, 2014). Here, we report on how grounded theory methodology was used to analyze student work in a dynamic virtual environment in order to understand how students communicate the direction and non-commutativity of the vector cross product.

The data for this study came from exploratory activities in the CalcPlot3D applet (Seeburger, 2017). The data set included electronic responses from 434 college-level, multivariable calculus students collected over four years to two embedded questions in the online cross product activity: *Considering the right-hand rule, what is true about the angle between the two vectors when the cross product vector points in the a) positive z-direction and b) negative z-direction?*

The responses were examined for emerging themes through a general inductive analysis (Thomas, 2006) using intercoder reliability where a single response to a single question was treated as the unit of analysis. Four main categories were identified and used for coding. Three were relevant properties to cross product: orthogonality, right-hand rule, cross product magnitude, and one was not: location of vectors. Ainsworth notes that one problem learners face in using multiple representations is retrieval of the relevant information and that this is strongly affected by a learners' familiarity with the topic (2008). Furthermore, developing ideas or "transitional conceptions" (Moschkovich, 1999) gleaned from student responses provided support for the categories created.

We report on the methodology, the findings, and limitations of the study as an initial step in developing a conceptual model of student understanding of cross product. This poster will provide visual displays and supporting evidence for the developed categorization system that allowed representation of both completely correct statements and statements that showed some thought in the category but that the idea expressed was neither correct nor precise.

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# Dynamic Textbooks and their Use in Teaching Linear and Abstract Algebra

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*In this poster we present two analyses of two dynamic textbooks. One analysis attends to their dynamic features, the mathematical practices embedded, and the scope of contents. The second analysis uses the documentational approach (Gueudet & Trouche, 2009) to investigate the ways in which these textbooks are used by instructors and their students. Data collection involves seven instructors and nearly 150 students across four states (New York, Texas, California, and Michigan; 50% female, 30% non-Caucasian or Asian) and surveys, logs, student tests, classroom observations, and clinical interviews. In both textbooks the interactive features are prominent via links and interactive computational cells (with Sage). They both include deduction, symbolization, and representation as mathematical practices. There are differences in the scope of contents. Regarding use we found that instructors took advantage of the features only when those could be integrated into their usual practices.*

*Keywords:* dynamic textbooks, textbook use, teaching, instruction

Even though the textbook continues to be one of the most important resources for instructors, textbooks enhanced with technologically advanced features are still in their infancy. In Undergraduate Teaching and Learning in Mathematics with Open Software and Textbooks (UTMOST, Beezer et al., 2016), we investigate whether and how instructors and students take advantage of features that are included in dynamic textbooks enhanced with Sage computational cells (Beezer, 2015; Judson, 2017). Data sources include bi-weekly logs, surveys, video recordings of the planning and the enactment of lessons, interviews, and tests of content knowledge with seven volunteer instructors (one female, five Caucasian), four teaching linear algebra and three teaching abstract algebra. The textbook analysis allowed us to discern textbook characteristics in terms of three emerging thematically connected categories: dynamic features, mathematical practices, and scope of contents. Using the documentational approach (Gueudet & Trouche, 2009) we analyzed two processes, instrumentation (how the textbooks "affect" the instructor) and instrumentalization (how the instructor "affects" the textbooks), present when instructors used the textbooks for planning and teaching. While the textbook analysis indicated that the potential for novel use is embedded in the design of the textbook features, we found that novel use was not as extended, in part, because the instructors lacked familiarity with, or experience using, the features embedded in the textbooks. In particular, we found that instructors took advantage of the features only when those can be integrated into their usual practices. All the participant instructors used their textbooks to create their lecture notes attending to the sequencing of topics presented in the textbooks and maintaining the notation, definitions, and theorems. Their lecture notes nevertheless included either different proofs (because the proofs provided were not satisfactory for the instructors) or additional examples (because the ones available in the textbook were not contextualized or had no geometric visualization). We explained those departures with instructors' personal and professional histories and experiences teaching a particular course, their understandings about how resources should be used, and their goals for teaching the course, according to the documentational approach.

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Benefits to Students of Team-Based Learning in Large Lecture Calculus

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*Team-Based Learning (TBL) is a specific form of active learning that utilizes the flipped classroom model. We implemented TBL in Calculus I in both large and small classes and investigated the impact of this form of instruction over two semesters. In the second semester, we observed many positive benefits to students, including exceptionally high class attendance, higher midterm and final exam scores, significantly lower DFW rates, and larger gains on the Calculus Concept Inventory when compared to students enrolled in non-TBL sections.*

*Keywords:* Team-Based Learning, flipped classroom, active learning, calculus, large class

Our ongoing study at Iowa State University addresses the following research question: Is Team-Based Learning Calculus I instruction more effective than non-TBL instruction? In Fall 2015 and Fall 2016, three members of our research group taught Calculus I in large (N~150 students) and small (N~35 students) classes using Larry Michaelsen's TBL approach (Michaelsen, Knight, & Fink, 2004). This teaching strategy is based on a constructivist learning theory and involves students first engaging with introductory material individually and then at a higher level in teams (Hrynchak & Batty, 2012). The students do preparatory work outside of class using reading guides and instructional videos before completing a five-question quiz individually and then again with a team. The majority of class time is spent working on application exercises in teams.

We investigated the impact of this form of instruction over two semesters and noticed steady improvement from the first implementation to the second. In our second implementation, we observed exceptionally high class attendance, including an overall attendance rate of 92% in one of our large classes. In analyzing midterm and final exam scores as well as DFW rates (percentage of students who finish the course with a D letter grade, F letter grade, or withdraw from the course), we compared our TBL students (N~370 students) to non-TBL students enrolled in the course (N~1500 students). The mean score on the departmental midterm exam (out of 100) for the TBL group exceeded that of the non-TBL group by 5.0 points, and the mean score on the departmental final exam (out of 100) for the TBL group exceeded that of the non-TBL group by 7.4 points. The DFW rate for the TBL group (19.1%) was significantly lower than the DFW rate for the non-TBL group (32.0%), with sufficient evidence at an alpha level of 0.01 using a two-sample t-test. We noticed lower DFW rates for TBL female students (24.7%) and TBL ethnic underrepresented students (34.8%) than their non-TBL counterparts (30.1% and 45.5%, respectively). By ethnic underrepresented, we mean African American, Hispanic, Native Hawaiian, Native American, or two or more ethnicities. Finally, the average of gains on the Calculus Concept Inventory (Epstein, 2013) for the TBL group (0.20 +/- 0.02) was larger than that of the non-TBL comparison group of 93 students (0.13 +/- 0.03), and this is statistically significant ($t(357)=2.16$, $p<0.05$).

Our findings have important implications for the way in which calculus is taught. The findings provide further evidence that active learning is an effective way to teach calculus, keeping in line with the study of Characteristics of Successful Programs in College Calculus undertaken by the Mathematical Association of America (Bressoud & Rasmussen, 2015).

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# Faculty Collaboration and its Impact on Instructional Practice in Undergraduate Mathematics

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*To reform instruction by moving towards student-centered approaches, research has shown that faculty need and could benefit from support and collaboration (Henderson, Beach, & Finkelstein, 2011; Speer & Wagner, 2009). In this qualitative instrumental case study I examine the ways in which a mathematician's instruction developed during his participation in a faculty collaboration geared towards reforming instruction and aligning it with inquiry oriented instruction (Kuster, Johnson, Keene, & Andrews-Larson, 2017; Rasmussen & Kwon, 2007). Preliminary results indicate ways in which student thinking was used as a discussion point in the faculty collaboration connected to the ways in which student thinking was used in the classroom to advance the mathematical agenda. Further, results indicate that the mathematical beliefs of the mathematician sometimes took precedence over the use of student work in the classroom.*

*Keywords:* faculty collaboration, instructional practice, differential equations

Research has shown that mathematicians may struggle implementing a new curriculum without support (Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007). Further, research has shown how summer workshops and online forums aid mathematicians in sustaining instructional change (Hayward, Kogan, & Laursen, 2015). Research is underway that explores diverse ways to engage mathematicians and support them in reaching their goal for instructional change (e.g., online video calls). This study focuses on one participant of an ongoing project to support mathematicians' instructional change, Dr. DM, and seeks to find links between his experiences in the online faculty collaboration and his instructional practice. The poster will address the following research question: How does one mathematician's instructional practice develop while participating in a faculty collaboration for inquiry-oriented differential equations (IODE)?

## Methods and Preliminary Results

Data for this case study comes from observations of the faculty collaboration online meetings, Dr. DM's classroom observations, and audio recordings of three interviews. Analysis of the classroom instruction uses the inquiry oriented instructional framework (Kuster et al., 2017) while analysis of the faculty collaboration and interviews uses a priori coding from our previous work (Keene, Fortune, & Hall, under review) as well as research on the roles of speakers and listeners in mathematics (Krummheuer, 2007, 2011).

Analysis is ongoing but preliminary results seem to indicate that Dr. DM's implementation of the IODE materials was influenced by his participation in the faculty collaboration. During the semester-long faculty collaboration Dr. DM shifts from discussing his students' thinking to evaluating and anticipating it. When that shift occurred, Dr. DM also shifted the way in which he used his students' thinking to advance the mathematical agenda. This use of student thinking, however, was stifled by his own mathematical beliefs when the content of the course aligned with his research agenda. While it is desirable for faculty to be passionate about their research and integrate it into their teaching for authentic learning experiences, in this case, it was sometimes at odds with the students thinking at given moments in the class. Further work will describe the connections and offer ideas to other facilitators of faculty online instructional support groups.

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# Student Intuition Behind the Chain Rule and How Function Notation Interferes

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*Recently, Speer and Kung (2016) informed the RUME community on what was missing from our research. In an effort to begin to fill these gaps in the literature, we explored students conceptual understanding of the chain rule in Calculus I classrooms taught by the first author. In this teaching experiment (Steffe & Thompson, 2000), our preliminary results indicate that if students are afforded opportunities to engage in experientially real tasks (Freudenthal, 1991; Rasmussen & King, 2000) on the chain rule, they understand the purpose it serves and can extend that understanding to varied contexts. However, the largest interference to this understanding was function notation, particularly nested function notation. Implications indicate that the instruction of chain rule could be enhanced by preempting a chain rule unit with nested function notation, while still maintaining tasks centered around a conceptual understanding of the chain rule.*

*Keywords:* chain rule, function notation, student mathematical thinking

While little is known about student understanding of the chain rule, much is known about student understanding of function and function notation. From this extensive body of research, we know that challenging activities, particularly constructive ones, aid in the development of the function concept (Carlson, 1998). Additionally, research highlights the importance of linking representations of functions and how that connects to learning function concepts (Even, 1998; Ronda, 2015). Research has also shed light on common misconceptions students have about functions. For example, students have been shown to not fully understand the use and meaning of parentheses in function notation (Carlson, 1998). In this study, we aim to answer the following research question: How do Calculus I students interpret various forms of notation when related to their understanding of the chain rule?

## Methods and Preliminary Results

In this teaching experiment, we filmed a full unit of chain rule from two Calculus I sections taught at a public university in the eastern United States. Focus of video data was always on small group work while students were solving tasks designed by the first author. The context for the tasks was as follows: A student, Mary, is taking a hike between two nearby towns. Students were given a graph of Mary's elevation height in terms of time and a table of the temperature of her location given an elevation height. Ultimately students were prompted to develop a need for the chain rule when ascertaining the change in Mary's temperature based on time.

Analysis is ongoing; yet, our preliminary results indicate that students were able to understand the need for the chain rule. That is, if a function $r$ depends on the function $s$, which itself depends on the variable $t$, the rate of change of $r(t)$ is the rate of change of $r$ in terms of $s$ times the rate of change of $s$ in terms of $t$, $\frac{dr}{dt} = \frac{dr}{ds} \cdot \frac{ds}{dt}$. However, when confronted with the parenthetical notation of nested functions, $r(s(t))$, students' ability to generalize the chain rule was impeded. Oftentimes students considered this parenthetical notation to be an indication of multiplication which lead them to misconceptions. Future work will consider ways to redesign instruction to preempt this pitfall in student thinking.

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Preservice Secondary Mathematics Teachers' Conceptions of the Nature of Theorems in Geometry

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*Proof plays an important role in school mathematics curriculum across grade levels and content areas. Being able to understand and apply the axiomatic system, such as with theorems, is considered as a high level of proof and reasoning ability in geometry. By adopting a collective case study design, I investigated preservice secondary mathematics teachers' (PSMTs) conceptions of theorems in geometry, in order to develop knowledge about PSMTs' current conceptions and provide mathematics educators and researchers with a possible means to unpack PSMTs' conceptions. This proposal focuses on one dimension of PSMTs' conceptions, the nature of theorems (NoT) in geometry. The Findings include interpretations of PSMTs' conceptions of the NoT, in terms of the ways they claimed the truth of mathematical statements, examined the validity of given proofs, and disproved given statements, as well as the role of task-based interviews in understanding their conceptions.*

*Keywords:* theorems, conceptions, proof and reasoning, geometry, teacher education

Proof and theorems form part of the core content of secondary geometry curriculum, and should be well grasped by secondary math teachers and their students (NCTM, 2000, 2003, 2012). Studies show that both secondary teachers and students have encountered challenges in teaching and learning proofs (Cirillo, 2009; Knuth, 2002; McCrone & Martin, 2004; NCES, 1998; Senk, 1985). In this study, I examined the essential elements of three PSMTs' conceptions of the NoT through research-informed task-based interviews, in order to answer the research questions: What conceptions do PSMTs hold regarding the NoT in geometry? And how do research-informed task-based interviews help unpack PSMTs' conceptions of the NoT in geometry?

I created a set of principles of the NoT that served as the conceptual framework for the development of the task-based interviews, including the elements *theorem has to be proved (NoT 1)*, *theorem is true for all instances (NoT 2)*, and *one counterexample is sufficient to disprove (NoT 3)* (Cirillo, 2014; Dreyfus & Hadas, 1987; Duval, 2007; McCrone & Martin, 2004). Each of the PSMTs participated in an individual task-based interview that addressed the above principles. The data analysis process started by "dividing the overall data set into categories or groups based on predetermined typologies" (Hatch, 2002, p. 152). An analytical framework was developed to identify the typologies of the data, including the definitions of PSMTs' *goals of the task*, *goal-directed activities (GAs)*, *sequence of actions* within the GAs, and *effects* of their GAs (Simon, Tzur, Heinz, & Kinzel, 2004; Tzur, 2007; Tzur & Simon, 2004).

The findings included interpretations about PSMTs' clarity of understanding about NoT 1, confusion about NoT 2 that the validity of the proving result and the validity of the proving process could be evaluated separately, and varied understandings about NoT 3 in terms of the definition of a counterexample and its role in disproving. In addition, the study discussed the role of the task-based interviews, in terms of providing an accessible problem-solving environment, encouraging free problem-solving, encouraging PSMTs' reflection, and letting the researcher be open to unforeseen activities during the interview (Goldin, 2000; Lin, Yang, Lee, Tabach, & Stylianides, 2012). The implications of the use of prompts in the interviews were also discussed.

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# Perceptions of Underrepresented Community College STEM Majors

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*Community College STEM majors from underrepresented groups were interviewed about their experiences in math classes and their motivations behind choosing a STEM major. The goal was to uncover events that may occur in math classrooms that serve to marginalize underrepresented students and contribute to the dearth of these demographic groups in these majors. As interview data was gathered, it became clear that these students did not suffer from feelings of marginalization. Results suggest that involvement in co-curricular activities, guidance and encouragement from faculty members, and support from family and peers may serve to mitigate feelings of alienation that can occur in students from these underrepresented groups.*

*Keywords:* Equity, Females, Minorities, STEM

The STEM fields in the United States have traditionally been dominated by white males. Females and minorities continue to be underrepresented in STEM occupations (NSF, 2017). These minority populations will comprise an increasingly larger percentage of the workforce moving forward, with the Hispanic population expected to increase by 115% between 2014 and 2060 (Colby & Ortman, 2015). If the United States hopes to remain competitive in these fields in the future, more candidates from these demographic groups must enter STEM degree programs at colleges and universities, earn degrees, and enter the STEM workforce.

The explanation for the dearth of females and minorities in STEM fields is multifaceted. However, the common denominator (pun intended) for STEM disciplines is their connection to math. Most of these disciplines require at least Calculus I, and many typically require through Calculus III. It is hypothesized that many underrepresented STEM majors may be lost due to racialized (McGee & Martin, 2007) and gendered (Hughes, 2000) encounters in these classes.

A group of underrepresented students in STEM majors at a suburban community college on the East Coast were interviewed about their experiences in math classes and the reasons behind their choosing a STEM major.

Research Questions:

1. Do specific events occur in math classes at the high school and college level that serve to marginalize females and minorities, affecting their retention in STEM fields?
2. What kinds of supports can help encourage the persistence of underrepresented students in STEM?

Interviews were conducted in the Spring and Fall semesters of 2017. Results suggested that these students did not suffer from feelings of marginalization in their math classes. On the contrary, encouragement they received from family, peers, and faculty members were instrumental in their success. Additionally, it is hypothesized that the community college context for this study impacted its results. Similar studies (Chavous et al, 2004; Cole & Espinoza, 2008; Wells, 2008; Espinosa, 2011) found varying student experiences based on the type of institution the student attended and its perceived campus climate. These results suggest that cultural capital (Bourdieu, 1986) may play a role in underrepresented student persistence and that the community college, with its unique positioning among institutes of higher education, may provide a more nurturing environment for underrepresented students to succeed in STEM.

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# Mathematical Reasoning and Proving for Prospective Secondary Teachers

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*The design-based research approach was used to develop and study a novel capstone course: Mathematical Reasoning and Proving for Secondary Teachers. The course aimed to enhance prospective secondary teachers' (PSTs) content and pedagogical knowledge by emphasizing reasoning and proving as an overarching approach for teaching mathematics at all levels. The course focused on four proof-themes: quantified statements, conditional statements, direct proof and indirect reasoning. The PSTs strengthened their own knowledge of these themes, and then developed and taught in local schools a lesson incorporating the proof-theme within an ongoing mathematical topic. Analysis of the first-year data shows enhancements of PSTs' content and pedagogical knowledge specific to proving.*

*Keywords:* Reasoning and Proving, Preservice Secondary Teachers, Design-Based Research

Our NSF-funded 3-year project addresses the limited practical and theoretical knowledge base on how to prepare PSTs to teach in ways that emphasize mathematical reasoning and proving (Ko, 2010; Stylianides & Stylianides, 2015). We designed, implemented and studied a novel capstone course: *Mathematical Reasoning and Proving for Secondary Teachers.* The focus of reasoning and proving was motivated by the persistent discrepancy between the value of proof as advocated by researchers (e.g., Hanna & deVillers, 2012) and policy documents (NCTM, 2009; CCSS, 2010) and the marginal place of proof in school mathematics, which is often viewed by students and teachers alike as redundant confirmation of known results, rather than a means for deepening understanding (Knuth, 2002; Kotelawala, 2016).

The course consists of modules corresponding to four proof-themes: quantified statements, conditional statements, direct proof and indirect reasoning, which were identified in the literature as challenging for students and PSTs (Antonini & Mariotti, 2008; Weber, 2010). Each module has activities to enhance PSTs' knowledge of a certain proof theme, followed by developing and teaching lessons at a local school integrating that proof-theme with current mathematical topics.

We used multiple sources of data to evaluate how PSTs' knowledge of content and pedagogy, and their dispositions towards proof evolved throughout the course. These included pre- and post- measures of mathematical knowledge for teaching proof and dispositions towards proving. We collected PSTs' lesson plans and 360° video-recordings of their lessons, which captured simultaneously the PSTs teaching performance and the school students' engagement with proof-oriented lessons. The PSTs also submitted self-reflections after each lesson, and cumulative teaching portfolios at the conclusion of the course.

Preliminary data analysis of the first-year course implementation shows improvement in PSTs' content knowledge of the four proof-themes. The repeated cycles of lesson development, implementation and video-supported reflection contributed to PSTs' pedagogical knowledge for proving. However, analysis reveals that integrating the proof-themes with pedagogical practices can be challenging for PSTs. To better support PSTs in this aspect, we plan to further conceptualize and enhance instructional scaffolding of the course in the subsequent iterations of the study. Through this process we seek to generate an evidence-based instructional model, and four proof-modules that can be adopted by other courses or institutions to improve preparation of secondary mathematics teachers.

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# The STEM Service Courses Initiative of Project PROMESAS:[1]

***Pathways with Regional Outreach and Mathematics Excellence for Student Achievement in STEM***

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*In this poster, we present preliminary results of how Project PROMESAS' STEM Service Courses (SSC) initiative assists collegiate instructors (n = 14) in transforming their teaching of Calculus I. These instructors participated in a one-week summer institute focused on integrating student-centered activities via rich tasks and promoting a sense of community in the classroom. During the fall 2017 semester, they adopted and adapted ideas from the summer institute and participated in monthly day-long follow-up meetings. The monthly meetings were an opportunity to continue learning about student-centered activities and to share newly created teaching materials. As part of the evaluation of this project, the instructors journaled during the summer institute and each month during the fall. These journal entries serve as our data: it suggests that although our instructors struggle balancing student-centered activities with teaching the required content, they are committed to transforming their teaching.*

*Key words:* collegiate PD, rich tasks, sense of community, student-centered learning

Nationally, lack of student persistence through the Calculus sequence is a significant contributor for students leaving their intended STEM disciplines (Bressoud, 2013). Research indicates that negative learning experiences "endured" during Calculus courses have the most effect on student retention (Seymore & Hewitt, 1997; Ulricksen, et al., 2010). Project PROMESAS is a regional STEM initiative where mathematics faculty from a 4-year Hispanic-Serving Institution (HSI) and 2-year HSI community colleges collaboratively address systemic change in teaching. The aim of Project PROMESAS' SSC initiative is to transform mathematics pathways into STEM and to strengthen the STEM student success pipeline.

The project emphasizes faculty development on cultural competency, inclusive pedagogy, and renewing the curriculum itself. Thus, we created a 2-year long professional development program for collegiate instructors. The curriculum for the PD focused on integrating student-centered activities via rich tasks while promoting a sense of community in the classroom as emphasized in the MAA *Instructional Practices Guide* (in press). The first cohort has completed a summer institute and follow-up monthly meetings for a semester, for all of which they responded to journal prompts. Preliminary analysis of these journal entries suggests that although our instructors struggle balancing student-centered activities with teaching the required content, they are committed to transforming their teaching. In our poster, we will detail the program, participants, data collection, and preliminary results of how this program is transforming our instructors' teaching of Calculus I. We will also share plans for the second cohort and our plans to assess the impact that this program has on STEM students' attitudes towards mathematics and their success in their STEM discipline.

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**Computational Thinking in Mathematics: Undergraduate Student Perspectives**


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|---|---|---|---|
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*Computational thinking (CT) is understood as the thinking, strategies, and approaches for solving complex problems with algorithmic considerations and in ways that can be executed by a computer. This survey study (n=104) reports on the conceptions of undergraduate mathematics majors and future mathematics teachers enrolled in a sequence of programming-based mathematics courses. Results suggest that students' emerging conceptions of CT became relatively well-aligned with expert views and that their characterization of CT included many computational practices (e.g. from modeling and simulation) and related affordances (e.g., creativity, agency) and outcomes (e.g., benefits learning, deeper mathematics understanding).*



*Keywords:* Computational Thinking; Concept Image/Definition; Taxonomy of Practices


Computational thinking (CT) has had a place in mathematics learning and education research since early experiments with Logo (e.g., Papert, 1980), and the push to introduce learners to CT practices has also recently increased in conjunction with the integration of computational applications into professional mathematical endeavors (e.g., Weintrop et al., 2016). Whereas research has been focusing primarily on school learning (e.g., Gadanidis, 2015) and more recently on ways computer programming can be used in undergraduate mathematics (Leron & Dubinsky, 1995), much work still needs to be done to understand how learners are engaging with CT practices, how they come to understand the mathematical content, procedures, and skills associated with computational applications, and what instructional interventions can best support student learning and achievement. With this in mind, we sought to investigate mathematics undergraduate students' conceptions of CT practices as they emerged during one of their three programming-based mathematics courses.

We addressed the following research questions:

1. How do undergraduate mathematics students characterize CT?
2. In what ways do undergraduate mathematics students' emerging understandings of CT align with expert categorizations of CT?

We use the framework of concept-image / concept-definition (Tall & Vinner, 1981) to analyze the evoked conceptions of undergraduates as they reflected on their understanding of CT during various stages of their course. For a concept definition of CT, we draw upon descriptions of CT mainly from the work of Wing (e.g., 2008, 2014) as well as the taxonomy of computational practices in mathematics and science proposed by Weintrop et al. (2016). Participants in our survey (3 times during the term) study were 104 undergraduate students enrolled in one of the three project-based mathematics courses at Brock University at which they learn to design, program, and use interactive computer environments to investigate mathematics conjectures, concepts, or real-world applications (Buteau et al., 2015). The specific items of the questionnaires were designed to elicit participants' personal conceptions of CT as related to their experiences coming into, and working through, the courses. Results suggest that overall by the middle of their CT-based mathematics course, students' emerging conceptions of CT became relatively well-aligned with expert views. Their characterization of CT included many computational practices, mainly from modeling and simulation, and computational problem solving; for example, a participant wrote: "[CT] is the ability to look at a problem and use models and computer simulations to solve and understand problems".

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# US and Chinese Prospective Elementary Teachers' Problem-posing Performance: A Comparison Study of Specific Problem-posing Processes

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*This study reports on 33 US and 55 Chinese prospective elementary teachers' problem-posing performance while engaged in five specific problem-posing processes. The study found that: (1) both US and Chinese participants were able to pose solvable mathematical problems, and (2) the US participants were more challenged by the Selecting and Comprehending processes, while their Chinese counterparts posed a higher percentage of solvable problems for these two processes than the other three processes. In the future, we might investigate the possible causes that triggered the differences and the impacts of the differences on mathematics learning.*

*Key words:* Problem Posing, Prospective Elementary Teachers, Comparison Study

## Literature Review

Teachers and students' problem-posing performance has been investigated in multiple countries (Chen, Van Dooren, Chen, & Verschaffel, 2011; Kojima & Miwa, 2008; Rosli, Goldsby, & Capraro, 2013; Siswono, 2014). However, only a few cross-national comparison studies have examined problem posing (Cai, 1998; Cai & Hwang, 2002; Yuan & Sriraman, 2011), and few were conducted with prospective teachers. Taking into account the unique contribution provided by international comparison study, which "allows us to see different things, and sometimes to see things differently" (Ma, 1999, p. xx), this study attempted to address the following research question: What are similar and different patterns of US and Chinese prospective elementary teachers' problem-posing performance while engaged in specific problem-posing processes?

## Guiding Framework and Methodology

This study utilized four problem-posing processes, i.e., Translating, Comprehending, Editing, and Selecting, as a guiding framework (Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman, 2005). Two sets of tasks involving fractions and geometric graphs were carefully designed. We first asked participants to pose problems for the above four processes, then to solve a given mathematical problem, and finally to pose two more mathematical problems. Each posed problem was classified into one of the following categories: (1) a solvable mathematical problem, (2) an unsolvable mathematical problem, and (3) not a mathematical problem.

## Results and Discussions

We found that both US and Chinese prospective elementary teachers were able to pose solvable mathematical problems. However, they showed quite a difference in performance in problem-posing processes. More specifically, the US participants were more challenged by the Selecting and Comprehending processes, while their Chinese counterparts posed a comparatively higher percentage of solvable mathematical problems for these two processes than the other three processes. We also noticed that both US and Chinese participants' best performance in problem posing did not occur during the problem posing that came after the problem-solving process. In the future, it would be important to investigate the causes that triggered such big differences as well as the impacts of such differences on their mathematics learning.

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Mathematics Through the Lens of Service-learning


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*In this poster, we report on a study conducted at a midwestern private liberal arts university where researchers incorporated service-learning into a non-major mathematics course. Data reveal students felt more confident learning mathematical concepts because of its real-world application in the community. Additionally, students reported an increase in the value of mathematics and its importance in societal reforms.*



*Keywords*: mathematical anxiety, service-learning, undergraduate students, civic engagement


Many undergraduate students experience math anxiety, which often results in their being unprepared (Nagy et al., 2010; Ashcraft, 2002) to pass graduation requirements for a bachelor's degree (Bound, Lovenheim, & Turner, 2010). This leads to students developing negative attitudes toward mathematics. One strategy to combat these issues is engaging students in service learning, which can increase confidence (Soria & Thomas-Card, 2014; Soria, Nobbe, & Fink, 2013; Soria, Troisi, & Stebleton, 2012). As students communicate math skills from college classrooms to community settings, they learn practical and applicable uses of mathematics in daily life. This is particularly beneficial for low-income and first-generation students who gain self-efficacy, persistence, and college retention (Yeh, 2010). Furthermore, Schulteis (2013) argues service learning is an "excellent way to enhance the extent of student learning" by helping students develop "greater mastery of classroom material and an increase in civic values and skills" (p. 582). Indeed, there are calls to train instructors to engage in learning beyond the classroom (National Task Force on Civic Learning and Democratic Engagement, 2012; Kielsmeier, 2010) so service learning can be a requirement of college education.

We wanted to explore these opportunities through a non-major mathematics course at our midwestern private liberal arts university. The course incorporated three service learning activities, worth 10 percent of the course grade, by partnering with a local elementary school. The course was offered for one month, Monday—Friday for 3.5 hours every day. Thirty-three undergraduate students were enrolled in the course and participated in our study. Students created and revised lesson plans from the topics discussed in class and then taught the lessons to about 50 third graders. After each teaching session, university students wrote self-reflections on their experience, which were collected, analyzed, and coded as data using open coding methods (Strauss & Corbin, 1998).

Three key themes emerged from this data. First, students expressed more confidence in mathematics communication and a better understanding of its role in society. Second, students found teaching through hands-on mathematical activities more applicable to the real world, which was different than prior experiences learning in a traditional university classroom setting. Third, students reflected on becoming more aware of future generations of young(er) students; they shared hopeful statements that these elementary children would grow up to make a difference in the world because of educational opportunities like this course/study. Ultimately, our data demonstrates that service learning opportunities can transform mathematics from something scary and disconnected to a more meaningful and civically engaged area of study.

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# The Distribution of the Mathematical Work during One-on-one Tutor Problem Solving

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*Undergraduate math tutoring is an important context for student learning, yet little empirical work has been done to understand tutor-student interactions. Using frameworks for problem-solving and socially mediated metacognition (Carlson and Bloom, 2005; Goos, et. al, 2002), this poster examines who guides the development of mathematical ideas throughout the problem solving process within a drop-in one-on-one tutoring context. We found that the majority of the tutoring interactions closely followed the Orienting-Planning-Executing-Checking phases of problem solving. The "Executing" phase had the highest degree of student contribution, while the "Checking" phase was least represented.*

*Keywords:* Undergraduate Tutoring, Problem Solving, Socially Mediated Metacognition

Peer undergraduate mathematics tutoring is widespread (Sonnert & Sadler, 2015) and has been shown to lead to significant learning gains for both tutors and students (Graesser, 2011; Lepper & Woolverton, 2002). However, empirical investigation of tutor-student interactions has been minimal (Roscoe & Chi, 2008), particularly at the undergraduate level. Given the lack of theoretical development in this area, our understanding of tutoring interactions can be framed by modifying lenses developed for other contexts. For example, we draw heavily on best practices for teaching, which strongly emphasize the importance of active learning (Larsen et. al, 2015; Freeman et. al, 2014; Topping 1996). In addition, the majority of math tutor interactions, particularly in a drop-in context, are based on solving homework problems. Thus, frameworks for problem solving, such as Carlson and Bloom's (2005) are useful for understanding the progression of the tutor-student interaction. We are interested in understanding the problem-solving process in a tutor-student interaction. In particular, this poster focuses on who guides the mathematics in the interaction, and how that shifts during the problem-solving process.

Data for this study was drawn from 18 undergraduate math tutors at two different universities in a drop-in tutoring environment. Tutoring sessions were recorded using video or scribe-cast. Data for this analysis was based on 6 episodes. The episodes were selected based on their clarity and focus on a problem-solving context. We coded transcripts according to two frameworks. First, we identified the problem solving phase: (1) Orienting, (2) Planning, (3) Executing, or (4) Checking (Carlson and Bloom, 2005). Next, within each phase we identified how the mathematics was being presented or developed. We modified Goos, Galbraith, and Renshaw's (2002) coding scheme for socially mediated metacognition to include kinds of interaction (Explain, Answer, Question, Correct, or Reflect) and types of mathematics for those interactions (Information, Strategy, Concept, or Computation).

We found that many of the tutor-student interactions closely followed the problem-solving cycle proposed by Carlson and Bloom (2005). The "Checking" phase was least represented in our episodes: commonly a single line or completely absent. Across the episodes, the "Planning" phase had the most variation in the level of student participation; either entirely planned by the tutor or planned cooperatively between the dyad. The "Executing" phase had the most consistent student mathematical contributions. This study indicates a need for tutor training that elicits student mathematical contributions at every stage of the problem solving process.

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Features of Tasks and Instructor Actions That Promote
Preservice Secondary Mathematics Teachers' Understanding of Functions

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*The Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers Project is developing research-based tasks and explorations as well as instructor materials to be used in mathematics courses for preservice secondary mathematics teachers. The project, now in year two, continues to develop and refine these items based on data collected in year one and the advice of an expert panel and advisory board. The goal of this poster presentation is to provide information on lesson development and methods used in determining key characteristics of instructor moves for building student understanding of functions as well as gather feedback and suggestions on further design and development.*

*Keywords:* Mathematical Knowledge for Teaching, Preservice Secondary Mathematics Preparation, Functions

High-quality mathematics teaching requires common, horizon, and specialized content knowledge (Ball et al., 2008; Hill, Ball, & Schilling, 2008). Developing this understanding of the content preservice teachers will teach can be done by engaging in tasks that illuminate mathematical concepts (Loucks-Horsley et al., 2003; Zaslavsky, 2008). This project aims to refine and supplement widely-used secondary teacher preparation materials used in a course on functions as well as develop an instructor's guide to scaffold the lessons and explorations.

## Research Questions

Currently, limited research exists on how to facilitate development of profound understanding of functions and key characteristics of mathematical tasks that can promote this specifically for preservice secondary mathematics teachers. Discussion with RUME attendees will assist us in identifying additional design issues that need to be accounted for in addressing the following research questions: (1) What are key characteristics of mathematical tasks that promote development of profound understanding of functions for secondary mathematics teachers in the first two years of undergraduate study? (2) What are key instructor moves and pedagogical strategies for facilitating development of profound understanding of functions for secondary mathematics teachers engaging in high-yield mathematical tasks?

## Discussion

Over two iterations of this course at a large urban university in the Southwest, this project has collected student artifacts including pre- and post-surveys on functions, classroom videos, pre-class instructor logs, post-class instructor interviews, observation notes, and 18 student interviews. This poster will provide details on lesson development as well as the methods used in determining key characteristics of instructor moves for building understanding of functions.

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## Connecting Physics Students' Conceptual Understanding to Symbolic Forms s Using a Conceptual Blending Framework

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*In an effort to understand physics students' construction of equations in terms of mathematical structures, previous work has employed a symbolic forms framework. To account for students' contextual physics understanding related to these structures for vector differentials, we mapped symbolic forms into the framework of conceptual blending to model students' construction of equations. This allows us to shed further light on recent literature in this area.*

Key words: Physics, Equation, Symbolic Forms, Conceptual Blending

Much of physics involves the construction and understanding of equations. Writing an equation to describe a physical system is a process that entails encoding conceptual meaning of the related physics using specific variables and mathematics symbolism to describe the ways in which the physical variables relate to one another. In many theoretical models used to frame how students use mathematics in physics this process is labeled "modeling" or "mathematization" (Redish & Kuo, 2015; Uhden et al., 2012; Wilcox et al., 2013).

Interpreting the equation as a construct of physical-mathematical language, we present a model for the construction of equations, developed from research on student understanding of non-Cartesian vector differentials (Schermerhorn & Thompson, 2017), that combines a symbolic forms framework addressing the structures through which students understand physics equations (Sherin, 2001) and formal conceptual blending theory from linguistics (Fauconnier & Turner, 2002). In this model the conceptual schema of symbolic forms, which describes the justification for the mathematical structures of an equation, serves as the underlying generic space in a conceptual blending framing of students' construction of equations and thus drives the blend of two input spaces: Sherin's symbol template (the externalized structure of the expression) and content understanding. Symbolic forms were designed as acontextual constructs, independent of content understanding. Therefore, by incorporating conceptual blending theory we can explicitly connect students' content understanding to the expression of terms in an equation.

The proposed model for equation construction allows us to reinterpret recent symbolic forms literature (Jones, 2013; Kuo et al., 2013; Meredith & Marrongelle, 2008) which has interpreted the conceptual schema to be on par with, rather than independent from, content understanding. Conceptual blending literature addressing the interwoven nature of mathematics in physics at both the introductory (Bing & Redish, 2007; Brahmia et al., 2016) and upper levels (Bollen et al., 2016; Hu & Rebello, 2013), has not included the generic space, which serves as an underlying structure for each of the conceptual input spaces and determines which pieces combine to form a new blended concept. Our approach uses features of one framework to fill in the missing analytical aspects of the other framework in these contexts.

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## Children's Topological Thinking

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**Abstract:** *This poster presents an ongoing investigation into children's topological thinking. Prior research identified and advanced children's informal ideas about topological equivalence – and equivalence and invariance more broadly. This investigation extends that research into children's thinking about related ideas such as order, separation, and coverings. Newly identified forms of geometric thinking have implications for the teaching and learning of geometry and for research into students' mathematical thinking.*

**Keywords:** Geometric reasoning, Topology, Equivalence, Teaching experiment

Children's experiences in geometry throughout elementary school are entirely Euclidean. However, research finds that they also possess intuitive topological ideas (Greenstein, 2014; Laurendeau & Pinard, 1970; Piaget & Inhelder, 1956). These findings lay the foundation for the claim that there are forms of topological reasoning available to young learners that can be identified as mathematical, are significant, and can be seen to develop in ways that would have implications both for research into students' mathematical reasoning and as a focus for further curriculum development and design.

The poster we are proposing will illustrate our current investigation into children's thinking about topological equivalence, as well as the foundations for this investigation in prior research (Greenstein, 2014). That research found that a microworld for topological equivalence (Greenstein, 2017) supported two children's constructions of ways of thinking about topological equivalence. They used those schemes (von Glasersfeld, 1995) to build equivalence classes of shapes and identify the properties of shapes within equivalence classes. Broadly speaking, it was evident from this investigation that engagement with topology provides learners with powerful forms of mathematical engagement that are not available to them in Euclidean geometry.

Our current investigation seeks to develop a superseding model (Steffe & Thompson, 2000) of children's thinking about topological equivalence, and extend the focus of our prior research into additional aspects of equivalence relations that arise in the context of topology, including notions of order, covering, and separation. For example, through their investigations of topological equivalence and invariance, children are also engaging with the ideas of order through examinations of points along a curve; of coverings through a task that provides the child with a square and calls for a collection of shapes that can adequately cover that square; and of separation through a task that provides a collection of distinct points and calls for shapes that can be used to confirm their separation.

Findings from this study are beneficial to students whose topological ideas have yet to be engaged in schools and also to the community of mathematics educators whose research has only nominally investigated them.

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# Development of Students' Mathematical Discourse through Individual and Group Work with Nonstandard Problems on Existence and Uniqueness Theorems

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*Research shows that students' learning is affected by the types of tasks. We explore how the use of nonstandard problems influences understanding of the Existence and Uniqueness Theorems (EUTs) by a group of engineering students. The focus is on the development of students' mathematical discourse during the individual and group work with nonstandard problems. We present the evidence indicating that students developed new mathematical routines gaining a deeper understanding of EUTs and appreciated the experience.*

*Key words:* ordinary differential equations, existence and uniqueness theorems, design research, mathematical discourse, individual work, group work, nonstandard problems.

## Description of the Study

Ordinary differential equations (ODEs) is one of important post-calculus courses in university STEM (Science, Technology, Engineering and Mathematics) education. Nevertheless, the recent review of the literature related to research on ODEs in undergraduate education during the last decade surveys only about twenty papers dealing with the understanding of the concepts of solution of an ODE, a system of ODEs and bifurcation (Rasmussen and Wawro, 2017). EUTs are among very few theoretical results included nowadays in standard ODE courses for engineering students. Understanding and the correct use of the EUTs present serious challenges for students (Raychaudhuri, 2007), as even the concept of a solution of an ODE itself (Arslan, 2010). In our study, students consecutively produced three scripts of solutions to the set of six nonstandard problems designed by the lecturer to challenge students' conceptual understanding of the EUTs: individual solutions obtained in the first tutorial, individual solutions submitted as a homework, and solutions submitted after the discussion with peers in small groups and group presentations of solutions during the second tutorial. We analyzed three scripts, pre- and post-activity surveys, and audio recordings of the peer discussions and of the presentations.

## Research Questions

1) How can nonstandard problems challenge students and help to develop analytical skills and further conceptual understanding of mathematical routines in an ODE course?
2) To what extent have individual work and group discussions contributed to the development of students' mathematical discourse?

## Conclusions

Working on the problems, students made use of theorems and definitions, generalized and designed examples, verified validity of statements and analyzed reasoning. All these practices promoted students' conceptual understanding and contributed to the development of a new mathematical discourse because, using a commognitive lens, "learning mathematics means modifying one's present discourse so that it acquires the properties of the discourse practiced by mathematical community." (Sfard, 2009). We believe that lecturers should ask more nonstandard questions that they know their students will find difficult and may not be able to answer, and do it more often. Our research has shown that students valued the experience and gained a deeper understanding of the EUTs.

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# Assessing the Development of Students' Mathematical Modeling Competencies: An Information Entropy Approach

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*We suggest a new scaling tool for converting big amounts of qualitative data into quantitative data based on the recent developments in the information theory. We believe that it can be used with reasonable efficiency to monitor the development of students' mathematical competencies, and not only. Discussing advantages and shortcomings of this tool along with the possibilities for further development, we invite to discussion of a new approach.*

Keywords: mathematical modeling, competencies, quantitative data, evaluation tool, information entropy.

## Description of the methodology

We introduced mathematical modeling tasks with biological content to engage biology students more actively into learning mathematics and created mathematical competencies profiles for individual learners to follow their development from session to session. From a wide selection of approaches to the notion of mathematical competencies reported in the literature (Maaß, 2006, Boesen et al, 2014, Weinert, 2001), we chose a competency framework from the Danish KOM project (Niss, 2003). Viewing a competency as an individual's ability to use mathematical concepts in a variety of situations, within and outside of the normal realm of mathematics (Niss, 2003), we retain five basic groups of mathematical competencies out of the eight suggested in KOM: thinking/acting mathematically, modeling mathematically, representing and manipulating symbolic forms, communicating/reasoning mathematically, and making use of aids and tools. Fifteen competencies in five groups are coded separately in a reliable manner. When the data collected in the sessions are coded, the record of each competency frequency and strength (beginning, intermediate, developed, exemplary) is being kept. This creates big data sets which we would like to analyze in order to assess and monitor students' competency development. To this end, we rely on a so-called Shannon entropy (information entropy), one of the central concepts in the information theory. Our approach to the competencies development evaluation combines Shannon's entropy and VIKOR method developed for finding closest to an ideal solution to decision problems with conflicting and noncommensurable criteria. Our research questions in this study are: 1) How can large amounts of qualitative data be converted into quantitative data by using the tools from modern information theory? 2) How reliable and efficient is this new scaling tool?

## Conclusions

We believe that a new entropy-based scaling tool opens new interesting opportunities for researchers who need a consolidated evaluation for big amounts of data; it could be efficiently used both to monitor the development of individual students' competencies and to compare their performance. It allows to assign different weights to competencies providing thus possibilities for monitoring the progress in the development of particular skills. However, using this method of data analysis inevitably leads to the loss of some essential information, the risk we are willing to undertake in order to study the potential of this new progress-tracking tool.

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Queer Students in STEM: The Voices of Amber, Charles, Jenny and Juan

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*Abstract: This report provides a rich narrative documenting the experiences of four queer students in STEM, which showcase both the challenges and power of being queer in STEM. Students viewed the nature of STEM through a paradoxical lens of a discipline that is objective and thus neutral to issues of identity, yet hostile and exclusionary to non-normative identities in STEM spaces. In response, queer students in undergraduate math courses described the difficulties in navigating the amount of personal information they reveal about themselves or be faced with the psychological burden or cognitive stress derived from presenting in non-normative ways.*

Keywords: Queer, STEM, LGBT, oSTEM, Narrative Analysis, Equity

Despite growing attention paid to student identities, when it comes to the topic of sexual orientation, the research literature remains largely in the closet; it neglects to address the impact or representation of queer individual in STEM. However, there is some evidence suggesting that marginalization due to sexuality might be felt more acutely within STEM-related courses (Bilimoria & Stewart, 2009).

We recruited students from two LGBTQ-friendly universities (Pride Index, 2017), that have active oSTEM student organizations. Four students, Amber, Charles, Jenny and Juan agreed to participate in the study. We used a semi-structured interview protocol (Ginsburg, 1997) to target information about their experiences as a queer student in STEM; how they perceive the nature of STEM; their favorite courses and instructors; description of the "coming out" processes; advice for other students; and the completion of two mathematical tasks. A narrative analysis based in grounded theory was utilized to identify emergent themes (Strauss & Corbin, 1994).

The students in this study described multiple ways in which they conceived of the nature of math and science, resulting in paradoxical experiences. The students made mention to STEM as an objective set of processes, focused on facts or rules. Yet, their queer identities are often at odds when viewed through a lens of precision within the STEM discipline. Furthermore, the students felt that the nature of STEM is removed from their personal identities, and described the classroom as a vacuum operating without consideration to the external world. As an example, Jenny characterized her bisexual identity in STEM as "silent," and felt that her math professor did not create space for processing traumatic events (e.g. impact of presidential election).

The students' stories further showcase challenges and the impact of being queer in STEM. Students described "coming out" in STEM spaces as either a form of information control or as a psychological distractor. For instance, Charles uses a form of "vetting." If Charles deems a person "safe enough," he will slowly engage the person in conversation to determine whether he will "come out" to the person. Charles also stated that he had "very few positive experiences coming out or being queer within my major." In contrast, Amber did not feel that they have a choice when conveying their gender fluid identity. Amber described feelings of psychological stress induced by presenting in gender non-conforming ways in math classrooms.

While this study seeks to capture and promote voices of queer students using narrative accounts as a means of centering queer identities in STEM discourses, it calls to attention the many voices often silenced by resistance. As an example of persistence in the face of resistance, Juan stated, "not everyone saw the rationale in creating a space for queer in STEM. So that was so discouraging. I was so angry. But that fortunately lit a fire under me."

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Red X's and Green Checks: A Preliminary Study of Student Learning from Online Homework

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*Homework is thought to play an important role in students' learning of mathematics, but few studies have addressed what, if anything, students learn from doing online homework. This study is a preliminary attempt to answer the question* do students learn what instructors intended they learn from an online homework assignment about sequences?

*Keywords:* homework, sequences, instructional triangle

Homework is thought to play an important role in students' learning of mathematics. University calculus students spend more time doing homework than they do in class (Ellis, Hanson, Nuñez, & Rasmussen, 2015; Krause & Putnam, 2016), meaning that homework accounts for the majority of students' interaction with content. Due to increasingly sophisticated technology, online homework has become more prevalent in mathematics courses. However, few studies have investigated student learning from online homework. Researchers have found that online homework format does not have a statistically significant effect on course or exam grades, or that it has a slight positive effect (Dedic, Rosenfield, & Ivanov, 2006; Halcrow & Dunnigan, 2012; Hauk, Powers, Safer, & Segalla, 2002; Hirsch & Wiebel, 2003; LaRose, 2010). A few studies have investigated qualitative factors related to homework, such as students' preferred format (Hauk & Segalla, 2002; Krause & Putnam, 2016) and the resources they use while doing homework (Krause & Putnam, 2016). However, we do not know much about what, if anything, students learn from doing online homework. This study is a preliminary attempt to answer the question *do students learn what instructors intended they learn from an online homework assignment about sequences?*

The study design reflected a theoretical perspective employed by Ellis et al. (2015), who positioned homework as a *task* in Herbst and Chazan's (2012) instructional triangle. In the triangle, edges represent the interactions between teacher, students who complete tasks, and the knowledge at stake. This study used a series of clinical interviews with professors to determine what they saw as the knowledge at stake within a particular online homework assignment, and observations and interviews with students to investigate whether their interaction with the homework supported their learning that knowledge. Specifically, the researcher conducted clinical interviews (Hunting, 1997) with two calculus II instructors about what they hoped students would learn from each of 14 problems on an online homework assignment about sequences. The researcher then video-recorded three students as they worked individually on the homework. In the third phase, the researcher and student watched the video together while the researcher asked questions about what the students did and why, if the student felt (s)he had learned anything from the problem, and if the student felt (s)he had learned or noticed what the professor intended for that problem.

Preliminary findings indicate that students learned some of what professors intended (e.g., how to generate terms in a recursive sequence, what notations like $a_n$ and $b_n = a_{n+1}$ mean) but not all (e.g., that a sequence is defined on a set of sequential integers, not on the set of real numbers). Analysis is ongoing, and should yield implications for designing homework tasks that engender desired understandings of ways of thinking.

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Essential Aspects of Mathematics as a Practice in Research and Undergraduate Instruction

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*A gap between mathematics as used by mathematicians and mathematics as experienced by undergraduate mathematics students has persistently been identified as problematic; A commonly proposed solution is to provide opportunities for students to* do *mathematics and* be *mathematicians (e.g., Whitehead, 1911; Harel, 2008). Conceptions or beliefs about what this means may vary depending on a mathematician's research and experience. The authors explore mathematicians' expressed conceptions of mathematics in their research and in their teaching.*

*Keywords:* conceptions, beliefs

A gap described between meanings of mathematics as used in mathematical research and mathematics as experienced by students has persistently been identified as problematic, potentially preventing students from knowing mathematics in the expected deep and complex ways (e.g., Whitehead, 1911; Harel, 2008). One solution proposed by mathematicians and mathematics educators for all levels is to provide opportunities to students to *do mathematics* and *be mathematicians* (e.g., Whitehead; Harel; Ball, Lubienski, & Mewborn, 2001; Stein, Grover, & Henningsen, 1996). This solution depends, however, on the conceptions or beliefs held about the nature of mathematics itself (e.g., Skemp, 1978). We asked: How do mathematicians at Georgia Southern University view mathematics as a practice in their own mathematical research? What aspects of mathematics do they try to teach their undergraduate students?

We developed 12 broad statements about aspects of mathematics based on descriptions of the nature of mathematics from mathematicians and mathematics educators (e.g., Ball et al., 2001; Ernest, 1989; Harel, 2008; Stein et al., 1996; Whitehead, 1911). Mathematics is: (a) a mass of details and procedures; (b) strategies and solutions with internal or external validity; (c) ideas that can be theoretically interesting, elegant, and beautiful; (d) ways of thinking systematically and analytically; (e) ways of precisely communicating; (f) a powerful tool for interacting with real world and everyday situations; (g) productive struggle through framing and solving problems; (h) experimentation through making and testing conjectures, examining constraints, and making inferences; (i) the study of patterns; (j) the abstraction of properties and characteristics apart from emotions or sensations; (k) a human endeavor, continually growing through the dynamic process of creating knowledge through purposeful activity; and (l) a crystalline structure existing in complete, static, pure form, discovered through logical reasoning.

We selected 10 mathematics faculty with different research interests and experience. We asked each participant to respond to two open-ended questions and two questions that involved a card-sorting task. We used separate open-ended questions to ask them to describe the nature of mathematics as it appears in their research and that they intend to teach to their undergraduate students. We used the 12 statements in separate think-aloud card-sorting tasks. In each, the participant chose to keep, discard, or edit each card or to add new cards. They selected four aspects of mathematics they felt were most critical in 1) their research and 2) their teaching.

We analyzed faculty participants' selections and reasoning to understand how they view the nature of mathematics and how they hope their students will view the nature of mathematics. In this poster, we present their views to explore more deeply what it would mean for students to *do mathematics* and to *be mathematicians* in different areas within undergraduate mathematics.

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# Reflections on a Peer-Led Mentorship Program for Graduate Teaching Assistants

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*This poster will describe a peer-led mentorship program offered to Graduate Teaching Assistants (GTAs) in a Canadian University. We present the creators' rationale for implementing this program, as well as the perspectives of the two graduate student peer mentors who have taken the lead in its development and implementation.*

*Keywords:* Graduate teaching assistants, peer-mentoring, professional development

Research concerning the professional development of mathematics graduate teaching assistants (GTAs) has been driven by two seemingly contradictory observations: (1) GTAs typically arrive at their graduate studies with little to no *formal* training in mathematics education (Kung & Speer, 2009); and yet (2) GTAs play a significant role in shaping the current and future state of undergraduate mathematics education (Speer, Gutmann, & Murphy, 2005). To assist GTAs in successfully assuming teaching-related positions like marking, one-on-one tutoring, leading tutorials, or instructing introductory courses, training programs of various natures have been developed and discussed (e.g. Belnap & Allred, 2009; DeFranco & McGivney-Burelle, 2001). With this poster, we aim to contribute to this discussion by presenting and reflecting on one possible approach to "training", in the form of a peer-led mentorship program. This represents the initial phase of a wider research project aimed at better understanding how to provide GTAs with *educational experience* (in the sense of Dewey, 1938). Our work aligns with the goal mentioned in multiple papers (Speer, Deshler, & Ellis, 2017; Speer, Murphy, & Gutmann, 2009) to use research to inform the design, improvement, and efficacy of professional development programs for GTAs.

For almost a decade, a retired professor volunteered at a mathematics department of a large, urban Canadian university to provide support to new GTAs. Following his retirement from all activities, the department decided to initiate a peer-led mentorship program in the fall of 2016. Although the initial motivation for shifting to peers as opposed to professors was mainly an issue of availability, the department grounded its choice on two main goals. First, to provide students with training and guidance from mentors who can closely relate to their current experience, both within and beyond their teaching roles. And second, to build a safe (i.e., confidential and non-evaluative) community within which graduate students come to see it as normal to receive feedback about their teaching, reflect on their teaching choices, and discuss various pedagogical and didactic issues. Two doctoral students were chosen to run the program, based on their previous experience and interest in teaching, certain characteristics of their personality (e.g., their likelihood to put a lot of effort into the development of the program, as well as to be empathetic, open, and constructive in interactions with their colleagues), and their complementarity (e.g., one researches pure mathematics, while the other researches mathematics education at the university level).

Our poster will provide details about the kinds of interactions that have occurred between the mentors and mentees, sometimes over multiple semesters, and reflect on struggles and successes, as perceived by the mentors. As a result, we hope to incite discussions about the participation of peers, who have studied both mathematics and mathematics education, in improving the experience of GTAs and the teaching and learning of mathematics at the undergraduate level.

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Capture of Virtual Environments for Analysis of Immersive Experiences

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*This poster addresses emerging technologies for the capture and reconstruction of participants experiences in immersive virtual environments. These methods might improve communication of participants experiences when solving mathematical problems in three-dimensional contexts.*

*Keywords:* Technology, Virtual-Reality, Mixed-Reality, Geometry

Virtual, mixed and augmented reality technologies provide novel opportunities for undergraduate students to investigate three-dimensional mathematical phenomena. For solid geometry, these immersive dynamic spatial displays support dynamic construction with virtual manipulatives in an immersive space without numeric measurements (Dimmel & Bock, 2017). Recent studies observing students solving mathematical problems use a two-dimensional projection of the first-person perspective in immersive (Lai et al, 2016; Bock & Dimmel, 2017) and augmented environments (Radu et al, 2015). However, these renderings might limit researchers' ability to analyze student's experiences in the virtual environment. This poster discusses the research question: How can participant's experiences solving mathematical problems in immersive three-dimensional mediums be understood through two-dimensional mediums? To address this question, this poster explores mixed-reality video capture, three-dimensional gesture capture, and figure logging provide as partial solutions for the reconstruction of a student's experience.

## Mixed Reality Video Capture

Mixed Reality video capture can be used to record and stream live video of the physical participants inside their virtual environment (Figure 1), using both physical and virtual external cameras (Blueprint Reality, 2017). This provides more context about the participant's environment than a first-person view (Figure 1), while it can still be managed within traditional 2-d mediums.

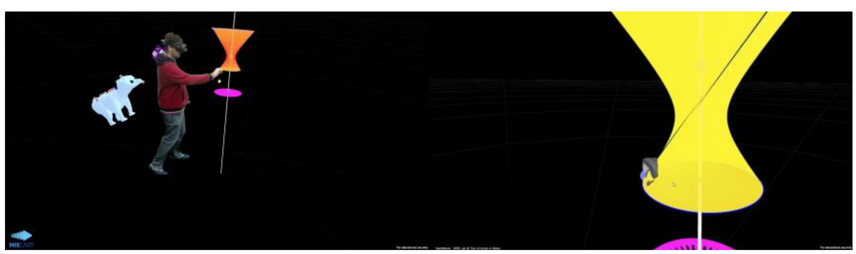

*Figure 1. First Person (left) and Mixed Reality (right) views of the virtual environment.*

## Applications to Future Research

These data capture methods might improve studies using immersive dynamic spatial displays in addition to the first-person screen captures, when researching small groups of participants or sharing research data between multiple researchers. Immersive renderings of dynamic figures might be relevant for future studies of the teaching and learning of college mathematics in contexts where three-dimensional properties are difficult to render in 2D, including solids of revolution, constructions in solid geometry and gradients of three-dimensional surfaces.

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Growth Mindset Assessments in Mathematics Classrooms

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*Recent scientific evidence shows the incredible potential of the brain to grow and change. Equally important are the observations of the positive impact that having growth mindset has upon students' achievement. Students with a growth mindset view errors and obstacles as opportunities for growth. These students welcome challenges and the opportunity to learn from their mistakes. Although some university instructors are incorporating growth mindset into their lectures and attitudes, unfortunately, the traditional exam method used in undergraduate mathematics classrooms is a fixed mindset model. This poster shows the implementation of a growth mindset structured exam in a multivariable calculus class. The implementation incudes structured opportunities to rework exam problems, give presentations, and papers. All of these focus on assessing the student's achievement of the objectives in the class.*

*Keywords:* Growth Mindset, Assessment, Exam, Attitudes, Multivariable Calculus

A mindset is a self-perception that people hold about themselves. In a fixed mindset, people believe that traits like intelligence and talent are unchanging. They spend time documenting their intelligence instead of trying to learn and grow. In a growth mindset people believe that intelligence and talents can be developed through hard work and dedication. Growth mindset has been shown to have a positive impact on student achievement (Dweck, 2007).

Four techniques that teachers can use to increase the growth mindset of their students are as follows. First, let students know what growth mindset is and teach them that their brains can grow. Second, praise them for their efforts and not for intelligence (Dweck, 2007). Third, tell students stories of people that achieved great things with hard work and a growth mindset (Aronson, Fried, & Good, 2002). Finally, teach students that mistakes are how our brains grow (Moser, 2011) to create an atmosphere in the classroom that leads students to look at mistakes (theirs or others) without any shame, but instead opportunities to improve. Like the other techniques, openness to mistakes can be fostered through changes to classroom instruction. Importantly, it can also usefully be addressed when assessing the students' achievement of the course objectives.

In this poster we describe the implementation of a growth mindset exam structure in a multivariable calculus class at a large public university. Each exam consisted of a traditional in-class portion and a take home portion built on principles of growth mindset. The structure of the assessment had three main elements: students were able to rework, and then orally defend their learning on individual exam problems from the in-class portion of the exam, they gave presentations, including worked out board problems, on a relevant topic, lastly they wrote papers that focused on comprehension, communication and understanding of the objectives for the course.

In this poster, we will describe the course and exam structure; present examples of the exams, paper requirements, presentation descriptions, and corresponding rubrics; exhibit student work; and give feedback from students about the growth mindset exam structure.

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An APOS Perspective of Meaning in Mathematics Teaching

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*The 'meaning of' mathematics can be thought of as mathematical understandings whereas the 'meaning for' mathematics can be understood as understanding the significance of math for non-mathematical purposes. Studies have suggested instructors have difficulty addressing both senses of meaning simultaneously while other studies have indicated factors that affect graduate teaching assistants' (GTA) instruction. Using APOS theory as a theoretical lense, this study examines how these factors affect GTA instruction of the derivative and in turn, how GTAs navigate differing senses of meaning. Through interviews, the researcher found many parallels between GTA instruction and proposed decompositions of the derivative. Regarding meaning, the researcher found when GTAs experience tension between the two senses of meaning, instructional decisions may be taken that anticipate GTA instructional concerns.*

*Keywords:* APOS Theory, Mathematical Meaning, Derivative, Graduate Teaching Assistants

Brownell (1947) defined the "meaning of" mathematics as mathematical understandings and the "meaning for" mathematics as understanding its significance. Studies on instruction in certain contexts, like service learning, have shown tension between these senses of meaning for instructors (Carducci, 2014; Connor, 2008; Donnay, 2014; Hadlock, 2013; Rousseau, 2004; Schulteis, 2013; Zack & Crow, 2013). Whether this occurs in a 'typical' math class needs further study. With respect to graduate teaching assistants' (GTAs') instruction, studies have identified factors such as content knowledge, responsibilities, and control (Addy & Blanchard, 2010; Bond-Robinson & Rodriques, 2006; Hill, Rowan & Ball, 2005). By adapting APOS theory as done Martin, Loch, Cooley, Dexter, and Vidakovic (2010), a decomposition of the derivative was used to categorize instruction while the framework of meaning categorized the 'why' behind those decisions. By doing so, this study aimed to see how affective factors effect instruction.

Participants were mathematics GTAs (Ann and Inigo). Data included interviews and emails which explored instruction of the derivative, beliefs, concerns, and instructional goals. Using a research-based genetic decomposition (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Hähkiöniemi, 2006), responses on instruction were coded as action, process, or object depending on level of elicited understanding while the reasons for choices were coded as "meaning of" or "meaning for"

Results showed the GTAs covered much of the decomposition, eliciting action up through object level understandings. On derivative rules, Inigo (lacking content control, but content with the set syllabi) would have students go through derivations while Ann (who took issue with the syllabi and mentioned competing responsibilities) only would in some cases to be able to stay on schedule. Cutting engagement with derivations subsequently cuts engaging with the limiting process and perhaps results in a pre-object understanding of derivatives as noted by Zandieh (2000). Interestingly, Ann was concerned students do not connect limits and calculus. If pressure to cover material is a case of focusing on the 'meaning for' and the strictly conceptual aims are cases of 'meaning of', attending to the 'meaning for' seems to have anticipated a teaching concern for Ann. While preliminary, perhaps understanding meaning for instructors may serve as an organizing framework of how affective factors reciprocally influence instruction.

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University Teachers' Meanings for Average Rate of Change: Impacts on Student Feedback

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*Previous research that has used Thompson's mathematical "meanings" framework has focused on secondary teachers' meanings for mathematics. We examine the meanings that graduate students and professors hold for average rate of change. Further, we attempt to connect meanings to the facets of student work that graduate students and professors notice. This work lets us start to extend the meanings framework and has implications for graduate student professional development.*

*Keywords:* Meanings, Mathematical Knowledge for Teaching, Average Rate of Change

Using two average rate of change items from the Mathematical Meanings for Teaching Secondary Mathematics (MMTsm) (Thompson, 2016; Yoon et al., 2015), we designed a task-based interview protocol and gathered written and interview data from graduate students and professors to answer these questions 1) what mathematical meanings do graduate students and professors hold for average rate of change (AROC)? 2) Based on these meanings, what do graduate students and professors notice about students' meanings of AROC displayed in written work? Goals for this poster are to present the interview design as well as the results from at least three pilot interviews, and for RUME attendees to discuss how the mathematical meanings framework (Thompson, 2016) connects to Ball and colleagues' framework for mathematical knowledge for teaching (Ball et al., 2008). These discussions will help shape connections we draw between the frameworks after further analysis and data collection.

Average rate of change is a concept commonly included in pre-calculus and calculus curricula in universities across the US. It has been identified as a concept that students *must* grasp to be prepared for calculus (Carlson et al., 2003). Recent work has identified meanings that secondary teachers hold for AROC (Yoon et al., 2015) and for underlying concepts that support AROC such as quotient, measure, covariation, and rate of change (Byerley & Thompson, 2017). In this vein of research, the focus is the teacher-centric personal meaning that each teacher holds for the mathematics they teach. By examining this construct, we can draw conclusions about the ways that student meanings may be supported or limited by teachers' meanings. By utilizing the mathematical meanings framework to examine university professor meanings, we aim to build on existing work by drawing connections between professors' and graduate teaching assistants' meanings and what they notice in student work on AROC tasks.

Results from pilot interviews suggest that graduate students may not have fully coherent meanings for AROC. Interviews were analyzed by coding for meanings identified in Yoon and colleagues' (2015) study. On the MMTsm items, one student conveyed a weak meaning for unit, and another student conveyed a formulaic meaning as the arithmetic mean. Both students were limited by a meaning for rate of change as the slope of a secant line, which made it difficult to convey their meaning in any way besides a graphical representation. Interestingly, in contrast to the professor, both graduate students' noticing of the mathematical meaning in student written work was framed by how similar their meaning was to that of the student. These findings can be used to further refine the meanings framework and inform efforts to design professional development for graduate students.

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**MPWR-ing Women in RUME: Continuing Support**

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|---|---|---|---|---|
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*For the past four years, we have run the MPWR Seminar, a daylong mentoring and networking event for women in RUME. Each year, we have hosted 60 – 90 women at various career stages (graduate students, postdoctoral fellows, faculty and professionals outside of academic positions). The seminar takes place the day before the annual RUME conference, allowing most of the participants to continue engaging with each other throughout the subsequent three days during RUME. In this poster, we address the motivation for the seminar, the structure and topics from the 2017 seminar, modifications in the structure for sustainability purposes, and research related to the efficacy and transferability of MPWR. Our aim in sharing this poster is to disseminate our experiences and gather input from the community on what more could be done.*

**Key Words:** Mentoring, Women, Support

Women in STEM fields are disproportionately underrepresented at all stages of a career in academia (Hill, Corbett, & St. Rose, 2010). Mentoring can serve as a mechanism to draw in and keep women in these positions, but this too is lacking (Beede et al., 2011). Preston (2004) highlighted that mentoring is underutilized, and other researchers suggest that not all types of mentoring are as effective among women (Allen, Day, & Lentz, 2005; Caldwell, Casto & Salazar, 2005). In particular, they point to informal mentoring as most helpful for women, but it may be difficult for women and marginalized groups to seek out and form these informal relationships (Ragins & Cotton, 1999).

Female mathematicians with a research concentration in undergraduate mathematics (RUME) are doubly (and sometimes more) marginalized, firstly for being females in mathematics and secondly for being math education researchers in math departments. The degree of mentoring women historically received varied drastically, primarily due to varied personal or academic networks, creating inequitable access to much needed support for success in this field. The inequity especially stood out for women coming from universities with no other RUME researchers, or for women coming into RUME from mathematics or non-undergraduate mathematics education. As such, we recognized a need for support and mentorship for this subpopulation of the RUME community. We thus began the Mentoring and Partnerships for Women in RUME (MPWR) Seminar in 2014 to establish the first formal mentoring structure for women in RUME, and for RUME participants in general (though there have been recent efforts to change this for the broader RUME community). It is thus the intent of MPWR to establish mechanisms that provide support for all women at all career stages in their academic development.

In this poster, we address the motivation for the seminar, the structure and topics from the 2017 seminar, modifications in the structure for sustainability purposes, and research related to the efficacy and transferability of MPWR. Our aim in sharing this poster is to disseminate our experiences and gather input from the community on what more could be done.

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Comparing Students' and Teachers' Descriptions of First Year STEM Instruction

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*The precalculus through single-variable calculus (P2C2) sequence is often viewed as a barrier for STEM intending students. Additionally, many students point to poor instruction as a primary reason for leaving STEM. This leads to many questions about student experiences in the P2C2 sequence. This study is part of a larger national project and draws on student and instructor survey data from three universities. We aim to lay groundwork for understanding student experiences in the P2C2 sequence by answering: (1) How do P2C2 students and instructors describe their class, (2) Do students and instructors describe them differently?*

*Keywords:* Precalulus, Calculus, Instruction, Student Perceptions, Instructor Perceptions

## Methods

Data for this project pulled from student and instructor surveys administered to all students and instructors in the P2C2 sequence at three universities. To answer our research questions, we replicated Ellis, Kelton, and Rasmussen's (2014) analyses of student and instructor surveys. We consider 430 student surveys and 14 instructor surveys. The student and instructor surveys included 16 parallel items regarding classroom experiences (e.g., 'I listen and take notes as the instructor guides me through major topics' and 'I guide students through major topics as they listen and take notes'). Responses were obtained on a 5-point scale, with five representing most descriptive of their class. We considered descriptive statistics and conducted a paired samples t-test for each of the 16 student-instructor responses.

## Sample Results

Both students and instructors responses indicated that the item related to the instructor knowing the student's name (i.e., 'The instructor knows my name' and 'I know most of my students by name') had the highest mean rating of the 16 items, $M_{students}=4.36$, $SD_{students}=1.08$, $M_{instructor}=4.76$, $SD_{instructor}=0.54$. Alternatively, student responses indicated that 'I explore or discuss my understanding of new concepts before formal instruction' had the lowest mean rating, M=2.75, SD=1.11, while instructor responses indicated that 'I structure class so that students constructively criticize one another's ideas' had the lowest mean rating, M=2.14, SD= 0.81.

Results suggested that the average student rating was significantly different than the average instructor rating for 13 of the 16 items. For instance, instructors indicated that 'students regularly talk with one another about course concepts' was significantly more descriptive of their class, M=3.59, SD=1.24, than students, M=3.20, SD=1.28, $t(421)=6.27$, $p<0.05$, $d=0.31$. Additionally, students indicated that 'I have enough time during class to reflect about the processes I use to solve problems' was significantly more descriptive, M=3.29, SD=1.13, than what instructors indicated for 'I provide time for students to reflect about the processes they use to solve problems', M=2.89, SD=0.91, $t(419)=5.96$, $p<0.05$, $d=0.29$. The Cohen's (1988) standardized effect size suggests that the both of these differences in means were medium.

Along with the results presented here, we will further investigate the differences by separating our data by class. We will compare each class' average student rating to their instructor's rating. Additionally, we will compare our results to Ellis, Kelton, and Rasmussen's (2014) findings.

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Support for Active Learning in Introductory Calculus: Exploring the Relationship Between Mathematics Identity and Pedagogical Approaches

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*Calculus I is a main gatekeeper course for STEM majors, so increasing student success in this course is imperative to retaining more students in STEM fields. Since students' mathematics identity is a strong predictor of pursing a STEM career, more information is needed about how students are developing these identities in introductory mathematics courses. This study reports a piece of a larger mixed methods approach to gain more insight into how instructional approaches in introductory calculus are related to students' mathematics identity development. Interviews were conducted to explore students' perceptions of the pedagogy used in their introductory calculus class. Students' descriptions of their mathematics identity, which includes the constructs of interest, recognition, and performance/competence beliefs are discussed and compared between an active learning environment and a traditional lecture classroom.*

*Keywords:* Calculus Success, Mathematics Identity, Active Learning

Student success in introductory calculus is imperative to obtaining a degree in any STEM field, with about 75% of students taking Calculus I intending to have a career in STEM (Bressoud 2015). Calculus I has been shown to be gatekeeper course, and research continues to show that Calculus I "lowers students' confidence, enjoyment of mathematics, and desire to continue in a field that requires further mathematics" (Bressoud 2015). One reason for this continued problem is a lack of information about how pedagogical choices influence the culture established in the calculus classroom and how this impacts students' mathematics identity development. Engineering identity has been shown to be significantly related to grade performance in introductory engineering courses, which merits taking a closer look at this relationship for introductory calculus (Schar 2017). Also, Cribbs (2012) found that students' mathematics identity strongly predicts their career choice in STEM fields.

Results from the 2015 MAA national calculus study showed that traditional lecture was the predominant instructional method used in Calculus I throughout the country. However, a meta-analysis of 225 studies comparing traditional lecture to active learning in STEM courses found that failure rates under traditional lecture increase by 55% over the rates observed under active learning (Freeman 2014). A smaller study at a large research university of student grade trends in Calculus I revealed that failure rates were significantly lower when an active learning model was implemented in the mathematics department (Norton et al. 2017). Since introductory calculus still functions as a gatekeeper role for STEM majors, more attention needs to be paid to the learning environments that are being provided for these students.

This study will report a piece of a larger mixed methods approach to gain more insight into how instructional practices in introductory calculus are related to students' mathematics identity development. Interviews were conducted to explore students' perceptions of the pedagogy used in their introductory calculus class. Students' descriptions of their mathematics identity, which includes the constructs of interest, recognition, and performance/competence beliefs (Cribbs 2015), will be discussed and compared between an active learning environment and a traditional lecture classroom. Preliminary analysis revealed that the group work provided in the active learning classroom supported students' performance/competence beliefs as well as their feelings of recognition as a mathematics student.

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DISA – Digital Self-Assessment for Large University Courses

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*We introduce a model for replacing the course exam with self-assessment on a large undergraduate mathematics course. In our course model, the self-assessment method is seen as a part of new learning environment that enhances the students' reflection skills and encourages them to foster their ownership of their own learning. Self-assessment skills are trained throughout the course and the students receive feedback from multiple sources, including teachers and peers. The DISA model is an important initiative to develop large course pedagogy in the university mathematics setting.*

*Keywords:* Self-Assessment, Reflection, Technology-Enhanced Assessment, Assessment Culture

Improving students' reflection and self-assessment skills is an important goal of university education, as these skills are vital for life-long learning and building a successful career (e.g. Boud, 2000). Self-assessment has been shown  to have a positive effect on learning: It can improve reflection skills (MacDowell, 1995), emphasise learner autonomy and communication skills (Stallings & Tascione, 1996) and be a more effective learning method than studying for an exam (Friess & David 2016). However, self-assessment skills are rarely explicitly taught. In the DISA project, we have replaced the final exam with self-assessment on a large undergraduate mathematics course. At the end of the course, the students assess their own skills based on a detailed assessment rubric, which contains not only content knowledge items but also generic skills such as mathematical writing and discussion. Cheating is controlled by an automatic verification process in which the student's self-assessment is compared to their performance in various tasks during the course. Self-assessment is supported by extensive formative feedback during the course, as well as several self-assessment exercises.

The DISA method has been implemented twice on a first-year course in linear algebra. Based on quantitative and qualitative studies on the course feedback, the self-assessment and removing the course exam have encouraged deeper learning approaches in the students. They also report having “studied for themselves” instead of the course exam. They also mention having been relieved from stress and anxiety related to the final exam. The grades the students assign for themselves have been comparable to a typical grade distribution on the course, with the exception that the students are reluctant to give themselves the lowest grades 1 and 2 (on the scale 1–5).

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Using Comparative Judgment to Analyze Precalculus Algebra Exam Tasks

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*Mathematics educators hold varying views on the teaching and learning of mathematics. Literature revealed inconsistencies in educators' interpretations of conceptual understanding and procedural fluency. To explore these differences in perspectives, the current study asked mathematics doctoral faculty, college instructors, graduate teaching assistants, and high school teachers to analyze undergraduate Precalculus Algebra final exam items. Participants were asked to compare exam items based on their effectiveness to gauge conceptual understanding, procedural fluency, or differences between students. The method of comparative judgment was used to yield an ordered rank of the items from least effective to most effective, as perceived by the judges. The weak ordinal association between the rankings of the different groups indicated disagreement in educators' judgments. The factors contributing to these differences in perceptions are unknown. Further research includes exploring these discrepancies in item rankings amongst different subgroups of mathematics educators*.

*Keywords*: Precalculus, Task Analysis, Comparative Judgment

College students are expected to develop proficient conceptual understanding and procedural fluency in their mathematics courses (MAA, 2015). Examinations should be designed to measure students' acquisition of such proficiency. Literature revealed inconsistencies in college mathematics instructors' implementation and perceptions of exam tasks according to the tasks' cognitive demand and conceptual orientation (e.g. Tallman, Carlson, Bressoud, & Pearson, 2016; White & Mesa, 2014). This study was designed to investigate the extent to which these differences in perspectives of tasks exist between different groups of math educators.

The study implemented the method of Comparative Judgment, founded on Thurstone's (1927) psychological principle, which asserts that judgments are comparative in nature. Using this principle, a series of pairwise judgments can be used to produce a linear measurement scale. The online comparative judgment application from No More Marking was used to create twelve surveys—three for each of the four groups of math educators: doctoral faculty, college instructors, graduate teaching assistants, and high school teachers. Each survey contained thirty prompts, through which the judges compared two of the forty-two Precalculus Algebra final exam items. The judges were asked which exam item was better at gauging students' conceptual understanding, students' procedural fluency, or differences between students.

No More Marking used the Bradley-Terry probability model to develop twelve rankings of the items, from least effective to most effective, according to their ability to gauge conceptual understanding, procedural fluency, and difference in student performance, based on the math educators' subjective judgment. Kendall's rank correlation coefficient, tau, was used to compare the ordinal association between the ranks. Weak correlations were found between the rankings of tasks by the different groups, indicating disagreement in the mathematics educators' classifications of the tasks. Further research could explore what factors contribute to these differing perceptions of tasks, and which characteristics of mathematical tasks deem them more conceptual versus procedural.

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The Development of a Video Coding Instrument for Assessing Instructional Quality in Community College Algebra Classrooms

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*This poster presents the evolution of a video coding protocol for mathematics classroom instruction. We highlight challenges encountered while analyzing 18 hours of pilot data from six community college algebra classrooms, entailing calibration of over 150 episodes.*

*Keywords:* community colleges, algebra, instruction, student success, video coding protocol

Community colleges prepare many students for STEM and other mathematics-based career options. Specifically, in 2010, more than 585,000 students were taking intermediate or college algebra in community colleges (Blair et al., 2013). It is shown that there is a relation between quality of instruction and student learning based on the instructor's knowledge of teaching, content knowledge, and instructional practices in K-12 education (Hill, Rowan, & Ball, 2005). However, there is limited information regarding characteristics of instruction that inform community college student success. Throughout 2016-2017 while working on a federally funded research project (Watkins, Duranczyk, Mesa, Ström, & Kohli*, 2016)*, we attempted to establish a protocol to codify instruction so that we can identify the conditions under which instruction in community college algebra courses associate well with student learning gains and performance. As we began to consider important features of mathematical instruction in community colleges, we looked to research in K-12 mathematics classrooms to identify features of quality instruction. We started by reviewing the Mathematical Quality of Instruction (MQI) (Learning Mathematics for Teaching Project, 2011), a video analysis tool for mathematics instruction in grades K-6. We then adapted the Quality of Instructional Practices in Algebra (QIPA) (Litke, 2015), a tool developed for 9th grade Algebra lessons, to add features that we could not capture with the MQI. Modifications in both protocols were warranted given that the tools (MQI and QIPA) did not clearly provide distinctions or delineations of practices we observed in the community college classroom associated with quality instruction. By adapting these two protocols, we created, developed, and refined a new protocol, Evaluating the Quality of Instruction in Post-secondary Mathematics (EQIPM) that addresses the complexity of community college mathematics instruction. The videos were segmented into 7.5-minute episodes and distributed among five teams. The teams (at least one community college faculty member per team) using an iterative process of review and calibration coded 18 hours of algebra instruction. We shifted from a 4-point scale with 21 codes to a 5-point scale with 15 codes to record components of quality instruction. This poster will chart our development process and seeks feedback from our peers.

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Geometric Reasoning in an Undergraduate on the Autism Spectrum: A Magic Carpet Case

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*In this report, I examine the unusually precise geometric reasoning of a student in linear algebra given the beginning of the Magic Carpet sequence outside of their normal curriculum. Analysis of possible reasons for taking this approach and implications for teaching are presented.*

*Keywords:* autism, linear algebra, geometric reasoning

My research attends to mathematical problem solving by adults on the autism spectrum (with a formal diagnosis), particularly those with a relatively strong background in mathematics. In this report, I focus particularly on the case of one student's work on one of the Magic Carpet problems of Wawro, Rasmussen, Zandieh, Sweeney, & Larson (2012).

Much of the research currently done on mathematics learning in people on the autism spectrum is focused on young children (e.g. Klin, Danovitch, Mers & Volkmar, 2010; Simpson, Gaus, Biggs & Williams, 2010; Iuculano et al., 2014) or looks at mostly arithmetic. There is also a notable strain of work done on the population of research mathematicians (e.g. James, 2003; Baron-Cohen, Wheelwright, Burtenshaw & Hobson, 2007), but very little attention is paid to groups in the middle (mainly high school and college students, or adults other than career mathematicians). This is a gap which I have sought to help fill with my own research, including the particular selection which I present here.

The theoretical framework that guides my research is rooted in the work of Vygotsky. I also start from a perspective of neurodiversity, generally referring to a positive and inclusive perspective on not only autism, but also other neurological differences (Silberman, 2015). More specifically, given my interest in focusing in-depth on interviews with a small number of people, I use case studies (Yin, 2009) from these perspectives. I also include Fischbein's notion of intuition (1979, 1982) and Grandin's work on geometric reasoning and autism (1995) to further my analysis.

The data for my study comes from a series of eleven clinical interviews with a university student on the autism spectrum that I conducted, each focusing on a different set of problems. In this report, I focus on the first of several Magic Carpet tasks, introduced by Wawro, Rasmussen, Zandieh, Sweeney, & Larson (2012).

In this specific portion of the data, I have found suggestions of a tendency toward higher precision than typically seen in geometric solutions and inclinations toward systematic rather than intuitive reasoning. The tendency toward geometric solutions generally is also a notable characteristic, but comparison to other participants suggests that this is only part of a tendency among people on the autism spectrum to gravitate toward favored types of solutions.

I also examine possible effects of the tendencies seen in the interview data for instruction, such as on the possibility to avoid intended approaches and topics (such as an algebraic view of vectors) as well as opportunities to take advantage of the unusual approaches of students on the autism spectrum to benefit instruction overall. This highlights the importance of being able to see validity in unusual student work and interacting with students without deficit-based preconceptions, something which holds particular importance across a variety of forms of disability-related education research.

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# What are the Functions of Proof in Introduction to Proof Textbooks?

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*The role of proof in mathematics is multifaceted. de Villiers (1990) argues students should be exposed to at least five essential roles: verifying, explaining, systematizing, discovering, and communicating. We analyzed six commonly-used Introduction to Proof (ITP) textbooks for their treatment of the function of proof. We used a thematic analysis approach leveraging de Villiers categorizations. We found that verification was heavily emphasized, while discovery was almost excluded. However, the textbooks emphasized verifying, explaining, systematizing, discovering, and communication in ways that de Villiers did not. For example, de Villiers focused on the global view of systematizing, where the books were more likely to emphasize the local structure of proof.*

*Keywords*: Introduction to Proof, textbook analysis, role of proof

Textbook analysis assesses and provides information of curriculum, which influences the instructor's lessons and thereby students' learning (Stein, Smith, & Remillard, 2007). Further, the textbook itself is a resource that impacts students directly. The ITP course is frequently students' first indoctrination into the mathematical community's proof practices. Each of the ITP books provided commentary and information for students to assimilate the function of proof in mathematics. We focus on the early portion of the texts that explicitly treat function of proof.

*Methods.* We selected textbooks reflecting market-share (David and Zazkis, 2017): Chartrand, Polimeni, and Zhang (2013), Hammack (2013), Smith, Eggen, & St Andre (2006), and Velleman (2006). Additionally, we selected a Mathematical Association of America publication (Hale, 2003) and an international series (Cullinane, 2013). Each book was coded using a thematic analysis (Braun & Clarke, 2006) using first de Villiers' (1990) functions of proof as an initial set of codes: *verification*, *explanation*, *systemization*, *discovery,* and *communication* with a sentence-level unit-of-analysis. For example, "You will learn and apply the methods of thought that mathematicians use to verify theorems [verification], explore mathematical truth [explanation] and create new mathematical theories [discovery]." (Hammack, 2013). Within these categories, we both expanded roles beyond de Villiers' framing and developed a set of sub-classifications capturing additional factors such as intended audience.

*Sample Results.* seen in the table 1, the role of verification dominated the textbook discussion while the role of explaining, systemizing, and discovery were inconsistently treated across textbooks. ITP textbooks may not address important roles of proof. If we want students to learn and appreciate these roles, the impetus may be on instructors to move beyond textbook treatment.

*Table 1. Instances of reference to de Villiers' roles of proof.*

| Book/function | Verify | Explain | Systemize | Discovery | Communicate | total |
|---|---|---|---|---|---|---|
| Smith, et al. (2006) | 4 | 0 | 4 | 0 | 1 | 8 |
| Chartrand et al. (2013) | 10 | 5 | 0 | 4 | 4 | 18 |
| Velleman (2006) | 12 | 1 | 2 | 0 | 5 | 18 |
| Hammack (2013) | 3 | 1 | 1 | 1 | 3 | 7 |
| Hale (2003) | 5 | 2 | 4 | 0 | 6 | 10 |
| Cullinane (2013) | 10 | 9 | 0 | 0 | 2 | 17 |

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Mathematics Teaching Assistant Preparation and Support: What Would Piaget, Vygotsky, and Dewey Have to Say?

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*The purpose of this paper is to analyze the philosophies of the three foundational learning theorists (Piaget, Vygotsky, and Dewey) and explore what can be learned from these theorists about mathematics teaching assistant-professional development (MTA-PD). I begin by reflecting on the professional development opportunities I personally received as an MTA and by analyzing how well my own experience aligns with each of the three theories. I conclude with an argument, based in the literature, why MTA-PD may best be served by Vygotsky's sociocultural theory.*

*Keywords:* Mathematics Teaching Assistant (MTA), Learning Theories, College Teaching, Professional Development

Since 2009, at the conferences hosted by The Special Interest Group of the Mathematics Association of America on Research in Undergraduate Mathematics Education, there has been a working group focused on the professional development of college mathematics instructors. One of the group's goals, consistent with my own for writing this paper, is the "development of materials, processes, and *theories* to support the professional development of collegiate mathematics instructors" (Hauk, Deshler, & Speer, 2015, emphasis added).

Piaget (1964/1997) asserts that individuals construct their own knowledge and build on prior knowledge. Dewey (1899/1964) claims that knowledge is generated through problem solving, inquiry, and experimentation. Vygotsky (1978) posits a strong role for social interaction in learning. Which of these theories is most useful for analyzing MTA preparation and support?

Characteristics of successful K-12 teacher professional development include an intensive initial experience, spaced-across-time follow-up, opportunities to analyze student thinking, collaboration in teams to learn about teaching, and working with a mentor through multiple classroom visits and follow-up (Blank & de las Alas, 2009). Research specific to college teaching has revealed important elements of MTA-professional development (MTA-PD). These include a focus on MTAs' mathematical knowledge for teaching (Musgrave & Carlson, 2017; Speer & Wagner, 2009), an overall culture of department support for good teaching (Latulippe, 2009), and opportunities for practice and feedback from a mentor (Ellis, 2014). Since each of these elements involves high levels of social interaction, Vygotsky's sociocultural theory aligns well with successful MTA-PD. Indeed, Vygotsky's (1978) mechanism for learning is *internalization*, or the concept that knowledge is built first *between* individuals and then moves inward to the intrapersonal plane. MTA-PD that involves collaboration in teams, opportunities for practice and feedback from mentors, and a department culture of support for good teaching, therefore aligns well with Vygotsky's theory. His notion of the *zone of proximal development* (Vygotsky, 1978, p. 86) is also salient here, with more capable peers helping MTAs reach higher levels of good teaching that, if left to their own means, they would not be able to reach.

Since many universities rely on MTAs to teach of a wide variety of undergraduate mathematics courses, and because MTAs are "the source of mathematics faculty of the future" (Speer, Gutmann, & Murphy, 2005, p. 76), further analysis of MTA-PD through the lens of the three foundational learning theorists will likely prove important for the success of both present and future undergraduate learners of mathematics.

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Research on Concept-based Instruction of Calculus

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*Abstract: This study, involving 254 college-level calculus students and 3 teachers, investigated the misunderstanding of concepts in calculus and designed concept-based instruction to help students understand concepts. Multiple achievement measures were used to determine the degree to which students from different instructional environments had mastered the concepts and the procedures. The midterm examination and the final examination results showed that the students enrolled in the concept-based learning environment scored higher than the students enrolled in the traditional learning environment and the investigation at the end of the semester showed that most of students like the concept-based learning environment.*

*Key words*: Concept-based Instruction, Misunderstandings, Teaching design

In the context of mass higher education, the ability of college freshmen is generally in a lower level than before. Many college students can do simple works on calculus, but they cannot understand the idea behind the concept, and as a result, usually have fuzzy understanding of the relationship between concepts. Therefore, to find the cognitive difficulties of the students on the concepts of calculus and to design the concept instruction are the keys to the reform of the teaching on Calculus.

This research presented a study on calculus course in three freshmen classes by carrying out the teaching design and teaching experiment. Research methods such as design research, questionnaires, interviews and classroom observation were adopted. There were 3 teachers and 254 students participated in the practice. Based on the findings of this study, the following conclusions could be drawn:

Firstly, college students' concept image of the fundamental concepts of calculus was one-sided, and some even wrong. Some students couldn't define the limit by correct words. Most of the students usually thought of the slope of the tangent when seeing the derivative, rather than the rate of change. There was confusion in the understanding of the geometrical meaning of differential and linear approximation. Some students know that the definite integral can express the area, but they can't make sure the area of what region; some students did not know which amount was sliced when they calculated the integral.

Secondly, we constructed principles on concept instruction in calculus as follows: (1) Concepts were introduced and demonstrated in a genetic way. (2) Help students understand the concepts by means of geometric or intuitive examples. (3) Paying attention to the elaboration of the relations of the concept between them. The results of teaching experiment showed that the students enrolled in the concept-based learning environment scored higher (M=34.42) than the students enrolled in the traditional learning environment (M=30.27) on the 40 point Conceptual Understanding Subscale and the students enrolled in the concept-based learning environment scored significantly higher (M=48.68) than the students enrolled in the traditional learning environment (M=42.65) on the 60 point Procedural Skill Subscale in the examination.

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Mathematics Tutors' Perceptions of Their Role

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*Undergraduate students who work as mathematics tutors completed surveys and interviews to assess their attitudes towards mathematics and their beliefs about the roles of a tutor and instructor. Tutors in this study viewed their role as supplementary to that of a teacher, and emphasized tutors' ability to tailor their mathematical content to the individual.*

*Keywords:* mathematics tutors, beliefs, roles

In a recent study, 97% of institutions surveyed offered mathematics tutoring to Calculus students (Bressoud, Mesa, & Rasmussen, 2015), so tutors are a prevalent resource for student learning outside of the classroom. Research on classroom teachers has investigated how teachers' perception of mathematics and their role as a teacher affects instructional practice (Thompson, 1984), and how teachers notice students' mathematical thinking (Jacobs, Lamb, & Philipp, 2010). Similar questions may be asked of mathematics tutors. This study will report on undergraduate mathematics tutors' views of mathematics and their perceptions of their role as a tutor and how that compares to the role of a mathematics instructor.

The participants in this study were undergraduate tutors from drop-in mathematics tutoring centers at one large research university in the Midwestern United States and one small private university in the northwestern United States. Twenty-six tutors were given surveys prior to the start of the Fall 2017 semester asking about their beliefs about mathematics, mathematics instructors, and mathematics tutors. The surveys were made up of items that were modified from the CSPCC math attitudes survey (Bressoud, et al., 2015) and the NCTM Teaching and Learning Beliefs Survey (NCTM, 2014). Additionally, tutors recorded tutoring sessions, answered reflection questions, and were interviewed to allow them to elaborate on their responses.

As an example of the results, one Likert scale survey item asked participants to choose whether an effective mathematics tutor "guides students step by step through problem solving" or "provides students with appropriate challenges, … and supports productive struggle". The same question was asked about effective mathematics instructors. Survey responses revealed that the tutors seem to be split between whether tutors should be a guide or provide challenges. However, most of the tutors believed that instructors should provide challenges to students.

In the interviews, 82% of the tutors made statements that indicated that the instructor is the one who presents the theoretical material and "lays the foundation" while the tutor's role is to work with specific examples and "fill in the gaps." Other themes were tutors' ability to personalize their instruction to meet individual student needs and to offer a different perspective than the instructor. One tutor said that tutors "slow-walk students through a problem by asking simple closed-ended questions." Another tutor said that tutors may have more insight into how the mathematics can be applied to classes in the students' specific majors. Several of the tutors mentioned that tutors are more equipped than instructors to encourage students and attend to affective issues.

Future studies will investigate how these perceived differences between the role of a tutor and instructor impact tutors' noticing of students' mathematical thinking and their attention to student affect, and other aspects of their tutoring practice.

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# Exploring Neural Correlates for Levels of Cognitive Load During Justifying Tasks

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*A recent special issue of ZDM (June 2016) made the case for increasing the interdisciplinary collaboration between researchers in the fields of mathematics education and cognitive neuroscience. Specifically, Ansari and Lyons (2016) argued for increasing the "ecological validity of the testing situations and specific [neurocognitive] tests used to measure mathematical processing" (pp. 379-380). The study reported on in this poster serves as a "proof-of-concept" for the use of Functional Near-Infrared Spectroscopy (fNIRS) to measure the level of cognitive load of the brain under mathematical justifying. The poster will address the pros and cons of using neurocognitive measures, such as the fNIRS, to measure and examine the physiological stresses of the brain under the complex mathematical process of proving.*

*Keywords:* Mathematical justification; Neurocognition; Cognitive load

In their introductory commentary to the *ZDM* special issue: *Cognitive neuroscience and mathematics learning – revisited after 5 years*, Ansari and Lyons (2016) posit that "much progress has been made in the diversity of topics being investigated that make connections between [the fields of] cognitive neuroscience and mathematics education" (p. 380). This claim is echoed by Norton and Bell (2017) in their chapter about mathematics educational neuroscience. They report on various studies (e.g. Ischebeck, Shocke, & Delazer, 2009; Waisman, Leikin, Shaul, & Leikin, 2014) that fall into the growing intersection between the fields of cognitive neuroscience and psychology, and the field of mathematics education. However, even allowing for the growth of this new intersectional field, there are numerous persistent and pervasive challenges that remain. One of these challenges, as outlined by Norton and Bell (2017), is with regard to the nature of cognitive demanding tasks within mathematics education.

This poster will report on preliminary findings from a study that investigates the following question: What neural correlates, if any, exist between the neurocognitive data that is collected from participants while engaged in justifying tasks, and the self-reported cognitive load requirements by the same participants after engaging in those justifying tasks? The participants in question for this study were undergraduate and graduate students of mathematics at a large public university in the Northwest United States.

The poster will be a methodological presentation that focuses on the design and development of the methodology used to study this question, including the justifying tasks used, the neuroimaging tools used such as the Functional Near-Infrared Spectroscopy (fNIRS), and a detailed description of the experimental procedure used and its rationale. We will also present a glimpse into the nature and impact of the qualitative and quantitative data that was collected. This report on the study aims to serve as a "proof-of-concept" for highlighting the usefulness, as well as the challenges inherent in the intersectionality of the seemingly disparate fields of cognitive neuroscience and psychology, and the field of mathematics education.

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**Multivariable Calculus Textbook Analysis Highlights a Lack of Representation for Non-Cartesian Coordinate Systems**


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| --- | --- | --- |
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*Upper-division undergraduate physics coursework necessitates a grasp of mathematical knowledge, including an understanding of non-Cartesian coordinate systems. To fully grasp what upper-division physics' students understanding of non-Cartesian coordinates is, it is worthwhile to study the mathematics course where non-Cartesian coordinate systems are taught most extensively, Multivariable Calculus. Seven Multivariable Calculus textbooks were examined using content analysis techniques. Additionally, textbook items in four textbooks were qualitatively coded by coordinate system. Results indicate that there were few instances where non-Cartesian coordinate systems were present. These findings suggest that before upper-division physics coursework, students' instruction on non-Cartesian coordinate systems is minimal and that it might be difficult for students to employ mathematical techniques that involve non-Cartesian coordinates in their upper-division courses.*



*Keywords:* Content Analysis, Non-Cartesian Coordinates, Multivariable Calculus


Understanding non-Cartesian coordinate systems is essential for upper-division physics courses. Published literature suggests that student understanding of non-Cartesian coordinate systems is weak; studies by Moore, Paoletti, and Musgrave (2014) observed mathematics students having continued difficulty with polar coordinates after taking mathematics through Calculus III (Multivariable), and studies by Sayre and Wittman (2007) of junior-level physics students also suggested that students' understanding of the polar coordinate system was still under formation when compared to their understanding of Cartesian coordinate systems. Multivariable Calculus textbooks typically introduce three-dimensional non-Cartesian coordinate systems and study polar coordinate systems at a greater depth. This study examines seven textbooks as sources that can potentially enable or obstruct student understanding of non-Cartesian coordinate systems. To capture a comprehensive examination of these textbooks, qualitative content analysis and quantitative content analysis were performed. Qualitative analysis techniques were used, for example, to examine the coordinate systems new topics were introduced in. Quantitative content analysis categorized examples, definitions, and problems/exercises according to their coordinate system(s). Results demonstrated that non-Cartesian coordinate system representation was minimal. New Multivariable Calculus topics were always introduced in Cartesian coordinates and sometimes did not utilize non-Cartesian coordinates at all. Further, only 21% of textbook chapters included any instance of non-Cartesian coordinates. Of those chapters, 73% of items qualitatively coded according to their coordinate systems were Cartesian. When present, these instances of non-Cartesian coordinate systems often involve simply converting from one coordinate system to another rather than posing questions that elicit a higher level of understanding of when to apply particular coordinate systems. This work implies that Multivariable Calculus textbooks do not require a high level of understanding of non-Cartesian coordinate systems, suggesting that textbooks, which serve as a resource for professors and students, could be part of what limits student understanding and application of non-Cartesian coordinate systems at higher levels of mathematics and physics.

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Perspectives in the Use of Primary Sources in Undergraduate Mathematics Education: A Triangulation of Author, Instructor, and Student

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*We report on a case study of two different university mathematics classes (both Linear Algebra courses) that implemented the same primary source project (PSP) as part of the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) project. One class was taught by the author of the PSP; the other was taught by an instructor at a second university. Data were collected from students in both courses via pre- and post-surveys with Likert items and open-ended items designed to assess their mathematical attitudes and perceived gains. Instructors completed a PSP implementation report and pre-/post-course surveys. In our poster presentation, we provide a triangulation of the data from the perspectives of the author as an instructor, a non-author instructor of a same course (Linear Algebra), and the students.*

*Keywords:* primary source projects, mathematical attitudes, instructor implementation

Mathematics faculty and educational researchers are increasingly recognizing the value of the history of mathematics as a support to student learning. The expanding body of literature in this area includes recent special issues of *Science & Education* and *Problems, Resources and Issues in Undergraduate Mathematics Education* (*PRIMUS*), both of which include direct calls for the use of primary historical sources in teaching mathematics. For many instructors, the current lack of classroom-ready materials poses an obstacle to the incorporation of history into the classroom. As noted by Jankvist (2009), "the 'urgent task' of developing critical implements for using history in the teaching and learning of mathematics" (p. 256) is also essential for further research on the benefits and effectiveness of using the history of mathematics to teach. The collection of PSPs being developed by TRIUMPHS addresses these related concerns.

The TRIUMPHS PSP "Solving a System of Linear Equations using Ancient Chinese Methods" (Flagg, 2017) was first implemented in fall 2017. Survey response data were collected from 11 students in the author's course and 7 in the non-author's course. The pre-course survey included items to determine students' beliefs about mathematics, prior experience with primary source materials, views about mathematics learning and general demographic information. The post-PSP survey contained questions intended to capture students' perceived gains in skills specifically related to linear algebra, general mathematical skills such as reading and writing about mathematics, and attitudes and confidence in mathematics. Other post-PSP survey questions asked about the interaction of students with peers, the instructor, and the primary source material inside and outside of class. Finally, several open-ended questions asked students to reflect upon their experience with the PSP, including their perception of benefits and obstacles of learning mathematics using primary sources, and their attitudes towards using primary sources in a linear algebra course. Implementation reports and pre-/post-course surveys were collected from both instructors, and an instructional guide containing implementation recommendations for instructors was provided by the PSP author.

We will discuss the successes from each implementation from the student and instructor perspectives, the ways in which the two course populations reported similar student gains, and the ways in which students' reported benefits and obstacles for learning with primary source materials can inform future implementations in the TRIUMPHS project.

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# Historical Analysis on Predictive Practices: The Case of Chaotic Dynamics

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*This poster focuses on the historical analysis of three main characters of the history of chaos: the Poincare's error in his memoire about the three-body problem, the ideas of Edward Lorenz about the deterministic non-periodic flow, and the work of Robert May about the logistic map. This into the Variational Thinking and Language research program from the Socioepistemological Theory. The results show a predictive practice characterized for four main actions: to search periodicities, to recognize the uncertain, to compare temporal states, and to classify kinds of behaviors. We assume that the promotion of landscapes and activities out of school are a way for construct specialized mathematical knowledge, and the incorporation of the ability to wait for the unexpected is necessary for teachers and students living this century.*

*Keywords:* Variation, Chaotic Dynamics, Historical Analysis, Socioepistemology,

## Introduction and motivation

Faced with the lack of meaning in the central ideas of the mathematics of change, mainly with the notion of variation (Carlson, Jacobs, Coe, & Hsu, 2003; Doorman, Drijvers, Gravemeijer, Boon, & Reed, 2012; Thompson, Byerley, & Hatfield, 2013; Tall, 2013; Cuevas, 2014; Moreno, 2014), this research looks at the dynamical systems, focus on chaotic ones, as source of mathematical objects that provides different kind of examples, behaviors and *practices* that, currently, are not part of the scholar context. This work assume that Education is not synonymous of schooling but only an aspect (UNESCO, 2012; Rosas Colin, 2014; Valero, 2015), and the promotion of landscapes and activities out of school are a way to construct specialized mathematical knowledge. In another hand we agree with the Morin's idea about the humans living in an uncertain world and the incorporation of the ability to wait for the unexpected (Morin, 1999), and Ghys' discourse concerning to the necessity of include (not only) in mathematics career specific training to teach topics like chaos theory in order communicate these ideas to other scientist or non-scientist (Ghys, 2015).

The fundamental objective of this study is the *search and characterization of the actions* in the transition from the stable periodicities to unstable ones (transition from predictable to unpredictable), first doing a historical analysis looking for *actions over mathematical objects* and the *description of the phenomena* where this kind of behavior appears, and second recollecting data that supports the actions subtracted from the precedent analysis. This poster focuses on the analysis of three main characters of the history of chaos: the Poincare's error in his memoire about the three-body problem (Poincaré, 1898; Barrow-Green, 1997), the ideas of Edward Lorenz about the deterministic non-periodic flow (Lorenz, 1963; 1993) and the methods and tools used by Robert May in the analysis of the logistic map traying to understand the complicated dynamic from a "simple" mathematical model (May, 1974; 1976).

From the Socioepistemological methodology (Cantoral, 2016) over the historical analysis some results show the evolution of a *predictive practice* that is characterized for four main emergent actions over mathematical and physical objects: to search periodicities, to recognize the uncertain, to compare temporal states, and to classify kinds of behaviors. These actions will lead the construction of an experimental instrument to get data about how students and teachers in the last semester of high school and the first of university in scientific and engineer careers face the chaotic behavior.

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# Graduate Teaching Assistants' Evolving Conceptualizations of Active Learning

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*Graduate teaching assistants (GTAs) play a critical role in undergraduate mathematics education, but most have no experience using active learning to promote higher-order thinking. This research investigates how beginning GTAs conceptualize active learning and how these understandings evolve as they engage in a teaching program. This poster describes the program, as well as the evolution of and variation in GTAs' conceptualizations and uses of active learning.*

*Keywords:* Graduate Teaching Assistants, Teacher Development, Active Learning

Graduate teaching assistants (GTAs) play a key role in lower-division undergraduate mathematics courses (Speer, Gutmann, & Murphy, 2005). With the changing context of education, GTAs need exposure to new pedagogical strategies that may fall outside their prior experiences in order to learn how to effectively teach and create valuable learning opportunities for their students (Deshler, Hauk, & Speer, 2015). As researchers and teacher educators, we need to understand how best to help GTAs develop a refined understanding of how to implement active learning effectively, and explore the ways in which they enact it in their own classrooms. Our objective was to introduce new GTAs to active learning strategies during professional development and to research how their views and uses of active learning evolve. Our research questions were: 1) How do beginning GTAs conceptualize active learning? and 2) How do new GTAs' understandings of active learning evolve during professional development?

We adopt the perception that active learning is an instructional method that engages students in mathematical thinking (CBMS, 2016). Our study draws on Bonwell and Sutherland's (1996) conceptual framework that portrays active learning as a continuum where strategies range in difficulty and engagement. They argue teachers should "consider their course objectives and teaching style and to determine through self-reflection what active learning strategies best meet their individual needs" (p. 4) and where these strategies lie on the continuum.

The participants in this study were new GTAs in the first year of their graduate programs (n=20); 35% were female, and 20% were international students. All participants were assigned to be sole instructors of undergraduate lower-division mathematics courses. They completed a week-long teaching orientation before classes began and attended weekly workshops throughout the fall. The program focused on active learning and engaging students in the classroom.

We administered free response surveys at the beginning, middle and end of the fall semester to collect data on GTAs' descriptions and uses of active learning. We also conducted semi-structured interviews with GTAs, asking them to reflect upon their teaching experiences and how their conceptualizations and uses of active learning changed (if at all) over the semester. Qualitative analyses using Bonwell and Sutherland's (1996) framework are ongoing.

Preliminary results indicate that beginning GTAs varied in how they conceptualized and used active learning throughout the semester. Many associated it with an activity and group work, but others had a more nuanced understanding of the term, discussing the process of engaging and involving students in learning. This suggests GTAs need time to develop as teachers and learn how to effectively incorporate active learning strategies in their classrooms. Further research is needed to examine how GTAs' self-identified views and uses of active learning align with their actual classroom practices and continue to evolve with experience.

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# Here's What You Do: Personalization and Ritual in College Students' Algebraic Discourse

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*We present results of a discourse analysis focused on college algebra students' uses of personal and impersonal language, references to endorsed mathematical routines, and inferences about mathematical objects in responses to a small-group problem-posing activity. We analyze students' responses with respect to selected dimensions of the arithmetical discourse profile of Ben-Yehuda et al., and provide evidence of a positive association between impersonal language and the presence of object-level mathematical statements and precise uses of algebraic terminology.*

*Key words:* College Algebra, Discourse Analysis, Mathematical Routines

Students' comprehension of mathematical ideas is inextricably linked to their processes of communication (Wittgenstein, 1953; Sfard, 2007). Ben-Yehuda et al. (2005) developed the *arithmetical discourse profile* as a tool for describing and analyzing students' use of words, mediators, and routines (Sfard, 2007; 2016) when communicating about mathematical tasks. Statements in mathematical narratives can be classified as *personal* (involving a human actor performing mathematical operations) or *impersonal* (describing mathematical structure without personalization) (Ben-Yehuda et al., 2005). Empirical studies have suggested a negative correlation between the use of personalized language, such as "I" and the past tense, and achievement levels in young children (Bills, 2002). Statements in mathematical narratives can additionally be classified into *object-level* statements about mathematical objects, and *meta-level* statements about the discourse itself (Sfard, 2007). In this study, we explore college algebra students' uses of language in mathematical narratives; in particular, we investigate associations between impersonal descriptions of routines and other features of literate mathematical discourse, such as correct uses of algebraic terminology and object-level statements (such as that when the division $P(x) / (x - a)$ leaves a remainder of zero, $a$ is a zero of $P(x)$).

For this study, students in three large sections of college algebra (total enrollment 327 students) participated in a small-group activity in which each group wrote an open-ended problem that could be used to review for an upcoming exam, and worked together to produce a written solution to the problem they created. Students were permitted to write a problem on any topic covered by the upcoming exam; however, most problems dealt with polynomial functions and their zeroes. We found that students' written solutions were largely governed by routines prescribed by the course text (Abramson, 2012) and endorsed by guided notes published by the course instructors; however, students' descriptions of these routines varied in their use of personal and impersonal language and in their uses of mathematical terminology and reasoning. We hypothesize that in some cases, the use of personal language was dictated by the topic selected for the problem; for example, narratives of synthetic division relied heavily on descriptions of human actions on mediators (e.g., "Make sure to bring down the first coefficient."). For many other topics, uses of personal language by students appeared to mimic uses of personal language in the textbook and guided notes. However, some topics led to greater variation in uses of personal and impersonal language. Our poster will report on this variation and present examples of both personal and impersonal descriptions of algebraic routines, and illustrate instances in which impersonal discourse is associated with greater precision in uses of mathematical terminology and algebraic reasoning.

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Adaption of Sherin's Symbolic Forms for the Analysis of Students' Graphical Understanding

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*We describe a methodological presentation of Sherin's (2001) symbolic forms, discussing adaptions made to the framework to analyze graphical reasoning. Symbolic forms characterize the ideas students associate with patterns in an expression. To expand symbolic forms beyond equations, we supplement it with another framework that considers modeling as discussing mathematical narratives. This affords the language to describe how students think about the process or "story" that could have given rise to a graph. By considering registrations in general terms as structural features students attend to (parts of the "story"), when students assign ideas to registrations (parts of an equation or regions of a graph), they are using symbolic forms.*

*Keywords*: mathematical reasoning, symbolic forms, rates, chemistry

Sherin (2001) developed symbolic forms as a means to characterize how students used mathematical ideas to reason about equations when solving problems in physics. This framework has its roots in the constructivist idea of "phenomenological primitives" (p-prims), which describe intuitive ideas developed based on experience (Bodner, 1986; diSessa, 1993). Symbolic forms can be seen as mathematical p-prims, involving students associating ideas (conceptual schema) with a pattern of symbols (symbol template); for example, students associating the idea of "balancing" with the symbolic form "$\square = \square$", where the boxes are generic placeholders for algebraic terms (Sherin, 2001). This is important because without explicit instruction students associate ideas with patterns that are productive when learning concepts (e.g., opposing forces in physics). This framework has been utilized across different discipline-based education research (DBER) fields to explore student understanding of integration, the differential (*dx)*, area and volume, and mathematical expressions in physics and chemistry (Becker & Towns, 2012; Jones 2013, 2015a, 2015b; Dorko & Speer, 2015; Marredith & Marrongelle, 2008; Von Korff & Rubello, 2014). We assert students have similar ideas about graphs and seek to expand symbolic forms to move beyond equations, which has broad applicability across DBER fields.

A central tenet of our adaption of symbolic forms to graphical reasoning is Nemirovsky's (1996) conceptualization of "mathematical narratives" as the integration of events with symbolic notations (i.e., modeling). Nemirovsky (1996) used mathematical narratives to focus on student descriptions of "stories" that could give rise to a particular graph in the context of graphical representations of velocity, distance, and time. Viewing modeling as "story-telling" is particularly useful when considering students' graphical reasoning because it provides the language to describe students' discussion of the series of events represented by a graph. In the literature "registrations" have been used to describe features students focus on in computer simulations; we adopt this terminology to describe structural features students attend to in representations, and when students "register" or associate specific ideas with these features, they are reasoning using symbolic forms (Roschelle, 1991; Sengupta and Willensky, 2009).

Although it has been suggested that symbolic forms can be adapted to graphical reasoning, in practice it has not yet been taken up in the literature (Izak, 2000; Lee & Sherin, 2006; Sherin, 2001). Our presentation will provide examples of how we functionalize this adapted framework, using chemistry as a rich context to study students' reasoning associated with graphs that describe the rate of change of chemical compounds over time, since research has shown that students have difficulty with ideas related to the derivative and rate (Orton, 1983; Rasmussen, Marrongelle, & Borba, 2014; White & Mitchelmore, 1996).

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Design Research in German Mathematics Tertiary Education Focusing on Profession-Specificity

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*The poster proposal presents design research projects in the context of German tertiary education for preservice secondary teachers and service mathematics courses. The approach of design research for university students with a content-specific focus on profession-specificity is exemplified by two concrete design research projects.*

*Keywords:* Design Research, Service Courses, Preservice Secondary Teachers, Calculus

In Germany, due to a growing heterogeneity among university students (Heublein et al., 2012) the need for instructional innovations in mathematics and mathematics-related studies is taken very seriously at the moment. Especially in mathematics, also high rates of drop out (Dieter & Törner, 2012) led to an increasing attention to more adaptive teaching as a means of reacting to students' heterogeneity as well as enhancing their motivation. The poster presents research projects of the researcher and her research group on instructional innovations within preservice secondary teacher education on the one hand and mathematics service courses on the other. Although the target groups of the innovations differ in several ways, Design Research as the common research approach is chosen to meet the innovation needs, which will be lined out and motivated.

Design Research is a widely-established research methodology for enhancing and investigating students' learning. It is especially strong when the two aims 'designing learning arrangements' and 'investigating the initiated learning processes and contributing to local instruction theories' are to be combined (Bakker & van Eerde, 2015). The research projects presented here follow a topic-specific approach with a focus on learning processes (Prediger & Zwetzschler, 2013) which is adapted to designing and researching teaching learning arrangements in mathematics tertiary education. The approach is exemplified by two research projects which foster university students' content knowledge and pedagogical content knowledge with a focus on profession-specificity.

*Example 1*: For preservice secondary teachers, the design research project focuses on pedagogical content knowledge of functional reasoning and calculus. The overarching research question "*How can profession-specific learning tasks be specified and structured and which learning pathways and obstacles can be identified*" is pursued. The Four Component Instructional Design Model by van Merriënboer & Kirschner (2007) builds the instructional framework being implemented in three cycles of design experiments (laboratory setting). At the moment, data analysis of n=26 students' pre- and post written answers of three learning tasks from the second and third design experiment cycle is ongoing by means of qualitative content analysis (Mayring, 2008).

*Example 2*: For fostering first-year students' understanding of functions and calculus (and related procedural knowledge) in mathematics service courses, adaptive online remediation modules are designed and investigated. Profession-specificity of the modules is realized by contexts of applications from the field of studies (natural sciences, engineering). Video-taped design experiments at the computer (laboratory one-on-one and partner setting) and the qualitative analysis of the initiated learning processes are much needed, since little is known about how students work with online remediation material and many questions, e.g. concerning adaptive feedback or relations of usability and conditions of success, are still open.

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Constant Rate of Change: The Reasoning of a Former Teacher and Current Doctoral Student

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*In this work, I provide brief illustrations of multiple ways of reasoning about constant rate of change that I observed in a mathematics education doctoral student's activity when tasked to draw graphs relating two varying quantities. These ways of reasoning suggest that textbook authors and instructors critically examine those illustrations and experiences provided to students in order for students to come away from mathematics courses with consistent and productive reasonings about rate of change.*

*Keywords:* Covariational Reasoning, Rate of Change, Calculus

Researchers have reported that a productive meaning for the idea of *rate of change* involves one to conceptualize relationships between two covarying quantities (Thompson, 1994). Covariational reasoning describes the mental actions involved in one coordinating such varying quantities (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). In recent decades, researchers have observed and characterized students' and teachers' mental actions while engaged in tasks to model covarying quantities (Carlson et al., 2002; Castillo-Garsow, 2012; Coe, 2007). Researchers have also identified students and teachers having difficulty with covariational reasoning (Carlson et al., 2002; Johnson, 2015). In particular, Musgrave and Carlson investigated mathematics graduate students' meanings of average rate of change (Musgrave & Carlson, 2016). They found that these students often held non-conceptual meanings for average rate of change that were primarily focused on computations or geometric interpretations involving secant lines on graphs. With the goal of developing students' and teachers' covariational reasoning, it is productive to construct models of individuals' thinking about rate of change as a means for creating rich experiences in which students and teachers can develop sophisticated ways of reasoning. The work of this study contributes to expanding and broadening models of individuals' covariational reasonings by providing insights into those reasoning processes that continued mathematics users (i.e., mathematics teachers turned graduate students) engage in.

In this poster, I present multiple and inconsistent ways of reasoning about constant rate of change that I observed in the activity of one mathematics education doctoral student with high school mathematics teaching experience. The study involved clinical interviews in which I asked the participant to draw graphs relating two varying quantities in an animated situation. The participant's reasonings resulted in inconsistent conclusions and suggest that he did not interpret or describe rate of change covariationally by imagining changes. I characterize these ways of reasoning as tangent line reasoning and constant ratio reasoning. *Tangent lines* involved the participant constructing and reasoning geometrically with tangent lines he constructed on his graph. *Constant ratio* involved the participant identifying that the two accumulated quantities he was graphing could be related computationally by a scale factor of some fixed unit magnitude of each quantity (which he identified as the "constant" in a constant rate of change relationship). These ways of reasoning did not seem productive for the participant and yet seem to be suggestive of certain non-quantitative curricular treatment of rate of change. These illustrations suggest that mathematics educators and textbook authors critically examine those reasonings of their students and the experiences they provide students in order for students to develop more consistent and productive reasoning abilities about rate of change.

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Investigating Student Learning and Sense-Making from Instructional Calculus Videos


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*Growing interest in "flipped" classrooms has made video lessons an increasingly prominent component of post-secondary mathematics curricula. This format, where students watch videos outside of class, can be leveraged to create a more active learning environment during class. Thus, for very challenging but essential classes in STEM, like calculus, the use of video lessons can have a positive impact on student success. However, relatively little is known about how students watch and learn from calculus instructional videos. This research generates knowledge about how students engage with, make sense of, and learn from calculus instructional videos.*

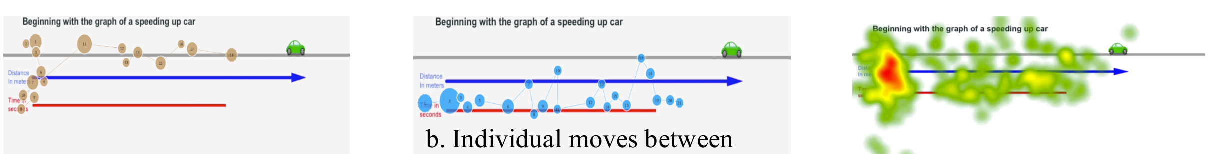


*Keywords:* Calculus, Eye-Tracking, Flipped Classrooms, Sense-Making, Quantitative Reasoning


To help instructors design videos for flipped classrooms, we have collected data from four different calculus classes using instructional videos. Videos used in this study have ranged from innovative approaches to calculus proven successful by supporting students' development of covariational reasoning (e.g., Martin & Oehrtman, 2015; Thompson, Byerley, & Hatfield, 2013) to videos of more traditional whiteboard type lecture. We investigate:

- The ways students interact with video lectures, including how they pause, skip, and re-watch portions of the videos;
- The aspects of the videos students attend to – and report attending to – as they watch;
- The ways students make sense of and learn from these videos, and how this relates the other aspects described above (e.g. Weinberg & Thomas 2016a, 2016b);
- How various ways of structuring the video-watching experience, such as providing an outline, can influence each of these aspects (e.g. Johnson & Mayer, 2009).

Data consists of student responses to mathematical content questions before and after watching videos, timestamps of students' interactions with videos (i.e. playing, pausing, and time-shifting videos), student responses to interview questions as they watch videos, and eye-tracking data from students watching videos. Our analysis yields knowledge about how students learn and interact with these videos. For example, Figure 1a. and b. demonstrates how eye-tracking data shows distinctions between student fixations (the brown and blue circles) while watching a video of a moving car. Figure 1c. indicates how the participant group tended to still be reading the labels when the car started traveling before moving their fixations between the representations for time elapsed, distance traveled, and the moving car.




a. Individual fixates on car　b. Individual moves between representations of quantities　c. Heat map of group fixations

*Figure 1. Eye-tracking data indicating individual and group fixations in the context of a car speeding up.*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Is Mathematics Important for Accounting Learning? – A Study on Students' Attitudes and Beliefs

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*This study examined students' attitudes toward mathematics, and beliefs in mathematics' influence on accounting learning. It also explored how these two factors correlate to students' scores in accounting courses. This study found that students believed that being good at math is a necessary, but not a sufficient, condition for performing well in accounting. Students who performed well in accounting usually showed a relatively positive attitude toward math.*

*Keywords:* accounting performance, attitude toward mathematics, belief in mathematics

## Introduction

Accounting majors usually need to have some mathematics ability before taking accounting courses and may be asked to take courses such as calculus I as prerequisites. Many college professors or researchers think prerequisites in mathematics for the accounting major are well-grounded (Brown, 1962; Collier & McGowan, 1989). However, mathematics' influence on a student's accounting learning is inconsistent. Burdick and Schwartz (1982) found that students' performance in mathematics courses couldn't significantly predict students' scores in accounting courses. Some other studies (Clark & Sweeney, 1985; Collier & McGowan, 1989) found that students' mathematics preparation is positively linked to the accounting coursework performance. It is unclear that in accounting students' perspectives, if mathematics is important for accounting learning. We address two research questions: 1. What are accounting students' attitudes toward mathematics? 2. What are accounting students' beliefs in mathematics' influence on accounting learning?

## Method

This study was conducted in a four-year college in the mid-west area of the United States. Our sample consists of 203 undergraduate students from introductory financial, introductory managerial and intermediate financial accounting courses. Participants completed a consent form to take the study and were asked to finish a survey (23 questions) online toward the end of a semester. Students' scores in the accounting course were collected when the semester ended.

## Data Analysis and Results

Students were asked to respond to questions on their attitudes toward mathematics, and their beliefs in mathematics. The results revealed a positive relationship between a student's attitude toward mathematics and accounting performance. However, no significant relationship was found between a student's belief in the usefulness of mathematics and accounting performance. Students agreed that mathematics was important in accounting learning. However, they did not believe that they were not doing well in accounting because they were not good at math.

## Conclusion

Based on our research, we find our students believe that being good at mathematics is a necessary, but not a sufficient, condition for performing well in accounting. In addition, students who perform well in accounting usually show a relatively positive attitude toward mathematics.

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Goals, Resources, and Orientations for Equity in Collegiate Mathematics Education Research

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*Though the terms equity, diversity, inclusion, and social justice have entered the research lexicon, we face significant challenges in gaining a nuanced understanding of the various ideas associated with these words and how those ideas are consequential for collegiate mathematics education research. This interactive poster presents a theoretical framework for making sense of (and making sense with) "equity" as an essential component of research. The poster offers tools for thinking and talking about equity and research design, implementation, and reporting. Poster visitors will have an opportunity to contribute questions and observations about the definitions of equity and proposed connections among approaches to courageous conversations about equity in research, self- and other-awareness, and aspects of equity in the mathematics content, curricula, and instruction at the heart of the research.*

*Keywords:* Equity, Social justice, RUME

As people trained in research in undergraduate mathematics education (RUME), we know that our work starts with diagnosing challenges in teaching and learning. As citizens of a first-world country in the 21st century, we are keenly aware of social, political, and economic inequity. And, as a community, we have an opportunity to guide how equity is defined, explored, and addressed in collegiate mathematics education research. Attention to equity has existed for a while (e.g., Aguirre & Civil, 2016; Adiredja, Alexander, & Andrews-Larson, 2015; D'Ambrosio et al., 2013; Davis, Hauk, & Latiolais, 2010; Gutiérrez, 2013; Nasir, 2016).

According to the TODOS-NCSM position paper (2016), three conditions are necessary to establish just and equitable mathematical education for all learners: (1) acknowledging that an unjust social system exists, (2) taking actions to eliminate inequities and establish effective policies, procedures, and practices that ensure just and equitable learning opportunities for all, and (3) being eager for accountability so changes are made and sustained. How do we increase researcher capacity to do these three things? We must address our needs – as researchers – for language, definitions, and awareness-building about equity. This will support us in the inevitable struggle to gain and use pertinent understandings in the design, conduct, and reporting of research. The poster offers key ideas and examples from communication for restorative justice (e.g., Singleton & Hays, 2008) and intercultural orientation development (Bennett, 1993; 2004).

**Questions driving poster conversation:** What questions and observations do RUME researchers have regarding definition(s) of equity and the role of equity in research in collegiate mathematics education? How does equity play into our decisions about who research participants are? How might research be designed to provide evidence that supports action to eliminate an inequity? How might engaging the population we wish to study in the research design and analysis provide new insights into phenomena? How might the research design and analysis be different if the results of the work are to be held accountable by research peers *and* judged in a court of stakeholder opinion that values equity as much as excellence in mathematics education? In what ways is the mathematics implicit in a given research project contributing to inequity and/or equity for participants? How do we pay attention to that in the research goals, resources, and orientations we bring to our work? What are some of the concepts and language from intercultural development that can help us address these questions?

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How Diagrams are Leveraged in Introduction to Proof Textbooks

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*According to research, diagrams can play a vital role in the constructing and understanding of proofs (Samkoff, Lai, & Weber, 2012). Introduction to Proof (ITP) courses are usually a student's first exposure to proofs. Therefore, the ITP curriculum reflects important opportunities for students to develop proof construction and proof understanding skills. We analyzed how diagrams were presented in the top four market share ITP textbooks across a set of standard topics. Through this process, we categorized the role and nature of diagrams in the curricula. We found that a majority of diagrams were used to illustrate statements and definitions. Other important roles such as supporting proof construction, building conjectures, or finding counterexamples were infrequent.*

*Keywords:* Introduction to Proof, diagrams, textbook analysis

Mathematicians find using diagrams beneficial when constructing and understanding proofs (Samkoff et al., 2012). ITP courses are usually a student's first exposure to proofs. During ITP courses, students develop their proof skills. Textbooks reflect an important component of the intended curriculum and opportunities for students to learn (e.g., Thompson, Senk, & Johnson, 2012). We conducted a textbook analysis to explore how diagrams are leveraged in the curricula. In particular, we investigated how diagrams are used in the proving process in ITP textbooks.

We selected four ITP textbooks to analyze: Chartrand, Polimeni, and Zhang (2012), Hammack (2013), Smith, Eggen, and St. Andre (2010), and Velleman (2006). David and Zazkis (2017) identified these four ITP textbooks as having the top market share use in ITP courses, as well as, covering standard topics for ITP courses: sets, logic, proof techniques, relations, functions, and cardinality. We identified the textbook sections corresponding to these topics then used a thematic analysis (Braun & Clarke, 2006) approach to open-code and develop themes related to the nature and role of diagrams. We coded 173 diagrams across the textbooks.

We identified the following roles of diagrams in ITP textbooks: conjecturing (CJ), identifying counterexamples (CE), instantiating definitions (DEF), instantiating statements (ST), illustrating a procedure (PRO), illustrating the key idea to a proof (KIP), organizing/synthesizing information (ORG), being part of the proof (PP), and other. Most diagrams were used to instantiate definitions and theorems, which only represents a small subset of how mathematicians use diagrams (Samkoff et al., 2012). The poster contains a detailed comparison of same topics across different textbooks.

*Table 1. Frequency of the role of diagrams in the standard ITP topics across four textbooks.*

| Topic | CJ | CE | DEF | ST | PRO | KIP | ORG | PP | OTHER |
|---|---|---|---|---|---|---|---|---|---|
| Sets | 0 | 1 | 38 | 11 | 2 | 3 | 0 | 1 | 5 |
| Cardinality | 3 | 2 | 13 | 5 | 2 | 13 | 0 | 0 | 0 |
| Relations | 0 | 0 | 34 | 1 | 0 | 2 | 3 | 1 | 0 |
| Functions | 2 | 1 | 17 | 5 | 1 | 0 | 2 | 4 | 1 |
| Total | 5 | 4 | 102 | 22 | 5 | 18 | 5 | 6 | 6 |
| Percentage | 2.89% | 2.31% | 58.96% | 12.72% | 2.89% | 10.4% | 2.89% | 3.47% | 3.47% |

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Student's Attention to the Conclusion During Proofs

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*This study investigates students' use of conclusions to structure their proofs for a standard statement in introductory Group Theory. We surveyed 65 students across three classes asking them to evaluate the truth of a statement and provide a proof. We found students tend to use hypothesis-driven second level proof framework (rather than conclusion-driven). These students were then less likely to produce a deductive argument that aligned with the original statement. We conclude with implications for the treatment of proof analysis and proof frameworks to support students' proving activity.*

*Keywords:* Group Theory, Proof Frameworks, Proof Analysis

In courses, such as group theory, students frequently prove statements about structure-preserving properties such as the following statement: Let *f* be an isomorphism from (G, o) to (H, *). If *G* is an abelian group, then *H* is an abelian group. In order to approach such statements, students must structure their proofs around the conclusion to argue about arbitrary elements of *H* rather than arguing about the image of elements in *G.* We designed a study to test the conjecture that students do not necessarily attend to the conclusion when proving. We surveyed 65 students across three group theory classes using either the isomorphism prompt ($n$=32) or an alternate false version with 1-1 homomorphism missing the necessary requirement of onto ($n$=33).

To analyze students' proof approaches, we use two framings: proof frameworks (Selden & Selden, 1995) and proof analysis (c.f., Marchi, 1980; Lakatos, 1976). The proof framework is the "representation of the 'top-level' logical structure of a proof" (p. 129) which is tied directly to the statement to be proven. In order to approach the isomorphism prompt above, one option is to employ the appropriate second-level proof framework (Selden & Selden, 2015): using the conclusion to structure proof (i.e. starting with elements in *H*). An alternate approach would be the selection of a second level proof framework beginning with elements in *G*, arriving at a statement about the images of these elements then using proof analysis (c.f., Marchi, 1980) to recognize that the deductive argument does not align with the statement. We coded surveys based on (1) second-level proof frameworks, (2) validity, and (3) proof corrections.

We found that students used a *G*-first proof framework ($n$=39) compared to *H*-first ($n$=17) at a rate significantly higher than chance ($p$=0.0016). This approach was consistent across the true and false prompt where students produced deductive arguments about the image of *G* rather than *H*. For the true statement, we further analyzed the likelihood of arriving at a valid deductive argument finding that only 2 of 16 *G*-first students arrived at a valid proof with 7 of 9 *H*-first students arriving at a valid proof, a statistically significant difference. Our results reflect that many students are not attending to the conclusion of statements when proving. Instructors may need to work with students to help the students understand the importance of using the conclusion to structure the proof. Further, proof analysis techniques (comparing the statement and deductive proofs, searching for counterexamples) could also support students in producing arguments that better align with original statements.

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What Would You Say You Do Here?
Metaphor as a Tool to Characterize Mathematical Practice

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*In the cognitive science literature, multiple researchers have pointed out the importance of metaphor as a cognitive mechanism for sense-making. In mathematics in particular, metaphor has been shown to be a valuable tool in making sense of and reasoning with mathematics. To our knowledge, there has been no research on the metaphors that professors use when communicating the nature of mathematical practice to students in advanced mathematics lectures. In this poster, we describe the metaphors the research team identified across 11 undergraduate mathematics lectures at the advanced level. We found metaphors used by many lecturers that convey ideas about the nature of mathematical practice. We identified the affordances of these metaphors to better understand the way that mathematicians describe mathematics as a practice to undergraduate students studying advanced mathematics at the undergraduate level.*

*Keywords*: Metaphor, Mathematics education, Advanced mathematics

In the language used in everyday thought and speech, there are large number of expressions whose literal interpretations suggest something other than the intended meaning (Lakoff & Johnson, 1980/2003; Reddy, 1979). In documenting occurrences of these expressions and analyzing their content, researchers have developed the modern theory of metaphor, which has been applied across many disciplines to analyze how we as a species make sense of the world and develop our ideas (Lakoff & Johnson, 2003/1980). In mathematics, it has been argued that metaphors form the foundation of sense making and that we can analyze many important mathematical concepts to identify their metaphorical foundations (Lakoff, 1998; Lakoff & Nuñez, 2000; Nuñez, Edwards, & Matos, 1999; Sfard, 1994)

As the theory of metaphor has developed in a mathematical context, there has been some research documenting the metaphors that mathematicians use to personally make sense of and reason about mathematics (Nathalie & Tabaghi, 2010; Sfard, 1994). Many different types of metaphors have been documented, such as those relying on our conceptions of motion and our experiences with manipulating physical objects (Lakoff & Nuñez, 2000; Nuñez, 2004). Nuñez (2004) found occurrences of motion metaphors in an advanced mathematics lecture.

Through the study of metaphors, we answer the question "How do mathematicians describe their practice to advanced undergraduate students?" To answer this question, we analyzed a corpus of 11 undergraduate mathematics lectures at the advanced level. We found several metaphors that characterize the practice of mathematics (e.g. mathematics as play "*You have to play around* slightly and get three disjoint sets", mathematics as a journey "We have some loose ends in the theory that we won't be able to deal with until we *get to a more advanced place"*). We analyzed the metaphors we found and describe the entailments of these metaphors. Through this lens, we can describe the ways that mathematicians convey the practice of their discipline to students. We may also use this analytical tool as a method of understanding how students conceptualize mathematics.

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# Active vs. Traditional Learning in Calculus I

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*In this poster, we describe an ongoing study on the effect of active learning in Calculus I. We compare the achievement gap between underprepared and prepared students in the active versus traditional setting. Data comes from 16 sections of Calculus I during the 2017 – 2018 academic year, targeting the concepts of limits, continuity, differentiability, and area. We present our study design and initial findings; we look forward to feedback as we enter the latter half of our project.*

*Keywords:* Calculus, Active Learning, STEM, Achievement Gap

Calculus I is a foundational class in the degree plan of nearly all science majors. Calculus is a crucial benchmark in the path to a STEM education; however, many students rely heavily on memorization and repetition as paths to success in mathematics. These techniques fail when they are asked to explore the abstract concepts *of limits, continuity of functions, differentiability,* and *area*. One pedagogical approach to increasing student understanding and mastery is active learning. Active learning activities provide a setting for students to learn in cooperation with others, thus placing them in an excellent environment to construct complex mental frameworks (Bransford et al., 1999; Vygotsky, 1978). Existing literature supports the idea that active learning techniques can increase student learning outcomes significantly (Freeman et. al, 2014; Bressoud, 2011; Haak et. al, 2011; Boaler & Greeno, 2000). In this project, we study active learning specific to the calculus classroom, and target the population of students who enter with deficiencies in algebra, trigonometry, and/or pre-calculus. We explore the following questions:

- Do students who are underprepared for calculus perform better than their calculus-ready peers after learning in an active classroom versus a traditional classroom?
- Does the performance gap between underprepared and calculus-ready students change to a different extent in an active classroom as compared to a traditional classroom?
- Do students identified as underprepared for calculus have a more favorable perception of mathematics after learning in an active classroom as compared to a traditional classroom?
- Do students who learned in an active classroom see more success in Calculus II than those learning in a traditional classroom?

In this study, we compare student learning outcomes in four classrooms employing active techniques to outcomes in four traditional lecture-based classrooms in each of Fall 2017 and Spring 2018. We administer a pre-test assessment and initial survey in each classroom. We use the pre-test to identify students with weak preparation and to gauge students' attitudes and mindsets towards mathematics. The active sections discuss each of our target concepts: limits, continuity, differentiability, and area, using an exploratory activity, discussion, and follow-up assignment. The traditional sections cover the same content, but from a lecture approach. We assess learning outcomes by scoring performance on in-class exams and administer a post-test and survey (Carlson, Oehrtman, & Engelke, 2010). The survey will assess the changes in students' attitudes and mindsets about mathematics, as well as ask them to self-assess their preparedness for Calculus II. We intend to collect data regarding participants' persistence and success in Calculus II. At the conclusion of this project, we hope to better inform teaching practices in calculus at our institution.

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# Do Prospective Elementary Teachers Notice Cultural Aspects of Mathematics in a Teaching Scenario?

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*Many teachers view mathematics as culture-free, which can result in difficulty attending to and valuing the cultural backgrounds of their students. We asked 23 prospective elementary teachers (PTs) to respond to a case that describes a teacher dismissing a third-grade student's solution to a multi-digit subtraction problem, due to the child's use of a nonstandard algorithm. Through a process of open-coding, three themes emerged in the PTs' responses: (1) the PTs would have responded differently/the teacher should have responded differently (22 PTs), (2) the PTs focused on the mathematics and/or on children's mathematical thinking (22 PTs), and (3) the PTs focused on the child's background/culture/family (3 PTs). We examine the differences in response levels between the first two themes and the third. Additional data has been collected and preliminary results show a strong focus on the first two themes, as well as a higher incidence of the third.*

*Keywords:* Prospective Elementary Teachers, Cultural Responsiveness, Elementary Teacher Education, Number and Operation

Although the elementary teacher population in the United States is largely homogenous, the student population is culturally diverse. As such, it can be difficult for teachers to connect to their students' cultural backgrounds (Turner et al., 2014). As mathematics teacher educators, we have a responsibility to help prospective elementary teachers (PTs) learn how to connect with all their students, but to do so we must first explore the ways in which PTs notice cultural aspects of mathematics, as "mathematics is not neutral, is not culture free, and is not value free" (Bishop, 1988). In this poster we will address the following research questions: (1) how do prospective elementary school teachers (in the United States) react to a case of a teacher dismissing a student's nonstandard algorithm?, and (2) do prospective elementary school teachers (in the United States) notice cultural aspects in a mathematics teaching scenario?

Through a process of open-coding, three main themes emerged from the PTs' responses: (1) the PTs would have responded differently/the teacher should have responded differently, (2) the PTs focused on the mathematics and/or children's mathematical thinking, and (3) the PTs focused on the child's background/culture/family. 22 PTs indicated that they would have responded differently and/or the teacher should have responded differently, 22 PTs focused on the mathematics and/or the child's mathematical thinking, and 3 PTs considered the background/culture/family aspect of the case. These results suggest that while PTs attend to and value the child's mathematical thinking, they are less apt to notice or respond to cultural aspects of mathematics in a teaching scenario.

Additional data was collected in Fall 2017 from PTs in the same university which will be analyzed using the same methods as above. This new data will serve to refine the previously found themes, as well as potentially illuminate new themes. Preliminary analysis suggests that a greater number of PTs focused on the child's background/culture/family, indicating that the PTs represented in the second round have more culturally responsive attitudes than those in the initial round of data collection.

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Exploring the Role of Active Learning in a Large-Scale Precalculus Class

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*In a large undergraduate mathematics classroom, introducing evidence based learning practices can be challenging. Due to persisting outdated methods of teaching, results of recent research call for more investigation of active-learning in all STEM classrooms, including large scale ones. Using Fraser's (1989) lens on perception, results from this study indicate that students who participated in Team Activities and other learner-centered activities in a large scale precalculus undergraduate class reported good experiences and are more positive in their attitudes towards mathematics.*

*Keywords*: Large-Scale Classrooms, Precalculus, Evidence-Based Practices

Researchers and curriculum developers have responded to the call for instructional improvements, developing numerous learner-centered curricular innovations particularly using collaborative and open-ended activities. Learner-centered instruction has been shown to support conceptual learning gains (e.g.; Kwon, Rasmussen, & Allen, 2005), diminish the achievement gap (Kogan & Laursen, 2013; Riordan & Noyce, 2001), and improve STEM retention rates (Hutcheson, Pampaka, & Williams 2011; Rasmussen, Ellis, & Bressoud, 2013; Seymour & Hewitt 1997). The objective of this research was to investigate student outcomes from the introduction of a small number of evidence-based active learning practices in a large size Precalculus classroom. For this study, outcomes were defined as students' attitudes towards mathematics and themselves as mathematics learners, interest in mathematics, and self-efficacy. The research question was: "What are students' experiences in a large-sized undergraduate Precalculus class when active learning strategies are present?

Fraser (1989) stated that "the strongest tradition in past classroom environment research has involved investigation of associations between students' cognitive and affective learning outcomes and their perceptions of psychosocial characteristics of their classrooms" (p. 315). We choose to use Fraser's lens to study associations between students' perceptions of a large-scale classroom environment and their cognitive and learning outcomes.

This mixed methodological exploratory research study (Creswell, 2013) was designed to introduce evidence-based instruction to students in order to study how these new practices are implemented and how they affect student outcomes. Data collection included interviews and surveys (pre-and post-) administered to the 14 participants at a very large, public southeastern university. Semi-structured interviews were conducted by the researches and were utilized for students to share their experiences. Video recordings of each interview were then transcribed and coded by the authors. Quantitative analysis was then completed to compare the Likert-scale scores from the pre- and post-surveys. Results will be shown on the poster.

Several themes arose as we analyzed the codes. Overall, (1) students were neutral about math in application, (2) collaboration was important for students, and (3) active learning was important for students. The results are noteworthy as we are finding that it makes a difference even to include just a few instructional strategies that are considered learner-centered even in large scale classrooms. This result leads to a significant question: How important is it to include learner centered instruction fully implemented or can a partial implementation work?

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Video Case Analysis of Students' Mathematical Thinking to Support Preservice Teacher Candidates' Functional Reasoning and Professional Noticing

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*Using a design-based research approach, we are developing a series of online video-based instructional modules to engage secondary mathematics teacher candidates in case analyses of students' functional reasoning and to improve their own mathematical and pedagogical understandings. We present our project framework for module development, implementation, and revision, with an end goal of identifying preliminary hypothetical learning trajectories for candidates' functional reasoning and professional noticing.*

*Keywords:* Functional Reasoning, Hypothetical Learning Trajectory, Preservice Teacher Preparation, Video-Based Learning, Professional Noticing

## Video Case Analysis of Students' Mathematical Thinking (VCAST) Module Development

To advance student understanding of mathematics, teachers must pay careful attention to and then interpret evidence of student thinking. This requires a specialized mathematical knowledge of common patterns in students' reasoning and how their ideas are related and represented (Ball, Thames, & Phelps, 2008; Stein & Smith, 2011). Video-based modules can offer purposefully selected student evidence (i.e. case studies) to highlight important mathematical ideas.

Our module development process is informed by the literature on functional reasoning (Cooney, Beckmann, Lloyd, & Wilson, 2010; Oerhtman, Carlson, & Thompson, 2008), the use of video to support preservice teacher learning (Coffey, 2014; van Es, Cashen, Barnhart & Auger, 2017), design-based research (Anderson &Shattuck, 2012; Reeves, Herrington, & Oliver, 2005), professional noticing (Jacobs, Lamb, & Philipp, 2010), and hypothetical learning trajectories (HLTs)(Lobato & Waters, 2017; Simon, 1995; Simon & Tzur, 2004).

## Project Framework

We approach our module design through iterative improvement of HLTs while leveraging video cases' affordances for presenting specific episodes of students' reasoning. This entails identifying learning goals for teacher candidates, hypothesizing increasing levels of sophistication in reasoning toward those goals, developing learning activities which target those goals, and iteratively refining each as candidates engage in the modules (see Figure 1).

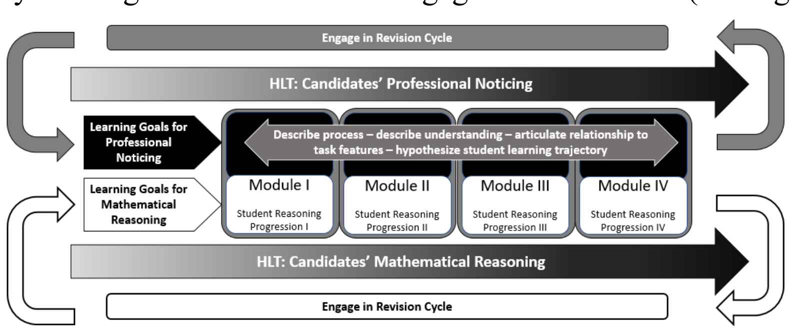


*Figure 1. HLT development in relation to the learning modules built around student reasoning progressions*

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# Exploring Pre-service Elementary Teacher's Relationships with Mathematics via Creative Writing and Survey

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*Thirty-two pre-service elementary teachers completed a survey regarding their beliefs and attitudes regarding learning and teaching mathematics and two creative writing tasks. In the writing tasks, participants described their relationship with personified mathematics and introduced personified mathematics to their future students. By interpreting the survey and writings, different aspect of attitudes towards mathematics were discovered.*

*Keywords:* Affect, pre-service teachers, creative writing

Pre-service teachers' relationship with mathematics is important because it can affect how teachers introduce mathematics to their students (Swars, Daane & Giesen, 2006). Existing research has used conventional methods, such as surveys and interviews, to measures preservice and in service teachers' attitude towards mathematics. (Brown, 2007; Raymond, 1997)

Zazkis (2015) used an unconventional method to assess preservice teachers' relationship with mathematics. Zazkis assessed pre-service teachers' attitudes towards mathematics via creative writing task in which they described their relationship with mathematics as though mathematics were a person. Then, he used conceptual blending to interpret participants' human description of mathematics, personification, by mapping their descriptions to corresponding mathematical character.

In this study, we will investigate whether interpreting personification writing tasks using conceptual blending yields the same results as conventional surveys about mathematics attitudes. Thirty-two pre-service teachers completed a 14 question survey assessing their beliefs and attitude regarding learning mathematics. The survey questions were modified from Mathematics Anxiety Rating Scale-abbreviated version (Alexander & Martray, 1989) and Mathematics Teaching Efficacy Belief Instrument (Enochs & Riggs, 2002). They also completed two creative writing tasks. The first one was the same as Zazkis' task, and in the second one, they introduced mathematics to their future students through personifying mathematics.

For one survey item, 68% of participants agreed that "teacher's own feeling about mathematics is related how well a teacher can teach mathematics to students". For example, Alex (pseudonym), who agreed with the statement wrote "sometimes math is a crazy monster that seems to try to make my life so much harder than it needs to be" on task 1 and on task 2 wrote "Math can be scary sometimes because we don't always understand what it is trying to show us" which indicate warning of math to students based on Alex's experience with mathematics.

On the same survey item, Casey selected that "teacher's own feeling about mathematics is independent of a teacher's practice". However, Casey wrote "I found myself face to face with someone(math) I hoped never to see again…" on task 1 and wrote "just remember, he(math) is really never going to be easy to talk to, so always prepare to think when you are around him" on task 2. These responses reflect Casey's relationship with mathematics affecting Casey's portrayal of mathematics to future students, and are contrary to what Casey selected on the survey. Therefore, two writing tasks can offer different view of participants' relationship of math and how it could affect their future teaching.

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# Transformers! More than Meets the Eye!

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*In this study, we characterize a conceptual model some students draw upon in their problem-solving activity when engaged in definite integral tasks. We call this model an Integral as a Transformer conception as it is invoked by students as a means to transform a quantitative relationship suitable for constant values into a structure appropriate for co-varying quantities.*

*Keywords:* Definite Integral, Adding Up Pieces, Quantification, Problem-Solving

This poster explores a conceptual model underlying two common misconceptions demonstrated by students attempting to apply definite integrals to problems in context. The literature often distinguishes between these two errors by the symbolic forms (Sherin, 2001) which cued the need for integration (Jones, 2013; Meredith & Marrongelle, 2008; Nguyen & Rebello, 2011). The first error, observed in students cued by the dependence symbolic form, is characterized by a student placing a given, or derived, quantity in for the integrand without consideration for its physical contextual relationship to the differential. The student might omit the differential entirely or only justify its presence as signifying the changing variable within the integrand. Meredith and Marrongelle described this reasoning as a dead end regarding successful student integration when the integrand is not a rate of change or density. When students were instead cued by the parts of a whole symbolic form some were observed to give quantitative meaning to the differential but only viewed the accumulation process as applying to the integrand; Jones described this as Adding up the Integrand. Through classroom observations, we noticed the dependence misunderstanding emerged in some students' reasoning even when they were cued to integrate by the parts of a whole symbolic form. It also appeared this error did not necessarily prevent students from making progress through definite integral tasks. In light of this, we hypothesized there might be an underlying tool (Dewey, 1938; Hickman, 1990) students utilize in their problem-solving process which motivates these misconceptions.

This tool, which we called an Integral as a Transformer conception, entails a student invoking a definite integral to convert a mathematical model that is appropriate for constant values of its constituents (e.g., distance = velocity · time) into a model applicable for contexts in which the constituent quantities co-vary. It should be noted that for a simple rate of change and density problems this conception often provides students with a heuristic for composing correct integral structure despite an incorrect quantitative interpretation.

Planning to challenge this heuristic, we developed our study using Dewey's theory of Inquiry which characterizes knowledge as a byproduct of the dialectic interplay between a student's selection, application, testing, and refinement of a conceptual tool when faced with a problematic situation. We videotaped interviews with nine students, eight in pairs and one alone, as they worked through a series of increasingly difficult contextual definite integral tasks. Our analysis found that every group in the study used the Integral as a Transformer conception at least once in their problem-solving process, despite many pairs also justifying the need for integration in terms of the parts of a whole symbolic form. In our poster presentation, we will discuss the numerous forms in which the Integral as a Transformer conception appeared throughout the interviews, when and how it proved problematic, and more importantly how it interacted with other conceptual tools in students' mathematical modeling activity.

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Characterizing Self-explanations for Undergraduate Proof Comprehension

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*A study was conducted with 11 undergraduate students in a real analysis course to further investigate important results reported by Hodds, Alcock, and Inglis (2014) on self-explanation and undergraduate proof comprehension, and by Ainsworth, S., & Burcham, S. (2007) on self-explanation and textual coherence. The main product of the current study is a framework of self-explanations in proof comprehension that takes into account students' questions as they self-explain both high and low coherence proofs.*

*Keywords:* Self-explanation, Proof Comprehension, Text Coherence

Self-explanations are explanations of textual material generated *by* the student *for* the student that attempt to aid comprehension. Students can be taught to self-explain and can realize the same benefits as those that do so spontaneously (Chi et al. 1994) especially in domains like mathematics (Rittle-Johnson, B., & Loehr, A. M. 2016). Hodds, Alcock, and Inglis (2014) replicated the self-explanation effect for proof comprehension in undergraduate mathematics. Based on previous research, Hodds et al. described three categories of self explanations (Principle-Based, Goal-Driven, and Noticing Coherence) and four categories of non-explanations (Paraphrasing, False Explanation, Positive Monitoring, and Negative Monitoring).

While these categories reveal important distinctions between undergraduate self-explanations, we argue that they are too broad. Although participants who were trained to self-explain saw greater scores on a proof comprehension test, they still produced the same amount of Paraphrasing and False Explanations as those who were not. Additionally, the proofs provided to participants did not vary in textual coherence (a text can be made more or less coherent depending on the degree to which it makes inferences between ideas and connections to textual goals explicit). While Ainsworth and Burcham (2007) showed that minimally coherent texts elicited different self explanations compared to maximally coherent texts, Hodds, Alcock, and Inglis (2014) used only minimally coherent proofs in their study. However, undergraduate students encounter proofs that are not minimally coherent, particularly in lectures and textbooks.

We explore the following questions: What are some of the different types of questions/self explanations that students generate when reading low- and high-coherence proofs, after going through a self-explanation training? How do these types relate to previous self explanation frameworks? In this study, 11 undergraduate students in a real analysis course received self-explanation training and were audio recorded as they modeled the self-explanation strategy out loud, with both low and high coherence proofs. Their self-explanations and questioning behaviors were used to create a framework based off of Hodds et al. (2014) that allows for a more nuanced consideration of self explanation types. For example, a statement in a proof such as *Since object O has properties A, B, and C,* was often followed by: "I see why we need properties A and B, but why was C necessary?" This kind of self-explanation has attributes of a Goal-Driven self-explanation, but it's not really related to the structure of the proof as a whole (as defined by Hodds et al., 2014). Furthermore, since more coherent proofs would be more likely to include this information, the level of coherence of the proof seemed to influence the extent to which this type of self-explanation was produced.

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# Students' Experiences in an Undergraduate Mathematics Class: Case Studies from one Student-Centered Precalculus Course

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*Using evidence-based practices in a large undergraduate mathematics classroom demands further investigation as there is still not significant work in this area. Results from this case study show that students perceived that their participation in student-centered instruction in an undergraduate Precalculus course, was helpful to their learning. The results also suggest that students demonstrated positive attitudes in regard to the collaborative efforts active learning components including Team Activities were utilized in this course and that the strategies were considered useful and important by the students interviewed.*

*Keywords*: Large Scale Classrooms, Precalculus, Evidence-Based Practices

In general, students have demonstrated greater gains in achievement and positive changes in affect when introduced to student-centered instructional strategies (Freeman et al., 2014; Kwon, Rasmussen, & Keene, 2005). However, there has not been significant work done to give a voice to students in these courses. Being able to show the effects these strategies have on students' perceived experiences in particularly large classes is an important contribution to the RUME community. The goal of this study was to understand the experiences of 5 students in a large-lecture Precalculus course where active learning strategies, such as the use of Team Activities and Clickers, were used. The research question was: What are students' perceptions of their experience in a large-scale Precalculus course where active learning strategies are used?

This is a multiple case study in a large Precalculus course at a southeastern university. The course included lecture and a computer lab component and introduced team activities and conceptual clicker questions designed to be completed in groups. The results are presented as 5 narratives to tell the stories of the students' perceptions of their experiences in this large-scale Precalculus course where student-centered instructional strategies are present (Creswell, 2013). Data collection included interviews and pre- and post-surveys. Interviews were transcribed and then coded using open coding techniques (Strauss & Corbin, 1998). Interview and survey responses were analyzed and reconstructed as a narrative to demonstrate these experiences.

For these 5 participants, it was most common that the Team Activities were classified as the favorite part of the course. Most of the students enjoyed collaborating and problem-solving with their peers. They also liked how it seemed relevant to real-life and was different than the procedural-type learning that they received from lectures and Smart Lab. The students in this study expressed mostly positive attitudes towards these student-centered learning strategies, in addition to demonstrating a similar change in attitude from a pre- to post-survey that was given to them at the beginning and end of the course.

Overall, we found that it is possible to bring student-centered learning strategies to a large-scale undergraduate class. To implement these strategies successfully, we suggest starting with a few activities and then slowly adding more activities to the course.

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Gender-based Analysis of Learning Outcomes in Inquiry-Oriented Linear Algebra

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*In order to better understand gender-based differences in learning experiences and outcomes in inquiry-oriented instructional settings, we analyze data from a common end-of-term assessment administered across 7 sections of linear algebra. This analysis focuses on data from 58 students, 22 of whom identified as female and 36 of whom identified as male. Distribution of the 58 students' scores was negatively skewed, similar to that of broader sample of 153 assessment scores (many of which did not have gender information available as that was collected separately from the assessment). A two-tailed t-test with independent samples was administered, revealing that the difference between the scores of the two groups is not statistically significant.*

*Key words:* inquiry-oriented instruction, learning outcomes, gender

Some research suggests that inquiry-based approaches to teaching undergraduate mathematics are likely to "[level] the playing field" between men and women (Laursen, Hassi, Kogan, & Weston, 2014, p. 412). Other research suggests that inquiry-oriented approaches may disproportionately advantage men (Johnson, Andrews-Larson, Keene, Keller, Fortune, & Melhuish, 2017). Inquiry-oriented instruction aims to support students in 'reinventing' important mathematical ideas by first posing challenging problems to students. The instructor's role is to elicit and build on these ideas so as to support the development of more formal langauge and notation that is rooted in students' initial, informal, and intuitive ideas (Rasmussen & Kwon, 2007). In our study, we ask: Are there gender-based difference in learning outcomes among students whose instructors received support to teach inquiry-oriented linear algebra?

Assessment data was collected from 153 students at the end of the term from 7 different classes; instructors of these classes received supports (curricular materials, a 16-hour summer workshop, and facilitated online workgroups for one hour per week during the semester of instruction) to teach inquiry-oriented linear algebra. The assessment consisted of 9 items, including both multiple choice and open-ended response questions, aimed to measure student understanding of key ideas in introductory linear algebra (solutions to linear systems, linear transformations, span and linear independence, and eigenvectors and eigenvalues). We were only able to match 58 of these assessments with gender data gathered as part of a student survey. Of those students, 22 identified as female and 36 identified as male. A team of 3 coders developed and used a rubric for scoring the open-ended assessment questions. After fine-tuning of the rubrics, 13% of the assessments were double coded, with each item requiring at least 80% inter-rater reliability (mean of 91%). Mean scores and standard deviations were computed for students who identified as male and as female; students who identified as other were omitted from this analysis. A two-tailed t-test with independent samples was conducted to determine if the difference of means of male and female TIMES students was statistically significant.

There is not a statistically significant difference between the scores of students who identified as male and female. With 51 points possible, the mean was 38.09 (SD=7.30) for female students and 38.59 (SD=7.67) for male students. There was not a significant effect for gender, $t(57) = .248$, $p = .805$. The effect size between male and female students was .065. A larger sample size is needed to support bolder claims about the generalizability of this finding.

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Examining Exams, Evaluating Evaluations: An Alternate Approach Assessed

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*In an attempt to bring a more realistic environment into the classroom during assessments, an alternative form of assessment was piloted during a mathematical modeling course at the United States Military Academy at West Point in the fall of 2017. The "alternate" assessments are primarily conceptual in nature and consist of three parts: a night before read-ahead introducing a new application, an in class individual portion, and an in-class group portion. Through the evaluation of this assessment technique, we hope to determine if it should be expanded to a larger audience in the future. Preliminary finding of this evaluation are presented.*

*Keywords:* Assessment, Innovation, Mathematical Modeling, Technology, Application

**Poster Proposal**

Our course has higher-order learning goals for students to learn to think critically, work collaboratively, use technology appropriately and effectively, and to work towards solving real-world problems creatively. While traditional exams are effective at assessing computational lesson objectives, they are limited in their ability to assess a student's growth with regards to these higher-order learning objectives. A desire to more meaningfully challenge students to grow in these ways by requiring them to engage these skills in an assessment setting led us to design and implement this alternate form of assessment.

A sub-movement of the flipped classroom movement that has been growing is team-based learning. The Team-Based Learning Collaborative maintains a list of related publications. These papers ( i.e. Rezaee, Moadeb, & Shokrpour, 2016; Huggins & Stamatel, 2015; Stein, Colyer, & Manning, 2016) focus on team-based learning techniques applied to classroom instruction and how effective it is in comparison with more traditional lecture techniques. While these studies are interesting, the assessment techniques used in these classrooms remain traditional exams. Eric Mazur, the Balkanski Professor of Physics and Applied Physics and Dean of Applied Physics at Harvard University provides a video of a talk on his website entitled *Assessment: The Silent Killer of Learning* in which he claims that the traditional method of assessment is outdated. He claims that we can create assessments in such a way as to encourage the higher-order thinking skills (creating, evaluating, and analyzing) of Anderson & Krathwohl's (2001) revision of Bloom's Taxonomy rather than the traditional assessment which required far more remembering, understanding, and applying.

While Mazur clearly has strong opinions and an abundance of ideas about this topic, we are unaware of any research studies that have addressed the feasibility and effectiveness of such assessments. Consequently, it is the goal of our pilot study to begin to address such questions of feasibility and effectiveness.

Our poster will provide sample assessments and some preliminary findings from our fall pilot. We look forward to discussing the project and potential ways to improve our assessments with conference attendees.

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# An Examination of Preservice Mathematics Teachers Using Ratios and Proportions in a Social Justice Context

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*This study examines ways in which preservice teachers use mathematics in a social justice context. Using a mathematical task and social justice activity adapted from Gutstein and Peterson (2005), participants were asked to respond to questions surrounding their experience with using mathematics topics such as ratios and proportions in a social justice context. Using the preservice teachers' responses from pre- and post-surveys, researchers compared participants initial conceptions of teaching mathematics using a social justice lens to their views after completing a mathematical task involving social justice topics of world wealth and population disparity using ratios and proportions.*

*Keywords*: Algebra, Ratios, Preservice Teachers, Social Justice, Equity and Diversity

When helping develop the knowledge and skills that preservice mathematics teachers need, part of what is needed are mathematical modeling tasks that are designed to elicit thinking and mathematical discourse (Blum & Ferri, 2009; Doerr & English, 2003). There has been a general call for increased quality in science, technology, engineering and math (STEM) undergraduate instruction in the United States due to a fear that the US is falling behind as a professional leader in STEM (Henderson, Beach, & Finkelstein, 2011; Jones & Johnston, 2010). Jones and Johnston (2010) propose that this heavily relies on improved mathematics instruction.

While diversity increases among the student population in public schools, the population of preservice teachers remains homogenous -- predominantly White, female, and middle class (Barnes, 2006; Swartz, 2003). One of the challenges for teacher education preparation programs is preparing preservice teachers to teach diverse student populations. An attitude of *naive egalitarianism* is prevalent among preservice teachers. Causey, Thomas, and Armento (2000) define this as, "[when preservice teachers] believe each person is created equal, should have access to equal resources, and should be treated equally" (p. 34). Preservice teachers with these beliefs may lack an understanding of multicultural issues, as well as disregard effects of past and present discrimination (Causey, Thomas, & Armento, 1999; Finney & Orr, 1995). This study looks at the ways in which preservice teachers viewed mathematics in the context of social justice related issues.

## Study Design

In this study, we provide mathematical tasks for preservice middle and high school teachers (N=40) that aligns with the ideals of teaching mathematics for social justice. We investigated the responses to the mathematical tasks centered around the topics of ratios and proportions. In this mixed methodological study, the researchers observed participants interacting with the tasks, and how they rationalized and reasoned with the material. There was a special emphasis on how the preservice teachers thought about students' misconceptions that the mathematics could possibly trigger and how they would guide these students. Through surveys, quantitative data was collected and will be reported on the poster. One major theme that arose was initial hesitation with the idea of teaching mathematics using a social justice lens, but this later evolved after having taken part in this study.

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Connecting Advanced Undergraduate Mathematics to School Mathematics

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*The Mathematical Education of Teachers as an Application of Undergraduate Mathematics (META Math) is a project to create, pilot, and field-test modules for use in undergraduate mathematics and statistics courses taken by pre-service teachers. Materials in calculus, discrete mathematics, algebra, and statistics showcase vital connections between college mathematics and the mathematics taught in high school. Drawing on recommendations in the Mathematical Education of Teachers II and the Statistical Education of Teachers, the project puts attending to the needs of pre-service teachers on par with attending to the needs of other undergraduate students by focusing on applications related to high school mathematics teaching.*

*Keywords:* Pre-service Secondary Teachers, Undergraduate Mathematics, Curriculum Modules

The Mathematical Education of Teachers II (MET II) report (CBMS, 2012) calls for a future in which secondary school students engage in substantive mathematical inquiry, solve non-routine problems, and make deep mathematical connections. The project, "The Mathematical Education of Teachers as an Application of Undergraduate Mathematics" (META Math), draws on the expertise of mathematicians and mathematics education researchers to address the content knowledge needs of undergraduate pre-service teachers, providing faculty with tools to better prepare teachers to contribute to the vision outlined in the MET II report.

The project team developed modules, focusing on four content areas: Calculus, Statistics, Algebra, and Discrete Mathematics. Each module is self-contained and textbook-independent, comprised of materials to assist faculty in seamlessly incorporating content into existing undergraduate courses to meet the specific needs of future teachers and foster deep examination of school mathematics content from the advanced perspective of undergraduate content. We plan to recruit and train a cohort of mathematics faculty to pilot test these modules during the 2018-2019 academic year, collecting research data from their implementation of the modules.

We will test the modules for effectiveness in promoting student understanding of the connections between 7-12 and undergraduate mathematics, investigate effective practices for using these modules, and provide insight into how module usage affects a faculty member's own understanding of school mathematics content from an advanced perspective. To do this, we will employ a qualitative case study approach, in which each content area is a case. We plan to conduct: (1) in-depth qualitative observations of faculty using modules in their classrooms, (2) a preliminary and a follow-up interview with faculty, and (3) cognitive interviews with students.

META Math not only focuses on enhancing pre-service teachers' understanding of the vertical connections from school mathematics through advanced undergraduate mathematics but also awareness of these connections among mathematics majors not intending to teach as a career. This poster presents an overview of the META Math project, including examples of the modules and ways faculty can participate in the field-testing of modules.

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# Content Analysis of Introductory Textbooks in Point-Set Topology

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*This study compared twelve point-set topology textbooks at the introductory level. The goal was to differentiate each textbook according to its overall conceptual approach to the field, as well as its mathematical approach to four fundamental topological ideas. The analysis indicated significant differences in the conceptual and mathematical presentation of those topics among the twelve textbooks. These findings highlight the need for researchers to distinguish between the conceptual and mathematical approaches found in textbooks for proof-intensive courses in undergraduate mathematics education.*

*Keywords:* Concept, Proof, Topology, Textbook analysis

Textbooks play an important role in outlining the learning progressions that students are expected to follow as they first construct their conceptual understanding for new ideas. The purpose of this study was to delineate some of the major conceptual approaches to point-set topology that are introduced by well-known textbooks on the subject, and therefore highlight the ways that students are expected to conceptualize ideas in their undergraduate topology courses.

## Methodology

Twelve introductory textbooks in point-set topology were selected for analysis (see References). The textbook choices were primarily based on the Mathematical Association of America's list of recommendations for undergraduate libraries (MAA, 2017), supplemented by a survey of recommendations published online by authors in the field. The textbooks were chosen to include a variety of conceptual approaches to the topic, but excluded specialized content, outline-style treatments, and out-of-print texts. One popular online textbook was also selected for comparison. The textbooks were examined and compared according to their overall presentation of point-set topology, as well as their approaches to four key analytical and topological concepts. These topics were: open sets, closed sets, sequence/limits, and continuous functions.

## Findings

There were significant differences found among the twelve textbooks, both in terms of the theoretical perspective taken on the field of topology, and in the conceptual progressions that were used to present individual topics. Differences in the textbooks' broad approaches to topology were often reflected in each author's choice of the definitions, theorems, and order of presentation of the concepts screened in the analysis. The textbooks' conceptual approaches to the introduction of topology were grouped into three categories, labelled the *metric-analytic*, *geometric-intuitive*, and *abstract-axiomatic* approaches. Textbooks that followed the *metric-analytic* approach tended to generalize concepts from the study of real analysis, often relying on sequential and metric-based techniques. The *geometric-intuitive* approach tended to focus on low-dimensional surfaces and spatial imagery to establish concepts based on physical intuition. Textbooks that followed the *abstract-axiomatic* approach established definitions and theorems from the topological axioms, employing abstract examples or counterexamples to illustrate concepts. Concept-specific comparisons will be presented in table format on the poster.

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Proportional Reasoning Using HLT Instead of Finding 'Missing' Value in Word Problems

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*Proportional reasoning sounds like finding the missing 'number' when the other three numbers are already given in the context. In this poster we are proposing a Hypothetical Learning Trajectory to achieve some learning goals leveraging proportional reasoning. We also propose effective strategies for solving proportional using graphs and conceptual analysis of proportional reasoning.*

*Keywords:* Proportional Reasoning, Conceptual Analysis, Hypothetical Learning Trajectory

## Introduction and Theoretical Framework

In proportional reasoning we are interested in comparing quantities in relation to one another instead of finding the 'missing' number of given situation. The fundamental concept we need for proportional reasoning is the idea of 'ratio'. A ratio is a binary relation which involves ordered pairs of quantities. (Lesh, Post, & Behr, 1988). According to Thompson (1994), A ratio is the result of comparing two quantities multiplicatively. When we discuss proportionality we not only consider one ratio, we compare two ratios with likely quantities. And the rate of change of both ratios remain same constant in this relationship. By Thompson (1994), a rate is a reflectively abstracted constant ratio. Both definitions of ratio and rate followed by Thompson's 1994 paper is fundamental perspective to look forward to proportional reasoning.

## Conceptual Analysis and Hypothetical Learning Trajectory

A conceptual analysis is a way to describe what students might understand about an idea to reason the way it should be understood (Thompson 2008). To conceptualize and reason proportionality I conjecture that the student will need to achieve seven learning goals I tried to identify in this poster. Simon's (1995) development of hypothetical learning trajectory(HLT) is consist of the goal for the student learning, and hypotheses of the students' learning (Simon, M. & Tzur, R., 2004). Generalizing conceptual analysis (Thompson 2008) and HLT (Simon 1995), this poster is going to present an HLT for proportional reasoning-

1. Students will draw a picture which represents the given situation
2. Students will identify quantities and determine whether they are varying or fixed quantities, and they will always verbalize them with corresponding units.
3. Students will be able to represent the situation graphically with scaled measurements.
4. Students will identify the varying quantities in the given situation and will be able to relate these quantities with the constant rate of change.
5. Students will understand that one quantity is as many times bigger or smaller as the second quantity. If there are more than two quantities in one situation they will be able to understand the relation among them as well.
6. Students will avoid seeing ratio and proportions only as a tool of performing calculations, applying rules and formulae and manipulating numbers and symbols in proportion equations.

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# Quantum Physics Students' Reasoning about Eigenvectors and Eigenvalues

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*Eigentheory is an important mathematical tool for modeling quantum mechanical systems, but little is known about how physics students reason about eigenvectors and eigenvalues as they transition from linear algebra courses into quantum mechanics. In this poster, we share examples of the resources (elements of students' knowledge) we have identified in physics students' reasoning about the eigenvectors and eigenvalues of real 2x2 matrices, as well as connections among these resources within and across students.*

*Keywords:* Linear Algebra, Eigentheory, Quantum Physics, Student Reasoning, Resources

This work is part of a larger research project that is examining the various ways students reason about and symbolize concepts related to eigentheory in quantum physics, as well as how students' language and symbols for concepts related to eigentheory compare and contrast across mathematics and quantum physics contexts. Expanding on the work done by Henderson, Rasmussen, Zandieh, Wawro, and Sweeney (2010), for this poster we focus on the following research question: What ways of reasoning about eigenvectors and eigenvalues of real 2x2 matrices exist for physics majors at the beginning of a quantum mechanics course?

To operationalize the research question, we use a Resources Framework, a type of fine-grained constructivism (Redish, 2004) initially proposed by Hammer (2000): "A resource is a basic cognitive network that represents an element of student knowledge or a set of knowledge elements that the student tends to consistently activate together" (Sabella & Redish 2007, p. 1018). Resources are activated depending on how individuals frame a given situation; resources can be linked to other resources, in which activation of one resource can promote or demote activation of others; and resources may internally consist of finer-grained resources linked in a particular structure (Sayre & Wittmann, 2008). We seek to identify resources that characterize how our participants reasoned about eigenvectors and eigenvalues of real 2x2 matrices.

Data come from semi-structured (Bernard, 1988) individual interviews conducted in the first week of the semester with eight students enrolled in a senior-level quantum mechanics course at a public research university in the Northeast United States. The linear algebra prerequisite at this university was either a linear algebra or combined differential equations and linear algebra course, both sophomore-level and taught in the mathematics department. The interview question analyzed asked students to reason about the equals sign and solutions to $A\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix}$, where $A$ is a 2x2 matrix, and to find the eigenvectors and eigenvalues of a specific 2x2 matrix.

Each interview was transcribed and then watched independently by the three authors (one physics and two mathematics education researchers) in an attempt to identify students' conceptual and procedural resources (Wittmann & Black, 2015). For example, conceptual resources include eigenvectors being stretched by eigenvalues, scalar multiples of eigenvectors also are eigenvectors, or that the result of the product $A\begin{bmatrix}x\\y\end{bmatrix}$ is the same object as the result of the product $2\begin{bmatrix}x\\y\end{bmatrix}$. Procedural resources include, for example, performing the mathematical steps to rewrite $Ax = \lambda x$ as a system of equations or finding the roots of a characteristic polynomial to find a matrix's eigenvalues. This presentation will feature several examples of these identified resources, along with connections between these resources within and across students.

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Bridge Programs for Engineering Calculus Success

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*Mathematics is often the gatekeeper for students aspiring for a college degree in any field. A precalculus bridge program to improve success in the engineering calculus sequence was initiated at Texas A&M University in summer of 2010. Students who placed into precalculus were offered the program. The program was revised and additional bridges were added. Surveys were administered to examine student beliefs about college level mathematics expectations and requirements. Overall, the programs benefited hundreds of students.*

*Keywords* - calculus, bridge program

Success in engineering programs is highly dependent on mathematics knowledge, but many freshmen entering college lack the preparation needed for success in their college mathematics coursework. This deficiency limits their future career opportunities (Achieve, 2008). Bridge programs have been used to support students' mathematics knowledge and skills during the summer prior to college coursework (Conley, 2008). These programs often cost institutions considerable revenue and staff resources (Kallison & Stader, 2012). Bridge programs were often used to specifically assist first generation college students and those of low socio-economic status (Grimes & David, 1999; Inkelas & McCarron, 2006). Results have been mixed (An, 2012), but positive results have been reported (Gamoran, Porter, Smithson, & White, 1997).

A precalculus bridge program was initiated at Texas A&M University to help engineering students prepare for college level mathematics workloads. The university enrolls about 3,300 freshman engineering students each fall (60% white; 26% Hispanic; 24% female; 76% male). Students who did not meet the cut score on the Mathematics Placement Exam (MPE) for Engineering Calculus I were offered the Personalized Precalculus Program (PPP). Studies for various cohorts have shown improvement on the MPE (Morgan, Nite, Allen, Capraro, Capraro, & Pilant, 2015; Nite, Allen, Morgan, Bicer, & Capraro, 2016) and performed as well in calculus as their peers with similar backgrounds (Nite, Allen, Capraro, Bicer, & Morgan, 2016; Nite, Morgan, Allen, Capraro, Capraro, & Pilant, 2015). Students who spent more time were more successful (Nite & Allen, 2014a; Nite, Allen, Bicer, & Morgan, 2016). Results from surveys identified areas in which students lacked confidence (Nite & Allen, 2014a) and student beliefs and expectations for college mathematics study (Nite, Allen, Bicer, Morgan, & Barroso, 2017).

A bridge to Engineering Calculus II was added for continued support (Nite & Allen, 2014b; Nite, Morgan, Allen, Capraro, & Capraro, 2015), and finally, a precalculus program for students who met the cut score on the MPE (Nite, Morgan, Capraro, Allen, & Capraro, 2014). The overarching question is "What impact did the three bridge programs for engineering calculus courses have on student success in their college mathematics courses?"

Results for the bridge programs have shown a positive impact overall, with statistical significance in MPE improvement ($p < .05$), and course pass rates have been acceptable. Students have difficulty with study behavior and time commitments for college mathematics. Therefore, future bridge programs should consider other factors in addition to mathematics knowledge and skills to improve success for students in college mathematics. Bridge programs to support non-STEM majors in their mathematics could be an effective strategy to improve perseverance toward completing college degrees for at-risk populations.

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**A Hypothetical Learning Trajectory (HLT) for Preservice Secondary Teachers' Construction of Congruence Proofs**

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*With the advent of the Common Core State Standards, there has been renewed interest in teaching geometry from a transformation perspective; however, most geometry teachers are unfamiliar with this approach as they learned geometry from a perspective based on Euclid's Elements. Consequently, there is little knowledge of how teachers who come from this traditional perspective learn geometry from a transformation approach. One major difference that teachers must reconcile is in the construction of congruence and similarity proofs. As such, there is a need to understand how teachers learn these proofs from a transformation perspective. We propose to present a poster reporting a hypothetical learning trajectory (HLT) for preservice teachers' construction of such congruence proofs, based on the coursework of 15 preservice secondary teachers and cognitive interview responses to geometry tasks.*

Keywords: hypothetical learning trajectory, transformational geometry, proofs

Simon (1995) defined a *hypothetical learning trajectory* (HLT) as "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (p. 133). HLTs can be influential in improving curriculum and instruction (e.g., Sztajn et al., 2012) and have been examined extensively for K-8 levels (e.g,, Daro et al., 2011).

We contend that HLTs could be similarly influential in improving the instruction of geometry from a transformation approach, including for pre-service secondary teachers who themselves learned geometry from a more traditional perspective and who may have to teach from a transformation perspective in the future. The central difference distinguishing these two perspectives on geometry is in the construction of congruence and similarity proofs. Thus we address: *What thinking and learning do pre-service teachers progress through while learning to construct congruence proofs?* We focus on congruence as it is typically viewed as a prerequisite for learning similarity.

Based on the mechanism of reflection on activity-effect relationships for generating an HLT (Simon & Tzur, 2004), we identified components of a HLT as specified by Simon (1995). The data for this study are 12 written assignments, selected using Simon and Tzur's (2004) framework, from 15 pre-service teachers (PSTs) (7 white females, 8 white males) enrolled in a course on geometry from a transformation perspective; and 4 students' responses to cognitive interviews. The PSTs are undergraduates at a research-intensive, doctoral granting institution.

The main result of our analysis is an HLT for pre-service secondary teachers' learning to construct congruence proofs. The HLT spans initial learning of rigid motions, to constructing proofs for triangle congruence criterion, to constructing proofs incorporating more complex and/or multi-component geometric objects. Results from this study can be applied to improving undergraduate education of pre-service secondary teachers and potentially informs task design to support concepts of congruence and similarity.

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Relationships Between Calculus Students' Ways of Coordinating Units and their Ways of Understanding Integration


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*This poster describes results from a paired-student teaching experiment focused on college calculus students' understandings of integration. Our aim was to model relationships between students' covariational reasoning, quantitative reasoning, and numerical reasoning as they were developing meanings for integration, via teaching sessions that were concurrent but independent from the students' "traditionally-taught" second-term calculus course. We will discuss commonalities between students' ways of reasoning multiplicatively, ways of reasoning about linear rates of change, and ways of understanding integration.*



*Keywords:* Covariational Reasoning, Integration, Numerical Reasoning, Quantitative Reasoning


A key aspect for conceptualizing the fundamental theorem of calculus, the *accumulation function*, $F(x) = \int_a^x f(t)dt$, requires coordinating three varying values: that of an independent variable, *t,* as it varies from *a* to *x,* that of a dependent variable, *f*(*t*), as *t* varies, and that of the accumulation of values of *f*(*t*) as *f*(*t*) and *t* co-vary (Swiden & Yerushalmy, 2016; Thompson, 1994; Thompson & Silverman, 2008). Research with K-12 students points to the necessity of students' construction of a structure for coordinating three *levels of units* for (a) reasoning flexibly with (im)proper fractions, e.g., for thinking of '9/7' as "containing" potential multiplicative relations with '1', '1/7', '1/9' and '7/9', and (b) reasoning flexibly with algebraic equations in the middle grades (Hackenberg & Lee, 2015). Students sometimes experience success in school mathematics if they learn to reason with three levels of units *in activity*, which means they "build" an ephemeral third level of units as part of their way of reasoning rather than assimilating situations with a units (of units (of units)) structure (Ulrich, 2015). Indeed, some students assimilating with two levels of units pursue STEM majors in college: Boyce and Wyld (2017) described constraints in two such differential calculus students' reasoning about function inverses and function composition, and Byerley (2016) described how students' reasoning with fractions was (and was not) associated with their success in different aspects of introductory calculus.

We report on an 8-session constructivist teaching experiment (Steffe & Thompson, 2000) exploring connections between students' units coordination and understandings of integral calculus. Our poster focuses on contrasting the reasoning of a pair of students, one who assimilated with two levels of units and one who could assimilate with three levels of units. Our poster will exemplify contrasts (and commonalities) in (a) their units coordination (b) their ways of reasoning about linear rates of change (c) their meanings for the quantities represented in the statement $F(x) = \int_0^x sin(t)dt$, and (d) their associated justifications for why $\int_0^\pi sin(t)dt = 2$. The results provide conjectures of how differences in the constraints students face in conceptualizing the accumulation function (and fundamental theorem of calculus) may be attributed to differences in their ways of coordinating units.

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Exploring the Efficacy of a Game-Based Learning Application in Undergraduate Mathematics: Functions of the Machine

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*Covariational reasoning is at the heart of many pre-calculus concepts and is vital for calculus readiness (Cottrill et al., 1996; cited in Carlson, Jacobs, Coe, Larsen & Hsu, 2002). To explore the efficacy of a game-based learning application to facilitate conceptual understanding of function concepts through covariational reasoning, the University of Oklahoma's Virtual Learning Experience Team developed a digital game titled "Functions of the Machine". In the game, the student plays the role of a scientist tasked with making a complex machine run. The student explores, tests and fixes the machine's moving parts that consist of gears, fluid tanks, and conveyor belt contraptions. Through a series of interactive scaffolded problems, students transition from proportional reasoning to complex covariational reasoning simulations.*

*Keywords:* Covariational Reasoning, Game-Based Learning, College Algebra

According to Carlson, Oehrtman, and Engelk (2010) it has been well established that pre-calculus level students' thought processes for function concepts are primarily procedural and indicative of an action view of functions (Carlson, 1998; cited Carlson et al., 2010). Furthermore, covariational reasoning, described as the ability to interpret the relationship between two varying quantities as they continuously change, has been documented to be a challenge for even "academically talented" undergraduate students (Carlson, 1998; cited in Carlson, Jacobs, Coe, Larsen & Hsu, 2002, p. 353). A meta-analysis conducted by Vogel et al. (2006), reported with high validity that interactive educational computer games were associated with more significant cognitive gains and more desirable learning attitudes compared to traditional teaching methods (Vogel et. al., 2006). The University of Oklahoma Virtual Learning Experience Team developed a digital game titled "Functions of the Machine" to address the following research question: Is a visually dynamic game-based learning environment associated with better cognitive outcomes in covariational reasoning compared to visually static traditional homework or non-game-based covariational homework? It is hypothesized that the visually dynamic game-based learning environment may be better equipped at helping students develop covariational reasoning. A randomized controlled design was used to test this hypothesis. Students enrolled in College Algebra and Pre-Calculus for Business, Life, and Social Sciences at the undergraduate level were recruited and randomly assigned to one of three conditions: digital game play, traditional problems, or covariational problems without a game environment. All three conditions completed demographic and engagement surveys, pre-post assessments, and a subset of items from The Attitudes Toward Mathematics Instrument (Tapia & Marsh, 2004). Data collection and analysis is still ongoing; results will be presented at a later time.

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Students' Understanding of Quadratic Equations

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*Action–Process–Object–Schema theory (APOS) was applied to study student understanding of quadratic equations with one variable. This requires proposing a detailed conjecture (called a genetic decomposition) of mental constructions students may use to understand quadratic equations. The genetic decomposition, which was proposed, can contribute to help students achieve an understanding of quadratic equations with improved interrelation of ideas and more flexible application of solution methods. Semi-structured interviews with eight beginning undergraduate students explored which of the mental constructions conjectured in the genetic decomposition students could do, and which they had difficulty doing. Two of the mental constructions that form part of the genetic decomposition are highlighted and corresponding further data was obtained from the written work of 121 undergraduate science and engineering students taking a multivariable calculus course. The results suggest the importance of explicitly considering these two highlighted mental constructions.*

*Keywords:* Quadratic Equations; APOS Theory; Genetic Decomposition; Calculus

Many secondary school students, and undergraduate students, do not truly understand quadratic equations or the rules they use to solve them (Didis, Bas, & Erbas, 2011). Some studies suggest that problems like these may stem from the lack of details in books, which, consequently, teachers tend to not emphasize (Sönnerhead, 2009). Some investigations on student understanding of quadratic equations refer to specific misconceptions (Bossé & Nandakumar, 2005; Ochoviet & Oktac¸ 2009¸ 2011; Vaiyavutjamai, Ellerton, & Clements, 2005). In Puerto Rico, the Department of Education established it is not until tenth grade that students learn how to solve quadratic equations. These quadratic equations are simpler by design, and can be solved using the following techniques: factoring, using square roots, completing the square, the quadratic formula, and by using technology. Given the described context held by incoming first year students attending a Puerto Rican university, this article investigated students' understanding of quadratic equations by: (1) establishing a conjecture of their mental constructions (stated in terms of the constructs of Action–Process–Object–Schema (APOS) theory) that beginning university students may do in order to understand how to solve quadratic equations; (2) using semi-structured interviews in order to investigate which of the conjectured mental constructions students can do and which they have difficulty doing; and (3) using written work from more advanced undergraduate students to investigate their use and understanding of two specific mental constructions conjectured in the genetic decomposition.

## Results and Conclusions

The genetic decomposition and results from this study highlight two specific mental constructions that play a key role in students' understanding of quadratic equations, that students have difficulty doing and that they seem to be overlooked in traditional instruction. Another result of the study underscores the importance of numerical and graphical explorations into the nature of the possible solutions of a quadratic equation.

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Using Everyday Examples to Understand the Concept of Basis

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*The purpose of this paper is to explore everyday examples given by students to explain the notion of basis. By exploring key aspects of the examples generated by the students we can see what roles and characteristics of basis the students attend to.*

*Keywords:* Examples, Linear Algebra, Basis, Generating Examples

In natural language, it is common to use an everyday example as a conceptual metaphor for abstract ideas. Research suggests that digesting formal definitions can be a stumbling block for students (Edwards & Ward, 2004; Knapp, 2009). Several researchers have studied how students come to understand the concepts of basis, span, and linear (in)dependence (e.g. Aydin, 2014; Stewart & Thomas, 2010; Trigueros & Possani, 2013; Plaxco &Wawro, 2015). Adiredja & Zandieh (2017) developed a framework for exploring students' generation of everyday examples for basis. We use a modified version of this framework to explore student everyday examples.

## Data Collection and Analysis

Data was collected by our second author in Germany with nine Applied Math upper division and masters university students. Interviews were held in English and videotaped by a speaker fluent in English and German. Adiredja & Zandieh (2018) studied ways that students understand the notion of basis in linear algebra. Two aspects of that understanding were how the basis vectors related to the space (*roles*) and the nature of the set of basis vectors (*characteristics*). The roles codes are Generating, Covering, Structuring, Traveling and Supporting. Characteristic codes are Minimal, Essential, Representative, Non-redundant, and Different. We will use the data to illustrate some of these roles and characteristics.

## Results and Discussion

After engaging in mathematical activities on basis, students were asked how they might describe the idea of basis to someone who had not yet learned the concept. The students developed everyday examples as part of their explanation. One student, Andreas, began with a description of a sailboat; however, he decided to think of a different example because the sailboat example had a flaw "[if] you can't assess the point already with one of the other vectors or a combination then it isn't actually a new vector and it can be crossed out or x'ed out from the basis." The student here was attending to the characteristic of non-redundancy. The sailboat example didn't have a mechanism to keep there from being redundant vectors in the basis.

Andreas then chose a star fish to describe basis. "You have sea stars; … they just walk straight in one of those directions." He thought of the star fish legs as the vectors in the basis. "So they just declare one of their legs as front. And then they march on. And so they can access any point on the sea floor which is our plane again." In using the verbs "march, access, move, and walk," Andreas's star fish example illustrates the Traveling code. He reflected on the characteristics of the star fish example, "by moving in those five directions, the question is, is this even necessary to have five directions." Before Andreas was focused on "x-ing out" redundant vectors, now his focus has shifted to consider the minimal amount of legs needed to access the entire sea floor. This illustrates the characteristic of minimal.

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**Math Help Centers: Factors that Impact Student Perceptions and Attendance**

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*Mathematics help centers have become more common in post-secondary education, but there is scant research on them. In this study we use data from 1088 students over six academic semesters and grounded theory analysis techniques to study and draw initial conclusions on student perceptions of and reasons for attending a math center.*

*Keywords:* tutoring, university mathematics, support services, help center

Mathematics help centers (or "math centers") typically aim to support undergraduate students in the mathematics courses they take during their freshman and sophomore years of study. These facilities are where students receive a) tutoring, b) access to print resources, c) guidance on use of digital devices and platforms used in mathematics courses, and d) pre-exam review sessions. The increased attention to math centers is evidenced by the new working group in RUME and by a recent handbook for math center directors, which had contributors from 31 institutions ranging from two-year community colleges to liberal arts institutions to large research universities (Coulombe, O'Neill & Shuckers, 2016). This study is an initial foray to consider students' perceptions of and reasons for attending a math center. More specifically, the following open research questions guided the study: *What do students expect from a math center? What are their perceptions of a math center? What impacts students' attendance at a math center?*

This study reports on responses to an online survey with items specifically related to a math center. It was administered over six academic semesters to 1088 students at a large, research university in the southwest United States. The research team used inductive techniques and constant comparison to consider factors that impact students' perceptions of and attendance at a math center.

All quantitative results of the study including information on the sample and the most frequent responses will be reported. Primary findings suggest that math center factors that impact students typically involve the number of staff and student interactions with staff. However, students also commented on issues related to their courses and instructors (e.g., desire for a high grade, perception of instructor deficiencies). Based on the results the research team will offer an initial explanation of the interrelated factors that impact student perceptions of and attendance at a math center in light of expectations of both a) the math center and b) students' courses and instructors.

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# Applying Cognitive Learning Theory to Design a Calculus Class for Engineers

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*We describe the methodology used to create a set of interactive online calculus resources for undergraduate engineering students. This content was designed using cognitive learning theory to actively engage students in doing calculus, transfer to engineering courses, and address misconceptions.*

*Keywords:* applied math, calculus, engineering students

Success in calculus has a large effect on success in later engineering courses (Boyajian, 2007). We created a set of interactive online calculus resources that we have used as a free calculus MOOC, as the main course material for a flipped classroom on campus, and for online homework (blended learning) for a traditional lecture course. Here we describe how we design this set of calculus resources to provide motivation, encourage transfer to later engineering courses, and keep students actively engaged with content.

Students learn more when they are motivated. According to Everett et al, 2000, Gagne et al 2004, and Merrill, 2013, motivation is achieved when an instructor provides context for a student to see how the content is relevant to their lives. For each lecture, we created a motivational video that either connected the lecture content with real-life or engineering design problems, or provided big picture connections among calculus concepts, as in Shah, D. et al, 2013. For example, to motivate parametric curves, we showed a student constructed roller coaster with a vertical segment, which cannot be described by a function, but can be by a parametric curve.

To enhance transfer to engineering courses, we introduce linear and quadratic approximations early, immediately after students learn how to differentiate. One of the first problems students work out in the lecture sequence involves using linear approximation and measurement error to design a zipline that is neither too fast nor leaves riders stranded in the middle. Similar problems tying approximations to new content is on almost every problem set throughout the course. By the lecture on Taylor Series, students already have a good sense of the utility of linear and quadratic approximations. This type of interleaving encourages long term retention (Roediger & Butler, 2011), which is necessary for transfer (Halpern & Hakel, 2003).

Unlike a traditional lecture or textbook, each lecture sequence was designed to engage a student in actively working through the content in a manner more similar to a tutorial (Freeman et al, 2014 and Hmelo, Gotterer, & Bransford, 1994). There are 1413 problems interspersed among the 395 short teaching videos and 25 motivational videos. These problems do much more then drill on concepts. Several concept checks are designed to address common misconceptions (Epstein, J. (2007).). Many build upon students' prior knowledge (Roediger & Pyc, 2012). Problems ask students to set up problems rather than do rote computations. These problems all offer immediate feedback (Bransford, Brown, & Cocking, 2000, and Hattie & Timperley, 2007), including feedback on hand drawn graphs (French et al, 2016). The solutions provided after a student has gotten a problem correct or has run out of attempts often contain much of the teaching content and offer an expert perspective. The student is truly learning by doing: *Mens et Manus,* the MIT motto.

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# Teachers' Reasoning with Frames of Reference in US and Korea

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*We gave approximately 180 US and 380 Korean teachers frame of reference tasks, and coded the open responses with rubrics intended to rank responses by the extent to which their responses demonstrated conceptualized and coordinated frames of reference. In both countries less than half of teachers scored at the highest level on almost every task, showing that teachers frequently struggle to keep track of quantities within a frame of reference in a meaningful way. Our US-Korean comparison also shows that US teachers struggle on most of these tasks significantly more than Korean teachers.*

*Keywords:* Frames of Reference, Secondary Teachers, International Comparison

## Theoretical Perspective

When we speak of frame of reference, we mean that an individual can think of a measure as merely reflecting the size of an object relative to a unit or he can think of a measure within a system of potential measures and comparisons of measures. An individual conceives of measures as existing within a *frame of reference* if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives meaning to a zero measure and all non-zero measures, and 3) committing to a directionality of measure comparison additively, multiplicatively, or both. […] An individual is coordinating two frames of reference if she conceives each frame as a valid frame, stays aware of the need to coordinate quantities' measures within them, and carries out the mental process of finding a relation between the frames while keeping all relative quantities and information in mind (Joshua, Musgrave, Hatfield, & Thompson, 2015).

## Methodology

From 2012 to 2015, the Project Aspire team created the 48-item assessment Mathematical Meanings for Teaching – secondary math (MMTsm). Two items were categorized as frame of reference items: "Willie Chases Robin" and "Nicole Chases Ivonne". We gave them to 177 US and 359 Korean teachers and coded their free-response answers; our poster has both the aggregate data and sample responses.

## Results & Discussion

It is our hope that the data collected will orient professional development leaders to consider their teacher's meanings for the mathematics that they teach, and guide their PD to focus on helping teachers build more productive meanings. While professional development projects continue to administer the MMTsm, the data discussed in our poster is telling. In both countries less than half of teachers scored at the highest level on every task except for Korean teachers on Part A of Willie Chases Robin. There was also a statistically significant difference between US and Korean teachers on every task. Additionally, the Korean data shows us that the US data cannot be ignored simply by arguing that these tasks are inappropriate to give to high school teachers; there are a lot of gains with US teachers that could be made simply up to the current Korean teacher levels. Our data show that the U.S. teachers in our sample are not prepared to help their students reason through tasks involving multiple frames of reference.

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**Calculus I Instructors' Desires to Improve Their Teaching**


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|---|---|---|
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*As calculus is a course required for many undergraduate programs, several studies over the past decade have examined aspects that create successful calculus programs in the United States. While many of these studies have looked at the teaching practices and beliefs of calculus instructors, none have focused on instructors' desires to improve their teaching. The goal of this research is to examine how desires to improve teaching vary among different types of instructors (GTAs, instructors, tenured faculty, etc.), and how institutional or departmental expectations might influence those desires.*

*Keywords:* Calculus, teaching practices, improving teaching, teacher beliefs and desires

Calculus is a common course required for many undergraduate programs, especially science, technology, engineering, and mathematics (STEM) majors, but is often connected to attrition from these programs. In light of this, mathematics education researchers have begun examining aspects that create successful Calculus programs throughout the United States (Bressoud, Mesa, Rasmussen, 2015). One important aspect that has been a focus of some of these studies is the calculus teacher, specifically their beliefs, teaching practices, and interest in teaching Calculus.

Several studies (e.g., Bressoud & Rasmussen, 2015; Sonnert & Saddler, 2015) have examined mathematics teachers' beliefs and instructional practices, including their impact on students' attitudes towards and success in mathematics, but none have looked at instructors' desires to improve their teaching, and the beliefs and institutional expectations that might influence those desires. For this study, we are particularly interested in calculus instructors' interest in improving their teaching and better helping students understand concepts in calculus. The specific research questions for this study are:

1. To what extent are graduate students, faculty, and instructors interested in improving their teaching of Calculus I, and their awareness of student learning of calculus?
2. What supports do they perceive from their institution and department in regards to the scholarship of teaching and learning?
3. Is there a relationship between institutional or departmental support and calculus I instructors' desires to improve their teaching?

To pursue these research questions, we make use of a 2010 national data set collected by the Mathematical Association of America (MAA), with support from the NSF, as part of the "Characteristics of Successful Programs in College Calculus" (CSPCC) study (Bressoud, Mesa, Rasmussen, 2015). We specifically use the pre- and post-surveys given to calculus instructors, focusing on the questions related to their desires to improve their teaching and questions that could conceivably influence those desires (such as institutional or departmental expectations). Our poster will share descriptive statistics to give an overview of which types of instructors of calculus I are most interested in improving their teaching, as well as correlations between variables that may affect instructors' interest in professional development. A future direction for this project is to run more statistical tests to investigate if there are connections between other components, such as gender and type of institution, on instructors' desires to improve their teaching of Calculus I.

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# Seminars to Support Learning Assistants in Mathematics

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*Abstract: This poster reports on the design and implementation of a series of seminars to support undergraduate learning assistants (ULAs) working in university mathematics courses. The ULAs participating in this study work as tutors and classroom assistants in early college mathematics courses through Calculus 2. The seminars support ULAs to more fully understand their roles and to consider ways that they can improve equity and access for all students in their classes. The work is grounded in the perspective that learning is a sociocultural process (Lave, 1991) and that students' learning is significantly impacted by opportunities to participate in actively doing mathematics (Laursen, 2014; Freeman et al, 2014). This project strives to answer research questions related to understanding how ULAs conceptualize teaching for equity and access, how conceptualizations change through participation in seminars, and how these conceptualizations are related to students' experiences in mathematics classes.*

*Keywords:* Learning assistants, Calculus, Equity, Active learning, Professional development

## Background and Conceptual Framework

Undergraduate Learning Assistants (ULAs) are employed at University of Colorado at Boulder in mathematics courses ranging from Mathematical Analysis in Business through Functions and Models, which is a post Calculus upper level course. Their roles include acting as classroom assistants and serving as tutors in the mathematics department's academic resource center. These undergraduate students enter their positions because they are interested in helping others learn mathematics. They begin their jobs with little or no prior experience in teaching or tutoring, and they may have limited foundational knowledge of design principles for active learning or strategies for teaching for equity and access. They are enrolled in a course which supports ULAs from a range of math and science subject areas. The seminars on which this study is based support the ULAs specifically within the context of their work in mathematics.

This work is grounded in research on design principles of active learning (Webb, 2016) which demonstrates the effectiveness of active learning for increasing persistence to subsequent mathematics courses (Laursen, 2014; Freeman et al, 2014). The work is also informed by sociocultural learning theory which explains the ways in which learning develops from the conversations and activities in which students take part, as well as their own roles within those conversations (Yackel & Cobb, 1996). The seminars on which this work is focused are designed to increase the degree to which ULAs are able to help cultivate opportunities for students to actively participate in doing mathematics in their college courses.

## Research Methodology, Results and Implications

This poster will report on the design and implementation of seminars to support ULAs in university mathematics classes to better understand their roles in increasing equitable opportunities for students to actively participate in doing mathematics. Results from a survey administered to the ULAs after the seminar will be reported, as well as analysis of those results. Results from interviews conducted with a small group of ULAs will also be included.

The results from this study will serve to inform the research community about ways to support ULAs to maximize their positive impact on opportunities to learn for all students in undergraduate mathematics classes. Specifically, these results will inform further research about how ULAs developing conceptualizations of teaching for equity and access, and how this relates to students' learning opportunities in undergraduate mathematics.

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An Instructional Resource for Improving Students' Conceptual Understanding of Functions through Reflective Abstraction

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Sepideh Stewart
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*It has been widely documented that undergraduate-level students' understanding of functions is rigid and indicative of an action view which constrains conceptual understanding (Carlson, Jacobs, Coe, & Hsu, 2002). Duval affirms that, "to understand the difficulties that many students have with comprehension of mathematics, we must determine the cognitive functioning underlying the diversity of mathematical processes" (2006, p.103). What are the underlying cognitive skills students need to gain a better conceptual understanding of functions? How should instruction of function content and training in these cognitive skills be combined? We propose the theoretical model "Structural-Schema Development for a Function", to address these questions. This model defines developmental stages students pass through to form a global view for the function concept, identifies underlying cognitive mechanisms involved in each stage, and develops instructional exercises that combine content with cognitive skills training for these cognitive mechanisms.*

*Keywords:* Schema, Functions, Understanding, Reflective Abstraction, Structuralism

The proposed model in this study is influenced by Duval's (2006) framework for *Treatments and Conversion* and Piaget's theory *Reflective Abstraction* (Arnon et. al., 2014; Dubinsky & Lewin, 1986). In Dubinsky and Lewin's view, "reflective abstraction includes the act of reflecting on one's cognitive action and coming to perceive a collection of thoughts as a structured whole" (1986, p.63). It can also be thought of as coordinating multiple lower-level structures and reflecting on these structures to combine them into a new higher-level structure (Dubinsky & Lewin). This is the glue that binds the four developmental stages of this model: Identification, Informal Classification, Formal Classification and Ordering. Students who reach the Identification stage can identify whether or not some object satisfies the formal definition of a function. Once students have identified enough objects that are elements of a function space they can begin to construct informal properties of these elements and classify them based on informal properties. Students who can describe a collection of functions as sharing the same informal properties (i.e. these functions are all smooth, these functions have jumps, etc.) have reached the Informal Classification Stage. These informal classifications can then become updated content on which formal classifications are either constructed or presented. These classifications are then coordinated for Ordering. The Ordering stage is reached when students can make substructures within the structure by ordering or nesting classifications using set relations and intuitions guided by informal classifications. Formation of a student's structural-schema for a function can occur in any of the developmental stages. The stage that a student reaches determines a threshold for how rich their structural schema has the potential to be. The student's structural-schema continually undergoes Maintenance through *assimilation* and *accommodation*. The proposed model anticipates to offer instructional resources designed to promote the skills students need to reach higher level developmental stages and ultimately gain a better conceptual understanding of functions.

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**Student reasoning with complex numbers in upper-division physics**

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Abstract: *Students encounter complex numbers in many physics courses. In particular physics uses complex exponentials to describe oscillatory phenomena and requires that students use multiple representations (algebraic, x vs t graphs, complex plane). In this poster we will examine student responses suggesting difficulties with the connection between complex numbers and oscillation, drawn from students in upper-division physics courses in math methods.*

## Description

This work is part of a collaboration to investigate student learning and application of mathematics in the context of upper-division physics courses. In particular the project focuses on a course required by most physics departments focusing on developing mathematical methods for upper-division physics. Throughout, we seek to go beyond procedures and to probe conceptual understanding and the development of quantitative reasoning skills.

Results suggest that procedural understanding of complex algebra is often not enough for students to connect mathematics with relevant physics contexts. Students had difficulty in relating complex numbers to oscillatory phenomena. It was not immediately clear whether incorrect responses reflected difficulties with procedures or conceptual understanding. For example, students were asked on a course exam to show, using expressions with complex exponentials, how two waves would destructively interfere given a $\pi$ phase difference. Of ten students answering after instruction, only three gave correct answers, none using polar form.

To probe student reasoning, we have used a variety of tasks including both procedural symbolic manipulations and more conceptual questions. While students were largely successful on procedural tasks, their responses suggested a key disconnect with the use of complex numbers to describe oscillations. For example, students were asked to sketch the real part of the function $Ae^{iwt}$ (8 sections, $N = 107$). Student written responses were examined and coded based on correctness and the explanation; the relevant codes after several iterations included the overall graph template (oscillatory / exponential / linear), the value of the function at t = 0, and, in the case of oscillatory sketches, whether the amplitude was constant or changing. Figure 1 shows a correct response and one showing exponential growth.

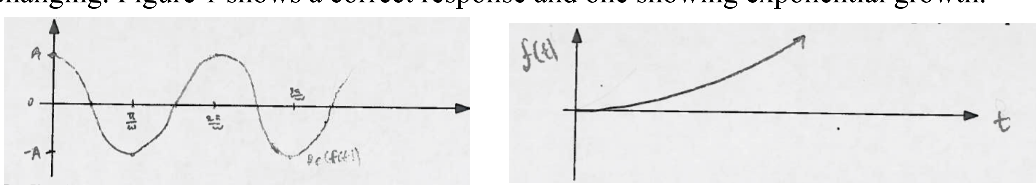


**Figure 1** Scans of student sketches of the real part of the function $Ae^{iwt}$. About a third of students sketched responses like the second example, showing exponential growth.

About 28% of responses were categorized as correct and another 15% showed an oscillatory function with incorrect features (e.g., phase shift or decreasing amplitude). Many responses, however, did not show oscillation; 33% of responses were categorized as showing exponential *growth*. Sadaghiani (2005) reported similar confusion between $e^{kx}$ and $e^{ikx}$ in quantum mechanics examples.

In this context and others, our data suggest that experience with mathematical procedures is not sufficient for students to make sense of the meaning of the procedures and apply them to physics contexts. The course text includes only procedural exercises with a handful of applications, including oscillations, separated into a later section at the end of the relevant chapter. We believe students need dedicated curricular materials focusing on these ideas.

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Using Catan as a Vehicle for Engaging Students in Mathematical Sense-Making

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*Catan is an increasingly popular board game which is rich with opportunities for mathematical applications. The research presented in this poster demonstrates how Catan served as an effective vehicle for engaging students enrolled in a freshman learning community. Students engaged with various mathematical concepts involving probability, combinatorics, and game theory.*

*Keywords:* student engagement, motivation, mathematical reasoning, combinatorics, games

Games provide unique opportunities for students to engage in mathematical reasoning (Canada and Goering, 2008; Capaldi and Kolba, 2017). Using games in teaching applies to all four components of Keller's ARCS model of Motivation Design Thoery (1987). Catan is a property development and trading-based game with a board which is set-up differently every time you play. This dynamic board and other aspects of the game provide several opportunities for students to engage in mathematical reasoning to improve their chances of winning (Austin and Molitoris-Miller, 2015). Players place settlements and roll dice to collect resources from resource tiles adjacent to their settlements. Resources are used to build more structures which increase card production or score points. In this study we explored the ways in which using Catan in a general education course for freshmen affected their engagement and motivation.

Participants were freshman students from a variety of majors enrolled in a freshman seminar course which was part of a learning community coupled with a section of pre-calculus. The course consisted of general college skills as well as opportunities to play Catan and discuss the related mathematics. Data were collected throughout the course of a semester in the form of written class work, homework, projects, exam items, and two online surveys.

Preliminary analysis of the data indicates that the game supported students' engagement in mathematical reasoning around a variety of ideas. The use of two different color dice in the game helped clarify justification behind the probability of rolling 2 through 12 using two standard six-sided dice. Determining which player to rob one resource card from in order to maximize the chance of obtaining a desired resource supported reasoning about probability and fraction comparison. Evaluating which locations to choose for initial settlements invoked various considerations of expected value including: the value of each resource based on production costs, the value of each resource based on rarity on a particular board, the probability of a settlement location producing a card, and the probability of a settlement location producing a rare resource. They also deduced results related to Bayes' Theorem by exploring if a particular resource is produced, what is the probability it came from a certain location. Survey results indicate that students saw value in learning mathematics in the context of Catan. 70% of students either agreed or strongly agreed that they enjoy using Catan to learn about mathematics. 75% of students either agreed or strongly agreed to "I understand mathematics best when I use an example or tool to figure it out."

These findings are aligned with other work which highlights the important links between games and mathematical reasoning. By utilizing Catan in this setting with freshman non-mathematics majors we have identified a creative way to make mathematics fun, engaging and accessible to students with a variety of mathematical backgrounds.

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# Teachers’ Knowledge of Fraction Arithmetic with Measured Quantities

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Abstract: *A national sample of 990 middle grades teachers completed a knowledge assessment aimed at measuring teachers’ knowledge of fraction arithmetic using measured quantities. Utilizing a simple measurement of the percentage of items answered correctly, middle grades teachers scored significantly different based on their undergraduate major. These findings reveal the importance of developing instruction for undergraduates focused on developing their multiplicative reasoning with measured quantities, especially fractions.*

**Keywords: Mathematical knowledge for teaching, Fractions, Multiplicative Reasoning**

The skills of multiplicative reasoning are important because their development can greatly influence success for students in later mathematics (Beckmann & Izsák, 2015). Common Core (2010) stresses the importance of reasoning with fractions as measured quantities. Middle grades teachers are required to reason multiplicatively using fractional amounts with designated units, and researchers have developed instruments to measure this specific content knowledge needed for teaching (Izsák, Jacobson & Bradshaw, in press; Jacobson & Izsák, 2015; Izsák, Jacobson, de Araujo & Orrill, 2012). The present study investigates how undergraduates with different majors who become middle grades teachers perform on a knowledge assessment of reasoning with fraction multiplication of measured quantities. Based on a sample of 990 middle grades teachers, undergraduate engineering majors scored higher on average than any other major, although this was not deemed statistically significant due to the small number (n=23) and variability. Mathematics majors had mean scores significantly higher than teachers who reported the category of “Other”. Izsák et al (in press) reports similar results. Since Business and other STEM majors comprised approximately a quarter and the “Other” category included almost half of the participants, the data were recoded into different categories (see Table 1).

*Table 1. Frequency of undergraduate majors*

| Undergraduate Major | Counts | Percentage |
|---|---|---|
| Business or STEM | 256 | 26% |
| Elementary Education | 240 | 24% |
| Mathematics Education | 169 | 17% |
| Other | 325 | 33% |
| **Total** | 990 | 100% |

At least one of the undergraduate majors has a statistically significant mean score according to a one-way analysis (p-value of less than 0.0001). Furthermore, a Tukey-HSD analysis reveals that teachers with an undergraduate major of Elementary Education and “Other” scored significantly lower than those with a Business or STEM degree. Teachers with an undergraduate degree in Mathematics Education cannot be differentiated from other majors. Middle grades content of multiplicative and proportional reasoning with fractions in context continues to be difficult for many people beyond their K-12 and college education. Business and STEM majors require more advanced mathematics than other majors and it follows they would have more opportunities to apply these skills across a wider variety of contexts and develop mastery.

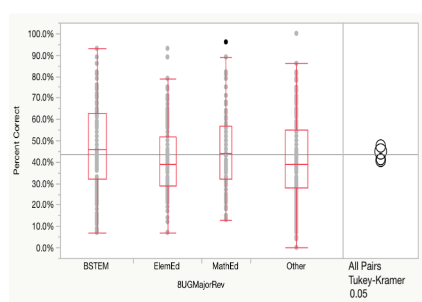

Figure 1

These findings reveal the importance of developing a deep understanding of multiplicative reasoning with measured quantities, especially fractions. Implications are especially important for the undergraduate teaching of our future teachers, who should possess a more robust understanding of this content in order to scaffold their future students learning.

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# Student Resources for Unit and Position Vectors in Cartesian and Non-Cartesian Coordinate Systems


| Warren Christensen | Brian Farlow | Marlene Vega | Michael Loverude |
|---|---|---|---|
| North Dakota St U | North Dakota St U | Cal St U – Fullerton | Cal St U - Fullerton |



*As part of a broader study into students' understanding of students' use of mathematics in upper-division physics courses, this study investigates how students conceptualize unit and position vectors in Non-Cartesian Coordinate Systems using a theoretical framework of resources. We present a case study of Mark, a Senior physics major, and identify the resources that Mark activates while answering conceptual questions without a direct physics context during a one-on-one interview protocol. This analysis identifies specific resources that Mark brings to bear when reasoning about vectors. The results of this case study provide a guide for analyzing additional interviews and allow us to pursue the long-term goal of curriculum development that can be used to improve students' use and understanding of non-Cartesian coordinate systems.*



*Keywords:* Vector Calculus, Resources, Student Reasoning, Physics


Using non-Cartesian coordinate systems is a challenge for undergraduate mathematics students [Paoletti, Moore, Gammaro, and Musgrave, 2013; Montiel, Vidakovic, and Kabael, 2008] and this challenge continues into upper-division physics courses where application becomes increasingly important [Hinrichs, 2010; Sayre and Wittman, 2008]. As part of a broader study to develop research-based curriculum materials for physics courses that bridge the gap between middle-division mathematics and upper-division physics courses, this work attempts to shed light on student thinking about vectors in Cartesian and Non-Cartesian coordinate systems through the use of a Resources Framework [Hammer, Elby, Scherr, and Redish, 2005]. The research presented explores the thinking of a single student—Mark–as revealed during a semi-structured, one-on-one interview, and identifies the student's thinking as activated conceptual and procedural resources. Mark is a high-achieving, senior-level physics major, and was selected for detailed analysis due to the clear explanation of his thinking. From Mark's data, identified resources can be grouped into clusters based on the content of the student's thinking [Vega, et al., 2016]. Examples of such clusters are basis unit vectors, position vectors, and velocity vectors. Connections between these resources are also observed, allowing a mapping of the student's thinking as the student works through a series of questions.

In this poster, we present a map of Mark's activated resources and, based on the data, the connections between those resources to understand Mark's complicated thought process. This detailed analysis provides insight into the kinds of ideas undergraduates might activate when faced with questions about unit vectors and position vectors, in Cartesian and non-Cartesian coordinate systems. For instance, Mark initially demonstrates a clear understanding of unit vectors in polar coordinates. He notes that unit vectors are of unit length, and point in the direction of increasing coordinate, in this case, increasing r and theta. These are both identified as specific resources within the unit vector cluster. Later in the interview, Mark activates additional resources that conflict with these unit vector resources, namely, that the unit vectors of r and theta can be written in terms of the unit vectors of Cartesian coordinates. This conflict demonstrates a potential area of interest for curriculum development.

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Development of Reasoning about Rate of Change, Based on Quantitative and Qualitative Analysis

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*Pre-calculus and Calculus are two big compartments as we consider their developmental and complemental attribute. I analyzed data quantitatively from a series of pre-calculus assessments conducted in a large public university 2017 fall, then investigated the result and its impacts qualitatively in calculus context focused on rate of change. The two-part analysis consists of discerning intrinsic factors in the assessment items that have a large effect on overall performance followed by clinical interviews about meaning of the Fundamental Theorem of Calculus and its applications. The results support my claim that the ability to conceptualize constant rate of change has a considerably positive effect on students' reasoning about rate of change and the Fundamental Theorem of Calculus as well.*

*Keywords:* Quantitative Analysis, Constant Rate of Change, Rate of Change, Fundamental Theorem of Calculus

Quantitative analysis on pre-calculus assessment with reform curricula (Carlson, Oehrtman et al. 2013) shows that 34 items in the assessment have 11 principal components by factor analysis and each item assesses students' understanding independently by regression analysis. I focused on items in the first component since they have a huge impact on students' overall performance, coding them with $R_i$(reasoning abilities), $F_i$(understandings of various function types), $U_i$(understandings of various concepts), and $A_i$(other abilities) suggested from CCR (The Calculus Concept Readiness) taxonomy (Carlson, Madison et al. 2015). It turns out that the combination of R3 (Quantitative and covariational Reasoning), U3(Constant rate of change) and A4(Understand and use function notation to express one quantity in terms of another) becomes a critical factor to students' conceptualization in terms of their further learning.

The result mentioned above and my experiences of teaching pre-calculus and working with DIRACC (Developing and Investigating A Rigorous Approach to Conceptual Calculus) project led me to ponder on what would be the most problematic part for students as connecting the two compartments of pre-calculus and calculus. I hypothesize that student struggles stem from fragmented interpretations on constant rate of change and the fragmented knowledge hinders comprehensive understanding of rate of change and the Fundamental Theorem of Calculus. Accordingly, I elaborately devised open-ended interview items so that each item could reflect developmental aspects by inquiring on constant rate of change, rate of change, net change, and meaning of fundamental theorem of calculus.

Qualitative analysis on the interview shows that the way of students' reasoning on constant rate of change has different layers in various contexts of graphical, verbal, physical, and symbolic representations based on the framework for the concept of derivative (Zandieth, M. 2000). Also, it supports that each layer becomes distinct pivots of their interpretation on rate of change, having a large effect on the way of reasoning the Fundamental Theorem of Calculus as a continuum of rate of change. I believe that the results from my two-part study will be of interest to the mathematic education community because students' conceptualization on rate of change will be a foundation to the next step of learning.

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If $f(2) = 8$ then $f'(2) = 0$: A Common Misconception, Part 2

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*This study reports calculus students' failure to differentiate the cubing function when represented piecewise: $f(x) = x^3$ if $x \neq 2$, $f(x) = 8$ if $x = 2$. The data reported here suggest that students did not fail simply due to inattention to the function definition; when reminded that 2 cubed is 8 and prompted to compare the graph of f to that of the cubing function, student performance did not increase in a statistically significant way, suggesting the presence of deep-seated misunderstandings.*

*Keywords:* function, derivative, calculus, representation

Harel and Kaput (1991) observed a troubling phenomenon among calculus students. Students claimed that for $g(x) = sin(x)$ if $x \neq 0$, $g(x) = 1$ if $x = 0$ , then $g'(0) = 0$ (due to the constant rule). To these students, the only aspect of the representation as relevant for determining the value of $g'(0)$ is the second line of the piecewise function definition. It seems reasonable to believe that if the definition of $g$ were modified to instead have $g(x) = 0$ if $x = 0$ (resulting in a nonstandard representation of the sine function) students would answer identically. However, given the anecdotal nature of Harel and Kaput's claim, there is no data available to substantiate how common such errors are or why they occur. A study I presented at RUME 2017 addresses how common this sort of error is (Mirin, 2017) by discussing student performance on the task of evaluating $f'(2)$ when $f(x) = x^3$ if $x \neq 2$, $f(x) = 8$ if $x = 2$. The poster describes a follow-up investigation that begins to address why students perform so poorly at that task (henceforth called "The Task").

A participant at RUME 2017 observed that the piecewise definition of *f* was rather contrived and therefore students might simply assume that *f* is a discontinuous function without realizing that *f* and the cubing function agree on $x = 2$. In the original study, several students provided a graph with a single point discontinuity and answered in a way consistent with presuming that 2 cubed is not 8. Hence, it seems that inattention, rather than a major conceptual misunderstanding, was at fault in *some* students' responses. Utilizing the data from the 2017 open-ended version, I re-administered The Task in multiple choice form in 2018, first prompting students to calculate $2^3$ , and then, next to a graph of the cubing function, provide a graph of *f*.

The data reveal no evidence to support that inattention could account for student responses. Although there was a slight improvement in correctness rate from 2017 to 2018, this improvement was not statistically significant ($\chi^2$=1.21, p>.05). In other words, prompting students to compare the graph of $y = f(x)$ to that of $y = x^3$ *did not* appear to cause improvement, suggesting that students did not err simply due to inattention to the function's graph. Moreover, the students in 2018 who answered "12" were no more likely than the students in 2017 who answered "12" to draw an explicit comparison between *f* and the cubing function (4.2% of year-2017 students who answered 12 did, whereas only 2.9% of year-2018 students did so).

These results suggest that prompting students to compare *f* to the cubing function did not appear to encourage them to infer that *f* and the cubing function share a derivative at 2.

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**Cooperative Learning and its Impact in Developmental Mathematics Courses: A Case Study in a Minority-Serving Institution**

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*In this poster, we report on the evolution of developmental students' mathematics background knowledge after a four-week long course that emphasized active learning. The research took place at a large Hispanic serving institution in the state of California. Students' progress or lack thereof was measured using a diagnostic test developed by the Mathematics Diagnostic Testing Project (MDTP). These students were initially considered not ready for college level mathematics course work and were subsequently enrolled, in a four-week summer course which is designed to prepare them for a college level math course. During each class, students would spend at least 30 minutes engaging in cooperative learning that utilizes active learning strategies such as think-pair share, peer lesson, and wait time. A pre/posttest analysis of SYART showed that these students showed a statistically significant growth, leading us to conclude that the four-week intervention in math remediation had a considerable impact.*

*Key words:* Developmental Mathematics, Cooperative Learning, College Algebra.

Far too many students began their postsecondary mathematics education in remedial mathematics (Bailey, 2009, Schwartz, 2007). Within the California State University (CSU) System, approximately a third of incoming freshmen are considered unprepared for college level mathematics courses (CSU, 2012). For some CSU students, a year will elapse before they can enroll in a college level mathematics course. Across the CSUs, unless exempted, every admitted student is required to take the Entry Level Mathematics (ELM) test, which aims to measure proficiency in basic skills need to succeed in a college level mathematics course. 50 on the ELM is a cutoff score that determines whether a student needs mathematics remediation or not.

It is important to note that the ELM is not a diagnostic test; as such, it does not shed light on specific contents that students are struggling with. To that end, in this study, we used Second Year Algebra Readiness Test (SYART) to understand the mathematical background knowledge of 1100 students who received a score below 50 on their ELM test. These students were enrolled in a two, four-week courses designed to prepare them for a college math course: beginning algebra and intermediate algebra. In both classes, students met their instructor five times a week, and every class, except exam days, they would spend approximately 30 minutes in cooperative learning that utilizes active learning strategies such as *think-pair share, peer lesson,* and *wait time*. A pre/posttest analysis of SYART showed that students' overall score improved significantly. On average, beginning algebra students' SYART score improved by approximately 39.5%. Using a two-sample t-test, session one witnessed a statistically significant growth with a *p-value of* $3.3 * 10^{-63}$.

To summarize, students improved their performance in several topics of the test. However, the biggest growth were observed in the following topics: Exponents and square roots; Scientific notation, Linear equations and inequalities, Polynomial and quadratic equations. However, several students were still below a critical level in some topics. Specifically, students continue to struggle in graphical representation of solution of equations and inequalities. Still, there is a strong evidence to conclude that the four-week intervention in math remediation has a considerable impact.

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